Use of Direct Numerical Simulations for Studies on Magnetohydrodynamics

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August 9, 2017

Overview

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  - Grid spacing Δx small enough to resolve dissipative scales.

Start with a reformulated Navier Stokes equation in Fourier space.

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right) u_\alpha(\mathbf{k}, t) = M_{\alpha\beta\gamma}(\mathbf{k}) \left[\sum_{\mathbf{j}+\mathbf{l}=\mathbf{k}} u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{l}, t)\right] \quad (3)$$

A discretisation in time and wavenumber is needed to obtain a numerical solution.

$$k_\alpha = \frac{2\pi}{L} n_\alpha \qquad n_\alpha = 0, 1, \ldots, N-1 \quad (4)$$

Non-linear term:

$$A_{\beta\gamma}(\mathbf{k}, t) = \sum_{\mathbf{j}+\mathbf{l}=\mathbf{k}} u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{l}, t) \quad (5)$$

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4. Use a suitable finite-differences method for evolving the velocity field in time [1].

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- Before FFT: $\mathcal{O}(N^6)$.
- After FFT: $\mathcal{O}(N^3 \log_2 N)$.

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Equations of fluid motion are obtained by combining Navier Stokes equations with Maxwell's equations.

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$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{b} = (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{u} + \eta \nabla^2 \mathbf{b} + \mathbf{f}_b \tag{8}$$

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$$Re = \frac{uL}{\nu} \qquad Re_m = \frac{uL}{\eta} \tag{9}$$

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$Pr_m < 1$ when most energy is dissipated through the magnetic channel.
$Pr_m > 1$ when most energy is dissipated through the kinetic (viscous) channel.

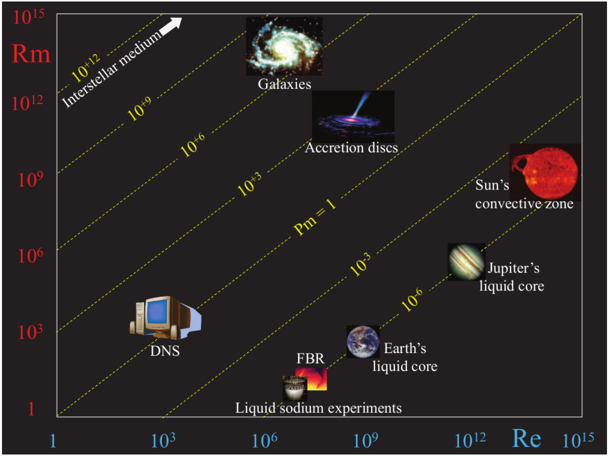

Figure: Map of "typical" objects in the plane (Re, Re_M). Yellow dashed lines are Pr_M isolines. [3].

A well known research on non-unity magnetic Prandtl numbers is:

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Range of achieved Magnetic Prandtl Numbers: [0.01 : 10].

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$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{b} = (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{u} + \eta \nabla^2 \mathbf{b} + \mathbf{f}_b \tag{12}$$

$$\mathbf{f}_u = \mathbf{0} \qquad \qquad \mathbf{f}_b = \mathbf{0} \tag{13}$$

Runs (f-h) have higher Reynolds numbers compared to their (a-e) equivalents.

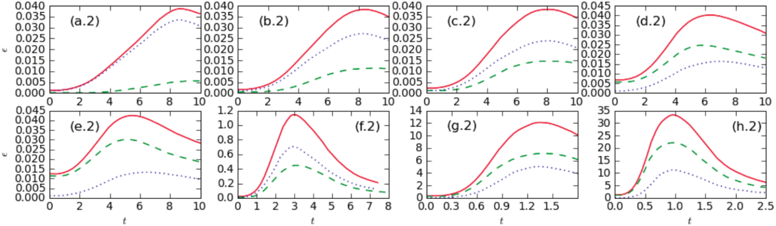

Figure: Total (red), kinetic (green), and magnetic (blue) energy dissipation ratios for a: $Pr_M = 0.1$, b: $Pr_M = 0.5$, c: $Pr_M = 1.0$, d: $Pr_M = 5.0$, e: $Pr_M = 10.0$, f: $Pr_M = 1.0$, g: $Pr_M = 5.0$, h: $Pr_M = 10.0$ [4].

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2. To avoid extremely high computational complexity ($\mathcal{O}(N^6)$), convolution theorem and FFTs can help.
3. Increases in computing power will allow broadening Pr_M ranges under study.

References

[1] W. D. McComb, "The physics of fluid turbulence," *Chemical Physics*, 1990.

[2] S. R. Yoffe, "Investigation of the transfer and dissipation of energy in isotropic turbulence," *arXiv preprint arXiv:1306.3408*, 2013.

[3] F. Plunian, R. Stepanov, and P. Frick, "Shell models of magnetohydrodynamic turbulence," *Physics Reports*, vol. 523, no. 1, pp. 1–60, 2013.

[4] G. Sahoo, P. Perlekar, and R. Pandit, "Systematics of the magnetic-prandtl-number dependence of homogeneous, isotropic magnetohydrodynamic turbulence," *New Journal of Physics*, vol. 13, no. 1, p. 013036, 2011.

[5] A. Brandenburg, "Magnetic prandtl number dependence of the kinetic-to-magnetic dissipation ratio," *The Astrophysical Journal*, vol. 791, no. 1, p. 12, 2014.

Aliasing Errors

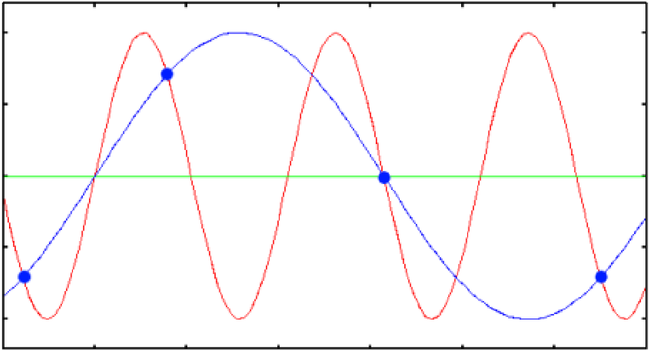

Figure: Aliasing exemplified.

Run Information

Runs	N	ν	Pr_M	δt	u_{rms}	ℓ_I	λ	Re_λ	t_c	$k_{max}\eta_d^u$	$k_{max}\eta_d^b$
R1	512	10^{-4}	0.1	10^{-3}	0.34	0.65	0.18	610	9.7	0.629	2.280
R2	512	5×10^{-4}	0.5	10^{-3}	0.34	0.67	0.27	187	9.1	1.752	2.389
R3	512	10^{-3}	1	10^{-3}	0.34	0.70	0.35	121	8.1	2.772	2.444
R4	512	5×10^{-3}	5	10^{-3}	0.33	0.76	0.60	39	7.0	8.224	2.692
R5	512	10^{-2}	10	10^{-3}	0.31	0.80	0.73	23	6.5	13.267	2.836
R3B	512	10^{-3}	1	10^{-4}	1.07	0.62	0.20	210	3.1	1.175	1.052
R4B	512	5×10^{-3}	5	10^{-4}	2.32	0.63	0.24	110	1.4	1.961	0.644
R5B	512	10^{-2}	10	10^{-4}	3.21	0.63	0.26	85	1.0	2.490	0.520
R1C	1024	10^{-4}	0.01	10^{-4}	0.35	0.65	0.23	810	8.0	1.431	22.12
R2C	1024	10^{-4}	0.1	10^{-4}	1.11	0.47	0.08	890	2.9	0.472	1.690
R3C	1024	10^{-3}	1	10^{-4}	1.14	0.49	0.15	172	2.5	1.996	1.779
R4C	1024	10^{-2}	10	10^{-4}	2.37	0.51	0.24	57	1.1	5.550	1.164
R1D	512	10^{-4}	0.01	10^{-4}	1.31	0.82	0.18	2367	–	0.320	5.364
R2D	512	10^{-4}	0.1	10^{-4}	0.99	0.74	0.14	1457	–	0.334	1.145
R3D	512	10^{-3}	1	10^{-4}	1.06	0.65	0.17	239	–	1.264	1.033
R4D	512	10^{-2}	10	10^{-4}	1.04	0.67	0.23	61	–	6.505	1.129

Figure: k_{max} is the magnitude of the largest-magnitude wave vectors resolved in these DNS studies which use the 2/3 dealiasing rule; $k_{max} \approx 170.67$ and 341.33 for $N = 512$ and 1024, respectively [4].

Shell Models

Energy profiles with shell model

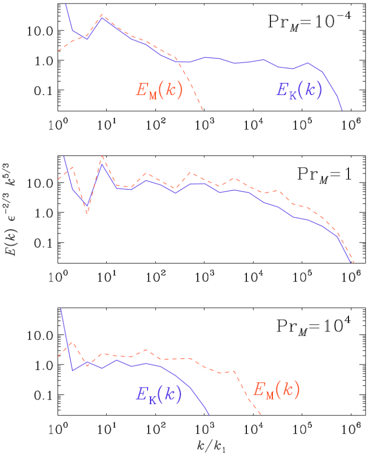

Figure: Compensated time-averaged kinetic and magnetic energy spectra for shell models at three values of Pr_M [5].