Subdividing Alpha Complex

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Basic Definitions

Let B be a set of balls and D_B be its Delaunay complex.

- A Delaunay simplex δ is an alpha simplex if its corresponding balls intersect with the dual of δ.

An Example of alpha complex

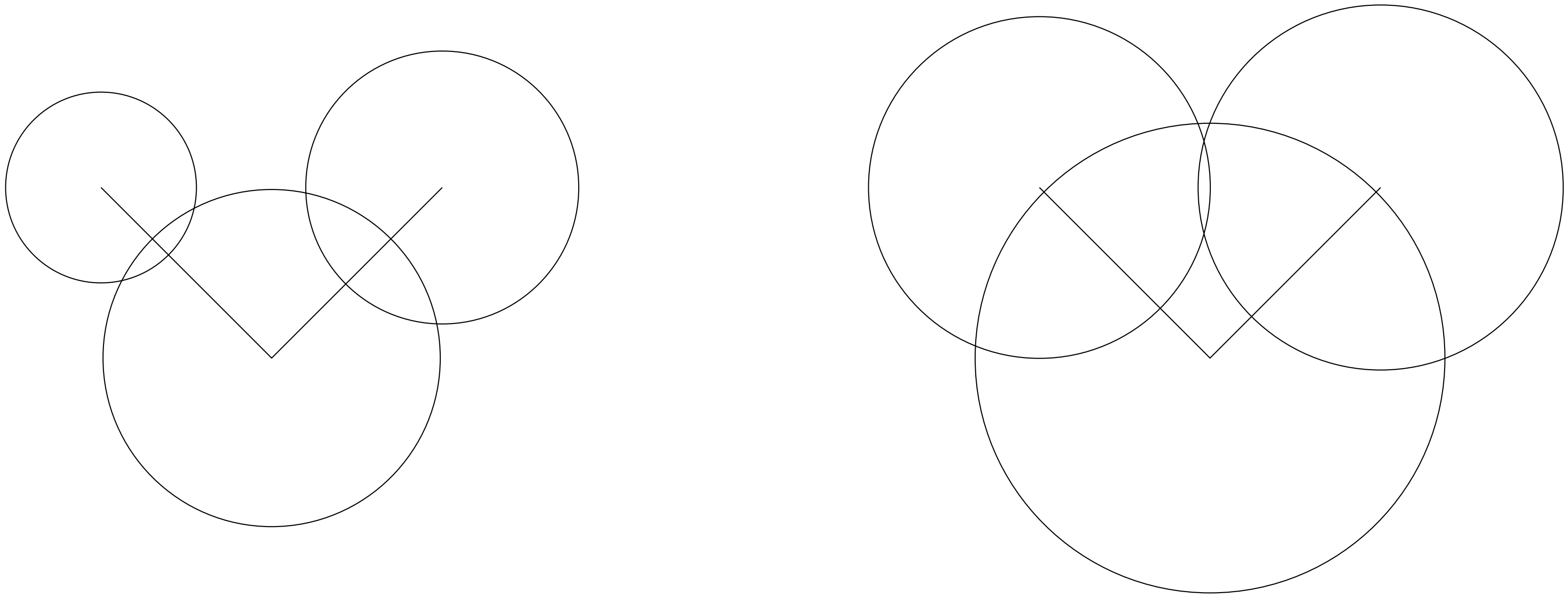

Figure 1:

Subdividing Alpha Complex

An alpha complex $\mathcal{K}_B$ *subdivides* a simplicial complex $\mathcal{C}$ if:

- $|\mathcal{K}_B| = |\mathcal{C}|$, and
- every simplex in $\mathcal{K}_B$ is contained in a simplex in $\mathcal{C}$

Problem: Given a simplicial complex $\mathcal{C}$, construct a set of balls B such that $\mathcal{K}_B$ subdivides $setC$

Assumption: We are given the constraint triangulation of $\mathcal{C}$, i.e. the triangulation of he convex hull of $\mathcal{C}$ which contains $\mathcal{C}$.

Main Theorem

Theorem. *Let B be a set of balls and $\mathcal{C}$ be a simplicial complex. If B satisfies the following Conditions C1 and C2:*

C1. *for a subset $X \subseteq B$, if $\bigcap X \neq \emptyset$ then $z(X) \subseteq \sigma$ for some $\sigma \in \mathcal{C}$, and,*

C2. *for each $\sigma \in \mathcal{C}$, define $B(\sigma) = \{b \in B \mid b \cap \sigma \neq \emptyset\}$. Then we have: $z(B(\sigma)) \subseteq \sigma \subseteq \bigcup B(\sigma)$,*

then $\mathcal{K}_B$ subdivides $\mathcal{C}$.

Satisfying Condition 1

Subdivision

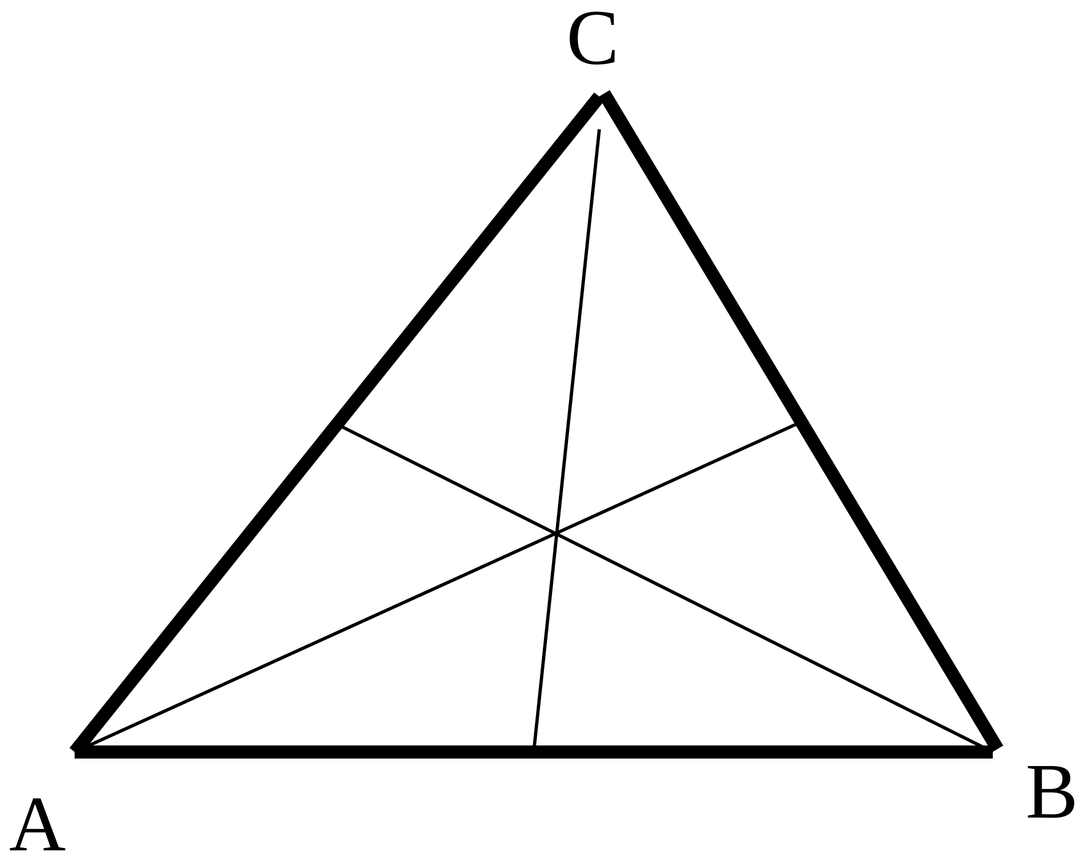

Figure 2:

Satisfying Condition 1

Protecting cells

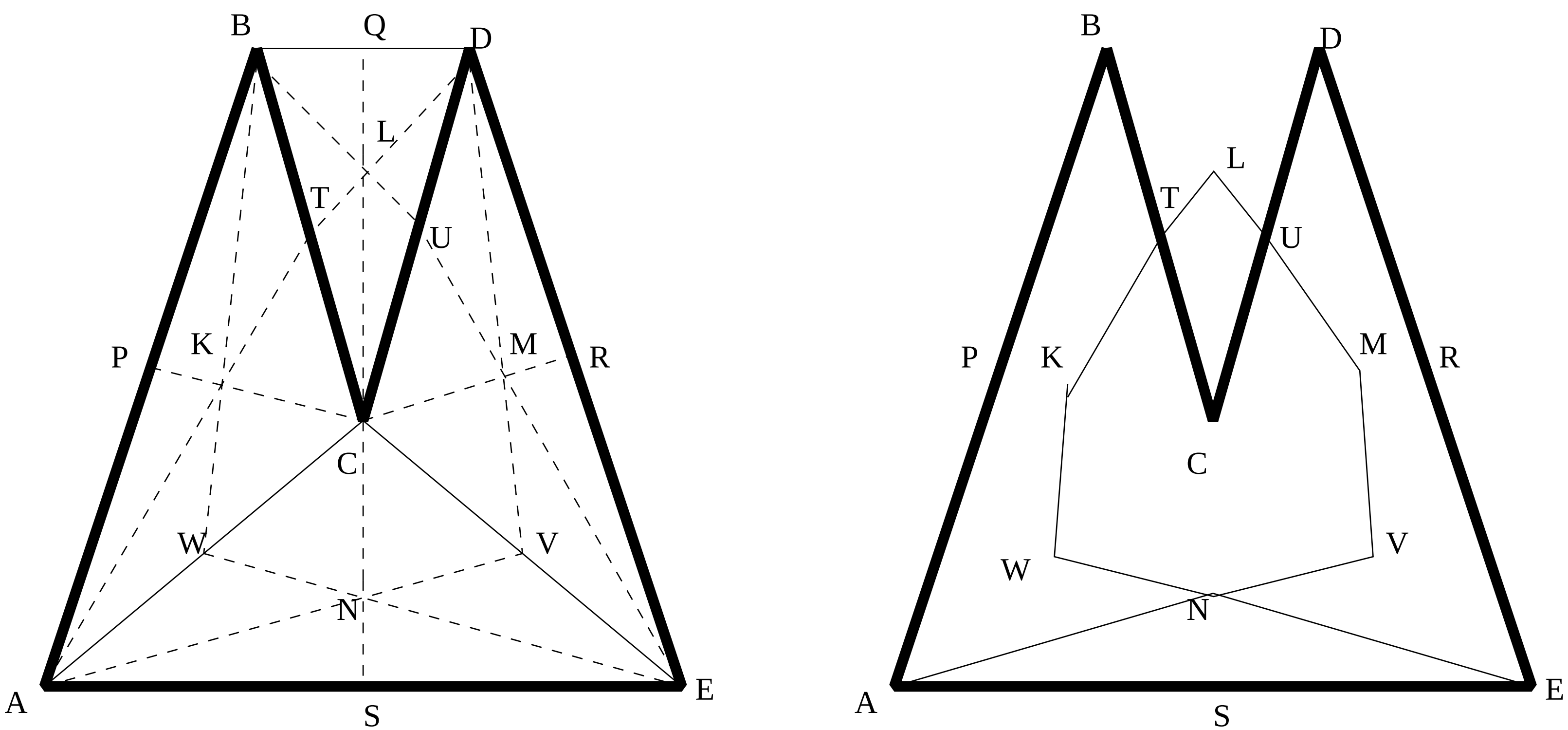

Figure 3:

Satisfying Condition 1

For a point $p \in \rho$, $\mathrm{MaxWeight}(p)$ is the weight of the smallest ball centered on p that intersects the protecting cell of ρ.

The Algorithm

Restricted Voronoi complex:

Let $\sigma \in \mathcal{C}$ and X a set of balls.

- Restricted Voronoi complex $V_X(\sigma)$ is the restriction of V_X on ρ.
- A Voronoi vertex v is called a *positive* vertex, if v is outside every ball in X.

The Main Algorithm:

`for` $i = 0, \ldots, d$

  `for all` $\rho \in \mathcal{C}$ `of dimension i`

    `ConstructBalls`(ρ)

The Procedure **ConstructBalls**(σ):

if dim $\sigma = 0$
  $B(\sigma) := (\sigma, \gamma \cdot \mathrm{MaxWeight}(\sigma))$
else
  Let $l :=$ dim σ
  Let $\tau_1, \ldots, \tau_{l+1}$ be the $(l-1)$-faces of σ.
  $X := B(\tau_1) \cup \cdots \cup B(\tau_{l+1})$
  while $\exists$ a positive vertex u in $\mathrm{V}_X(\sigma)$
    $w := \gamma \cdot \mathrm{MaxWeight(u)}$
    $X := X \cup \{(u, w)\}$
  endwhile
  $B(\sigma) := X$
endif

Proof of Correctness:

Condition C2 follows from the following proposition:

Proposition. *Let X be a set of balls. Suppose $z(X) \subseteq \sigma$. Then $\sigma \subseteq \bigcup X$ if and only if there is no positive vertex in $V_x(\sigma)$.*

Proof of Termination:

Fact. *Let Λ be a subset of σ whose boundary lies entirely in the interior of σ. Then there exists a constant $c > 0$ such that for all $p \in \Lambda$, $\mathrm{MaxWeight}(p) > c$.*

Termination follows from the fact and the compactness of ρ.

Object Approximation