University of Edinburgh

An Analysis of the Turbulence Spectrum at Varying Magnetic Prandtl Number

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05/04/17

Aims of the Project

- Magnetohydrodynamics (MHD) describes the movement of an electrically conducting fluid, including plasmas and liquid metals.
- Numerical simulations of MHD turbulence can improve our understanding of the behaviour of these fluids in the centre of the earth or the solar wind.

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- Numerical simulations of MHD turbulence can improve our understanding of the behaviour of these fluids in the centre of the earth or the solar wind.
- This project focused on varying the magnetic Prandtl number of the fluid and studying the affects that had on the system.

Contents

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- The motion is governed by the Navier-Stokes equation, and depends on various parameters:

$\mathbf{u}$	velocity	ρ	density
P	pressure	ν	viscosity

- The fluid is also assumed to be incompressible:

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

Fluid Dynamics

Navier-Stokes Equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} P + \nu \nabla^2 \mathbf{u} + \mathbf{F_u}$$

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Reynold's Number:

$$Re = \frac{UL}{\nu}$$

Magnetohydrodynamics

- Magnetohydrodynamics describes the flow of a fluid which conducts an electric charge.
- The movement of the fluid, and thus the charge, creates a magnetic field.
- Dynamics are thus governed by Navier-Stokes equations combined with Maxwell's Equations.

Magnetic diffusivity

$$\eta = \frac{1}{\mu_0 \sigma_0}$$

- Magnetic diffusivity is proportional to the resistivity of the fluid.
- It is a measure of how strongly the material opposes the flow of current.

MHD equations

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$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} P + \frac{1}{\rho}(\boldsymbol{\nabla} \times \mathbf{b}) \times \mathbf{b} + \nu \nabla^2 \mathbf{u} + \mathbf{F_u}$$

$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{b} = (\mathbf{b} \cdot \nabla)\mathbf{u} + \eta \nabla^2 \mathbf{b} + \mathbf{F_b}$$

Magnetic Prandtl Number

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Magnetic Prandtl number:

$$Pr_m = \frac{\nu}{\eta} = \frac{Re_m}{Re}$$

Turbulence

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- Occurs when the Reynolds number is above a critical value.

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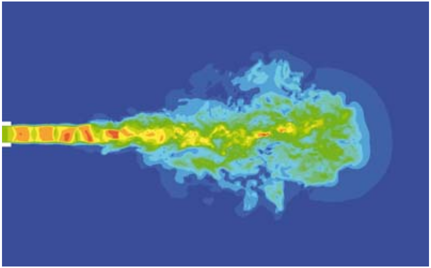

[D. F. Harlacher, H. Klimach, and S. Roller. Turbulence simulation at large scale. inSiDe, 10, 2012]

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- This length scale is called the **integral length scale**, and features in our definition of the Reynold's number.

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- Energy is transferred to smaller scales in turbulent motion.
- Kinetic energy is dissipated at small scales due to the viscosity, and the magnetic energy is dissipated due to the resistivity.

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Energy Cascade

- The transfer of energy from low to high k is known as the Energy Cascade
- The rate of transfer is given by Kolmogorov's $\frac{5}{3}$ law for the energy spectrum:

$$E(k) = C\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$$

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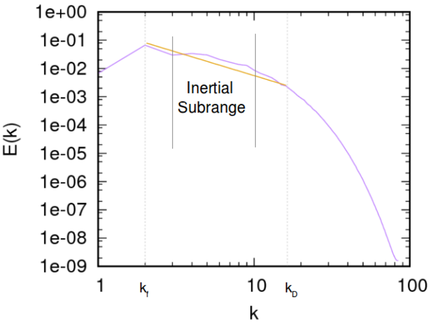

Energy Transfer

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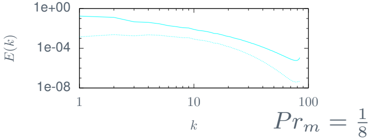

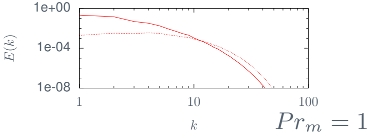

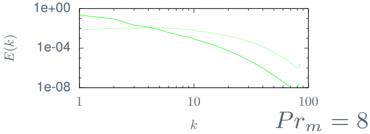

Homogeneous and Isotropic turbulence

- **Homogeneous**: invariant under translations in space
- **Isotropic**: invariant under rotations and reflections

Direct Numerical Simulations

- Direct numerical simulations simulate all scales of the motion down to the kolmogorov microscale.
- The code used in this Project was written by S. Yoffe in 2012 and extended to MHD by M. Linkmann in 2015.

Forcing Regimes

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- Stationary turbulence can be obtained by adding a forcing term

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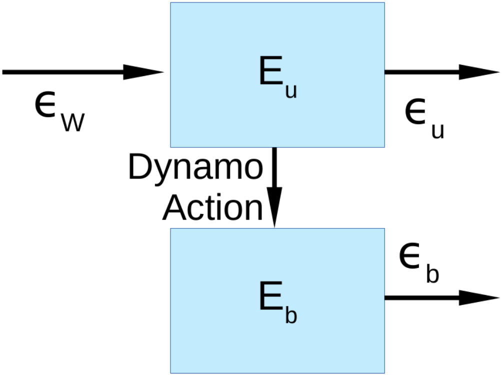
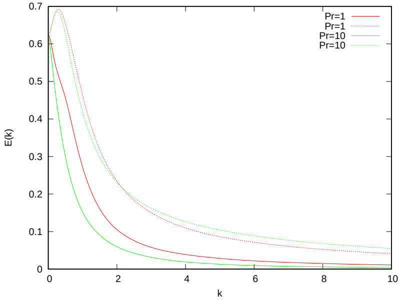
$$\mathbf{f}_u(\mathbf{k},t) = \begin{cases} \left(\dfrac{\epsilon_W}{2E_f}\right)\hat{\mathbf{u}}(\mathbf{k},t) & 0 < |\mathbf{k}| \leqslant k_f \\ 0 & \text{otherwise} \end{cases}$$

Forcing Regimes



Energy Time Graphs



(a) Decaying Simulation

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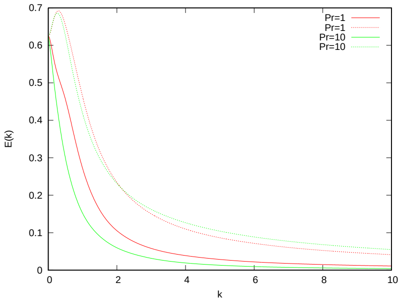

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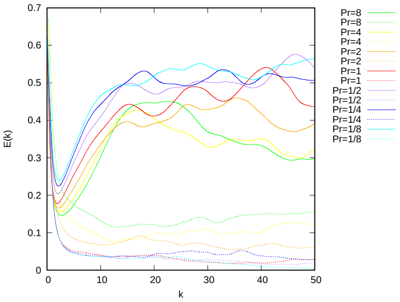
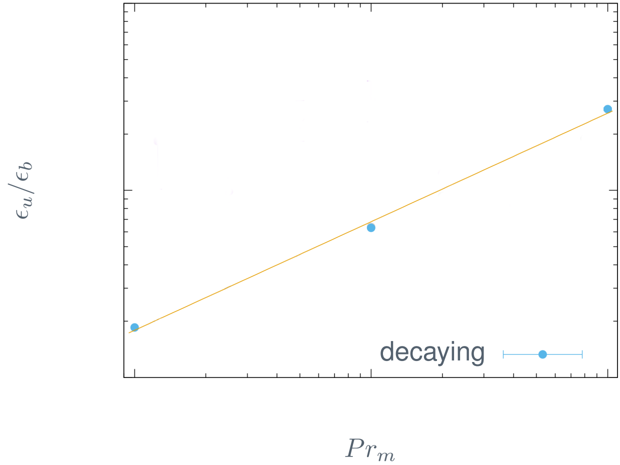


(b) Stationary Simulation

Dissipation Ratio



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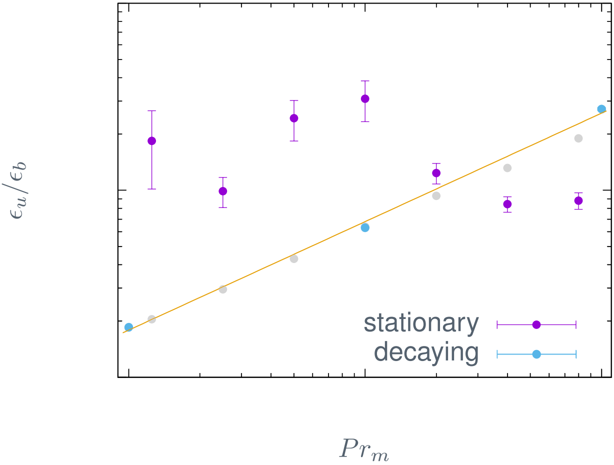

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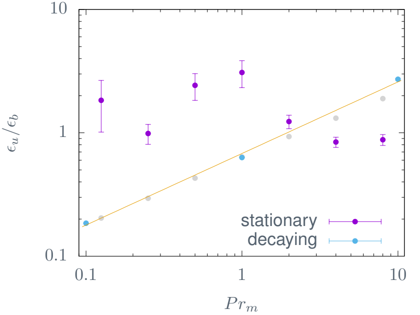

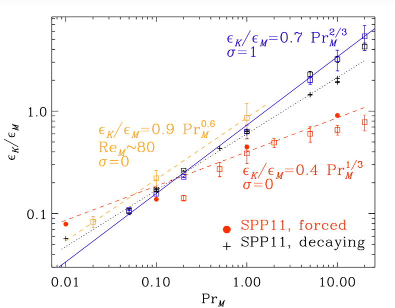

[A. Brandenburg. Magnetic prandtl number dependence of the kinetic-to-magnetic dissipation ratio. The Astrophysical Journal, 791(1):12, 2014.]

Dissipation Ratio

A relationship of $\frac{\epsilon_u}{\epsilon_b} = 0.7Pr_m^{0.64}$ was found which is in good agreement with the relationship of $\frac{\epsilon_u}{\epsilon_b} = 0.6Pr_m^{0.55}$ found in Brandenburg et al. for decaying simulations.

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No relationship between $\frac{\epsilon_u}{\epsilon_b}$ and Pr_m could be found for the stationary simulations

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- Unlikely due to the forcing regimes used, as we used the same method as Sahoo et. al.

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- Stationary simulations were run at a higher viscosity than the decaying simulations. It is possible that there was not enough turbulent motion for the relationship to arise.
- Further studies at different viscosity and magnetic diffusivity would need to be dne to determine if this is the case.

Summary