
The Lottery Ticket Hypothesis: Finding Small, Trainable Neural Networks

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Abstract

Neural network compression techniques are able to reduce the parameter counts of trained networks by over 90%—decreasing storage requirements and improving inference performance—without compromising accuracy. However, contemporary experience is that it is difficult to train small architectures from scratch, which would similarly improve training performance.

We articulate a new conjecture to explain why it is easier to train large networks: the *lottery ticket hypothesis*. It states that large networks that train successfully contain subnetworks that—when trained in isolation—converge in a comparable number of iterations to comparable accuracy. These subnetworks, which we term *winning tickets*, have won the initialization lottery: their connections have initial weights that make training particularly effective.

We find that a standard technique for pruning unnecessary network weights naturally uncovers a subnetwork which, at the start of training, comprised a winning ticket. We present an algorithm to identify winning tickets and a series of experiments that support the lottery ticket hypothesis. We consistently find winning tickets that are less than 20% of the size of several fully-connected, convolutional, and residual architectures for MNIST and CIFAR10. Furthermore, winning tickets at moderate levels of pruning (20-50% of the original network size) converge up to 6.7x faster than the original network and exhibit higher test accuracy.

1 Introduction

Neural networks tend to be dramatically over-parameterized [51]. Techniques for eliminating unnecessary weights (*pruning*) [31, 18, 17, 38, 34] and training small networks to mimic large ones (*distillation*) [2, 21]) demonstrate that the number of parameters can be reduced by more than 90% while maintaining accuracy. Doing so diminishes the size [17, 21] or energy consumption [50, 33, 39, 36] of trained networks, making inference more efficient. If a network can be so compressed, then the function it learned can be represented by a far smaller network than that used during training. Why, then, do we train large networks when we could improve efficiency by training smaller networks instead? In practice, large networks are easier to train from the start than small ones [17, 5, 21, 51].

In this paper, we add to the body of evidence and theory about why large networks are easier to train by articulating the *lottery ticket hypothesis*. It states that any large network that trains successfully contains a subnetwork that is initialized such that—when trained in isolation—it can match the accuracy of the original network in at most the same number of training iterations.

We designate these subnetworks *winning tickets* since they have won the initialization lottery with a combination of weights and connections capable of training. We find that a standard pruning technique [17] automatically uncovers winning tickets. When randomly reinitialized, the winning tickets that we discover no longer match the performance of the original network, suggesting the importance of the original initialization.

Returning to our motivating question, we extend the lottery ticket hypothesis into the conjecture (that we do not empirically test) that large networks are easier to train because, when randomly initialized, they have more combinations of subnetworks from which training can recover a winning ticket.

Methodology. To evaluate the lottery ticket hypothesis experimentally, we identify winning tickets in large networks by training and subsequently pruning the smallest-magnitude weights [17]. The set of connections that survives this pruning process is the architecture of a winning ticket. Unique to our work, the winning ticket’s weights are the values to which these connections were initialized *before training began*. This forms our central experiment:

1. Randomly initialize a neural network.
2. Train the network until it converges.
3. Prune a fraction of the network.
4. To extract the winning ticket, reset the weights of the remaining portion of the network to their values from (1)—the initializations they received before training began.

If large networks contain winning tickets and pruning reveals them, then the lottery ticket hypothesis predicts that the pruned network—when reset to the original initializations—will maintain competitive accuracy and convergence times at sizes too small for a randomly-initialized or a randomly-configured network to do the same.

Research questions and results. In this paper, we investigate the following questions:

How does the lottery ticket hypothesis manifest for different networks? We identify winning tickets in a fully-connected architecture for MNIST and convolutional and residual architectures for CIFAR10 [29], including networks with dropout [48], Xavier initialization [12], and batch normalization [26]. On a resnet trained with momentum, we could not find a winning ticket.

How large are winning tickets? When we find winning tickets, they are 20% (or less) of the size of the original network and match its convergence times and accuracy.

How effectively do winning tickets train compared to the original network? Winning tickets pruned by 50% to 70% converge in 1.2x-6.7x fewer iterations while surpassing the original network’s accuracy.

What is the structure and initialization of a winning ticket? When randomly reinitialized or rearranged, winning tickets perform far worse than the original network, meaning neither structure nor initialization alone is responsible for a winning ticket’s success. Fully-connected winning ticket initializations are the extremes of the truncated normal distribution from which they were sampled.

Contributions.

- We propose the *lottery ticket hypothesis* as a new perspective on neural network training.
- We demonstrate that pruning uncovers the winning tickets predicted by the lottery ticket hypothesis, making it possible to extract small, trainable networks from larger networks.
- We apply this technique to empirically evaluate the lottery ticket hypothesis on fully-connected, convolutional, and residual networks. The evidence we find supports both the lottery ticket hypothesis and our contention that pruning can extract winning tickets.
- We show that winning tickets at moderate levels of pruning converge in fewer iterations and reach higher accuracy than the original network.

Implications. In this paper, we mainly measure the lottery ticket hypothesis, which requires repeatedly training and pruning a network. Now that we have demonstrated the existence of winning tickets, we hope to design training schemes that exploit this knowledge. Since winning tickets can be trained from the start in isolation, it would be beneficial to identify them as early as possible in the training process or to integrate lessons from winning tickets directly into the design of new architectures and initialization schemes. Techniques that prune networks during training [35, 38, 41] might already identify winning tickets and could benefit from making this an explicit goal.

Network	lenet [30]	conv2 [30]	conv4 [43, 6]	resnet18 [19]
			3x3x64, 3x3x64	3x3x16, 3x[3x3x16, 3x3x16]
Convolutions		5x5x64, pool 5x5x64, pool	pool, 3x3x128 3x3x128, pool	3x[3x3x32, 3x3x32] 3x[3x3x64, 3x3x64]
FC Layers	300, 100, 10	384, 192, 10	256, 256, 10	avg-pool, 10
All/Conv Weights	266K	1.75M / 107K	1.15M / 260K	271K / 270K
Init. / Epochs	$\mathcal{N}(0, 0.1)$ / 50	$\mathcal{N}(0, 0.1)$ / 30	Normal Xavier / 20	Normal Xavier / 85
Optimizer	SGD(0.05)	SGD(0.01)	SGD(0.003)	SGD(0.1, 0.01, 0.001)

Figure 1: Architectures tested in Sections 2, 3, 4, and 5. Square brackets are residual layers. Normal distributions are truncated at two standard deviations from the mean.

2 Winning Tickets in Fully-Connected Networks

In this section, we assess the lottery ticket hypothesis as applied to fully-connected networks trained on MNIST. We use the lenet-300-100 architecture [30] as described in Figure 1. We follow the outline from Section 1: after randomly initializing and training a network, we prune the network and reset the remaining connections to their original initializations. We use a simple pruning heuristic: remove a percentage of the weights with the lowest magnitudes within each layer (as in [17]). Connections to outputs are pruned at half of the rate of the rest of the network to avoid severing connectivity.

We test two pruning strategies: *one-shot pruning* and *iterative pruning*. One-shot pruning prunes all at once in a single step after training. Iterative pruning repeatedly trains, prunes, and resets the weights, removing more of the network on each iteration of the process.

2.1 Results

One-shot pruning. Winning tickets found via one-shot pruning converge faster than the original network. Figure 2 plots the test set accuracy and convergence behavior during training of winning tickets pruned to different levels.¹ Each curve is the average of five runs starting from randomly initialized networks; error bars are the minimum and maximum of any run. A dot with error bars shows the average, minimum, and maximum convergence time for the curve in the same color.

For the first few pruning steps, convergence times decrease and accuracy increases (left graph in Figure 2). A winning ticket comprising 90% of the weights from the original network converges faster than the original network but slower than when pruned to 75%. This pattern continues until

¹We define convergence as the moment at which the 1000-iteration moving average of test loss changed by less than 0.001 for 1000 consecutive iterations. We measured test accuracy every 100 iterations. Measuring convergence is an imprecise art, but this metric seems to adequately characterize the behavior for our purposes.

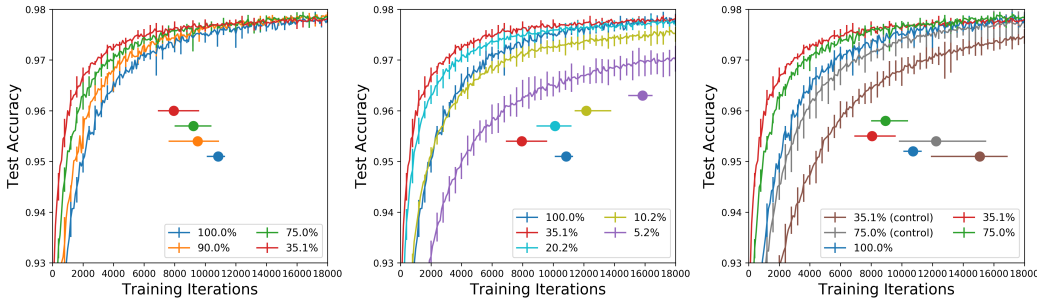


Figure 2: Test accuracy on the fully-connected network (one-shot pruning) as training proceeds. Each curve is the average of five trials. Percents are the fraction of weights remaining in each layer after pruning. Error bars are the minimum and maximum of any trial. Dots are the time when the corresponding colored curve converges; error bars are the earliest and latest convergence times.

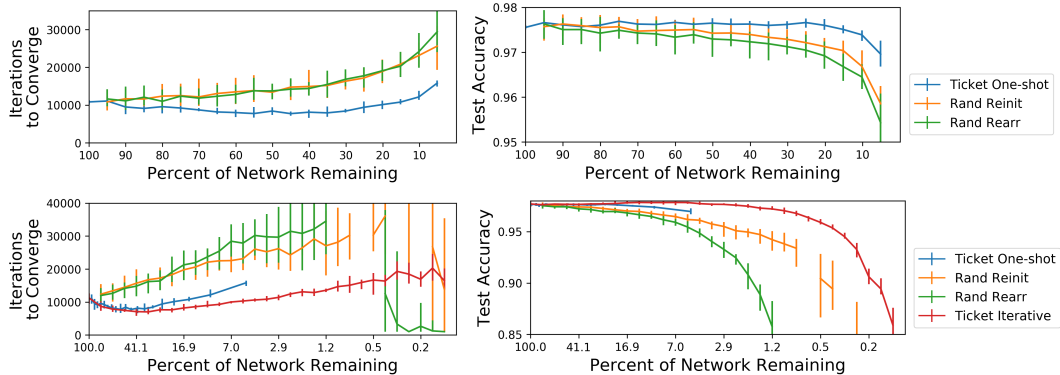


Figure 3: Convergence behavior and accuracy of the fully-connected network under one-shot (top) and iterative (bottom) pruning. Error bars are the minimum and maximum value any trial took on.

55%, after which convergence times flatten and, after 35%, increase (middle graph). When pruned to between 10% and 15%, a winning ticket regresses to the performance of the original network.

The top of Figure 3 summarizes this behavior for all pruning levels. On the left are convergence times in relation to pruning level; on the right is accuracy at convergence. When pruned to between 70% and 35%, the average winning tickets converge at least 22% faster and are more accurate than the original network. Further pruning causes convergence times and accuracy to degrade.

Iterative pruning. Figure 3 (bottom) shows the results of iteratively pruning by 20% per iteration (red). One-shot data from Figure 3 (top) is reproduced in blue. Iteratively-pruned winning tickets converge faster and reach higher accuracy at smaller network sizes than one-shot pruned networks. Convergence times flatten when pruned down to between 41% (38% faster than the original network) and 21% (33% faster). The average winning ticket returns to the original convergence time when pruned to 2.9% and the original accuracy when pruned to 3.6%.

Although iterative pruning extracts smaller winning tickets, repeated training means they are costlier to find. However, we aim to analyze the behavior of winning tickets rather than to find them efficiently. Iterative pruning’s advantage is that it puts a tighter upper-bound on the size of a winning ticket.

2.2 Control Experiments

The lottery ticket hypothesis predicts that winning tickets train effectively due to a combination of initialization and structure. To compare the importance of these factors, we ran two control experiments: (1) retain structure but randomize initializations and (2) retain initializations but randomly rearrange structure. We trained three controls for each winning ticket (15 in total). We find that structure seems more important, but neither factor alone explains the efficacy of a winning ticket.

Random reinitialization. This experiment evaluates the importance of the winning ticket’s initialization by reusing its structure but randomly reinitializing its weights from the original distribution. The right graph in Figure 2 shows this experiment for one-shot pruning. In addition to the original network and winning tickets at 75% and 35%, it has two curves for the controls. The broader results of this experiment are the orange lines in Figure 3. Unlike winning tickets, the controls converge increasingly slower than the original network and lose accuracy after little pruning. The average iterative control’s accuracy drops off from the original accuracy when the network is pruned to about 51%, compared to 3.6% for the winning ticket. This experiment supports the lottery ticket hypothesis’ emphasis on initialization: the original initialization withstood and benefited from pruning, while the random reinitialization’s performance immediately suffered and diminished steadily.

Random rearrangement. This experiment evaluates the importance of structure by randomizing the locations of a winning ticket’s connections within each layer while retaining the original initializations (green lines in Figure 3). The rearranged networks perform slightly worse than the previous experiment—convergence times increase even faster and accuracy drops off earlier—suggesting that

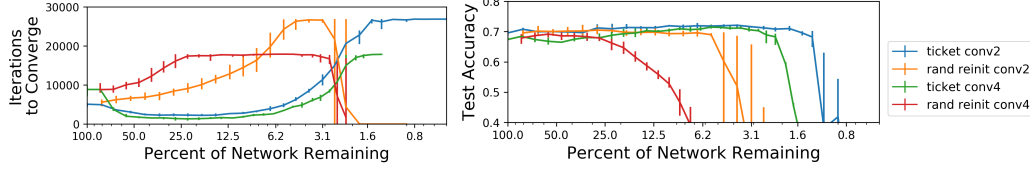


Figure 4: Convergence times and accuracy for Questions 1 and 2 (pruning and random reinitialization).

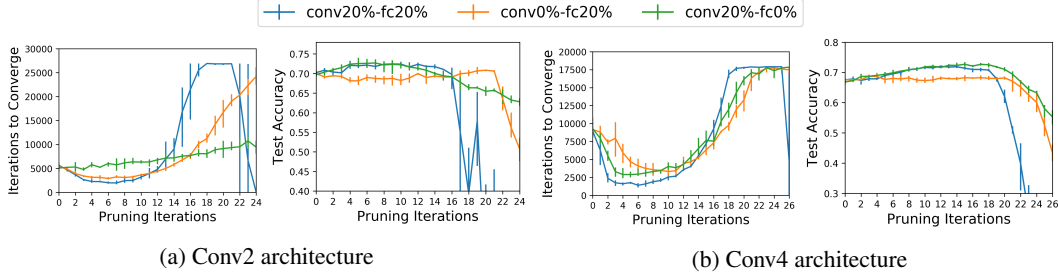


Figure 5: Convergence times and accuracy for Question 3 (pruning convolutions vs. fully-connected).

structure is more important than initialization. However, in general, the control experiments indicate that winning tickets emerge from a combination of initialization and structure; neither initialization nor structure alone explained the better performance of winning tickets.

3 Winning Tickets in Convolutional Networks

In this section, we apply the lottery ticket hypothesis to convolutional networks. We consider the middle two architectures in Figure 1. Conv2 is a variation of lenet-5 [30] with two convolutional layers and three fully-connected layers. Conv4, which derives from [43] and [6], has four convolutional layers and three fully-connected layers. We investigate three questions:²

Question 1: How does the lottery ticket hypothesis manifest for convolutional networks?

Question 2: What is the effect of randomly reinitializing winning tickets?

Question 3: What is the effect of pruning convolutions and fully-connected layers alone and together?

Question 1. We trained and iteratively pruned the conv2 and conv4 architectures with a 20% pruning rate for fully-connected layers and a 10% pruning rate for convolutional layers; on a holdout validation set, these hyperparameters allowed the network to maintain accuracy and low convergence times at a smaller size. The results of doing so are the blue and green lines in Figure 4. The pattern from Section 2 repeats: as the network is pruned, convergence times drop and accuracy rises as compared to the original network. In this case, the results are more pronounced. Winning tickets converge 2.28x faster for conv2 at 15% pruning and 6.7x faster for conv4 at 22.7% pruning.

Question 2. We repeated the random reinitialization trial from Section 2 (Figure 4 in orange and red). The controls again take longer to converge upon continued pruning. The control networks initially match the accuracy of the original network; accuracy drops off precipitously when 5.5% (conv2) and 22.7% (conv4—convergence takes longer than allotted training time) of weights remain. The corresponding winning tickets maintain their accuracy until 1.2% (conv2) and 1.9% (conv4).

Question 3. We pruned the conv2 and conv4 architectures with a 20% pruning rate for (1) convolutions, (2) fully-connected layers, and (3) both. Figure 5 shows the results. Pruning convolutions in conv2 has substantial impact on accuracy but less impact on convergence times; the reverse is true for fully-connected layers. When both are pruned in concert, the overall network seems to inherit

²These experiments use iterative pruning so as to find the smallest winning tickets possible. Convergence was defined as the iteration of minimum validation/test loss. Mini-batches contained 50 samples.

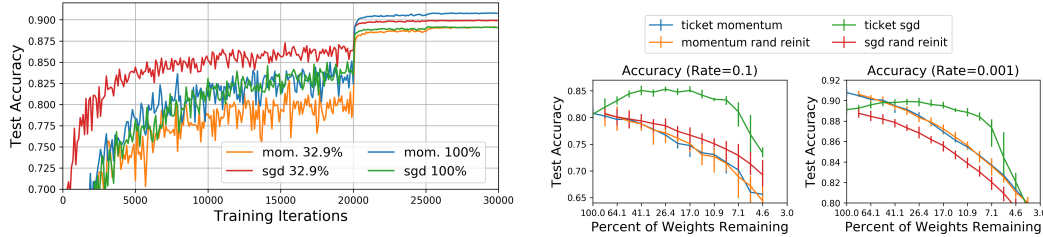


Figure 6(a): Resnet test accuracy as training proceeds. Percents are the fraction of weights remaining. Error bars are elided for legibility. Figure 6(b): Resnet test accuracy averaged over the last 1,000 iterations of using the specified learning rate.

both sets of benefits. Conv4 follows a similar pattern for accuracy. When pruned to between 16.2% and 10.0%, average accuracy increases by more than 4 percentage points. Pruning the convolutional layers alone to 7.3% causes average accuracy to rise by 5.8 percentage points.

4 Winning Tickets in Residual Networks

Here, we investigate the lottery ticket hypothesis as applied to residual networks, specifically resnet18 [19] (see Figure 1) on CIFAR10. We match the training conditions of [19], with mini-batch size 128, augmented data (random flips, translations), batch normalization, weight decay (0.0001), and momentum (0.9). We follow a similar learning rate schedule: 0.1 for the first 20,000 iterations, 0.01 until iteration 25,000, and 0.001 to iteration 30,000. In this configuration, pruning does not find winning tickets. Figure 6a, which plots test accuracy as the networks train, shows that the network pruned to 32.9% converges more slowly to lower accuracy than the original. Figure 6b (right) summarizes accuracy across all iterative winning ticket sizes. Pruning never improves the accuracy of the network; its accuracy is no better than that of the randomly-reinitialized control trials.

However, when we use SGD without momentum, the usual lottery ticket pattern reemerges. The average unpruned SGD-trained network’s accuracy is 89.1% as compared to 90.8% for the momentum-trained network. However, when pruned to 32.9%, the average SGD-trained network initially learns faster (Figure 6a) and reaches 90% accuracy. Winning tickets pruned by up to 83% surpass the accuracy of the unpruned SGD-trained network; randomly-reinitialized controls follow the usual pattern. Interestingly, the SGD-trained winning tickets outperform the momentum-trained network by up to 4.5 percentage points (26.4% pruning) before the first learning rate change (Figure 6b left).

These results demonstrate that resnets can contain winning tickets. They also imply a nuanced relationship between training strategies and our technique for finding winning tickets. It is possible that training with momentum created a network for which pruning failed to find winning tickets, or that the winning tickets it found could not train productively with momentum. Likewise, it is possible that, had we custom-developed a learning schedule for the SGD-trained winning tickets, they might have outperformed the momentum-trained network for the entirety of training.

5 Winning Tickets and Dropout

Dropout [48, 22] is a technique that improves network accuracy by randomly disabling a fraction of the units (i.e., randomly sampling a subnetwork) on each training iteration. Baldi and Sadowski [3] characterize dropout as simultaneously training the ensemble of all possible subnetworks. Since the lottery ticket hypothesis suggests that one of these subnetworks comprises a winning ticket, it is natural to ask whether dropout and our strategy for finding winning tickets interact.

Figure 7 shows the result of training the conv4 network with a dropout rate of 0.5 (orange). The blue line is the baseline performance for conv4 (the same as the green line in Figure 4). When training with dropout, we continue to identify winning tickets that converge faster than the unpruned network (2.52x when pruned to 19.1%). Moreover, these winning tickets reach higher accuracy; a winning ticket pruned to 13.8% bests the accuracy of the unpruned dropout network by 5.5 percentage points and the unpruned network without dropout by 9.6 percentage points.

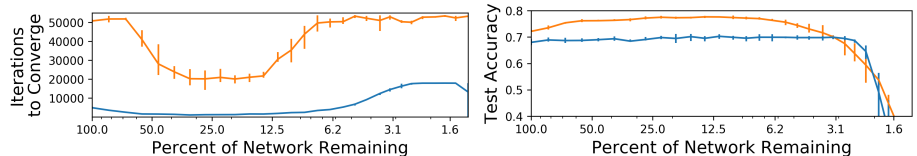


Figure 7: Convergence times and accuracies for dropout on conv4. + dropout + conv4

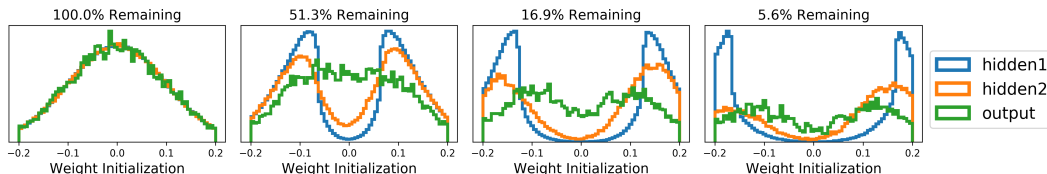


Figure 8: Histogram (normalized) of remaining initializations for each layer at four pruning levels.

These improvements suggest that the lottery ticket hypothesis continues to hold in the context of dropout and, moreover, that the lottery ticket hypothesis and dropout may interact in a complementary way. For example, Srivastava et al. [48] observed that dropout induces sparse activations in the final network; it is possible that dropout-induced sparsity primes a network to be pruned. If this is the case, dropout techniques that specifically target weights [49] or learn per-weight dropout probabilities [38, 35] could make winning tickets even easier to find with our pruning-based method.

6 Examining Winning Tickets

In this section, we briefly examine the winning ticket initializations resulting from iteratively pruning fully-connected networks trained on MNIST. Figure 8 shows the winning ticket initializations (weights *before* training) at four levels of pruning. The left graph contains the initial truncated normal distribution of weights in the unpruned network. To the right are the initial weights after pruning to 51.3%. The initializations are the left and right extremes of the original distribution. Nearly all weights that began near zero remained the smallest weights after training and were pruned during the first few iterations; they were considered least important by our pruning heuristic, and the network performed better without them. (The output layer maintains its structure for longer, likely because we pruned it at a slower rate.) Han et al. [17] find this pattern for weights *after* training, but Figure 8 shows that it originates before training. This pattern continues at 16.9% and 5.6%. While these distributions differ significantly from typical initializations, these results suggest that randomly initializing from a normal distribution may—in effect—sample a sparse architecture in which small weights are infrequently part of winning tickets.

7 Related Work

In practice, neural networks tend to be dramatically over-parameterized. Distillation [2, 21] and pruning [31, 17] rely on the fact that parameters can be reduced while preserving accuracy. Even with sufficient capacity to memorize training data, networks naturally tend to learn simpler functions [51, 42, 1]. Contemporary experience [5, 21, 51] and our random reinitialization experiments suggest that large networks are easier to train than small ones. Our contribution is to show that large networks contain subnetworks capable of learning on their own starting from their original initializations. Several other research directions aim to train small or sparse networks.

Prior to training. Squeezenet [25] and MobileNets [23] are specifically engineered image-recognition networks that are an order of magnitude smaller than standard architectures. Denil et al. [8] represent weight matrices as products of lower-rank factors. Li et al. [32] restrict optimization to a small, randomly-sampled subspace of the parameter space (meaning all parameters can still be updated); they successfully train networks under this restriction. We show that one need not even update all parameters to optimize a network, and we find winning tickets through a principled search

process involving pruning. Our contribution to this class of approaches is to demonstrate that small, trainable networks exist within larger networks in the form of winning tickets.

After training. Distillation [2, 21] trains small networks to mimic the behavior of large networks; small networks are easier to train in this paradigm. Recent pruning work aims to compress large models into forms that run with limited resources (e.g., on mobile devices). Although pruning is central to our experiments, we aim to gain insight into why training needs the overparameterized networks that make pruning necessary. LeCun et al. [31] and Hassibi et al. [18] first explored pruning based on second derivatives. More recently, Han et al. [17, 15, 14] showed per-weight magnitude-based pruning substantially reduces the size of image-recognition networks. Han et al. iteratively train to convergence, prune, and continue training. Guo et al. [13] restore pruned connections as they become relevant again. Han et al. [16] and Jin et al. [27] restore pruned connections to increase network capacity after small weights have been pruned and surviving weights fine-tuned. Other proposed pruning heuristics include pruning based on activations [24], redundancy [37, 44], per-layer second derivatives [9], and energy/computation efficiency [50] (e.g., pruning convolutional filters [33, 39, 36] or channels [20]). Cohen et al. [7] observe that convolutional filters are sensitive to initialization (“The Filter Lottery”); after training, they randomly reinitialize unimportant filters.

During training. Bellac et al. [4] train with sparse networks and replace weights that reach zero with new random connections. Srinivas et al. [47] and Louizos et al. [35] learn gating variables that minimize the number of nonzero network parameters. Narang et al. integrate magnitude-based pruning into training [40]. Gal and Ghahramani show that dropout approximates Bayesian inference in Gaussian processes [10]. Bayesian perspectives on dropout learn the dropout probabilities during training [11, 28, 46]. Techniques that learn per-weight, per-unit [46], or structured dropout probabilities naturally [38, 41] or explicitly [34, 45] prune and sparsify networks during training as dropout probabilities for some weights reach 1. In contrast, we train networks at least once to find winning tickets. These techniques might also find winning tickets, or, by inducing sparsity, they might beneficially interact with our methods for finding winning tickets (as discussed in Section 5).

8 Conclusions

We articulate the *lottery ticket hypothesis* to offer insight into why large networks are easier to train. It states that, when training succeeds for a large network, a subnetwork (a *winning ticket*) has been randomly initialized such that it can be trained in isolation to the same level of accuracy as the original network in a comparable number of iterations. We showed that pruning naturally uncovers winning tickets in fully-connected, convolutional, and residual networks, including when trained with techniques like dropout and batch normalization. In nearly all cases, we could find winning tickets that were more than 80% smaller than the original network. When pruned to moderate levels, these winning tickets converged faster than the original network and reached higher accuracy. Both initialization and structure play a role in the efficacy of these winning tickets. Based on these results, we return to our motivating question and conjecture that large networks are easier to train because they contain more combinations of subnetworks from which training can recover a winning ticket.

Limitations. We only consider vision-centric datasets and network architectures. We only show a single method for finding winning tickets: pruning. We do not explain why the winning tickets we find train effectively. Although we show that many networks contain winning tickets, we do not show that containing a winning ticket is necessary or sufficient for a network to learn successfully.

Future directions. This paper documents empirical evidence for the lottery ticket hypothesis. There are numerous opportunities to further understand and make practical use of this paradigm.

Examining winning tickets. Understanding the properties that make subnetworks winning tickets.

Training. Exploring the interaction between training strategies (e.g., momentum) and winning tickets.

Finding other winning tickets. Designing other strategies beyond pruning for finding winning tickets.

Improving training. Exploiting winning tickets to develop new network architectures and initialization strategies. Modifying strategies that prune during training to explicitly search for winning tickets.

Theoretical foundations. Developing a formal understanding of the lottery ticket hypothesis.

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10 Units		8 Units		6 Units		4 Units		2 Units	
DB	ZL	DB	ZL	DB	ZL	DB	ZL	DB	ZL
98.5	92.9	96.8	87.5	92.5	76.8	78.3	55.3	49.1	17.6

Figure 9: Success rates of 1000 random XOR networks, each with the specified number of hidden units. DB = percent of trials that found the correct decision boundary. ZL = percent of trials that reached zero loss.

Pruning Strategy	10 Units		4 Units (Pruned)		2 Units (Pruned)	
	DB	ZL	DB	ZL	DB	ZL
One-shot Product	99.2	93.3	98.0	90.3	82.4	65.3
Input Magnitude	98.9	93.5	97.9	92.2	83.8	76.5
Output Magnitude	99.0	93.3	96.9	85.9	78.6	56.1
Product	98.5	92.9	97.6	90.3	91.5	79.4

Figure 10: Success rates of different pruning strategies on 1000 trials each. DB and ZL defined as in Figure 9. The pruned columns include only those runs for which both the original ten-unit network and the pruned winning ticket found the right decision boundary or reached zero loss. The first row of the table was obtained by pruning in one shot; all subsequent rows involved pruning iteratively

A Winning Tickets and the XOR Function

Many of the same behaviors we observed in image-recognition networks also surface when learning XOR with exceedingly simple networks.

The XOR function is among the simplest examples that distinguish neural networks from linear classifiers. The XOR function has four data points: the coordinates (0, 0), (0, 1), (1, 0), and (1, 1). The first and last points should be placed in class 0 and the middle two points in class 1. Geometrically, this problem requires a nonlinear decision boundary. In this experiment, we consider the family of fully connected networks for XOR with two input units, one hidden layer (ReLU activation), and one output unit (sigmoid activation). Even with this simple level, the lottery ticket hypothesis surfaces.

Although a network of this form with two hidden units is sufficient to perfectly represent the XOR function,³ the probability that a standard training approach—one that randomly initializes the network’s weights and then applies gradient descent—correctly learns XOR for a network with two hidden units is low relative to that for a larger network.

Figure 9 contains the overall success rates (percent of networks that found the right decision boundary or reached zero loss). In 1000 training runs, a network with two hidden units learned a correct decision boundary in only 49.1% of trials. Cross-entropy loss reached 0 (meaning the network learned to output a hard 0 or 1) in only 17.6% of trials. Meanwhile, an otherwise identical network outfitted with ten hidden units learned the decision boundary in 98.5% of trials and reached 0 loss in 92.9% of trials. Figure 9 charts the loss for these and other hidden layer sizes.⁴

To put the central question of this paper in the concrete terms of the XOR problem, why do we need to start with a neural network with ten hidden units to ensure that training succeeds when a much smaller neural network with two hidden units can represent the XOR function perfectly?

According to the lottery ticket hypothesis, successful networks with a large number of parameters (e.g., the XOR network with ten hidden units) should contain winning tickets comprising a small number of fortuitously-initialized weights on which training will still succeed.

³Example satisfying weights for the first layer: $\begin{bmatrix} n & -n \\ -n & n \end{bmatrix}$. Satisfying weights for the output unit: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Satisfying bias for the output unit: $-n/2$. $n \geq 1$. As n grows, the output approaches a hard 1 or 0.

⁴Weights were sampled from a normal distribution centered at 0 with standard deviation 0.1; all values more than two standard deviations from the mean were discarded and resampled. Biases were initialized to 0. The network was trained for 10,000 iterations.

A.1 Methodology

To test the lottery ticket hypothesis with the XOR function, we instantiated the experiment from Part 1 with the following details:

1. Randomly initialize a network with ten hidden units.
2. Train for 10,000 iterations on the entire training set.
3. Prune a certain number of hidden units according to a particular pruning heuristic.
4. To extract the winning ticket, reset the pruned network to the original initializations.

The first three steps extract the architecture of the winning ticket; the crucial final step extracts the corresponding initializations. We ran this experiment with two different classes of pruning strategies. *One-shot* pruning involves pruning the network in a single pass. For example, one-shot pruning a network by 80% would involve removing 80% of its units after it has been trained. In contrast, *iterative pruning* involves repeating steps (2) through (4) several times, removing a small portion of the units (in our case, two units) on each iteration. We find iterative pruning to be more effective for extracting smaller winning tickets; Han et al. [17] found the same for compressing large networks while maintaining accuracy.

We consider three different heuristics for determining which hidden units should be pruned:

- *Input magnitude*: remove the hidden unit with the smallest average input weight magnitudes.
- *Output magnitude*: remove the hidden unit with the smallest output weight magnitude.
- *Magnitude product*: remove the hidden unit with the smallest product of the magnitude of its output weight and the sum of the magnitudes of its input weights.

The magnitude product heuristic achieved the best results, so we use it unless otherwise stated.

A.2 Results

One-shot pruning. We generated 1000 networks with ten hidden units and pruned them down to both four and two hidden units using the magnitude product heuristic. The results of doing so appear in the first row of Figure 10. The winning tickets with two hidden units found the correct decision boundary 82.4% of the time (up from 49.1% for randomly-initialized networks with two hidden units) and reached zero loss 65.3% of the time (up from 17.6% of the time for a random network).

Iterative pruning. We conducted the iterative version of the pruning experiment 1000 times, starting with networks containing ten hidden units that were eventually pruned down (in two unit increments) to networks containing a candidate winning ticket of just two hidden units. Of the 93.5% of ten hidden unit networks that reached zero loss, 84.9%⁵ had a two hidden unit winning ticket that also reached zero loss (as compared to 17.6% of randomly-initialized two hidden unit networks). Likewise, of the 98.9% of ten hidden unit networks that found the correct decision boundary, 92.8% had a two-unit winning ticket that did the same (as compared to 49.1% of randomly-initialized two hidden unit networks). The four hidden unit winning tickets almost identically mirror the performance of the original ten hidden unit network. They found the correct decision boundary and reached zero loss respectively in 99% and 97% of cases where the ten hidden unit network did so. Both of these pruned trials appear in Figure 10 (under the *Magnitude Product* row).

These experiments indicate that, although iterative pruning is more computationally demanding than one-shot pruning, it finds winning tickets at a higher rate than one-shot pruning. More importantly, they also confirm that networks with ten hidden units can be pruned down to winning tickets of two hidden units that, when initialized to the same values as they were in the original network, succeed in training far more frequently than a randomly initialized network with two hidden units. The winning tickets with four hidden units succeed nearly as frequently as the ten unit networks from which they derive. Both of these results support the lottery ticket hypothesis—that large networks contain smaller, fortuitously-initialized winning tickets amenable to successful optimization.

⁵These numbers are derived from the last row of Figure 10. 93.5% of networks with ten hidden units reached zero loss. 79.4% of networks started with ten units, reached zero loss, and were pruned to into two-unit networks that also reached zero loss. 79.4% of 93.5% is 84.9%.

In addition to the magnitude-product pruning heuristic, we also tested the input magnitude and output magnitude heuristics. The results of doing so appear in Figure 10. The magnitude product heuristic outperformed both. We posit that this success is due to the fact that, in the XOR case when all input values are either 0 or 1, the product of input and output weight magnitudes should mimic the activation of the unit (and therefore with its influence on the output).