The mechanism is going to ask hospitals to report their graphs $G'_1, \ldots, G'_m$. Based on these reports define $G'_\oplus$ as the combined graph. Let $R'_{\oplus,AB}$ and $R'_{\oplus,BA}$ denote the total number of AB and BA pairs in graph $G'_\oplus$. Let $R'_{i,AB}$ and $R'_{i,BA}$ denote the number of AB and BA pairs respectively in G'_i. Similarly, let $S'_{\oplus,T}$ denote the total number of self-demanded blood type T pairs in graph $G'_\oplus$, and similarly for $S'_{i,T}$.

The *idealized* xCM mechanism builds up a matching μ, and is defined as:

1. Receive graphs $G'_1, \ldots, G'_m$ from each hospital
2. Compute regular matching $\mu'_1, \ldots, \mu'_m$ for each hospital
3. Adopt in the output matching μ the parts of the matchings $\mu'_1, \ldots, \mu'_m$ that involve OD-UD pairs
4. For R pairs:
   (a) Assume WLOG that $R'_{\oplus,AB} \geq R'_{\oplus,BA}$ (just reverse the role of AB and BA otherwise), and let $\delta_R = R'_{\oplus,AB} - R'_{\oplus,BA}$
   (b) Initialize H' to the set of hospitals with $R'_{i,AB} > R'_{i,BA}$. Do δ_R times:
      i. Select a hospital i in set H' at random and select at random one of its AB pairs in graph $G'_\oplus$, updating $G'_\oplus$ to remove this pair.
      ii. Remove i from H' if i no longer has more AB pairs than BA pairs in $G'_\oplus$.
   (c) Adopt in the output matching μ a maximum matching on the R subgraph remaining in $G'_\oplus$
5. For S pairs:
   (a) For each blood type $T \in \{$O, A, B, AB$\}$:
      i. Let $S'_{\oplus,T}$ denote the number of pairs of self-demanded blood type T in the combined graph $G'_\oplus$
      ii. If $S'_{\oplus,T}$ is odd then select one hospital i at random from the hospitals with an odd number of $S'_{i,T}$ pairs in graph G'_i, and select at random one of its self-demanded T pairs in graph $G'_\oplus$, updating $G'_\oplus$ to remove this pair.
   (b) Adopt in the output matching μ a maximum matching on the S subgraph remaining in $G'_\oplus$
6. Return the matching μ

This is an idealized description of the mechanism because it assumes the existence of a regular matching in step (2), and thus it is not well defined without regularity. The mechanism is defined to construct a regular matching on graph $G'_\oplus$ with the property that each agent matches at least as many pairs, of every type, as in its own regular match.

To gain some intuition for the behavior of xCM on R pairs, consider the following simple example.

Example 1. *Suppose there are two hospitals, hospital 1 with 10 and 6 and hospital 2 with 8 and 10 of pairs reciprocal AB and reciprocal BA respectively, then there is an overall excess of AB. A regular matching on R in the combined graph will match 16 AB with 16 BA (under the perfect matching assumption). Certainly hospital 1 and 2 will have 6 and 10 BA pairs matched, respectively. Moreover, to respect the constraint that each hospital should do as well as in its individual regular matching, hospital 1 and 2 should match at least 6 and 8 AB pairs, respectively (again, under the PM assumption). The xCM mechanism ensures this by dropping 2 of hospital 1's reciprocal AB pairs from consideration in the matching.*

The theoretical analysis is done for the idealized mechanism, and is followed with brief comments on a practical xCM design.

Lemma 1. *The idealized xCM mechanism satisfies the following properties under the regularity and PM assumptions:*

1. *The matching μ is regular on $G'_\oplus$.*
2. *The matching μ provides every hospital i with at least as many OD, UD, S_T (for all types T), R_{AB} and R_{BA} pairs matched as in the matching μ'_i. In particular, no OD pairs of one hospital ever match with the UD pairs of another hospital.*
3. *If $R'_{\oplus,AB} \geq R'_{\oplus,BA}$, such that there are more reciprocal AB than BA in the combined graph, then a hospital i with $R'_{i,AB} \leq R'_{i,BA}$ matches all of its R pairs. Moreover, the expected number of R pairs matched by hospital i with $R'_{i,AB} > R'_{i,BA}$ weakly increases in $R'_{i,AB}$.*

3' Symmetrically for $R'_{\oplus,AB} \geq R'_{\oplus,BA}$

4. *If there is an even number of self-demanded type T in graph $G'_\oplus$, or hospital i submits an even number $S'_{i,T}$ then i matches all its self-demanded type T pairs in μ. Otherwise, hospital i matches every self-demanded type T pair with probability $1 - \frac{1}{n_{odd,T}}$ where $n_{odd,T}$ is the number of hospitals submitting an odd number of self-demanded type T, and matches all but one of its self-demanded type T pairs otherwise.*

Proof. (1) The matchings $\mu'_1, \ldots, \mu'_m$ are each regular, and therefore all OD pairs in matching μ must match with UD pairs in order to satisfy the constraint that as many UD pairs must be matched in μ as in $\mu'_1, \ldots, \mu'_m$ (recall that UD pairs can only match with OD pairs). This is part of what is required for regularity. In addition, we must show that the matching μ is maximum on the R and S subgraphs. For the R subgraph, and considering WLOG the case of $R'_{\oplus,AB} \geq R'_{\oplus,BA}$, we only drop pairs while there are more AB than BA pairs. Eventually, by the PM assumption the maximum matching matches all remaining BA pairs to AB pairs and thus is maximum amongst all possible matches on the initial set of AB and BA pairs. For the S subgraph, consider the component defined on blood type T. All but one pair is matched in the maximum matching by the PM assumption, and the only time one pair is not matched is when there is an odd number of pairs. Thus, this matching is maximium amongst all possible matches on the initial set of S pairs of blood type T.

(2) For OD and UD this follows by construction. For self-demanded, if $S'_{\oplus,T}$ is even then all pairs remain in graph $G'_\oplus$ and by the PM assumption all pairs are matched. If $S'_{\oplus,T}$ is odd then one pair is dropped and the remaining number of pairs is even and all match bt the PM assumption. All hospitals except one match all type T self-demanded pairs, and the one whose pair is dropped is one with an odd number in G'_i and thus with one pair not matching in μ'_i. For reciprocal pairs, consider the case that AB is long and $R'_{\oplus,AB} \geq R'_{\oplus,BA}$. A symmetric argument holds for BA long. Now, some pairs are dropped from $G'_\oplus$ but all pairs that remain are matched by the PM assumption. For hospitals with $R'_{i,AB} \leq R'_{i,BA}$, no pairs are dropped and they match all pairs in μ. For other hospitals, only AB pairs are dropped, and only while the hospital has more AB pairs than BA pairs. Since only $2\min(R'_{i,AB}, R'_{i,BA})$ such pairs are matched in μ'_i (by PM assumption) then it is guaranteed to match at lease this many given that at least $2\min(R'_{i,AB}, R'_{i,BA})$ remain and all remaining pairs are matched. Moreover, the more pairs $R'_{i,AB}$ the more pairs $R'_{i,AB}$ survive the random deletion process in expectation and thus the expected number of matched pairs increases in $R'_{i,AB}$, fixing $R'_{i,BA}$.

(3') Symmetric proof

(4) Consider self-demanded type T. If the number of pairs $S'_{\oplus,T}$ in $G'_\oplus$ is odd then one pair will be dropped from a hospital with an odd number $S'_{i,T}$ in its input G'_i. By the PM assumption all remaining pairs match in matching μ. Thus if the number $S'_{\oplus,T}$ is even or i has an even number in its input then

it will match all its pairs in μ. Otherwise, if $S'_{\oplus,T}$ is odd then one pair will be matched, and selected uniformly at random from those submitting an odd number. This completes the proof. $\square$

Lemma 2. *The idealized xCM mechanism is efficient if every hospital is truthful given the regularity and PM assumptions.*

Proof. By Lemma 1 we have that the matching μ is regular on $G'_{\oplus}$ and thus maximum on $G'_{\oplus}$ by the regularity assumption. Given that hospitals are truthful the matching is maximum on the true combined graph. $\square$

Say that the residual pairs are those that a hospital hides when submitting pairs and those that are returned unmatched. The recourse of a hospital is to find a maximum matching on this residual, to augment the matching μ from the mechanism. The following follows from the standard result about regular matchings being maximum:

Lemma 3. *A regular matching on the residual pairs is the optimal resource for a hospital, when it exists.*

Note that the correctness of this lemma does not rely on the PM or regularity assumptions (and so is not sensitive to the size of the residual).

Lemma 4. *The idealized xCM mechanism is DSIC given the regularity and PM assumptions.*

Proof. Fix the graphs G'_{-i} reported by other hospitals. For hospital i with true graph G_i, we argue:

(i) By sending all UD and OD pairs it matches all OD pairs with its own UD pairs, whatever its reports in regard to S and R pairs, and given regularity this is the same outcome of its optimal recourse if it holds back some UD or OD pairs by Lemma 3. From this we have that sending all UD and OD pairs is weakly dominant.

(ii) Given that all OD pairs match with UD pairs, then the effect of its strategy in regard to S type T can be studied in isolation to S type T. By Lemma 1 (4) the hospital can definitely do at least as well as its recourse by being truthful when it has an even number of pairs or the other hospitals submit an odd number. In the case where the hospital has an odd number and the reports of others send an odd number, then the hospital might be unlucky and match all but 1 pair. But the probability of this occurring is invariant to holding back any even number, and holding back an odd number is worse in expectation because it suffers a sure loss of 1 pair.

(iii) Given that all OD pairs match with UD pairs, then the effect of strategy in regard to R can be studied in isolation to R. Suppose $R_{i,AB} \geq R_{i,BA}$ WLOG. By Lemma 1 (3) if there is a net excess of BA then i matches all its R pairs, and this is at least as good as it could do by hiding and then using recourse. On the other hand, if it is on the long-side then some of its pairs may be unmatched. But by Lemma 1 (3), the expected number of pairs weakly increases as it reports more $R_{i,AB}$, for any fixed $R'_{i,BA}$ it reports. [**slightly delicate to complete the argument from here as to why want to send all of both, but not too bad. didn't get time yet**] □

Comment: the proof currently assumes regularity on the residual for step (i) although it seems there should be a workaround here. But it is prob OK as is, anyway. It doesn't assume PM on small graphs, and regularity (all OD can match to UD) should be an OK assumption even for a small number of OD given the large number of UD.

Theorem 1. *The idealized xCM mechanism is DSIC and efficient under the regularity and PM assumptions*

Proof. By Lemmas 2 and 5 □

0.1. A practical, xCM-based mechanism

A practical, xCM-based mechanism should be equivalent to the idealized mechanism when regularity and PM holds on realized graphs, but be well defined for general graphs.

A sketch design for such a practical, xCM-based mechanism is:

1. Compute an almost-regular matching $\mu'_1, \ldots, \mu'_m$ for each hospital
2. Adopt in the output matching μ the parts of these matchings μ' that involve OD pairs (whether matched with UD or other pairs). Remove these pairs from $G'_\oplus$
3. For R pairs: compute a constrained maximum matching on the R pairs in $G'_\oplus$, imposing a constraint that every hospital matches at least as many AB and BA pairs as in its matching μ'_i, and breaking ties to select a maximum constrained matching uniformly at random

4. For S pairs: for each blood type T, compute a constrained maximum matching on the S pairs in $G'_{\oplus}$, imposing a constraint that every hospital matches at least as many pairs as in its matching μ'_i, and breaking ties to select a maximum constrained matching uniformly at random

Conjecture: The tie-breaking procedure defined as uniform random in this way replicates that for the idealized mechanism in the case that the regularity and PM assumptioms hold.

xx still need to work on this