Approximating Polygonal Objects by Deformable Smooth Surfaces

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Abstract. We propose a method to approximate a polygonal object by a deformable smooth surface, namely the t-skin defined by Edelsbrunner for all $0 < t < 1$. We guarantee that they are homeomorphic and their Hausdorff distance is at most $\epsilon > 0$. Such construction makes it possible for fully automatic, smooth and robust deformation between two polygonal objects with different topologies. En route to our results, we also give an approximation of a polygonal object with a union of balls.

1 Introduction

Geometric deformation is a heavily studied topic in disciplines such as computer animation and physical simulation. One of the main challenges is to perform deformation between objects with different topologies, while at the same time maintaining a good quality mesh approximation of the deforming surface.

Edelsbrunner defines a new paradigm for the surface representation to solve these problems, namely the *skin surface* [5] which is a smooth surface based on a finite set of balls. It provides a robust way of deforming one shape to another without any constraints on features such as topologies [2]. Moreover, the skin surfaces possess nice properties such as curvature continuity which provides quality mesh approximation of the surface [3].

However, most of the skin surface applications are still mainly on molecular modeling. The surface is not widely used in other fields because general geometric objects cannot be represented by the skin surfaces easily. This leaves a big gap between the nicely defined surfaces and its potential applications. We are trying to fill this gap in this paper.

1.1 Motivation and Related Works

One of the main goals of the work by Amenta et. al in [1] is to convert a polygonal object into a skin surface. We can view our work here as achieving this goal and the purpose of doing so is to perform deformation between polygonal objects. As noted earlier in some previous works [2, 5], deformation can be performed robustly and efficiently if the object is represented by the skin surface.

Moreover, our work here can also be viewed as a step toward converting an arbitrary smooth object into a provably accurate skin surface. In this regard, previous work has been done by Kruithof and Vegter [8]. For input their method requires a so-called r-admissible set of balls B which approximate the object well. Then, it expands all the weights of the balls by a carefully computed constant t, before taking the $\frac{1}{t}$-skin of the expanded balls to approximate the smooth object.

However, we observe that there are at least two difficulties likely to occur in such approach. First, such an r-admissible balls are not trivial to obtain. Furthermore, when the computed factor t is closed to 1, the skin surface is almost the same as the union of balls, thus, does not give much improvement from the union of balls. On the other hand, the approach discussed here allows the freedom to choose any constant $0 < t < 1$ for defining the skin surface.

On top of the skin approximation, we also give an approximation of a polygonal object with a union of balls. Such approximation has potential applications in computer graphics such as collision detection and deformation [7, 9, 10]. Ranjan and Fournier [9] proposed using a union of balls for object interpolation. Sharf and Shamir [10] also proposed using the same representation for shape matching. Those algorithms require a union of balls which accurately approximate the object as an input and to provide such a good set of balls at the beginning is still not trivial.

A comparison with our previous work. In [4], we proposed a method to construct a set of weighted points whose alpha shape is the same as the input simplicial complex in $\mathbb{R}^d$, which we call the *subdividing alpha complex.* Given such alpha complex it is quite straightforward to obtain a set of balls which can be used to approximate the object. However, to construct the subdividing alpha complex, we need to make the assumption that the constrained triangulation of the input is given too.

In this paper the input is a piecewise linear complex which constitutes the boundary of the object. To avoid assuming we are given the constrained triangulation, we make use of the notion of *local gap size*(lgs) in the construction of the subdividing alpha complex.

1.2 Approach and Outline

The first step is to construct a set of balls whose alpha shape is the same as the boundary of the polygonal object, namely, the subdividing alpha complex. The radii of the balls constructed are at most ϵ, for a given real number $\epsilon > 0$.

In the second step we fill the interior of the object with balls according to the Voronoi complex of the balls constructed in the first step, namely, the balls that make up the subdividing alpha complex. Specifically, we consider all the Voronoi vertices which are inside the object. Each Voronoi vertex determines an orthogonal ball. We use the set of all such orthogonal balls to approximate the object. It is shown that that the union of such balls is homeomorphic to the object and furthermore, the Hausdorff distance between them is at most ϵ.

To obtain the skin approximation, we invert the weights of the balls that make up the subdividing alpha complex of the boundary. Those inverted balls, together with the balls in the interior of the object(computed in the second step), generate a skin surface which is homeomorphic to the object. It is also shown that the Hausdorff distance between them is at most ϵ.

Outline. This paper is organized as follows. In the next section we introduce some basic terminologies on piecewise linear complex(PLC) and alpha complex. In Section 3 we describe our method in constructing the subdividing alpha complex of a given PLC and the approximation of a polygonal object with a union of balls. Then we briefly review the definition of the skin surface in Section 4. The object approximation by the skin surface is described in Section 5. Finally, we end with some discussions in Section 6.

2 Notations and Basic Definitions

In this section we introduce a few basic definitions that we use throughout this paper: polygonal objects, piecewise linear complexes and alpha complexes.

Polygonal objects. A polygonal object $\mathcal{O} \subseteq \mathbb{R}^3$ is a compact 3-manifold whose boundary is a piecewise linear 2-manifold. Our algorithm takes as an input a *piecewise linear complex*(PLC) which constitutes the boundary of $\mathcal{O}$.

Piecewise linear complexes. In $\mathbb{R}^3$, a piecewise linear complex is a set $\mathcal{P}$ of vertices, line segments and polygons with the following conditions:
i) all elements on the boundary of an element in $\mathcal{P}$ also belong to $\mathcal{P}$, and,
ii) if two elements intersect, the intersection is a lower dimensional element in $\mathcal{P}$.
The underlying space of $\mathcal{P}$ is denoted by $|\mathcal{P}| = \bigcup_{\sigma \in \mathcal{P}} \sigma$.

The *local gap size* is a function $lgs : |\mathcal{P}| \mapsto \mathbb{R}$ where $lgs(x)$ is the radius of the smallest ball centered on x that intersects an element of $\mathcal{P}$ that does not contain x. We remark that lgs is continuous on the interior of every element $\sigma \in \mathcal{P}$.

Alpha complexes. We describe a *weighted point* $b \in \mathbb{R}^3 \times \mathbb{R}$ by its *location* $z_b \in \mathbb{R}^3$ and its *weight* $w_b \in \mathbb{R}$, written also as $b = (z_b, w_b)$. A weighted point b can also be viewed as a *ball* with center z_b and radius $\sqrt{w_b}$, that is, the *set of points* $\{p \in \mathbb{R}^3 \mid \|p - z_b\|^2 \leq w_b\}$. If w_b is negative then b is an imaginary ball, which is, an empty set. In this paper, we will use the terms *ball* and *weighted point* interchangeably.

The *weighted distance* of a point $p \in \mathbb{R}^3$ to a ball b is defined as

$$\pi_b(p) = \|p - z_b\|^2 - w_b.$$

Two balls b_1 and b_2 are *orthogonal* to each other if $\|z_{b_1} - z_{b_2}\|^2 = w_{b_1} + w_{b_2}$.

Given a finite set of balls B, each ball $b \in B$ defines a *Voronoi cell* ν_b which consists of the points in $\mathbb{R}^3$ with weighted distance to b less than or equal to any other ball in B. For $X \subseteq B$, the *Voronoi cell* of X is

$$\nu_X = \bigcap_{b \in X} \nu_b.$$

If ν_X consists of only one point then it is called a *Voronoi vertex*.

Let $\nu_X = \{p\}$ be a Voronoi vertex. We can associate ν_X with the ball b' where $z_{b'} = p$ and $w_{b'} = \pi_b(p)$ for some $b \in X$. Note that b' is orthogonal to every ball $b \in X$. For this reason, we call b' the *associated orthogonal ball* of the Voronoi vertex ν_X.

The collection of all Voronoi cells is called the *Voronoi complex* of B,

$$V_B = \{\nu_X \mid X \subseteq B \text{ and } \nu_X \neq \emptyset\}.$$

In this paper, we make an important but standard assumption regarding V_B:

General Position Assumption. *Let $B \subseteq \mathbb{R}^3 \times \mathbb{R}$ be a finite number of set of balls and let $X \subseteq B$. Suppose $\nu_X \neq \emptyset$ with respect to the Voronoi complex V_B. Then $1 \leq \mathrm{card}(X) \leq 4$ and the dimension of ν_X is $4 - \mathrm{card}(X)$.*

Such assumption can be achieved by small perturbation on either one of the weights or positions of the balls in X. (See, for example, [6])

For a set of balls X, we abuse the notation z_X to denote the set of the ball centers of X. The *Delaunay complex* of B is the collection of simplices,

$$D_B = \{\mathrm{conv}(z_X) \mid \nu_X \in V_B\}.$$

Note that by the *general position assumption*, the number of tetrahedra in D_B is the same as the number of Voronoi vertices in V_B.

The *alpha complex* of B is a subset of the Delaunay complex D_B which is defined as follow,

$$\mathcal{K}_B = \{\mathrm{conv}(z_X) \mid \left(\bigcup X\right) \cap \nu_X \neq \emptyset\}.$$

The *alpha shape* of B is the underlying space of $\mathcal{K}_B$, namely, $|\mathcal{K}_B|$. Note that if $\mathrm{conv}(z_X) \in \mathcal{K}_B$ then $\bigcap X \neq \emptyset$. Conversely, if $\bigcap X = \emptyset$ then $\mathrm{conv}(z_X) \notin \mathcal{K}_B$.

3 Subdividing Alpha Complex

Given a PLC $\mathcal{P}$ and a set of balls B, we say $\mathcal{K}_B$ subdivides $\mathcal{P}$ if $|\mathcal{K}_B| = |\mathcal{P}|$. In this section, we show how to construct B such that $\mathcal{K}_B$ subdivides $\mathcal{P}$. For this we need the following Lemma 1 which is a straightforward generalization of Theorem 1 in [4]. The proof is very similar, thus, we omit it.

Lemma 1. *Let $\mathcal{P}$ be a PLC. If B is a set of balls that satisfies the following two conditions:*

C1. *For $X \subseteq B$, if $\bigcap X \neq \emptyset$ then $\mathrm{conv}(z_X) \subseteq \sigma$ for some $\sigma \in \mathcal{P}$, and,*
C2. *For each $\sigma \in \mathcal{P}$, define $B(\sigma) = \{b \in B \mid b \cap \sigma \neq \emptyset\}$.*
Then we have: $z_{B(\sigma)} \subseteq \sigma \subseteq \bigcup B(\sigma)$,

then $\mathcal{K}_B$ subdivides $\mathcal{P}$.

We call $\mathcal{K}_B$ a *subdividing alpha complex*, or in short SAC, of $\mathcal{P}$. Furthermore, if all the weights in B are less than a real value ϵ, then $\mathcal{K}_B$ is called an ϵ-SAC of $\mathcal{P}$.

The aim is to construct a set of balls B that satisfies Conditions C1 and C2 in Lemma 1 and at the same time all the weights of the balls are bounded above by an input real number $\epsilon > 0$. In the first step we fix a real number $0 < \gamma < 0.5$. Then we construct the set of balls $B(\sigma)$ for each $\sigma \in \mathcal{P}$, starting with those of dimension 0, then dimension 1 and ending with those of dimension 2. Algorithm 1 outlines the sequence of computational steps.

Algorithm 1 Construction of a set of balls B such that $\mathcal{K}_B$ subdivides $\mathcal{P}$

1: Fix a real number $0 < \gamma < 0.5$
2: **for** $i = 0, 1, 2$ **do**
3: Construct $B(\sigma)$ for all $\sigma \in \mathcal{P}$ of dimension i.
4: **end for**
5: Output $B = \bigcup_{\sigma \in \mathcal{P}} B(\sigma)$.

The construction of $B(\sigma)$ where $\dim(\sigma) = 0$ is trivial. For each vertex v in $\mathcal{P}$, we add a ball with center v and radius $r = \min(\gamma \cdot lgs(v), \sqrt{\epsilon})$. So, $B(v) = \{(v, r^2)\}$. For completeness, we present it as Algorithm 2.

Algorithm 2 To construction $B(\sigma)$ for all $\sigma \in \mathcal{P}$ with dimension 0

1: **for** each vertex $\sigma \in \mathcal{P}$ **do**
2: $r := \min(\gamma \cdot lgs(v), \sqrt{\epsilon})$
3: $B(\sigma) := \{(v, r^2)\}$
4: **end for**

To describe the construction of $B(\sigma)$ with σ is of dimension 1 or 2, we need the notations of *restricted Voronoi complex*. The restricted Voronoi complex of a set of balls X on $\sigma \in \mathcal{P}$, denoted by $V_X(\sigma)$, is the complex which consists of $\nu_X \cap \sigma$, for all $\nu_X \in V_X$. A Voronoi vertex u in $V_X(\sigma)$ is called a *positive* vertex if $\pi_b(u) > 0$, for all $b \in X$. Note that such a vertex is outside every ball in X. To determine whether a vertex is positive, it suffices to compute $\pi_{b'}(u)$ where u is the Voronoi vertex in the Voronoi cell of b'.

We construct $B(\sigma)$ where $\dim(\sigma) = 1$ according to Algorithm 3. The basic idea is to add a ball to a positive vertex in an edge until the edge is covered by the balls. To avoid unwanted elements other than the edge itself, we set the radius of every ball to be less than both $\sqrt{\epsilon}$ and γ times the *lgs* of the ball center. The construction of $B(\sigma)$ where σ is of dimension 2 is similar. For completeness, we present it as Algorithm 4 here.

We claim that our algorithms terminate and the output $B = \bigcup_{\sigma \in \mathcal{P}} B(\sigma)$ satisfies both Conditions C1 and C2. It should be clear that all weights in B are

Algorithm 3 To construct $B(\sigma)$ for all $\sigma \in \mathcal{P}$ with dimension 1

for all the edge $\sigma \in \mathcal{P}$ **do**
  Let v_1, v_2 be the two vertices of σ.
  $X := B(v_1) \cup B(v_2)$
  while there exists a positive vertex u in $V_X(\sigma)$ **do**
    $r := \min(\gamma \cdot \mathit{lgs}(u), \sqrt{\epsilon})$
    $X := X \cup \{(u, r^2)\}$
  end while
  $B(\sigma) := X$
end for

Algorithm 4 To construct $B(\sigma)$ for all $\sigma \in \mathcal{P}$ with dimension 2

for all each polygon $\sigma \in \mathcal{P}$ **do**
  Let $\tau_1, \ldots, \tau_m$ be the edges of σ.
  $X := B(\tau_1) \cup \cdots \cup B(\tau_m)$
  while there exists a positive vertex u in $V_X(\sigma)$ **do**
    $r := \min(\gamma \cdot \mathit{lgs}(u), \sqrt{\epsilon})$
    $X := X \cup \{(u, r^2)\}$
  end while
  $B(\sigma) := X$
end for

at most ϵ. Since every ball with center p has radius less than $0.5 \times \mathit{lgs}(p)$, it is obvious that Condition C1 is satisfied. Condition C2 follows from Proposition 1 below. Theorem 2 establishes the termination of our algorithm.

Proposition 1. *Let X be a set of balls. Suppose $z_X \subseteq \sigma$. Then $\sigma \subseteq \bigcup X$ if and only if there is no positive vertex in* $\mathrm{V}_X(\sigma)$*.*

Proof. The "only if" part is immediate. We will show the "if" part. Suppose there is no positive Voronoi vertex in $V_X(\sigma)$. We claim that $\nu_b(\sigma) \subseteq b$ for all $b \in X$. This claim follows from the fact that $\nu_b(\sigma)$ is the convex hull of its Voronoi vertices and bounded. Thus, by our assumption that all the Voronoi vertices are not positive, it is immediate that $\nu_b(\sigma) \subseteq b$ for any $b \in X$. Since σ is partitioned into $\nu_b(\sigma)$ for all $b \in X$, it follows that $\sigma \subseteq \bigcup X$.

To establish the termination of the algorithm, we need the following fact.

Proposition 2. *Let $\rho \in \mathcal{P}$. Suppose $\Gamma \subset \sigma$ is a closed region such that it does not intersect the boundary of σ. Then there exists a constant $c > 0$ such that for every point $p \in \Gamma$, $\mathrm{lgs}(p) > c$.*

Proof. We observe that lgs is a continuous function on Γ. Moreover, Γ is compact. Thus, there exists $p_0 \in \Gamma$ such that $\mathit{lgs}(p_0) = \min_{p \in \Gamma} \mathit{lgs}(p)$. The value $\mathit{lgs}(p_0) \neq 0$ since p_0 is in the interior of σ. Thus, we can choose $c = \frac{1}{2}\mathit{lgs}(p_0)$ to establish our proposition.

Lemma 2. *Both algorithms 3 and 4 terminate.*

Proof. We prove that Algorithm 3 terminates. It suffices to show that the `while`-loop does not iterate infinitely many times. The proof is by contradiction and it follows from the fact that each element ρ in $\mathcal{P}$ is compact.

Assume to the contrary that for some edge $\sigma = (v_1, v_2) \in \mathcal{P}$ the `while`-loop iterates infinitely many times. That is, it inserts infinitely many balls to $B(\sigma)$ whose centers are in the region $\sigma - (b_1 \cup b_2)$ where $b_i \in B(v_i)$ for $i = 1, 2$. The region $\sigma - (b_1 \cup b_2)$ is a closed region which does not intersect with the boundary of σ. By Proposition 2, there exists a constant $c > 0$ such that all the radii of the balls are greater than c.

Moreover, $\sigma - (b_1 \cup b_2)$ is compact, so if $B(\sigma)$ contains infinitely many balls, then there are two balls b and b' whose centers are at the distance less than c. Without loss of generality, we assume that b was inserted before b'. This is impossible, because at the time b' was inserted, its center would be a negative vertex. Therefore, the `while`-loop iterates only finitely many times. The proof of the termination of Algorithm 4 is similar.

3.1 Approximating polygonal object with a union of balls

Let $\mathcal{O}$ be a polygonal object and $\mathcal{P}$ be its boundary, given in the form of piecewise linear complex. Our method to approximate the object $\mathcal{O}$ with a union of balls can be summarized as follows.

1. Construct a set of balls B such that $\mathcal{K}_B$ is an ϵ^2-SAC of $\mathcal{P}$.
2. Compute the Voronoi complex of B.
3. Denote by T, the set of all the Voronoi vertices which are located inside the object $\mathcal{O}$.
4. Let $B^{\perp}$ be the set of all orthogonal balls associated with all the Voronoi vertices in T.
5. Output $B^{\perp}$.

Remark 1. We remark that every ball in $B^{\perp}$ has positive weight, thus, is a real ball. The reasoning is as follows. Because $|\mathcal{K}_B| = |\mathcal{P}|$, there is no tetrahedron in $\mathcal{K}_B$. This means each Voronoi vertex is not inside any ball $b \in B$, thus, has positive weighted distance to the each ball in B. Therefore, the associated orthogonal ball of each Voronoi vertex has positive weight.

We claim that $\bigcup B^{\perp}$ can be used to approximate the object $\mathcal{O}$ well.

Theorem 1. *The union of balls $\bigcup B^{\perp}$ is contained inside the object $\mathcal{O}$ and homeomorphic to $\mathcal{O}$. Moreover, the Hausdorff distance between them is at most ϵ.*

Proof. (Sketch) Consider the Delaunay complex D_B. Let Δ be the set of all Delaunay tetrahedra which are located inside the object $\mathcal{O}$. The object $\mathcal{O}$ is decomposable into the tetrahedra of Δ. By the general position assumption, $\mathrm{card}(\Delta) = \mathrm{card}(B^{\perp})$.

Furthermore, $\bigcup B^{\perp}$ is decomposable into $\mathrm{conv}(z_X) \cap b$ where b is the associated orthogonal ball of ν_X for all $\mathrm{conv}(z_X) \in \Delta$. We can establish a homeomorphism between $b \cap \mathrm{conv}(z_X)$ and $\mathrm{conv}(x)$ for each $\mathrm{conv}(z_X) \in \Delta$. By combining all such homeomorphisms, we obtain a homeomorphism between $\bigcup B^{\perp}$ and $\mathcal{O}$.

The Hausdorff nearness part can be established via the fact that each $b \in B^{\perp}$ is contained inside the object $\mathcal{O}$. Furthermore, the ball b is orthogonal to some balls $b' \in B$ and all the weights of the balls in B are less than or equal to ϵ.

4 Skin Surface

In this section we briefly review both the algebra of balls and the definition of the skin surface which is based on the algebra of balls [5].

Algebra of balls. The algebra of balls is based on a bijection $\phi : \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^4$ defined as

$$\phi(b) = (z_b, \|z_b\|^2 - w_b).$$

The space $\mathbb{R}^4$ together with the usual componentwise addition and scalar multiplication forms a vector space. The addition and scalar multiplication operations are defined on $\mathbb{R}^3 \times \mathbb{R}$ in such a way that ϕ is a vector space isomorphism, that is,

$$\phi(b_1 + b_2) = \phi(b_1) + \phi(b_2),$$
$$\phi(\gamma \cdot b) = \gamma \cdot \phi(b),$$

where $b_1, b_2, b \in \mathbb{R}^3 \times \mathbb{R}$ and $\gamma \in \mathbb{R}$. One can easily verify that

$$b_1 + b_2 = (z_{b_1} + z_{b_2}, w_{b_1} + w_{b_2} + 2\langle z_{b_1}, z_{b_2}\rangle), \tag{1}$$
$$\gamma b = (\gamma z_b, \gamma w_b + (\gamma^2 - \gamma)\|z_b\|^2). \tag{2}$$

By the two operations above, the convex hull of a set of balls $B = \{b_1, \ldots, b_n\}$ is the set of balls $\mathrm{conv}(B) = \{\sum_i \gamma_i b_i \mid \sum_i \gamma_i = 1 \text{ and } \gamma_i \geq 0 \text{ for all } i = 1, \ldots, n\}$. It is straightforward to verify that if a ball b is orthogonal to every ball $b_i \in \{b_1, \ldots, b_n\}$, then b is orthogonal to every ball $b' \in \mathrm{conv}(b_1, \ldots, b_n)$.

Skin surfaces. Let b be a weighted point and $t \in \mathbb{R}$, we define $b^t = (z_b, tw_b)$. For a set of balls B, B^t is defined as $B^t = \{b^t \mid b \in B\}$.

For $0 \leq t \leq 1$, the skin body of a set of balls B is defined as

$$\mathrm{body}^t(B) = \bigcup \mathrm{conv}(B)^t,$$

that is, the set of points obtained by shrinking all balls in the convex combination of B. The skin surface is the boundary of the skin body of B, denoted by $\mathrm{skin}^t(B)$. Note that $\bigcup B = \mathrm{body}^1(B)$. We cite here an important relation between a union of balls $\bigcup B$ and the skin body that it generates.

Theorem 2. [5] *The union of balls $\bigcup B$ is homeomorphic to* $\mathrm{body}^t(B)$, *for* $0 < t < 1$.

5 Approximating a Polygonal Object with the Skin Surface

To approximate a polygonal object with a union of balls, we start by constructing a set of balls B such that $\mathcal{K}_B$ is an ϵ^2-SAC of the boundary of the object. Then we use the associated orthogonal balls $B^\perp$ to approximate the object $\mathcal{O}$.

In this section we will show that the set of balls $B^\perp \cup B^{-1}$ will generate a skin body that approximates the object well too, as stated in Theorem 3 below.

Theorem 3. *For all $0 \le t \le 1$, the skin body $\mathrm{body}^t(B^\perp \cup B^{-1})$ is homeomorphic to the object $\mathcal{O}$. Moreover, the Hausdorff distance between them is at most ϵ.*

Proof. All balls in B^{-1} have negative weights, thus, $\bigcup(B^\perp \cup B^{-1}) = \bigcup B^\perp$. By Theorem 1, $\bigcup B^\perp \subseteq \mathcal{O}$, thus, it follows that $\mathrm{skin}^t(B^\perp \cup B^{-1}) \subseteq \bigcup(B^\perp \cup B^{-1}) = \bigcup B^\perp \subseteq \mathcal{O}$.

The homeomorphism follows from Theorem 2 that $\mathrm{skin}^t(B^\perp \cup B^{-1})$ is homeomorphic to $\bigcup(B^\perp \cup B^{-1}) = \bigcup B^\perp$ which is homeomorphic to $\mathcal{O}$(Theorem 1). The proof for the Hausdorff nearness part is presented in the next subsection.

5.1 Proof of the Hausdorff Nearness in Theorem 3

Note that for every point p in the object $\mathcal{O}$, there is a weighted point $b \in \mathrm{conv}(B^\perp \cup B^{-1})$ such that $z_b = p$. In other words, $\mathcal{O} \subseteq \mathcal{Z}$ where $\mathcal{Z} = \{z_b \mid b \in \mathrm{conv}(B^\perp \cup B^{-1})$. In view of this, it suffices to prove the following lemma.

Lemma 3. *For every ball $b \in \mathrm{conv}(B^\perp \cup B^{-1})$ where $z_b \in \mathcal{O}$, if $w_b < 0$ then there exists a ball $b' \in \mathrm{conv}(B^\perp \cup B^{-1})$ such that $w_{b'} > 0$ and $\|z_b - z_{b'}\| \le \epsilon$.*

We note that the object $\mathcal{O}$ can be partitioned into tetrahedra of Delaunay complex $D_{B^\perp \cup B}{}^\star$. We made a few simple observations concerning the tetrahedron of $D_{B^\perp \cup B}$ which is contained inside $\mathcal{O}$.

Fact 1. *Let $X = \{b_1, \ldots, b_4\}$ such that $\mathrm{conv}(z_X)$ is a tetrahedron in $D_{B^\perp \cup B}$ and is contained inside $\mathcal{O}$. Then,*

1. *At least one of the balls in X is a ball in $B^\perp$.*
2. *If $b_i \in X \cap B^\perp$ and $b_j \in X \cap B$ then b_i and b_j are orthogonal to each other.*
3. *The simplex $\mathrm{conv}(z_{B\cap X})$ is a simplex in $\mathcal{K}_B$, i.e. $\mathrm{conv}(z_{B\cap X}) \subseteq |\mathcal{P}|$.*

Statements 1 and 2 are pretty straightforward. The intuition of Statement 3 is as follows. Let $X' = X \cap B$. It is clear when $\mathrm{card}(X') = 1$. For $\mathrm{card}(X') = 2$ or 3, assume to the contrary that $\mathrm{conv}(z_{X'}) \notin \mathcal{K}_B$. Since $|\mathcal{K}_B| = |\mathcal{P}|$, the simplex $\mathrm{conv}(z_{X'})$ is in the interior of $\mathcal{O}$. Then, there exist at least $5 - \mathrm{card}(X')$ balls of $B^\perp$ which are orthogonal to every ball in $X'^{\star\star}$. These balls of $B^\perp$ make $\nu_X = \emptyset$,

* Note that $D_{B^\perp \cup B}$ may not be the same as $D_{B^\perp \cup B^{-1}}$. The object $\mathcal{O}$ may not be partitioned into tetrahedra of $D_{B^\perp \cup B^{-1}}$.

** That is, if $\mathrm{card}(X') = 2$, then $\dim(\mathrm{conv}(z_{X'})) = 1$. So, $\mathrm{conv}(z_{X'})$ is incident to at least three tetrahedra in D_B and each tetrahedron corresponds to one ball in $B^\perp$. Similarly, if $\mathrm{card}(X') = 3$, then $\mathrm{conv}(z_{X'})$ is incident to two tetrahedra in D_B and each tetrahedron correspond to one ball in $B^\perp$.

thus, yields a contradiction that $\mathrm{conv}(z_X)$ is a Delaunay tetrahedron. Therefore, $\mathrm{conv}(z_{X'}) \in \mathcal{K}_B$, where $X' = X \cap B$.

In view of Statement 3 in Fact 1, we categorize the tetrahedra of $D_{B^\perp \cup B}$ within $\mathcal{O}$ into four types according to $\mathrm{card}(X \cap B)$. We illustrate it in Figure 1.

1. Tetrahedron type I is a tetrahedron where $\mathrm{card}(X \cap B) = 1$.
   In Figure 1, $b_1 \in B$ and $b_2, b_3, b_4 \in B^\perp$.
2. Tetrahedron type II is a tetrahedron where $\mathrm{card}(X \cap B) = 2$.
   In Figure 1, $b_1, b_2 \in B$ and $b_3, b_4 \in B^\perp$.
3. Tetrahedron type III is a tetrahedron where $\mathrm{card}(X \cap B) = 3$.
   In Figure 1, $b_1, b_2, b_3 \in B$ and $b_4 \in B^\perp$.
4. Tetrahedron type IV is a tetrahedron where $\mathrm{card}(X \cap B) = 0$.
   In Figure 1, all $b_1, b_2, b_3, b_4 \in B^\perp$.

Fig. 1. The bold point in type I, the bold edge in type II and the the shaded triangle in the type III indicate that they are in $\mathcal{K}_B$, thus in the boundary of the object. None of the vertices in the type IV tetrahedron belongs to B.

In view of this, to prove Lemma 3 it is sufficient to prove the following.

Claim. Let $\mathrm{conv}(z_X) \in D_{B^\perp \cup B}$ and located inside $\mathcal{O}$. For every ball $b \in \mathrm{conv}(X)$, if $w_b < 0$ then there exists a ball $b' \in \mathrm{conv}(X)$ such that $w_{b'} > 0$ and $\|z_b - z_{b'}\| \leq \epsilon$.

We divide the proof of the claim according to $\mathrm{card}(X \cap B)$, that is, the type of the tetrahedron that contains z_b. If $\mathrm{card}(X \cap B) = 4$ then all balls $b \in \mathrm{conv}(X)$ have weights $w_b > 0$.

The following Lemma 4 states that all points in $\mathrm{conv}(b_1^{-1}, b_2, b_3, b_4)$ (i.e. in tetrahedron type I) with negative weights are located within the ϵ-neighborhood of $z_{b_1^{-1}}$. This immediately implies the validity of claim for tetrahedron type I.

Lemma 4. *Let* $(p, w) \in \mathrm{conv}(b_1^{-1}, b_2, b_3, b_4)$*. If* $w \leq 0$ *then* $\|p - z_{b_1^{-1}}\| \leq \epsilon$*.*

Proof. Let

$$\begin{aligned}(p, w) &= \gamma_1 b_1^{-1} + \gamma_2 b_2 + \gamma_3 b_3 + \gamma_4 b_4 \\ &= \gamma_1 b_1^{-1} + (1 - \gamma_1) b',\end{aligned}$$

where $b' = \frac{1}{1-\gamma_1} \sum_{i=2}^{4} \gamma_i b_i$ and $\sum \gamma_i = 1$ and $\gamma_i \geq 0$, for $i = 1, \ldots, 4$. Since b_2, b_3, b_4 are all orthogonal to b_1, then b' is also orthogonal to b_1, i.e. $w_{b'} + w_{b_1} = \|z_{b_1} - z_{b'}\|^2$. We apply the formula of combination of weighted points:

$$w = (1 - \gamma_1) w_{b'} + \gamma_1 w_{b_1^{-1}} + (\gamma_1^2 - \gamma_1) \|z_{b'} - z_{b_1^{-1}}\|^2.$$

Since $w \leq 0$, we arrange the terms into

$$(\gamma_1^2 - \gamma_1)\|z_{b'} - z_{b_1^{-1}}\|^2 - \gamma_1(w_{b'} + w_{b_1}) + w_{b'} \leq 0 \tag{3}$$

$$\gamma_1^2\|z_{b'} - z_{b_1^{-1}}\|^2 - 2\gamma_1\|z_{b'} - z_{b_1^{-1}}\|^2 \leq -w_{b'} \tag{4}$$

$$\gamma_1^2\|z_{b'} - z_{b_1^{-1}}\|^2 - 2\gamma_1\|z_{b'} - z_{b_1^{-1}}\|^2 + \|z_{b'} - z_{b_1^{-1}}\|^2 \leq \|z_{b'} - z_{b_1^{-1}}\|^2 - w_{b'} \tag{5}$$

$$(\gamma_1 - 1)^2\|z_{b'} - z_{b_1^{-1}}\|^2 \leq w_{b_1} \tag{6}$$

$$(1 - \gamma_1)^2\|z_{b'} - z_{b_1^{-1}}\|^2 \leq \epsilon^2 \tag{7}$$

$$\|p - z_{b_1^{-1}}\| \leq \epsilon \tag{8}$$

From Inequality 3 to Inequality 4 and Inequality 5 to Inequality 6, we apply $w_{b'} + w_{b_1} = \|z_{b_1} - z_{b'}\|^2$. From Inequality 7 to Inequality 8, we apply $\|p - z_{b_1^{-1}}\| = (1 - \gamma_1)\|z_{b'} - z_{b_1^{-1}}\|$.

The validity of the claim for tetrahedra types II and III is presented as Lemmas 5 and 6 below. Lemma 5 states that all points in $\text{conv}(b_1^{-1}, b_2^{-1}, b_3, b_4)$ (i.e. in tetrahedron type II) with negative weights are located within the ϵ-neighborhood of $\text{conv}(z_{b_1^{-1}, b_2^{-1}})$. Similarly, Lemma 6 states that all points in $\text{conv}(b_1^{-1}, b_2^{-1}, b_3^{-1}, b_4)$ (i.e. in tetrahedron type III) with negative weights are located within the ϵ-neighborhood of $\text{conv}(z_{b_1^{-1}, b_2^{-1}, b_3^{-1}})$. Both proofs are just a slight twist of the proof of Lemma 4 and we omit them.

Lemma 5. *Let* $(p, w) = \text{conv}(b_1^{-1}, b_2^{-1}, b_3, b_4)$*. If* $w \leq 0$ *then there exists* $b' \in \text{conv}(b_1^{-1}, b_2^{-1})$ *such that* $\|p - z_{b'}\| \leq \epsilon$.

Lemma 6. *Let* $(p, w) = \text{conv}(b_1^{-1}, b_2^{-1}, b_3^{-1}, b_4)$*. If* $w \leq 0$ *then there exists* $b' \in \text{conv}(b_1^{-1}, b_2^{-1}, b_3^{-1})$ *such that* $\|p - z_{b'}\| \leq \epsilon$.

6 Discussion

One future direction is to implement the same idea in approximating smooth objects with skin surfaces. Amenta et.al [1] showed that given a sufficiently dense sample points on a smooth surface, the set of polar balls obtained can be used to approximate the object well. There is an analogy between such approach with our method here. We can view the ϵ-SAC constructed as the sample points and $B^\perp$ as the polar balls.

By appropriately assigning certain weights to the sample points and taking the polar balls, we hope to be able to approximate the smooth object by a skin surface. At this point, the usefulness of this idea is still under investigation.

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