Φροντιστήριο 1 ΦΥΣ112

11/9/2024

1) Βρείτε τα ολοκληρώματα: a) $\int_{-a}^{a} \frac{dx}{(x^2+d^2)^{1/2}}$, b) $\int_{-a}^{a} \frac{dx}{(x^2+d^2)^{3/2}}$, c) $\int_{-a}^{a} \frac{xdx}{(x^2+d^2)^{3/2}}$.

2) Να δείξετε ότι: a) $\frac{1}{1+a} = 1 - a + \mathcal{O}(a^2)$, b) $\frac{1}{\sqrt{1+a}} = 1 - \frac{a}{2} + \mathcal{O}(a^2)$, για $a \to 0$.

3)Έστω $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$. Βρείτε τις παραγώγους: a) $\frac{\partial}{\partial x} r$, b) $\frac{\partial}{\partial x} r^2$, c) $\frac{\partial}{\partial x} \frac{\hat{r}}{r^2}$.

4)Έστω τα διανύσματα: $\vec{r}_1 = 5\hat{x} - 4\hat{y} + 3\hat{k}$, $\vec{r}_2 = 1\hat{x} - 3\hat{y} + 2\hat{k}$. a)Βρείτε την γωνία μεταξύ των δύο διανυσμάτων. b)Βρείτε όλα τα διανύσματα που είναι παράλληλα με το διάνυσμα $\vec{r}_1$.

5)Έστω $\vec{r} = x\hat{x} + y\hat{y}$. Βρείτε τα ολοκληρώματα: a) $\int_{-a}^{a} \frac{dx}{r}$, b) $\int_{-a}^{a} \frac{\hat{r}dx}{r^2}$, c) $\int_{0}^{a} \frac{\hat{r}dx}{r^2}$.

6)Να δείξετε ότι: $\frac{\partial}{\partial y} \int_{-a}^{a} \frac{dx}{r} = -\hat{y} . \int_{-a}^{a} \frac{\hat{r}dx}{r^2}$

1.a) $$\int_{-a}^{a}\frac{dx}{\sqrt{x^2+d^2}} = \int_{-\theta_a}^{\theta_a}\frac{d\sec^2\theta d\theta}{\sqrt{d^2\tan^2\theta+d^2}} = \int_{-\theta_a}^{\theta_a}\frac{\sec^2\theta d\theta}{|\sec\theta|} = \ln|\sec\theta_a+\tan\theta_a|\Big|_{-\theta_a}^{\theta_a}$$

$x = d\tan\theta$

$\Rightarrow dx = d\sec^2\theta d\theta$

$a = d\tan\theta_a$

$$= \ln\left(\frac{\sec\theta_a+\tan\theta_a}{\sec\theta_a-\tan\theta_a}\right) = \ln\left(\frac{\frac{\sqrt{a^2+d^2}}{d}+\frac{a}{d}}{\frac{\sqrt{a^2+d^2}}{d}-\frac{a}{d}}\right)$$

$\tan\theta_a = \frac{a}{d} \Rightarrow \cos\theta_a = \frac{d}{\sqrt{a^2+d^2}}$

$$= \ln\left(\frac{\sqrt{a^2+d^2}+a}{\sqrt{a^2+d^2}-a}\right)$$

$\Rightarrow \sec\theta_a = \frac{\sqrt{a^2+d^2}}{d}$

1.b) $$\int_{-a}^{a}\frac{dx}{(x^2+d^2)^{3/2}} = \int_{-\theta_a}^{\theta_a}\frac{d\sec^2\theta d\theta}{(d^2\tan^2\theta+d^2)^{3/2}} = \int_{-\theta_a}^{\theta_a}\frac{d\sec^2\theta d\theta}{d^3\sec^3\theta d\theta}$$

$x = d\tan\theta$

$dx = d\sec^2\theta d\theta$

$$=\frac{1}{d^2}\int_{-\theta_a}^{\theta_a}\cos\theta d\theta = \frac{1}{d^2}(\sin\theta_a - \sin(-\theta_a))$$

$\tan\theta_a = \frac{a}{d}$, $\cos\theta_a = \frac{d}{\sqrt{a^2+d^2}}$

$$= \frac{2}{d^2}\sin\theta_a = \frac{2a}{d^2\sqrt{a^2+d^2}}$$

$\sin\theta_a = \frac{a}{\sqrt{a^2+d^2}}$

1.c) $$\int_{-a}^{a}\frac{x dx}{(x^2+d^2)^{3/2}} = -\frac{1}{\sqrt{x^2+d^2}}\Big|_{-a}^{a} = -\frac{1}{\sqrt{a^2+d^2}} + \frac{1}{\sqrt{a^2+d^2}} = 0$$

2a) $$1-a = \frac{1+a}{1+a} - a = \frac{1}{1+a} + \frac{a}{1+a} - a = \frac{1}{1+a} + \frac{a-a+a^2}{1+a} = \frac{1}{1+a} + \frac{a^2}{1+a}$$

$$= \frac{1}{1+a} + O(a^2)$$

2.b) $$\frac{1}{\sqrt{1+a}} = \frac{1}{\sqrt{1+a+\frac{a^2}{4}-\frac{a^2}{4}}} = \frac{1}{\sqrt{(1+\frac{a}{2})^2-\frac{a^2}{4}}} = \frac{1}{(1+\frac{a}{2})\sqrt{1-\frac{a^2}{4(1+\frac{a}{2})^2}}}$$

$$= \frac{1-\frac{a}{2}+O(a^2)}{\sqrt{1-\frac{a^2}{4(1+\frac{a}{2})^2}}} = 1-\frac{a}{2}+O(a^2)$$

Σημ: $f(a) = 1+a+O(a^2) \Leftrightarrow \lim_{a\to 0}\frac{f(a)-1}{a} = 1.$

3.a) $r = (x^2+y^2+z^2)^{1/2}$

$$\frac{\partial}{\partial x} r = \frac{x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{r}$$

3.b) $\frac{\partial}{\partial x} r^2 = 2x$

3.c) $$\frac{\partial}{\partial x} \frac{\hat{x}}{r^2} = \frac{\partial}{\partial x} \frac{\vec{r}}{r^3} = \frac{\partial}{\partial x} \frac{r_x \hat{x}}{r^3} + \frac{\partial}{\partial x} \frac{r_y \hat{y}}{r^3} + \frac{\partial}{\partial x} \frac{r_z \hat{z}}{r^3}$$

$$= \frac{\partial}{\partial x} \frac{x}{(x^2+y^2+z^2)^{3/2}} \hat{x} + \frac{\partial}{\partial x} \frac{y\hat{y} + z\hat{z}}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{1 \hat{x}}{(x^2+y^2+z^2)^{3/2}} + \frac{-3x^2 \hat{x}}{(x^2+y^2+z^2)^{5/2}} - \frac{3x}{(x^2+y^2+z^2)^{5/2}} \cdot (y\hat{y} + z\hat{z})$$

$$= \frac{\hat{x}}{r^3} - \frac{3x\vec{r}}{r^5}$$

4.a) $\vec{r_1} \cdot \vec{r_2} = r_1 r_2 \cos\theta \Rightarrow \cos\theta = \frac{\vec{r_1} \cdot \vec{r_2}}{r_1 r_2} = \frac{5.1 + 4.3 + 3.2}{5\sqrt{2} \cdot \sqrt{14}} = \frac{23}{10\sqrt{7}} \Rightarrow \theta = 29{,}6°$

$r_1 = \sqrt{\vec{r_1} \cdot \vec{r_1}} = \sqrt{25+16+4} = 5\sqrt{2}$

$r_2 = \sqrt{\vec{r_2} \cdot \vec{r_2}} = \sqrt{1+9+4} = \sqrt{14}$

4.b) $\vec{r_3} \parallel \vec{r_1} \Rightarrow \vec{r_3} = \lambda \vec{r_1}, \ \lambda \in \mathbb{R}.$

$$\cos\theta = \frac{\vec{r_1} \cdot \vec{r_3}}{r_1 r_3} = \frac{\vec{r_1} \cdot \lambda \vec{r_1}}{r_1 \cdot \lambda r_1} = \frac{\lambda r_1^2}{\lambda r_1^2} = 1 \Rightarrow \theta = 0$$ ✓

③

5.a) $\int_{-a}^{a} \frac{dx}{r} = \int_{-a}^{a} \frac{dx}{\sqrt{x^2+y^2}} \overset{(1.a)}{=} \ln\left(\frac{\sqrt{a^2+y^2}+a}{\sqrt{a^2+y^2}-a}\right)$

5.b) $\int_{-a}^{a} \frac{\hat{r}\,dx}{r^2} = \int_{-a}^{a} \frac{\vec{r}}{r^3}\,dx = \int_{-a}^{a} \frac{x\,dx}{(x^2+y^2)^{3/2}}\,\hat{x} + \int_{-a}^{a} \frac{y\,dx}{(x^2+y^2)^{3/2}}\,\hat{y}$

$\overset{(1.c)\ (1.b)}{=} 0 + y \cdot \frac{2a}{y^2\sqrt{a^2+y^2}}\,\hat{y} = \frac{2a}{y\sqrt{a^2+y^2}}\,\hat{y}$

5.c) $\int_{0}^{a} \frac{\hat{r}\,dx}{r^2} = \int_{0}^{a} \frac{x\,dx}{(x^2+y^2)^{3/2}}\,\hat{x} + \int_{0}^{a} \frac{y\,dx}{(x^2+y^2)^{3/2}}\,\hat{y}$

$\overset{(1.c)}{=} -\frac{1}{\sqrt{x^2+y^2}}\Big|_0^a\,\hat{x} + \frac{1}{2}\int_{-a}^{a} \frac{y\,dx}{(x^2+y^2)^{3/2}}\,\hat{y} \overset{(5.b)}{=} \left(\frac{1}{y} - \frac{1}{\sqrt{a^2+y^2}}\right)\hat{x} + \frac{a}{y\sqrt{a^2+y^2}}\,y^2$

6.a) $\frac{\partial}{\partial y}\int_{-a}^{a} \frac{dx}{r} = \frac{\partial}{\partial y}\left(\ln\left(\sqrt{a^2+y^2}+a\right) - \ln\left(\sqrt{a^2+y^2}-a\right)\right)$

$= \frac{y}{\sqrt{a^2+y^2}\left(\sqrt{a^2+y^2}+a\right)} - \frac{y}{\sqrt{a^2+y^2}\left(\sqrt{a^2+y^2}-a\right)} = \frac{\cancel{y}}{\sqrt{a^2+y^2}} \cdot \frac{\sqrt{a^2+y^2}-a-\sqrt{a^2+y^2}-a}{\cancel{a^2}+y^{\cancel{2}}\cancel{-a^2}}$

$= -\frac{2a}{y\sqrt{a^2+y^2}}$

$-\hat{y}\int_{-a}^{a} \frac{\hat{r}\,dx}{r^2} = -\frac{2a}{y\sqrt{a^2+y^2}}$ ✓