HOMOLOGY OF THE SPACE OF CURVES ON BLOWUPS

RONNO DAS AND PHILIP TOSTESON

ABSTRACT. We consider the space of algebraic maps from a compact Riemann surface to a projective space blown up at finitely many points. We show that the homology of this mapping space equals that of the space of continuous maps that intersect the exceptional divisors positively, once the degree of the maps is sufficiently positive compared to the degree of homology. The proof uses a version of Vassiliev's method of simplicial resolution. As a consequence, we conclude a homological stability result for rational curves the degree 5 del Pezzo surface, which is analogous to the Batyrev–Manin conjecture on rational points.

CONTENTS

1. INTRODUCTION

Let X be a projective algebraic variety, and C be a projective algebraic curve over $\mathbb{C}$. What is the homology of the space of algebraic (or equivalently holomorphic) maps $\mathrm{Alg}(C, X)$ from C to X?

Segal [Seg79] discovered a remarkable phenomenon: when $X = \mathbb{P}^n$, the homology of the space of degree d algebraic maps approximates the homology of the space of degree d continuous maps as $d \to \infty$. More precisely, for all $i \in \mathbb{N}$ we have that

$$H_i(\mathrm{Alg}_d(C, X)) \to H_i(\mathrm{Top}_d(C, X))$$

is an isomorphism for $d \gg 0$. Here Alg_d (resp. Top_d) denote the clopen subset consisting of degree d algebraic (resp. continuous) maps. Since Segal's work this result has been extended to rational curves on Toric varieties, Grassmannians and Flag varieties, and more generally varieties with a dense solvable group action [Kir85; Gue95; BHM01; BMHM94; Gra89].

In this paper, we consider the case where X is the blowup of $\mathbb{P}(V)$ at a finite number of points $p_1, \ldots, p_r$. One reason this case is notable is that when the number of points is large and in general position, X is not a homogeneous variety (it does not admit a solvable group action with a dense orbit). Classes in $H_2(X)$ are parameterized by tuples $(d, n_1, \ldots, n_r) \in \mathbb{Z}^{r+1}$, where n_i records the intersection number with the exceptional divisor E_i and d records the intersection number with the hyperplane class H. We have decompositions

$$\mathrm{Alg}(C, X) = \coprod_{(d,n) \in \mathbb{Z}^{r+1}} \mathrm{Alg}_{d,n}(C, X) \qquad \mathrm{Top}(C, X) = \coprod_{(d,n) \in \mathbb{Z}^{r+1}} \mathrm{Top}_{d,n}(C, X),$$

where the summand associated to (d, n) consists of maps $C \to X$ such that $f_*[C] = (d, n)$. By composing with the blowdown map, the space $\mathrm{Alg}_{d,n}(C, X)$ of algebraic maps of multi-degree (d, n) can be identified with degree d algebraic maps $C \to \mathbb{P}(V)$ that intersect p_i with multiplicity n_i.

Our most general result relates $\mathrm{Alg}_{d,n}(C, X)$ to a variant of the continuous mapping space consisting of maps with that intersect E_i **positively**. We let $\mathrm{Top}^+(C, X) \subseteq \mathrm{Top}(C, X)$ be the subspace of continuous maps f such that

- $f^{-1}(E_i)$ is discrete
- f has positive local intersection multiplicity with E_i at every point $c \in f^{-1}(E_i)$

As above we have a decomposition $\mathrm{Top}^+_{d,n}(C) = \coprod_{d,n} \mathrm{Top}^+_{d,n}(C, X)$. We have the following theorem, which we state for both pointed and unpointed mapping spaces. When X and C have a distinguished base point, we will write $\mathrm{Top}_*(C, X)$, (resp. $\mathrm{Alg}_*(C, X)$ etc.), for the space of pointed continuous (resp. algebraic etc.) maps.

Theorem 1.1. *Let $X = \mathrm{Bl}_{p_1,\ldots,p_r} \mathbb{P}(V)$. Suppose $\alpha = (d, n_1, \ldots, n_r)$ with $M_\alpha = d - \sum_{i=1}^r n_i > 0$. Then the maps*

$$H_i(\mathrm{Alg}_{k\alpha}(C, X)) \to H_i(\mathrm{Top}^+_{k\alpha}(C, X)), \quad H_i(\mathrm{Alg}_{k\alpha,*}(C, X)) \to H_i(\mathrm{Top}^+_{k\alpha,*}(C, X))$$

are isomorphisms for $i < M_\alpha k - 2g - 2$.

If $\{p_j \mid j \le \dim V\}$ are in linearly general position, then we can weaken the inequality on (d, n) to $d > -n_j + \sum_{i=1}^r n_i$ for all $j \le \dim V$, and take $M_\alpha = \min_{j \le \dim V}(d - \sum_{i \ne j} n_i)$. (Here we define $n_j = 0$ if $r < j \le \dim V$).

It is natural to ask to whether the previous theorem holds when Top^+ is replaced by Top. In general the answer is no: the homology of the space of positive maps differs from that of all continuous maps. For instance, when $n = 0$ we have that $\mathrm{Top}^+_{d,0}(C, X)$ equals the space of continuous maps to the complement of the exceptional divisors which is homeomorphic to $\mathbb{P}^n - \{p_1, \dots, p_r\}$. However, we suspect that when $n \gg 0$ the homology of the space of positive maps approximates the homology of the space of continuous maps, see §1.5.2.

In the pointed case when $C = \mathbb{P}^1$ we may pass from Top^+ to Top provided that $n_i > 0$ for all i.

Corollary 1.2. *Let $X = \mathrm{Bl}_{p_1,\dots,p_r}\mathbb{P}(V)$, with $\{p_j \mid j \le \dim V\}$ in linearly general position. Let $\alpha = (d, n_1, \dots, n_r)$ with $n_i > 0$ for all $i = 1, \dots r$. Further suppose $d > -n_j + \sum_{i=1}^r n_i$ for all $j \le \dim V$ (again we set $n_j = 0$ for $r < j \le \dim V$). Then for $k \gg 0$, the map*

$$H_i(\mathrm{Alg}_{k\alpha,*}(\mathbb{P}^1, X)) \to H_i(\mathrm{Top}_{k\alpha,*}(\mathbb{P}^1, X))$$

is an isomorphism.

Note that $\mathrm{Top}_{k\alpha,*}(\mathbb{P}^1, X)$ is independent of α (it is isomorphic to the degree 0 component of the double loop space of X), so Corollary 1.2 implies a homological stability statement for the space of algebraic maps.

Corollary 1.2 is obtained from Theorem 1.1 by deforming the points to lie on a line, and applying previous results on mapping spaces for varieties admitting a dense solvable group action. Surprisingly, despite this proof method, the corollary does not hold for points which are not in linearly general position (unless we strengthen the bound to $d > \sum_i n_i$ as in Theorem 1.1). For instance, if $\dim V = 3$ and p_1, p_2, p_3 lie on a line and $(d, n) = (5, 2, 2, 2)$ then by Bezout's theorem any degree $5k$ algebraic map $C \to \mathbb{P}(V)$ with $\mathrm{mult}(f^{-1}(p_i)) = 2k$ must lie on the line. It follows that $\mathrm{Alg}_{*,k\alpha}(\mathbb{P}^1, X) = \emptyset$, while the topological mapping space is nonempty.

1.1. **del Pezzo surfaces and Batyrev–Manin conjectures.** Work by Ellenberg–Venkatesh–Westerland [VE10; EVW16; ETW17] has drawn attention to the relationship between the homological stability phenomena sequences of algebraic varieties, and the asymptotic behavior of associated arithmetic statistics. In our case, if C and X were defined over a finite field $\mathbb{F}_q$ then set of $\mathbb{F}_q$-valued points of $\mathrm{Alg}_{d,n}(C, X)$ would correspond to the set of K-valued points of X satisfying certain height conditions, where $K = \mathbb{F}_q(C)$ is the function field of C. The large-height asymptotics of the number of K-valued points on a Fano variety X/K (for K a global field) is the subject of conjectures due to Batyrev–Manin [BM90].

In dimension 2, these conjectures concern del Pezzo surfaces. Over $\mathbb{C}$, a del Pezzo surface X of degree $\deg(X) < 9$ is isomorphic to the blowup of $\mathbb{P}^2$ at $9 - \deg X$ general points. In this case, the Batyrev–Manin conjectures suggest that the following statement holds. There is a subset of the ample cone $U \subseteq A \subseteq H_2(\mathbb{R})$ which is arithmetically dense in the sense that

$$\lim_{R\to\infty} \frac{\#(U \cap H_2(X,\mathbb{Z}) \cap B(0,R))}{\#(A \cap H_2(X,\mathbb{Z}) \cap B(0,R))} = 1$$

and such that for every class $\alpha \in U \cap H_2(X,\mathbb{Z})$ the homology of $\mathrm{Alg}_{k\alpha}(C,X)$ approximates the the homology of $\mathrm{Top}_{k\alpha}(C,X)$ for $k \gg 0$.

Remark 1.3. The reader may wonder why the *a priori* transcendental $\mathrm{Top}(C,X)$ should appear in an analog of the arithmetic Batyrev–Manin conjectures. A full justification is outside the scope of this paper, but as motivation we note that (1) $\mathrm{Top}(C,X)$ appears as the limiting object in previous homological stability results for algebraic mapping spaces, and (2) its rational homology admits purely algebraic models [Hae82]. $\lozenge$

In this direction, we have the following result for $\deg(X) = 5$:

Theorem 1.4. *Let X be the degree 5 del Pezzo surface, and let $\alpha = (d, n_1, n_2, n_3, n_4)$ be an ample class satisfying $n_i \neq n_j$ for any $i \neq j$. Then there exists a constant N_α such that*

$$H_i(\mathrm{Alg}_{k\alpha,*}(\mathbb{P}^1, X)) \to H_i(\mathrm{Top}_{k\alpha,*}(\mathbb{P}^1, X))$$

is an isomorphism for all $i \leq N_\alpha k - 2g - 2$.

Remark 1.5. The condition $n_i \neq n_j$ for $i \neq j$, may be weakened to the condition that $\min(n_i)$ is distinct from the other values of n_i, and the constant N_α may be computed as $\max_{\sigma\in S_5} M_{\sigma\alpha}$, where the symmetric group S_5 acts on α via its usual action on X by Cremona transformations. $\lozenge$

We view Theorem 1.4 as a homological analog of the Batyrev–Manin conjectures for the degree 5 del Pezzo surface, proved by de La Bretéche [dLa 02] in the case of a split surface over $\mathbb{Q}$. Our methods also work for higher degree del Pezzo surfaces (the toric cases).

For lower degree del Pezzo surfaces, Corollary 1.2 still applies to prove that for certain classes α the map

$$H_i(\mathrm{Alg}_{k\alpha,*}(\mathbb{P}^1, X)) \to H_i(\mathrm{Top}_{k\alpha,*}(\mathbb{P}^1, X))$$

is an isomorphism. However, because the ample cone of a lower degree del Pezzo surface is larger, the classes for which Corollary 1.2 applies are not dense in the ample cone.

1.2. **Proof strategy.** Our approach to establishing Theorem 1.1 is through a version of Vassiliev's method for computing the cohomology of discriminant complements (a topological version of an inclusion-exclusion argument). As mentioned above, $\mathrm{Alg}_{d,n}(C, X)$ is identified with the space of degree d maps to $C \to \mathbb{P}(V)$ passing through p_i with multiplicity n_i. Thus there is a principle $\mathbb{C}^*$-bundle $\widetilde{\mathrm{Alg}}_{d,n}(C, X) \to \mathrm{Alg}_{d,n}(C, X)$ which parametrizes the data of a line bundle $L \in \mathrm{Pic}^d(C)$ and a non-vanishing algebraic section $s \in \Gamma(L \otimes V)$, which intersects $L \otimes \ell_i$ with multiplicity exactly n_i.

Let $W_n \subseteq \prod_{i=1}^r \mathrm{Sym}^{n_i} C$ be the space of pairwise disjoint divisors. Then there is a map $\widetilde{\mathrm{Alg}}_{d,n}(C, X) \to W_n \times \mathrm{Pic}^d(C)$, taking a section $s \in \Gamma(V \otimes L)$ to the pair $(s^{-1}(\ell_i))_{i=1}^r,\ L$. In fact, $\widetilde{\mathrm{Alg}}_{d,n}(C, X)$ is an open subset of the linear space over $E_{d,n} \to W_n \times \mathrm{Pic}^d(C)$ whose fiber over $(U_i)_{i=1}^r, L$ consists of sections $s \in \Gamma(L \otimes V)$ such that $s(U_i) \subseteq \ell_i$. When d is large enough relative to n, these fibers have constant rank and $E_{d,n}$ is a vector bundle. The complement of $\widetilde{\mathrm{Alg}}_{d,n}(C, X)$ is a closed subset of $\Delta_{d,n} \subseteq E_{d,n}$ consisting of sections which vanish or intersect some ℓ_i to order greater order.

We stratify $E_{d,n}$, using a poset $\mathrm{Hilb}(C)^{Q_r}$ whose elements are collections of divisors $D_{\ell_1}, \ldots, D_{\ell_r}, D_0 \in \mathrm{Hilb}(C)$, with $D_0 \subseteq D_{\ell_i}$ for all i. Given such an element, we can consider the space of sections $s \in \Gamma(L \otimes V)$ satisfying the incidence conditions $s(D_i) \subseteq \ell_i$ and $s(D_0) \subseteq 0$. As D_{ℓ_i}, D_0 vary, the subspaces of sections satisfying these incidence conditions form a vector bundle over an appropriate configuration space (at least if $d \gg 0$).

We use this stratification to produce a bar complex that resolves the "discriminant locus" $\Delta_{d,n}$ in the sense that it comes with a map to $\Delta_{d,n}$ that induces isomorphisms on homology in a range of degrees (governed by how large d is compared to the degrees of the divisors). This resolution produces a spectral sequence whose terms are the compactly supported cohomology of vector bundles over configuration space, which converges to the compactly supported cohomology $\widetilde{\mathrm{Alg}}_{d,n}(C, X)$.

We then perform a similar construction for a 'semi-topological' version of $\mathrm{Alg}_{d,n}(C, X)$. Specifically, we introduce a moduli space $\mathcal{M}_{d,n}$ which is intermediate between the space of holomorphic and continuous maps. The points of $\mathcal{M}_{d,n}$ parameterize isomorphism classes of the following data:

- a degree d holomorphic line bundle L on C

- a collection of disjoint divisors $D_i \in \mathrm{Sym}^{n_i}(C), i = 1, \dots, r$
- a **continuous** section $s \in \Gamma_{\mathrm{top}}(C, L \otimes V)$

such that if $p \in C$ is a multiplicity k point of D_i, the section s intersects ℓ_i holomorphically to order exactly k in the following sense:

- the section $\overline{s} \in \Gamma(L \otimes V/\ell_i)$ takes the form $s = az^k + o(|z|^k)$, where z is a holomorphic local coordinate of L at p and $a \in V/\ell_i - 0$.

Taking a similar approach to the homology of $\mathcal{M}_{d,n}$, we show that $\mathrm{Alg}_{d,n}(C, X) \to \mathcal{M}_{d,n}$ induces an isomorphism on homology in a range of degrees (depending on d, n). Then we show that $\mathcal{M}_{d,n}$ is weakly homotopy equivalent to $\mathrm{Top}^+_{d,n}(C, X)$, in order to establish Theorem 1.1.

1.3. **Summary of paper.** In §2 we review Gysin maps, stratifications and bar constructions. In §3, we introduce the posets of divisors $\mathrm{Hilb}(C)^Q$ that we use to resolve the complement of $\widetilde{\mathrm{Alg}}_{d,n}(C, X)$. In §4, we introduce a stratification of $\mathrm{Hilb}(C)^Q$. In §5, we use this stratification to compute the homology of bar constructions over $\mathrm{Hilb}(C)^Q$. We also prove an important criterion for the homology of a simplicial resolution to agree with the homology of the discriminant locus, Theorem 5.4. In §6, we describe how to topologize $\mathcal{M}_{d,n}$ and similar function spaces. We also construct finite dimensional approximations to these function spaces, in order to be able to apply compactly supported cohomology. In §7, we use these finite dimensional approximations and Theorem 5.4 establish Theorem 7.1 a criterion for the map $\mathrm{Alg}_{d,n}(C, X) \to \mathcal{M}_{d,n}$ to induce an isomorphism on a range of homology groups. In §8, we show that semi-topological model is weakly homotopy equivalent to the space of positive maps. In §9, we show that certain spaces of sections are unobstructed in order to verify the hypotheses of Theorem 7.1 and deduce our main results.

1.4. **Relation to other work.** As far as we are aware, all previous results showing that the integral homology of $\mathrm{Alg}_*(\mathbb{P}^1, X)$ stabilizes to $\mathrm{Top}_*(\mathbb{P}^1, X)$ concern varieties X which are *homogeneous* in the sense that they admit an action of a solvable algebraic group N with a dense orbit. In these cases, there is a description of the algebraic mapping space in terms of certain labelled configuration spaces of zeroes and poles. In our case, when $r \geq \dim V + 2$ there is no analogous description. (Configuration spaces play a key role in our argument, but they appear in a different way). On the other hand, work of Browning–Sawin [BS20] on a geometric version circle method yields method for comparing the rational homology of pointed algebraic mapping spaces to low-degree affine hypersurfaces (which are not homogeneous) and double loop spaces.

There have been a number of applications of variants of the Vassiliev method to spaces of algebraic/holomorphic maps to homogeneous targets (often for higher dimensional source varieties) by Mostovoy, Koszlowski, Yamaguchi, [KY18; Ban22; Mos12] and related work by Banerjee [Ban22].

Aumonier has used the Vassiliev method to establish a general h-principle that relates holomorphic sections of a vector bundle satisfying incidence conditions to continuous sections of a jet bundle (satisfying the same incidence conditions) [Aum21]. Our results do not fit into this framework. Although the fiber of $\widetilde{\mathrm{Alg}}_{d,n}(C,X)$ above $L \in \mathrm{Pic}^d(C)$ equals the space of holomorphic sections $L \otimes V$ satisfying incidence conditions, these conditions are not locally closed subset of the jet bundle $J(L \otimes V)$. In fact, sections of $J(L \otimes V)$ satisfying these incidence conditions differ from the semi-topological model $\mathcal{M}_{d,n}$ that we compare $\mathrm{Alg}_{d,n}(C,X)$ with. (It is for this reason that we introduce $\mathcal{M}_{d,n}$, see Remark 7.2 for further discussion). Additionally, to obtain a stability range sharp enough to conclude Theorem 1.4, we need a weaker condition than jet ampleness.

Lehmann–Tanimoto have introduced a geometric analog of Batyrev–Manin's conjecture [LT19], concerning the number of irreducible components of the space of rational curves (related to degree 0 homology). In the del Pezzo surface case, this conjecture holds by the work of Testa [Tes09]. From the point of view of the analogy with the arithmetic conjectures, stability for higher order homology groups is related to the constant that appears in Peyre's version of the Manin conjectures [Pey95].

1.5. **Further questions.**

1.5.1. *Lower degree del Pezzo surfaces.* As mentioned above, our approach only yields partial information about degree ≤ 4 del Pezzo surfaces. For instance, we do not know the answer to the following question. For a degree 4 del Pezzo which is isomorphic to $\mathbb{P}^2$ blown up at 5 general points, when α is the anti-canonical class $(3,1,1,1,1,1)$ is the map

$$H_i(\mathrm{Alg}_{k\alpha,*}(\mathbb{P}^1,X)) \to H_i(\mathrm{Top}_{k\alpha,*}(\mathbb{P}^1,X))$$

is an isomorphism for $k \gg 0$? (More generally, we can consider the same question for classes α lying the cone spanned by an ϵ-neighborhood of the anti-canonical class).

1.5.2. *Positive mapping spaces.* Motivated by the appearance of the space of positive maps in Theorem 1.1, we ask the following question. Let M be a manifold, and let $E \subseteq M$ be a codimension 2 submanifold with connected components $E_1,\dots,E_r$, equipped with complex orientation of its normal bundle N_E

Let C be an oriented surface. We consider the space $\mathrm{Top}^+(C,M) \subseteq \mathrm{Top}(C,M)$ consisting of maps f with $f^{-1}(E)$ discrete and all local intersection multiplicities positive. Given $n \in \mathbb{Z}^r$ we let $\mathrm{Top}_n(C,M)$ denote the space of maps $f : \Sigma \to M$

such that $f_*(c_1(N_{E_i}) \cap [C] = n_i$. (When C is not compact, by definition $[C] = 0$ so this intersection number is always zero).

- Does the inclusion $\mathrm{Top}_n^+(C, M) \to \mathrm{Top}_n(C, M)$ induce an isomorphism on H_i for all $n \gg 0$?

Here by $n \gg 0$ we mean that $n_i \gg 0$ for every component n_i of n. A positive answer would be interesting, even in the specific case of a blowup of projective space, as it would immediately yield a generalization of Corollary 1.2 to the unpointed and higher genus case.

1.5.3. *Higher genus maps to solv-varieties.* To obtain Corollary 1.2, we apply the results of [BHM01] relating the spaces $\mathrm{Alg}_*(\mathbb{P}^1, X) \to \mathrm{Top}_*(\mathbb{P}^1, X)$, whenever X admits an action by a solvable algebraic variety with dense orbit. Is it possible to adapt these methods to the case where $\mathbb{P}^1$ is replaced by a higher genus curve, or to the unpointed case? The generality of Segal's results in the case $X = \mathbb{P}^n$ suggests a positive answer, which would yield a generalization of Corollary 1.2 to the unpointed and higher genus case.

1.5.4. *Arithmetic and motivic analogs.* Our approach to computing the homology of $\mathrm{Alg}_{d,n}(C, X)$ (the Vassiliev method) is a categorification of a strategy for counting the number of points $\mathrm{Alg}_{d,n}(C, X)(\mathbb{F}_q)$ (an inclusion-exclusion or sieve argument). Compared with de La Bréteche's argument establishing the Batyrev–Manin conjecture for a split degree 5 del Pezzo, it is surprising to us that a direct inclusion-exclusion approach to Theorem 1.4 is successful.

Can the inclusion-exclusion approach be used to establish the Batyrev–Manin conjectures (either over function fields or $\mathbb{Q}$)? The main difficulty in adapting our approach lies in bounding the error term arising from truncating the inclusion-exclusion term at finite level. In the setting considered in this paper, it suffices to bound the dimension of the homological contribution of the error term–so we expect our arguments to translate most easily to a "motivic" setting where we consider convergence of the classes $[\mathrm{Alg}_{d,n}(C, X)]$ in the Grothendieck ring of varieties with respect to a dimension filtration.

1.6. **Acknowledgements.** The authors would like to thank Alexis Aumonier, Sylvain Douteau, Peter Haine, and Brian Lehmann for discussions related to the topic of this paper.

2. Gysin maps and poset topology

2.1. **Gysin maps.** For a locally compact topological space X, we use $H_c^*(X)$ to denote the compactly supported cohomology. We take $H_c^*(X)$ to be defined in terms of sheaf cohomology: given any injective resolution of sheaves on X, $\mathbb{Z}_X \xrightarrow{\sim} \mathcal{F}$, the compactly supported cohomology of X is computed by $H^*(\pi_!\mathcal{F})$, where $\pi_!$ denotes the compactly supported global sections.

We recall that compactly supported cohomology is functorial for proper maps. If $U \to X$ is an open embedding, with complement Z then there is a canonical isomorphism $H^*_c(U) \cong H^*_c(X, Z)$ induced by the short exact sequence

$$0 \to j_!\mathbb{Z}_U \to \mathbb{Z}_X \to i_!\mathbb{Z}_Z \to 0.$$

More generally, for a finite filtration by closed subspaces

$$Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n = X,$$

the induced co-filtration

$$\mathcal{F} \to i_{n-1!}\mathcal{F}|_{Z_{n-1}} \to \cdots \to i_{0!}\mathcal{F}|_{Z_0}$$

gives rise to a spectral sequence with E_1 page

$$\bigoplus_{i=0}^{N} H^*_c(Z_i, Z_{i-1}) = \bigoplus_{i=0}^{N} H^*_c(U_i),$$

converging to $H^*_c(X)$. Here $U_i := Z_i - Z_{i-1}$, and the differentials of the spectral sequence are induced by the long exact sequences in cohomology for the pairs (Z_i, Z_{i-1}).

Now suppose that $i : Y \subseteq X$ is a closed embedding, and $\theta \in H^j(X, X - Y)$ is a class. Following Fulton–Macpherson [FM81], we think of θ as a generalized orientation class for i to which we may associate a Gysin map as follows. We have that

$$H^j(X, X - Y) = H^j(\pi_* i^! \mathbb{Z}_X) = \mathrm{Ext}^j_{\mathrm{Sh}(X)}(i_!\mathbb{Z}_Y, \mathbb{Z}_X),$$

and so θ induces a map in the derived category

$$\theta : i_!\mathbb{Z}_Y \to \mathbb{Z}_X[j].$$

Choosing an injective resolution $\mathbb{Z}_X \xrightarrow{\sim} \mathcal{F}$, we may choose a representative $\widetilde{\theta}$: $i_!\mathcal{F}|_Y \to \mathcal{F}[j]$. Applying $\pi_!$ we obtain a Gysin map $\theta_! : H^*_c(Y) \to H^{*+j}_c(X)$, independent of the choice of $\widetilde{\theta}$. Considering the filtration $Z_i \cap Y$, the representative $\widetilde{\theta}$ induces a map of co-filtered sheaves, so we obtain a Gysin map of spectral sequences:

$$\theta_! : \bigoplus_{i=0}^{N} H^*_c(U_i \cap Y) \to \bigoplus_{i=0}^{N} H^{*+j}_c(U_i).$$

In particular, if θ induces an isomorphism $H^*_c(U_i \cap Y) \to H^*_c(U_i)$ for all i, then we have $\theta_! : H^*_c(Y) \cong H^*_c(X)[j]$.

More generally, suppose we are given a poset P and collection of closed subsets $Z_p \subseteq X$, for $p \in P$ satisfying $Z_p \supseteq Z_q$ for $p \leq q$.

Definition 2.1. We refer to the data of P and $\{Z_p\}_{p\in P}$ as a *stratification of* X. The *closed strata* are the sets Z_p, and the *locally closed stratum* associated to $p \in P$ are the sets $S_p := Z_p - \bigcup_{p>q} Z_q$. $\Diamond$

Assume that P is finite, and choose a homomorphism of posets $r : P \to (\mathbb{N}, \leq)$ such that $p < q$ implies $r(p) < r(q)$. From r, we obtain an associated filtration

$$Z_0 \supseteq \cdots \supseteq Z_n, \qquad Z_i := \bigcup_{p,\ r(p)\geq i} Z_p.$$

Then $Z_i - Z_{i-1} = \bigsqcup_{p,\ r(p)=i} S_p$, and so we obtain a spectral sequence with E_1 page $\bigoplus_p H^*_c(S_p)$, converging to $H^*(X)$. Again, a closed embedding $i : Y \to X$ and orientation class $\theta \in H^*(X, X-Y)$ induces a Gysin map of spectral sequences.

The following compatibility between Poincaré duality and Gysin maps is standard (see eg [MS74, Problem 11-C]), we state it here for convenience.

Proposition 2.2. *Let $E \subseteq F$ be an inclusion of complex vector bundles over a manifold M. Write $\dim F$, $\dim E$ for the real dimension of the total spaces of F, E respectively. Let $\theta \in H^{\dim F - \dim E}(F, F - E)$ be the associated Thom class. Then for every open subsets $U \subseteq F$, the diagram*

$$\begin{tikzcd}
H_i(U \cap E) \arrow[r] \arrow[d] & H_i(U) \arrow[d] \\
H_c^{\dim E - i}(U \cap E) \arrow[r, "\theta_!"] & H_c^{\dim F - i}(U)
\end{tikzcd}$$

commutes, where the vertical arrows are the Poincaré duality isomorphisms.

2.2. **Topological posets and bar complexes.** In this section, we recall several facts about simplicial resolutions arising from topological posets.

A *topological poset* is a topological space P and a relation $\mathcal{R}_P \subseteq P \times P$, that is reflexive, antisymmetric, and transitive. We say that the relation $\mathcal{R}_P$ is *proper* if both projections $\mathcal{R}_P \to P$ are proper.

A P-*space*, consists of a continuous map $Z \to P$ together with continuous map of spaces over P

$$a : \mathcal{R}_P \times_{\pi_2} Z \to Z$$

where $\pi_2 : \mathcal{R}_P \to P$ denotes the map $(p \leq q) \mapsto q$ and $\mathcal{R} \times_{\pi_2} Z$ is considered as a space over P via π_1. We write Z_p for the fiber of Z over $p \in P$.

Given a P-space, we may form the bar construction:

Definition 2.3. Let P be a topological poset such that $\Delta_P \subseteq \mathcal{R}_P$ is open. Let Z be a P-space. The (semi-simplicial) *bar construction* is

$$\mathrm{B}_t(Z,P) := \{p_0 < p_1 \cdots < p_t \in P,\ z \in Z_{p_t}\} = (\mathcal{R}_P - \Delta_P)^{\times_P (t+1)} \times_P Z.$$

The boundary maps are given by forgetting elements in the chain, and applying the action map a when p_t is forgotten. We write $|\mathrm{B}(Z,P)|$ for the geometric realization. ◊

We use topological posets to define a continuous stratifications of a topological space. Let X be a topological space, and P be a topological poset.

Definition 2.4. A *continuous stratification of* X *by* P is a closed subspace $Z \subseteq P \times X$ satisfying $Z_p \supseteq Z_q$ if $p \leq q$. ◊

Let $Z \subseteq P \times X$ be a continuous stratification of X, and let $W \subseteq P$. Since Z is canonically a P space, we may use the bar construction to resolve the complement of the union of strata $\bigcup_{w \in W} S_w \subseteq Z \times_P W$, in terms of the following poset associated to W and P.

Definition 2.5. Let $W < P$ be the following topological poset. We let

$$W < P := \{(w,p) \in W \times P \mid w < p\} = W \times_P (\mathcal{R}_P - \Delta_P),$$

and define the relation

$$\mathcal{R}_{W<P} := \{(w,p_1),(w,p_2) \mid w < p_1 \leq p_2\} = (\Delta_W \times \mathcal{R}_P) \cap (W < P)^{\times 2}. \qquad ◊$$

There is a a morphism of posets $\pi : (W < P) \to P$, and a map of topological spaces $(W < P) \to W$. Pulling Z back to $W < P$, we obtain a $(W < P)$ space, denoted $\pi^* Z$. Applying the Vassiliev bar construction, we obtain a semi-simplicial space

$$\mathrm{B}(W,P,Z) := \mathrm{B}(W < P, \pi^* Z),$$

together with an augmentation map $a : \mathrm{B}(W,P,Z) \to W \times_P Z$ that takes a chain $w < p_1 < \cdots < p_r, z \in Z_{p_r}$ to the pair $w \in W, z \in Z_w$.

3. Posets of divisors and section spaces

In this section, we introduce a family of topological posets and use them to resolve spaces of algebraic sections satisfying certain incidence conditions. The level of generality taken here is greater than necessary for our main application, but we have found the general framework helpful, and include it with an eye towards later work. We will include the case relevant for applications as a running example.

3.1. **Divisors labeled by Q.** Let C be an algebraic curve. We write $\overline{\mathrm{Hilb}}(C)$ for the poset of all closed subschemes of C (including C), and $\mathrm{Hilb}(C)$ for the poset of finite length subschemes.

Definition 3.1. Let Q be a finite poset with top element $\hat{1}$. We put $\widetilde{Q} := Q - \hat{1}$ We define $\overline{\mathrm{Hilb}}(C)^Q$ to be the poset of homomorphisms $Q \to \overline{\mathrm{Hilb}}(C)$ which take $\hat{1}$ to C. We let $\mathrm{Hilb}(C)^Q$ denote the subposet consisting of homomorphisms such that the preimage of C is precisely $\hat{1}$. $\lozenge$

In other words, an element of $\mathrm{Hilb}(C)^Q$ consists of a collection of finite closed subschemes

$$\{D_q \subseteq C\}_{q\in\widetilde{Q}},$$

satisfying the property $D_p \subseteq D_q$ if $p \leq q$. We have that

$$\mathrm{Hilb}(C)^Q \subseteq \prod_{q\in\widetilde{Q}} \mathrm{Hilb}(C)$$

is a closed subscheme.

We may equivalently describe an element of $\mathrm{Hilb}(C)^Q$ as a function which assigns to each $c \in C$ a family of closed subschemes $\{D_q\}_{q\in\widetilde{Q}}$, which are supported at c and satisfy the containment relation $D_p \subseteq D_q$ for all $p \leq q$. Since finite subschemes supported at c are determined by their multiplicity (i.e. $\mathrm{Hilb}(C)$ can be identified with finitely supported functions $C \to \mathbb{N}$), this yields a correspondence between points of $\mathrm{Hilb}(C)^Q$ and finitely supported functions

$$C \to \mathrm{Hom}(\widetilde{Q}, \mathbb{N}).$$

Here $\mathbb{N} \cup \infty$ is the poset of non-negative numbers and Hom denotes the poset of (poset homomorphisms $Q \to \mathbb{N}$. The *trivial homomorphism* is the homomorphism $q \mapsto 0$ for all $q \in \widetilde{Q}$, and the *support* of a function $C \to \mathrm{Hom}(\widetilde{Q}, \mathbb{N})$ is the set of $c \in C$ which map to a nontrivial homomorphism.

3.2. **Stratifying sections.** In this subsection, we use $\mathrm{Hilb}(C)^Q$ to continuously stratify spaces of sections over C.

For the remainder of the paper, we will assume that Q has all meets. Let $\mathcal{X} \to C \times \mathcal{B}$ be a family of schemes of finite presentation over $C \times \mathcal{B}$, together collection of closed subschemes $\mathcal{K}_q \subseteq \mathcal{X}$ for all $q \in Q$ satisfying:

- $\mathcal{K}_{\hat{1}} = X$
- $\mathcal{K}_{p\wedge q} = \mathcal{K}_p \cap \mathcal{K}_q$ for all $p, q \in Q$.

There is an algebraic space of sections $\Gamma_{\mathrm{alg}}(C, \mathcal{X}) \to \mathcal{B}$, representing the functor

$$(f : T \to B) \mapsto \Gamma(C \times T, f^*\mathcal{X}).$$

(c.f. [**stacks-project**]). (In the cases of interest $\Gamma_{\mathrm{alg}}(C, \mathcal{X})$ will be representable by a scheme).

There are universal algebraic subspaces

$$\mathcal{D}_q \subseteq \mathrm{Hilb}(C)^Q \times C$$

for $q \in Q$, which satisfy $\mathcal{D}_q \subseteq \mathcal{D}_{q'}$ if $q \leq q'$. By restricting sections of C to sections of D_q, we may form the scheme $\Gamma_Q(C, \mathcal{X})$ parameterizing sections $s \in \Gamma(C, \mathcal{X}_b)$ such that $s(D_q) \subseteq (\mathcal{K}_q)_b$ as a fiber product:

$$\begin{tikzcd}
\Gamma_Q(C, \mathcal{X}) \arrow[r] \arrow[d] & \prod_{q \in \widetilde{Q}} \Gamma(\mathcal{D}_q, \mathcal{K}_q) \arrow[d] \\
\Gamma(C, \mathcal{X}) \times \mathrm{Hilb}(C)^Q \arrow[r] & \prod_{q \in \widetilde{Q}} \Gamma(\mathcal{D}_q, \mathcal{X}).
\end{tikzcd}$$

Note that the underlying topological space of $\Gamma_Q(C, \mathcal{X})$ is an $\mathrm{Hilb}(C)^Q \times B$-space: if $D_q \subseteq D'_q$ for all $q \in Q$ then the set of sections mapping Z'_q to $\mathcal{K}_q$ is contained contained in the set of sections mapping D_q to $\mathcal{K}_q$. Thus $\Gamma_Q(C, \mathcal{X})$ defines a continuous stratification of $\Gamma(C, \mathcal{X})$ by $\mathrm{Hilb}(C)^{\mathrm{Q}} \times \mathcal{B}$, according the preimages of $\mathcal{K}_q$.

Example 3.2. The following is our main example. Let V be a vector space, and $\ell_1, \ldots, \ell_r \subseteq V$ be lines. Let $\mathcal{B} = \mathrm{Pic}^d(C)$ and $\mathcal{X} = \mathcal{L}_d \otimes V$. Then $\mathcal{X}$ is stratified by the poset $Q_r := \{V, \ell_1, \ldots, \ell_r, 0\}$ consisting of the subspaces ℓ_i, V and 0 ordered by containment: the stratum corresponding to a subspace $S \subseteq V$ is $S \otimes \mathcal{L}_d$. $\lozenge$

3.3. **Saturated elements.** Many distinct elements of $\mathrm{Hilb}(C)^Q$ impose the same incidence conditions on a space of sections, and are in a sense "redundant." Using the lattice structure of Q, we may construct a smaller poset that removes these redundancies and is combinatorially simpler, keep in mind that Q is finite and assumed to have meets.

Definition 3.3. We say that an element $(Z_q)_{q \in Q} \in \mathrm{Hilb}(C)^Q$ is *saturated* if for every $S \subseteq Q$ the natural containment

$$Z_{\bigwedge_{s \in S} s} \subseteq \bigcap_{s \in S} Z_s$$

is an equality. We write Q^{JC} for the *subposet of saturated elements*. Note that Q^{JC} is a Zariski open subset of $\mathrm{Hilb}(C)^Q$. $\lozenge$

We defined an element of $\mathrm{Hilb}(C)^Q$ to be saturated if and only if the associated homomorphism $Q \to \mathrm{Hilb}(C)$ preserves meets. In terms of finitely supported functions $C \to \mathrm{Hom}(\widetilde{Q}, \mathbb{N})$, saturated elements are functions which assign to every $c \in C$ a meet-preserving homomorphism $\widetilde{g}_c : \widetilde{Q} \to \mathbb{N}$.

We write $g_c : Q \to \mathbb{N} \cup \infty$ for the extension of $\widetilde{g}_c$ defined by $g_c(\hat{1}) = \infty$. Because g_c preserves meets, it admits a left adjoint $f_c : \mathbb{N} \cup \infty \to Q$ given by

$$f_c(n) := \inf\{q \in Q \mid g_c(q) \geq n\}.$$

Conversely the adjoint f_c, which must be join preserving, uniquely determines g_c. More precisely we obtain a bijection:

$$\begin{aligned} &\{f_c : \mathbb{N} \cup \infty \to Q \mid f_c \text{ join preserving and } f_c(\infty) = \hat{1}\} \\ &\qquad\qquad \longleftrightarrow \{\widetilde{g}_c : \widetilde{Q} \to \mathbb{N} \text{ meet preserving}\} \end{aligned} \tag{3.4}$$

Here we have used that for an adjoint pair (f_c, g_c) the condition that $f_c(\infty) = \hat{1}$ is equivalent to $g_c^{-1}(\infty) = 1$.

The correspondence (3.4) is inequality reversing: we have that $f_c \leq f'_c$ if and only if $g_c \geq g'_c$. It motivates the following definitions.

Definition 3.5. We call a function $f : \mathbb{N} \cup \infty \to Q$ a *chain of* Q if $f(\infty) = \hat{1}$ and it preserves joins, i.e. satisfies $f(0) = 0$ and $f(\infty) = f(n) = \hat{1}$ for all $n \gg 0$. We write $\mathrm{Ch}(Q)$ for the set of all chains of Q. The *trivial chain* is the largest element of $\mathrm{Ch}(Q)$: the function $0 \mapsto 0$ and $i \mapsto \hat{1}$ for $i > 0$. ♢

We have the following immediate consequence of (3.4).

Proposition 3.6. *There is a canonical bijection between $Q^{\mathrm{J}C}$ and the set of finitely supported functions $C \to \mathrm{Ch}(Q)$. (Here the support of f is the set of c for which the chain $f(c)$ is nontrivial.)*

Remark 3.7. We use the notation $Q^{\mathrm{J}C}$ because it is possible to think of the chain f_c as describing the incidence relations of a map from the jet at c to Q: the ith element of the chain represents the smallest stratum that ith order jet is contained in. ♢

Definition 3.8. We define the *saturation function*

$$\mathrm{sat} : \mathrm{Hilb}(C)^Q \to Q^{\mathrm{J}C}$$

to be the left adjoint to the inclusion of the subposet of saturated elements. In other words, sat takes an element of $x \in \mathrm{Hilb}(C)^Q$ to the smallest saturated element which is greater than or equal to x. The function sat exists because $Q^{\mathrm{J}C}$ is closed under meets. ♢

Remark 3.9. Note that sat is *not* continuous if $Q^{\mathrm{J}C}$ is given the subspace topology as a Zariski open subset of $\mathrm{Hilb}(C)^Q$. However (for our purposes) it would be more natural to think of $Q^{\mathrm{J}C}$ as a quotient poset of $\mathrm{Hilb}(C)^Q$ (making sat a quotient map), from which the stratification of the space of sections is pulled back. Then bar constructions over $\mathrm{Hilb}(C)^Q$ would retract onto bar constructions over $Q^{\mathrm{J}C}$, simplifying them considerably.

Unfortunately, when one attempts to topologize $Q^{\mathrm{J}C}$ as a quotient, one obtains a non-Hausdorff space which is difficult to understand. For this reason, we will work directly with $\mathrm{Hilb}(C)^Q$ but show that (after stratification) bar constructions over $\mathrm{Hilb}(C)^Q$ are closely related to $Q^{\mathrm{J}C}$. This indirect relationship between $Q^{\mathrm{J}C}$ and $\mathrm{Hilb}(C)^Q$ is a key technical point for us. $\Diamond$

Example 3.10. We continue our main example from Example 3.2, with the poset $Q_r = \{V, \ell_1, \dots, \ell_r, 0\}$. In this case, an element of $\mathrm{Hilb}(C)^{Q_r}$ consists of a collection of divisors $D_{\ell_1}, \dots, D_{\ell_r}, D_0 \in \mathrm{Hilb}(C)$, such that $D_0 \subseteq D_{\ell_i}$ for all i. An element is saturated if and only if $D_0 = D_{\ell_i} \cap D_{\ell_j}$ for all $i \neq j$. The saturation operation replaces D_0 by $D_0' = \bigcup_{i \neq j} D_{\ell_i} \cap D_{\ell_j}$ and every other D_{ℓ_i} by $D_{\ell_i} \cup D_0'$.

For $c \in C$, the associated function $\widetilde{g}_c : \widetilde{Q}_r \to \mathbb{N}$ consists of values $\widetilde{g}_c(\ell_i), \widetilde{g}_c(0)$ recording the multiplicity of D_{ℓ_i} and D_0 at c. This function is saturated if and only if $\widetilde{g}_c(0)$ equals all but the maximum of $\{\widetilde{g}_c(\ell_i)\}_{i=1}^r$. In this case, suppose that the maximum is achieved by ℓ_j, and let $m_0 = \widetilde{g}_c(0)$ and $m_{\ell_j} = \widetilde{g}_c(\ell_j) - \widetilde{g}_c(0)$. The associated chain $f_c : \mathbb{N} \cup \infty \to Q_r$ is given by

$$f_c(n) = \begin{cases} 0 & \text{if } n < m_0 \\ \ell_j & \text{if } m_0 \leq n < m_{\ell_j} + m_0 \\ \infty & \text{if } m_{\ell_j} + m_0 \leq n. \end{cases}$$

We denote this chain by $m_0 0 + m_{\ell_j} \ell_j$ (because it takes the value q, m_q times). $\Diamond$

3.4. **Essential saturated elements.** Because Q has meets, the poset of chains $\mathrm{Ch}(Q)$ has joins: the join of the chains $q_0 \leq q_1 \dots$ and $p_0 \leq p_1 \dots$ is $p_0 \wedge q_0 \leq \dots$. (Meets and joins are exchanged because our convention is that the ordering $\mathrm{Ch}(Q)$ is opposite to Q). There is a canonical injective homomorphism $Q^{\mathrm{op}} \to \mathrm{Ch}(Q)$ given by taking q to the chain $q \leq \hat{1} \leq \hat{1} \cdots$.

Definition 3.11. Let $f_0 \in \mathrm{Ch}(Q)$ be a chain, and let S be the collection of chains f_s such that $f_0 \prec f_s$, ie such that $f_0 < f_s$ and there is no f' with $f_0 < f' < f_s$. We say that a pair of elements $f_0 \leq f \in \mathrm{Ch}(Q)/\mathrm{Ch}(Q)$ is *essential* if f can be obtained from f_0 by taking the join of some subset of S. We say that $(w \leq x) \in (W \leq Q^{\mathrm{J}C})$ is *essential* if it is pointwise essential. (In other words if for every $c \in \mathrm{supp}(w \leq x)$ we have that $f_c^w \leq f_c^x$ is essential). $\Diamond$

Example 3.12. In our main example, we take $Q = Q_r = \{0, \ell_1, \dots, \ell_r, V\}$ and let $W \subseteq \prod_{i=1}^k \mathrm{Sym}^{k_i} C$ be the set of elements of the product with disjoint support. We use $\ell_1, \dots, \ell_r$ as our distinguished generators. Then a saturated element $(w \le x) \in (W \le Q^{\mathrm{J}C})$ is *essential* if for every $c \in \mathrm{supp}(w \le x)$ we have that $f_c^w \le f_c^x$ is one of the following pairs of chains:

- $d\ell_i \le (d+1)\ell_i$
- $d\ell_i \le (d-1)\ell_i + 0$
- $d\ell_i \le d\ell_i + 0$
- $V \le \ell_i$ (Note that here V denotes the chain $V \le \dots$ and ℓ_i denotes the chain $\ell_i \le V \le \dots$).
- $V \le 0$ (if $r \ge 2$) $\Diamond$

Our main motivation for introducing essential pairs is the following version of the crosscut theorem.

Proposition 3.13. *For any $f_0 \in \mathrm{Ch}(Q)$, the essential pairs in $f_0 < \mathrm{Ch}(Q)$ form an initial subposet of $f_0 < \mathrm{Ch}(Q)$. In particular $N(f_0 < \mathrm{Ch}(Q))$ deformation retracts onto the nerve of the subposet of essential pairs.*

Proof. By definition, we need to show that for any (f_0, f) there is a maximal essential pair $(f_0, \widetilde{f}) \le (f_0, f)$. We take $\widetilde{f}$ to be the join of every f_s such that $f \prec f_s \le f$. Then the inclusion of the subposet of essential pairs admits a right adjoint, and so the statement follows. $\square$

3.5. **Combinatorial functions on $Q^{\mathrm{J}C}$.** Let h be a function $h : Q \to \mathbb{N}$, satisfying $h(\hat{1}) = 0$.

We may extend h to a function $Q^{\mathrm{J}C} \to \mathbb{N}$, as follows. First we may extend h to a function $h : \mathrm{Ch}(Q) \to \mathbb{N}$ by defining $h(f) = \sum_{n \ge 1} f(n)$, which is a finite sum since $f(n) = \hat{1}$ for $n \gg 0$. Next recall that an element $x \in Q^{\mathrm{J}C}$ corresponds to a finitely supported function

$$C \to \mathrm{Ch}(Q), \qquad c \mapsto f_c,$$

so we define $h(x) := \sum_{c \in C} h(f_c)$.

We may further extend h to a function on $\mathrm{Hilb}(C)^Q$ by precomposing with sat, and extend to pairs of elements by declaring $h(w \le x) = h(w) - h(x)$.

Example 3.14. Let $q_0 \in \widetilde{Q}$, and let m_{q_0} be the indicator function defined by $m_{q_0}(q) = 1$, if $q = q_0$ and $m_{q_0}(q) = 0$ otherwise. We refer to $m_{q_0} : \mathrm{Hilb}(C)^Q \to \mathbb{N}$ as the *multiplicity of* q. $\Diamond$

Example 3.15. Let $\mathrm{rank} : \mathrm{Ch}(Q) \to \mathbb{N}$ be the rank function, where $\mathrm{rank}(x)$ is the length of the longest chain from $\hat{1}$ to x. We extend rank to a function on $Q^{\mathrm{J}C}$ as above. The value $\mathrm{rank}(x)$ is the length of the maximal chain starting at $\hat{1}$ and ending at x, or equivalently the dimension of the simplicial complex $\mathrm{N}([\hat{1}, \mathrm{sat}(x)] \cap Q^{\mathrm{JC}})$. We note that when Q is a *graded poset* in the sense that every maximal chain has the same length, then the rank function on $\mathrm{Ch}(Q)$ can be obtained from the rank function on Q by the extension procedure above. $\lozenge$

Example 3.16. Given $x \in \mathrm{Hilb}(C)^Q$ we define $\mathrm{Supp}(x)$ to be the cardinality of the support of x. This function can be obtained from the above extension procedure from the function of $\mathrm{Ch}(Q)$ which is 0 on the trivial chain and 1 otherwise. The value $|\mathrm{Supp}(x)|$ is the dimension of the subspace of elements $x' \in Q^{\mathrm{JC}}$ with the same *combinatorial type* as x (to be defined in the next section). $\lozenge$

Example 3.17. . Let $Q_r = \{0, \ell_1, \dots, \ell_r, V\}$ be the poset in our main example. We have two important examples of functions.

(1) We have that $\mathrm{rank}(\ell_i) = 1$ and $\mathrm{rank}(0) = 2$. This poset is graded, so these values determine the others.
(2) Let $\gamma : Q_r \to \mathbb{N}$ be the function defined by $\gamma(\ell_i) = 2(\dim V - 1)$ and $\gamma(0) = 2 \dim V$. We extend γ to a function on Q_r^{JC}. Heuristically $\gamma(x)$, is the expected real codimension of the incidence conditions imposed by x. $\lozenge$

4. Combinatorial stratifications of $\mathrm{Hilb}(C)^Q$ and Q^{JC}

Set theoretically, both Q^{JC} and $\mathrm{Hilb}(C)^Q$ can be decomposed into a disjoint union of labeled configuration spaces. To describe these decompositions, we introduce some terminology. We will use these decompositions to stratify bar constructions in the next section.

4.1. **Combinatorial types.** Recall that an element $x \in \mathrm{Hilb}(C)^Q$ corresponds to a finitely supported function

$$C \to \mathrm{Hom}(\widetilde{Q}, \mathbb{N}) \qquad c \mapsto \widetilde{g}_c.$$

The *combinatorial type* of such a element is the **multiset** of functions

$$\mathrm{type}(x) := \{\widetilde{g}_{c_1}, \dots, \widetilde{g}_{c_r} : \widetilde{Q} \to \mathbb{N}\},$$

where $c_i \in C$ are the elements such that $\widetilde{g}_{c_i}$ is nontrivial (i.e. not the constant function 0 on $\widetilde{Q}$). The *set of combinatorial types* is the set of all multisubsets of functions $g : \widetilde{Q} \to \mathbb{N}$.

Note that x is an element of Q^{JC} if and only if each $\widetilde{g} \in \mathrm{type}(x)$ is meet preserving, in which case each g corresponds to a chain $f \in \mathrm{Ch}(Q)$. In this case, we say the multi-subset, $\mathrm{type}(x)$, is *saturated.* There is a canonical bijection between saturated types and multisubsets of $\mathrm{Ch}(Q)$; we will use this bijection to specify saturated types. Furthermore we obtain a saturation function from combinatorial types to saturated types, by applying $\mathrm{sat} : \mathrm{Hilb}(C)^Q \to Q^{\mathrm{J}C}$ pointwise.

Definition 4.1. Given a multiset T of functions $Q \to \mathbb{N}$, we let $\mathcal{N}_T$ denote the subspace of all $x \in \mathrm{Hilb}(C)^Q$ such that $\mathrm{type}(x) = T$. $\Diamond$

We have that $\mathcal{N}_T$ is homeomorphic to $\mathrm{Conf}(C;T)$, the space of configurations of C with labels in the multiset T. And $\mathcal{N}_T \subseteq Q^{\mathrm{J}C}$ if and only if T is a saturated type.

4.2. **Stratifications.** We now define a partial order on the set of combinatorial types, taking into account both the closure relations of $\mathcal{N}_T$ and the poset structure of $\mathrm{Hilb}(C)^Q$. Using this order we will stratify $\mathrm{Hilb}(C)^Q$.

Definition 4.2. We say that $\{g_1, \dots, g_l\} \geq_+ \{h_1, \dots, h_s\}$ if there is a function $F : [l] \to [s]$ such that $h_j \leq \sum_{i \in F^{-1}(j)} g_i$ for all $j \in [s]$. $\Diamond$

The set $\bigcup\limits_{S \leq_+ T} \mathcal{N}_S$ is both closed and downward-closed in the poset structure on $\mathrm{Hilb}(C)^Q$.

Definition 4.3. Given a saturated combinatorial type S, we define

$$\mathcal{S}_S := \bigcup_{T,\ \mathrm{sat}(T)=S} \mathcal{N}_T,$$

where the union is over combinatorial types whose saturation is S. $\Diamond$

We note that the subspace topology on $\mathcal{S}_S$ agrees with the disjoint union topology, $\bigsqcup\limits_{T,\ \mathrm{sat}(T)=S} \mathcal{N}_T$. (Because $|\mathrm{sat}(T)| = |T|$, if both T_1, T_2 saturate to S, no element of $\mathcal{N}_{T_1}$ is in lies in the closure of $\mathcal{N}_{T_2}$).

We define a new stratification of $\mathrm{Hilb}(C)^Q$ whose locally closed strata are given by $\mathcal{S}_S$, where S ranges over saturated types. This stratification is a coarsening using of the previous one, obtained by constructing a poset structure on the set of saturated types as follows.

Proposition 4.4. *There exists a poset structure $\leq_{+,\mathrm{sat}}$ on the set of saturated types, which is the coarsest one such that the saturation function from the set of types to the set of saturated types is a morphism of posets with intertwining*

$\leq_+$ *and* $\leq_{+,\mathrm{sat}}$.

Motivated by Proposition 4.4, for any saturated type S we define

$$\mathcal{Z}_S := \bigcup_{T \text{ saturated},\ T \leq_{+,\mathrm{sat}} S} \mathcal{S}_T = \bigcup_{\widetilde{T},\ \mathrm{sat}(\widetilde{T}) \leq_{+,\mathrm{sat}} S} \mathcal{N}_{\widetilde{T}}.$$

It follows that $\mathcal{Z}_S$ is both closed and downward closed in $\mathrm{Hilb}(C)^Q$, because

$$\{\widetilde{T} \mid \mathrm{sat}(\widetilde{T}) \leq_{+,\mathrm{sat}} S\}$$

is downward closed with respect to $\leq_+$.

In addition, we have that $\mathcal{Z}_{S_1} \subseteq \mathcal{Z}_{S_2}$ if and only if $S_1 \leq_{+,\mathrm{sat}} S_2$ and

$$\mathcal{Z}_S - \bigcup_{T <_{+,\mathrm{sat}} S} \mathcal{Z}_T = \mathcal{S}_S.$$

Proof of Proposition 4.4. We define $\leq_{+,\mathrm{sat}}$ to be the transitive closure of the relation

$$S_1 \preceq_+ S_2 \text{ if there exist types } T_1, T_2 \text{ with } T_1 \leq_+ T_2 \text{ and } \mathrm{sat}(T_i) = S_i.$$

To prove that $\leq_{+,\mathrm{sat}}$ is a poset relation, it suffices to show antisymmetry.

Consider the function $|-|$ assigning to a type T the number of elements of T. This function is decreasing and we have that $|\mathrm{sat}(T)| = |T|$.

Thus given chains of elements $S \preceq_+ \cdots \preceq_+ S'$ and $S' \preceq_+ \cdots \preceq_+ S$ it follows that $|S| = |S'|$. So for each $S_1 \preceq'_+ S_2$ in the chain, there exist types T_1, T_2 with $\mathrm{sat}(T_i) = S_i$ and $T_1 \leq'_+ T_2$. We have $|T_1| = |T_2|$ so we have $T_1 = \{e_1, \ldots, e_l\}$ and $T_2 = \{h_1, \ldots, h_l\}$ and there is a permutation $\sigma \in S_l$ such that $e_{\sigma i} \leq h_i$ for all $i = 1, \ldots l$. Then $\mathrm{sat}(e_{\sigma i}) \leq \mathrm{sat}(h_i)$ for all $i = 1, \ldots, l$. Thus writing $S = \{g_1, \ldots, g_l\}$ and $S' = \{g'_1, \ldots, g'_l\}$, we obtain permutations such that τ and τ' such that $g_{\tau i} \leq g'_i$ and $g'_{\tau' i} \leq g'_i$ for all i, as elements of $\mathrm{Hom}(Q, \mathbb{N})$. Iterating, we obtain that $g'_i \geq g_{\tau i} \geq g_{(\tau'\tau)^n i}$ for all $n \in \mathbb{N}$. Since $\tau'\tau$ has finite order, we obtain that $g'_i = g_{\tau i}$ for all $i = 1, \ldots, l$. So $S_1 = S_2$ as desired. $\square$

4.3. **Relative variant.** We may extend all of the definitions in this section to pairs of elements

$$(w \leq x) \in (\mathrm{Hilb}(C)^Q \leq \mathrm{Hilb}(C)^Q)$$

by observing that $(\mathrm{Hilb}(C)^Q \leq \mathrm{Hilb}(C)^Q) = \mathrm{Hilb}(C)^{Q'}$ where Q' is the poset $(\cdot \to \cdot) \times Q$.

For convenience, we expand out these definitions. The *combinatorial type* of an element $w \leq x$ is the multi-set

$$\mathrm{type}(w \leq x) = \{g^w_{c_1} \leq g^x_{c_1}, \ldots, g^w_{c_k} \leq g^x_{c_k}\},$$

of pairs of elements of $\mathbb{N}^Q$, where c_i ranges over the elements of C for which g^x_c or g^w_c is nontrivial. We obtain a partial order $\leq_+$ on the set of types, and a stratification of

$(\mathrm{Hilb}(C)^Q \leq \mathrm{Hilb}(C)^Q)$ by subvarieties, with open strata $\mathcal{N}_T$ the set of pairs $w \leq x$ with $\mathrm{type}(w \leq x) = T$. And we obtain a notion saturated type, and poset structures $\leq_{+,\mathrm{sat}}$ on saturated types such that $\mathcal{Z}_S = \bigcup_{\mathrm{sat}(T) \leq_{+,\mathrm{sat}} S} \mathcal{N}_T$ is both closed and downward closed, and satisfies

$$\mathcal{S}_S = \mathcal{Z}_S - \bigcup_{T <_{+,\mathrm{sat}} S} \mathcal{Z}_T.$$

4.4. **Pointed variant.** In dealing with pointed maps from a pointed curve C, it is useful to work with a pointed version of combinatorial types where the label of the basepoint is specified. To that end, we define *pointed combinatorial type* T to be a pair consisting of $g_0 \in \mathbb{N}^Q$ and T' an ordinary combinatorial type.

When C has a distinguished base point $*$ we may associate a pointed combinatorial type to an element $x \in \mathrm{Hilb}(C)^Q$ by taking $g_0 = x_*$ and taking T to be the multiset of labels of points $C - *$.

In this setting, we define the stratum $\mathcal{N}_T$ associated to to T to be the set of all x such that $\mathrm{type}(x) = T$. In this case $\mathcal{N}_T$ is homeomorphic to $\mathrm{Conf}(C-*, T')$. There is a partial order $\leq_+$ on pointed combinatorial types, defined analogously to the partial order on combinatorial types. We say that $(x_0, \{g_1, \ldots, g_l\}) \geq_+ (h_0, h_1, \ldots, h_s)$ if there is a map of pointed sets $F : \{0\} \cup [l] \to \{0\} \cup [s]$ such that $h_i \leq \sum_{i \in F^{-1}(j)} g_i$. There an associated poset relation $\leq_{+,\mathrm{sat}}$ and stratification with strata $\mathcal{S}_T$.

5. Stratifying bar constructions

In this section, we will use the stratification introduced in §4 to prove two important theorems about bar constructions associated to subposets of $\mathrm{Hilb}(C)^Q$. Throughout, we let C be an algebraic curve, possibly with a chosen basepoint. We will treat the pointed and unpointed case simultaneously. In the pointed case, all combinatorial types are assumed to be pointed.

Let $W \subseteq \mathrm{Hilb}(C)^Q$ be a locally closed union of subsets $\mathcal{N}_T$ where T is a saturated combinatorial type of $\mathrm{Hilb}(C)^Q$. Let $P \subseteq (W < \mathrm{Hilb}(C)^Q)$ be a downward closed union of finitely many many $\mathcal{S}_T$ for T a saturated combinatorial type of $W < \mathrm{Hilb}(C)^Q$, such that $P \to W$ is proper. Let $\mathcal{B}$ be a locally compact topological space and let $U \subseteq \mathcal{B} \times W$ be an open subset. We write $U \leq \mathrm{Hilb}(C)^Q$ and $P_U \to U$ for the poset obtained by pulling back P to U.

Suppose we are given a space $E \to U$ with a stratification by $U \leq \mathrm{Hilb}(C)^Q$, denoted by $Z \subseteq E \times_U (U \leq \mathrm{Hilb}(C)^Q)$. We assume this stratification satisfies the additional condition that the inclusion $Z_{b,w<x} \supseteq Z_{b,w<\mathrm{sat}(x)}$ is an equality for all

b, w, x. We write $\mathrm{B}(U, P, Z)$ for the bar construction with r-simplices

$$\{(b, w) \in U, w < x_0 < \cdots < x_r \in \mathrm{Hilb}(C)^Q, s \in Z_{(b,w<x_r)} \mid w < x_i \in P\}.$$

5.1. **Spectral sequence.** First we will describe a spectral sequence associated to a stratification of the bar construction.

If T is a combinatorial type of P so that $\mathcal{Z}_T \subseteq P$, we can apply the bar construction to $\mathcal{Z}_T$ to obtain a closed semi-simplicial subspace $\mathrm{B}(U, \mathcal{Z}_T, Z) \subseteq \mathrm{B}(U, P, Z)$. Thus we obtain a stratification of the geometric realization $|\mathrm{B}(U, P, Z)|$ by closed subspaces. Let $\mathcal{Y}_T$ be the locally closed stratum associated to T. A point of $\mathcal{Y}_T$ is specified by the following data

- an element $(b, w) \in U$,
- a point t in the interior of Δ^r
- a chain $w < x_0 < \cdots < x_r$ with $w < x_i \in P$ for all i and $w < x_r \in \mathcal{S}_T$
- an element $s \in Z_{b,w<x_r}$

The space $Z|_{U \times_W \mathcal{N}_T}$ parameterizes the data of

- $(b, w) \in U$
- $(w < x) \in \mathcal{N}_T$
- $s \in Z_{b,w<x}$.

There is a map $\mathcal{Y}_T \to Z|_{U \times_W \mathcal{N}_T}$ taking $(b, w, t, x_\bullet, s)$ to $(b, w, \mathrm{sat}(x_r), s)$. (The map is well defined because we have assumed that $J_{b,w<\mathrm{sat}(x_r)} = J_{b,w<\mathrm{sat}(x)}$. It is continuous because if $y \leq x \in \mathcal{S}_T$, then $\mathrm{sat}(y) = \mathrm{sat}(x) \in \mathcal{N}_T$.)

We have that $\mathcal{Y}_T \to Z|_{U \times_W \mathcal{N}_T}$ is a fiber bundle which is pulled back from $\mathcal{N}_T$. The fiber over $(b, w < x, s)$ only depends on $w < x \in \mathcal{N}_T$ and is homeomorphic to

$$\mathrm{N}(w, \mathrm{sat}^{-1}(x)] - \mathrm{N}(w, \mathrm{sat}^{-1}(x)).$$

Here $(w, \mathrm{sat}^{-1}(x)]$ denotes the subposet of $\mathrm{Hilb}(C)^Q$ consisting of elements x' such that $w < x' \leq x$ and $\mathrm{sat}(x') \leq x$. And $(w, \mathrm{sat}^{-1}(x)) \subseteq (w, \mathrm{sat}^{-1}(x)]$ denotes the subposet of x' satisfying the additional condition that $\mathrm{sat}(x') < x$.

Proposition 5.1. *The saturation operation* $\mathrm{sat} : \mathrm{Hilb}(C)^Q \to \mathrm{Q}^{\mathrm{JC}}$ *induces an homotopy equivalence of pairs*

$$(\mathrm{N}(w, \mathrm{sat}^{-1}(x)]\ , \mathrm{N}(w, \mathrm{sat}^{-1}(x))) \simeq (\mathrm{N}((w, x] \cap \mathrm{Q}^{\mathrm{JC}}),\ \mathrm{N}((w, x) \cap \mathrm{Q}^{\mathrm{JC}})).$$

Furthermore, the cohomology of this pair vanishes unless $w < x$ is essential.

Proof. This follows because the saturation operation, being a right adjoint, provides a retract of the first pair onto the second. $\square$

The bundle of pairs over $\mathcal{N}_T$, whose fiber over $w < x$ is $(\mathrm{N}(w, \mathrm{sat}^{-1}(x)], \mathrm{N}(w, \mathrm{sat}^{-1}(x)))$ gives rise to a pushforward sheaf of chain complexes on $\mathcal{N}_T$ whose stalks compute the cohomology of the pair. We write $\mu(T)[1]$ for this complex of sheaves.

Remark 5.2. We choose this notation because the complex of sheaves $\mu(T)$ is a categorification of the Möbius function of the poset of combinatorial types. More precisely, the Euler characteristic of the stalk of $\mu(T)$ at $w < x$ equals the Möbius function of the interval: $\chi(\mu(T)_{w<x}) = \mu(w, x)$. $\lozenge$

Consequently, we obtain a description of the spectral sequence associated to the stratification of $\mathrm{B}(U, P, Z)$.

Theorem 5.3. *There is a spectral sequence converging to $H^*_c(|\mathrm{B}(U, P, Z)|)$ with total E_1 page*

$$E_1^{*,*} = \bigoplus_{T \text{ essential, saturated}} H^*_c(Z|_{U \times_W \mathcal{N}_T}\ ,\ \mu(T)[1]).$$

Here the sum is over all saturated essential types of P.

5.2. **Approximation criteria.** We now assume $\mathcal{B}$ is a finite dimensional cell complex. We also assume that $E \to U$ is a finite dimensional real vector bundle of constant rank.

We further assume that

- for every $(b, w \leq x) \in (U \leq \mathrm{Hilb}(C)^Q)$ the fiber $Z_{b,w\leq x} \subseteq E_{b,w}$ is a linear subspace
- the natural containment $Z_{b,w\leq x} \subseteq Z_{b,w\leq \mathrm{sat}(x)}$ is an equality
- if $w \leq x_1$ and $w \leq x_2$ are two saturated elements corresponding to configurations in C with disjoint support, then $Z_{b,w\leq x_1} \cap Z_{b,w\leq x_2} = Z_{b,w\leq x_1 \vee x_2}$.

We suppose we are given an *expected codimension* function $\gamma : (W \leq \mathrm{Hilb}(C)^Q) \to \mathbb{N}$ which only depends on the combinatorial type of $w \leq \mathrm{sat}(x)$. We say that $Z_{b,w\leq x}$ is *unobstructed* if its codimension equals $\gamma(w \leq x)$. Using γ, we define a function κ on relative combinatorial types by

$$\kappa(T) := \gamma(T) - \mathrm{rank}(T) - \mathrm{Supp}(T).$$

Here $\mathrm{rank}(T)$ is defined as in Example 3.15 and $\mathrm{Supp}(T)$ is defined as in Example 3.16. (Note that for these functions $f(w \leq x)$ only depends on the combinatorial type of $w \leq x$, so that $f(T)$ is well defined). Our convention in the pointed and relative case is that, if $T = (g_0 \geq h_0, \{g_1 \geq h_1, \ldots, g_r \geq h_r\})$ then $\mathrm{Supp}(T)$ is the cardinality of $\{i > 0 \mid g_i > h_i\}$. Heuristically, κ records the expected codimension of the contribution of the stratum $\mathcal{S}_T$ to the bar construction.

Let $R \subseteq (W < \mathrm{Hilb}(C)^Q)$ be closed and downwards closed union of strata $\mathcal{S}_T$ for T a saturated combinatorial type, which is initial in its upwards closure in the following sense:

- if $w < y$ is in the upward closure of R, then there exists $\iota_R(w < y) \in \mathrm{Hilb}(C)^Q$ such that $w < \iota_R(w < y) \in R$ the maximal element of R with $\iota_R(w < y) \leq y$.

We will say that a saturated type T is a *type of* R (or P) if $\mathcal{S}_T \subseteq R$ (or $\mathcal{S}_T \subseteq P$).

In this setting, we have the following theorem relating the compactly supported cohomology of the bar construction of P to its image under the projection map $\mathrm{im}(Z_P \to E)$. (Recall that $P \subseteq W < \mathrm{Hilb}(C)^Q$ is a poset with the properties assumed in the beginning of the section).

Theorem 5.4. *Let $I \in \mathbb{N}$. Suppose that the following conditions hold.*

(1) We have that $P \subseteq R$ and P contains every $\mathcal{S}_T$ such that T is a type of R with $\kappa(T) \leq I$.
(2) For every pair $(b, w < x)$ in P_U such that x is saturated, and every $y \in Q^{\mathrm{JC}}$ such that $x \prec y$ and $w < y$ is essential the fiber $Z_{b,w<y}$ is unobstructed.

Then the map induced by $B(P, Z) \to \mathrm{im}(Z|_P)$ on compactly supported cochains is connected in codimension $I + 2$. In other words, for all $i > \dim E - I - 2$ the map $H^i_c(B(P, Z)) \leftarrow H^i_c(\mathrm{im}(Z|_P))$ is an isomorphism and for $i = \dim E - I - 2$ it is a surjection.

Proof. We let $P' \supseteq P$ be the closure and downward closure to the set of successors of elements of P (I.e. the closure and downward closure of the set of $w < x$ such that there is a $w < x \in P$ with $x \prec y$). Then P' is closed and downward closed, proper over W, and is a union of finitely many combinatorial types. We use the poset of saturated combinatorial types of P' to stratify E as follows.

Using the proper projection map $f : Z|_{P'_U} \to E$, for every combinatorial type S of P' we define $\overline{J}_S \subseteq E$ be the closed subset $f(Z|_{\mathcal{Z}_S})$. (In other words, for $(b, w) \in U$ we have that $\overline{J}_S \cap E_{b,w}$ is the set of all $e \in E_{b,w}$ such that there exists $(w < x) \in \mathcal{S}_S$ with $e \in Z_{b,w<x}$). We write J_S for the locally closed stratum associated to this stratification. (So the complement of $J_S \cap E_{b,w} \subseteq \overline{J}_S \cap E_{b,w}$ is the set of $e \in \overline{J}_S \cap E_{b,w}$ for which there exists $w < x < x'$ with $w < x' \in P'$ and $e \in Z_{b,w<x'}$).

Using the compactly supported cohomology spectral sequence for the above stratification, it suffices to prove that for every combinatorial type S of P', that the map induced by $f^{-1}(J_S) \to J_S$ is connected in codimension $I + 2$ on compactly supported cochains. Given $(b, w) \in U$ and $s \in E_{b,w} \cap J_S$, the fiber $f^{-1}(s)$ is the geometric realization of the semi-simplicial space

$$[t] \mapsto \{w < x_0 < \cdots < x_t \in P \mid s \in Z_{b,w<x_t}\}.$$

If S is a saturated type such that $\iota_R(S)$ is a type of P, we claim that this fiber is a contractible simplicial complex. To establish the claim, let x_s be a saturated element of $\mathcal{S}_S$ such that $s \in Z_{b,w<x_s}$. (Such an x_s exists because by definition $s \in Z_{b,w<x}$ for some $w < x \in \mathcal{S}_S$ and we may take $x_s = \operatorname{sat}(x)$). If $x \in P'$ is such that $s \in Z_{b,w<x}$, then $s \in Z_{b,w<x} \cap Z_{bw<x_s} = Z_{b,w<(x \vee x_s)}$. Hence $x \leq x_s$, because otherwise there exists an element $y \in P'$ with $y > x_s$ and $s \in Z_{w<y}$ contradicting the fact that $s \in s^{-1}(T)$. Therefore the fiber $f^{-1}(s)$ is the nerve of the poset of elements of P' which are $\leq x_s$. By adjointness, this is equivalent to the nerve of the poset of elements of P which are $\leq \iota_R(x_s)$. Since $\iota_R(S)$ is a type of P we have that $\iota_R(x_s) \in P$, and therefore the fiber is the nerve of a poset with a terminal element $\iota_R(x_S)$, hence it is contractible. Therefore if $\iota_R(S)$ is a type of P, then $f^{-1}(J_S) \to J_S$ is a proper equivalence.

Thus suffices to prove that if $\iota_R(S)$ is a type of $P' - P$ then $C_c^*(f^{-1}(J_S)) \to C_c^*(J_S)$ is connected in codimension $I+2$. We will in fact prove that $H_c^i(f^{-1}(J_S)) = H^i(J_S) = 0$ for $i \geq \dim E - I - 2$. (We will show this for $f^{-1}(J_S)$, the proof for J_S is similar but simpler). We stratify $f^{-1}(J_S)$, which is the geometric realization of the semi-simplicial space $\mathrm{B}(U, P, J_S \cap Z)$ defined by

$$[r] \mapsto \{(b,w) \in U, w < x_0 < \cdots < x_r \in P, s \in J_S \cap Z_{b,w<x_r}\},$$

using saturated types of P as in §5.1.

Given a combinatorial type T of P, we let $Y_{T,S}$ denote the simplicial set $[r] \mapsto$

$$\{(b,w) \in U, w < x_0 < \cdots < x_r \in \mathrm{Q}^{\mathrm{JC}} \cap P, s \in J_S \cap Z_{b,w<p_r} \mid w < x_r \in \mathcal{N}_T \text{ essential }\}.$$

From the spectral sequence Theorem 5.3, it suffices to establish that $\dim |Y_{T,S}| < \dim E - I - 2$, because then all of the nonvanishing terms of the spectral sequence will have compactly supported cohomology concentrated in codimension $> I+2$.

Lemma 5.5. *There is a saturated combinatorial type T' of P' such that T' is essential, $\kappa(T') > I$ and $T \leq_{+,\mathrm{sat}} T \leq_{+,\mathrm{sat}} S$.*

Assuming the Lemma, there is a surjection $|Y_{T,T',S}| \twoheadrightarrow |Y_{T,S}|$, where $Y_{T,T',S}$ is the semi-simplicial space with r-simplices

$$\{(b,w) \in U, w < p_0 < \cdots < p_r < q \in \mathrm{Q}^{\mathrm{JC}},\ s \in J_S \cap Z_q \mid p_r \in \mathcal{S}_T, w < q \in \mathcal{S}_{T'}\},$$

given by forgetting q. Because T' is an essential combinatorial type of P', by (2) we have that every $w < q \in \mathcal{N}_{T'}$ is unobstructed. So we may bound the dimension of $Y_{T,T',S}$: it is less than or equal to

$$\dim E - \gamma(T') + (\operatorname{rank}(T') - 2) + |T'| = \dim E - \kappa(T') - 2,$$

because $Y_{T,T',S}$ maps to $Z|_{\mathcal{N}_{T'}}$ with fibers of dimension $\leq \operatorname{rank}(T') - 2$ and $Z_{\mathcal{N}_{T'}} \subseteq E \times_W \mathcal{N}_{T'}$ has codimension $\geq \gamma(T')$. Therefore $\dim(|Y_{T,S}|) \leq \dim(E) - \kappa(T') - 2 < \dim(E) - I - 2$, completing the proof. $\square$

Proof of Lemma 5.5. Take a maximal chain $T \prec T_1 \prec T_2 \prec \cdots \prec \mathrm{ess}(\iota_R(S))$, and choose the smallest T_k which is not a type of P. Then $\kappa(T_k) > I$ by (1) and $T_k \in P' - P$ is essential. □

In the same setting as above, we have the following theorem on Gysin maps.

Theorem 5.6. *Let $E' \subseteq E$ be an inclusion of vector bundles over U, and let $Z' \to Z$ be a compatible map of stratifications by P. Suppose that for every essential element $p \in P$ that $Z'_p \to Z_p \times_E E'$ is an isomorphism. Then there is an induced Gysin isomorphism*

$$H_c^{*-\dim(E/E')}(\mathrm{B}(U, P, Z')) \to H_c^*(\mathrm{B}(U, P, Z)).$$

Proof. Write i^*Z for the pullback representation of P defined by $(i^*Z)_p = Z_p \times_E E'$. There is a factorization $Z' \to i^*Z \to Z$. The pullback square induces a Gysin map

$$H_c^{*-\dim(E/E')}(\mathrm{B}(U, P, i^*Z)) \to H_c^*(\mathrm{B}(U, P, Z)),$$

and there is an ordinary pullback map

$$H_c^*(\mathrm{B}(U, P, i^*Z)) \to H_c^*(\mathrm{B}(U, P, Z')).$$

We prove that both these maps are isomorphisms by stratifying as in §5.1 and applying Theorem 5.3. □

6. Semi-topological model and finite dimensional approximations

To compare algebraic maps $C \to \mathrm{Bl}_{p_1,\ldots,p_r}\mathbb{P}^n$ with continuous ones, we pass through an intermediate model, which we call a semi-topological model.

6.1. **Continuous sections vanishing with multiplicity.**

Definition 6.1. Let $U \subseteq \mathbb{C}$ be an open subset

and let $w : K \to \mathrm{Sym}^n U$ be a family of divisors of C, where K is a compact set. We say that a family of functions $f_k : U \to \mathbb{C}, k \in K$ *vanishes to order* $\geq w$ if

- for every $u \in U, k \in K$ there is an $a_{u,k} \in \mathbb{C}$ such that $f_k = a_{u,k}\rho_{w(k)} + o(|\rho_{w(k)}|)$ at u uniformly in K. (Such an $a_{u,k}$ is necessarily unique)

If $a_{u,k} \neq 0$ for all u, k we say that the family vanishes to order *exactly* w. Finally, if f_k equals $q_k + g_k$ where q_k is a continuous family of degree n polynomials and g_k vanishes to order $\geq w$, then we say that f_k is a *w-holomorphic family.* ♢

Proposition 6.2. *Let $w : K \to \mathrm{Sym}^n(U)$ be given. Then:*

(1) Multiplication by $\rho_{w(k)}$ induces a bijection

$$\{g : K \times U \to \mathbb{C} \text{ continuous }\} \to \{f_k \text{ family vanishing to order } \geq w\}$$

(2) *Given a biholomorphism $h : U' \to U$, precomposition with h induces a bijection*

$$\{f_k : U \to \mathbb{C} \textit{ vanishing to order} \geq w\} \to \{f'_k : U' \to \mathbb{C} \textit{ vanishing to order} \geq h^*(w)\}$$

Proof. A function $K \times U \to \mathbb{C}$ denoted $g_k(u)$ is continuous if and only if for every $u \in U$ there exists a constant $a_{u,y}$ (which must equal $g_k(u)$) such that $g_k = a_{u,k} + o(1)$ at u. Hence if g_k is continuous $\rho_{w(k)} g_k$ vanishes to order $\geq w$, and multiplication by $\rho_{w(k)}$ defines an injection from the RHS to the LHS. Conversely, if f_k satisfies (*) then

$$g_k(u) := \begin{cases} f_k(u)/\rho_{w(k)} & \text{if } \rho_{w(k)} \neq 0 \\ a_{u,k} & \text{if } \rho_{w(k)} = 0 \end{cases}$$

is continuous by the same criterion.

To obtain (2) from (1), we write $c_k = \rho_{w(k),U} \circ h$ and $b_k = \rho_{h^* w(k), U'}$ and use that

$$C^0(U', \mathbb{C}) \times K \xrightarrow{(c_k/b_k)\cdot} C^0(U', \mathbb{C}) \times K$$

is a homeomorphism because c_k/b_k does not vanish on U'. $\square$

Corollary 6.3. *Consider the space of pairs*

$$X := \{(w \in \mathrm{Sym}^n(U), f : U \to \mathbb{C} \textit{ vanishing to order } \geq w\}.$$

There is a unique compactly generated topology on X such that continuous maps $f : K \to X$ for K compact correspond to families of functions indexed by K vanishing to order $\geq w$. Furthermore, this topology is invariant under biholomorphisms $h : U \to U'$.

Proof. Endow X with the topology induced by the bijection $\mathrm{Sym}^n(U) \times C^0(U, \mathbb{C}) \to X$ given by multiplication by ρ_w, and use Proposition 6.2. $\square$

We also have a variant of this corollary for w-holomorphic maps.

Proposition 6.4. *There is a unique compactly generated topology on the set*

$$Y = \{(w \in \mathrm{Sym}^n(U), f : U \to \mathbb{C} \textit{ } w\textit{-holomorphic})\}$$

such that continuous maps $f : K \to X$ for K compact correspond to w-holomorphic families of functions indexed by K. This topology is invariant under biholomorphisms $h : U \to U'$.

Proof. There is an bijection $Y \to X \times \mathrm{Poly}_n(\mathbb{C})$, where

$$X := \{(w \in \mathrm{Sym}^n(U), f : U \to \mathbb{C} \text{ vanishing to order } \geq w)\},$$

because by definition any w-holomorphic f can be written as a sum of a degree n polynomial q and a function g vanishing to order w. (The Taylor expansion of f at the points of w determines q uniquely). We give Y the product topology.

To prove invariance, suppose that U has two holomorphic coordinates z, z'. Let f_k be a family of w-holomorphic functions with respect to z, and q_k, q_k' be the family of degree n polynomials in z, z' respectively whose Taylor expansion agrees with f_k at $w(k)$. By definition, we have that $f_k - q_k$ vanishes to order $\geq w$. And $f_k - q_k' = f_k - q_k + (q_k - q_k')$. Since $q_k - q_k'$ is the difference of two holomorphic functions with the same residues at $w(k)$ it vanishes to order $\geq w$, so $f_k - q_k'$ must as well. $\square$

Now let C be a Riemann surface, and let $w : K \to \operatorname{Sym}^n C$ be a family of divisors. For every $k_0 \in K$ there is

- a neighborhood of $w(k_0)$ of the form $\prod_{i=1}^t \operatorname{Sym}^{n_i} U_i$ where the U_i are disjoint open neighborhoods which are biholomorphic open subsets of C and $n = n_1 + \cdots + n_t$
- a neighborhood $V \ni k_0$ such that $w|_V$ factors through $\prod_{i=1}^t \operatorname{Sym}^{n_i} U_i$ as $\prod_{i=1}^t w_i$

We say that $f : K \times C \to \mathbb{C}$ is a family of functions vanishing to order $\geq w$ (resp. exactly w) if for every $k_0 \in K$ and every i we have that $f|_{V \times U_i}$ vanishes to order $\geq w_i$ (resp. exactly w_i) for some (equivalently any) choice of $\prod_{i=1}^t \operatorname{Sym}^{n_i}(U_i)$ and V as above.

Definition 6.5. Let M be a complex manifold and let $N \subseteq M$ be a complex submanifold. Let C be a Riemann surface, let $w : K \to \operatorname{Sym}^n C$ be a family of divisors, and let $f : K \times C \to M$ be a continuous map such that $f(k, p) \in N$ for every p in the support of $w(k)$. Then for every $k_0 \in K$ we have the following:

- for every p in the support of $w(k_0)$ there exist holomorphic local coordinates $s_1^p, \ldots, s_{\dim M - \dim N}^p$ on a neighborhood $\mathcal{U}_p \ni f_{k_0}(p)$, whose vanishing locus cuts out $N \cap \mathcal{U}_p$
- there are disjoint closed balls $B_p \ni p$ contained in $f^{-1}(\mathcal{U}_p)$ and an open neighborhood of $w(k_0)$ of the form $\prod_p \operatorname{Sym}^{n_p} U_p$ whose support is contained in the union of the balls B_p
- there is a neighborhood V of k_0 such that for every $k \in V$ we have that $f(B_p) \subseteq \mathcal{U}_p$ and w_V factors through $\prod_p \operatorname{Sym}^{n_p} U_p$ as $\prod_p w_p$

We say that $f : K \times C \to M$ intersects N to order $\geq w$ if for some (equivalently every) choice of $\mathcal{U}_p, B_p, U_p, V$ as above we have that $s_i^p|_{V \times U_p}$ vanishes to order $\geq w_p$ for all p, i. We say that it intersects *to order exactly* w if $f_k^{-1}(N)$ equals the support of w_k and for every p there is at least one i such that $s_i^p|_{V\times U_p}$ vanishes to order exactly w_p. $\Diamond$

Remark 6.6. Note that in the above definition, the holomorphicity conditions are only imposed on the coordinates whose vanishing defines N. $\Diamond$

Definition 6.7. For a single function $f : U \to \mathbb{C}$ and $w \in \mathrm{Sym}^n(U)$, we can define each of the above notions by taking $K = \mathrm{pt}$. If further $w = nu$ for some $u \in U$ then we will write f vanishes to order $\geq k$ at p, f vanishes to order exactly k at p, and so on. $\Diamond$

Remark 6.8. Note that if f vanishes to order $\geq w$ but does not vanish to order exactly w (i.e. $a_{u,k}$ vanishes somewhere), it is not necessary that f vanishes to order $\geq w'$ for some $w' > w$, even if $K = \mathrm{pt}$ and $w = nc$ for some $c \in U$. There just might not be a polynomial that approximates f to the required precision. $\Diamond$

Proposition 6.9. *Let $N_1, \dots, N_r \subseteq M$ be complex submanifolds of M. Let $n_1, \dots, n_r \in \mathbb{N}$. There is unique compactly generated topology on the set*

$$T = \left\{ (D_i)_{i=1}^r \in \prod_{i=1}^r \mathrm{Sym}^{n_i} C, f : C \to M \mid f \textit{ intersects } N_i \textit{ to order} \geq D_i \forall i \right\}$$

such that for all compact K continuous maps $f : K \to T$ correspond to the data of a continuous map $f : K \times C \to M$ and maps $w_i : K \to \mathrm{Sym}^{n_i} C$ for every i such that f_k is a family of maps intersecting N_i to order $\geq w_i$.

Proof. By taking fiber products, we reduce to the case where $r = 1$. Then we may apply the following general statement about the relation between local representability and global representability Lemma 6.10 in the case where

$$X = \{ w \in \mathrm{Sym}^n C, f : C \to M \mid h \text{ intersects } N \text{ to order} \geq w \},$$

with basis of open subsets associated to choices of $w_0 \in \mathrm{Sym}^n C$ and $(\mathcal{U}_p, B_p, U_p)_{p \in \mathrm{Supp}(w_0)}$ as in Definition 6.5. $\square$

Lemma 6.10. *Let X be a topological space. Suppose that X has a basis of open subsets $\mathcal{U}$ and for every $U \in \mathcal{U}$ there is a subsheaf $\mathcal{F}(K, U) \subseteq C^0(K, U)$ on the category of compact Hausdorff spaces such that for all $U \subseteq V \in \mathcal{U}$ we have that $\mathcal{F}(K, U) = \mathcal{F}(K, V) \cap C^0(K, U)$.*

There is a unique extension of $\mathcal{F}(K, W)$ to all open subsets $W \subseteq X$ such that $\mathcal{F}$ is a sheaf in K and for all $W_1 \subseteq W_2$ we have $\mathcal{F}(K, W_1) = \mathcal{F}(K, W_1) \cap C^0(K, W_2)$. Moreover, if $\mathcal{F}(-, U)$ is representable by a compactly generated topology on U for all $U \in \mathcal{U}$ then $\mathcal{F}(-, W)$ is representable by a compactly generated topology on W.

Proof. Let $f \in C^0(K, W)$. Writing $W = \cup_\alpha U_\alpha$ for $U_\alpha \in \mathcal{U}$ we have that there is a cover of K by K_α such that $f|_{K_\alpha} \in C^0(K_\alpha, U_\alpha)$. We declare that $f \in \mathcal{F}(K, W)$ if each f_{K_α} lies in $\mathcal{F}(K_\alpha, U_\alpha)$. This definition is independent of the choice of covers: for any other choice of K_β, U_β such that $f|_{K_\beta} \in C^0(K_\beta, U_\beta)$ there is a refinement by K_γ, U_γ and the sheaf property implies that $f|_{K_\beta}$ lies in $\mathcal{F}$ if and only if $\mathcal{F}|_{K_\gamma}$ does. Moreover, by definition $\mathcal{F}(K, W)$ is the unique sheaf satisfying the inclusion property which extends the definition on the base.

Finally, we declare a subset $L \subseteq W$ to be open if and only if it is open for every intersection $U \cap L$ with $U \in \mathcal{U}$. Then $f : K \to W$ is continuous if and only if $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is continuous for all $U \in \mathcal{U}$ if and only if $f|_K \in \mathcal{F}(K, U)$ for all $K \subseteq f^{-1}(U)$. □

6.1.1. *Case of interest.* We now consider the case where $M = V$ and $N_i = \ell_i \subseteq V$. Choose complements $\ell_i^\perp$ to ℓ_i.

Let $W \subseteq \prod_{i=1}^r \operatorname{Sym}^{n_i} C$ be the open subset of the product consisting of divisors with pairwise disjoint support. Let $(D_i)_{i=1}^r$ be the universal family of pairwise disjoint divisors, so that the fiber of D_i above $w \in W$ is w_i. There is a vector bundle $V(-D)$ on $U \times C$, such that for every $w \in W$ the fiber of $V(-D)$ over w is given by multiplying together the transition maps for $\ell_i \oplus \ell_i^\perp(-w_i)$ on U_i. If $\mathcal{L}_d$ is the Poincaré bundle on $\operatorname{Pic}^d C$, we can further consider the vector bundle $\mathcal{L}_d \otimes V(-D)$ on $U \times \operatorname{Pic}^d(C) \times C$.

Now fix $w' \in W$. Let a $\mathcal{V}$ be a neighborhood of w' of the form $\prod_{i=1}^t \operatorname{Sym}^{n_i} U_i$ where the U_i are disjoint open neighborhoods of C equipped with holomorphic local coordinates identifying them with open subsets of $\mathbb{C}$, and $n = n_1 + \cdots + n_t$. Using these local coordinates, we have a continuous family of polynomials $\rho_i = \rho_{w_i}$ parameterized by $w \in \mathcal{V}$. Consider the projection map $\pi : \mathcal{V} \times \operatorname{Pic}^d C \times C \to \mathcal{V} \times \operatorname{Pic}^d C$. We write $\Gamma(\pi, V(-D))$ for the space of relative sections, consisting of pairs $w \in \mathcal{V}$ and $s \in \Gamma(C, V(-D)|_w)$.

Proposition 6.11. *The map from $\Gamma(\pi, V(-D))$ to*

$$\{(w \in \mathcal{V}, L \in \operatorname{Pic}^d C, \ \textit{sections } C \to L \otimes V \ \textit{intersecting } L \otimes \ell_i \textit{ to order} \geq w_i)\}$$

given by multiplying by $\mathrm{id}_{\ell_i} \oplus \rho_i$ on U_i is a homeomorphism where the codomain is endowed with the above topology (and the domain is endowed with the compact generation of the compact-open topology).

Proof. It suffices to show that for any compact K, and any choice of $w_i : K \to \operatorname{Sym}^{n_i}(U_i)$ the map from $\Gamma(K \times C, V(-D))$ to

$$\{\text{families of sections } K \times C \to V \text{ intersecting } \ell_i \text{ to order} \geq w_i\}$$

given by multiplication by $\mathrm{id}_{\ell_i} \oplus \rho_{i,w_i(k)}$ is a bijection.

Given a vector bundle $E \to C$ with a trivialization $E \to V \times U$ on $U \subseteq C$, a subspace $V_0 \subseteq V$ and a divisor $A \in \mathrm{Sym}^k(B)$ where $B \subseteq U$ we may form the twist $E(-A, V_0)$ by gluing $E|_{W-B}$ to $(V_0 \oplus V_0^\perp(-A)) \times \overline{B}$, where $V_0^\perp$ is a complement of V_0 in V, keeping in mind that $V_0^\perp(-A)$ is trivial away from the support of A. Then $E(-A, V_0)$ is a vector bundle on C with a trivialization on $U - B$ together with a map $\rho_A \oplus \mathrm{id}_{\ell^\perp} : E(-A, \ell) \to E$ such that sections of $E(-A, \ell)$ correspond to sections of E intersecting ℓ to order $\geq A$.

If we start with $\mathcal{L}_d \otimes V$ and apply the above procedure iteratively with $V_0 = L \otimes \ell_i$, $A = w_i$, $B = \mathcal{V}_i$ for each i and $V_0 = V$, $A = w_0$ then we the bundle $\mathcal{L}_d \otimes V(-D)$. Following the correspondences of sections applied to families over compact K establishes the claimed homeomorphism. □

Remark 6.12. The topology of each fiber $\Gamma(\pi, \mathcal{L}_d \otimes V(-D))_w$ is the space of sections of a vector bundle over C. Since C is compact, the compact-open topology on it is given by a norm which makes it a separable Banach space. Further, since a convergent sequence along with its limit forms a compact set, any first-countable space is compactly generated: if $x \in A$ is not an interior point then there is a sequence $x_n \notin A$ converging to x, but then x is not an interior point of $(\{x_n\} \cup \{x\}) \cap A$. So each fiber remains a (separable) Banach space in the given topology. ◊

Let

$$E = E^{\text{top}} = \{(w \in \mathcal{V}, L \in \mathrm{Pic}^d C, \text{section } C \to L \otimes V \text{ intersecting } L \otimes \ell_i \text{ to order} \geq w_i)\}$$

as above and let $E_{w,L}$ be its fiber over $(w, L) \in \mathcal{V} \times \mathrm{Pic}^d C$. For $(w \leq x) \in (U \leq \mathrm{Hilb}(C)^Q)$ let $Z_{w \leq x}$ be the subspace of E_w consisting of those f that intersect ℓ_i to order $> k$ (i.e. to order $\geq k$ but not exactly to order k) at any point $p \in C$ such that $x_i(p) > w_i(p) = k$.

Proposition 6.13. *The incidence*

$$Z^{top} = \{((w \leq x) \in (\mathcal{V} \leq \mathrm{Hilb}(C)^Q), f : C \to V) \mid f \in Z_{w \leq x}\}$$

is a closed subset of $(\mathcal{V} \leq \mathrm{Hilb}(C)^Q) \times_{\mathcal{V}} E$ *and therefore defines a stratification of* E.

Proof. Given the above proposition, we want to check that for any continuous choice $(w \leq x) : K \to (\mathcal{V} \leq \mathrm{Hilb}(C)^Q)$ and section $f : K \times C \to V(-D)$, the subset $\{k \mid f_k \text{ takes values in } \ell_i \text{ on } x_{k,i} - w_{k,i} \text{ for each } i\}$ is closed in K. Fixing i, we can assume $x_{k,i} - w_{k,i} \in \mathrm{Sym}^n U$ for some fixed n and neighborhood U of C over which $V(-D)$ trivializes. Then f_k induces (a continuous choice of) a map $\mathrm{Sym}^n f_k : \mathrm{Sym}^n U \to \mathrm{Sym}\, nV$ and therefore the evaluation $(\mathrm{Sym}^n f_k)(x_{k,i} - w_{k,i})$ is a continuous choice $K \to \mathrm{Sym}^n V$. It suffices to note that $\mathrm{Sym}^n \ell_i \subset \mathrm{Sym}^n V$ is closed. □

Lemma 6.14. *$E^{top} \to W_n \times \operatorname{Pic}^d C$ is a locally trivial bundle of (infinite dimensional) separable Banach spaces.*

Proof. For a contractible neighborhood $\mathcal{U}$ of $(w_*, L_*) \in W_n \times \operatorname{Pic}^d C$, the restriction $V \otimes \mathcal{L}_d(-D)|_{\mathcal{U}\times C}$ is pulled back from a (topological) vector bundle $\mathcal{E}_{d,n} \cong (L_* \otimes V)(-n)$ on C. Therefore by Proposition 6.11 $E|_{\mathcal{U}} \cong \Gamma(\pi, V \otimes \mathcal{L}_d(-D))$ is trivial, and the fiber can be identified with $\Gamma^{\mathrm{top}}(C, \mathcal{E}_{d,n})$ as a Banach space (by Remark 6.12). $\square$

6.2. **Finite dimensional approximation.** As above, let $\mathcal{L}_d$ be the Poincaré bundle on $\operatorname{Pic}^d C$. We defined $E = E^{\mathrm{top}}$ to be the space parameterizing the data of $L \in \operatorname{Pic}^d C$, $w \in W_n$ and $s \in \Gamma^{\mathrm{top}}(L \otimes V)$ such that at a multiplicity k point of w labelled by i, the section s passes through ℓ_i to multiplicity $\geq k$. E comes with a map to $W_n \times \operatorname{Pic}^d$ and for $L \in \operatorname{Pic}^d$ fixed we denote the fiber over (w, L) by $E_{w,L}$. We want to approximate this bundle of Banach spaces by vector bundles of finite rank, in the sense of Proposition 6.17 below. To this end, we will need the following elementary lemma, which says that every embedding of a finite rank vector bundle into a bundle of Banach spaces can be locally trivialized. We provide a proof instead of locating a reference.

Lemma 6.15. *Suppose B is a Banach space, F is a finite dimensional vector space, $s : X \to \operatorname{Emb}(F, B)$ is a continuous family of embeddings of F in B and $x_0 \in X$ is a chosen basepoint. Then there is a neighborhood U of x_0 and a continuous choice of automorphisms $t : U \to \operatorname{Aut}(B)$ such that $t(x) \cdot s(x) = s(x_0)$ for each $x \in U$.*

Proof. Since F is finite dimensional, by the Hahn–Banach theorem there is a continuous splitting $p : B \to F$ of $s(x_0)$, ie p is continuous and $p \cdot s(x_0) = \mathrm{id}_F$. Now $x \mapsto p \cdot s(x)$ is a continuous map $X \to \operatorname{End}(F)$ taking the value id_F at x_0. So there is some neighborhood U_1 of x_0 and a continuous map $r : U_1 \to \mathbf{GL}(F)$ such that $r(x) \cdot p \cdot s(x) = \mathrm{id}_F$ for each $x \in U_1$. Now let $t(x) = \mathrm{id}_B + (s(x_0) - s(x)) \cdot r(x) \cdot p \in \operatorname{End}(B)$. Then $t(x_0) = \mathrm{id}_B$ and for each $x \in U_1$,

$$t(x)s(x) = s(x) + (s(x_0) - s(x)) \cdot r(x) \cdot p \cdot s(x) = s(x) + s(x_0) - s(x) = s(x_0).$$

Since B is Banach, $\operatorname{Aut}(B)$ is open in $\operatorname{End}(B)$ so it suffices to take $U = t^{-1}(\operatorname{Aut}(B))$. $\square$

Proposition 6.16. *For each $(w_*, L_*) \in \operatorname{Hilb}(C)^Q$, we can find a (distinguished) neighborhood U of (w_*, L_*) and a chain $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ of finite rank subbundles of $E|_U$ such that for each $(w, L) \in U$:*

1. *$E^{alg} := \Gamma^{alg}(L \otimes V) \cap E^{top}$ is contained in $\mathcal{F}_1$ and hence each $\mathcal{F}_m$ as a subbundle;*
2. *$\bigcup_m \mathcal{F}_{m,(w,L)}$ is dense in $E_{(w,L)}$.*

Proof. Consider the inclusion $E^{\mathrm{alg}} \subset E^{\mathrm{top}}$. Since each is locally trivial (E^{top} by Lemma 6.14), we can find a neighborhood U of (w_*, L_*) such that the inclusion is isomorphic to $F \times U \subset E \times U$ for a separable Banach space E and a finite dimensional vector space F. Now for a chain F_i of finite dimensional subspaces of E such that $F \subset F_1$ and $\cup F_i$ is dense, set $\mathcal{F}_i = F_i \times U$. Then these clearly satisfy (1) and (2). □

The next proposition shows the homology of finite dimensional approximations approximates to the homology of the space of all sections.

Proposition 6.17. *Let B be a normed vector space and suppose F_n is a chain of finite dimensional subspaces such that $\bigcup F_n$ is dense in B. Suppose X is metrizable and let $G \subset X \times B$ be open. Then $G' := \bigcup G \cap (X \times F_n) \to G$ is a weak homotopy equivalence.*

Proof. We prove that for every pair of finite complexes (Y, Y_0) and choice of map $f : (Y, Y_0) \to (G, G \cap (X \times F_n))$, there is some $m \geq n$ and a map $\overline{f} : Y \to G \cap (X \times F_m)$, homotopic to f rel Y_0 and such that $\mathrm{pr}_1 \circ f = \mathrm{pr}_1 \circ \overline{f} : Y \to X$.

Since Y is compact, let $\varepsilon > 0$ be such that the 2ε neighborhood of the image of f is contained in G. Let $f = (f_X, f_B)$. Subdivide (Y, Y_0) such that f_B is within ε of its linearization (ie the linear extension of f_B restricted to $Y_0 \cup$ vertices of Y) and each simplex of Y not contained in Y_0 has at least one vertex not in Y_0. Since $\bigcup F_m$ is dense in B, for m sufficiently large and for each vertex $v \in Y \setminus Y_0$ we can find,

$$\overline{f}_B(v) \in \mathrm{pr}_2(G \cap (\{f_X(v)\} \times B)) \cap B(f_B(v), \varepsilon) \cap F_m.$$

Then extend $\overline{f}_B$ to Y linearly, and set $\overline{f} = (f_X, \overline{f}_B)$ which lands in G since $||\overline{f} - f||_\infty < 2\varepsilon$.

Now without loss of generality, choose a base point $g_0 \in G \cap F_1$. The above argument shows that for each i, the composition

$$\operatorname{colim} \pi_i(G \cap (X \times F_n), g_0) \to \pi_i(G', g_0) \to \pi_i(G, g_0)$$

is an isomorphism. Thus it suffices to show that $\operatorname{colim} \pi_i(G \cap (X \times F_n), g_0) \to \pi_i(G', g_0)$ is surjective, which follows from the same argument with (B, G) replaced by $(\bigcup F_n, G')$. □

7. Comparing algebraic maps and the semi-topological model

In this section, we specialize to our main case of interest: V is a finite dimensional vector space and $\ell_1, \ldots, \ell_r$ are lines corresponding to points $p_1, \ldots, p_r \in \mathbb{P}(V)$. We let $X = \mathrm{Bl}_{p_1,\ldots,p_r} \mathbb{P}(V)$, and consider the space of algebraic maps from $C \to X$, where C is a fixed algebraic curve. We fix a tuple $(d, n) = (d, n_1, \ldots, n_r) \in \mathbb{N}^{r+1}$.

7.1. **Unpointed comparison theorem.** Let $W_n \subseteq \prod_{i=1}^r \mathrm{Sym}^{n_i}(C)$ be the open subset consisting of tuples of pairwise disjoint divisors. We let $\widetilde{\mathcal{M}}_{d,n}$ be the space parameterizing the data of $L \in \mathrm{Pic}^d(C)$ and $w \in W_n$ and $s \in \Gamma(L \otimes V)$ a continuous section intersecting ℓ_i to order exactly w_i. We will prove the following theorem comparing the space of algebraic maps with $\mathcal{M}_{d,n} := \widetilde{\mathcal{M}}_{d,n}/\mathbb{C}^*$. As described in §3.2, there is a stratification of the space of algebraic sections of the universal degree d line bundle, $\Gamma_{\mathrm{alg}}(C, \mathcal{L} \otimes V)$ by $\mathrm{Hilb}(C)^{Q_r}$. For $y \in \mathrm{Hilb}(C)^{Q_r}$ we define the expected codimension $\gamma(y) := \sum_{i1}^r 2(\dim_{\mathbb{C}} V - 1)m_{\ell_i}(y) + 2\dim_{\mathbb{C}}(V)m_0(y)$. As in §3.5 have an associated definition of $\kappa(w < y)$.

The following is our main criterion for establishing that the map from $\mathrm{Alg}_{d,n}(C, X) \to \mathcal{M}_{d,n}$ induces isomorphisms on homology groups. To state it, recall that a map $f : X \to Y$ is *homology I-connected* if $H_i(f) : H_i(X) \to H_i(Y)$ is an isomorphism for $i < I$ and a surjection for $i = I$.

Theorem 7.1. *Let $I \in \mathbb{N}$. Suppose there is a poset $P \subseteq (W_n < \mathrm{Hilb}(C)^{Q_r})$ which is a closed and downward closed union of finitely many combinatorial types such that*

(1) P is proper over W
(2) For every pair $w < x$ in P, and every $y \in (W \leq Q^{\mathrm{JC}})$ such that $x \prec y$ and y is essential the fiber $\Gamma_{\mathrm{alg}}(C, V \otimes L)_y$ is unobstructed, for every line bundle $L \in \mathrm{Pic}^d(C)$.
(3) P contains all types T with $\kappa(T) \leq I$ and all the minimal types of $W < \mathrm{Hilb}(C)$.

Then the map $\mathrm{Alg}_{d,n}(C, X) \to \mathcal{M}_{d,n}$ is homology I-connected.

Proof. To establish Theorem 7.1, we first make a series of reductions. (For brevity we write $\widetilde{\mathrm{Alg}}_{d,n}$ instead of $\widetilde{\mathrm{Alg}}_{d,n}(C, X)$.)

(1) By the Leray spectral sequence, it suffices to prove that the map of principal $\mathbb{C}^*$ bundles from $\widetilde{\mathrm{Alg}}_{d,n}(C, X)$ to $\widetilde{\mathcal{M}}_{d,n}$ is homology I-connected.
(2) Both $\widetilde{\mathcal{M}}_{d,n}$ and $\widetilde{\mathrm{Alg}}_{d,n}$ map to $W_n \times \mathrm{Pic}^d(C)$. Again by Leray, it suffices to prove, for basis of open subsets $U \subseteq W_n \times \mathrm{Pic}^d(C)$, that $\widetilde{\mathrm{Alg}}_{d,n}|_U \to \widetilde{\mathcal{M}}_{d,n}|_U$ is homology I-connected.
(3) $\widetilde{\mathcal{M}}_{d,n}$ is an open subset of bundle of Banach spaces $E^{\mathrm{top}} \to W_n \times \mathrm{Pic}^d(C)$, where the fiber above (w, L) is the space of sections of L intersecting ℓ_i holomorphically to order w_i. This bundle contains the space of all algebraic sections E^{alg} that intersect ℓ_i to order $\geq w_i$. By Proposition 6.16 there is a basis of open subsets $U \subseteq W_n \times \mathrm{Pic}^d(C)$ such that on every U there is a sequence of finite dimensional sub-bundles $E^\beta \subseteq E^{\mathrm{top}}|_U$ containing $E^\beta|_U$ with

$\cup_\beta E^\beta$ fiberwise dense. By Proposition 6.17, it suffices to prove that map

$$\widetilde{\mathrm{Alg}}_{d,n}|_U \to \widetilde{\mathcal{M}}^\beta_{d,n}|_U := E^\beta \cap \widetilde{\mathcal{M}}_{d,n}|_U$$

is homology I-connected all β sufficiently large.

(4) We have that $\widetilde{\mathrm{Alg}}_{d,n}|_U = E^{\mathrm{alg}} \cap \widetilde{\mathcal{M}}^\beta_{d,n}|_U$. Hence by Proposition 2.2, the push-forward map

$$H_i(\widetilde{\mathrm{Alg}}_{d,n}|_U) \to \widetilde{H}_i(\widetilde{\mathcal{M}}^\beta_{d,n}|_U)$$

is Poincaré dual to the Gysin map

$$H_c^{\dim E^{\mathrm{alg}}-i}(\widetilde{\mathrm{Alg}}_{d,n}|_U) \to H_c^{\dim E^\beta - i}(\widetilde{\mathcal{M}}^\beta_{d,n}|_U)$$

induced by the Thom class for the inclusion of vector bundles $E^{\mathrm{alg}}|_U \subseteq E^\beta$. So it suffices to prove that this Gysin map is an isomorphism for $i < I$ and surjection for $i = I$.

(5) Let $K^{\mathrm{alg}} = E^{\mathrm{alg}} - \widetilde{\mathrm{Alg}}_{d,n}|_U$ be the complement of the space of algebraic maps. Similarly let $K^\beta := E^\beta - \mathcal{M}^\beta_{d,n}$. By the long exact sequence in compactly supported cohomology, it suffices to prove that the Gysin map $C^*_c(K^{\mathrm{alg}})[\dim E_\beta] \to C^*_c(K^\beta)[\dim E_\beta]$ is $(I+1)$-connected.

Now we are in a position to apply the results of the previous sections. The stratification Z^{top} of E^{top} described in Proposition 6.13 restricts to a stratification of E^β by $(U \leq \mathrm{Hilb}(C)^{Q_r})$, denoted by Z^β. As described in the running example 3.2, we have a corresponding stratification of the space of algebraic sections $\Gamma(C, \mathcal{L}_d \otimes V)$ by $\mathrm{Hilb}(C)^{Q_r}$ inducing a stratification of $E^{\mathrm{alg}}|_U$ by $\mathrm{Hilb}(C)^{Q_r}$.

Restricting to P_U, we have a commutative diagram of bar constructions

$$\begin{tikzcd} K^\beta & K^{\mathrm{alg}} \arrow[l] \\ \mathrm{B}(U,P,Z^\beta) \arrow[u] & \mathrm{B}(U,P,Z^{\mathrm{alg}}) \arrow[l] \arrow[u] \end{tikzcd}$$

inducing a commutative diagram on compactly supported cochains

$$\begin{tikzcd} C^*_c(K^\beta)[\dim E^\beta] \arrow[d] & C^*_c(K^{\mathrm{alg}})[\dim E^{\mathrm{alg}}] \arrow[l] \arrow[d] \\ C^*_c(\mathrm{B}(U,P,Z^\beta))[\dim E^\beta] & C^*_c(\mathrm{B}(U,P,Z^{\mathrm{alg}}))[\dim E^{\mathrm{alg}}], \arrow[l] \end{tikzcd}$$

where the horizontal arrows are Gysin maps. We obtain that the lower horizontal map is a quasi-isomorphism by applying Theorem 5.6 and that the vertical maps are $(I+2)$-connected by applying Theorem 5.4 twice (with $R = (W < \mathrm{Hilb}(C)^Q)$). Hence the upper horizontal map is $(I+1)$-connected, and so we are done. $\square$

Remark 7.2. In an earlier stage of this project, we believed that our methods would allow us to compare the fiber of $\widetilde{\mathrm{Alg}}_{d,n}(C,X)$ over $(D_i)_{i=1}^r \in W_n$ to a space of sections of a jet bundle; namely the space consisting of pairs $L \in \mathrm{Pic}^d C$ and continuous non-vanishing sections $s \in \Gamma(\mathrm{J}(L \otimes V))$ satisfying the condition:

- if $c \in C$ is a multiplicity e point of D_i, then the jet $s(c)$ intersects ℓ_i to order exactly e (in the sense that $s(c)$ is a Taylor expansion of such a function).

However, this is not the case. To illustrate the difference between the space of sections of the jet bundle and the model $\mathcal{M}_{d,n}$ we use, let us specialize the case $C = \mathbb{C}, V = L = \mathbb{C}$ and D is the divisor of degree one supported at the origin. (Here $r = 1$ and $\ell_1 = 0 \subseteq \mathbb{C}$).

Let $\mathcal{I} \subseteq J^1\mathbb{C}$ be the incidence condition stating that a function s vanishes only at the origin to order exactly 1. We have that $J^1\mathbb{C} = \mathbb{C}^3$ and

$$\mathcal{I} = ((\mathbb{C} - 0) \times (\mathbb{C} - 0) \times \mathbb{C}) \cup (0 \times 0 \times \mathbb{C} - 0),$$

which is not open or closed (or even locally compact). Furthermore $\Gamma(\mathbb{C}, \mathcal{I})$ consists of a pair of continuous functions $a : \mathbb{C} \to \mathbb{C}$ and $b : \mathbb{C} \to \mathbb{C}$ such that a vanishes exactly at 0 and b is non-vanishing at 0. The data of a and b are independent, and a is determined up to homotopy by its restriction to the unit circle, while b is determined by its value at 0: so we have that $\Gamma(\mathbb{C}, \mathcal{I}) \simeq \mathrm{Top}(S^1, \mathbb{C} - 0) \times (\mathbb{C} - 0)$.

On the other hand, consider the space of functions $s : \mathbb{C} \to \mathbb{C}$ which intersect 0 holomorphically to order exactly one at the origin. By a sort of Alexander trick (see (8.2)), this space retract to the subspace of functions which agree with a nonzero $\mathbb{C}$-linear function on the unit disk. We may further retract to the subspace of nonzero $\mathbb{C}$-linear functions, identified with $\mathbb{C} - 0$.

The key difference between these two cases is that in the first there is no relationship between the "formal Taylor expansion" of s at 0 and its actual values. In contrast, the second space is equivalent to the space of pairs consisting of an injective linear function $z \mapsto bz$ and a function $f : S^1 \to \mathbb{C} - 0$ and a homotopy between them: via the map that takes s to its Taylor coefficient at 0 and its restriction to S^1 and the homotopy provided by its restriction to the unit disk.

The situation does not significantly improve for C compact. For instance, if $C = \mathbb{P}^1$ and $L = \mathcal{O}(d)$ with d large, the space of holomorphic sections with exactly one zero of order 1 at $0 \in \mathbb{P}^1$ is empty whereas the relevant subspace of sections of the jet bundle is always non-empty. $\Diamond$

7.2. **Pointed variant.** We now consider the pointed case where C and X are equipped with distinguished base points $*_C \in C$ and $*_X \in X - \cup_{i=1}^r E_i$. We take the same approach, but with a modified poset. Accordingly let

$$W_{n,*} := \left(\prod_{i=1}^{r} \operatorname{Sym}^{n_i} C\right) \times *_C \subseteq \left(\prod_{i=1}^{r} \operatorname{Sym}^{n_i} C\right) \times C \subseteq \operatorname{Hilb}(C)^{Q_{r+1}}.$$

Let $\overline{R} \subseteq \operatorname{Hilb}(C)^{Q_r}$ be the closed subposet consisting of divisors $D_{\ell_1}, \ldots, D_{\ell_{r+1}}, D_0$ such that $D_{\ell_{r+1}} - D_0$ is supported at $*_C$ with multiplicity ≤ 1. We again have a stratification $\Gamma_{\mathrm{alg}}(C, \mathcal{L} \otimes V)$ by $\operatorname{Hilb}(C)^{Q_{r+1}}$ restricting to a stratification by $\overline{R}$ The expected codimension of $y \in \operatorname{Hilb}(C)^{Q_{r+1}}$ is $\gamma(y) := \sum_{i1}^{r+1} 2(\dim_{\mathbb{C}} V - 1)m_{\ell_i}(y) + 2\dim_{\mathbb{C}}(V)m_0(y)$.

We define $R := (W_{n,*} < \operatorname{Hilb}(C)^{Q_{r+1}}) \cap (W_{n,*} \leq \overline{R})$. Then we have the following pointed variant of Theorem 7.1.

Theorem 7.3. *Let $I \in \mathbb{N}$. Suppose there is a poset $P \subseteq R$ which is a closed and downward closed union of finitely many combinatorial types such that*

1. *For every pair $w < x$ in P, and every $y \in Q_{r+1}^{\mathrm{JC}}$ such that $x \prec y$ and y is essential the fiber $\Gamma_{\mathrm{alg}}(C, V \otimes L)_y$ is unobstructed, for every line bundle $L \in \operatorname{Pic}^d(C)$.*
2. *P contains all types T of R with $\kappa(T) \leq I$ and all the minimal types of R*

Then the map $\operatorname{Alg}_{d,n,*}(C, X) \to \mathcal{M}_{d,n,*}$ is homology I-connected.

The proof of Theorem 7.3 is identical to that of Theorem 7.1. We note that R is initial in $W_{n,*} < \operatorname{Hilb}(C)^{Q_{r+1}}$: if $x = (D_{\ell_1}, \ldots, D_{\ell_{r+1}}, D_0)$ then $\iota_R(x) = (D_{\ell_1}, \ldots, D'_{\ell_{r+1}}, D_0)$ where the divisor $D'_{\ell_{r+1}}$ equals D_0 away from $*_C$ and has multiplicity $\min(1, \operatorname{mult}_{*_C}(D_{\ell_{r+1}} - D_0)$ at $*_C$.

8. Comparing the semi-topological model and spaces of positive maps

Let $X = \operatorname{Bl}_{p_1,\ldots,p_r} \mathbb{P}(V)$. We defined $\mathcal{M}_{d,n}$ to be the space parameterizing the data of $L \in \operatorname{Pic}^d(C)$, $w \in W_n$ and a choice up to scalar multiple of a continuous section $s \in \Gamma(L \otimes V)$ intersecting ℓ_i to order exactly w_i. The results of the previous section can be used establish that the canonical map

$$\operatorname{Alg}_{d,n}(C, X) \to \mathcal{M}_{d,n}$$

induces an isomorphism on homology in a given range. (For simplicity, we will only consider the unpointed case in this section, but all of the results hold with the same arguments in the pointed case).

The purpose of this section is to prove that $\mathcal{M}_{d,n}$ is weakly equivalent to a space parameterizing continuous maps $C \to X$ of positive intersection multiplicity. (For notational convenience, we fix $d \in \mathbb{N}, n \in \mathbb{N}^r$ for the remainder of the section and simply write $\mathcal{M}$). We let $\mathcal{T}$ be the subspace of $W_n \times \mathrm{Top}(C, X)$, in fact of $W_n \times \mathrm{Top}^+_{d,n}(C, X)$ using notation from the introduction, consisting of $(w \in W_n, f : C \to X)$ satisfying:

- the projection of f to $\mathbb{P}(V)$ is degree d
- $f^{-1}(E_i)$ is discrete, and f has positive local intersection multiplicity at every point of $f^{-1}(E_i)$
- at a multiplicity k point of w_i, the map f has intersection multiplicity k with E_i.

The main result of this section is the following.

Theorem 8.1. *There is a canonical weak homotopy equivalence $\mathcal{M} \to \mathcal{T}$.*

We construct the map $\mathcal{M} \to \mathcal{T}$ in two stages. First there is a projection map, $\mathcal{M} \to \mathcal{M}'$ where $\mathcal{M}'$ consists pairs $w \in W_n$ and continuous degree d maps $f : C \to \mathbb{P}^n$ such that f intersects p_i to order exactly w_i.

Next we construct a homeomorphism $\mathcal{M}' \to \mathcal{T}'$, where $\mathcal{T}' \subseteq \mathcal{T}$ consists of continuous degree d maps $f : C \to X$ such that f intersects E_i holomorphically to order w_i. The map $\mathcal{M}' \to \mathcal{T}'$ is induced by the strict transform operation defined in the following subsection.

8.1. **Strict transform.** Let $U \subseteq \mathbb{C}$. Consider the blowdown map $\pi : \mathrm{Bl}_0 \mathbb{C}^v \to \mathbb{C}^v$.

Proposition 8.2. *Let $w \in \mathrm{Sym}^m U$. Then postcomposition with π defines a homeomorphism between the following spaces of functions:*

- *continuous functions $U \to \mathrm{Bl}_0 \mathbb{C}^v$ intersecting the exceptional divisor holomorphically to order exactly w.*
- *continuous functions $U \to \mathbb{C}^v$ intersecting the origin holomorphically to order exactly w.*

Proof. Given $(f_1, \dots, f_v) : U \to \mathbb{C}^v$ vanishing to order exactly w we may write f_i uniquely as $f_i = \rho_w \widetilde{f}_i$, where ρ_w is the unique monic degree m polynomial vanishing at w. Then

$$(f_1, \dots, f_v) \times [\widetilde{f}_1 : \cdots : \widetilde{f}_v]$$

is a map to the blowup. Recall that the blowup $\mathrm{Bl}_0 \mathbb{C}^v \subseteq \mathbb{C}^v \times \mathbb{P}(\mathbb{C}^v)$ has standard coordinate charts for $i = 1, \dots n$ given by

$$\mathrm{Bl}_0(\mathbb{C}^v) - \mathbb{V}(\widetilde{x}_i) \to \mathbb{C}^v \qquad (x_1, \dots, x_v) \times [\widetilde{x}_1 : \cdots : \widetilde{x}_v] \mapsto (\widetilde{x}_1/\widetilde{x}_i, \dots, x_i, \dots, \widetilde{x}_v/\widetilde{x}_i).$$

Thus locally at any point of U where $\widetilde{f}_i$ does not vanish, in the ith chart U is given by

$$(\widetilde{f}_1/\widetilde{f}_i, \dots, f_i, \dots, \widetilde{f}_v/\widetilde{f}_i),$$

hence intersects the exceptional divisor holomorphically to order exactly w. □

The proposition immediately globalizes to the case where $(\mathbb{C}^n, 0)$ is replaced by a complex manifold and finite set of points $(M, \{p_1, \dots, p_r\})$, yielding a homeomorphism between spaces of:

- continuous functions $C \to \mathrm{Bl}_{p_1,\dots,p_r} M$ intersecting the exceptional divisor E_i holomorphically to order exactly w_i.
- continuous functions $U \to M$ intersecting the point p_i holomorphically to order exactly w_i.

The inverse homeomorphism is called the *strict transform*, yielding a homeomorphism $\mathcal{M}' \to \mathcal{T}'$.

8.2. **Comparing $\mathcal{M}$ and $\mathcal{T}$.** So far, we have constructed a map $\mathcal{M} \to \mathcal{T}$. We now prove that this map is a weak homotopy equivalence. To do so, we will cover $\mathcal{T}$ and $\mathcal{M}$ by open subsets of the following form.

We will use a basis of open neighborhoods of $W_n \subseteq \prod_{i=1}^r \mathrm{Sym}^{n_i}(C)$ which we call distinguished opens.

Definition 8.3. Fix a metric on C. A point $w \in W_n$ corresponds to a finite configuration of points in $T \subseteq C$, together with a label for each $t \in T$ of the form (i_t, m_t), where $i_t = 1, \dots, r$ and $m_t \in \mathbb{N}_{>0}$. For any $\epsilon > 0$ with $\epsilon < \min\limits_{t \neq t' \in S} d(t, t')$, we say that the *distinguished open subset* of radius ϵ around w is the open subset

$$\mathrm{B}(\epsilon, w) := \prod_{t \in T} \mathrm{Sym}^{m_t}(\mathrm{B}(\epsilon, t)) \subseteq W_n,$$

where $\mathrm{B}(\epsilon, t)$ is the open ball of radius ϵ around s. ◊

Definition 8.4. Given a finite union of closed balls $K \subseteq C$ labelled by $\{1, \dots, r\}$, we let $\mathcal{T}(K)$ denote the subspace consisting of functions $f : C \to Y$ which map the balls of K labelled by i to the tubular neighborhood $\mathrm{Tub}(E_i)$. For a subset $R \subseteq W_n$, we let $\mathcal{T}_R(K) := \mathcal{T}_R \cap \mathcal{T}(K)$, where $\mathcal{T}_R$ is the preimage of R under the projection $\mathcal{T} \to W_n$.

We define $\mathcal{M}_R(K)$ to be the pre-image of $\mathcal{T}_R(K)$. For a distinguished open subset $V \subseteq W_n$, centered on w of radius ϵ, we write $V \subseteq K$ if the radius ϵ neighborhood of each point of w labelled by i is contained in a ball of K labelled by i. ◊

First we establish a local to global principle.

Proposition 8.5. *To prove that $\mathcal{M} \to \mathcal{T}$ is a weak equivalence, it suffices to prove that $\mathcal{M}_V(K) \to \mathcal{T}_V(K)$ is a weak equivalence for every $V \subseteq K$ such that V is a distinguished open and K is a finite union of closed balls labelled by $\{1, \dots, r\}$.*

Proof. The subspace $\mathcal{T}_V(K) \subseteq \mathcal{T}$ is open by definition of the compact open topology. Furthermore, for any $(w, f) \in \mathcal{M}$, there is an ϵ such that the image of the closed ball of radius ϵ around every point of $f^{-1}(E_i)$ contained in $\mathrm{Tub}(E_i)$. Taking K to be the union of these balls and V to be a distinguished neighborhood of radius $\epsilon/2$ of w, we have that $(w, f) \in \mathcal{T}_V(K)$. So $\mathcal{T}_V(K)$ form an open cover.

We have the identities $\mathcal{T}_{V_1}(K_1) \cap \mathcal{T}_{V_2}(K_2) = \mathcal{T}_{V_1 \cap V_2}(K_1 \cup K_2)$ and $\cup_\alpha \mathcal{T}_{V_\alpha}(K) = \mathcal{T}_{\cup_\alpha V_\alpha}(K)$. Since the distinguished open subsets form a basis of W_n, it follows from these identities that any finite intersection of subsets of the form $\mathcal{T}_V(K)$ is covered by opens of the same form. Therefore, we may form a hypercover $\mathcal{U}_\mathcal{T}$ of $\mathcal{T}$ by disjoint unions of the open subsets $\mathcal{T}_V(K)$. Taking preimages, we obtain a hypercover $\mathcal{U}_\mathcal{M}$ of $\mathcal{T} = N$ by disjoint unions of $\mathcal{M}_V(K)$.

By the main result of Dugger-Isaksen [DI04], we have that $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{U}_\mathcal{M} \to \mathcal{M}$ and $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} \mathcal{U}_\mathcal{T} \to \mathcal{T}$ are weak equivalences. Therefore, if $\mathcal{M}_K(V) \to \mathcal{T}_K(V)$ are weak equivalences for every K, V then $\mathcal{M} \to \mathcal{T}$ is a weak equivalence. $\square$

Next we retract onto fibers.

Proposition 8.6. *Let $K \subseteq C$ be a finite union of closed balls labelled by $1, \dots, r$. Let V be a distinguished open neighborhood of $w \in W_n$ satsifying $V \subseteq K$. Then the inclusions $\mathcal{T}_w(K) \to \mathcal{T}_V(K)$ and $\mathcal{M}_w(K) \to \mathcal{M}_V(K)$ are homotopy equivalences.*

Proof. First we construct a deformation retraction of $\mathcal{T}_V(K)$ onto $\mathcal{T}_w(K)$. Recall that w corresponds to a labelled subset $T \subseteq C$ and V is determined by a radius ϵ. Let $t \in T$ be labelled by i with multiplicity m. We will construct a deformation retraction from the space of pairs $w' \in \mathrm{Sym}^m(B(\epsilon, t))$, $f : \overline{\mathrm{B}(\epsilon, t)} \to \mathrm{Tub}(E_i)$ with multiplicities of $f^{-1}(E_i)$ prescribed by w', to the subspace of pairs that satisfy $w' = mt$. This deformation retraction will fix the values of f on the boundary of $\overline{B(\epsilon, t)}$, so that by gluing together the retractions for each element of T we obtain the desired retraction of $\mathcal{T}_V(K)$.

To construct the retraction, we use a variant of the Alexander trick. Identify $\mathrm{Tub}(E_i)$ with the normal bundle $\pi : N_{E_i} \to E_i$, and $B(\epsilon, t)$ with the unit disk. We proceed in three steps. First, given w', f make f radially constant on the points of absolute value $\geq 1/2$, via

$$f_u(x) = \begin{cases} f(\frac{x}{1-u}) & \text{if } |x| \leq 1-u \\ f(x/|x|) & \text{if } |x| \geq 1-u \end{cases}$$

for $u \in [0, 1/2]$. (We define w'_u to be the scaling of w' by $1-u$).

Second we make $\pi \circ f(x) = \pi \circ f(0)$ for all x with $|x| \leq 1/2$, using the radial path from $\pi(f(x))$ to $\pi(f(0))$ and lifting to N_{E_i}. More precisely we lift the homotopy

$$\overline{f}_u(x) = \begin{cases} \pi(f((1-u)x) & \text{if } |x| \leq 1/2 \\ \pi(f((1-u)x + xu(1-|x|/2)) & \text{if } |x| \geq 1/2 \end{cases}$$

for $u \in [0,1]$ to N_{E_i}. (To perform the lift we choose a metric on N_{E_i} and write N_{E_i} as the cone on the unit circle bundle. Then the universal path lifting for the unit circle bundle provides a continuous way of lifting paths of maps $X \to E_i$ to paths of maps $X \to N_{E_i}$ such that the distance from the zero is preserved section).

Finally we apply the Alexander trick to the radius 1/2 disk

$$f_u(x) = \begin{cases} (1-u)f(\frac{x}{1-u}) & \text{if } |x| \leq 1/2(1-u) \\ 2|x|f(\frac{x}{2|x|}) & \text{if } 1/2(1-u) \leq |x| \leq 1/2 \\ f(x) & \text{if } |x| \geq 1/2 \end{cases}$$

for $u \in [0,1]$ and $f_1(0) = 0$. This three step process yields the desired retraction, because in the second step w' is fixed and in the third step w' is scaled down to the origin.

Next we construct a deformation retraction of $\mathcal{M}_V(K)$ onto $\mathcal{M}_w(K)$. Similarly to above, we will construct a retraction from the space of pairs (w', s), where $w' \in \mathrm{Sym}^m(B(\epsilon, t))$, and s is a section of $L \otimes V$ over $\overline{B(\epsilon, t)}$ which is w'-holomorphic and such that the multiplicity of $f^{-1}(E_i)$ is prescribed by w', to the subspace of pairs that satisfy $w' = mt$. Gluing together these local retractions then yields a global retraction.

To describe the retraction we choose coordinates $V \cong \mathbb{C}^v$ such that $\ell_i = \mathbb{C} \times 0^{v-1}$, and s corresponds to sections $s_1, \ldots, s_v$ of L. Because the map to $\mathbb{P}^{v-1}$ associated to s takes $\overline{B(\epsilon, t)}$ to the neighborhood $\pi(\mathrm{Tub}(E_i))$ of p_i, we have that s_1 nonvanishing. Therefore s_1 defines a continuous trivialization of $L|_{\overline{B(t,\epsilon)}}$ (which varies continuously in w and s_1) that we use to identify $s_1, \ldots, s_v$ with functions $U \to \mathbb{C}$. (Here s_1 is the constant function 1.) Then because s_i vanish to order w for $i \geq 2$, we have that $s_i = \widetilde{s}_i p_w$ for unique nonvanishing functions $\widetilde{s}_i$. Recall that $p_w(z)$ is the polynomial in z with roots at w, namely $\prod_{c \in w}(z - c)$.

We retract w to the origin radially, letting $w_u = (1-u)w$ for $u \in [0,1]$, and defining $s_{i,u} = \widetilde{s}_i \; p_{w,u}$ for $i > 1$ where

$$p_{w,u}(z) := \begin{cases} \prod_{c \in w}(z - (1-u)c) & \text{if } |z| \leq \max(1-u, 1/2) \\ \prod_{c \in w}(z - |z|c) & \text{if } u \leq 1/2,\ |z| \geq 1-u \\ \prod_{c \in w}(z - (1 + 2(1-u)(|z|-1))c) & \text{if } u \geq 1/2,\ |z| \geq 1/2. \end{cases} \tag{8.7}$$

In words, we break up the unit disk into two regions: the interior disk of radius $\max(1-u, 1/2)$ and the exterior ring. On the interior, we scale down the roots of p_w to the origin, and on the exterior we interpolate the roots linearly so that $p_{w,u}$ is unchanged on the boundary of the ball. □

Proposition 8.8. *Let $w \in W_n$, and K be a union of labelled closed balls satisfying $w \in K$. Then $\mathcal{M}_w(K) \to \mathcal{T}_w(K)$ is a homotopy equivalence.*

Proof. We show that each of the maps $\mathcal{M}_w(K) \to \mathcal{M}'_w(K)$ and $\mathcal{T}'_w(K) \to \mathcal{T}_w(K)$ are weak homotopy equivalences. (This suffices because by Proposition 8.2 the map $\mathcal{M}'_w(K) \to \mathcal{T}'_w(K)$ is a homeomorphism with inverse given by post-composition with the projection $Y \to \mathbb{P}(V)$).

First consider $\mathcal{M}_w(K) \to \mathcal{M}'_w(K)$. The target parameterizes w-holomorphic maps $C \to \mathbb{P}(V)$ intersecting p_i to the order prescribed by w. Now w-holomorphic maps $C \to \mathbb{P}(V)$ correspond to pairs consisting of a w-holomorphic line bundle L and $s \in \Gamma(L \otimes V)$ a non-vanishing w-holomorphic section, considered up to isomorphism. (Here (L, s) is isomorphic to (L', s') if there is a continuous, and therefore w-holomorphic, isomorphism of line bundles $L \to L'$ taking s to s'). The fiber F over an element $\mathcal{M}'_w$ corresponding to a w-holomorphic line bundle L and section s parameterizes the data of:

- $\widetilde{L}$ a holomorphic line bundle
- $\widetilde{s} \in \Gamma(V \otimes \widetilde{L})$ a section intersecting ℓ_i holomorphically order precisely w_i
- $f : \widetilde{L} \to L$ a continuous isomorphism of $\mathbb{C}$ line bundles, taking $\widetilde{s}$ to s,

considered up holomorphic isomorphism of pairs $(\widetilde{L}, \widetilde{s})$. This data is uniquely determined by the isomorphism f, so F parameterizes pairs consisting of $\widetilde{L}$ a holomorphic bundle and $f : \widetilde{L} \to L$ a continuous isomorphism (considered up to holomorphic isomorphisms of $\widetilde{L}$). In other words it is the space of holomorphic structures on L. The map $\mathcal{M}_w(K) \to \mathcal{M}'_w(K)$ is a fiber bundle because there is a cover of $\mathrm{Top}(C, \mathbb{P}(V))$ by open subsets U such that the universal bundle $\mathrm{ev}^* \mathcal{O}_{\mathbb{P}(V)}(1)|_{U \times C}$ is isomorphic to $\pi^* \mathcal{O}_C(d)$. Here ev is the evaluation map $\mathrm{Top}(C, \mathbb{P}(V)) \times C \to \mathbb{P}(V)$ and π is the projection to C and $\mathcal{O}_C(d)$ is the unique degree d continuous line bundle on C. A choice of isomorphism between the universal bundle over $U \times C$ and $\mathcal{O}_C(d)$ yields a trivialization over $U \cap \mathcal{M}'_w(K)$, identifying the fiber of every point with the space of holomorphic structures on $\mathcal{O}_C(d)$.

To show that $\mathcal{M}_w(K) \to \mathcal{M}'_w(K)$ is an equivalence, we prove that the fiber F is contractible. First F fibers over the space of C^∞ structures on L, which is isomorphic to $C^0(C, \mathbb{C}^*)/C^\infty(C, \mathbb{C}^*)$ because there is one C^∞ degree d complex line bundle up to isomorphism. Since the inclusion of C^∞ functions to continuous functions is an equivalence so the space of C^∞ structures on L is contractible. Equipping L with a

Riemannian metric, we have that holomorphic structures on L correspond to unitary connections on L. Therefore the fiber over a given C^∞ structure on L is identified with the space of unitary connections (a torsor for the space of $(1,0)$ forms with values in L). (See the related discussion at [AB83, p. 565]). This establishes that $\mathcal{M}_w(K) \to \mathcal{M}'_w(K)$ is an equivalence.

Second we show $\mathcal{T}'_w(K) \to \mathcal{T}_w(K)$ is a weak equivalence. Choosing disjoint disks around each point of w inside of the balls of K (i.e. the closure of a distinguished neighborhood $V \subseteq K$), we reduce to the following local statement. Let $i \in \{1, \ldots, r\}$ $m \in \mathbb{N}$. For a map $D \subseteq \mathbb{C}$ the unit disk, and a degree m map $g : \partial D \to \mathrm{Tub}(E_i) - E_i$ the inclusion of mapping spaces between

$$\{f : D \to \mathrm{Tub}(E_i),\ f|_{\partial D} = g,\ f^{-1}(E_i) = 0 \mid f \text{ intersects } E_i \text{ holomorphically to order } m \text{ at } 0\}$$

and

$$\{f : D \to \mathrm{Tub}(E_i),\ f|_{\partial D} = g,\ f^{-1}(E_i) = 0\}$$

is an equivalence.

We fiber both spaces over E_i via the map $f \mapsto f(0)$ to reduce to spaces of maps satisfying the additional condition that $f(0) = p$. Accordingly we analyze the homotopy type of

$$G := \{f : D \to \mathrm{Tub}(E_i),\ f|_{\partial D} = g,\ f^{-1}(E_i) = 0, f(0) = p\}.$$

As in the proof of Proposition 8.6 we may deformation retract onto the subspace consisting of maps satisfying $\pi(f(x)) = p$ for $|x| \leq 1/2$. By the Alexander trick applied to the normal slice N_p to p, we may retract further to the space of functions which are radial for $|x| \leq 1/2$ (in other words functions satsifying $f(rx) = rf(x)$ for $|x| \leq 1/2$ and $r \in [0,1]$). This shows that G is equivalent to

$$\{g' : S^1 \to N_p - 0,\ h \text{ homotopy between } g' \text{ and } g\ \}.$$

Next we consider the subspace of $H \subseteq G$ consisting of functions that intersect E_i holomorphically to order m at 0. Identifying N_p with an open ball in $\mathbb{C}$, we will show that this subspace is equivalent to

$$\{g' : S^1 \to N_p - 0,\ h \text{ homotopy between } g' \text{ and } g\ \mid g'(z) = az^d \text{ for some } a \in \mathbb{C}^*\}$$

and the inclusion is the natural one. Because the space of degree m maps $\mathrm{Top}_m(S^1, \mathbb{C}-0)$ retracts to the subspace of maps of the form $z \mapsto az^m$ where $a \in \mathbb{C}^*$, establishing this will complete the proof.

Since the holomorphicity condition is only imposed in the normal direction to E_i, as above we may first deformation retract to the subspace of maps such that $\pi(f(x)) = p$ for $|x| \leq 3/4$. For such a map, the restriction of f to the radius $3/4$ ball $D_{3/4}$ is determined by the induced map to the normal slice $\widetilde{f} : D_{3/4} \to N_p$. Then we retract further to the subspace of functions f which satisfy $\pi(f(x)) = p$ for $|x| \leq 1/2$

and such that the induced map $\widetilde{f} : D_{1/2} \to N_p \subseteq \mathbb{C}$ agrees with its degree m Taylor polynomial. To do this, we use a rescaling of the following deformation retraction from the space

$$\{j : D \to \mathbb{C} \mid f^{-1}(0) = 0,\ j \text{ intersects zero holomorphically to order exactly } m\ \}$$

to the subspace of functions that agree with their degree m Taylor polynomial at the origin on the ball of radius $1/2$,

$$j_u(z) := \begin{cases} \frac{1}{(1-u)^m} j((1-u)z) & \text{if } |z| \leq \max(1-u, 1/2) \\ \frac{1}{(1-|z|)^m} j((1-|z|)z) & \text{if } u \leq 1/2,\ |z| \geq 1-u \\ \frac{1}{(1+2(1-u)(|z|-1))^m} j((1+2(1-u)(|z|-1))z) & \text{if } u \geq 1/2,\ |z| \geq 1/2. \end{cases}$$

This retraction is very similar to (8.7). We break the disk into an interior region of radius $\max(1-u, 1/2)$ and an exterior annulus. On the interior disk use the $j_u(z) := (1-u)^{-m} j((1-u)z)$, and on the exterior annulus we interpolate between j and the values on the interior. The key point is that, because j vanishes to order exactly m at the origin, $\lim_{t\to 0} t^{-m} j(tz)$ equals the degree m Taylor polynomial of j. □

8.3. **Relating $\mathcal{T}$ and Top^+.** In this subsection, we recall the dependence of the constructions on a choice of (d, n). Recall that $\mathcal{T}_{d,n}$ was defined to be a subset of $W_n \times \mathrm{Top}_{d,n}(C, X)$ given by forgetting the choice of divisors $w \in W_n$.

Proposition 8.9. *The projection map $\mathcal{T}_{d,n} \to \mathrm{Top}_{d,n}(C, X)$ is a homoemorphism onto its image.*

Proof. The projection is bijective, because given a function $f : C \to Y$ that intersects E_i positively for $i = 1, \ldots, r$ we can recover D_i as the unique divisor supported at $f^{-1}(E_i)$, whose multiplicity at a point of $f^{-1}(E_i)$ is the local intersection multiplicity of f with E_i.

To show that the inverse is continuous, fix $w \in W_n$ corresponding to a finite labelled subset $T \subseteq C$ and a distinguished open subset of radius ϵ containing w. Let $f \in \mathrm{Top}_+(C, Y)_{d,n}$ be a function whose intersection multiplicities with E_i are those specified by w. Consider the union of circles of radius $\epsilon/2$ around each $t \in T$, and the open subspace consisting of functions $f' \in \mathrm{Top}_+(C, Y)_{d,n}$ such that f' differs from f by at most δ on each circle. For δ sufficiently small, we have that for every f' in this subspace, its restriction to each radius $\epsilon/2$ circle is homotopic to the restriction of f' (considered as a map to $Y - \cup_i E_i$). In particular, the intersection multiplicity of f' on the interior of each circle agrees with that of f'. Since f' only intersects E_i

positively, it follows that the divisor associated to f' is contained in the distinguished open subset of w of radius $\epsilon/2$. Hence the inverse is continuous. $\square$

Since by definition, the image of $T_{d,n}$ under the projection map is $\mathrm{Top}^+_{d,n}(C, X)$, we have that Proposition 8.9 and Theorem 8.1 yield the following. (We also include the pointed variant, which follows from arguments identical to the ones in this section.)

Theorem 8.10. *There is a weak equivalence $\mathcal{M}_{d,n} \simeq \mathrm{Top}^+_{d,n}(C, X)$ and $\mathcal{M}_{d,n,*} \simeq \mathrm{Top}^+_{d,n,*}(C, X)$.*

9. Unobstructedness

In this section, we verify the hypotheses of Theorem 7.1 in several cases, and consequently deduce our main results. Establishing these hypotheses boils down to proving that certain section spaces are unobstructed. There is a range where section spaces are always unobstructed by a simple application of Riemann–Roch, which we consider in §9.1. If the points $p_1, \dots, p_{\dim V}$ are in linearly general position, then this range can be improved slightly: we do this in §9.2. Finally in §9.4 we consider the case of the degree 5 del Pezzo surface, where we apply the results of §9.2 and the symmetry under S_5 to improve the range to a dense open subset of the ample cone.

Throughout this section we let V be a vector space containing lines $\ell_1, \dots, \ell_r$ corresponding to $p_1, \dots, p_r \in \mathbb{P}(V)$. We let $v := \dim V$, and let Q_r denote the poset $\{0, \ell_1, \dots, \ell_r, V\}$, and let C be a fixed smooth proper genus g curve.

9.1. Riemann–Roch.

First we apply Riemann–Roch directly to construct a poset to which we can apply Theorem 7.1. We state it using the multiplicity functions m_{ℓ_j}, m_0 of §3.5. We will set $m_\ell := \sum_{j=1}^r m_{\ell_j}$. Recall that given $x \in {Q_r}^{\mathrm{JC}}$ the expected codimension of $\Gamma(C, L \otimes V)_x$ in $\Gamma(C, L \otimes V)$ is $\gamma(x) = 2(\dim V m_0(x) + (\dim V - 1)m_\ell(x))$. We have the following simple consequence of Riemann–Roch.

Proposition 9.1. *Let C be a smooth projective genus g curve. Let $x \in {Q_r}^{\mathrm{JC}}$. If $d - m_0(x) - m_\ell(x) \geq 2g - 1$, then $\Gamma_{Q_r}(C, L \otimes V)_x$ is unobstructed.*

Proposition 9.2. *Let $d \in \mathbb{N}, n \in \mathbb{N}^r$, and let*

$$I = d - \sum_{i=1}^r n_i - 2g$$

Let $W \subseteq \prod_{i=1}^r \mathrm{Sym}^{n_i} C$ be the locus of pairwise disjoint divisors. Consider the subset $P \subseteq (W < \mathrm{Hilb}(C)^{Q_r})$ consisting of $w < x$ which satisfy $m_0(x) + m_\ell(x) \leq I + \sum_{i=1}^r n_i$, then

(1) P is proper over W and downward closed
(2) for all $w < x \prec y$, with $(w < x) \in P$ and every $L \in \mathrm{Pic}^d(C)$ the fiber $\Gamma_{Q_r}(C, L \otimes V)_y$ is unobstructed.
(3) P contains every type T with $\kappa(T) \leq I$.

Proof. For (1), it suffices to show that the subset of $\mathrm{Hilb}(C)^{Q_r}$ consisting of x satisfying $m_0(x) + m_\ell(x) \leq I$ is closed for every $j = 1, \dots v$. This follows from Lemma 9.3.

Condition (2) follows from Proposition 9.1. For (3) suppose that $w < x \in \mathcal{S}_T$ with $\kappa(T) \leq I$. Then $r(w < x) \leq I$ by Lemma 9.4. Now $r(w < x) = 2m_0(x) + m_\ell(x) - \sum_{i=1}^r n_i$. Since $m_0(x) \geq 0$, we obtain that $m_0(x) + m_\ell(x) \leq I - \sum_{i=1}^r n_i$. $\square$

Lemma 9.3. *Let $J \in (1/2, 1]$, and $I \in \mathbb{R}$. Then the set of $x \in \mathrm{Hilb}(C)^{Q_r}$ such that $m_0(x) + Jm_\ell(x) \leq I$ is compact and downward closed.*

Proof. Downward closure is immediate, because the function $E := m_0 + Jm_\ell$ is increasing. To show compactness it suffices to show that the set of x such that $E(x) \leq I$ is closed: since if $m_0(x) + Jm_\ell(x) \leq I$ then $x \in (\mathrm{Hilb}(C)_{\leq \max(I, I/J)})^{r+1}$.

For closedness, suppose that $g_1, g_2 : Q_r \to \mathbb{N}$ are two functions. We claim that

$$m_0(g_1) + Jm_\ell(g_1) + Jm_0(g_2) + m_\ell(g_2) \geq m_0(g_1 + g_2) + Jm_\ell(g_1 + g_2),$$

(i.e. that $E(g_1 + g_2) \geq E(g_1) + E(g_2)$). This claim establishes closedness, because it implies that collection of combinatorial types satsifying the the condition $m_0(T) + Jm_\ell(T) \leq I$ is downward closed with respect to the partial order $\leq_+$.

Because $g \leq \mathrm{sat}(g)$ for all g, we have that $\mathrm{sat}(g_1 + g_2) \leq \mathrm{sat}(\mathrm{sat}(g_1) + \mathrm{sat}(g_2))$. Since $E(g) = E(\mathrm{sat}(g))$, we have $E(g_1 + g_2) \leq E(\mathrm{sat}(g_1) + \mathrm{sat}(g_2))$, so it suffices to establish the claim with g_1, g_2 replaced by their saturations.

So suppose that g_1, g_2 are saturated and correspond to the chains $(a_1 - a_2)\ell_i + a_2$ and $(b_1 - b_2)\ell_j + b_2$ for $a_1 \geq a_2$ and $b_1 \geq b_2$ and $i, j \in \{1, \dots r\}$. If $i = j$, then $E(g_1 + g_2) = E(g_1) + E(g_2)$. Otherwise, suppose without loss of generality that $a_2 + b_1 \geq a_1 + b_2$. Then the saturation of $g_1 + g_2$ corresponds to the chain

$$(a_1 + b_1 - a_2 - b_1)\ell_k + (a_1 + b_2)0,$$

where $k = i$ or j (depending on $\max(a_1, b_1)$). Thus we have

$$E(g_1 + g_2) = (a_1 + b_1 - a_2 - b_1)J + (a_1 + b_2)$$

and

$$E(g_1) + E(g_2) = (a_2 - a_1 + b_2 - b_1)J + a_2 + b_2.$$

Simplifying, the inequality $E(g_1+g_2) \leq E(g_1)+E(g_2)$ reduces to $(2J-1)(a_1-a_2) \geq 0$, which holds by our assumption on J. $\square$

Lemma 9.4. *We have that $r(x) \leq \kappa(x)$ for all $x \in \mathrm{Hilb}(C)^{Q_r}$.*

Proof. If $x_1 \prec x_2 \in {Q_r}^{\mathrm{JC}}$, then it follows from case analysis that $\kappa(x_2) \geq \kappa(x_1) + 1$. Choosing a maximal chain of elements of ${Q_r}^{\mathrm{JC}}$ ending in $\mathrm{sat}(x)$ (which must have length $r(x)$), we obtain that $\kappa(x) \geq r(x)$. □

9.2. **Points in general position.** Write $v = \dim V$. In this subsection, we assume that the points $p_1, \ldots, p_{\min(v,r)} \in \mathbb{P}(V)$ are in linearly general position. In this case, we have the following improvement on Riemmann–Roch.

Proposition 9.5. *Let $d \in \mathbb{N}$ and let $x \in {Q_r}^{\mathrm{JC}}$, with $m_{\ell_i}(x) = n_i$. If $-m_{\ell_j}(x) + m_\ell(x) \leq d - m_0(x) + 2 - 2g$ for all $j = 1, \ldots, \min(v, r)$, then $\Gamma_{Q_r}(C, L^v)_x$ is unobstructed for any $L \in \mathrm{Pic}^d(C)$.*

Proof. For notational simplicity we assume that $r \geq v$, the case where $r < v$ is similar.

Let L be a degree d line bundle on C, and let $D_{\ell_1}, \ldots D_{\ell_r}, D_0$ be divisors specifying an element $x \in {Q_r}^{\mathrm{JC}}$. Because x is saturated, $U_i := D_{\ell_i} - D_0 \in \mathrm{Sym}^{m_{\ell_i}(x)}(C)$ are a collection of pairwise disjoint divisors.

We must show that $\Gamma_{Q_r}(C, L^v)_x$ has the expected dimension. Without loss of generality, we take $p_1, \ldots, p_v$ to be the points corresponding to the lines generated by the standard coordinate vectors $e_i \in \mathbf{C}^v, i = 1, \ldots, v$. Then $\Gamma_{Q_r}(C, L^v)_x$ consists of tuples of sections $s = (s_i)_{i=1}^v \in \Gamma(L^v(-D_0))$ such that

$$s_i(U_j) = 0, \quad i, j \in \{1, \ldots, v\},\ i \neq j$$

and $s(U_k) \in \ell_k$ for $k = v+1, \ldots, r$. Thus $\Gamma_{Q_r}(C, L^v)_x = \Gamma(\mathcal{F})$ where $\mathcal{F}$ is the fiber product of the following diagram of sheaves

$$\begin{tikzcd} & \prod_{b=v+1}^{r} \ell_b \arrow[d] \\ \prod_{j=1}^{v} L\left(-D_0 - \sum_{i=1, i\neq j}^{v+1} U_i\right) \arrow[r, "p"] & \prod_{b=v+1}^{r} L^v(-D_0)|_{U_b}. \end{tikzcd}$$

Now there is a short exact sequence of sheaves

$$0 \to \prod_{j=1}^{v} L(-D_0 - \sum_{i=1, i\neq j}^{r} U_i) \to \mathcal{F} \to \prod_{b=v+1}^{r} \ell_b \to 0,$$

because the left hand term is the kernel of p, and p is surjective as a map of sheaves. If $-m_{\ell_j}(x) + m_{\ell(x)} \leq d - m_0(x) + 2 - 2g$ for all $j = 1, \ldots, r$, then by Riemann–Roch we have that $H^1(L(-D_0 - \sum_{i=1, i\neq j}^{r} U_i)) = 0$ for all $j = 1, \dots, v$, so by the long exact sequence in cohomology $H^1(\mathcal{F}) = 0$ hence $H^0(\mathcal{F})$ has the expected dimension.

In general, we have that $\dim H^1(\mathcal{F}) \leq \sum_{j=1}^{\min(v,r)} \left(d - 2g + 2 - \sum_{i=1, i\neq j}^{r} n_i\right)$. □

Because of Proposition 9.5, we may construct the following poset.

Proposition 9.6. *Let $d \in \mathbb{N}, n \in \mathbb{N}^r$, and let*

$$I = d - \sum_{i=1}^{r} n_i - 2g + \min\{n_1, \dots, n_v\}$$

where we take $n_j = 0$ if $r < j \leq v$. Let $W \subseteq \prod_{i=1}^{r} \mathrm{Sym}^{n_i} C$ be the locus of pairwise disjoint divisors. Consider the subset $P \subseteq (W < \mathrm{Hilb}(C)^{Q_r})$ consisting of $w < x$ which satisfy

$$m_0(w < x) + \sum_{i \neq j} m_{\ell_i}(w < x) \leq I$$

for all $j = 1, \dots, v$. Then

(1) P proper over W and downward closed
(2) for all $w < x \prec y$, with $w < x \in P$ and every $L \in \mathrm{Pic}^d(C)$ the fiber $\Gamma_{Q_r}(C, L \otimes V)_y$ is unobstructed.
(3) P contains every type T with $\kappa(T) \leq I$.

Proof. For (1), it suffices to show that the subset of $\mathrm{Hilb}(C)^{Q_r}$ consisting of x satisfying $m_0(x) + m_\ell(x) - m_{\ell_j}(x) \leq I$ is closed for every $j = 1, \dots v$. Considering the projection forgetting points labelled by ℓ_j, this follows from Lemma 9.3.

Condition (2), follows from Proposition 9.5, together with the fact that if $y \succ x$ then

$$d - m_0(y) + m_\ell(y) - m_{\ell_j}(y) \geq d - m_0(x) + m_\ell(x) - m_{\ell_j}(x) - 1 \geq d - I - 1 \geq 2g - 2,$$

since $d - I$ is nonnegative.

For (3) suppose that $w < x \in \mathcal{S}_T$ where $\kappa(T) \leq I$. Then $r(w < x) \leq I$ by Lemma 9.4. Now $r(w < x) = 2m_0(w < x) + m_\ell(w < x)$. Since $m_{\ell_i}(x) - n_i \geq 0$ for all i, it follows that $m_0(x) + \sum_{i\neq j, i=1}^{r}(m_{\ell_i}(x) - n_i) \leq I$ for every $j = 1, \dots, v$. □

9.3. **Proof of main theorem and corollary.** With Proposition 9.2 and Proposition 9.2 in place, we are now in a position to establish Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. We first consider the case where the points $p_1, \dots p_{\min(v,r)}$ are assumed to be in general position. Given $(d, n) \in \mathbb{Z}^{r+1}$, by Proposition 9.6 there is a subposet $P \subseteq (W < \mathrm{Hilb}(C)^{Q_r})$ which satisfies the hypotheses of Theorem 7.1 for

$I(d,n) = \min_{j=1,\dots,\min(v,r)}(d+n_j-\sum_{i=1}^r n_i)-2g$. Therefore the map $\mathrm{Alg}_{d,n}(C,X) \to \mathcal{M}_{d,n}$ is homology $I(d,n)$-connected. So by Theorem 8.10, the map $\mathrm{Alg}_{d,n}(C,X) \to \mathrm{Top}^+_{d,n}(C,X)$ is homology $I(d,n)$-connected.

In the pointed case, we apply the Proposition 9.6 to the tuple $(d,n_1,\dots,n_r,1)$. Let Q_{r+1} be the poset $\{0,\ell_1,\dots,\ell_r,\ell_{r+1}\}$ where $\ell_{r+1} \subseteq V$ is the line corresponding to the basepoint of X. Then Proposition 9.6 yields a subposet of $P \subseteq (W_{n+1} < \mathrm{Hilb}(C)^{Q_{r+1}})$, which is unobstructed and contains all combinatorial types T such that $\kappa(T) \le I(d,n)-1$. Now consider the intersection of P with $R \subseteq (W_{n,*} < \mathrm{Hilb}(C)^{Q_{r+1}})$ (see §7.2 for Definition). Given a saturated combinatorial type T of P we have that $\mathcal{S}_T \cap R$ is a (possibly empty) union of strata of pointed combinatorial types T'. Furthermore, $\kappa(T') \ge \kappa(T)$ because the expected codimension γ is unchanged while $\dim \mathcal{S}_{T'} \le \dim \mathcal{S}_T$ and $r(T') \le r(T)$. Therefore the hypotheses of Theorem 7.3 apply to $P \cap R$ with $I = I(d,n)-1$. So as above, $\mathrm{Alg}_{d,n,*}(C,X) \to \mathrm{Top}^+_{d,n,*}(C,X)$ is $I(d,n)-1$ connected.

To rephrase these connectivity statements in the form stated in the introduction, suppose that $\alpha = (d',n') \in \mathbb{N}^{r+1}$ is a class satisfying $d'+n'_j-\sum_{i=1}^r n'_i > 0$ for all $j = 1,\dots,\min(v,r)$. Let $M_\alpha = \min(d'+n'_j-\sum_{i=1}^r n'_i)$. Then for $k \in \mathbb{N}$ we have that $I(kd',kn') = M_\alpha k - 2g$ hence for $i \le M_\alpha k - 2g - 2$ the induced maps on H_i are isomorphism.

When the points $p_1,\dots,p_{\min(v,r)}$ are not assumed to be in general position, we apply the same argument, using Proposition 9.2 instead of Proposition 9.6. ☐

Proof of Corollary 1.2. Let C be a curve. Up to homotopy, the space $\mathrm{Top}^+_{(e,l_1,\dots,l_r)}(C,X)$ is independent of e. More precisely fix a disc around $*_C$ to obtain a map $\pi : C \to S^2 \vee C$ (given by collapsing the boundary of the disk), and choose a map $\eta : S^2 \to X - (\cup_{i=1}^r E_i)$ that projects to a degree 1 map to $\mathbb{P}^m$ and satisfies $\eta(0) = \eta(\infty) = *_X$. Wedging with η defines a stabilization map

$$\mathrm{Top}^+_{(e,l_1,\dots,l_r)}(C,X) \to \mathrm{Top}^+_{(e+1,l_1,\dots,l_r)}(C,X) \qquad f \mapsto (\eta \vee f)\circ \pi.$$

We may think of η as a homotopy from constant map $S^1 \to *_X$ to itself. Let $\overline{\eta}$ be the reversed homotopy. Then wedging with $\overline{\eta}$ defines a destabilization map that is homotopy inverse to η, because the the map $S^2 \to S^2 \vee S^2 \to^{\eta\vee\overline{\eta}} X - \cup_i E_i$ is null-homotopic.

Furthermore, up to homeomorphism $\mathrm{Top}^+_{(e,l_1,\dots,l_r)}(C,X)$ is independent of the choice of points $p_1,\dots,p_r$.

When $C = \mathbb{P}^1$ and all of the points $p_1,\dots,p_r$ lie on a hyperplane, and

$$\min_j(e-l_1-\cdots-l_r,l_j) > i$$

the work of Boyer–Hurtubise–Milgram [BHM01] shows that

$$\mathrm{Alg}_{(e,l_1,\ldots,l_r)}(\mathbb{P}^1, X) \to \mathrm{Top}_{(e,l_1,\ldots,l_r)}(\mathbb{P}^1, X)$$

induces an isomorphism on homology in degrees $\leq i$. (Take N to be the group of translations of affine space which is the complement of the hyperplane. Then $c_0(X) = 1$ because X_∞ has normal crossings). Considering the diagram induced on homology by

$$\begin{tikzcd}
\mathrm{Alg}_{(e+j,l_1,\ldots,l_r),*}(\mathbb{P}^1, X) \arrow[r] & \mathrm{Top}^+_{(e+j,l_1,\ldots,l_r),*}(\mathbb{P}^1, X) \arrow[r] & \mathrm{Top}_{(e+j,l_1,\ldots,l_r),*}(\mathbb{P}^1, X) \\
& \mathrm{Top}^+_{(e,l_1,\ldots,l_r),*}(\mathbb{P}^1, X) \arrow[u, "-\vee\eta^{\vee j}"] \arrow[r] & \mathrm{Top}_{(e,l_1,\ldots,l_r),*}(\mathbb{P}^1, X), \arrow[u, "-\vee\eta^{\vee j}"]
\end{tikzcd}$$

we obtain that $\mathrm{Top}^+_{(e,l_1,\ldots,l_r),*}(\mathbb{P}^1, X) \to \mathrm{Top}_{(e,l_1,\ldots,l_r),*}(\mathbb{P}^1, X)$ induces an isomorphism on H_i for $i \leq \min(l_j)$. By Theorem 1.1 we obtain the result. $\square$

9.4. **The degree 5 del Pezzo Surface.** Let X be the unique degree 5 del–Pezzo surface over $\mathbb{C}$. In this section we show that for almost every ample class $\alpha \in \mathrm{N}_1(X)$, that as $k \to \infty$ there is a bar construction model for the homology of $\mathrm{Alg}_{k\alpha}(C, X)$. First we show that there is a way of writing X as a blowup, such that the ample class α satisfies certain equalities.

Proposition 9.7. *Let $\alpha \in H_2(X, \mathbb{Z})$ be an ample class. Then there are four distinct points $p_1, p_2, p_3, p_4 \in \mathbb{P}^2$, no three lying on an line and an isomorphism $X \cong \mathrm{Bl}_{p_1,\ldots,p_4}\mathbb{P}^2$ such that $\alpha = (d, n_1, n_2, n_3, n_4)$ with $n_i + n_j + n_4 \leq d$ for all choices of distinct $i, j \in \{1, 2, 3\}$. If α is such that there are no disjoint -1-curves E, E' with $E' \cdot \alpha = E \cdot \alpha$, we may choose this isomorphism such that $n_i + n_j + n_4 < d$*

Proof. Fix four points in $p_1, \ldots p_4 \in \mathbb{P}^2$, no three lying on a line. It is classical that X is isomorphic to $\mathrm{Bl}_{p_1,\ldots,p_4}\mathbb{P}^2$. In terms of this isomorphism, write $\alpha = (e, u_1, u_2, u_3, u_4)$. By relabelling the points, we may assume that $u_1 \geq u_2 \geq u_3 \geq u_4$. Now consider the Cremona transformation on p_1, p_2, p_3 defining an automorphism of $\mathrm{Bl}_{p_1,\ldots,p_4}\mathbb{P}^2$. On $H_2(X)$ it acts by

$$(e, u_1, u_2, u_3, u_4) \mapsto (d, n_1, n_2, n_3, n_4) := (2e-u_1-u_2-u_3, e-u_2-u_3, e-u_1-u_3, e-u_2-u_3, u_4).$$

Now direct calculation shows that $u_i \geq u_4$ implies that $d \geq n_j + n_k + n_4$ for $\{i, j, k\} = \{1, 2, 3\}$, with equality if and only if α lies in the hyperplane $u_3 = u_4$. $\square$

Applying Theorem 1.1, we immediately obtain the following theorem, as well as Theorem 1.4.

Theorem 9.8. *Let C be a compact Riemann surface. For any ample class α such that there are no two disjoint -1 curves on X with $E' \cdot \alpha = E \cdot \alpha$ and any basepoint $*_X$ not contained in the locus of exceptional curve. Then the maps*

$$H_i(\mathrm{Alg}_{k\alpha}(C,X)) \to H_i(\mathrm{Top}^+_{k\alpha}(C,X)) \qquad H_i(\mathrm{Alg}_{k\alpha,*}(C,X)) \to H_i(\mathrm{Top}^+_{k\alpha,*}(C,X))$$

induce isomorphisms for $i \leq kN_\alpha - 2g - 2$ where N_α is a constant depending on α

References

[AB83] M. F. Atiyah and R. Bott, "The Yang-Mills equations over Riemann surfaces," *Philos. Trans. Roy. Soc. London Ser. A*, vol. 308, no. 1505, pp. 523–615, 1983, ISSN: 0080-4614. DOI: 10.1098/rsta.1983.0017 (cit. on p. 42).

[Aum21] A. Aumonier, "An h-principle for complements of discriminants," *arXiv preprint arXiv:2112.00326*, 2021 (cit. on p. 7).

[Ban22] O. Banerjee, "Moduli of curves on toric varieties and their stable cohomology," *arXiv preprint arXiv:2210.05826*, 2022 (cit. on p. 7).

[BHM01] C. P. Boyer, J. C. Hurtubise, and R. J. Milgram, "Stability theorems for spaces of rational curves," *Internat. J. Math.*, vol. 12, no. 2, pp. 223–262, 2001, ISSN: 0129-167X,1793-6519. DOI: 10.1142/S0129167X01000721 (cit. on pp. 2, 8, 49).

[BM90] V. V. Batyrev and Y. I. Manin, "Sur le nombre des points rationnels de hauteur borné des variétés algébriques," *Mathematische Annalen*, vol. 286, no. 1, pp. 27–43, 1990 (cit. on p. 4).

[BMHM94] C. Boyer, B. Mann, J. Hurtubise, and R. Milgram, "The topology of the space of rational maps into generalized flag manifolds," *Acta Mathematica*, vol. 173, no. 1, pp. 61–101, 1994 (cit. on p. 2).

[BS20] T. Browning and W. Sawin, "A geometric version of the circle method," *Annals of Mathematics*, vol. 191, no. 3, pp. 893–948, 2020 (cit. on p. 6).

[DI04] D. Dugger and D. C. Isaksen, "Topological hypercovers and 1-realizations," *Mathematische Zeitschrift*, vol. 246, pp. 667–689, 2004 (cit. on p. 39).

[dLa 02] R. de La Bretèche, "Nombre de points de hauteur bornée sur les surfaces de del pezzo de degré 5," *Duke Mathematical Journal*, vol. 113, no. 3, pp. 421–464, 2002 (cit. on p. 4).

[ETW17] J. S. Ellenberg, T. Tran, and C. Westerland, "Fox-Neuwirth-Fuks cells, quantum shuffle algebras, and Malle's conjecture for function fields," *arXiv preprint arXiv:1701.04541*, 2017 (cit. on p. 3).

[EVW16] J. S. Ellenberg, A. Venkatesh, and C. Westerland, "Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields," *Annals of Mathematics*, pp. 729–786, 2016 (cit. on p. 3).

[FM81] W. Fulton and R. MacPherson, "Categorical framework for the study of singular spaces," *Mem. Amer. Math. Soc.*, vol. 31, no. 243, pp. vi+165, 1981, ISSN: 0065-9266,1947-6221. DOI: 10.1090/memo/0243 (cit. on p. 9).

[Gra89] J. Gravesen, "On the topology of spaces of holomorphic maps," *Acta Math.*, vol. 162, no. 3-4, pp. 247–286, 1989, ISSN: 0001-5962,1871-2509. DOI: 10.1007/BF02392839 (cit. on p. 2).

[Gue95] M. A. Guest, "The topology of the space of rational curves on a toric variety," *Acta Mathematica*, vol. 174, no. 1, pp. 119–145, 1995 (cit. on p. 2).

[Hae82] A. Haefliger, "Rational homotopy of the space of sections of a nilpotent bundle," *Transactions of the American Mathematical Society*, vol. 273, no. 2, pp. 609–620, 1982 (cit. on p. 4).

[Kir85] F. Kirwan, "On spaces of maps from Riemann surfaces to Grassmannians and applications to the cohomology of moduli of vector bundles," *Arkiv för matematik*, vol. 24, no. 1-2, pp. 221–275, 1985 (cit. on p. 2).

[KY18] A. Kozlowski and K. Yamaguchi, "The homotopy type of spaces of rational curves on a toric variety," *Topology and its Applications*, vol. 249, pp. 19–42, 2018 (cit. on p. 7).

[LT19] B. Lehmann and S. Tanimoto, "Geometric Manin's conjecture and rational curves," *Compositio Mathematica*, vol. 155, no. 5, pp. 833–862, 2019 (cit. on p. 7).

[Mos12] J. Mostovoy, "Truncated simplicial resolutions and spaces of rational maps," *Q. J. Math.*, vol. 63, no. 1, pp. 181–187, 2012, ISSN: 0033-5606,1464-3847. DOI: 10.1093/qmath/haq031 (cit. on p. 7).

[MS74] J. W. Milnor and J. D. Stasheff, *Characteristic classes* (Annals of Mathematics Studies). Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1974, vol. No. 76, pp. vii+331 (cit. on p. 10).

[Pey95] E. Peyre, "Hauteurs et mesures de Tamagawa sur les variétés de Fano," *Duke Math. J.*, vol. 79, no. 1, pp. 101–218, 1995, ISSN: 0012-7094,1547-7398. DOI: 10.1215/S0012-7094-95-07904-6 (cit. on p. 7).

[Seg79] G. Segal, "The topology of spaces of rational functions," *Acta Mathematica*, vol. 143, no. 1, pp. 39–72, 1979 (cit. on p. 1).

[Tes09] D. Testa, "The irreducibility of the spaces of rational curves on del Pezzo surfaces," *J. Algebraic Geom.*, vol. 18, no. 1, pp. 37–61, 2009, ISSN: 1056-3911,1534-7486. DOI: 10.1090/S1056-3911-08-00484-0 (cit. on p. 7).

[VE10] A. Venkatesh and J. S. Ellenberg, "Statistics of number fields and function fields," *Proceedings of the International Congress of Mathematicians 2010 (ICM 2010) (In 4 Volumes) Vol. I: Plenary Lectures* *and Ceremonies Vols. II–IV: Invited Lectures*, World Scientific, 2010, pp. 383–402 (cit. on p. 3).