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Zentrum Mathematik

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Exercise sheet 1 (April 9, 2018)

1D Poisson solver with finite differences

The aim is to numerically solve the Poisson equation

$$-\phi''(x) = \rho(x)\,, \qquad x \in (a,b) \subset \mathbb{R}\,, \tag{1}$$

for given ρ, continuous on the interval (a,b), and suitable boundary conditions. We use finite differences to approximate the equation. The domain $[a,b]$ (for generality with boundary points) is divided into N equally large cells, yielding $N+1$ grid points $x_i \in [a,b]$, $i = 0,\ldots,N$, on which the discrete solution is defined.

1. **Dirichlet boundary conditions.** The values of ϕ are given at the boundary:

$$\phi(a) = \alpha\,, \qquad \phi(b) = \beta\,. \tag{2}$$

   (a) Show that the problem (1),(2) has a unique solution. (Hint: suppose ϕ and ψ are both solutions of (1), then formulate the Laplace equation for $\eta := \phi - \psi$ (with boundary conditions). Multiply this equation by η and integrate over $[a,b]$).

   (b) Write a finite difference solver for (1),(2) for arbitrary a, b, α, β, N and ρ by approximating the Laplacian at the grid points $i = 1,\ldots,N-1$ via

$$\phi''(x_i) \approx \frac{1}{h^2}\Big(\phi_{i+1} - 2\phi_i + \phi_{i-1}\Big)\,, \tag{3}$$

   where $\phi_i = \phi(x_i)$ and h is the cell size (grid spacing). For given N, check the eigenvalues of the system matrix using the MATLAB command `eig`.

   (c) Set $a = 0$, $b = 2\pi$, $\rho(x) = 2\sin(x) + x\cos(x)$, $\alpha = 0$, $\beta = 2\pi$ and solve (1),(2) for $N = 8, 16, 32, 64, 128, 256$. In each run, compute and save the errors with respect to the true solution $\phi(x) = x\cos(x)$ in the L^1-, the L^2- and the L^∞-norm. Plot the numerical solution along with the true solution using the command `plot`.

   (d) Convergence tests: once you completed all the runs, plot the errors as a function of h. Does the solution converge as $h \to 0$? If so, at the expected rate? Can you compute the expected convergence rate from (3)? (Hint: insert Taylor expansion).

2. **Mixed boundary conditions.** The derivative of ϕ is now given at $x = b$:

$$\phi(a) = \alpha\,, \qquad \phi'(b) = \gamma\,. \tag{4}$$

   There are now N unknowns to be determined, since $\phi(b)$ is not given. Does the problem (1),(4) have a unique solution (see 1.a)?

(a) Approximate the derivative at the boundary via $(\phi_{N+1} - \phi_{N-1})/(2h) = \gamma$ and implement a finite difference solver for (1),(4). Check the eigenvalues of the system matrix. Launch your solver with the same parameters as in 1.c (except for the β-value) and with $\gamma = 1$. Perform convergence tests as in 1.d and plot the results.

(b) Approximate the derivative at the boundary via $(\phi_{N+1} - \phi_N)/h = \gamma$ and repeat the same tasks as in 2.a. Is there a difference in the convergence rate? If so, explain why.

3. **Periodic boundary conditions.** The solution is now defined on $\mathbb{R}$, assumed in C^∞ and periodic with period $L = b - a$:

$$\phi(x + L) = \phi(x)\,, \qquad \forall\, x \in \mathbb{R}\,. \tag{5}$$

It is sufficient to compute the solution in $[a, b)$ to know it on whole $\mathbb{R}$. There are now N unknowns to be determined, since $\phi(b) = \phi(a)$ and thus only one boundary point has to be computed.

(a) Is there a solvability condition on ρ? Does the problem (1),(5) have a unique solution (see 1.a)?

(b) Discretize the problem (1), (5) assuming $\phi_N = \phi_0$ and $\phi_{-1} = \phi_{N-1}$. Check the eigenvalues of the system matrix A. Is it invertible?

(c) Render the problem well-posed by imposing $\phi(0) = 0$, which leads to a modified system matrix A'. Check the eigenvalues of A'.

(d) Set $a = 0$, $b = 2\pi$ and $\rho = 4\sin(2x)$, launch your solver for different values of N and perform the usual convergence tests. The true solution is obviously $\phi(x) = \sin(2x)$. Plot the results.