Structure-function hierarchies and von Kármán–Howarth relations for turbulence in magnetohydrodynamical equations

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(Received 2 September 2013; published 14 January 2014)

We generalize the method of A. M. Polyakov, [Phys. Rev. E **52**, 6183 (1995)] for obtaining structure-function relations in turbulence in the stochastically forced Burgers equation, to develop structure-function hierarchies for turbulence in three models for magnetohydrodynamics (MHD). These are the Burgers analogs of MHD in one dimension [Eur. Phys. J. B **9**, 725 (1999)], and in three dimensions (3DMHD and 3D Hall MHD). Our study provides a convenient and unified scheme for the development of structure-function hierarchies for turbulence in a variety of coupled hydrodynamical equations. For turbulence in the three sets of MHD equations mentioned above, we obtain exact relations for third-order structure functions and their derivatives; these expressions are the analogs of the von Kármán–Howarth relations for fluid turbulence. We compare our work with earlier studies of such relations in 3DMHD and 3D Hall MHD.

DOI: 10.1103/PhysRevE.89.012117 PACS number(s): 05.70.Ln, 47.35.Tv, 47.65.−d, 47.27.−i

I. INTRODUCTION

The elucidation of the statistical properties of magnetohydrodynamic turbulence is a challenging problem in plasma physics, astrophysics, nonequilibrium statistical mechanics, and extended dynamical systems that show spatiotemporal chaos. Most studies of this problem begin with the equations of three-dimensional magnetohydrodynamics (3DMHD) [1–6]. The 3DMHD equations are far more complicated than the Navier-Stokes (NS) equations for a 3D fluid, so the investigation of the statistical properties of 3DMHD turbulence, especially power-law regimes in structure functions (see below) in the inertial range of length scales, is considerably more difficult than it is in hydrodynamic turbulence [7–10]. It is useful, therefore, to construct and study simple models for 3DMHD before embarking on studies of the complete 3DMHD equations or generalizations thereof. These simple models include shell models [11,12] for 3DMHD and Burgers-model analogs [13–15] of MHD. The former have been studied numerically by many groups [11,12], but the latter only to a limited extent [13–15]. The one-dimensional (1D) Burgers-model analog of MHD (henceforth referred to as BMHD [15]) is by far the simplest set of nonlinear, coupled partial differential equations (PDEs) with symmetries and conservation laws akin to those in 3DMHD; furthermore, the dissipation terms and wavelike propagation, in the presence of a mean magnetic field, are similar in these two PDEs. Therefore, studies of the BMHD equations can teach us as much about 3DMHD as we have learned about turbulence in the Navier-Stokes equation from investigations of the Burgers equation. With this background in mind, we generalize the methods of Ref. [16] (see also Ref. [17] for a similar study on the 3D Burgers equation together with a continuity equation), for turbulence in the stochastically forced Burgers equation, to develop a hierarchy of equations for the structure functions in the BMHD equations [15]. We then show how to initiate similar studies for (a) 3DMHD, where we concentrate on third-order structure functions, and (b) the 3D Hall-MHD equations [18–22], which are of relevance in developing an understanding of turbulence in the solar wind and the interplanetary medium [23], because recent measurements in these systems have shown that the magnetic-field spectrum has two power-law regimes, one at low and another at high wave numbers (or frequencies) [23,24]; there is a general consensus that the existence of these two power-law ranges can be attributed to nonlinear Hall-MHD processes [18–22,25], which are described by the 3D Hall-MHD equations.

The Burgers equation can be converted, by means of the Hopf-Cole transformation [26,27], into a linear diffusion equation; thus, it cannot display turbulence with sensitive dependence on initial conditions and is, in this sense, qualitatively different from the Navier-Stokes equation, for which the consensus is that turbulence is associated with a sensitive dependence on initial conditions. (The BMHD equations share the integrability property of the Burgers equation only when the magnetic Prandtl number $P_m \equiv \nu/\mu = 1$, where ν and μ are, respectively, the kinematic viscosity and the magnetic diffusivity.) Studies of Burgers turbulence [26,27] are, therefore, studies of the statistical properties of solutions of the Burgers equation that arise either (a) from initial conditions with random velocity fields or (b) because of stochastic forcing. The stochastically forced Burgers equation is often used to test statistical theories of turbulence because it has the same quadratic nonlinearity as the Navier-Stokes equation for a fluid. The development of such theories, which often begin with Feynman-graph-based perturbation expansions about the linear theory, is easier for the Burgers equation than for the Navier-Stokes case because in the former we do not have a pressure term and we do not have to enforce incompressibility [27].

Thus, we consider the stochastically forced BMHD equations. Our principal goal here is to develop a structure-function

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hierarchy for BMHD turbulence and to use it to explore the interplay of the coupled velocity and magnetic fields, the role of cross correlations in the stochastic forces, and the effects of a mean magnetic field that leads to propagating waves, the BMHD analogs of Alfvén waves.

The method introduced in Ref. [16] yields an infinite hierarchy of relations between velocity structure functions of different orders for turbulence in the stochastically forced Burgers equation. We show how to generalize this method to obtain structure-function hierarchies for turbulence first in the BMHD equations [15] in one dimension for pedagogical clarity. We then turn to the physically important problems of turbulence in the three-dimensional equations of magnetohydrodynamics (3DMHD) [1–6] and the equations of Hall MHD [18–22] and outline the derivation of structure-function hierarchies here; our study provides a unified scheme for the development of structure-function hierarchies for turbulence in coupled hydrodynamical equations in general [28] and MHD turbulence in particular. We show explicitly, for all these three types of MHD equations, how these structure-function hierarchies lead, at order $p = 3$, to the analogs of the von Kaŕmań–Howarth relation for fluid turbulence; these relations are important because they are among the few exact relations in a field that must resort, for other values of p, to approximate closures of the infinite hierarchy of structure-function equations. Such relations have not been obtained for the BMHD system hitherto. For 3DMHD and Hall-MHD turbulence, analogs of the von Kaŕmań–Howarth have been obtained earlier [21,29]; we compare our results with these earlier studies. Subsequently, we develop the BMHD counterparts of the von Kármán–Howarth relation for the third-order structure function in fluid turbulence [7] and then show how to generalize our studies to 3DMHD and 3D Hall-MHD. Furthermore, we explore the similarities and differences between such relations in the BMHD, 3DMHD, and 3D Hall-MHD turbulence.

The remaining part of this paper is organized as follows: In Sec. II we introduce the four models that concern us here, namely, the 1D Burgers, 1DBMHD, 3DMHD, and 3D Hall-MHD equations; we also summarize results on Burgers turbulence and the properties of the BMHD equations that are of relevance to our work. In Sec. III we generalize the generating-functional method of Ref. [16] to obtain the structure-function hierarchy for 1DBMHD. In Sec. III A we explore the equations for third-order structure functions in detail. Then we consider, in Sec. IV, analogous results for third-order structure functions for 3DMHD. In Sec. V, we derive the third-order structure functions for 3D Hall-MHD. Section VI contains a discussion of our results.

II. EQUATIONS AND OVERVIEW

We begin with the equations of incompressible 3DMHD [1–6] because we consider low-Mach-number flows:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{\nabla p}{\rho} + \frac{(\nabla\times\mathbf{b})\times\mathbf{b}}{4\pi\rho} + \nu_0\nabla^2\mathbf{v} + \mathbf{f}_v, \quad (1)$$

with incompressibility enforced by $\nabla\cdot\mathbf{v} = 0$; henceforth we set the uniform density $\rho = 1$; Ampère's law for a conducting fluid becomes

$$\frac{\partial \mathbf{b}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{v} + \mu_0\nabla^2\mathbf{b} + \mathbf{f}_b. \quad (2)$$

In Eqs. (1) and (2), $\mathbf{v}$ and $\mathbf{b}$ denote, respectively, velocity and magnetic fields; the latter are scaled such that $\mathbf{b}$ and $\mathbf{v}$ have the same dimensions; the effective pressure $p^* \equiv [p + b^2/(8\pi)]$, in CGS units, with p the pressure, can be eliminated as usual by using the incompressibility condition $\nabla\cdot\mathbf{v} = 0$ and the Maxwell equation $\nabla\cdot\mathbf{b} = 0$; ν_0 and μ_0 are, respectively, the fluid kinematic viscosity and the magnetic diffusivity, and $\mathbf{f}_v$ and $\mathbf{f}_b$ denote forcing terms. To simplify the algebra below, we replace $\mathbf{b}$ by the *normalized magnetic field* $\mathbf{b}/\sqrt{4\pi\rho}$. Often it is convenient to use the Elsässer variables

$$\mathbf{z}^{\pm} = \mathbf{v} \pm \mathbf{b}. \quad (3)$$

We use periodic boundary conditions because we consider, principally, homogeneous, isotropic 3DMHD turbulence; we also study anisotropic 3DMHD turbulence with a mean magnetic field $\mathbf{B}_0$, which is the source of the anisotropy.

In the 3D Hall-MHD equations, $\mathbf{v}$ continues to be governed by the generalized Navier-Stokes given in the MHD equations (1), but the induction equation (2) is augmented by an additional Hall term as given in the following equation:

$$\frac{\partial \mathbf{b}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{b} = \mathbf{b}\cdot\nabla\mathbf{v} - d_I\nabla\times[(\nabla\times\mathbf{b})\times\mathbf{b}] + \mu_0\nabla^2\mathbf{b} + \mathbf{f}_b, \quad (4)$$

where d_I is the ion-inertial length. If we define the fluid and magnetic dissipation length scales $\eta_d^u = (\nu^3/\langle\epsilon_u\rangle)^{1/4}$ and $\eta_d^b = (\eta^3/\langle\epsilon_b\rangle)^{1/4}$, with $\langle\epsilon_u\rangle$ and $\langle\epsilon_b\rangle$ the average values of the fluid- and magnetic-energy dissipation rates, respectively, then the Hall term has a significant effect if $d_I \gg \eta_d^u,\eta_d^b$. As we have mentioned above, the magnetic spectrum in 3D Hall-MHD turbulence displays two power-law ranges, one at low wave numbers k and the other at high k; these power-law regimes are beginning to be resolved in direct-numerical-simulation (DNS) studies [19–22,30], which have limited spatial resolution because of significant numerical challenges, and in shell-model investigations [25].

Before we introduce the BMHD equations, we consider the 1D Burgers equation because the BMHD equations decouple into two Burgers equations in terms of the BMHD analogs of the Elsässer variables if the fluid and magnetic viscosities are equal [15], i.e., the magnetic Prandtl number $P_m \equiv \nu/\mu = 1$ (see below). In this case, the large number of results, which have been obtained for Burgers turbulence, can be borrowed directly. Studies of Burgers turbulence deal either with decaying turbulence or driven states that are turbulent but statistically steady. We concentrate on the latter here. The 1D Burgers equation is

$$\partial_t u + u\partial_x u = \nu\partial_x^2 u + f(x,t), \quad (5)$$

where ∂_t is a shorthand notation for $\partial/\partial t$, u is the Burgers velocity field, ν is the viscosity, and f the external force. If we use the Hopf-Cole transformation, i.e., we set $u = \partial_x \psi$, $f \equiv -\partial_x F$, and $\psi \equiv 2\nu \ln \Theta$, we obtain $\partial_t \Theta = \nu \partial_x^2 \Theta + F\Theta/(2\nu)$, which can be solved explicitly in the absence of any boundary [26,27]. Of course, there is no incompressibility condition in this 1D equation.

Studies of decaying Burgers turbulence investigate the statistical properties of solutions of Eq. (5) with $f(x,t) = 0$ and random initial conditions. Random, homogeneous, but smooth, initial conditions lead, after some time, to the development of discontinuities, called shocks, in u at random locations x. For smooth initial conditions, these shocks do not cluster; and the velocity structure functions show *bifractal* scaling [26,27]. If the initial conditions are scale-free, i.e., $\langle \tilde{u}(k,0)\tilde{u}(-k,0)\rangle \sim k^{-\phi}$, $\phi > 0$, where the tilde denotes a spatial Fourier transform and k the wave vector, the velocity field develops shocks. If $\phi \geqslant 3$ then all the shocks eventually merge to give a single shock (or a few shocks in special cases) and bifractal scaling follows as described above. The case of a Brownian initial velocity field $\phi = 2$ has been studied in great detail [26,27,31]: A dense set of shocks appears at arbitrarily short times; however, it turns out that the Lagrangian map [27] is a devil's staircase [32] and bifractal scaling is found.

Statistically steady Burgers turbulence arises when Eq. (5) is forced stochastically, say with a zero-mean, Gaussian force $f(x,t)$ that is delta correlated in time [27,33,34]. It is important to distinguish between (a) forcing at large spatial scales and (b) power-law forcing; in the former case, the variance of the stochastic force is considerable only at large length scales or, in Fourier space, at small k; in the latter case, this variance depends on a power of k. For case (a) Polyakov [16,27] has developed a generating-functional method to obtain a hierarchy of equations for velocity structure functions in the former case; for the third-order structure function this yields the Burgers-equation analog of the von Kármán–Howarth relation [7] (see also Ref. [35] for an application of this technique in the context of turbulence in a rotating fluid). The multiscaling of structure functions has been studied in case (b) by several groups [27,33,34]; if the variance of the stochastic force in the Burgers equation scales as $\sim k^{-1}$, velocity structure functions exhibit bifractal scaling [27,34]; we do not describe multiscaling or bifractal scaling in detail here because they are not required for the rest of our paper. For the deterministically forced Burgers equation we refer the reader to [27].

The 1DBMHD model [13–15], with periodic boundary conditions, is constructed by demanding that (a) it should conserve the 1D analogs of the total energy and cross helicity in the inviscid, unforced limit and (b) it should be Galilean invariant. The simplest such model that is bilinear in the 1D velocity u and magnetic b fields is

$$\partial_t u + B_0 \partial_x b + u\, \partial_x u + b\, \partial_x b = \nu \partial_x^2 u + f_u(x,t), \tag{6}$$

$$\partial_t b + B_0 \partial_x u + \partial_x (u\, b) = \mu \partial_x^2 b + f_b(x,t), \tag{7}$$

where ν and μ are, respectively, the fluid and magnetic viscosities, B_0 is a mean magnetic field, and f_u and f_b are zero-mean, stochastic, stirring forces, which are Gaussian, white-in-time, and have the covariances

$$\langle f_u(x,t) f_u(x',t')\rangle = \delta(t-t') K_u(x-x'), \tag{8}$$

$$\langle f_b(x,t) f_b(x',t')\rangle = \delta(t-t') K_b(x-x'), \tag{9}$$

$$\langle f_u(x,t) f_b(x',t')\rangle = \delta(t-t') \tilde{K}(x-x'). \tag{10}$$

The BMHD equations (6) and (7) are invariant [15] under the $u \to -u, b \to b$ when $x \to -x$. Note that $u,b \in \mathbb{R}$, $u(x) = -u(-x)$, and $b(x) = b(-x)$, i.e., b is even under a parity transformation; thus [15], $K_u(x)$ and $K_b(x)$ must be real and even in x, but $\tilde{K}(x)$ must be real and odd in x. If we define the BMHD analogs of the Elsässer variables, namely,

$$z^{\pm}(x,t) \equiv u(x,t) \pm b(x,t), \tag{11}$$

the BMHD equations become

$$\partial_t z^+ + B_0 \partial_x z^+ + z^+ \partial_x z^+ = \nu_1 \partial_x^2 z^+ + \nu_2 \partial_x^2 z^- + f^+, \tag{12}$$

$$\partial_t z^- - B_0 \partial_x z^- + z^- \partial_x z^- = \nu_1 \partial_x^2 z^- + \nu_2 \partial_x^2 z^+ + f^-, \tag{13}$$

where $\nu_{1,2} \equiv (\nu \pm \mu)/2$ and $f^{\pm} \equiv f_u \pm f_b$. Thus, if $\nu = \mu$,

$$\partial_t z^+ + B_0 \partial_x z^+ + z^+ \partial_x z^+ = \nu \partial_x^2 z^+ + f^+, \tag{14}$$

$$\partial_t z^- - B_0 \partial_x z^- + z^- \partial_x z^- = \nu \partial_x^2 z^- + f^-. \tag{15}$$

Here

$$\begin{aligned}\langle f^+(x,t) f^+(x',t')\rangle &= \langle f^-(x,t) f^-(x',t')\rangle \\ &= \delta(t-t')[K_u(x-x') + K_b(x-x')], \\ \langle f^+(x,t) f^-(x',t')\rangle &= \delta(t-t')[\hat{K}(x-x') + \tilde{K}(x-x')],\end{aligned} \tag{16}$$

and $\hat{K}(x-x') = K_u(x-x') - K_b(x-x')$. Thus, the BMHD equations (6) and (7) decouple, in this Elsässer representation, into independent, 1D Burgers equations, for z^+ and z^-, if (a) $\nu = \mu$ and (b) the cross correlation $\langle f^+(x,t) f^-(x',t')\rangle = 0$; if these conditions are met, the results for the stochastically forced Burgers equation, which we have summarized above, can be used directly.

To develop the hierarchy of structure functions for 1DBMHD turbulence, it is useful first to derive energy-balance equations as follows. We multiply Eqs. (14) and (15) by z^+ and z^-, respectively, and average over the stochastic forces to get

$$\partial_t \left\langle \tfrac{1}{2}(z^+)^2 \right\rangle = -\nu_1 \langle (\partial_x z^+)^2\rangle - \nu_2 \langle (\partial_x z^+)(\partial_x z^-)\rangle + \langle f^+ z^+\rangle, \tag{17}$$

$$\partial_t \left\langle \tfrac{1}{2}(z^-)^2 \right\rangle = -\nu_1 \langle (\partial_x z^-)^2\rangle - \nu_2 \langle (\partial_x z^+)(\partial_x z^-)\rangle + \langle f^- z^-\rangle, \tag{18}$$

where the angular brackets denote averages over the probability distribution functions (PDFs) of the stochastic forces. Spatial homogeneity implies that averages of all one-point quantities, e.g., $\langle Q(x)\rangle$, are constant in space, whence we obtain $\partial_x^2 \langle (z^+)^2\rangle = \langle(\partial_x z^+)^2\rangle + \langle(z^+\partial_x^2 z^+)\rangle = 0$, and, therefrom, the following energy-balance equations for the Elsässer variables:

$$\partial_t E^\pm = -\langle\varepsilon^\pm\rangle - \langle\tilde\varepsilon\rangle + \langle f^\pm z^\pm\rangle, \tag{19}$$

where $E^\pm \equiv \langle\frac{1}{2}(z^\pm)^2\rangle$, $\varepsilon^\pm \equiv \nu_1[\partial_x z^\pm(x)]^2$, and $\tilde\varepsilon \equiv \nu_2[\partial_x z^+(x)][\partial_x z^-(x)]$; i.e., the rates of the changes of the energies $E^\pm$ are given by balances between the dissipation rates $-\langle\varepsilon^\pm\rangle$ and $-\langle\tilde\varepsilon\rangle$ and the power inputs $\langle f^\pm z^\pm\rangle$. Clearly, $\varepsilon^+ = \nu_1(\partial_x u)^2 + 2\nu_1\partial_x u\partial_x b + \nu_1(\partial_x b)^2$ and $\varepsilon^- = \nu_1(\partial_x u)^2 - 2\nu_1\partial_x u\partial_x b + \nu_1(\partial_x b)^2$. Note that $\langle\partial_x u\partial_x b\rangle = 0$ because u and b transform differently under $x\to -x$. Similarly, by using Eqs. (6) and (7), we obtain the following balance equations for the *total energy* $E^T \equiv \langle(u^2+b^2)\rangle/2 = (E^+ + E^-)/2$:

$$\partial_t E^T = -\langle\varepsilon^{\rm tot}\rangle + \langle f_u\, u + f_b\, b\rangle, \tag{20}$$

where the superscript tot stands for total, $\langle\varepsilon^{\rm tot}\rangle \equiv \langle(\varepsilon_u + \varepsilon_b)\rangle$; clearly, $\varepsilon^{\rm tot} = \varepsilon^+ + \varepsilon^-$; and $\varepsilon_u \equiv \nu(\partial_x u)^2$ and $\varepsilon_b \equiv \mu(\partial_x b)^2$ are, respectively, the local kinetic- and magnetic-energy dissipation rates per unit mass.

III. STRUCTURE-FUNCTION HIERARCHY IN BMHD TURBULENCE

To find the hierarchy of equations for the equal-time structure functions, in the nonequilibrium, statistically stationary state of the stochastically forced 1DBMHD equations, we define the two-point generating functions for the variables $z_i^\pm \equiv z^\pm(x_i,t)$, where $i = 1,2$ labels the points x_1 and x_2:

$$\begin{aligned}\mathcal{Z}(\lambda_1^\pm,\lambda_2^\pm,x_1,x_2,t_1,t_2) &\equiv \langle\exp(\lambda_1^+ z_1^+ + \lambda_2^+ z_2^+ + \lambda_1^- z_1^- + \lambda_2^- z_2^-)\rangle = \langle\mathcal{Z}_1\mathcal{Z}_2\rangle\\ &= \int\int dz_1^\pm\, dz_2^\pm\, \mathcal{P}(z_1^\pm,x_1,t_1;z_2^\pm,x_2,t_2)\mathcal{Z}_1\mathcal{Z}_2,\end{aligned} \tag{21}$$

where $\mathcal{Z}_1 = \exp(\lambda_1^+ z_1^+ + \lambda_2^+ z_2^+)$, $\mathcal{Z}_2 = \exp(\lambda_1^- z_1^- + \lambda_2^- z_2^-)$, $\lambda_i^\pm$ is the variable conjugate to $z_i^\pm$, and $\mathcal{P}(z_1^\pm,x_1,t_1;z_2^\pm,x_2,t_2)$ is a joint PDF, averaging over which is equivalent to the averaging over random force realizations in a stochastically forced ergodic system; we restrict ourselves to equal-time structure functions so we set $t_1 = t_2 \equiv t$. By using the BMHD equations, we obtain

$$\partial_t\mathcal{Z} = \langle[\lambda_1^\pm\partial_t z_1^\pm + \lambda_2^\pm\partial_t z_2^\pm]\exp(\lambda_1^\pm z_1^\pm + \lambda_2^\pm z_2^\pm)\rangle, \tag{22}$$

$$= \sum_{i=1,2}\big\langle\big[\lambda_i^\pm\big(-z_i^\pm\partial_i z_i^\pm \mp B_0\partial_i z_i^\pm + \nu_1\partial_i^2 z_i^\pm + \nu_2\partial_i^2 z_i^\mp + f_i^\pm\big)\big]\mathcal{Z}_1\mathcal{Z}_2\big\rangle, \tag{23}$$

where $\partial_i \equiv \frac{\partial}{\partial x_i}$ and $f_i \equiv f(x_i,t)$. The definitions of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ now yield the master equation (Appendix)

$$\begin{aligned}\partial_t\mathcal{Z} &+ \left\langle\sum_{j=1,2}\lambda_j^+\frac{\partial}{\partial\lambda_j^+}\left(\frac{1}{\lambda_j^+}\frac{\partial\mathcal{Z}_1}{\partial x_j}\right)\mathcal{Z}_2\right\rangle + \left\langle\sum_{j=1,2}\lambda_j^-\frac{\partial}{\partial\lambda_j^-}\left(\frac{1}{\lambda_j^-}\frac{\partial\mathcal{Z}_2}{\partial x_j}\right)\mathcal{Z}_1\right\rangle + B_0\left\langle\sum_{j=1,2}\frac{\partial\mathcal{Z}_1}{\partial x_j}\mathcal{Z}_2\right\rangle - B_0\left\langle\sum_{j=1,2}\frac{\partial\mathcal{Z}_2}{\partial x_j}\mathcal{Z}_1\right\rangle\\ &= \sum_{i,j}\lambda_i^+\lambda_j^+K(x_i-x_j)\mathcal{Z} + \sum_{i,j}\lambda_i^-\lambda_j^-K(x_i-x_j)\mathcal{Z} + \sum_{i,j}\lambda_i^+\lambda_j^-[\hat K(x_i-x_j) + \tilde K(x_i-x_j)]\mathcal{Z}\\ &\quad + \sum_{i,j}\lambda_j^+\lambda_i^-[\hat K(x_j-x_i) + \tilde K(x_j-x_i)]\mathcal{Z} + D_1 + D_2,\end{aligned} \tag{24}$$

where

$$\begin{aligned}D_1 &= \nu_1\left\langle\sum_j\lambda_j^+\frac{\partial^2 z_j^+}{\partial x_j^2}\mathcal{Z}_1\mathcal{Z}_2\right\rangle + \nu_2\left\langle\sum_j\lambda_j^+\frac{\partial^2 z^-}{\partial x_j^2}\mathcal{Z}_1\mathcal{Z}_2\right\rangle,\\ D_2 &= \nu_1\left\langle\sum_j\lambda_j^-\frac{\partial^2 z_j^-}{\partial x_j^2}\mathcal{Z}\right\rangle + \nu_2\left\langle\sum_j\lambda_j^-\frac{\partial^2 z^+}{\partial x_j^2}\mathcal{Z}_1\mathcal{Z}_2\right\rangle,\\ \Rightarrow D_1 + D_2 &= \nu\left\langle\sum_j\lambda_j^u\frac{\partial^2 u}{\partial x_j^2}\mathcal{Z}_1\mathcal{Z}_2\right\rangle + \mu\left\langle\sum_j\lambda_j^b\frac{\partial^2 b}{\partial x_j^2}\mathcal{Z}_1\mathcal{Z}_2\right\rangle;\end{aligned} \tag{25}$$

these are the anomaly terms; here $\lambda_j^u = \lambda_j^+ + \lambda_j^-$ and $\lambda_j^b = \lambda_j^+ - \lambda_j^-$. The master equation (24) can be simplified by using the Galilean invariance of Eq. (13); this leads to $\Sigma_j\lambda_j^+ = 0 = \Sigma_j\lambda_j^-$ and the resulting equation is (see the Appendix for details)

$$\begin{aligned}\partial_t\mathcal{Z} &+ \left\langle\lambda_1\mathcal{Z}_2\frac{\partial}{\partial\lambda_1}\left(\frac{1}{\lambda_1}\frac{\partial\mathcal{Z}_1}{\partial x}\right)\right\rangle + \left\langle\lambda_2\mathcal{Z}_1\frac{\partial}{\partial\lambda_2}\left(\frac{1}{\lambda_2}\frac{\partial\mathcal{Z}_2}{\partial x}\right)\right\rangle\\ &= 2\big(\lambda_1^2+\lambda_2^2\big)[K(0)-K(x)]\langle\mathcal{Z}_1\mathcal{Z}_2\rangle + 4\lambda_1\lambda_2[\hat K(0) + \tilde K(0) - \hat K(x) - \tilde K(x)]\langle\mathcal{Z}_1\mathcal{Z}_2\rangle + D_1 + D_2.\end{aligned} \tag{26}$$

Note that $\tilde{K}(0)=0$, because of the odd parity of $\tilde{K}(x)$.

Note that Eq. (26) is not closed because of the anomaly contributions (25); these contributions can be rewritten in terms of the kinetic- and magnetic-energy dissipation rates as follows (see the Appendix for details):

$$D_1+D_2=-2(\lambda_1^2+\lambda_2^2)\langle(\varepsilon_u+\varepsilon_b)\mathcal{Z}_1\mathcal{Z}_2\rangle - 4\lambda_1\lambda_2\langle(\varepsilon_u-\varepsilon_b)\mathcal{Z}_1\mathcal{Z}_2\rangle, \tag{27}$$

where $\lambda_1=\lambda_1^+-\lambda_2^+$, $\lambda_2=\lambda_1^--\lambda_2^-$, $\lambda_u=\lambda_1+\lambda_2$, and $\lambda_b=\lambda_1-\lambda_2$. Equations (24) and (27) allow us to obtain a set of hierarchical relations between the *generalized* structure functions

$$\mathcal{S}_{m,n}^{\alpha,\beta}(x)=\left\langle\frac{\partial}{\partial x^\alpha}(\Delta z^+)^m\frac{\partial}{\partial x^\beta}(\Delta z^-)^n\right\rangle = \left\langle\left(\frac{\partial^{m+n}}{\partial\lambda_1^m\partial\lambda_2^n}\frac{\partial}{\partial x^\alpha}\mathcal{Z}_1\frac{\partial}{\partial x^\beta}\mathcal{Z}_2\right)\right\rangle, \tag{28}$$

where $\alpha,\beta=0$ or 1 and

$$\Delta z^\pm \equiv z_1^\pm - z_2^\pm. \tag{29}$$

The conventional structure functions $S_{m,n}\equiv\langle(\Delta z^+)^m(\Delta z^-)^n\rangle$ are related to $\mathcal{S}_{m,n}^{\alpha,\beta}(x)$ by $S_{m,n}(x)=\mathcal{S}_{m,n}^{0,0}(x)$. This hierarchy of equations (28) is obtained by expanding different terms in the master equation (24) in powers of λ_1 and λ_2 and then equating the coefficients of terms with the same powers of λ_1 and λ_2. (For the application of such techniques to fluid turbulence see Refs. [16,36–40].) The coefficients of the terms of $O(\lambda_1)^0$, $O(\lambda_1)^1$, $O(\lambda_2)^0$, and $O(\lambda_2)^1$ yield trivial identities; by equating the coefficients of the terms of $O(\lambda_1^2\lambda_2^0)$ we get

$$\frac{1}{6}\partial_x S_{3,0}(x)-\frac{1}{2}\left\langle(\Delta z^+)^2\frac{\partial}{\partial x}(\Delta z^-)\right\rangle = -2\Delta\kappa(x)-2(\langle\varepsilon_u\rangle+\langle\varepsilon_b\rangle); \tag{30}$$

and from the coefficients of the term of $O(\lambda_1^0\lambda_2^2)$ we obtain

$$\frac{1}{6}\partial_x S_{0,3}(x)-\frac{1}{2}\left\langle(\Delta z^-)^2\frac{\partial}{\partial x}(\Delta z^+)\right\rangle = -2\Delta\kappa(x)-2(\langle\varepsilon_u\rangle+\langle\varepsilon_b\rangle), \tag{31}$$

where

$$\Delta\kappa\equiv K(x)-K(0). \tag{32}$$

Similarly, the terms of $O(\lambda_1\lambda_2)$ yield

$$\begin{aligned}&\left\langle\left[\frac{\partial}{\partial x}(\Delta z^+)^2\right]\Delta z^-\right\rangle+\left\langle\left[\frac{\partial}{\partial x}(\Delta z^-)^2\right]\Delta z^+\right\rangle\\&\quad=-2\Delta\hat{\kappa}(x)-2\Delta\tilde{\kappa}(x)-4(\langle\varepsilon_u\rangle-\langle\varepsilon_b\rangle);\\&\Delta\hat{\kappa}(x)\equiv\hat{K}(x)-\hat{K}(0);\\&\Delta\tilde{\kappa}(x)\equiv\tilde{K}(x)-\tilde{K}(0).\end{aligned} \tag{33}$$

In general, from the coefficients of the terms of $O(\lambda_1^m,\lambda_2^n)$, with $m+n\geqslant 3$, we obtain

$$\begin{aligned}&\frac{m-1}{(m+1)!}\frac{1}{n!}\mathcal{S}_{m+1,n}^{1,0}+\frac{1}{m!}\frac{n-1}{(n+1)!}\mathcal{S}_{m,n+1}^{0,1}\\&=-\frac{2}{(m-2)!n!}\langle(\Delta z^+)^{m-2}(\Delta z^-)^n(\varepsilon_u+\varepsilon_b)\rangle\\&\quad-\frac{2}{m!(n-2)!}\langle(\Delta z^+)^m(\Delta z^-)^{n-2}[\varepsilon_u+\varepsilon_b+2\Delta\kappa(x)]\rangle\\&\quad-\frac{4}{(m-1)!(n-1)!}\langle(\Delta z^+)^{m-1}(\Delta z^-)^{n-1}\\&\quad\times[\varepsilon_u-\varepsilon_b+\Delta\hat{\kappa}(x)+\Delta\tilde{\kappa}(x)]\rangle.\end{aligned} \tag{34}$$

Clearly, $S_{0,0}(x)=1$; this is just the normalization condition for the PDF $\mathcal{P}(\Delta z^+,\Delta z^-,x)$; and $S_{1,0}(x)=S_{0,1}(x)=0$ is the constraint imposed by the condition of spatial homogeneity. Equations (34) relate structure functions of different orders to each other. Note that the cross structure functions $S_{m,n}$, with $m,n\geqslant 1$ and $m+n=3$, are nonzero if either (a) the noise cross correlations $\hat{K}(x)$ and $\tilde{K}(x)$ are nonzero or (b) the magnetic Prandtl number $P_m\neq 1$.

A. Third-order structure functions

Equations (30) and (31) can be solved for the third-order structure functions in the inertial range ($x\ll L=1$), where the correlation functions can be expanded in powers of x. [We assume that the forcing term is significant around the integral scale $L\equiv 1$ (see the Introduction); so, in the inertial range, $x\ll L=1$.] The leading, nontrivial term in this expansion [16,36,41–43] is $\sim x^2$. If we neglect this term $\sim x^2$ relative to $\langle\varepsilon_u\rangle$ and $\langle\varepsilon_b\rangle$, we obtain

$$\begin{aligned}&\frac{1}{6}\partial_x\mathcal{S}_{3,0}^{0,0}(x)-\frac{1}{2}\mathcal{S}_{2,1}^{0,1}(x)=-2(\langle\varepsilon_u\rangle+\langle\varepsilon_b\rangle),\\&\frac{1}{6}\partial_x\mathcal{S}_{0,3}^{0,0}(x)-\frac{1}{2}\mathcal{S}_{1,2}^{1,0}(x)=-2(\langle\varepsilon_u\rangle+\langle\varepsilon_b\rangle),\\&\left\langle\left[\frac{\partial}{\partial x}(\Delta z^+)^2\right]\Delta z^-\right\rangle+\left\langle\left[\frac{\partial}{\partial x}(\Delta z^-)^2\right]\Delta z^+\right\rangle\\&\quad=-4(\langle\varepsilon_u\rangle-\langle\varepsilon_b\rangle).\end{aligned} \tag{35}$$

Equations (35) are the BMHD analogs of the von Kármán–Howarth relation for third-order velocity structure functions in 3D fluid turbulence. Notice that these equations are *not closed* as they are in the case of fluid turbulence. However, if the force cross correlations vanish and $P_m=1$, the first two of Eqs. (35) reduce to

$$\tfrac{1}{6}\partial_x\mathcal{S}_{3,0}^{0,0}=-2(\langle\varepsilon_u\rangle+\langle\varepsilon_b\rangle), \tag{36}$$

$$\tfrac{1}{6}\partial_x\mathcal{S}_{0,3}^{0,0}=-2(\langle\varepsilon_u\rangle+\langle\varepsilon_b\rangle), \tag{37}$$

because the mean magnetic field can be eliminated by two, different Galilean transformations [15]; thus, in this case, both $\mathcal{S}_{3,0}^{0,0}\equiv\langle(\Delta z^+)^3\rangle$ and $\mathcal{S}_{0,3}^{0,0}\equiv\langle(\Delta z^-)^3\rangle$ scale as follows:

$$\begin{aligned}&\langle(\Delta z^+)^3\rangle=-12(\langle\varepsilon_u\rangle+\langle\varepsilon_b\rangle)x;\\&\langle(\Delta z^-)^3\rangle=-12(\langle\varepsilon_u\rangle+\langle\varepsilon_b\rangle)x;\end{aligned} \tag{38}$$

these equations imply that the PDFs $\mathcal{P}(\Delta z^{\pm},x)$ are non-Gaussian and left-skewed. From Eqs. (38) we can obtain the following analogous, and well-known, results for the Burgers equation (30) when (a) the mean magnetic field $B_0 = 0$, (b) the magnetic field $b = 0$, and (c) there are no cross correlations between the forcing terms, i.e., we set $\Delta z^{+} = \Delta z^{-} = \Delta u$, $\langle \varepsilon_b \rangle = 0$, and neglect the aforementioned cross correlations:

$$-\tfrac{1}{6}\partial_x \langle \Delta u(x)^3 \rangle = -2\langle \varepsilon_u \rangle \Rightarrow \langle \Delta u(x)^3 \rangle = -12\langle \varepsilon_u \rangle x. \quad (39)$$

Let us return to the statistics of velocity- and magnetic-field increments in BMHD turbulence. From Eq. (11) we get

$$\Delta z^{\pm} = \Delta u \pm \Delta b, \quad (40)$$

where $\Delta u = u(x_1) - u(x_2)$, $\Delta b = b(x_1) - b(x_2)$, and $\Delta z^{\pm} = z^{\pm}(x_1) - z^{\pm}(x_2)$, whence we obtain

$$\frac{1}{6}\partial_x \langle (\Delta u)^3 \rangle + \frac{1}{2}\partial_x \langle \Delta u (\Delta b)^2 \rangle - \left\langle (\Delta u)^2 \frac{\partial}{\partial x} \Delta u \right\rangle + \frac{1}{2}\left\langle \Delta u \Delta b \frac{\partial}{\partial x} \Delta b \right\rangle - \left\langle (\Delta b)^2 \frac{\partial}{\partial x} \Delta u \right\rangle = -2(\langle \varepsilon_u \rangle + \langle \varepsilon_b \rangle), \quad (41)$$

$$\tfrac{1}{6}\partial_x \langle (\Delta b)^3 \rangle + \tfrac{1}{2}\partial_x \langle (\Delta u)^2 \Delta b \rangle - \tfrac{1}{2}\langle (\Delta u)^2 \partial_x \Delta b \rangle - \langle \Delta u \Delta b \partial_x \Delta u \rangle + \tfrac{1}{2}\langle (\Delta b)^2 \partial_x \Delta b \rangle = 0. \quad (42)$$

The right-hand side of Eq. (42) is zero because b is an even-parity function, i.e., under $x \to -x$, $b(x) \to b(-x)$, and, therefore, $\langle (\Delta b)^3 \rangle$ does not change sign under parity. [If we assume that u and b have the same canonical dimensions, then (dimensionally) $\langle (\Delta b)^3 \rangle \sim x$; but this is not possible because x is odd under parity; and $|x|$ cannot appear here because we assume analyticity, and hence Taylor expand the noise correlations about $x = 0$.] Furthermore, the mean magnetic field B_0 does not appear in any of the hierarchical relations discussed above; this is consistent with the corresponding results for 3DMHD (see Ref. [44] and below) and also with the symmetries of the BMHD equations [45].

We have assumed so far that, in the statistically steady state, there is a constant, wave-number-independent energy cascade in the inertial range. This is true if the Fourier transforms of the covariances of our stochastic forces (8)–(10) decay faster than q^{-d}, as the wave-vector magnitude $q \to \infty$ in dimension d. In contrast, forces with variances $\sim q^{-d}$ yield energy cascades that have a log q dependence [34,46], which has to be regularized by an ultraviolet cutoff on the force; in the inertial range and in the statistically steady state, the energy flux Π_q, through wave numbers of magnitude q, is equal to the energy dissipation rate $\varepsilon(q)$ at that scale. Energy cascades with a $\log(q)$ dependence imply $\Pi_q \sim \log(qL) \sim \log(x/L)$, where $q \sim 1/x$, whence we obtain $\langle \varepsilon \rangle \sim \log x$ that yields, in turn, a von Kármán–Howarth-type relation for the order-three structure functions with $\log x$ corrections to the normal linear dependence on x. We do not consider such logarithmic cascades in the remaining part of this paper.

IV. STRUCTURE FUNCTIONS IN 3DMHD TURBULENCE

We now generalize our discussion of BMHD structure functions in 1D to the case of 3DMHD, whose equations (1) and (2) are invariant under

$$\mathbf{r} \to -\mathbf{r}, \qquad \mathbf{v} \to -\mathbf{v}, \qquad \mathbf{b} \to \mathbf{b}. \quad (43)$$

It is convenient to rewrite Eqs. (1) and (2) in terms of the 3DMHD Elsässer variables (3) as follows (with $\nabla \cdot \mathbf{v} = 0$ and $\nabla \cdot \mathbf{b} = 0$, i.e., $\nabla \cdot \mathbf{z}^{\pm} = 0$):

$$\frac{\partial \mathbf{z}^{\pm}}{\partial t} + \mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm} \mp B_0 \cdot \nabla \mathbf{z}^{\pm} = -\nabla p^* + \nu_1 \nabla^2 \mathbf{z}^{\pm} + \nu_2 \nabla^2 \mathbf{z}^{\mp} + \mathbf{f}^{\pm}, \quad (44)$$

where $\nu_1 = (\nu_0 + \mu_0)/2$, and $\nu_2 = (\nu_0 - \mu_0)/2$. Equations (44) are invariant under the Galilean transformation

$$\mathbf{r} \to \mathbf{r} - \mathbf{u}_0 t, \qquad t \to t, \qquad \mathbf{z}^{\pm} \to \mathbf{z}^{\pm} + \mathbf{u}_0, \quad (45)$$

where $\mathbf{u}_0$ is a constant velocity. We assume that the stochastic external forces are such that each component has zero mean and is Gaussian distributed with covariances that are significant only at large length scales. These covariances can be written conveniently in Fourier space as follows:

$$\begin{aligned} \langle \tilde{f}_i^{+}(\mathbf{k},t) \tilde{f}_m^{+}(-\mathbf{k},0) \rangle &= 2\mathcal{D}(k)\mathcal{P}_{im}(\mathbf{k})\delta(t); \\ \langle \tilde{f}_i^{-}(\mathbf{k},t) \tilde{f}_m^{-}(-\mathbf{k},0) \rangle &= 2\mathcal{D}(k)\mathcal{P}_{im}(\mathbf{k})\delta(t); \\ \langle \tilde{f}_i^{+}(\mathbf{k},t) \tilde{f}_m^{-}(-\mathbf{k},0) \rangle &= 2\hat{\mathcal{D}}(k)\mathcal{P}_{im}(\mathbf{k})\delta(t) + 2i\bar{\mathcal{D}}_{im}(\mathbf{k})\delta(t); \end{aligned} \quad (46)$$

here the tildes denote spatial Fourier transforms, $\mathbf{k}$ is a wave vector with components k_i and magnitude k, and the transverse projector $\mathcal{P}_{im}(\mathbf{k}) \equiv \delta_{ij} - k_i k_m/k^2$, with δ_{ij} the Kronecker delta, enforces the conditions $\nabla \cdot \mathbf{z}^{\pm} = 0$; the functions $\mathcal{D}(k)$ and $\hat{\mathcal{D}}(k)$ are significant only at small values of k, and $\bar{\mathcal{D}}_{im}(\mathbf{k}) = D_a(\mathbf{k})\epsilon_{imn}k_n/k + D_s(\mathbf{k})\mathcal{P}_{im}(\mathbf{k})$, with ϵ_{imn} the fully antisymmetric tensor, a sum over repeated indices, and $D_a(\mathbf{k})$ and $D_s(\mathbf{k})$ odd functions of $\mathbf{k}$ that are significant only at small values of k.

We start by defining the generating functional $\mathcal{Z}_{\mathrm{3DMHD}}$ for the joint PDFs of the Elsässer fields at the points $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mathcal{Z}_{\mathrm{3DMHD}} = \langle \exp[\boldsymbol{\lambda}_1^{+} \cdot \mathbf{z}^{+}(\mathbf{x}_1,t) + \boldsymbol{\lambda}_2^{+} \cdot \mathbf{z}^{+}(\mathbf{x}_2,t) + \boldsymbol{\lambda}_1^{-} \cdot \mathbf{z}^{-}(\mathbf{x}_1,t) + \boldsymbol{\lambda}_2^{-} \cdot \mathbf{z}^{-}(\mathbf{x}_2,t)] \rangle = \langle \mathcal{Z}_M^{+} \mathcal{Z}_M^{-} \rangle, \quad (47)$$

where $\mathcal{Z}_M^{\pm} = \langle \exp[\boldsymbol{\lambda}(1)^{\pm} \cdot \mathbf{z}^{\pm}(\mathbf{x}_1,t) + \boldsymbol{\lambda}(2)^{\pm} \cdot \mathbf{z}^{\pm}(\mathbf{x}_2,t)] \rangle$. Derivatives of $\mathcal{Z}_{\mathrm{3DMHD}}$ with respect to the components of $\boldsymbol{\lambda}^{\pm}(1)$ and $\boldsymbol{\lambda}^{\pm}(2)$ generate different correlation functions of $\mathbf{z}^{+}$ and $\mathbf{z}^{-}$. We are, in particular, interested in structure functions of the longitudinal and transverse equal-time increments of $\mathbf{z}^{\pm}$. For convenience we choose $\boldsymbol{\lambda}^{+}(j) = \gamma \boldsymbol{\lambda}^{-}(j)$, where $j = 1$ or 2 and γ is a constant, i.e., $\boldsymbol{\lambda}(i)^{+}$ and $\boldsymbol{\lambda}(i)^{-}$ are parallel [47]. This is the *simplest* choice of $\boldsymbol{\lambda}(i)^{+}$ and $\boldsymbol{\lambda}(i)^{-}$ that allow us to calculate different longitudinal and transverse structure functions, which use increments of $\mathbf{z}^{+}$ and $\mathbf{z}^{-}$, from $\mathcal{Z}_M^{\pm}$ as we show below. Equations (1) and (2) are invariant under the transformations (43), so the PDFs of Elsässer field increments must also be invariant under the same transformations. Given the Galilean invariance (45), and because we are interested in the equal-time structure functions of Elsässer field increments, we can use the generating functional

$$\mathcal{Z}_{3\mathrm{DMHD}} = \langle \exp\{\boldsymbol{\lambda}^+ \cdot [\mathbf{z}^+(1) - \mathbf{z}^+(2)] + \boldsymbol{\lambda}^- \cdot [\mathbf{z}^-(1) - \mathbf{z}^-(2)]\}\rangle \equiv \langle \mathcal{Z}_M^+ \mathcal{Z}_M^- \rangle, \tag{48}$$

where $\mathcal{Z}_M^+ \equiv \exp\{\boldsymbol{\lambda}^+ \cdot [\mathbf{z}^+(1) - \mathbf{z}^+(2)]\}$ and $\mathcal{Z}_M^- \equiv \exp\{\boldsymbol{\lambda}^- \cdot [\mathbf{z}^-(1) - \mathbf{z}^-(2)]\}$, with $\boldsymbol{\lambda}^\pm = \boldsymbol{\lambda}(1)^\pm - \boldsymbol{\lambda}(2)^\pm$. If we use the Galilean invariance (45), we obtain the following equation of motion for $\mathcal{Z}_{3\mathrm{DMHD}}$, in the presence of a mean magnetic field $\mathbf{B}_0$:

$$\frac{\partial \mathcal{Z}_{3\mathrm{DMHD}}}{\partial t} + \left\langle \frac{\partial \mathcal{Z}_M^+}{\partial r_i} \frac{\partial \mathcal{Z}_M^-}{\partial \lambda_i^-} \right\rangle + \left\langle \frac{\partial \mathcal{Z}_M^-}{\partial r_i} \frac{\partial \mathcal{Z}_M^+}{\partial \lambda_i^+} \right\rangle \equiv I_{Mp} + I_{Mf} + D_M, \tag{49}$$

where r_i and $\lambda_i^\pm$, are, respectively, the Cartesian components of $\mathbf{r} \equiv (\mathbf{x}_1 - \mathbf{x}_2)$ and $\boldsymbol{\lambda}^\pm$ and there is a sum over the repeated Cartesian index i; Eq. (49) must be supplemented by the conditions $\boldsymbol{\nabla} \cdot \mathbf{z}^\pm = 0$; the effective-pressure and forcing contributions, I_{Mp} and I_{Mf}, respectively, are given below; and D_M is the sum of the following dissipative-anomaly terms:

$$\begin{aligned} D_M = {} & \nu_1 \langle \{\lambda_i^+ [\nabla^2 z_i^+(\mathbf{x}_1) - \nabla^2 z_i^+(\mathbf{x}_2)] + \lambda_i^- [\nabla^2 z_i^-(\mathbf{x}_1) - \nabla^2 z_i^-(\mathbf{x}_2)]\} \mathcal{Z}_M^+ \mathcal{Z}_M^- \rangle \\ & + \nu_2 \langle \{\lambda_i^+ [\nabla^2 z_i^-(\mathbf{x}_1) - \nabla^2 z_i^-(\mathbf{x}_2)] + \lambda_i^- [\nabla^2 z_i^+(\mathbf{x}_1) - \nabla^2 z_i^+(\mathbf{x}_2)]\} \mathcal{Z}_M^+ \mathcal{Z}_M^- \rangle, \end{aligned} \tag{50}$$

which we can express in terms of $\mathbf{v}$ and $\mathbf{b}$ to obtain

$$D_M = \nu_0 \lambda_i^v \big\langle \big[\nabla^2_{\mathbf{x}_1} v_i(\mathbf{x}_1) - \nabla^2_{\mathbf{x}_2} v_i(\mathbf{x}_2)\big] \mathcal{Z}_M^v \mathcal{Z}_M^b \big\rangle + \mu_0 \lambda_i^b \big\langle \big[\nabla^2_{\mathbf{x}_1} b_i(\mathbf{x}_1) - \nabla^2_{\mathbf{x}_2} b_i(\mathbf{x}_2)\big] \mathcal{Z}_M^v \mathcal{Z}_M^b \big\rangle, \tag{51}$$

where $\boldsymbol{\lambda}^v \equiv \boldsymbol{\lambda}^+ + \boldsymbol{\lambda}^-$, $\boldsymbol{\lambda}^b \equiv \boldsymbol{\lambda}^+ - \boldsymbol{\lambda}^-$, $\mathcal{Z}_M^v \equiv \exp(\boldsymbol{\lambda}^v \cdot [\mathbf{v}(\mathbf{x}_1) - \mathbf{v}(\mathbf{x}_2)])$, and $\mathcal{Z}_M^b \equiv \exp(\boldsymbol{\lambda}^b \cdot [\mathbf{b}(\mathbf{x}_1) - \mathbf{b}(\mathbf{x}_2)])$. The contributions from the effective pressure and force correlations are

$$I_{Mp} = -\langle (\boldsymbol{\lambda}^+ + \boldsymbol{\lambda}^-) \cdot [\boldsymbol{\nabla}_{\mathbf{x}_1} p^*(\mathbf{x}_1) - \boldsymbol{\nabla}_{\mathbf{x}_2} p^*(\mathbf{x}_2)] \mathcal{Z}_M^+ \mathcal{Z}_M^- \rangle, \tag{52}$$

$$I_{Mf} = \langle \boldsymbol{\lambda}^+ \cdot [\mathbf{f}^+(\mathbf{x}_1) - \mathbf{f}^+(\mathbf{x}_2)] + \boldsymbol{\lambda}^- \cdot [\mathbf{f}^-(\mathbf{x}_1) - \mathbf{f}^-(\mathbf{x}_2)] \mathcal{Z}_M^+ \mathcal{Z}_M^- \rangle, \tag{53}$$

where the subscripts on $\boldsymbol{\nabla}$ refer, respectively, to differentiation with respect to $\mathbf{x}_1$ or $\mathbf{x}_2$. The contributions I_{Mp} and I_{Mf} may be calculated as in fluid turbulence [48] (see also Ref. [49], where hierarchical relations among the structure functions in fluid turbulence are derived without the generating functional approach); the external force acts on large length scales $l_f \gg r$, where r lies in the inertial range, so the forcing contributions can be neglected. Thus, the anomaly terms D_M can be expressed, as in the BMHD case above, in terms of the fluid and magnetic dissipation rates. We now consider the cases without and with mean magnetic fields separately and write down the relations between various third-order structure functions below.

A. Third-order structure functions in the isotropic case: $\mathbf{B}_0 = 0$

For statistically homogeneous and isotropic 3DMHD turbulence, which we obtain far from boundaries and forcing scales, and if the mean magnetic field $B_0 = 0$, it is natural to assume $\boldsymbol{\lambda}^+ \parallel \boldsymbol{\lambda}^-$, with both of them along the z axis and $\boldsymbol{\lambda}^- = \alpha \boldsymbol{\lambda}^+$. Homogeneity and isotropy dictate that $\mathcal{Z}_M^\pm$ depends only on $\eta_1 = r$, $\eta_2^\pm = \boldsymbol{\lambda}^\pm \cdot \hat{r} = \lambda^\pm \cos\theta$, and $\eta_3^\pm = \lambda^\pm \sin\theta$, where $\lambda^\pm = \mid \boldsymbol{\lambda}^\pm \mid$, we choose $\boldsymbol{\lambda}^\pm \parallel \hat{z}$, and θ is the polar angle in spherical-polar coordinates; clearly, $\eta_2^+ = \alpha \eta_2^-$ and $\eta_3^+ = \alpha \eta_3^-$. For notational simplicity, we define $\eta_2 \equiv \eta_2^+$ and $\eta_3 \equiv \eta_3^+$ and obtain [48]

$$\mathcal{Z}_M^+ = \exp[\eta_2 \Delta z_\parallel^+ + \eta_3 \Delta z_\perp^+]; \quad \mathcal{Z}_M^- = \exp[\alpha \eta_2 \Delta z_\parallel^- + \alpha \eta_3 \Delta z_\perp^-]. \tag{54}$$

Here, $\Delta z_\parallel^\pm = [\mathbf{z}^\pm(\mathbf{x} + \mathbf{r}) - \mathbf{z}^\pm(\mathbf{x})] \cdot \hat{\mathbf{r}}$ and $\Delta z_\perp^\pm = |[\mathbf{z}^\pm(\mathbf{x} + \mathbf{r}) - \mathbf{z}^\pm(x)] \times \hat{\mathbf{r}}|$, are, respectively, the longitudinal and transverse components of the Elsässer-field increments, $\mathbf{r}$ is the separation vector, and $\hat{\mathbf{r}}$ is the unit vector along $\mathbf{r}$. In terms of these variables, the generating functional obeys the following steady-state equation:

$$\begin{aligned} & \left\langle \frac{\partial \mathcal{Z}_M^+}{\partial \eta_1} \frac{\partial \mathcal{Z}_M^-}{\partial \eta_2} \right\rangle + \frac{1}{r} \left\langle \frac{\partial \mathcal{Z}_M^-}{\partial \eta_3} \left(-\eta_3 \frac{\partial}{\partial \eta_2} + \eta_2 \frac{\partial}{\partial \eta_3} \right) \mathcal{Z}_M^+ \right\rangle + \alpha \left\langle \frac{\partial \mathcal{Z}_M^-}{\partial \eta_1} \frac{\partial \mathcal{Z}_M^+}{\partial \eta_2} \right\rangle + \frac{\alpha}{r} \left\langle \frac{\partial \mathcal{Z}_M^+}{\partial \eta_3} \left(-\eta_3 \frac{\partial}{\partial \eta_2} + \eta_2 \frac{\partial}{\partial \eta_3} \right) \mathcal{Z}_M^- \right\rangle \\ & \quad \equiv \alpha (I_{Mp} + I_{Mf} + D_M); \end{aligned} \tag{55}$$

D_M can be expressed as follows:

$$\begin{aligned} D_M = 2 \big\langle \big\{ & \eta_2^2 (1 + \alpha^2)(\epsilon_{v\parallel} + \hat{\epsilon}_{b\parallel}) + \eta_3^2 (1 + \alpha^2)(\epsilon_{v\parallel} + \epsilon_{b\parallel}) + 2\alpha \eta_2^2 (\epsilon_{v\perp} - \epsilon_{b\perp}) + 2\alpha \eta_3^2 (\epsilon_{v\perp} - \epsilon_{b\perp}) \\ & + 2\eta_2 \eta_3 (1 + \alpha^2)[\nu_0 (\partial_i v_\parallel)(\partial_i v_\perp) + \mu_0 (\partial_j b_\parallel)(\partial_j b_\perp)] + 4\alpha \eta_2 \eta_3 [\nu_0 (\partial_l v_\parallel)(\partial_l v_\perp) - \mu_0 (\partial_m b_\parallel)(\partial_m b_\perp)] \big\} \mathcal{Z}_M^+ \mathcal{Z}_M^- \big\rangle, \end{aligned} \tag{56}$$

where we sum over the repeated Cartesian indices i, j, l, and m, the local fluid- and magnetic-dissipation rates for longitudinal ($\parallel$) and transverse ($\perp$) parts are, respectively, $\epsilon_{v\parallel} = \nu_0 (\nabla v_\parallel)^2$, $\epsilon_{v_\perp} = \nu_0 (\nabla v_\perp)^2$, $\epsilon_{b\perp} = \mu_0 (\nabla b_\perp)^2$, $\epsilon_{b\perp} = \mu_0 (\nabla b_\perp)^2$, and $v_\parallel =$

$\mathbf{v} \cdot \hat{\mathbf{r}}$, $v_\perp = |\mathbf{v} \times \hat{\mathbf{r}}|$, $b_\parallel = \mathbf{b} \cdot \hat{\mathbf{r}}$, and $b_\perp = |\mathbf{b} \times \hat{\mathbf{r}}|$; furthermore, $\epsilon_v = \epsilon_{v\parallel} + \epsilon_{v\perp}$, $\epsilon_b = \epsilon_{b\parallel} + \epsilon_{b\perp}$, and the total dissipation rate per unit mass is $\epsilon = \epsilon_v + \epsilon_b$.

We proceed now, as we did in Sec. III for the BMHD case, to obtain the following relation between third-order structure function for 3DMHD, with $\nabla \cdot \mathbf{z}^\pm = 0$:

$$\nabla_j \langle [(\Delta z_\parallel^+)^2 + (\Delta z_\perp^+)^2] \Delta z_j^- \rangle = -4\langle \epsilon \rangle; \tag{57}$$

$$\nabla_j \langle [(\Delta z_\parallel^-)^2 + (\Delta z_\perp^-)^2] \Delta z_j^+ \rangle = -4\langle \epsilon \rangle; \tag{58}$$

there is a sum over the repeated Cartesian index j. For isotropic, 3DMHD turbulence we can integrate Eqs. (57) and (58) to get

$$\langle [(\Delta z_\parallel^+)^2 + (\Delta z_\perp^+)^2] \Delta z_\parallel^- \rangle = -\tfrac{4}{3}\langle \epsilon \rangle r, \tag{59}$$

$$\langle [(\Delta z_\parallel^-)^2 + (\Delta z_\perp^-)^2] \Delta z_\parallel^+ \rangle = -\tfrac{4}{3}\langle \epsilon \rangle r. \tag{60}$$

Equations (59) and (60) are the analogs of the von Kármán–Howarth relations for 3DMHD; they have been reported earlier, in slightly different forms, in Refs. [29,44,50,51]; the relations in Refs. [29,44,50] are equivalent to the ones we give above; the relation in Ref. [51] has an extra term, which should be zero because $\langle \mathbf{v} \cdot \mathbf{b} \rangle$ must vanish in the statistically steady state given the symmetries of $\mathbf{v}$ and $\mathbf{b}$, in the absence of any externally imposed explicit parity breaking effects (e.g., rotation). Indeed, recent simulations for 3DMHD turbulence [10] do find that $\langle \mathbf{v} \cdot \mathbf{b} \rangle = 0$, within error bars, but the probability distribution of $\mathbf{v} \cdot \mathbf{b}$ has broad tails; BMHD simulations [52] also yield $\langle ub \rangle = 0$. Equations (59) and (60) can be written in terms of the velocity $\mathbf{v}$ and magnetic field $\mathbf{b}$ as follows:

$$\begin{aligned}\langle [(\Delta v_\parallel)^3 + (\Delta v_\parallel)^2 \Delta b_\parallel + \Delta v_\parallel (\Delta b_\parallel)^2 + (\Delta v_\perp)^2 \Delta v_\parallel + 2\Delta v_\parallel \Delta v_\perp \Delta b_\perp + (\Delta b_\perp)^2 \Delta v_\parallel - (\Delta b_\parallel)^3 - (\Delta v_\perp)^2 \Delta b_\parallel \\ - 2\Delta v_\perp \Delta b_\perp \Delta b_\parallel - (\Delta b_\perp)^2 \Delta b_\parallel] \rangle = -\tfrac{4}{3}\langle \epsilon \rangle r,\end{aligned} \tag{61}$$

$$\begin{aligned}\langle [(\Delta v_\parallel)^3 - (\Delta v_\parallel)^2 \Delta b_\parallel + \Delta v_\parallel (\Delta b_\parallel)^2 + (\Delta v_\perp)^2 \Delta v_\parallel - 2\Delta v_\parallel \Delta v_\perp \Delta b_\perp + (\Delta b_\perp)^2 \Delta v_\parallel + (\Delta b_\parallel)^3 + (\Delta v_\perp)^2 \Delta b_\parallel \\ - 2\Delta v_\perp \Delta b_\perp \Delta b_\parallel + (\Delta b_\perp)^2 \Delta b_\parallel] \rangle = -\tfrac{4}{3}\langle \epsilon \rangle r.\end{aligned} \tag{62}$$

Equations (61) and (62) may be combined and written as

$$\langle (\Delta v_\parallel)^3 \rangle + \langle \Delta v_\parallel (\Delta b_\parallel)^2 \rangle + \langle (\Delta v_\perp)^2 \Delta v_\parallel \rangle + \langle (\Delta b_\perp)^2 \Delta v_\parallel \rangle - 2\langle \Delta v_\perp \Delta b_\perp \Delta b_\parallel \rangle = -\tfrac{4}{3}\langle \epsilon \rangle r, \tag{63}$$

$$\langle (\Delta v_\parallel)^2 \Delta b_\parallel \rangle + 2\langle \Delta v_\parallel \Delta v_\perp \Delta b_\perp \rangle - \langle (\Delta b_\parallel)^3 \rangle - \langle (\Delta v_\perp)^2 \Delta b_\parallel \rangle - \langle (\Delta b_\perp)^2 \Delta b_\parallel \rangle = 0. \tag{64}$$

Our analysis yields the following additional relations between third-order structure functions in isotropic 3DMHD turbulence:

$$\nabla_j \langle \Delta z_\parallel^+ \Delta z_\perp^+ \Delta z_j^- \rangle + \frac{1}{r}[\langle (\Delta z_\parallel^+)^2 \Delta z_\perp^- \rangle - \langle (\Delta z_\perp^+)^2 \Delta z_\perp^- \rangle] = 0, \tag{65}$$

$$\nabla_j \langle \Delta z_\parallel^- \Delta z_\perp^- \Delta z_j^+ \rangle + \frac{1}{r}[\langle (\Delta z_\parallel^-)^2 \Delta z_\perp^+ \rangle - \langle (\Delta z_\parallel^-)^2 \Delta z_\perp^+ \rangle] = 0, \tag{66}$$

$$\left\langle (\Delta z_\parallel^-)^2 \frac{\partial}{\partial r} \Delta z_\parallel^+ \right\rangle + \left\langle (\Delta z_\parallel^+)^2 \frac{\partial}{\partial r} \Delta z_\parallel^-) \right\rangle + \frac{2}{r} \langle \Delta z_\perp^+ \Delta z_\perp^- \Delta z_\parallel^- \rangle + \frac{2}{r} \langle \Delta z_\perp^- \Delta z_\perp^+ \Delta z_\parallel^+ \rangle = -\frac{4}{3}(\langle \epsilon_v \rangle - \langle \epsilon_b \rangle), \tag{67}$$

$$\left\langle \Delta z_\perp^- \Delta z_\parallel^- \frac{\partial}{\partial r} \Delta z_\perp^+ \right\rangle + \left\langle \Delta z_\perp^+ \Delta z_\parallel^+ \frac{\partial}{\partial r} \Delta z_\perp^- \right\rangle - \frac{2}{r} \langle \Delta z_\parallel^+ (\Delta z_\perp^-)^2 \rangle - \frac{2}{r} \langle \Delta z_\parallel^- (\Delta z_\perp^+)^2 \rangle = -\frac{8}{3}(\langle \epsilon_v \rangle - \langle \epsilon_b \rangle), \tag{68}$$

$$\begin{aligned}&\left\langle \Delta z_\perp^- \Delta z_\parallel^- \frac{\partial}{\partial r} \Delta z_\parallel^+ \right\rangle + \left\langle (\Delta z_\parallel^-)^2 \frac{\partial}{\partial r} \Delta z_\perp^+ \right\rangle + \left\langle \Delta z_\perp^+ \Delta z_\parallel^+ \frac{\partial}{\partial r} \Delta z_\parallel^- \right\rangle \\ &\quad + \left\langle (\Delta z_\parallel^+)^2 \frac{\partial}{\partial r} \Delta z_\perp^- \right\rangle - \frac{1}{r} \langle \Delta z_\parallel^- \Delta z_\parallel^+ \Delta z_\perp^+ - \Delta z_\perp^- (\Delta z_\perp^+)^2 \rangle - \frac{1}{r} \langle \Delta z_\parallel^+ \Delta z_\parallel^- \Delta z_\perp^- - \Delta z_\perp^+ (\Delta z_\perp^-)^2 \rangle = 0.\end{aligned} \tag{69}$$

Equations (65)–(69) cannot be integrated in any straightforward way. Thus, as in the BMHD model, different third-order structure functions are *not* separable. Hence, a simple analog of the von Kármán–Howarth relation in fluid turbulence does not exist in 3DMHD turbulence. The relations (59)–(69) reduce to their fluid-turbulence analogs in the absence of all magnetic fields.

B. Third-order structure functions in the anisotropic case: $\mathbf{B}_0 \neq 0$

We now consider third-order structure functions in the presence of a mean magnetic field $\mathbf{B}_0$. Most natural or laboratory plasmas, e.g., in the solar wind or tokamaks, contain such a mean magnetic field. Thus, it is important to develop structure-function hierarchies for this case. If $\mathbf{B}_0 = B_0\hat{z}$, with $B_0 \neq 0$, the full 3D rotational symmetry of the 3DMHD system is lost, but, in the x-y plane, this system has two-dimensional, rotational invariance. We are interested in the hierarchical relations between the structure functions consisting of velocity and magnetic-field differences along the z direction and the planar (XY) radial direction. As before, we choose $\boldsymbol{\lambda}^+$ and $\boldsymbol{\lambda}^-$ to be parallel to each other, i.e., $\boldsymbol{\lambda}^- = \alpha\boldsymbol{\lambda}^+$, where α is a constant; furthermore, we define $\boldsymbol{\lambda}^+ = (\lambda_z^+, \boldsymbol{\lambda}_\perp)$, where $\boldsymbol{\lambda}_\perp^+$ lies in the x-y plane. Thus, in this anisotropic (subscript A) version of 3DMHD, the remaining two-dimensional isotropy in the x-y plane then dictates that the generating functional $\mathcal{Z}_A$ for structure functions can be written as a function of $z, \hat{\eta}_1 = \mathbf{r}_\perp$, where $\mathbf{r}_\perp$ is the component of $\mathbf{r}$ along the x-y plane, whose magnitude $r_\perp = |\mathbf{r}_\perp|$, $\hat{\eta}_2 = \boldsymbol{\lambda}_\perp^+ \cdot \hat{\mathbf{r}}_\perp$, $\hat{\mathbf{r}}_\perp = \mathbf{r}_\perp/r_\perp$, $\hat{\eta}_3 = \boldsymbol{\lambda}_\perp^+ \cdot \hat{\theta}$, and $\boldsymbol{\lambda}_z^+$.

Given these definitions, we can write

$$\mathcal{Z}_A^+ = \exp[\lambda_z \Delta z_z^+ + \hat{\eta}_2 \Delta z_r^+ + \hat{\eta}_3 \Delta z_\theta^+], \tag{70}$$

$$\mathcal{Z}_A^- = \exp[\alpha\lambda_z \Delta z_z^- + \alpha\hat{\eta}_2 \Delta z_r^- + \alpha\hat{\eta}_3 \Delta z_\theta^-], \tag{71}$$

where $\Delta z_z^\pm, \Delta z_r^\pm, \Delta z_\theta^\pm$ are, respectively, the z, r, and θ components of Elsässer-field increments. In the statistically steady state we get

$$\begin{aligned}&\left\langle \frac{\partial \mathcal{Z}_A^+}{\partial z}\frac{\partial \mathcal{Z}_A^-}{\partial \lambda_z}\right\rangle + \alpha\left\langle \frac{\partial \mathcal{Z}_A^-}{\partial z}\frac{\partial \mathcal{Z}_A^+}{\partial \lambda_z}\right\rangle + \left\langle \frac{\partial \mathcal{Z}_A^+}{\partial \hat{\eta}_1}\frac{\partial \mathcal{Z}_A^-}{\partial \hat{\eta}_2}\right\rangle + \frac{1}{r_\perp}\left\langle \frac{\partial \mathcal{Z}_A^-}{\partial \hat{\eta}_3}\left(-\hat{\eta}_3\frac{\partial}{\partial \hat{\eta}_2} + \hat{\eta}_2\frac{\partial}{\partial \hat{\eta}_3}\right)\mathcal{Z}_A^+\right\rangle\\ &+ \alpha\left\langle \frac{\partial \mathcal{Z}_A^+}{\partial \hat{\eta}_2}\frac{\partial \mathcal{Z}_A^-}{\partial \hat{\eta}_1}\right\rangle + \left\langle \frac{\alpha}{r_\perp}\frac{\partial \mathcal{Z}_A^+}{\partial \hat{\eta}_3}\left(-\hat{\eta}_3\frac{\partial}{\partial \hat{\eta}_2} + \hat{\eta}_2\frac{\partial}{\partial \hat{\eta}_3}\right)\mathcal{Z}_A^-\right\rangle = \alpha(I_{Af} + I_{Ap} + D_A).\end{aligned} \tag{72}$$

I_{Ap}, I_{Af}, and D_A are, respectively, the contributions from the pressure, forcing, and anomaly terms and the subscript A denotes the anisotropic case we are considering now. The anomaly term D_A is best evaluated in terms of the fluid- and magnetic-energy dissipation rates; the final expression is

$$\begin{aligned}D_A = -2\big\langle\big\{&\lambda_z{}^2(1+\alpha^2)(\epsilon_{zv}+\epsilon_{zb}) + 2\alpha\lambda_z^2(\epsilon_{zv}-\epsilon_{zb}) + \hat{\eta}_2^2(1+\alpha^2)(\epsilon_{2v}+\epsilon_{2b})\\ &+2\alpha\hat{\eta}_2^2(\epsilon_{2v}-\epsilon_{2b}) + \hat{\eta}_3^2(1+\alpha^2)(\epsilon_{3v}+\epsilon_{3b}) + 2\alpha\hat{\eta}_3^2(\epsilon_{3v}-\epsilon_{3b})\\ &+2\lambda_z\hat{\eta}_2(1+\alpha^2)\,[\nu_0(\nabla v_z)\cdot(\nabla v_r) + \mu_0(\nabla b_z)\cdot(\nabla b_r)] + 2\hat{\eta}_2\hat{\eta}_3(1+\alpha^2)\,[\nu_0(\nabla v_r)\cdot(\nabla v_\theta) + \mu_0(\nabla b_r)\cdot(\nabla b_\theta)]\\ &+2\hat{\eta}_2\lambda_z(1+\alpha^2)\,[\nu_0(\nabla v_\theta)\cdot(\nabla v_z) + \mu_0(\nabla b_\theta)\cdot(\nabla b_z)] + 4\alpha\lambda_z\hat{\eta}_2\,[\nu_0(\nabla v_z)\cdot(\nabla v_r) - \mu_0(\nabla b_z)\cdot(\nabla b_r)]\\ &+4\alpha\hat{\eta}_2\hat{\eta}_3\,[\nu_0(\nabla v_r)\cdot(\nabla v_\theta) - \mu_0(\nabla b_r)\cdot(\nabla b_\theta)] + 4\alpha\hat{\eta}_3\lambda_z\,[\nu_0(\nabla v_\theta)\cdot(\nabla v_z) - \mu_0(\nabla b_\theta)\cdot(\nabla b_z)]\big\}\mathcal{Z}_A^+\mathcal{Z}_A^-\big\rangle,\end{aligned} \tag{73}$$

where $\epsilon_{zv} = \nu(\partial_j v_z)^2$, $\epsilon_{2v} = \nu(\partial_j v_r)^2$, $\epsilon_{3v} = \nu(\partial_j v_\theta)^2$, $\epsilon_{zb} = \mu(\partial_j b_z)^2$, $\epsilon_{2b} = \mu(\partial_j b_r)^2$, and $\epsilon_{3b} = \mu(\partial_j b_\theta)^2$. Equation (72) allows us to obtain the relations between different third-order structure functions; as before, there are no contributions from I_{Af} and I_{Ap}. If we define the total fluid-energy dissipation rate $\langle\epsilon_v\rangle = \langle\epsilon_{zv}\rangle + \langle\epsilon_{2v}\rangle + \langle\epsilon_{3v}\rangle$, the total magnetic-energy dissipation rate $\langle\epsilon_b\rangle = \langle\epsilon_{zb}\rangle + \langle\epsilon_{2b}\rangle + \langle\epsilon_{3b}\rangle$, and the total dissipation $\langle\epsilon\rangle = \langle\epsilon_v\rangle + \langle\epsilon_b\rangle$, we obtain

$$\nabla_j\langle\Delta z_j^-(\Delta z^+)^2\rangle = -4\langle\epsilon\rangle, \tag{74}$$

$$\nabla_j\langle\Delta z_j^+(\Delta z^-)^2\rangle = -4\langle\epsilon\rangle, \tag{75}$$

where $(\Delta z^+)^2 = (\Delta z_z^+)^2 + (\Delta z_r^+)^2 + (\Delta z_\theta^+)^2$ and $(\Delta z^-)^2 = (\Delta z_z^-)^2 + (\Delta z_r^-)^2 + (\Delta z_\theta^-)^2$. Equations (74) and (75) are identical to those derived in Ref. [44]. These are also identical to Eqs. (59) and (60), the corresponding relations for isotropic 3DMHD; even though Eqs. (74) and (75) have the same form as Eqs. (59) and (60), it is not obvious how to integrate the former, because of the anisotropy of the underlying system. In addition, we obtain

$$\begin{aligned}&\left\langle(\Delta z_r^-)^2\frac{\partial}{\partial r}\Delta z_r^+\right\rangle + \left\langle(\Delta z_r^+)^2\frac{\partial}{\partial r}\Delta z_r^-\right\rangle + \frac{2}{r}\langle\Delta z_\theta^+\Delta z_\theta^-\Delta z_\theta^-\rangle + \frac{2}{r}\langle\Delta z_\theta^-\Delta z_r^+\Delta z_\theta^+\rangle + \left\langle\Delta z_r^-\Delta z_z^-\frac{\partial}{\partial z}\Delta z_r^+\right\rangle + \left\langle\Delta z_r^+\Delta z_z^+\frac{\partial}{\partial z}\Delta z_r^-\right\rangle\\ &+\left\langle\Delta z_r^-\Delta z_\theta^-\frac{\partial}{\partial r}\Delta z_\theta^+\right\rangle + \left\langle\Delta z_r^+\Delta z_\theta^+\frac{\partial}{\partial r}\Delta z_\theta^-\right\rangle + \left\langle\Delta z_\theta^-\Delta z_z^-\frac{\partial}{\partial z}\Delta z_\theta^+\right\rangle + \left\langle\Delta z_\theta^+\Delta z_z^+\frac{\partial}{\partial z}\Delta z_\theta^-\right\rangle - \frac{2}{r}\langle\Delta z_r^+(\Delta z_\theta^-)^2\rangle - \frac{2}{r}\langle\Delta z_r^-(\Delta z_\theta^+)^2\rangle\\ &+\left\langle\Delta z_r^+\Delta z_z^+\frac{\partial}{\partial r}\Delta z_z^-\right\rangle + \left\langle\Delta z_r^-\Delta z_z^-\frac{\partial}{\partial r}\Delta z_z^+\right\rangle + \left\langle(\Delta z_z^-)^2\frac{\partial}{\partial z}\Delta z_z^+\right\rangle + \left\langle(\Delta z_z^+)^2\frac{\partial}{\partial z}\Delta z_z^-\right\rangle = -4(\langle\epsilon_v\rangle - \langle\epsilon_b\rangle).\end{aligned} \tag{76}$$

From the method we have outlined above, we can obtain similar relations between structure functions of order $p > 3$; we do not present these here. In all such relations, the mean magnetic field appears explicitly. The structure-function hierarchies and our relations (59)–(69) or (74)–(76) hold regardless of the value of the magnetic Prandtl number P_m and the cross correlations; of course, the structure functions depend on P_m and these cross correlations implicitly.

V. STRUCTURE FUNCTIONS IN 3D HALL-MHD TURBULENCE

Our method for obtaining the structure-function hierarchy for 3DMHD turbulence can be extended to obtain the analog of this hierarchy for 3D Hall-MHD turbulence. We note first that there is an extra term $-d_I\nabla\times[(\nabla\times\mathbf{b})\times\mathbf{b}]$ in Eq. (4), in addition to those in the induction equation (2). The Elsässer variables $\mathbf{z}^{\pm}$ now satisfy the equations

$$\frac{\partial\mathbf{z}^{\pm}}{\partial t}+\mathbf{z}^{\mp}\cdot\nabla\mathbf{z}^{\pm}\mp B_0\cdot\nabla\mathbf{z}^{\pm}=-\nabla p^{*}\mp d_I\nabla\times[(\nabla\times\mathbf{b})\times\mathbf{b}]+\nu_1\nabla^2\mathbf{z}^{\pm}+\nu_2\nabla^2\mathbf{z}^{\mp}+\mathbf{f}^{\pm}. \tag{77}$$

The master equation for the 3D Hall-MHD-turbulence generating functional $\mathcal{Z}_{\rm Hall}$ has the following form, in the statistically steady state:

$$\partial_t\mathcal{Z}_{\rm Hall}=0=\cdots\ -d_I\boldsymbol{\lambda}_+\cdot\langle\{[\nabla_1\times\mathbf{E}(\mathbf{x}_1)]-[\nabla_2\times\mathbf{E}(\mathbf{x}_2)]\}\mathcal{Z}^+_{\rm Hall}\mathcal{Z}^-_{\rm Hall}\rangle+d_I\boldsymbol{\lambda}_-\cdot\langle\{[\nabla_1\times\mathbf{E}(\mathbf{x}_1)]-[\nabla_2\times\mathbf{E}(\mathbf{x}_2)]\}\mathcal{Z}^+_{\rm Hall}\mathcal{Z}^-_{\rm Hall}\rangle, \tag{78}$$

where the ellipsis denotes terms that we have shown already in the 3DMHD master equation (49) and $\mathbf{E}=(\nabla\times\mathbf{b})\times\mathbf{b}$. In order to establish Eq. (78), we have used the Galilean invariance of Eqs. (1) and (4). As in the 3DMHD case, $\mathcal{Z}_{\rm Hall}$ depends only on $\eta_1=r$, $\boldsymbol{\lambda}_{\pm}\cdot\hat{\mathbf{r}}=\lambda_{\pm}\cos\theta=\eta_{2\pm}$, $\lambda_{\pm}\sin\theta=\eta_{3\pm}$. Thus, the new master equation for 3D Hall-MHD turbulence is

$$\begin{aligned}&\left\langle\frac{\partial\mathcal{Z}^+_{\rm Hall}}{\partial\eta_1}\frac{\partial\mathcal{Z}^-_{\rm Hall}}{\partial\eta_2}\right\rangle+\frac{1}{r}\left\langle\frac{\partial\mathcal{Z}^-_{\rm Hall}}{\partial\eta_3}\left(-\eta_3\frac{\partial}{\partial\eta_2}+\eta_2\frac{\partial}{\partial\eta_3}\right)\mathcal{Z}^+_{\rm Hall}\right\rangle+\alpha\left\langle\frac{\partial\mathcal{Z}^-_{\rm Hall}}{\partial\eta_1}\frac{\partial\mathcal{Z}^+_{\rm Hall}}{\partial\eta_2}\right\rangle+\frac{\alpha}{r}\left\langle\frac{\partial\mathcal{Z}^+_{\rm Hall}}{\partial\eta_3}\left(-\eta_3\frac{\partial}{\partial\eta_2}+\eta_2\frac{\partial}{\partial\eta_3}\right)\mathcal{Z}^-_{\rm Hall}\right\rangle\\&\quad-\alpha d_I\langle[\eta_2\{\nabla_1\times\mathbf{E}(\mathbf{x}_1)-\nabla_2\times\mathbf{E}(\mathbf{x}_2)\}_r+\eta_3\{\nabla_1\times\mathbf{E}(\mathbf{x}_1)-\nabla_2\times\mathbf{E}(\mathbf{x}_2)\}_\theta]\mathcal{Z}^+_{\rm Hall}\mathcal{Z}^-_{\rm Hall}\rangle\\&\quad+\alpha^2 d_I\langle[\eta_2\{\nabla_1\times\mathbf{E}(\mathbf{x}_1)-\nabla_2\times\mathbf{E}(\mathbf{x}_2)\}_r+\eta_3\{\nabla_1\times\mathbf{E}(\mathbf{x}_1)-\nabla_2\times\mathbf{E}(\mathbf{x}_2)\}_\theta]\mathcal{Z}^+_{\rm Hall}\mathcal{Z}^-_{\rm Hall}\rangle=\alpha(I_{Hf}+I_{Hp}+D_{\rm Hall}),\end{aligned} \tag{79}$$

where α, I_{Hf}, I_{Hp}, and $D_{\rm Hall}$ are the 3D Hall-MHD analogs of the forcing, pressure, and dissipation-anaomaly terms in Eq. (49). The dissipation anomaly here is defined as

$$\begin{aligned}D_{\rm Hall}=2\big\langle\big\{&\eta_2^2(1+\alpha^2)(\epsilon_{v\parallel}+\hat{\epsilon}_{b\parallel})+\eta_3^2(1+\alpha^2)(\epsilon_{v\parallel}+\epsilon_{b\parallel})+2\alpha\eta_2^2(\epsilon_{v\perp}-\epsilon_{b\perp})+2\alpha\eta_3^2(\epsilon_{v\perp}-\epsilon_{b\perp})\\&+2\eta_2\eta_3(1+\alpha^2)[\nu_0(\partial_i v_\parallel)(\partial_i v_\perp)+\mu_0(\partial_j b_\parallel)(\partial_j b_\perp)]+4\alpha\eta_2\eta_3[\nu_0(\partial_l v_\parallel)(\partial_l v_\perp)-\mu_0(\partial_m b_\parallel)(\partial_m b_\perp)]\big\}\mathcal{Z}^+_{\rm Hall}\mathcal{Z}^-_{\rm Hall}\big\rangle.\end{aligned} \tag{80}$$

We can now use Eq. (79) to obtain the hierarchical relations between the structure functions of various orders for 3D Hall-MHD turbulence. We present below the relations between various third-order structure functions, which are the analogs of the von Kármán–Howarth relations here.

A. Third-order structure functions in 3D Hall-MHD turbulence

From the $O(\eta_2^2\alpha)$ and $O(\eta_3^2\alpha)$ terms in Eq. (79) we obtain

$$\begin{aligned}&\left\langle\Delta z_\parallel^-\frac{\partial}{\partial r}(\Delta z_\parallel^+)^2\right\rangle-\frac{2}{r}\langle\Delta z_\parallel^+\Delta z_\perp^+\Delta z_\perp^-\rangle-2d_I\langle[\nabla_1\times\mathbf{E}(\mathbf{x}_1)-\nabla_2\times\mathbf{E}(\mathbf{x}_2)]_\parallel\Delta z_\parallel^+\rangle=-4(\langle\epsilon_{v\parallel}\rangle+\langle\epsilon_{b\parallel}\rangle),\\&\left\langle\Delta z_\parallel^-\frac{\partial}{\partial r}(\Delta z_\perp^+)^2\right\rangle+\frac{2}{r}\langle\Delta z_\parallel^+\Delta z_\perp^+\Delta z_\perp^-\rangle-2d_I\langle[\nabla_1\times\mathbf{E}(\mathbf{x}_1)-\nabla_2\times\mathbf{E}(\mathbf{x}_2)]_\perp\Delta z_\perp^+\rangle=-4(\langle\epsilon_{v\perp}\rangle+\langle\epsilon_{b\perp}\rangle),\end{aligned} \tag{81}$$

where the symbols $\epsilon_{v\parallel},\epsilon_{b\parallel},\epsilon_{v\perp},\epsilon_{b\perp}$ are as defined in Sec. IV A. By adding the two equations in (81) we find

$$\nabla\cdot\langle[(\Delta z_\parallel^+)^2+(\Delta z_\perp^+)^2]\Delta z_\parallel^-\rangle-2d_I\langle[\nabla_1\times\mathbf{E}(\mathbf{x}_1)-\nabla_2\times\mathbf{E}(\mathbf{x}_2)]\cdot\Delta\mathbf{z}^+\rangle=-4\langle\epsilon\rangle. \tag{82}$$

Similarly, by combining contributions of $O(\eta_2^2\alpha^3)$ and $O(\eta_3^2\alpha^3)$, we obtain

$$\nabla\cdot\langle[(\Delta z_\parallel^-)^2+(\Delta z_\perp^-)^2]\Delta z_\parallel^+\rangle+2d_I\langle[\nabla_1\times\mathbf{E}(\mathbf{x}_1)-\nabla_2\times\mathbf{E}(\mathbf{x}_2)]\cdot\Delta\mathbf{z}^-\rangle=-4\langle\epsilon\rangle. \tag{83}$$

By using the condition $\nabla\cdot\mathbf{z}^{\pm}=0$, noting that

$$\langle[\nabla_1\times\mathbf{E}(\mathbf{x}_1)-\nabla_2\times\mathbf{E}(\mathbf{x}_2)]\cdot\Delta\mathbf{z}^{\pm}\rangle=2\langle[\nabla_1\times\mathbf{E}(\mathbf{x}_1)]\cdot\Delta\mathbf{z}^{\pm}\rangle, \tag{84}$$

and by relabeling $\mathbf{x}_2$ as $\mathbf{x}_1$, we obtain

$$\nabla\cdot\langle[(\Delta z_\parallel^+)^2+(\Delta z_\perp^+)^2]\Delta\mathbf{z}^-\rangle-4d_I\nabla\cdot\langle\mathbf{E}(\mathbf{x}_1)\times\Delta\mathbf{z}^+\rangle=-4\langle\epsilon\rangle, \tag{85}$$

$$\nabla\cdot\langle[(\Delta z_\parallel^-)^2+(\Delta z_\perp^-)^2]\Delta\mathbf{z}^+\rangle+4d_I\nabla\cdot\langle\mathbf{E}(\mathbf{x}_1)\times\Delta\mathbf{z}^-\rangle=-4\langle\epsilon\rangle. \tag{86}$$

Equations (85) and (86) are the differential analogs of the Kármán-Howarth relation for Hall MHD equations. By using the isotropy of the turbulence here, the corresponding *integral forms* of (85) and (86) are

$$\langle[(\Delta z_\parallel^+)^2 + (\Delta z_\perp^+)^2]\Delta z_\parallel^-\rangle - 4d_I\langle[\mathbf{E}(\mathbf{x}_1)\times\Delta\mathbf{z}^+]_\parallel\rangle = -\tfrac{4}{3}\langle\epsilon\rangle r, \tag{87}$$

$$\langle[(\Delta z_\parallel^-)^2 + (\Delta z_\perp^-)^2]\Delta z_\parallel^+\rangle + 4d_I\langle[\mathbf{E}(\mathbf{x}_1)\times\Delta\mathbf{z}^-]_\parallel\rangle = -\tfrac{4}{3}\langle\epsilon\rangle r. \tag{88}$$

Equations (87) and (88) are the integrated forms of the analogs of the von Kármán–Howarth relations for 3D Hall-MHD turbulence. By summing the two equations (87) and (88), we can obtain Eq. (52) of Ref. [21], which is an integral form of a von Kármán–Howarth relation for 3D Hall-MHD turbulence. In addition to the two differential von Kármán–Howarth relations (86), we are able to obtain the following additional hierarchical relations between different third-order structure functions:

$$\begin{aligned}
&\nabla_j\langle\Delta z_\parallel^+\Delta z_\perp^+\Delta z_j^-\rangle + \frac{1}{r}[-\langle(\Delta z_\parallel^+)^2\Delta z_\perp^-\rangle + \langle(\Delta z_\perp^+)^2\Delta z_\perp^-\rangle] - 2d_I[\{\nabla\times\mathbf{E}\}_\parallel(\mathbf{x}_1)\Delta z_\perp^+ + \{\nabla\times\mathbf{E}\}_\perp(\mathbf{x}_1)\Delta z_\parallel^+] = 0,\\
&\nabla_j\langle\Delta z_\parallel^-\Delta z_\perp^-\Delta z_j^+\rangle + \frac{1}{r}[-\langle(\Delta z_\parallel^-)^2\Delta z_\perp^+\rangle + \langle(\Delta z_\perp^-)^2\Delta z_\perp^+\rangle] + 2d_I[\{\nabla\times\mathbf{E}\}_\parallel(\mathbf{x}_1)\Delta z_\perp^- + \{\nabla\times\mathbf{E}\}_\perp(\mathbf{x}_1)\Delta z_\parallel^-] = 0,\\
&\left\langle(\Delta z_\parallel^-)^2\frac{\partial}{\partial r}\Delta z_\parallel^+\right\rangle + \left\langle(\Delta z_\parallel^+)^2\frac{\partial}{\partial r}\Delta z_\parallel^-\right\rangle + \frac{2}{r}\langle\Delta z_\perp^+\Delta z_\perp^-\Delta z_\parallel^-\rangle + \frac{2}{r}\langle\Delta z_\perp^-\Delta z_\perp^+\Delta z_\perp^+\rangle\\
&\qquad - 2d_I[\langle(\nabla\times\mathbf{E})_\parallel(\mathbf{x}_1)\Delta z_\parallel^-\rangle - \langle(\nabla\times\mathbf{E})_\parallel(\mathbf{x}_1)\Delta z_\parallel^+\rangle] = -\frac{4}{3}(\langle\epsilon_v\rangle - \langle\epsilon_b\rangle),\\
&\left\langle\Delta z_\perp^-\Delta z_\parallel^-\frac{\partial}{\partial r}\Delta z_\perp^+\right\rangle + \left\langle\Delta z_\perp^+\Delta z_\parallel^+\frac{\partial}{\partial r}\Delta z_\perp^-\right\rangle - \frac{2}{r}\langle\Delta z_\parallel^+(\Delta z_\perp^-)^2\rangle - \frac{2}{r}\langle\Delta z_\parallel^-(\Delta z_\perp^+)^2\rangle\\
&\qquad - 2d_I[\langle(\nabla\times\mathbf{E})_\perp(\mathbf{x}_1)\Delta z_\perp^-\rangle - \langle(\nabla\times\mathbf{E})_\perp(\mathbf{x}_1)\Delta z_\perp^+\rangle] = -\frac{8}{3}(\langle\epsilon_v\rangle - \langle\epsilon_b\rangle).
\end{aligned} \tag{89}$$

To the best of our knowledge, Eqs. (89) have not been obtained until now. Equations (87)–(89), upon setting $d_I = 0$, reduce to their 3DMHD counterparts.

We have discussed above the existence of two power-law-scaling ranges, at small k and large k, where k is the wave number [18–22,25], in energy spectra $E(k)$ for 3D Hall-MHD turbulence; the inverse of the ion-inertial length scale d_I sets the wave-number scale at which we see a crossover from low- to high-k power-law ranges. Therefore, in the hierarchical relations (85)–(89), if we take the limit $d_I \to 0$, we recover the hierarchical relations for isotropic 3DMHD given by Eqs. (57) and (58); these yield the analogs of the von Kármán–Howarth relation for 3D Hall-MHD turbulence in the low-k scaling regime, i.e., for large separations r. In contrast, for large d_I, the Hall term in Eq. (4) dominates over other terms; hence, if we retain only the terms with d_I in Eqs. (85)–(89), we obtain the hierarchical relations for the large-k (or small-r) scaling regime in 3D Hall-MHD turbulence.

VI. CONCLUSIONS

We have shown how to generalize the method of Ref. [16], for structure-function relations in turbulence in the stochastically forced Burgers equation, to develop structure-function hierarchies for turbulence in three magnetohydrodynamical equations, namely, the BMHD equations [15] in one dimension, and, in three dimensions, the equations of magnetohydrodynamics (3DMHD) [1–6], and the equations of Hall MHD [18–22]. Our methods provide a convenient and unified scheme for the development of structure-function hierarchies for turbulence in a variety of coupled hydrodynamical equations [28]. In particular, as we have shown explicitly for the three magnetohydrodynamical equations mentioned above, these structure-function hierarchies lead, at order $p = 3$, to the analogs of the von Kármán–Howarth relation for turbulence in these equations. Such relations have not been obtained for the BMHD system until now. For 3DMHD turbulence, both isotropic and anisotropic (by virtue of a mean magnetic field), the analogs of such relations have been obtained, in slightly different forms, in Refs. [29,44,50,51]; the relations in Refs. [29,44,50] are equivalent to the ones we give above; the relation in Ref. [51] has an extra term, which should be zero; however, our method yields the additional relations (65)–(69). Similarly, for 3D Hall-MHD turbulence, the simplest analog of the von Kármán–Howarth relation has been obtained earlier in Ref. [21]; again, our method yields additional relations. The structure-function hierarchy for BMHD turbulence has not been studied until now.

For both the BMHD and 3DMHD turbulence, a mean, external field B_0 does not appear explicitly in the master equation. This is consistent with the discussions in Ref. [45]. Although B_0 does not appear in the master equations explicitly, its presence or absence dictates the degree of spatial isotropy, and hence the final form of the steady-state master equations in 3D. In the BMHD case this isotropy is not relevant because we consider only 1DBMHD.

Note that in 3DMHD turbulence, the number of possible third-order structure functions is larger than in 3D fluid turbulence because we have two fields, namely, $\mathbf{v}$ (velocity) and $\mathbf{b}$ (magnetic) fields (or, equivalently, the $\mathbf{z}^\pm$ fields). Equations (59) and (60) are the direct 3DMHD analog of the von Kármán–Howarth relations for 3D fluid turbulence. In addition, because we have both $\mathbf{z}^\pm$ fields in 3DMHD, there are additional relations involving several other structure functions, namely, Eqs. (65)–(69); together with Eqs. (59) and (60), they provide the full set of relations involving different third-order structure functions. This discussion holds true for Hall-MHD structure functions as well. Clearly, the number of third-order structure functions in 3DMHD and Hall MHD is larger than in 3D fluid turbulence. Therefore, the master equation yields additional relations between the different structure functions in 3DMHD and Hall-MHD; the number of such relations turns out to be smaller than the number of structure functions. Hence, we are unable to close these relations. Despite this difficulty, each of these relations must be obeyed in the nonequilibrium statistically steady states of 3DMHD or Hall-MHD turbulence. Experimental results should be able to obtain these structure functions.

The analogs of the von Kármán–Howarth relations, which we have studied here, are exact and must, of course, emerge from DNSs. However, the explicit verification of von Kármán–Howarth-type relations can pose significant challenges; they require very-high-resolution and careful DNS as can be seen, for fluid-turbulence DNS, by examining Fig. 3(b) in Ref. [53]. Although DNS studies of 3DMHD turbulence have begun to examine order-p structure functions [10] with $p > 2$, they have not tested von Kármán–Howarth-type relations in as much detail as their fluid-turbulence counterparts [53]. Direct numerical simulations of 3D Hall-MHD pose even greater challenges [18–22,25] than their 3DMHD counterparts. Numerical studies of BMHD [54] might well be able to test von Kármán–Howarth-type relations in great detail. We hope our study will stimulate both DNS and experimental studies that can begin to examine the analogs of the von Kármán–Howarth relations in all the types of magnetohydrodynamical turbulence we have examined here.

ACKNOWLEDGMENTS

We thank D. Mitra, S. S. Ray, D. Banerjee, and S. Ramaswamy for discussions. A.B. acknowledges the Max-Planck-Gesellschaft (Germany) and Department of Science and Technology/Indo-German Science and Technology Centre (India) for support through the Partner Group programme (2009). R.P. acknowledges support from UGC, CSIR, and DST (India); he is a member of the International Collaboration for Turbulence Research (ICTR).

APPENDIX

1. Master equation for the 1DBMHD model

The BMHD generating functions

We describe here the generating-function method, which has been proposed in Ref. [16] to study the statistical properties of Burgers turbulence, for the BMHD system. This nonperturbative method has been used to the study the statistical properties of turbulence in the stochastically forced Burgers and Navier-Stokes equations [16,36,43]. This is a natural way to study turbulence generated by a large-scale stirring mechanism because perturbative approaches are not useful here [55].

All the statistical information of the random quantities $z^{\pm}(x,t)$ [see Eqs. (14) and (15)] is contained in the PDFs $\mathcal{P}(z^{\pm},x,t)$. This is the probability density for $z^{\pm}(x,t)$ to have the value $z^{\pm}$ at point x and at time t. To investigate multipoint properties, we need to consider multipoint PDFs, e.g., the two-point PDF $\mathcal{P}(z_1^{\pm},x_1,t_1;z_2^{\pm},x_2,t_2)$, which is the joint probability density for $z^{\pm}(x,t)$ to have the values z_i^{+} at point x_i and at time t_i for $i = 1,2$.

We define the two-point generating functions for the variables $z_i^{\pm}$ as follows:

$$\begin{aligned}\mathcal{Z}(\lambda_1^{\pm},\lambda_2^{\pm},x_1,x_2,t_1,t_2) &\equiv \langle \exp(\lambda_1^{+}z_1^{+} + \lambda_2^{+}z_2^{+} + \lambda_1^{-}z_1^{-} + \lambda_2^{-}z_2^{-})\rangle = \langle \mathcal{Z}_1\mathcal{Z}_2\rangle \\ &= \int\int dz_1^{\pm}\,dz_2^{\pm}\,\mathcal{P}(z_1^{\pm},x_1,t_1;z_2^{\pm},x_2,t_2)\mathcal{Z}_1\mathcal{Z}_2, \end{aligned} \quad \text{(A1)}$$

where $\mathcal{Z}_1 = \exp(\lambda_1^{+}z_1^{+} + \lambda_2^{+}z_2^{+})$, $\mathcal{Z}_2 = \exp(\lambda_1^{-}z_1^{-} + \lambda_2^{-}z_2^{-})$; $\lambda_i^{\pm}$ are the variables conjugate to $z_i^{\pm}$ for $i = 1,2$. In the absence of cross correlations (i.e., zero cross helicity) $\mathcal{Z} = \langle\mathcal{Z}_1\rangle\langle\mathcal{Z}_2\rangle$. Here, $\langle\cdots\rangle$ denotes an ensemble average with the two-point PDF $\mathcal{P}(z_1^{\pm},x_1,t_1;z_2^{\pm},x_2,t_2)$, which is equivalent to the averaging over realizations of the random force in an ergodic system. Given ergodicity, the ensemble average is equal either to the average over space, in statistically homogeneous BMHD turbulence, or the average, over long time intervals, in statistically stationary BMHD turbulence [7]. In the following calculations, we restrict ourselves to *equal-time* statistics; hence, hereafter, we use $t_1 = t_2 = t$. It is easily seen that differentiation with respect to $\lambda_i^{\pm}$ leads to various correlation functions. In particular, we are interested in the following structure functions:

$$S_{m,n} \equiv \langle(\Delta z^{+})^m(\Delta z^{-})^n\rangle \quad \text{(A2)}$$

and

$$\Sigma_{m,n} \equiv \langle(\Delta u)^m(\Delta b)^n\rangle, \quad \text{(A3)}$$

where m and n are positive integers and

$$\Delta z^{\pm} \equiv z_1^{\pm} - z_2^{\pm}. \quad \text{(A4)}$$

We can also define the generalized structure functions

$$\mathcal{S}_{m,n}^{\alpha,\beta} \equiv \left\langle\partial_x^{\alpha}(\Delta z^{+})^m\partial_x^{\beta}(\Delta z^{-})^n\right\rangle, \quad \text{(A5)}$$

where $\alpha,\beta = 1,0$; note that $\alpha,\beta = 0$ means that there is no derivative. Thus, in our notations, $S_{m,n} = \mathcal{S}_{m,n}^{0,0}$. Similarly, we can define

$$\Sigma_{m,n}^{\alpha,\beta} \equiv \left\langle\partial_x^{\alpha}(\Delta u)^m\partial_x^{\beta}(\Delta b)^n\right\rangle. \quad \text{(A6)}$$

2. Master equations for equal-time, two-point, generating functions

We now examine the two-point statistics of velocity or magnetic fields; our goal is to extract such information from the two-point statistics of the Elsässer variables $z^{\pm}$. We first consider the forced 1DBMHD equations

$$\partial_t z^{+} + B_0\,\partial_x z^{+} + z^{+}\,\partial_x z^{+} = \nu_1\,\partial_x^2 z^{+} + \nu_2\,\partial_x^2 z^{-} + f^{+}, \quad \text{(A7)}$$

$$\partial_t z^{-} - B_0\,\partial_x z^{-} + z^{-}\,\partial_x z^{-} = \nu_1\,\partial_x^2 z^{-} + \nu_2\,\partial_x^2 z^{+} + f^{-}, \quad \text{(A8)}$$

where $f^{\pm}(x,t)$ are determined from f_u and f_b from Eqs. (16).

We now derive the master equations for the equal-time, two-point generating functions $\mathcal{Z}$. We start with $\partial_t \mathcal{Z}$ and use Eqs. (A7) and (A8) to observe that

$$\partial_t \mathcal{Z} = \langle [\lambda_1^{\pm}\partial_t z_1^{\pm} + \lambda_2^{\pm}\partial_t z_2^{\pm}] \exp(\lambda_1^{\pm} z_1^{\pm} + \lambda_2^{\pm} z_2^{\pm}) \mathcal{Z}_1 \mathcal{Z}_2 \rangle, \tag{A9}$$

$$= \sum_{i=1,2} \langle [\lambda_i^{\pm}(-z_i^{\mp}\partial_i z_i^{\pm} \mp B_0 \partial_i z_i^{\pm} + \nu_1 \partial_i^2 z_i^{\pm} + \nu_2 \partial_i^2 z_i^{\mp} + f_i^{\pm})] \mathcal{Z}_1 \mathcal{Z}_2 \rangle, \tag{A10}$$

where i labels the points 1 and 2. Hereafter we use the notations

$$\partial_i \equiv \frac{\partial}{\partial x_i}, \tag{A11}$$

$$f_i \equiv f(x_i,t), \tag{A12}$$

and, as noted before,

$$z_i^{\pm} \equiv z^{\pm}(x_i,t). \tag{A13}$$

By using the equations of motion we get

$$\begin{aligned}
\partial_t \mathcal{Z} &+ \sum_{j=1,2} \left\langle \lambda_j^+ \frac{\partial}{\partial \lambda_j^+} \left(\frac{1}{\lambda_j^+} \frac{\partial \mathcal{Z}_1}{\partial x_j} \right) \mathcal{Z}_2 \right\rangle + \sum_{j=1,2} \left\langle \lambda_j^- \frac{\partial}{\partial \lambda_j^-} \left(\frac{1}{\lambda_j^-} \frac{\partial \mathcal{Z}_2}{\partial x_j} \right) \mathcal{Z}_1 \right\rangle + B_0 \sum_{j=1,2} \left\langle \frac{\partial \mathcal{Z}_1}{\partial x_j} \mathcal{Z}_2 \right\rangle - B_0 \sum_{j=1,2} \left\langle \frac{\partial \mathcal{Z}_2}{\partial x_j} \mathcal{Z}_1 \right\rangle \\
&= \sum_{j=1,2} \lambda_j^+ \left\langle \left[f_+(x_j,t) + \nu_1 \frac{\partial^2 z^+}{\partial x_j^2} + \nu_2 \frac{\partial^2 z^-}{\partial x_j^2} \right] \mathcal{Z}_1 \mathcal{Z}_2 \right\rangle + \sum_{j=1,2} \lambda_j^- \left\langle \left[f_-(x_j,t) + \nu_1 \frac{\partial^2 z^-}{\partial x_j^2} + \nu_2 \frac{\partial^2 z^+}{\partial x_j^2} \right] \mathcal{Z}_1 \mathcal{Z}_2 \right\rangle.
\end{aligned} \tag{A14}$$

We can proceed further by applying the Furutsu-Novikov-Donsker [56,57] formalism to calculate the random-forcing terms. For Gaussian distributed, white-in-time random forces $f^{\pm}$, which are additive terms in the BMHD equations for $z^{\pm}$, we obtain the final form of *the master equation* for the equal-time, two-point generating functions $\mathcal{Z}$:

$$\begin{aligned}
\partial_t \mathcal{Z} &+ \sum_{j=1,2} \left\langle \lambda_j^+ \frac{\partial}{\partial \lambda_j^+} \left(\frac{1}{\lambda_j^+} \frac{\partial \mathcal{Z}_1}{\partial x_j} \right) \mathcal{Z}_2 \right\rangle + \sum_{j=1,2} \left\langle \lambda_j^- \frac{\partial}{\partial \lambda_j^-} \left(\frac{1}{\lambda_j^-} \frac{\partial \mathcal{Z}_2}{\partial x_j} \right) \mathcal{Z}_1 \right\rangle + B_0 \sum_{j=1,2} \left\langle \frac{\partial \mathcal{Z}_1}{\partial x_j} \mathcal{Z}_2 \right\rangle - B_0 \sum_{j=1,2} \left\langle \frac{\partial \mathcal{Z}_2}{\partial x_j} \mathcal{Z}_1 \right\rangle \\
&= \sum_{i,j} \lambda_i^+ \lambda_j^+ K(x_i - x_j) \mathcal{Z} + \sum_{i,j} \lambda_i^- \lambda_j^- K(x_i - x_j) \mathcal{Z} + \sum_{i,j} \lambda_i^+ \lambda_j^- [\hat{K}(x_i - x_j) + \tilde{K}(x_i - x_j)] \mathcal{Z} \\
&\quad + \sum_{i,j} \lambda_i^- \lambda_j^+ [\hat{\kappa}(x_j - x_i) + \tilde{K}(x_j - x_i)] \mathcal{Z} + D_1 + D_2,
\end{aligned} \tag{A15}$$

where $D_1 = \nu_1 \langle \sum_j \lambda_j^+ \frac{\partial^2 z_j^+}{\partial x_j^2} \mathcal{Z}_1 \mathcal{Z}_2 \rangle + \nu_2 \langle \sum_j \lambda_j^+ \frac{\partial^2 z^-}{\partial x_j^2} \mathcal{Z}_1 \mathcal{Z}_2 \rangle$; $D_2 = \nu_1 \langle \sum_j \lambda_j^- \frac{\partial^2 z_j^-}{\partial x_j^2} \mathcal{Z} \rangle + \nu_2 \langle \sum_j \lambda_j^- \frac{\partial^2 z^+}{\partial x_j^2} \mathcal{Z}_1 \mathcal{Z}_2 \rangle$. Thus, $D_1 + D_2 = \nu \langle \lambda_j^u \frac{\partial^2 u}{\partial x_j^2} \mathcal{Z}_1 \mathcal{Z}_2 \rangle + \mu \langle \lambda_j^b \frac{\partial^2 b}{\partial x_j^2} \mathcal{Z}_1 \mathcal{Z}_2 \rangle$, $\lambda_u = \lambda_1 + \lambda_2$, $\lambda_b = \lambda_1 - \lambda_2$ are the anomaly terms. Note that the master equation (A15) is not closed because of these anomaly terms. This nontrivial point has been treated by a variety of approaches ranging from approximate techniques to rigorous studies [16,58]. A problem arises when we look at the master equation in the limit of vanishing viscosity and magnetic diffusivity $\nu,\mu \to 0$; here the anomaly term produces a finite effect. For instance, the finiteness of the dissipation in the limit of vanishing viscosity (discussed above) is just produced by the anomaly terms $D^{\pm}$. In what follows, we specify the effects of the anomaly term in detail.

It is useful to apply the basic symmetries of the dynamical equations to simplify the structure of the master equation. We assume that statistically homogeneous and stationary turbulence has been produced under the dynamics of the stochastically forced BMHD equations (A7) and (A8) for $z^{\pm}$. Stationarity implies

$$\partial_t \mathcal{Z} = 0 \tag{A16}$$

and homogeneity yields

$$\mathcal{Z} = \mathcal{Z}(\lambda_1^{\pm},\lambda_2^{\pm},x_1 - x_2). \tag{A17}$$

Equivalently, if we define

$$x_{\pm} \equiv x_1 \pm x_2, \tag{A18}$$

we can write

$$\frac{\partial}{\partial x_1}\mathcal{Z} = \frac{\partial}{\partial x_+}\mathcal{Z} + \frac{\partial}{\partial x_-}\mathcal{Z}, \tag{A19}$$

$$\frac{\partial}{\partial x_2}\mathcal{Z} = \frac{\partial}{\partial x_+}\mathcal{Z} - \frac{\partial}{\partial x_-}\mathcal{Z}. \tag{A20}$$

Homogeneity now implies $(\partial/\partial x_+)\mathcal{Z} = 0$.

The 1DBMHD equations are also invariant under the Galilean transformation

$$x = x' + u_0 t', \qquad t = t', \qquad u(x,t) = u'(x',t') + u_0, \qquad b(x,t) = b'(x',t'); \tag{A21}$$

because $\mathcal{Z} = \langle \exp(\lambda_1^{\pm} z_1^{\pm} + \lambda_2^{\pm} z_2^{\pm})\rangle$ and $z^{\pm} = u \pm b$, this Galilean invariance implies translation invariance in the space of $z^{\pm}$ and hence $\lambda_1^+ + \lambda_2^+ = 0 = \lambda_1^- + \lambda_2^-$. Thus,

$$\mathcal{Z} = \mathcal{Z}(\lambda_1^{\pm} - \lambda_2^{\pm}, x_1 - x_2). \tag{A22}$$

If we introduce the variables

$$\lambda_+ \equiv \lambda_1^+ + \lambda_2^+, \qquad \lambda_1 \equiv (\lambda_1^+ - \lambda_2^+)/2, \qquad \lambda_- \equiv \lambda_1^- + \lambda_2^-, \qquad \lambda_2 \equiv \lambda_1^- - \lambda_2^- \tag{A23}$$

and use the same considerations as in Eqs. (A19) and (A20) for $x_{\pm}$, we see that Galilean invariance is equivalent to demanding $(\partial/\partial\lambda_1)\mathcal{Z} = 0$, $(\partial/\partial\lambda_2)\mathcal{Z} = 0$. Therefore, the master equation can be conveniently rewritten in terms of the variables $x_- \equiv x$ and λ_1,λ_2; this master equation for the generating function

$$\mathcal{Z}(\lambda_1,\lambda_2,x) = \langle \exp(\lambda_1 \Delta z^+ + \lambda_2 \Delta z^-)\rangle = \langle \mathcal{Z}_1 \mathcal{Z}_2\rangle \qquad (\text{with } \Delta z^{\pm} = z_1^{\pm} - z_2^{\pm}) \tag{A24}$$

is

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial\lambda_1}\frac{\partial}{\partial x}\mathcal{Z}_1\mathcal{Z}_2\right\rangle - \left\langle \frac{2}{\lambda_1}\frac{\partial}{\partial x \mathcal{Z}_1}\mathcal{Z}_2\right\rangle + \left\langle \frac{\partial}{\partial\lambda_2}\frac{\partial}{\partial x}\mathcal{Z}_2\mathcal{Z}_1\right\rangle - \left\langle \frac{2}{\lambda_2}\frac{\partial}{\partial x \mathcal{Z}_2}\mathcal{Z}_1\right\rangle \\
&\quad = 2\lambda_1^2[K(0) - K(x)]\mathcal{Z}_1\mathcal{Z}_2 + 2\lambda_2^2[K(0) - K(x)]\mathcal{Z} \\
&\qquad + 4\lambda_1\lambda_2[\hat{K}(0) - \hat{K}(x) + \tilde{K}(0) - \tilde{K}(x)]\mathcal{Z} + D_1 + D_2.
\end{aligned} \tag{A25}$$

It is convenient to write the contribution of the anomaly in terms of the local dissipation fields $\varepsilon^{\pm}(x)$: By starting with $\langle \nu \sum_{i=1,2} \mathcal{Z}_b \partial_i^2 \mathcal{Z}_u\rangle$ and $\langle \mu \sum_{i=1,2} \mathcal{Z}_u \partial_i^2 \mathcal{Z}_b\rangle$ for $i = 1,2$, we get, by using $\mathcal{Z}_u = \exp[\lambda_u \Delta u]$ and $\mathcal{Z}_b = \exp[\lambda_b \Delta b]$,

$$\begin{aligned}
\nu \sum_{i=1,2} \mathcal{Z}_b \partial_i^2 \mathcal{Z}_u &= \nu\lambda_u \langle [\partial_1^2 u_1 - \partial_2^2 u_2]\mathcal{Z}_u\mathcal{Z}_b\rangle + \nu\lambda_u^2 \left\langle \left[\left(\frac{\partial u(x_1)}{\partial x_1}\right)^2 + \left(\frac{\partial u(x_2)}{\partial x_2}\right)^2\right]\mathcal{Z}_u\mathcal{Z}_b\right\rangle \\
&= D_1 - 2\lambda_u^2\langle \varepsilon_u \mathcal{Z}_1\mathcal{Z}_2\rangle
\end{aligned} \tag{A26}$$

and

$$\begin{aligned}
\mu \sum_{i=1,2} \mathcal{Z}_u \partial_i^2 \mathcal{Z}_b &= \mu\lambda_b \langle [\partial_1^2 b_1 - \partial_2^2 b_2]\mathcal{Z}_u\mathcal{Z}_b\rangle + \mu\lambda_b^2 \left\langle \left[\left(\frac{\partial b(x_1)}{\partial x_1}\right)^2 + \left(\frac{\partial b(x_2)}{\partial x_2}\right)^2\right]\mathcal{Z}_u\mathcal{Z}_b\right\rangle \\
&= D_2 - 2\lambda_b^2\langle \varepsilon_b \mathcal{Z}_1\mathcal{Z}_2\rangle,
\end{aligned} \tag{A27}$$

where

$$\varepsilon_u \equiv \nu(\partial_i u^2), \qquad \varepsilon_b \equiv \mu(\partial_u b^2) \tag{A28}$$

are the local kinetic- and magnetic-energy dissipation rates, respectively, at the point x_i, and we have used $\mathcal{Z}_u\mathcal{Z}_b = \mathcal{Z}_1\mathcal{Z}_2$. In the limit $\nu,\mu \to 0$, the BMHD system shows some features of fluid and MHD turbulence. For instance, an inertial range is obtained in which the structure functions $S_{m,n} = \langle (\Delta z^+)^m (\Delta z^-)^n\rangle$ display power-law scaling (or multiscaling). The structure functions are all finite in the inertial range and, therefore, the generating function is finite in the limits $\nu,\mu \to 0$. Hence, the left-hand sides of Eqs. (A26) and (A27) tend to zero in this limit and we finally get the following expression for the anomaly terms:

$$\begin{aligned}
D_1 + D_2 &= -2\lambda_u^2\langle \varepsilon_u \mathcal{Z}_1\mathcal{Z}_2\rangle - 2\lambda_b^2\langle \varepsilon_b \mathcal{Z}_1\mathcal{Z}_2\rangle \\
&= -2(\lambda_1^2 + \lambda_2)^2(\langle \varepsilon_u \mathcal{Z}_1\mathcal{Z}_2\rangle + \langle \varepsilon_b \mathcal{Z}_1\mathcal{Z}_2\rangle) - 4\lambda_1\lambda_2(\langle \varepsilon_u \mathcal{Z}_1 \lambda Z_2\rangle - \langle \varepsilon_b \mathcal{Z}_1\mathcal{Z}_2\rangle).
\end{aligned} \tag{A29}$$

Here we have used the translational invariance of single-point functions.

3. Dissipative anomalies in 3DMHD

The dissipative anomaly in 3DMHD D_M is given by

$$\begin{aligned}D_M &= \langle \nu_1[\boldsymbol{\lambda}^+\cdot\nabla^2\mathbf{z}^+(\mathbf{x}_1)-\boldsymbol{\lambda}^+\cdot\nabla^2\mathbf{z}^+(\mathbf{x}_2)+\boldsymbol{\lambda}^-\nabla^2\mathbf{z}^-(\mathbf{x}_1)-\boldsymbol{\lambda}^-\cdot\nabla^2\mathbf{z}^-(\mathbf{x}_2)]\mathcal{Z}_M^+\mathcal{Z}_M^-\rangle\\ &\quad+\nu_2\langle[\boldsymbol{\lambda}^+\cdot\nabla^2\mathbf{z}^-(\mathbf{x}_1)-\boldsymbol{\lambda}^+\cdot\nabla^2\mathbf{z}^-(\mathbf{x}_2)+\boldsymbol{\lambda}^-\cdot\nabla^2\mathbf{z}^+(\mathbf{x}_1)-\boldsymbol{\lambda}^-\nabla^2\mathbf{z}^+(\mathbf{x}_2)]\mathcal{Z}_M^+\mathcal{Z}_M^-\rangle\\ &= \frac{\nu_0+\mu_0}{4}\langle\{(\boldsymbol{\lambda}^v+\boldsymbol{\lambda}^b)\cdot[\nabla^2\mathbf{z}^+(\mathbf{x}_1)-\nabla^2\mathbf{z}^+(\mathbf{x}_2)]+(\boldsymbol{\lambda}^v-\boldsymbol{\lambda}^b)\cdot[\nabla^2\mathbf{z}^-(\mathbf{r})_1-\nabla^2\mathbf{z}^-(\mathbf{x}_2)]\}\mathcal{Z}_M^+\mathcal{Z}_M^-\rangle\\ &\quad+\frac{\nu_0-\mu_0}{4}\{(\boldsymbol{\lambda}^v+\boldsymbol{\lambda}^b)\cdot[\nabla^2\mathbf{z}^-(\mathbf{x}_1)-\nabla^2\mathbf{z}^-(\mathbf{x}_2)]+(\boldsymbol{\lambda}^v-\boldsymbol{\lambda}^b)\cdot[\nabla^2\mathbf{z}^+(\mathbf{x}_1)-\nabla^2\mathbf{z}^+(\mathbf{x}_2)]\}\mathcal{Z}_M^+\mathcal{Z}_M^-\rangle\\ &= \nu_0\boldsymbol{\lambda}^v\cdot\langle[\nabla^2\mathbf{v}(\mathbf{x}_1)-\nabla^2\mathbf{v}(\mathbf{x}_2)]\mathcal{Z}_v\mathcal{Z}_b\rangle+\mu_0\boldsymbol{\lambda}^b\cdot\langle[\nabla^2\mathbf{b}(\mathbf{x}_1)-\nabla^2(\mathbf{x}_2)]\mathcal{Z}_v\mathcal{Z}_b\rangle=\mathcal{D}_v+\mathcal{D}_b,\end{aligned}\tag{A30}$$

where $\boldsymbol{\lambda}^v=\boldsymbol{\lambda}^++\boldsymbol{\lambda}^-$, $\boldsymbol{\lambda}^b=\boldsymbol{\lambda}^+-\boldsymbol{\lambda}^-$ and $\mathcal{Z}_v=\exp\{\boldsymbol{\lambda}^v\cdot[\mathbf{v}(\mathbf{x}_1)-\mathbf{v}(\mathbf{x}_2)]\}=\exp[\boldsymbol{\lambda}^v\cdot\Delta\mathbf{v}]$, $\mathcal{Z}_b=\exp[\boldsymbol{\lambda}^b\cdot(\mathbf{b}(\mathbf{x}_1)-\mathbf{b}(\mathbf{x}_2))]=\exp[\boldsymbol{\lambda}^b\cdot\Delta\mathbf{b}]$. In the above operators ∇^2 refers to $\frac{\partial^2}{\partial x_m^2}$ with $m=1,2$, depending upon the argument of the field that ∇^2 acts upon. Furthermore, we have defined

$$\mathcal{D}_v=\nu_0\boldsymbol{\lambda}^v\cdot\langle[\nabla^2\mathbf{v}(\mathbf{x}_1)-\nabla^2\mathbf{v}(\mathbf{x}_2)]\mathcal{Z}^u\mathcal{Z}^b\rangle,\qquad \mathcal{D}_b=\mu_0\boldsymbol{\lambda}^b\cdot\langle[\nabla^2\mathbf{b}(\mathbf{x}_1)-\nabla^2(\mathbf{x}_2)]\mathcal{Z}^u\mathcal{Z}^b\rangle=\mathcal{D}_v+\mathcal{D}_b.\tag{A31}$$

Consider now

$$\begin{aligned}\nu_0\left\langle\sum_{i=1,2}\nabla_i^2\mathcal{Z}^v\mathcal{Z}^b\right\rangle_{\lambda_b=0} &= \nu_0\boldsymbol{\lambda}^v\cdot\langle[\nabla^2\mathbf{v}(\mathbf{x}_1)-\nabla^2\mathbf{v}(\mathbf{x}_2)]\mathcal{Z}^v\mathcal{Z}^b\rangle\\ &\quad+\nu_0\langle\lambda_m^v\lambda_n^v[(\nabla_j v_m)(\nabla_j v_n)(\mathbf{x}_1)+(\nabla_j v_m)(\nabla_j v_m)(\nabla_j v_n)(\mathbf{r}_2)]\mathcal{Z}^v\mathcal{Z}^b\rangle.\end{aligned}\tag{A32}$$

All the structure functions are finite in the limit $\nu,\mu\to 0$, therefore, the left-hand side of Eq. (A32) vanishes, when $\nu_0,\mu_0\to 0$, and we find

$$\mathcal{D}_v=-\nu_0\langle\lambda_m^v\lambda_n^v[(\nabla_j v_m)(\nabla_j v_n)(\mathbf{x}_1)+(\nabla_j v_m)(\nabla_j v_n)(\mathbf{r}_2)]\mathcal{Z}^v\mathcal{Z}^b\rangle.\tag{A33}$$

Similarly, we get

$$\mathcal{D}_b=-\mu_0\lambda_m^b\lambda_n^b[(\nabla_j b_m)(\nabla_j b_n)(\mathbf{x}_1)+(\nabla_j b_m)(\nabla_j b_n)(\mathbf{r}_2)]\mathcal{Z}^v\mathcal{Z}^b\rangle.\tag{A34}$$

Expressions (A33) and (A34) may be used to obtain the contributions of the anomaly terms in the third-order structure functions of 3DMHD. Notice that the cross terms of the form $\langle(\nabla_j a_m)(\mathbf{r})(\nabla_j a_n)(\mathbf{r})\rangle$, $a=v,b$ is zero. This follows from $\nabla\cdot\mathbf{a}=0$, and hence $\langle(\nabla_j a_m)(\mathbf{r})(\nabla_j a_n)(\mathbf{r})\rangle=\int d^3k\,k_m k_n\phi_a(k^2)$, which is proportional to δ_{mn}, and is, therefore, zero for $m\neq n$. Thus, if we ignore the mixed terms in Eq. (A32), which suffices as long as we are interested in the third-order structure functions only, we obtain

$$\mathcal{D}_v+\mathcal{D}_b=-2\big\langle\big[\tfrac{1}{3}\eta_2^2(1+\alpha^2)(\epsilon_v+\epsilon_b)+\tfrac{2}{3}\eta_3^2(1+\alpha^2)(\epsilon_v+\epsilon_b)+\tfrac{2}{3}\eta_2^2\alpha(\epsilon_v+\epsilon_b)+\tfrac{4}{3}\eta_3^2\alpha(\epsilon_v-\epsilon_b)\big]\mathcal{Z}_M^+\mathcal{Z}_M^-\big\rangle.\tag{A35}$$

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[47] In our formulation, $\boldsymbol{\lambda}_+$ and $\boldsymbol{\lambda}_-$ allow us to define the Laplace transforms of the joint probability distribution of two vector fields, say, $\mathbf{a}_1$ and $\mathbf{a}_2$ (e.g., these could be the differences of $\mathbf{z}^{\pm}$ at two different points) as follows:

$$Z(\boldsymbol{\lambda}_+,\boldsymbol{\lambda}_-) = \int D\mathbf{a}_1 D\mathbf{a}_2 \exp[-\boldsymbol{\lambda}_+ \cdot \mathbf{a}_1 - \boldsymbol{\lambda}_- \cdot \mathbf{a}_2] P(\mathbf{a}_1,\mathbf{a}_2),$$

where $D\mathbf{a}_1$ and $D\mathbf{a}_2$ are integrations over the field $\mathbf{a}_1$ and $\mathbf{a}_2$; $P(\mathbf{a}_1,\mathbf{a}_2)$ is the joint probability distrbution of a_1 and a_2. Clearly, Z depends upon the choice for λ_+ and λ_-, which may be chosen freely. In particular, we are free to choose them to be parallel for convenience of the calculation that follows. This does not constrain the physical fields $\mathbf{a}_1$ and $\mathbf{a}_2$ in any way; and different moments of $\mathbf{a}_1$ and $\mathbf{a}_2$ can be obtained independently. For a different choice of $\boldsymbol{\lambda}_+$ and $\boldsymbol{\lambda}_-$ (where they may not be parallel), one may still write $\boldsymbol{\lambda}_- = \alpha\boldsymbol{\lambda}_+ + \delta\boldsymbol{\lambda}$, such that the moments of $\mathbf{a}_2$ may be obtained by taking derivatives with respect to $\delta\boldsymbol{\lambda}$ or $\alpha\boldsymbol{\lambda}_+$ separately, and taking both $\delta\boldsymbol{\lambda}$ and $\alpha\boldsymbol{\lambda}_+$ to zero. Thus, as far as obtaining the moments of $\mathbf{b}$ is concerned, both are equivalent, and so, without any loss of generality, we can set $\delta\boldsymbol{\lambda} = 0$. We can, in principle, work with nonparallel $\boldsymbol{\lambda}_+$ and $\boldsymbol{\lambda}_- = \alpha\boldsymbol{\lambda}_+ + \delta\boldsymbol{\lambda}$ to obtain the master equation, which then assumes a more complicated form than the one in our paper. From this more complicated master equation, we can also obtain the same hierarchical relations, as in our paper, by taking derivatives with respect to either $\delta\boldsymbol{\lambda}$ or $\alpha\boldsymbol{\lambda}_+$.
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