

## Systematics of the magnetic-Prandtl-number dependence of homogeneous, isotropic magnetohydrodynamic turbulence

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## Systematics of the magnetic-Prandtl-number dependence of homogeneous, isotropic magnetohydrodynamic turbulence

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**Abstract.** We present the results of our detailed pseudospectral direct numerical simulation (DNS) studies, with up to  $1024^3$  collocation points, of incompressible, magnetohydrodynamic (MHD) turbulence in three dimensions, without a mean magnetic field. Our study concentrates on the dependence of various statistical properties of both decaying and statistically steady MHD turbulence on the magnetic Prandtl number  $Pr_M$  over a large range, namely  $0.01 \leqslant Pr_M \leqslant 10$ . We obtain data for a wide variety of statistical measures, such as probability distribution functions (PDFs) of the moduli of the vorticity and current density, the energy dissipation rates, and velocity and magnetic-field increments, energy and other spectra, velocity and magnetic-field structure functions, which we use to characterize intermittency, isosurfaces of quantities, such as the moduli of the vorticity and current density, and joint PDFs, such as those of fluid and magnetic dissipation rates. Our systematic study uncovers interesting results that have not been noted hitherto. In particular, we find a crossover from a larger intermittency in the magnetic field than in the velocity field, at large  $Pr_M$ , to a smaller intermittency in the magnetic field than in the velocity field, at low  $Pr_M$ . Furthermore, a comparison of our results for decaying MHD turbulence and its forced, statistically steady analogue suggests that we

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have strong universality in the sense that, for a fixed value of  $Pr_M$ , multiscaling exponent ratios agree, at least within our error bars, for both decaying and statistically steady homogeneous, isotropic MHD turbulence.

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### 1. Introduction

The hydrodynamics of conducting fluids is of great importance in many terrestrial and astrophysical phenomena. Examples include the generation of magnetic fields via dynamo action in the interstellar medium, stars and planets [1]–[11] and in liquid–metal systems [12]–[18] that are studied in laboratories. The flows in such settings, which can be described at the simplest level by the equations of magnetohydrodynamics (MHD), are often turbulent [5]. The larger the kinetic and magnetic Reynolds numbers,  $Re = UL/\nu$  and  $Re_M = UL/\eta$ , respectively, the more turbulent is the motion of the conducting fluid; here  $L$  and  $U$  are typical length and velocity scales in the flow,  $\nu$  is the kinematic viscosity and  $\eta$  is the magnetic diffusivity. The statistical characterization of turbulent MHD flows, which continues to pose challenges for experiments [19], direct numerical simulations (DNS) [20] and theory [21], is even harder than its analogue in fluid turbulence, because (i) we must control both  $Re$  and  $Re_M$ , and (ii) we must obtain the statistical properties of both the velocity and the magnetic fields.

The kinematic viscosity  $\nu$  and the magnetic diffusivity  $\eta$  can differ by several orders of magnitude, so the magnetic Prandtl number  $Pr_M \equiv Re_M/Re = \nu/\eta$  can vary over a large range. For example,  $Pr_M \simeq 10^{-5}$  in the liquid–sodium system [15, 16],  $Pr_M \simeq 10^{-2}$  at the base of the Sun’s convection zone [22] and  $Pr_M \simeq 10^{14}$  in the interstellar medium [8, 20]. Furthermore, two dissipative scales play an important role in MHD; they are the Kolmogorov scale  $\ell_d$  ( $\sim \nu^{3/4}$  at the level of Kolmogorov 1941 (K41) phenomenology [23, 24]) and the magnetic-resistive scale  $\ell_d^M$  ( $\sim \eta^{3/4}$  in K41). A thorough study of the statistical properties of MHD turbulence must resolve both of these dissipative scales. Given current computational resources, this is a daunting task at large  $Re$ , especially if  $Pr_M$  is significantly different from unity. Thus, most

DNSs of MHD turbulence [25]–[30] have been restricted to  $Pr_M \simeq 1$ . Some DNS studies have now started moving away from the  $Pr_M \simeq 1$  regime, especially in the context of the dynamo problem [31, 32].

Here, we initiate a detailed DNS study of the statistical properties of incompressible, homogeneous and isotropic MHD turbulence for a large range of the magnetic Prandtl number, namely  $0.01 \leqslant Pr_M \leqslant 10$ . There is no mean magnetic field in our DNS [33]; and we restrict ourselves to Eulerian measurements (for representative Lagrangian studies of MHD turbulence see [34]). Before we give the details of our DNS study, we highlight a few of our qualitative, principal results. Elements of some of our results, for the case  $Pr_M = 1$  and for quantities such as energy spectra, exist in the MHD-turbulence literature, as can be seen from the representative references [5, 6, 25, 35] [37]–[39]. However, to the best of our knowledge, no study has attempted as detailed and systematic an investigation of the statistical properties of MHD turbulence as we present here, especially with a view to elucidating their dependence on  $Pr_M$ . Our study uncovers interesting trends that have not been noted hitherto. These emerge from our detailed characterization of intermittency, via a variety of measures that include probability distribution functions (PDFs), such as those of the modulus of the vorticity and the energy dissipation rates, velocity and magnetic-field structure functions that can be used to characterize intermittency, isosurfaces of quantities, such as the moduli of the vorticity and current and joint PDFs, such as those of fluid and magnetic dissipation rates. Earlier DNS studies [30] have suggested that intermittency, as characterized, say, by the multiscaling exponents for velocity- and magnetic-field structure functions, is more intense for the magnetic field than for the velocity field when  $Pr_M = 1$ . Our study confirms this and suggests, in addition, that this result is reversed as we lower  $Pr_M$ . This crossover from larger intermittency in the magnetic field than in the velocity field, at large  $Pr_M$ , to smaller intermittency in the magnetic field than in the velocity field, at low  $Pr_M$ , shows up not only in the values of multiscaling exponent ratios, which we obtain from a detailed local-slope analysis of extended-self-similarity (ESS) plots [40, 41] of one structure function against another, but also in the behaviors of tails of PDFs of dissipation rates, the moduli of vorticity and current density, and velocity and magnetic-field increments. Furthermore, a comparison of our results for decaying MHD turbulence and its forced, statistically steady analogue suggests that, at least given our conservative errors, the homogeneous, isotropic MHD turbulence that we study here displays strong universality [42, 43] in the sense that multiscaling exponent ratios agree for both the decaying and the statistically steady cases.

The remaining part of this paper is organized as follows. In section 2, we describe the MHD equations, the details of the numerical schemes we use (section 2.1) and the statistical measures we use to characterize MHD turbulence (section 2.2). In section 3, we present our results; these are described in the seven subsections 3.1–3.7 that are devoted, respectively, to (a) a summary of well-known results from fluid turbulence that are relevant to our study; (b) the temporal evolution of quantities such as the energy and energy-dissipation rates; (c) energy, dissipation-rate, Elsässer-variable and effective-pressure spectra; (d) various PDFs that can be used, *inter alia*, to characterize the alignments of vectors, such as the vorticity with, say, the eigenvectors of the rate-of-strain tensor; (e) velocity and magnetic-field structure functions that can be used to characterize intermittency; (f) isosurfaces of quantities such as the moduli of the vorticity and current; and (g) joint PDFs, such as those of fluid and magnetic dissipation rates. Section 4 contains a discussion of our results.

## 2. Magnetohydrodynamic (MHD) equations

The hydrodynamics of a conducting fluid is governed by the MHD equations [1]–[5], [7], in which the Navier–Stokes equation for a fluid is coupled to the induction equation for the magnetic field,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla \bar{p} + (\mathbf{b} \cdot \nabla) \mathbf{b} + \mathbf{f}_u, \quad (1)$$

$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{b} = (\mathbf{b} \cdot \nabla) \mathbf{u} + \eta \nabla^2 \mathbf{b} + \mathbf{f}_b. \quad (2)$$

Here,  $\mathbf{u}$ ,  $\mathbf{b}$ ,  $\boldsymbol{\omega} = \nabla \times \mathbf{u}$  and  $\mathbf{j} = \nabla \times \mathbf{b}$  are, respectively, the velocity field, the magnetic field, the vorticity and the current density at the point  $\mathbf{x}$  and time  $t$ ;  $\nu$  and  $\eta$  are the kinematic viscosity and the magnetic diffusivity, respectively, and the effective pressure is  $\bar{p} = p + (b^2/8\pi)$ , where  $p$  is the pressure;  $\mathbf{f}_u$  and  $\mathbf{f}_b$  are the external forces; while studying decaying MHD turbulence, we set  $\mathbf{f}_u = \mathbf{f}_b = 0$ . The MHD equations can also be written in terms of the Elsässer variables  $\mathbf{z}^\pm = \mathbf{u} \pm \mathbf{b}$  [7, 25]. We restrict ourselves to low-Mach-number flows, so we use the incompressibility condition  $\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0$ ; and we must, of course, impose  $\nabla \cdot \mathbf{b}(\mathbf{x}, t) = 0$ . By using the incompressibility condition, we can eliminate the effective pressure and obtain the velocity and magnetic fields via a pseudospectral method that we describe in section 2.1. The effective pressure then follows from the solution of the Poisson equation,

$$\nabla^2 \bar{p} = \nabla \cdot [(\mathbf{b} \cdot \nabla) \mathbf{b} - (\mathbf{u} \cdot \nabla) \mathbf{u}]. \quad (3)$$

### 2.1. Direct numerical simulation

Our goal is to study the statistical properties of homogeneous and isotropic MHD turbulence, so we use periodic boundary conditions and a standard pseudospectral method [44] with  $N^3$  collocation points in a cubical simulation domain with sides of length  $L = 2\pi$ ; thus, we evaluate spatial derivatives in Fourier space and local products of fields in real space. We use the 2/3 dealiasing method [44] to remove aliasing errors; after this dealiasing,  $k_{\max}$  is the magnitude of the largest-magnitude wave vectors resolved in our DNS studies. We have carried out extensive simulations with  $N = 512$  and  $N = 1024$ ; the parameters that we use for different runs are given in table 1 for decaying and statistically steady turbulence.

We use a second-order, slaved, Adams–Bashforth scheme, with a time step  $\delta t$ , for the time evolution of the velocity and magnetic fields; this time step is chosen such that the Courant–Friedrichs–Lowy (CFL) condition is satisfied [45].

In our decaying-MHD-turbulence studies, we have taken the initial (superscript 0) energy spectra  $E_u^0(k)$  and  $E_b^0(k)$ , for velocity and magnetic fields, respectively, to be the same; specifically, we have chosen

$$E_u^0(k) = E_b^0(k) = E^0 k^4 \exp(-2k^2), \quad (4)$$

where  $E^0$ , the initial amplitude, is chosen in such a way that we resolve both fluid and magnetic dissipation scales  $\eta_d^u$  and  $\eta_d^b$ , respectively: in all, except for a few, of our runs,  $k_{\max} \eta_d^u \gtrsim 1$  and  $k_{\max} \eta_d^b \gtrsim 1$ . The initial phases of the Fourier components of the velocity and magnetic fields are taken to be different and chosen such that they are distributed randomly and uniformly between 0 and  $2\pi$ . In such studies, it is convenient to pick a reference time at which various statistical

**Table 1.** List of parameters for our 16 DNS runs R1–R5, R3B–R5B, R1C–R4C and R1D–R4D:  $N^3$  is the number of collocation points in our simulation,  $\nu$  is the kinematic viscosity,  $Pr_M$  is the magnetic Prandtl number,  $\delta t$  is the time step; and  $u_{\text{rms}}$ ,  $\ell_I$ ,  $\lambda$  and  $Re_\lambda$  are the root-mean-square velocity, the integral scale, the Taylor microscale and the Taylor-microscale Reynolds number, respectively. These are obtained at  $t_c$  for our decaying-MHD turbulence runs R1–R5, R3B–R5B and R1C–R4C; and for statistically steady MHD turbulence (runs R1D–R4D), these are averaged over the statistically steady state; here,  $t_c$  (iteration steps multiplied by  $\delta t$ ) is the time at which the cascades for both the fluid and the magnetic fields are completed (see text);  $\eta_d^u$  and  $\eta_d^b$  are, respectively, the Kolmogorov dissipation length scales for the fluid and magnetic fields.  $k_{\max}$  is the magnitude of the largest-magnitude wave vectors resolved in our DNS studies which use the 2/3 dealiasing rule;  $k_{\max} \simeq 170.67$  and 341.33 for  $N = 512$  and 1024, respectively.

Runs	$N$	$\nu$	$Pr_M$	$\delta t$	$u_{\text{rms}}$	$\ell_I$	$\lambda$	$Re_\lambda$	$t_c$	$k_{\max} \eta_d^u$	$k_{\max} \eta_d^b$
R1	512	$10^{-4}$	0.1	$10^{-3}$	0.34	0.65	0.18	610	9.7	0.629	2.280
R2	512	$5 \times 10^{-4}$	0.5	$10^{-3}$	0.34	0.67	0.27	187	9.1	1.752	2.389
R3	512	$10^{-3}$	1	$10^{-3}$	0.34	0.70	0.35	121	8.1	2.772	2.444
R4	512	$5 \times 10^{-3}$	5	$10^{-3}$	0.33	0.76	0.60	39	7.0	8.224	2.692
R5	512	$10^{-2}$	10	$10^{-3}$	0.31	0.80	0.73	23	6.5	13.267	2.836
R3B	512	$10^{-3}$	1	$10^{-4}$	1.07	0.62	0.20	210	3.1	1.175	1.052
R4B	512	$5 \times 10^{-3}$	5	$10^{-4}$	2.32	0.63	0.24	110	1.4	1.961	0.644
R5B	512	$10^{-2}$	10	$10^{-4}$	3.21	0.63	0.26	85	1.0	2.490	0.520
R1C	1024	$10^{-4}$	0.01	$10^{-4}$	0.35	0.65	0.23	810	8.0	1.431	22.12
R2C	1024	$10^{-4}$	0.1	$10^{-4}$	1.11	0.47	0.08	890	2.9	0.472	1.690
R3C	1024	$10^{-3}$	1	$10^{-4}$	1.14	0.49	0.15	172	2.5	1.996	1.779
R4C	1024	$10^{-2}$	10	$10^{-4}$	2.37	0.51	0.24	57	1.1	5.550	1.164
R1D	512	$10^{-4}$	0.01	$10^{-4}$	1.31	0.82	0.18	2367	—	0.320	5.364
R2D	512	$10^{-4}$	0.1	$10^{-4}$	0.99	0.74	0.14	1457	—	0.334	1.145
R3D	512	$10^{-3}$	1	$10^{-4}$	1.06	0.65	0.17	239	—	1.264	1.033
R4D	512	$10^{-2}$	10	$10^{-4}$	1.04	0.67	0.23	61	—	6.505	1.129

properties can be compared. One such reference time is the peak that occurs in a plot of the energy dissipation versus time; this reference time has been used in studies of decaying fluid turbulence [46, 47], decaying fluid turbulence with polymer additives [48, 49] and decaying MHD turbulence [25, 26, 50]. Such peaks are associated with the completion of the energy cascade from large length scales, at which energy is injected into the system, to small length scales, at which viscous losses are significant. In the MHD case, these peaks occur at slightly different times,  $t_u$  and  $t_b$ , respectively, in plots of the kinetic ( $\epsilon_u$ ) and magnetic ( $\epsilon_b$ ) energy-dissipation rates. In our decaying-MHD-turbulence studies, we store velocity and magnetic fields at time  $t_c$ ; if  $t_u > t_b$ ,  $t_c = t_u$ ; and  $t_c = t_b$  otherwise; from these fields we calculate the statistical properties that we present in the next section.

In the simulations in which we force the MHD equations to obtain a nonequilibrium statistically steady state (NESS), we use a generalization of the constant-energy-injection method described in [51]. We do not force the magnetic field directly, so we choose  $f_b = 0$ .

The force  $f_u(\mathbf{x}, t)$  is specified most simply in terms of  $\tilde{f}_u(\mathbf{k}, t)$ , its spatial Fourier components, as follows,

$$\tilde{f}_u(\mathbf{k}, t) = \frac{\mathcal{P}\Theta(k_f - k)}{2E_u(k_f, t)}\tilde{\mathbf{u}}(\mathbf{k}, t), \quad (5)$$

where  $\Theta(k_f - k)$  is 1 if  $k \leq k_f$  and 0 otherwise,  $\mathcal{P}$  is the power input, and  $E_u(k_f, t) = \sum_{k \leq k_f} E_u(\mathbf{k}, t)$ ; in our DNS we use  $k_f = 2$ . This yields a statistically steady state in which the mean value of the total energy dissipation rate per unit volume balances the power input, i.e.

$$\langle \epsilon \rangle = \mathcal{P}; \quad (6)$$

once this state has been established, we save 50 representative velocity- and magnetic-field configurations over  $\simeq 36.08t_I$ ,  $29.29t_I$ ,  $32.61t_I$  and  $30.95t_I$ , for R1D, R2D, R3D and R4D, respectively, where  $t_I = \ell_I/u_{rms}$  is the integral-scale eddy-turnover time. We use these configurations to obtain the statistical properties we describe below.

For decaying MHD turbulence, we have carried out eight simulations with  $512^3$  collocation points and four simulations with  $1024^3$  collocation points. The parameters used in these simulations, which we have organized into three sets, are given in table 1.

In the first set of runs, R1–R5, we set the magnetic diffusivity  $\eta = 10^{-3}$  and use five values of  $\nu$ , namely  $10^{-4}$ ,  $5.0 \times 10^{-4}$ ,  $10^{-3}$ ,  $5.0 \times 10^{-3}$  and  $10^{-2}$ , which yield  $Pr_M = 0.1, 0.5, 1, 5$  and 10. These runs have been designed to study the effects, on decaying MHD turbulence, of an increase in  $Pr_M$ , *with the initial energy held fixed*: in particular, we use  $E_u^0 = E_b^0 \simeq 0.32$  in equation (4) for runs R1–R5. Given that this initial energy and  $\eta$  are both fixed, an increase in  $Pr_M$  leads to a decrease in  $Re$  and thus an increase in  $k_{\max}\eta_d^u$  and  $k_{\max}\eta_d^b$ , as we discuss in detail later.

In our second set of decaying-MHD-turbulence runs, R3B, R4B and R5B, we increase  $E^0$  in equation (4) as we increase  $\nu$  and thereby  $Pr_M$ , so that  $k_{\max}\eta_d^u \simeq 1$  and  $k_{\max}\eta_d^b \simeq 1$ . Thus, in these runs, the inertial ranges in energy spectra extend over comparable ranges of the wavevector magnitude  $k$ .

Our third set of decaying-MHD-turbulence runs, R1C, R2C, R3C and R4C, uses  $1024^3$  collocation points and  $Pr_M = 0.01, 0.1, 1$  and 10, respectively. By comparing the results of these runs with those of R1–R5, R3B, R4B and R5B, we can check whether our qualitative results depend significantly on the number of collocation points that we use.

We have carried out another set of four runs, R1D, R2D, R3D and R4D, in which we force the MHD equations, as described above, until we obtain a NESS. These runs help us to compare the statistical properties of decaying and statistically steady turbulence. In these runs, we use  $512^3$  collocation points and  $\nu$  and  $\eta$  such that  $Pr_M = 0.01, 0.1, 1$  and 10, respectively.

## 2.2. Statistical measures

We use several statistical measures to characterize homogeneous, isotropic MHD turbulence. Some, but not all, of these have been used in earlier DNS studies [25, 29, 35, 38], [52]–[54] and solar-wind turbulence [55]–[57].

We calculate the kinetic, magnetic and total energy spectra  $E_u(k) = \sum_{\mathbf{k}:|\mathbf{k}|=k} |\tilde{\mathbf{u}}(\mathbf{k})|^2$ ,  $E_b(k) = \sum_{\mathbf{k}:|\mathbf{k}|=k} |\tilde{\mathbf{b}}(\mathbf{k})|^2$  and  $E(k) = E_u(k) + E_b(k)$ , respectively, the kinetic, magnetic and total energies  $E_u = \sum_k E_u(k)/2$ ,  $E_b = \sum_k E_b(k)/2$  and  $E = E_u + E_b$  and the ratio  $E_b/E_u$ .

Spectra for the Elsässer variables, energy dissipation rates and the effective pressure are, respectively,  $E_{z^\pm}(k) = \sum_{\mathbf{k} \ni |\mathbf{k}|=k} |\tilde{\mathbf{z}}^\pm(\mathbf{k})|^2$ ,  $\epsilon_u(k) = v k^2 E_u(k)$ ,  $\epsilon_b(k) = v k^2 E_b(k)$  and  $P(k) = \sum_{\mathbf{k} \ni |\mathbf{k}|=k} |\tilde{\mathbf{p}}(\mathbf{k})|^2$ .

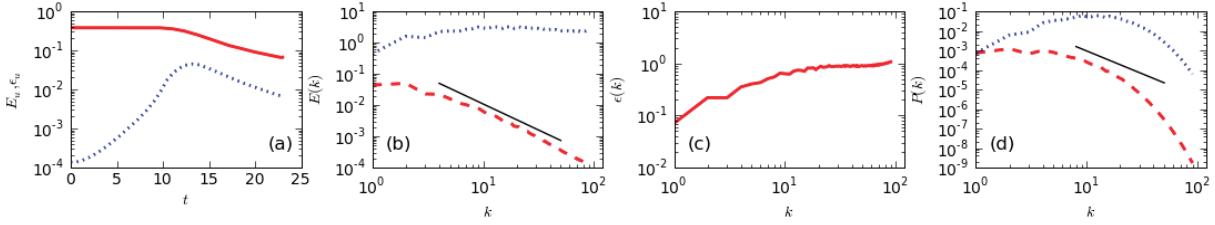
Our MHD simulations are characterized by the Taylor-microscale Reynolds number  $Re_\lambda = u_{\text{rms}} \lambda / v$ , the magnetic Taylor-microscale Reynolds number  $Rm_\lambda = u_{\text{rms}} \lambda / \eta$  and the magnetic Prandtl number  $Pr_M = Rm_\lambda / Re_\lambda = v / \eta$ , where the root-mean-square velocity  $u_{\text{rms}} = \sqrt{2E_u/3}$  and the Taylor microscale  $\lambda = [\sum_k k^2 E(k)/E]^{-1/2}$ . We also calculate the integral length scale  $\ell_I = [\sum_k E(k)/k]/E$ , the mean kinetic energy dissipation rate per unit mass,  $\epsilon_u = v \sum_{i,j} (\partial_i u_j + \partial_j u_i)^2 = v \sum_k k^2 E_u(k)$ , the mean magnetic energy dissipation rate per unit mass  $\epsilon_b = \eta \sum_{i,j} (\partial_i b_j + \partial_j b_i)^2 = \eta \sum_k k^2 E_b(k)$ , the mean energy dissipation rate per unit mass  $\epsilon = \epsilon_u + \epsilon_b$  and the dissipation length scales for velocity and magnetic fields  $\eta_d^u = (v^3/\epsilon_u)^{1/4}$  and  $\eta_d^b = (\eta^3/\epsilon_b)^{1/4}$ , respectively.

We calculate the eigenvalues  $\Lambda_n^u$  and the associated eigenvectors  $\hat{e}_n^u$ , with  $n = 1, 2$  or  $3$ , of the rate-of-strain tensor  $\mathbb{S}$  whose components are  $S_{ij} = \partial_i u_j + \partial_j u_i$ . Similarly  $\Lambda_1^b$ ,  $\Lambda_2^b$  and  $\Lambda_3^b$  denote the eigenvalues of the tensile magnetic stress tensor  $\mathbb{T}$ , which has components  $T_{ij} = -b_i b_j$ ; the corresponding eigenvectors are, respectively,  $\hat{e}_1^b$ ,  $\hat{e}_2^b$  and  $\hat{e}_3^b$ .

For incompressible flows  $\sum_n \Lambda_n^u = 0$ , so at least one of the eigenvalues  $\Lambda_n^u$  must be positive and another negative; we label them in such a way that  $\Lambda_3^u$  is positive,  $\Lambda_1^u$  is negative and  $\Lambda_2^u$  lies in between them; note that  $\Lambda_2^u$  can be positive or negative. We obtain PDFs of these eigenvalues; furthermore, we obtain PDFs of the cosines of the angles that the associated eigenvectors make with vectors such as  $\mathbf{u}$ ,  $\boldsymbol{\omega}$ , etc. These PDFs and those of quantities such as the local cross helicity  $H_C = \mathbf{u} \cdot \mathbf{b}$  help us to quantify the degree of alignment of pairs of vectors such as  $\mathbf{u}$  and  $\mathbf{b}$  [53]. We also compare PDFs of magnitudes of local vorticity  $\omega$ , the current density  $j$  and local energy dissipation rates  $\epsilon_u$  and  $\epsilon_b$  to obtain information about intermittency in velocity and magnetic fields.

We also obtain several interesting joint PDFs; to the best of our knowledge, these have not been obtained earlier for MHD turbulence. We first obtain the velocity-derivative tensor  $\mathbb{A}$ , also known as the rate-of-deformation tensor, with components  $A_{ij} = \partial_i u_j$ , and then the invariants  $Q = -\frac{1}{2}\text{tr}(\mathbb{A}^2)$  and  $R = -\frac{1}{3}\text{tr}(\mathbb{A}^3)$ , which have been used frequently to characterize fluid turbulence [58]–[60]. The zero-discriminant line  $D \equiv \frac{27}{4}R^2 + Q^3 = 0$  and the  $Q$  and  $R$  axes divide the  $QR$  plane into qualitatively different regimes. In particular, regions in a turbulent flow can be classified as follows: when  $Q$  is large and negative, local strains are high and vortex formation is not favoured; furthermore, if  $R > 0$ , fluid elements experience axial strain, whereas if  $R < 0$ , they feel biaxial strain. In contrast, when  $Q$  is large and positive, vorticity dominates the flow; if, in addition,  $R < 0$ , vortices are compressed, whereas if  $R > 0$ , they are stretched. Thus, some properties of a turbulent flow can be highlighted by making contour plots of the joint PDF of  $Q$  and  $R$ ; these  $QR$  plots show a characteristic, tear-drop shape. We explore the forms of these and other joint PDFs, such as joint PDFs of  $\epsilon_u$  and  $\epsilon_b$ , in MHD turbulence.

To characterize intermittency in MHD turbulence, we calculate the order- $p$ , equal-time, longitudinal structure functions  $S_p^a(l) \equiv \langle |\delta a_{\parallel}(\mathbf{x}, l)|^p \rangle$ , where the longitudinal component of the field  $\mathbf{a}$  is given by  $\delta a_{\parallel}(\mathbf{x}, l) \equiv [\mathbf{a}(\mathbf{x}+\mathbf{l}, t) - \mathbf{a}(\mathbf{x}, t)] \cdot \frac{1}{l}$ , where  $\mathbf{a}$  can be  $\mathbf{u}$ ,  $\mathbf{b}$  or one of the Elsässer variables. From these structure functions, we also obtain the hyperflatness  $F_6^a(l) = S_6^a(l)/[S_2^a(l)]^3$ . For separations  $l$  in the inertial range, i.e.  $\eta_d^u, \eta_d^b \ll l \ll L$ , we expect  $S_p^a(l) \sim l^{\zeta_p^a}$ , where  $\zeta_p^a$  are the inertial-range multiscaling exponents for the field  $\mathbf{a}$ ; the Kolmogorov phenomenology of 1941 [23]–[25], henceforth referred to as K41, yields the simple scaling



**Figure 1.** Plots from our DNS of decaying fluid turbulence in the Navier–Stokes equation with  $512^3$  collocation points. (a) Plots of the energy  $E$  (red full line) and mean energy dissipation rate  $\epsilon$  (blue dotted line) versus time  $t$  (given as a product of the number of iterations and the time step  $\delta t$ ). (b) Log–log (base 10) plots of the energy spectrum  $E(k)$  (red dashed line) and the corresponding compensated spectrum  $k^{5/3}E(k)$  (blue dotted line) versus  $k$ . The black solid line shows the K41 result  $k^{-5/3}$  for comparison. (c) Log–log (base 10) plot of the spectrum of the energy-dissipation spectrum (or enstrophy spectrum)  $\epsilon(k)$ . (d) Log–log (base 10) plots of the pressure spectrum  $P(k)$  (red dashed line) and the compensated pressure spectrum  $k^{7/3}P(k)$  (blue dotted line). The black solid line shows the K41 result  $k^{-7/3}$  for comparison.

result  $\zeta_p^{aK41} = p/3$ ; but multiscaling corrections are significant with  $\zeta_p^a \neq \zeta_p^{aK41}$  (section 3). From the increments  $\delta a_{\parallel}(\mathbf{x}, l) \equiv [\mathbf{a}(\mathbf{x}+\mathbf{l}, t) - \mathbf{a}(\mathbf{x}, t)] \cdot \frac{\mathbf{l}}{l}$ , we also obtain the dependence of PDFs of  $\delta a_{\parallel}$  on the scale  $l$ .

### 3. Results

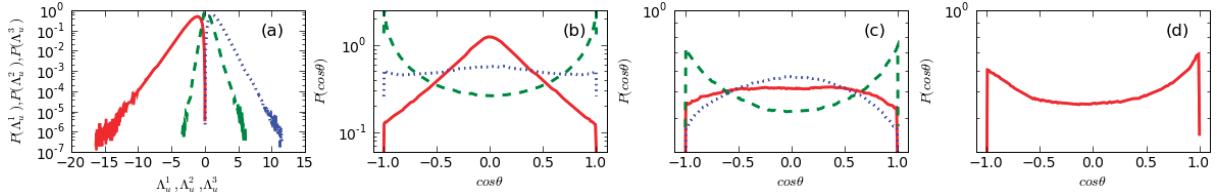
To set the stage for the types of studies we carry out for MHD turbulence, we begin with a very brief summary of similar and well-known results from studies of homogeneous, isotropic Navier–Stokes turbulence, which can be found, e.g., in [24, 46, 47], [61]–[67].

#### 3.1. Overview of fluid turbulence

For ready reference, we show here illustrative plots from a DNS study that we have carried out for the three-dimensional Navier–Stokes equation by using a pseudospectral method, with  $512^3$  collocation points and the  $2/3$  rule for removing aliasing errors; here,  $\nu = 0.001$ ,  $Re_{\lambda} \simeq 340$  and  $k_{\max} \eta_d^u \simeq 0.3$ .

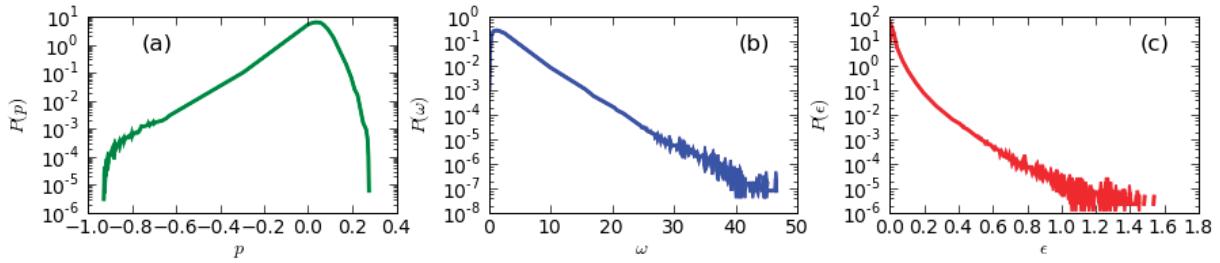
In decaying fluid turbulence, energy is injected at large spatial scales, as described in the previous section for the MHD case. This energy cascades down till it reaches the dissipative scale at which viscous losses are significant. We study various statistical properties; these are given in points (i)–(vi) below.

(i) Plots of the energy  $E$  and the mean energy dissipation rate  $\epsilon$  versus time show, respectively, a gentle decay and a maximum, as shown, e.g., by the full red and dotted blue curves in figure 1(a). This maximum in  $\epsilon$  is associated with the completion of the energy cascade at a time  $t_c$ ; the remaining properties (ii)–(vi) are obtained at  $t_c$ . (ii) If  $Re_{\lambda}$  is sufficiently large and we have a well-resolved DNS (i.e.  $k_{\max} \eta_d^u > 1$ ), then at  $t_c$ , the spectrum  $E(k)$  shows a well-developed inertial range, where at the K41 level  $E(k) \sim k^{-5/3}$ , and a dissipation range,

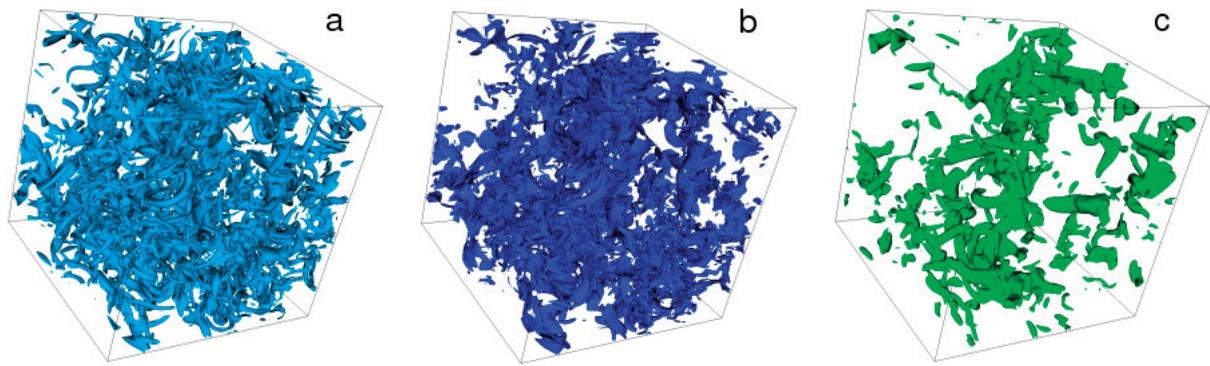


**Figure 2.** PDFs from our DNS of decaying fluid turbulence in the Navier–Stokes equation with  $512^3$  collocation points. (a) Semilog (base 10) plots of PDFs of eigenvalues of the rate-of-strain tensor  $\mathbb{S}$ , namely  $\Lambda_1^u$  (red full line),  $\Lambda_2^u$  (green dashed line) and  $\Lambda_3^u$  (blue dotted line). (b) Semilog (base 10) plots of PDFs of cosines of angles between the vorticity  $\omega$  and eigenvectors of  $\mathbb{S}$ , namely  $\hat{e}_1^u$  (red full line),  $\hat{e}_2^u$  (green dashed line) and  $\hat{e}_3^u$  (blue dotted line). (c) Semilog (base 10) plots of PDFs of cosines of angles between the velocity  $\mathbf{u}$  and eigenvectors of  $\mathbb{S}$ , namely  $\hat{e}_1^u$  (red full line),  $\hat{e}_2^u$  (green dashed line) and  $\hat{e}_3^u$  (blue dotted line). (d) Semilog (base 10) plots of PDFs of cosines of angles between the velocity  $\mathbf{u}$  and vorticity  $\omega$ .

in which the behavior of the energy spectrum is consistent with  $E(k) \sim k^\alpha \exp(-\beta k)$ , where  $\alpha$  and  $\beta$  are non-universal, positive constants [62, 65] and  $5k_d < k < 10k_d$ , with  $k_d = 1/\eta_d^u$ . An illustrative energy spectrum is shown by the dashed red line in figure 1(b); the blue dotted curve shows the compensated spectrum  $k^{5/3}E(k)$ ; the associated dissipation or enstrophy spectrum  $\epsilon(k)$  is shown in figure 1(c) and the inertial-range pressure spectrum [68],  $P(k) \sim k^{-7/3}$  at the K41 level, is shown in figure 1(d). (Note that our DNS for the Navier–Stokes equation, which suffices for our purposes of illustration, does not have a well-resolved dissipation range because  $k_{\max}\eta_d^u \simeq 0.3 < 1$ ; this is also reflected in the lack of a well-developed maximum in the enstrophy spectrum of figure 1(c).) (iii) Illustrative PDFs of the eigenvalues  $\Lambda_n^u$  of the rate-of-strain tensor  $\mathbb{S}$  are given for  $n = 1, 2$  and  $3$ , respectively, by the full red, dashed green and dotted blue curves in figure 2(a); PDFs of the cosines of the angles that the vorticity  $\omega$  and the velocity  $\mathbf{u}$  make with the associated eigenvectors  $\hat{e}_n^u$  are given, respectively, in figures 2(b) and (c) via full red ( $n = 1$ ), dashed green ( $n = 2$ ) and dotted blue ( $n = 3$ ) curves; these show that both  $\omega$  and  $\mathbf{u}$  have a tendency to be preferentially aligned parallel or antiparallel to  $\hat{e}_2^u$  [60]; the PDF of the cosine of the angle between  $\mathbf{u}$  and  $\omega$  also indicates preferential alignment or antialignment of these two vectors, but with a greater tendency towards alignment, as found in experiments with a small amount of helicity [69] and as illustrated in figure 2(d). Finally, we give representative PDFs of the pressure  $p$ , the modulus of vorticity  $\omega = |\omega|$  and the local energy dissipation  $\epsilon$  in figures 3(a)–(c), respectively; note that the PDF of the pressure is negatively skewed. (iv) Inertial-range structure functions  $S_p^u(l) \sim l^{\zeta_p^u}$  show significant deviations [24] from the K41 result  $\zeta_p^{uK41} = p/3$ , especially for  $p > 3$ . From these structure functions, we can obtain the hyperflatness  $F_6^u(l)$ ; this increases as the length scale  $l$  decreases, a clear signature of intermittency, as shown, e.g., in [49, 66]. This intermittency also leads to non-Gaussian tails, especially for small  $l$ , in PDFs of velocity increments (see e.g. [66, 70, 71]), such as  $\delta u_{\parallel}(l)$ . (v) Small-scale structures in turbulent flows can be visualized via isosurfaces [72] of, say,  $\omega$ ,  $\epsilon$  and  $p$ , illustrative plots of which are given in figures 4(a)–(c); these show that regions of large  $\omega$  are organized into slender tubes, whereas isosurfaces of  $\epsilon$  look like shredded sheets; pressure isosurfaces also show tubes [37, 47] but some studies have suggested the term ‘cloud-like’ for



**Figure 3.** PDFs from our DNS of decaying fluid turbulence in the Navier–Stokes equation with  $512^3$  collocation points. Semilog (base 10) plots of the PDFs of (a) the pressure  $p$ , (b) the modulus of vorticity  $\omega$  and (c) the local energy-dissipation rate  $\epsilon$ .



**Figure 4.** Isosurfaces of (a) the modulus of vorticity  $\omega$ , (b) the local energy-dissipation rate  $\epsilon$  and (c) the local pressure  $p$ , from our DNS of decaying fluid turbulence in the Navier–Stokes equation with  $512^3$  collocation points. The isovalues used in these plots are two standard deviations more than the mean values of the quantities.

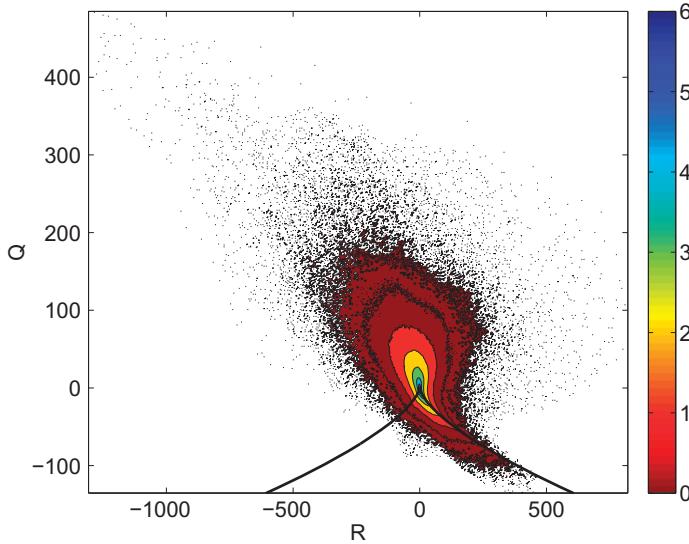
them [61]. (vi) Joint PDFs also provide useful information about turbulent flows; in particular, contour plots of the joint PDF of  $Q$  and  $R$ , as in the representative figure 5, show a characteristic tear-drop structure.

The properties of statistically steady, homogeneous, isotropic fluid turbulence are similar to those described in points (ii)–(vi) in the preceding paragraph for the case of decaying fluid turbulence at cascade completion at  $t_c$ . In particular, the strong-universality [42] hypothesis suggests that the multiscaling exponents  $\zeta_p^u$  have the same values in decaying and statistically steady turbulence.

The remaining part of this section is devoted to our detailed study of the MHD-turbulence analogues of the properties (i)–(vi) summarized above; these are discussed, respectively, in the six sections 3.2–3.7.

### 3.2. Temporal evolution

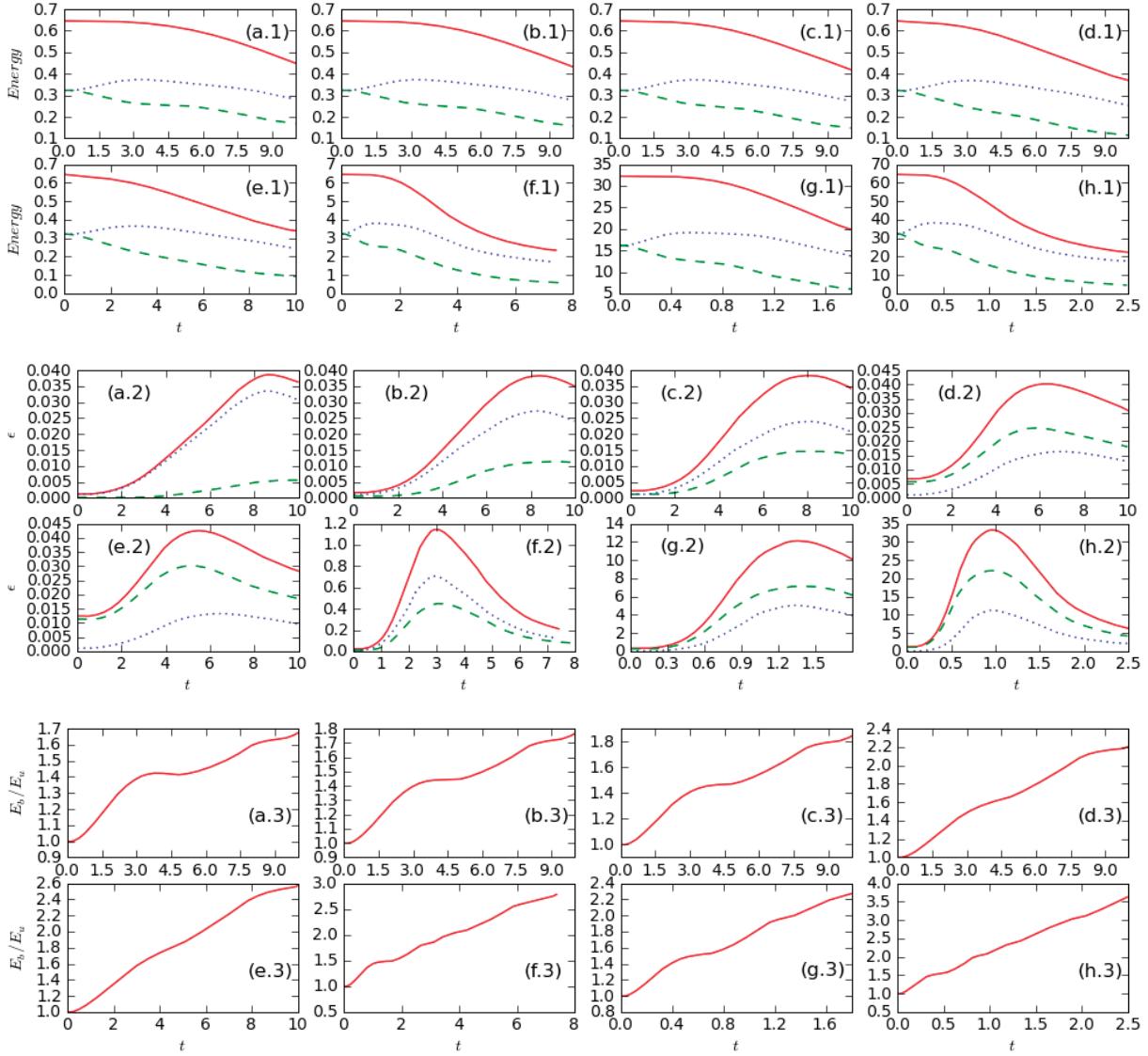
We examine the time evolution of the energy, the energy-dissipation rates and related quantities, first for decaying and then for statistically steady MHD turbulence.



**Figure 5.**  $QR$  plot, i.e. the joint PDF of  $Q$  and  $R$  (see text) shown as a filled contour plot in our log–log (base 10) scale, obtained from a DNS of decaying fluid turbulence in the Navier–Stokes equation with  $512^3$  collocation points.

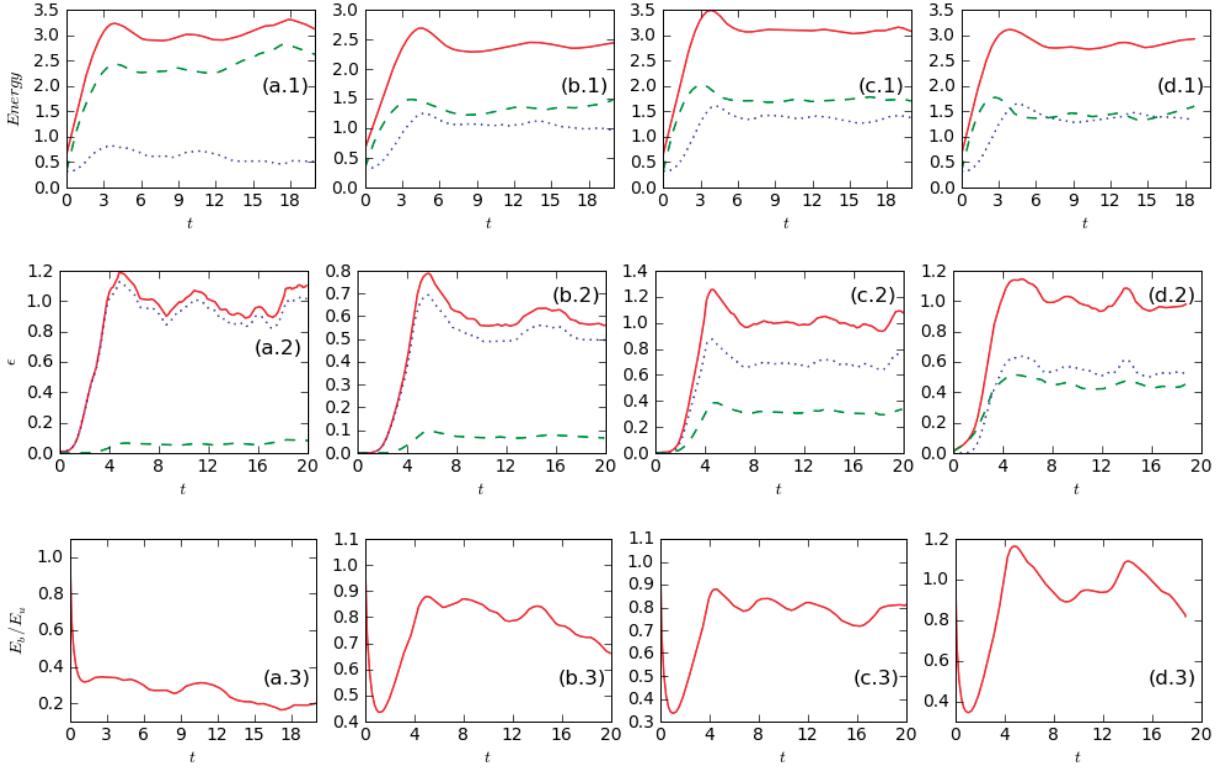
Figure 6 shows how the total energy  $E$  (red full line), the kinetic energy  $E_u$  (green dashed line) and the magnetic energy  $E_b$  (blue dotted line) evolve with time  $t$  (given as a product of the number of iterations and the time step  $\delta t$ ) for runs R1–R5 (figures 6(a.1)–(e.1)) and runs R3B–R5B (figures 6(f.1)–(h.1)) for decaying MHD turbulence. Figure 6 also shows similar plots for the mean energy dissipation rate  $\epsilon$  (red full line), the mean kinetic-energy dissipation rate  $\epsilon_u$  (green dashed line) and the mean magnetic-energy dissipation rate  $\epsilon_b$  (blue dotted line) versus time  $t$  for runs R1–R5 (figures 6(a.2)–(e.2)) and runs R3B–R5B (figures 6(f.2)–(h.2)). In addition, figure 6 depicts the time evolution of the ratio  $E_b/E_u$  for runs R1–R5 (figures 6(a.3)–(e.3)) and runs R3B–R5B (figures 6(f.3)–(h.3)). We see from these figures that, for all the values of  $Pr_M$  we have used, the energies  $E$  and  $E_u$  decay gently with  $t$  but  $E_b$  rises initially such that the ratio  $E_b/E_u$  rises, nearly monotonically, with  $t$  over the times we have considered; this is an intriguing trend that does not seem to have been noted earlier. The times over which we have carried out our DNS are comparable to the cascade-completion time  $t_c$  that can be obtained from the peaks in the plots of  $\epsilon$ ,  $\epsilon_u$  and  $\epsilon_b$  versus  $t$  (figures 6(a.2)–(h.2)); by comparing these plots we see that, as we move from  $Pr_M = 0.1$  to  $Pr_M = 10$ , with fixed  $\eta$ , we find that  $(\epsilon_u - \epsilon_b)$  and  $(t_b - t_u)$  grow from negative values to positive ones because  $\epsilon_u$  increases with  $Pr_M$ , where  $t_b$  and  $t_u$  are the positions of the cascade-completion maxima in  $\epsilon_b$  and  $\epsilon_u$ , respectively. We do not pursue the time evolution of our system well beyond  $t_u$  and  $t_b$  because the integral scale begins to grow thereafter and, eventually, can become comparable to the linear size of the simulation domain [46].

Figures 7(a.1)–(d.1) show how the total energy  $E$  (red full line), the total kinetic energy  $E_u$  (green dashed line) and the total magnetic energy  $E_b$  (blue dotted line) evolve with time  $t$  (given as a product of the number of iterations and the time step  $\delta t$ ) for, respectively, runs R1D–R4D for forced and statistically steady MHD turbulence. Figures 7(a.2)–(d.2) show similar plots for the mean energy dissipation rate  $\epsilon$  (red full line), the mean kinetic-energy dissipation rate



**Figure 6.** Plots versus time  $t$  (given as a product of the number of iterations and the time step  $\delta t$ ) of energies (a.1)–(h.1): total energy  $E$  (red full line), kinetic energy  $E_u$  (green dashed line) and magnetic energy  $E_b$  (blue dotted line); of energy-dissipation rates (a.2)–(h.2): mean energy dissipation rate  $\epsilon$  (red full line), kinetic-energy dissipation  $\epsilon_u$  (green dashed line) and magnetic-energy dissipation rate  $\epsilon_b$  (blue dotted line); and of the ratio  $E_b/E_u$  (a.3)–(h.3), generically, for decaying simulations (a)  $Pr_M = 0.1$  (R1), (b)  $Pr_M = 0.5$  (R2), (c)  $Pr_M = 1.0$  (R3), (d)  $Pr_M = 5.0$  (R4), (e)  $Pr_M = 10.0$  (R5), (f)  $Pr_M = 1.0$  (R3B), (g)  $Pr_M = 5.0$  (R4B) and (h)  $Pr_M = 10.0$  (R5B).

$\epsilon_u$  (green dashed line) and the mean magnetic-energy dissipation rate  $\epsilon_b$  (blue dotted line) versus time  $t$  for, respectively, runs R1D–R4D. Figures 7(a.3)–(d.3) depict the time evolution of the ratio  $E_b/E_u$  for these runs. We see from these figures that a statistically steady state is established in which the energies  $E$ ,  $E_u$  and  $E_b$ , the dissipation rates  $\epsilon$ ,  $\epsilon_u$  and  $\epsilon_b$  and the



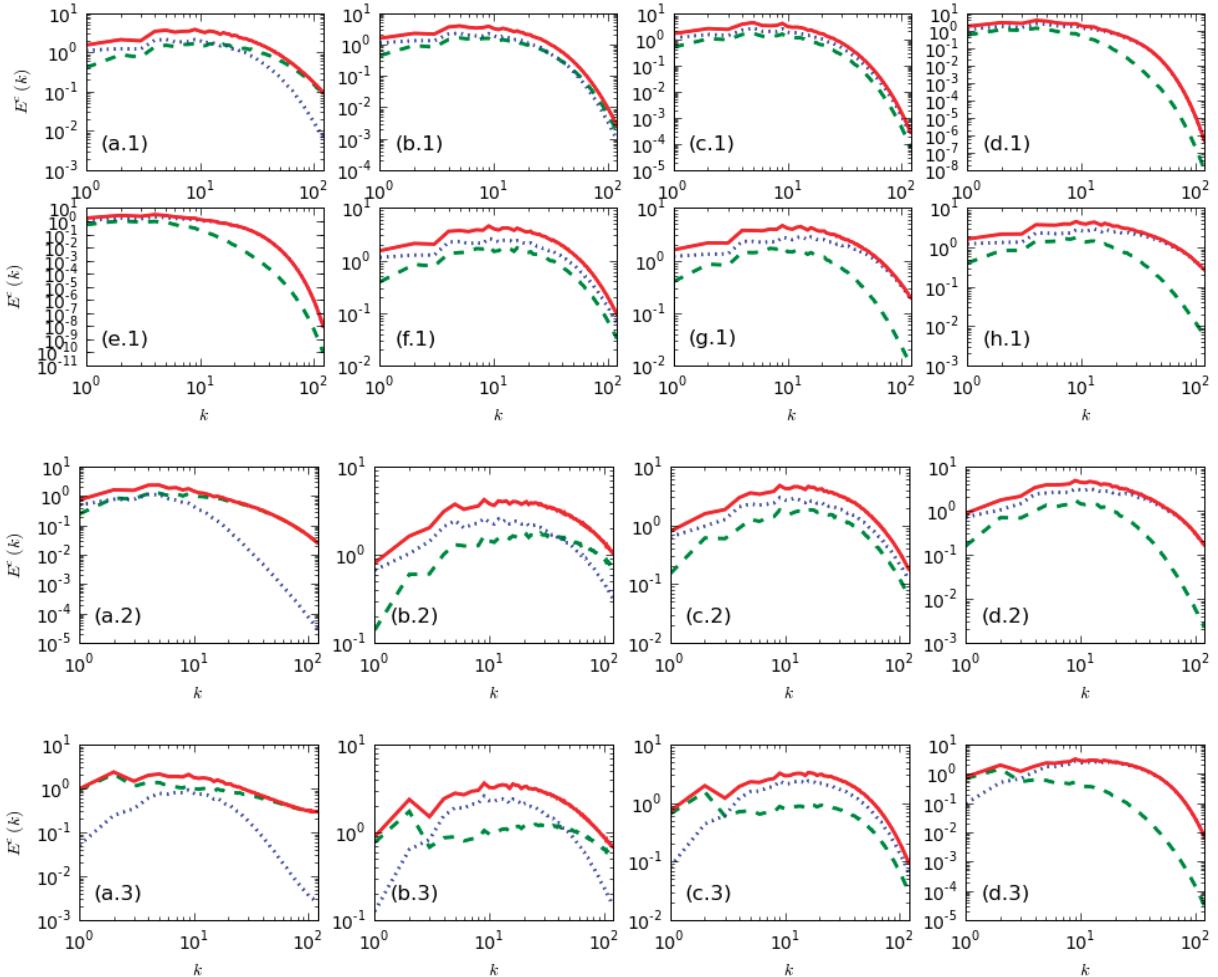
**Figure 7.** Plots versus time  $t$  (given as a product of the number of iterations and the time step  $\delta t$ ) of energies (a.1)–(d.1): total energy  $E$  (red full line), kinetic energy  $E_u$  (green dashed line) and magnetic energy  $E_b$  (blue dotted line); of energy-dissipation rates (a.2)–(d.2): mean energy dissipation rate  $\epsilon$  (red full line), kinetic-energy dissipation rate  $\epsilon_u$  (green dashed line); and magnetic-energy dissipation rate  $\epsilon_b$  (blue dotted line) and of the ratio  $E_b/E_u$  (a.3)–(d.3), generically, for forced simulations (a)  $Pr_M = 0.01$  (R1D), (b)  $Pr_M = 0.1$  (R2D), (c)  $Pr_M = 1.0$  (R3D) and (d)  $Pr_M = 10$  (R4D).

ratio  $E_b/E_u$  fluctuate about their mean values (after initial transients have died out). The mean value of  $E_b/E_u$  increases from about 0.2–0.3 to  $E_b/E_u \simeq 1$  as  $Pr_M$  increases from 0.01 to 10. Furthermore, the mean values of the dissipation rates  $\epsilon_u$  and  $\epsilon_b$  are such that  $(\epsilon_u - \epsilon_b)$  grows from a negative value  $\simeq -1$  to a value close to zero as  $Pr_M$  increases from 0.01 to 10.

### 3.3. Spectra

We now discuss the behaviors of the energy, kinetic-energy, magnetic-energy, Elsässer variable, dissipation-rate and effective-pressure spectra, first for decaying and then for statistically steady MHD turbulence. In the former case, spectra are obtained at the cascade-completion time  $t_c$ ; in the latter, they are averaged over the statistically steady state that we obtain.

We present compensated spectra of the total energy  $E_c(k) = \epsilon^{-2/3} k^{5/3} E(k)$  (red full line), the kinetic energy  $E_c^u(k) = \epsilon^{-2/3} k^{5/3} E_u(k)$  (green dashed line) and the total magnetic energy  $E_c^b(k) = \epsilon^{-2/3} k^{5/3} E_b(k)$  (blue dotted line) at  $t_c$  for runs R1–R5 (figures 8(a.1)–(e.1)), R3B–R5B (figures 8(f.1)–(h.1)) and R1C–R4C (figures 8(a.2)–(d.2)) for decaying MHD turbulence; and



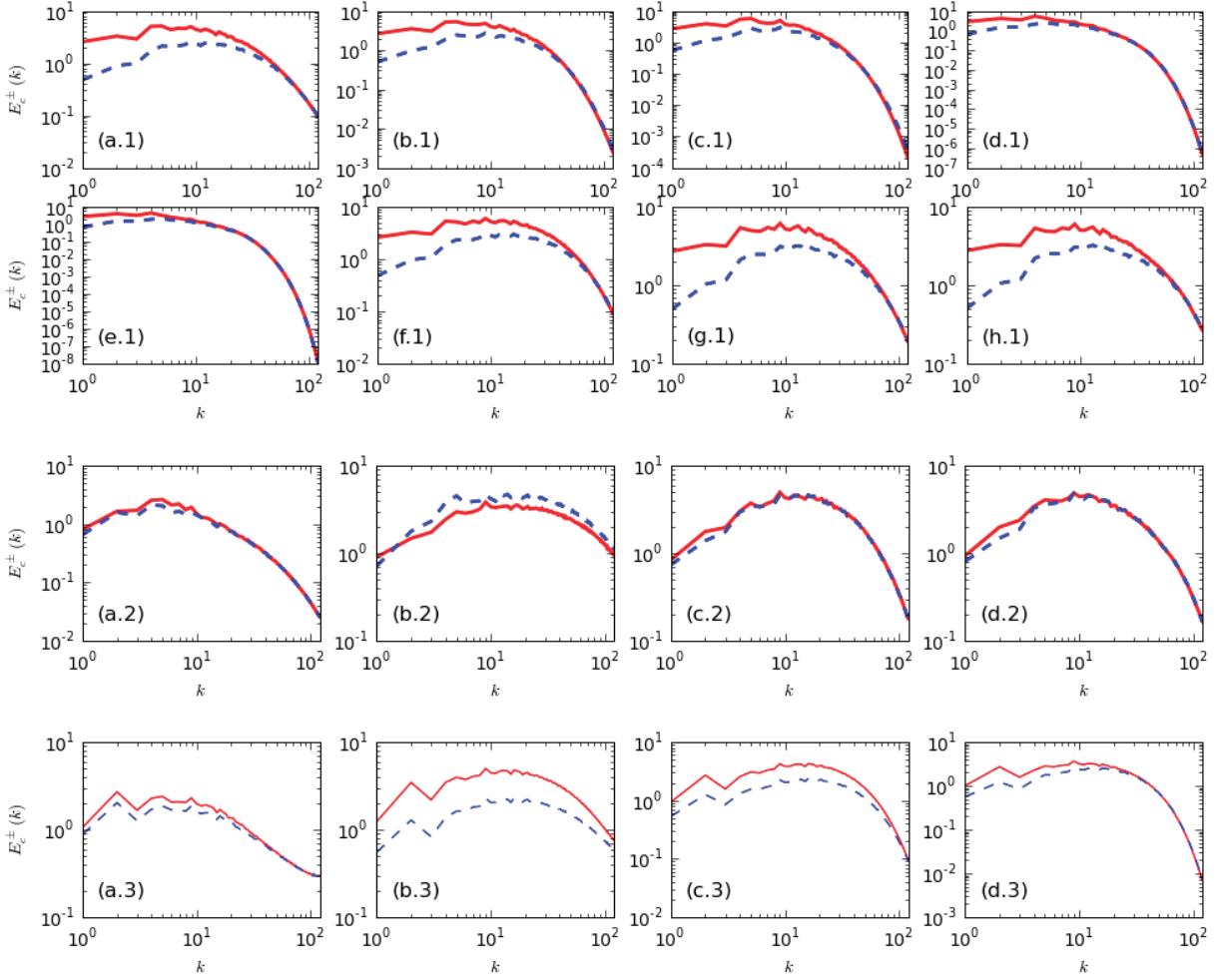
**Figure 8.** Log–log (base 10) plots of the compensated energy spectra  $\epsilon^{-2/3}k^{5/3}E(k)$  (red full lines),  $\epsilon_{-2/3}k^{5/3}E_u(k)$  (green dashed lines) and  $\epsilon_{-2/3}k^{5/3}E_b(k)$  (blue dotted lines); on the vertical axes these are denoted generically as  $E^c(k)$ : (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), (a.2)  $Pr_M = 0.01$  (R1C), (b.2)  $Pr_M = 0.1$  (R2C), (c.2)  $Pr_M = 1.0$  (R3C) and (d.2)  $Pr_M = 10.0$  (R4C) for decaying MHD turbulence; and for statistically steady MHD turbulence (a.3)  $Pr_M = 0.01$  (R1D), (b.3)  $Pr_M = 0.1$  (R2D), (c.3)  $Pr_M = 1.0$  (R3D) and (d.3)  $Pr_M = 10.0$  (R4D).

runs R1D–R4D (figures 8(a.3)–(d.3)) show these for statistically steady MHD turbulence. From figures 8(a.1)–(e.1) and table 1, we see that  $\eta_d^u$  increases as we increase  $\nu$  to increase  $Pr_M$ , because the initial energy is the same for runs R1–R5, so the dissipation tail in  $E_c^u(k)$  is drawn in towards smaller and smaller values of  $k$  as we move from  $Pr_M = 0.1$  to  $Pr_M = 10$ ; between  $Pr_M = 0.5$  and  $Pr_M = 1$ , the tails of  $E_c^u(k)$  and  $E_c^b(k)$  and eventually  $E_c^b(k)$  dominate and become indistinguishable from  $E_c(k)$  on the scales of figures 8(d.1) and (e.1). A comparison of figures 8(f.1)–(h.1) shows that, if we increase  $Pr_M$  from 1 to 10, we can keep both  $k_{\max}\eta_d^u$

and  $k_{\max} \eta_d^b$  close to 1, so the dissipation ranges of these spectra span comparable ranges of  $k$ ; however, as  $Pr_M$  increases, more and more of the energy is concentrated in the magnetic field. These trends are not affected (a) if we increase the number of collocation points, as can be seen from the compensated spectra in figures 8(a.2)–(d.2) for runs R1C–R4C, which use  $1024^3$  collocation points, or (b) if we study energy spectra for statistically steady MHD turbulence, as can be seen from the compensated spectra in figures 8(a.3)–(d.3) for runs R1D–R4D. Figures 8(c.1), (g.1), (c.2) and (c.3), for runs R3 ( $Re_\lambda = 121$ ), R3B ( $Re_\lambda = 210$ ), R3C ( $Re_\lambda = 172$ ) and R3D ( $Re_\lambda = 239$ ), respectively, all but one lie in one column and all have  $Pr_M = 1$ ; so they provide a convenient way of comparing the  $Re_\lambda$  dependence of these spectra with a fixed value of  $Pr_M = 1$ . All of the spectra in the subfigures of figure 8 have been compensated for by the  $5/3$  power of  $k$  and, to the extent that they show small, flat parts, their inertial-range, energy-spectra scalings are consistent with  $k^{-5/3}$  behaviors; other powers, such as  $-3/2$ , can also give small, flat parts in compensated spectra. A detailed error analysis is required to decide which power is most consistent with our data; we defer such an error analysis to section 3.5, where we carry it out for structure functions.

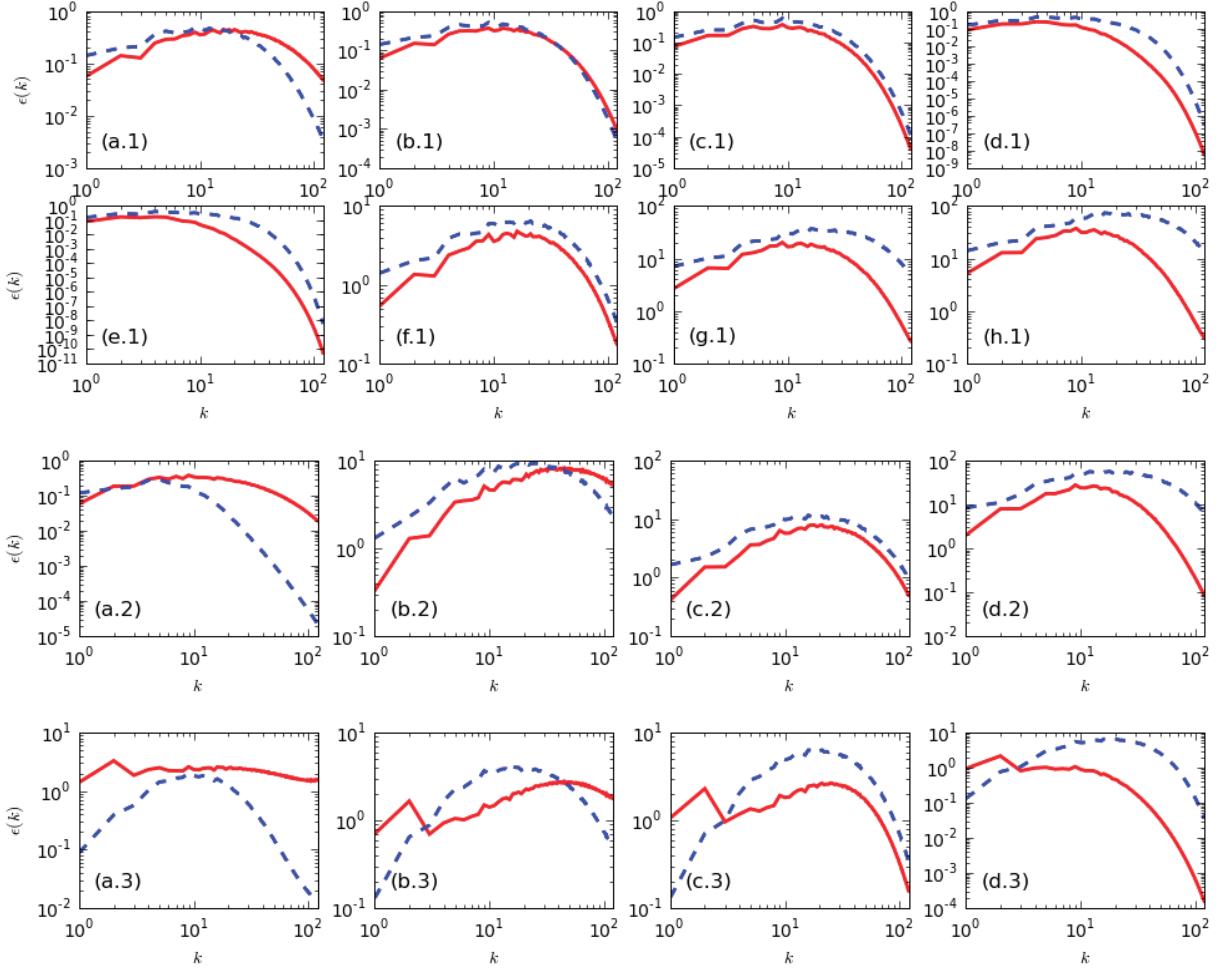
Compensated spectra of the Elsässer variables, namely  $E_c^+(k) = \epsilon^{-2/3} k^{5/3} E^+(k)$  (red full lines) and  $E_c^-(k) = \epsilon_u^{-2/3} k^{5/3} E^-(k)$  (blue dashed lines), are shown, at the cascade-completion time  $t_c$ , for the decaying-MHD turbulence runs R1–R5 in figures 9(a.1)–(e.1), R3B–R5B in figures 9(f.1)–(h.1) and R1C–R4C in figures 9(a.2)–(d.2); and figures 8(a.3)–(d.3) show these spectra for statistically steady MHD turbulence in runs R1D–R4D, respectively. Note that the dissipation ranges of  $E_c^+(k)$  and  $E_c^-(k)$  overlap nearly on the scales of these figures. Differences between these are most pronounced at small  $k$ , where, typically,  $E_c^-(k)$  lies below  $E_c^+(k)$ ; these differences decrease with increasing  $Pr_M$  if we hold the initial energy fixed as in figures 9(a.1)–(e.1) for runs R1–R5.

Next we come to the energy-dissipation (or enstrophy) spectra  $\epsilon_u(k) = k^2 E_u(k)$  (red full line) and  $\epsilon_b(k) = k^2 E_b(k)$  (blue dashed line) at  $t_c$ . These are shown, at the cascade-completion time  $t_c$ , for the decaying-MHD turbulence runs R1–R5 in figures 10(a.1)–(e.1), R3B–R5B in figures 10(f.1)–(h.1) and R1C–R4C in figures 10(a.2)–(d.2); and figures 10(a.3)–(d.3) depict these spectra for statistically steady MHD-turbulence runs R1D–R4D. To the extent that most of these spectra show maxima at values of  $k$  at the beginning of the dissipation range, most of our runs have well-resolved dissipation ranges; this also follows from the values of  $k_{\max} \eta_d^u$  and  $k_{\max} \eta_d^b$  in table 1. Runs R1D and R2D have slightly under-resolved fluid-dissipation ranges with  $k_{\max} \eta_d^u \simeq 0.32$  and 0.33, respectively; and, for the former, a barely discernible, dissipation-range maximum in  $\epsilon_u(k)$ ; however, as shown in our Navier–Stokes DNS in section 3.1, reasonable results can be obtained for various statistical properties with  $k_{\max} \eta_d^u \simeq 0.3$ . The elucidation of the behaviors of dissipation-range spectra of course requires large values of  $k_{\max} \eta_d^u$  or  $k_{\max} \eta_d^b$ ; in particular, runs R5 and R1C, with  $k_{\max} \eta_d^u \simeq 13.3$  and  $k_{\max} \eta_d^b \simeq 22.1$ , respectively, are well suited for uncovering the functional forms of  $E_u(k)$  and  $E_b(k)$  in their dissipation ranges. In figures 11(a) and (b), we show, respectively, the kinetic- and magnetic-energy spectra  $E_u(k)$  and  $E_b(k)$  deep in their dissipation ranges for runs R5 and R1, respectively; our data for these spectra can be fitted to the form  $\sim k^\alpha \exp(-\beta k)$  for  $k$  deep in the dissipation range and  $\alpha$  and  $\beta$  non-universal numbers that depend on the parameters of the simulation; similar results have been obtained for fluid turbulence [62, 65]. In particular, our data (figures 11(a) and (b)) for runs R5 and R1C are consistent with  $E_u(k) \sim k^{2.68} \exp(-0.235k)$ , for  $5k_d^u < k < 10k_d^u$  with  $k_d^u = 1/\eta_d^u$ , and  $E_b(k) \sim k^{-5.24} \exp(-0.014k)$ , for  $5k_d^b < k < 10k_d^b$  with  $k_d^b = 1/\eta_d^b$ , respectively.



**Figure 9.** Log–log (base 10) plots of compensated energy spectra,  $E_c^\pm(k) = k^{5/3} E^\pm(k)$ , with  $k$  being the magnitude of the wave vector, for the Elsässer variables fields  $z^+$  (red full line) and  $z^-$  (blue dashed line): (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), (a.2)  $Pr_M = 0.01$  (R1C), (b.2)  $Pr_M = 0.1$  (R2C), (c.2)  $Pr_M = 1.0$  (R3C) and (d.2)  $Pr_M = 10.0$  (R4C) for decaying MHD turbulence; and for statistically steady MHD turbulence (a.3)  $Pr_M = 0.01$  (R1D), (b.3)  $Pr_M = 0.1$  (R2D), (c.3)  $Pr_M = 1.0$  (R3D) and (d.3)  $Pr_M = 10.0$  (R4D).

We now turn to the spectra for the effective pressure  $P(k)$  (red full lines) and their compensated versions  $k^{7/3} P(k)$  (blue dashed lines) that are shown at  $t_c$  for runs R1–R5 (figures 12(a.1)–(e.1)) and R3B–R5B (figures 12(f.1)–12(h.1)) for decaying MHD turbulence; and for statistically steady MHD turbulence they are shown in figures 12(a.2)–(d.2) for runs R1D–R4D. Pressure spectra have been studied for fluid turbulence as, e.g., in [47, 68]; to the best of our knowledge they have not been obtained for MHD turbulence. The compensated spectra here show that, for all of our runs, the inertial-range behaviors of these effective-pressure spectra are consistent with the power law  $k^{-7/3}$ ; this is consistent with the  $k^{-5/3}$  behaviors of the

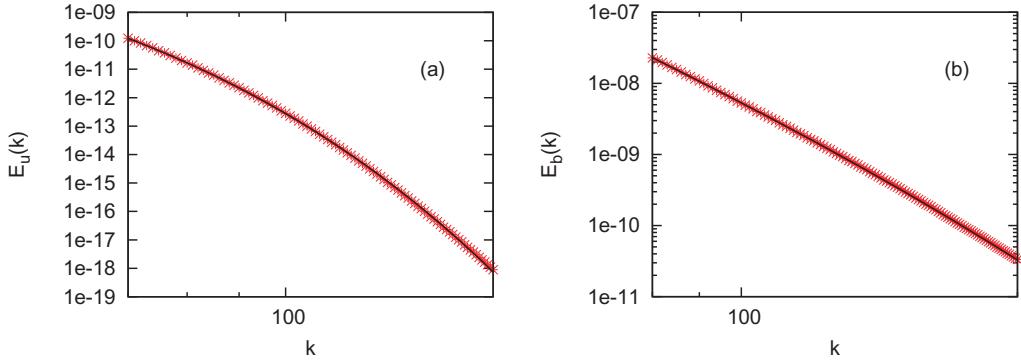


**Figure 10.** Log–log (base 10) plots of energy-dissipation spectra for the fluid (red full lines) and magnetic (blue dashed lines) fields, with  $k$  being the magnitude of the wavevector: (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), (a.2)  $Pr_M = 0.01$  (R1C), (b.2)  $Pr_M = 0.1$  (R2C), (c.2)  $Pr_M = 1.0$  (R3C) and (d.2)  $Pr_M = 10.0$  (R4C) for decaying MHD turbulence; and for statistically steady MHD turbulence (a.3)  $Pr_M = 0.01$  (R1D), (b.3)  $Pr_M = 0.1$  (R2D), (c.3)  $Pr_M = 1.0$  (R3D) and (d.3)  $Pr_M = 10.0$  (R4D).

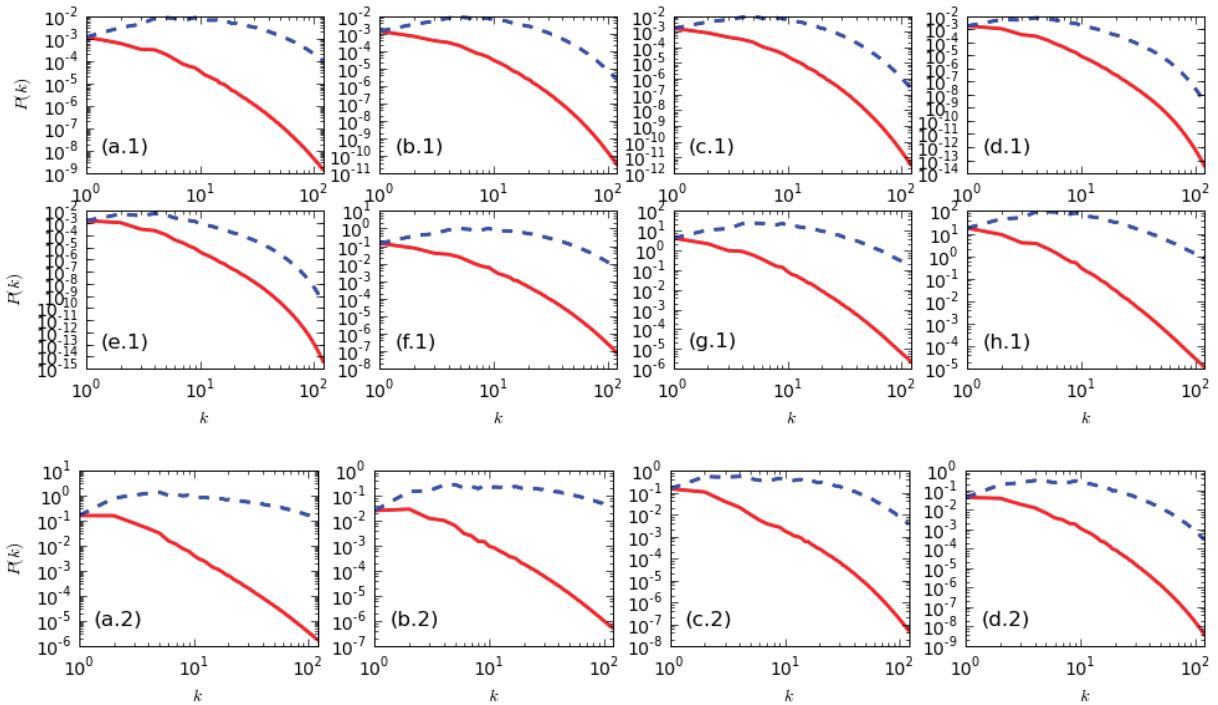
energy spectra discussed above. Furthermore, as  $Pr_M$  increases from 0.1 to 10 in runs R1–R5,  $P(k)$  falls more and more rapidly, as can be seen from the vertical scales in figures 12(a.1)–(e.1).

### 3.4. Probability distribution functions

We calculate several PDFs to characterize the statistical properties of decaying and statistically steady MHD turbulence. In the former case, PDFs are obtained at the cascade-completion time  $t_c$ ; in the latter, they are averaged over the statistically steady state that we obtain. The PDFs we consider are of two types: the first type are PDFs of the cosines of angles between various



**Figure 11.** (a) The kinetic energy spectrum  $E_u(k)$  (red asterisks) deep in the dissipation range for run R5; the black line indicates the fit  $E_u(k) \sim k^{2.68} \exp(-0.235k)$  for  $5k_d^u < k < 10k_d^u$ , where  $k_d^u = 1/\eta_d^u$ . (b) The magnetic energy spectrum  $E_b(k)$  (red asterisks) deep in the dissipation range for run R1C; the black line indicates the fit  $E_b(k) \sim k^{-5.24} \exp(-0.014k)$  for  $5k_d^b < k < 10k_d^b$ , where  $k_d^b = 1/\eta_d^b$ .



**Figure 12.** Log–log (base 10) plots of effective pressure spectra  $P(k)$  (red full lines), with  $k$  the magnitude of the wave vector, and the corresponding compensated spectra  $P(k)k^{7/3}$  (blue dashed lines): (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), for decaying MHD turbulence; and for statistically steady MHD turbulence (a.2)  $Pr_M = 0.01$  (R1D), (b.2)  $Pr_M = 0.1$  (R2D), (c.2)  $Pr_M = 1.0$  (R3D) and (d.2)  $Pr_M = 10.0$  (R4D).

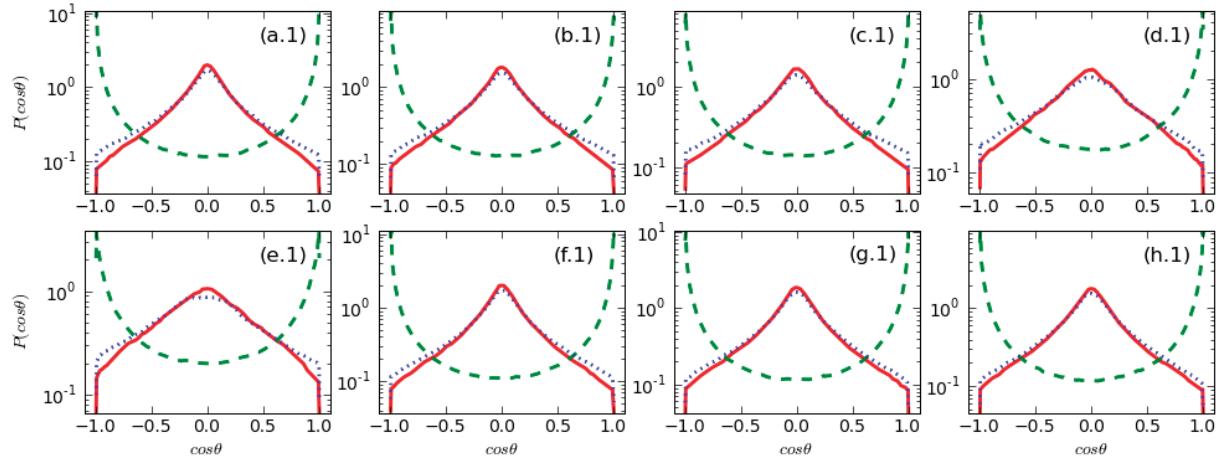
vectors, such as  $\mathbf{u}$  and  $\boldsymbol{\omega}$ ; these help us to quantify the degrees of alignment between such vectors; the second type are PDFs of quantities such as  $\epsilon_u$ ,  $\epsilon_b$  and the eigenvalues of the rate-of-strain tensor.

In figure 13, we show plots of the PDFs of cosines of the angles between the vorticity  $\boldsymbol{\omega}$  and the eigenvectors of the fluid rate-of-strain tensor  $\mathbb{S}$ , namely  $\hat{e}_u^1$  (red full line),  $\hat{e}_u^2$  (green dashed lines) and  $\hat{e}_u^3$  (blue dotted lines) for runs R1–R5 and R3B–R5B at the cascade-completion time  $t_c$  for the case of decaying MHD turbulence. In figure 14, we show similar plots of the PDFs of cosines of the angles between the current density  $\mathbf{j}$  and the eigenvectors of the fluid rate-of-strain tensor  $\mathbb{S}$ . The most important features of these figures are sharp peaks in the green dashed lines; these show that there is a marked tendency for the alignment or antialignment of  $\boldsymbol{\omega}$  and  $\hat{e}_u^2$ , as in fluid turbulence, and of a similar tendency for the alignment or antialignment of  $\mathbf{j}$  and  $\hat{e}_u^2$ ; these features do not depend very sensitively on  $Pr_M$ . Furthermore, the PDFs of cosines of the angles between  $\boldsymbol{\omega}$  and  $\hat{e}_u^1$  (blue dotted lines) and  $\boldsymbol{\omega}$  and  $\hat{e}_u^3$  (red full lines) show peaks near zero in figure 13; in contrast, analogous PDFs for the cosines of the angles between  $\mathbf{j}$  and  $\hat{e}_u^1$  (red full lines) and  $\boldsymbol{\omega}$  and  $\hat{e}_u^3$  (blue dotted lines) show nearly flat plateaux in the middle with very gentle maxima near  $-0.5$  and  $0.5$  (figure 14). Runs R1C–R4C and R1D–R4D yield similar PDFs, for the cosines of these angles, so we do not give them here.

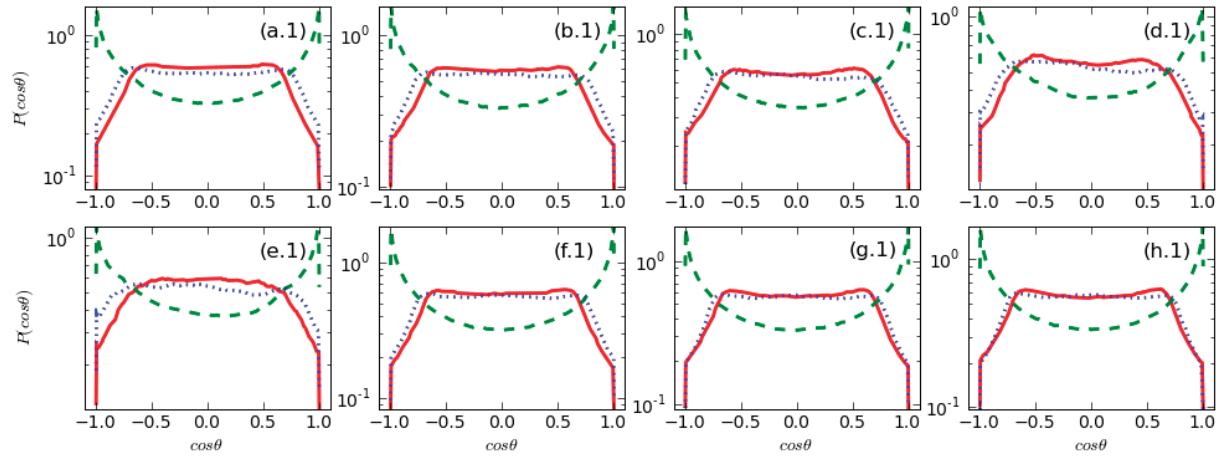
Plots of the PDFs of cosines of the angles between the velocity  $\mathbf{u}$  and the eigenvectors of the fluid rate-of-strain tensor  $\mathbb{S}$  are given in figure 15; their analogues for  $\mathbf{b}$  are given in figure 16. Again, the most prominent features of these figures are sharp peaks in the green dashed lines; these show that there is a marked tendency for the alignment or antialignment of  $\mathbf{u}$  and  $\hat{e}_u^2$  and of a similar tendency for the alignment or antialignment of  $\mathbf{j}$  and  $\hat{e}_u^2$ ; these features do not depend very sensitively on  $Pr_M$ . The PDFs of cosines of the angles between  $\mathbf{u}$  and  $\hat{e}_u^1$  (red full line) and  $\mathbf{u}$  and  $\hat{e}_u^3$  (blue dotted lines) show gentle, broad peaks that imply a weak preference for angles close to  $45^\circ$  or  $135^\circ$ ; these peaks are suppressed as we increase  $Pr_M$  (figures 15(a.1)–(e.1) for runs R1–R5) with fixed initial energy, but they reappear if we compensate for the increase in  $Pr_M$  by increasing the initial energy (figures 15(f.1)–(h.1)). Similar, but sharper, peaks appear in the PDFs of cosines of the angles between  $\mathbf{u}$  and  $\hat{e}_u^1$  (red full lines) and  $\mathbf{u}$  and  $\hat{e}_u^3$  (blue dotted lines); these show a weak preference for angles close to  $47^\circ$  or  $133^\circ$  (figure 16). Some simulations of compressible MHD turbulence have noted the presence of such peaks [35] for  $Pr_M = 1$ . The PDFs of figures 13–16 have a marginal dependence on  $Pr_M$ . Furthermore, they look quite similar to those obtained earlier in convection-driven dynamos (see figure 15 of [36]).

Only one of the eigenvalues  $\Lambda_1^b$  of the tensile magnetic stress tensor  $\mathbb{T}$  is non-zero; and the corresponding eigenvector  $\hat{e}_b^1$  is identically aligned with  $\mathbf{b}$ . Thus PDFs of cosines of angles between  $\mathbf{u}$ ,  $\boldsymbol{\omega}$ ,  $\mathbf{j}$  and  $\mathbf{b}$  and the eigenvectors of  $\mathbb{T}$  are simpler than their counterparts for  $\mathbb{S}$  and are not presented here.

Figure 17 shows plots of PDFs of cosines of angles, denoted generically by  $\theta$ , between (a)  $\mathbf{u}$  and  $\mathbf{b}$ , (b)  $\mathbf{u}$  and  $\boldsymbol{\omega}$ , (c)  $\mathbf{u}$  and  $\mathbf{j}$ , (d)  $\boldsymbol{\omega}$  and  $\mathbf{j}$ , (e)  $\mathbf{b}$  and  $\boldsymbol{\omega}$  and (f)  $\mathbf{b}$  and  $\mathbf{j}$  for runs R1 (red lines), R2 (green lines), R3 (blue lines), R4 (black lines) and R5 (cyan lines). These figures show the following: (a)  $\mathbf{u}$  and  $\mathbf{b}$  are more aligned than antialigned (this is related to the small, positive, mean values of  $H_C$  (see below) in our runs R1–R5); (b)  $\mathbf{u}$  and  $\boldsymbol{\omega}$  are more antialigned than aligned, as noted for decaying fluid turbulence with slight helicity in [47, 69]; (c)  $\mathbf{u}$  and  $\mathbf{j}$  show approximately equal tendencies for alignment and antialignment; (d)  $\boldsymbol{\omega}$  and  $\mathbf{j}$  display a greater tendency for alignment than antialignment; (e)  $\mathbf{b}$  and  $\boldsymbol{\omega}$  have approximately equal tendencies for alignment and antialignment and (f)  $\mathbf{b}$  and  $\mathbf{j}$  are more antialigned than aligned.

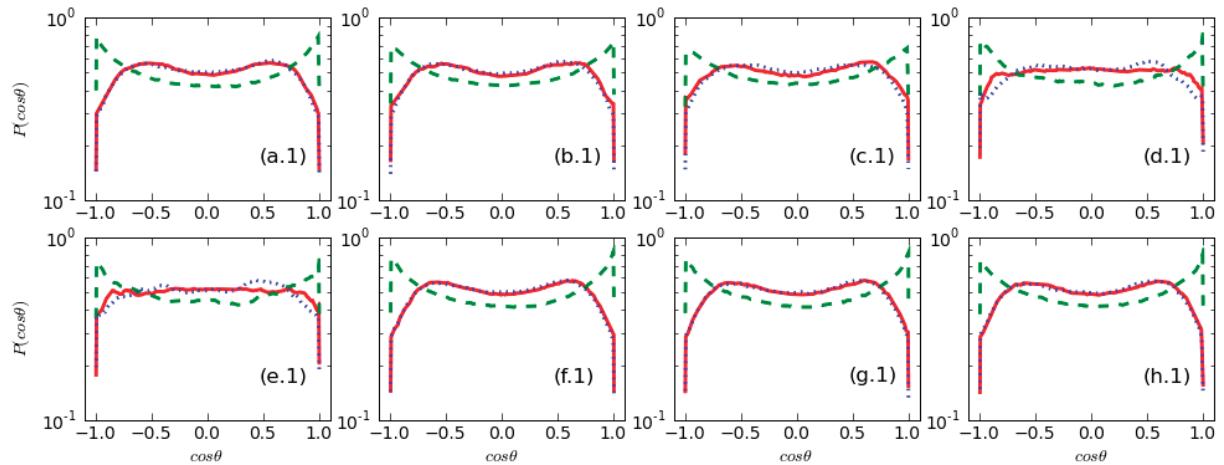


**Figure 13.** Semilog (base 10) plots of the PDFs of cosines of the angles, denoted generically by  $\theta$ , between the vorticity  $\omega$  and the eigenvectors of the fluid rate-of-strain tensor  $\mathbb{S}$ , namely  $\hat{e}_u^1$  (red full line),  $\hat{e}_u^2$  (green dashed line) and  $\hat{e}_u^3$  (blue dotted line): (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B) and (h.1)  $Pr_M = 10.0$  (R5B) for decaying MHD turbulence.

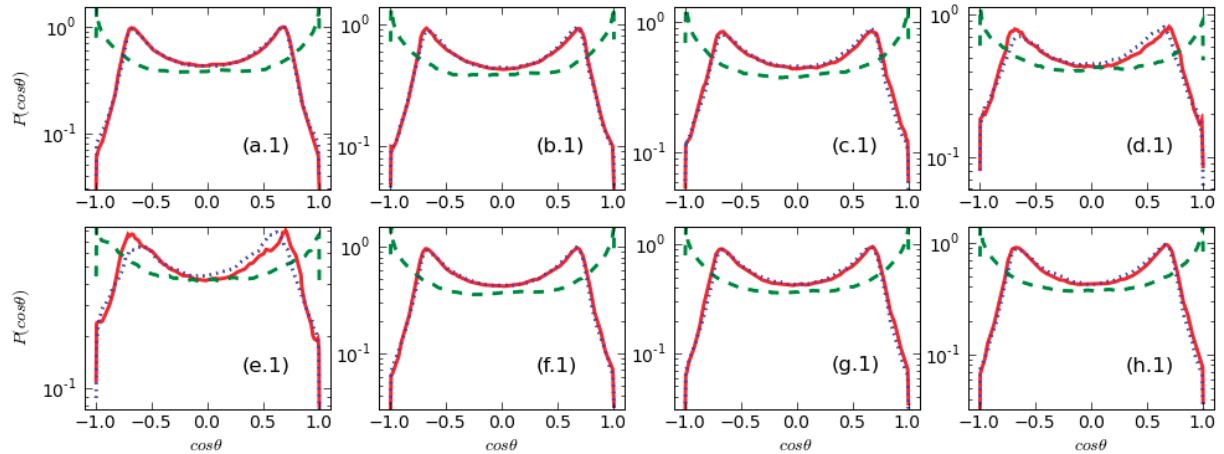


**Figure 14.** Semilog (base 10) plots of the PDFs of cosines of angles, denoted generically by  $\theta$ , between the current density  $\mathbf{j}$  and the eigenvectors of fluid rate-of-strain tensor  $\mathbb{S}$ , namely  $\hat{e}_u^1$  (red full line),  $\hat{e}_u^2$  (green dashed line) and  $\hat{e}_u^3$  (blue dotted line): (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B) and (h.1)  $Pr_M = 10.0$  (R5B) for decaying MHD turbulence.)

Probability distribution functions of the local cross helicity  $H_C = \mathbf{u} \cdot \mathbf{b}$  are shown via green full lines in figure 18. The arguments of these PDFs are scaled by their standard deviations, namely  $\sigma_{H_C}$ ; data for the PDFs are obtained at  $t_c$  for runs R1–R5 in figures 18(a.1)–(e.1),

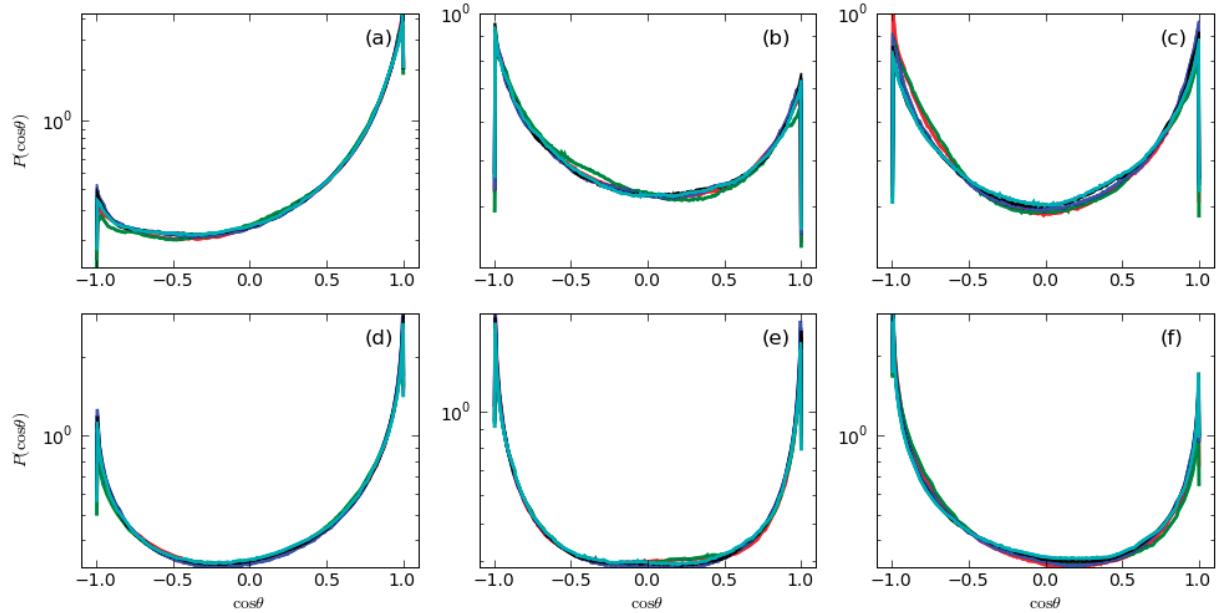


**Figure 15.** Semilog (base 10) plots of the PDFs of cosines of angles, denoted generically by  $\theta$ , between the velocity  $\mathbf{u}$  and the eigenvectors of the fluid rate-of-strain tensor  $\mathbb{S}$ , namely  $\hat{e}_u^1$  (red full line),  $\hat{e}_u^2$  (green dashed line) and  $\hat{e}_u^3$  (blue dotted line): (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B) and (h.1)  $Pr_M = 10.0$  (R5B) for decaying MHD turbulence.



**Figure 16.** Semilog (base 10) plots of the PDFs of cosines of angles, denoted generically by  $\theta$ , between the magnetic field  $\mathbf{b}$  and the eigenvectors of the fluid rate-of-strain tensor  $\mathbb{S}$ , namely  $\hat{e}_b^1$  (red full line),  $\hat{e}_b^2$  (green dashed line) and  $\hat{e}_b^3$  (blue dotted line): (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B) and (h.1)  $Pr_M = 10.0$  (R5B).

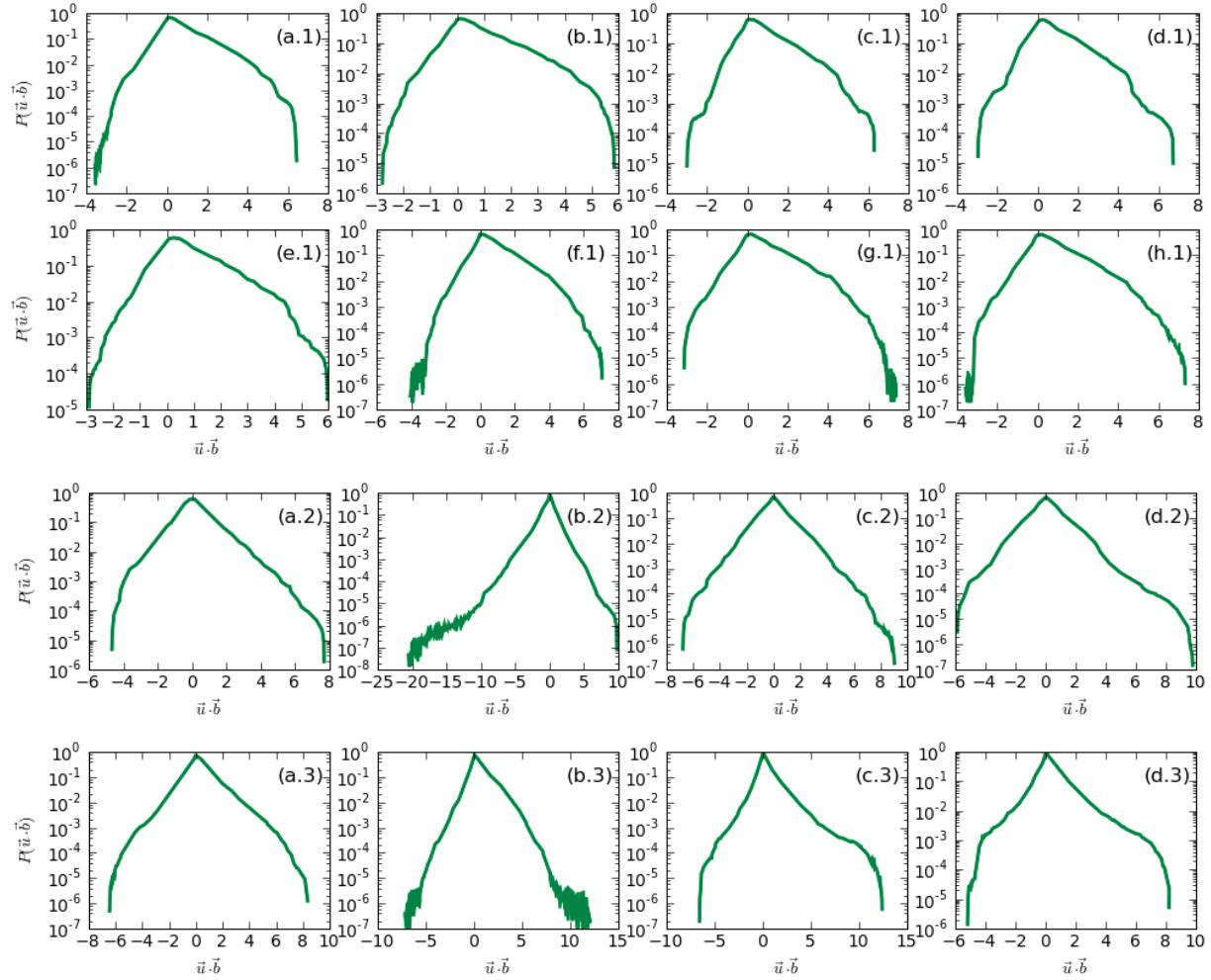
runs R3B–R5B in figures 18(f.1)–(h.1) and runs R1C–R4C in figures 18(a.2)–(d.2) for decaying MHD turbulence. For statistically steady MHD turbulence, these PDFs are shown in figures 18(a.3)–(d.3) for runs R1D–R4D. All of these PDFs have peaks close to  $H_C = 0$ ; this reflects the tendency for  $\mathbf{u}$  and  $\mathbf{b}$  to be aligned or antialigned that we have discussed above.



**Figure 17.** Semilog (base 10) plots of PDFs of cosines of angles, denoted generically by  $\theta$ , between (a)  $\mathbf{u}$  and  $\mathbf{b}$ , (b)  $\mathbf{u}$  and  $\boldsymbol{\omega}$ , (c)  $\mathbf{u}$  and  $\mathbf{j}$ , (d)  $\boldsymbol{\omega}$  and  $\mathbf{j}$ , (e)  $\mathbf{b}$  and  $\boldsymbol{\omega}$ , and (f)  $\mathbf{b}$  and  $\mathbf{j}$  for runs R1 (red lines), R2 (green lines), R3 (blue lines), R4 (black lines) and R5 (cyan lines).

However, these PDFs are quite broad and distinctly non-Gaussian; this can be seen easily from the values of the mean  $\mu_{H_C}$ , standard deviation  $\sigma_{H_C}$ , skewness  $\gamma_{3,H_C}$  and kurtosis  $\gamma_{4,H_C}$  given in table 2. Thus fluctuations of  $H_C$  away from the mean are very significant. Table 2 also gives the value of the mean energy  $E$  and the ratio  $E/\mu_{H_C}$ , which does not appear to be universal; for runs R1–R5 and R3B–R5B it lies in the range 0.23–0.26, for R1C–R2C in the range –0.04–0.04 and for R1D–R4D in the range 0.05–0.2. For all of our runs, with the exception of R2C, the mean  $\mu_{H_C}$  and the skewness  $\gamma_{3,H_C}$  are positive. Even if the PDF of  $H_C$  had been a Gaussian, its mean value would have been within one standard deviation of 0; the actual PDF is much broader than a Gaussian. On symmetry grounds, there is no reason for the system to display a non-zero value for  $\mu_{H_C}$  unless there is some bias in the forcing or in the initial condition (the latter for the case of decaying turbulence). In any given run, if there is some residual  $H_C$ , it is reflected in a slight asymmetry in alignment (or antialignment) of  $\mathbf{u}$  and  $\mathbf{b}$ , which we have studied above via the PDF of the cosine of the angle between  $\mathbf{u}$  and  $\mathbf{b}$ . When we consider the ratio  $\mu_{H_C}/E$ , it seems to be substantial in some runs but, given the arguments above, we expect it to vanish in runs with a very large number of collocation points; indeed, it is very small in runs R1C–R4C.

Consider now the PDFs of the eigenvalues  $\Lambda_u^1$  (blue dotted line),  $\Lambda_u^2$  (green dashed line) and  $\Lambda_u^3$  (red full line) of the rate-of-strain tensor  $\mathbb{S}$  shown in figures 19(a.1)–(e.1) for R1–R5 and figures 19(f.1)–(h.1) for runs R3B–R5B. Recall that these eigenvalues provide measures of the local stretching and compression of the fluid; also we label the eigenvalues such that  $\Lambda_u^1 > \Lambda_u^2 > \Lambda_u^3$ . The incompressibility condition yields  $\sum_{n=1}^3 \Lambda_u^n = 0$ , when it follows that  $\Lambda_u^1 > 0$  and  $\Lambda_u^3 < 0$ ; the intermediate eigenvalue  $\Lambda_u^2$  can be either positive or negative. The illustrative plots in figures 19(a.1)–(h.1) from our decaying-MHD-turbulence runs show that the



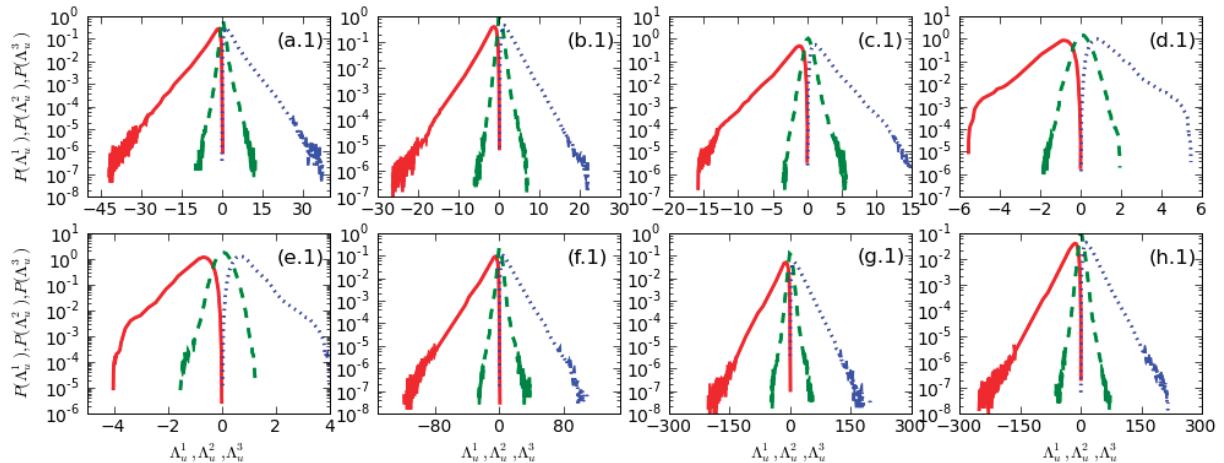
**Figure 18.** Semilog (base 10) plots of the PDFs of the cross helicity  $H_C = \mathbf{u} \cdot \mathbf{b}$  for (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), (a.2)  $Pr_M = 0.01$  (R1C), (b.2)  $Pr_M = 0.1$  (R2C), (c.2)  $Pr_M = 1.0$  (R3C) and (d.2)  $Pr_M = 10.0$  (R4C) for decaying MHD turbulence; and for statistically steady MHD turbulence (a.3)  $Pr_M = 0.01$  (R1D), (b.3)  $Pr_M = 0.1$  (R2D), (c.3)  $Pr_M = 1.0$  (R3D) and (d.3)  $Pr_M = 10.0$  (R4D); the arguments of the PDFs are scaled by their standard deviations  $\sigma_{H_C}$ .

PDFs of  $\Lambda_u^1$  and  $\Lambda_u^3$  have long tails on the right- and left-hand sides, respectively. These tails shrink as we increase  $Pr_M$  (figures 19(a.1)–(e.1) for runs R1–R5, respectively), by increasing  $\nu$  while holding the initial energy fixed; thus, there is a substantial decrease in regions of large strain. However, if we compensate for the increase in  $\nu$  by increasing the energy in the initial condition such that  $k_{\max} \eta_d^u$  and  $k_{\max} \eta_d^b$  are both  $\simeq 1$ , we see that these tails stretch out, i.e. regions of large strain reappear.

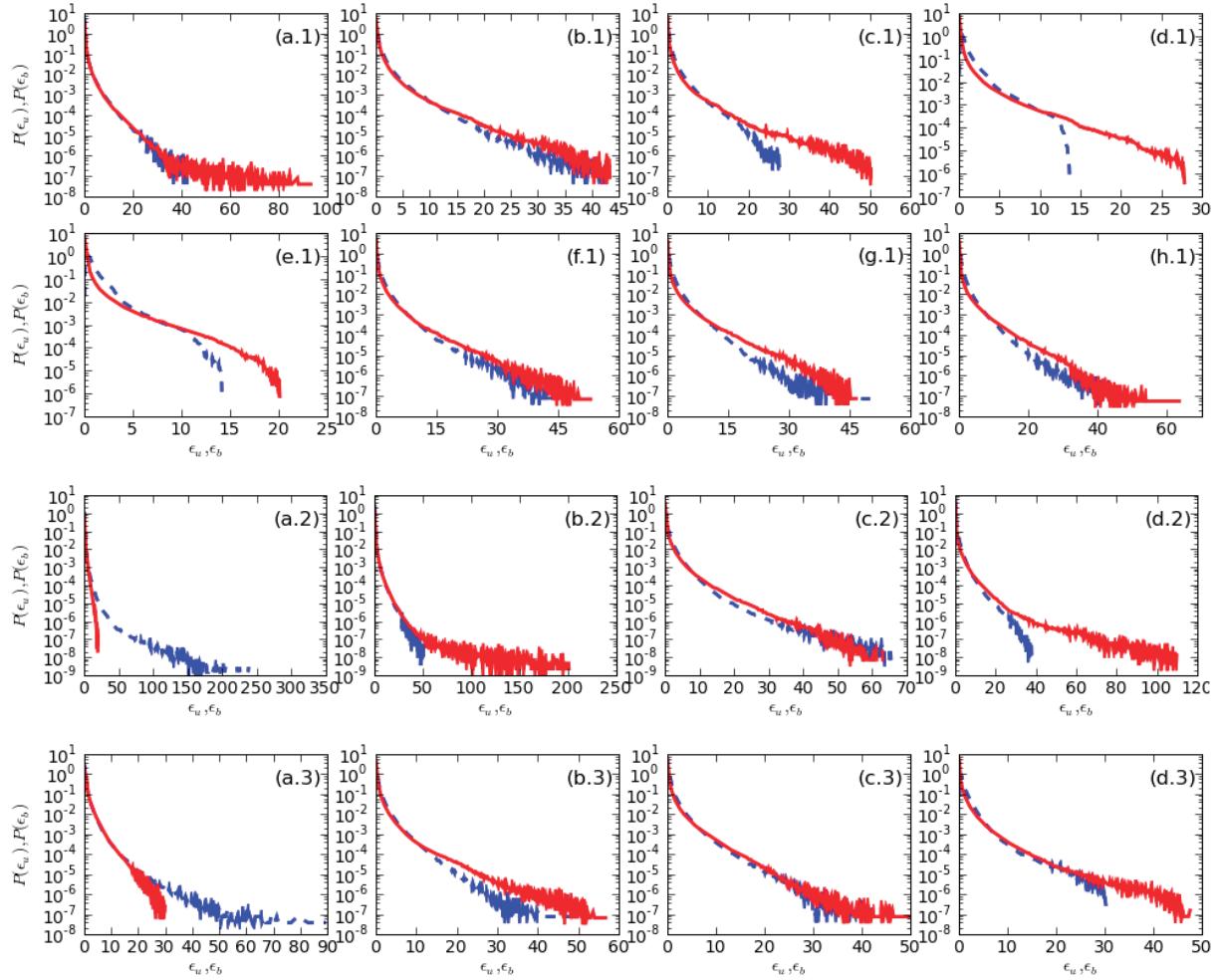
We show PDFs of the kinetic-energy dissipation rate  $\epsilon_u$  (blue dashed lines) and the magnetic-energy dissipation rate  $\epsilon_b$  (red full lines) that are obtained at  $t_c$  for runs R1–R5

**Table 2.** The mean  $\mu_{H_C}$ , standard deviation  $\sigma_{H_C}$ , skewness  $\mu_{3,H_C}$  and kurtosis  $\mu_{4,H_C}$  of the PDF of the cross helicity  $H_C$  for our runs R1–R5 and R3B–R5B for decaying MHD turbulence at cascade completion; columns 6 and 7 give, respectively, the mean energy  $E$  and ratio of the means of the cross helicity and the energy, i.e.  $\mu_{H_C}/E$ .

Run	$\mu_{H_C}$	$\sigma_{H_C}$	$\mu_{3,H_C}$	$\mu_{4,H_C}$	$E$	$\mu_{H_C}/E$
R1	0.118	0.173	1.103	4.901	0.461	0.256
R2	0.118	0.169	1.096	4.685	0.467	0.252
R3	0.120	0.170	1.096	4.679	0.490	0.245
R4	0.112	0.153	1.003	4.579	0.477	0.235
R5	0.105	0.141	0.934	4.324	0.460	0.228
R3B	1.217	1.804	1.100	4.912	4.909	0.248
R4B	5.915	8.766	1.097	4.917	24.50	0.241
R5B	11.50	17.05	1.102	5.000	48.32	0.238
R1C	0.014	0.113	0.615	5.748	0.358	0.041
R2C	-0.224	1.994	-0.698	8.441	5.440	-0.041
R3C	0.130	2.005	0.313	5.637	5.969	0.022
R4C	0.859	9.156	0.364	5.747	29.05	0.029
R1D	0.169	0.724	0.737	6.813	3.090	0.055
R2D	0.478	0.886	1.126	5.954	2.405	0.199
R3D	0.454	1.244	1.904	13.75	3.039	0.149
R4D	0.389	1.110	1.207	9.225	2.767	0.140



**Figure 19.** Semilog (base 10) plots of PDFs of the eigenvalues  $\Lambda_u^1$  (blue dotted line),  $\Lambda_u^2$  (green dashed line) and  $\Lambda_u^3$  (red full line) of the rate-of-strain tensor  $\mathbb{S}$  for (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B) and (h.1)  $Pr_M = 10.0$  (R5B); the arguments of the PDFs are scaled by their standard deviations.



**Figure 20.** Semilog (base 10) plots of PDFs of the local kinetic-energy dissipation rate  $\epsilon_u$  (blue dashed line) and the magnetic-energy dissipation rate  $\epsilon_b$  (red full line), with the arguments scaled by their standard deviations, for (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), (a.2)  $Pr_M = 0.01$  (R1C), (b.2)  $Pr_M = 0.1$  (R2C), (c.2)  $Pr_M = 1.0$  (R3C) and (d.2)  $Pr_M = 10.0$  (R4C) for decaying MHD turbulence; and for statistically steady MHD turbulence (a.3)  $Pr_M = 0.01$  (R1D), (b.3)  $Pr_M = 0.1$  (R2D), (c.3)  $Pr_M = 1.0$  (R3D) and (d.3)  $Pr_M = 10.0$  (R4D).

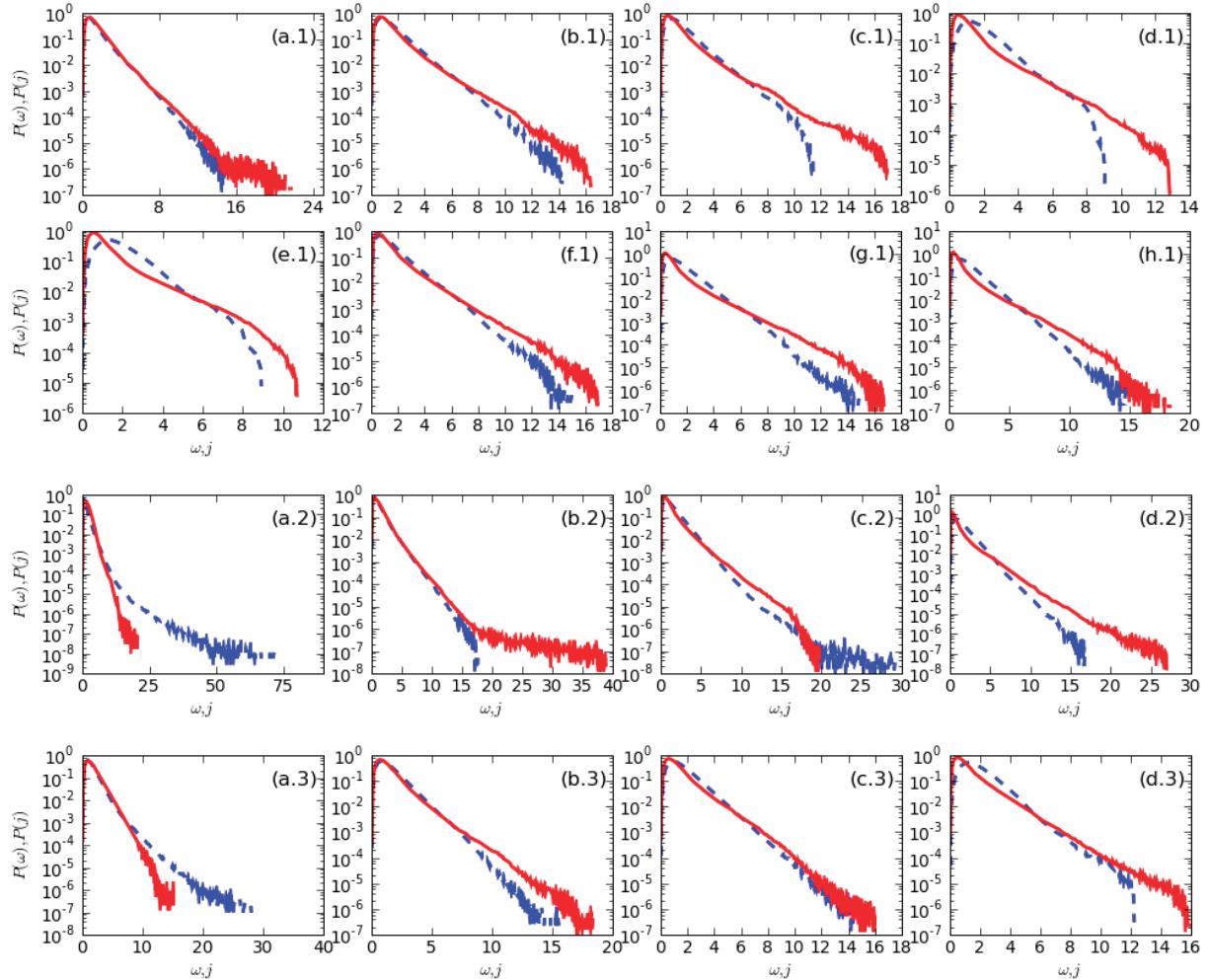
in figures 20(a.1)–(e.1), runs R3B–R5B in figures 20(f.1)–(h.1) and runs R1C–R4C in figures 20(a.2)–(d.2) for decaying MHD turbulence; and for statistically steady MHD turbulence they are shown in figures 20(a.3)–(d.3) for runs R1D–R4D. All of these PDFs have long tails; the tail of the PDF for  $\epsilon_b$  extends further than the tail of that for  $\epsilon_u$  for all except the smallest values of  $Pr_M$  (figures 20(a.1), (a.2) and (a.3) for runs R1, R1C and R1D, respectively). This indicates that large values of  $\epsilon_b$  are more likely to appear than large values of  $\epsilon_u$  and, given the long tails of these PDFs, suggests that, except at the smallest values of  $Pr_M$  we have used, we might obtain more marked intermittency for the magnetic field than for the velocity field.

**Table 3.** The mean  $\mu_{\epsilon_u}$ , standard deviation  $\sigma_{\epsilon_u}$ , skewness  $\gamma_{3,\epsilon_u}$  and kurtosis  $\gamma_{4,\epsilon_u}$  of the PDFs of the local fluid energy dissipation  $\epsilon_u$ , and their analogues for  $\epsilon_b$ , for runs R1–R5, R3B–R5B, R1C–R4C and R1D–R4D.

Run	$\mu_{\epsilon_u}$	$\sigma_{\epsilon_u}$	$\gamma_{3,\epsilon_u}$	$\gamma_{4,\epsilon_u}$	$\mu_{\epsilon_b}$	$\sigma_{\epsilon_b}$	$\gamma_{3,\epsilon_b}$	$\gamma_{4,\epsilon_b}$
R1	0.0048	0.0096	6.382	75.069	0.0302	0.0550	7.611	144.86
R2	0.0109	0.0187	6.053	70.566	0.0255	0.0527	8.182	121.47
R3	0.0141	0.0226	5.450	52.884	0.0233	0.0566	10.46	204.37
R4	0.0231	0.0284	4.042	28.306	0.0160	0.0397	7.955	97.662
R5	0.0273	0.0302	3.684	24.559	0.0130	0.0315	6.682	64.206
R3B	0.4165	0.7345	5.941	70.070	0.6440	1.5881	9.029	147.71
R4B	6.7843	10.898	5.343	55.676	4.4541	13.377	9.672	155.71
R5B	21.164	32.438	5.353	59.163	9.8332	31.177	9.621	151.64
R1C	0.0031	0.0076	18.45	1620.0	0.0566	0.0632	3.340	22.270
R2C	0.2354	0.5177	7.599	112.92	1.5655	3.5169	13.99	981.17
R3C	0.8349	1.6375	6.841	105.85	1.3186	3.5524	10.41	205.54
R4C	14.208	22.900	5.535	66.496	7.1624	24.974	13.33	406.21
R1D	0.0448	0.0630	5.799	89.678	0.8087	1.1004	4.587	40.328
R2D	0.0601	0.0933	5.180	51.311	0.4995	0.9808	8.488	142.97
R3D	0.2886	0.4233	5.230	55.366	0.6389	1.3120	6.755	80.154
R4D	0.4498	0.5536	5.055	58.832	0.5037	1.1391	8.077	129.65

Furthermore, as we expect, the tail of the PDF of  $\epsilon_u$  is drawn in towards small values of  $\epsilon_u$  as we increase  $Pr_M$  (figures 20(a.1)–(e.1) for runs R1–R5, respectively) while holding  $\eta$  and the initial energy fixed. However, if we compensate for the increase in  $\nu$  by increasing the initial energy so that  $k_{\max} \eta_d^u$  and  $k_{\max} \eta_d^b$  are both  $\simeq 1$ , we see that the tails of the PDFs of  $\epsilon_b$  and  $\epsilon_u$  get elongated as we increase  $Pr_M$ , e.g. in figures 20(f.1)–(h.1) for runs R3B–R5B, respectively. The values of the mean  $\mu_{\epsilon_u}$ , standard deviation  $\sigma_{\epsilon_u}$ , skewness  $\gamma_{3,\epsilon_u}$  and kurtosis  $\gamma_{4,\epsilon_u}$  of the PDFs of the local fluid energy dissipation  $\epsilon_u$  are given for all our runs, and their counterparts for  $\epsilon_b$  are given in table 3. From these values, we see that the right tails of these distributions fall much more slowly than the tail of a Gaussian distribution.

Similar trends emerge if we examine the PDFs of the moduli of the vorticity and the current density,  $\omega$  (blue dashed lines) and  $j$  (red full lines), respectively: these are presented at  $t_c$  for runs R1–R5 in figures 21(a.1)–(e.1), runs R3B–R5B in figures 21(f.1)–(h.1) and runs R1C–R4C in figures 21(a.2)–(d.2) for decaying MHD turbulence; and for statistically steady MHD turbulence they are shown in figures 21(a.3)–(d.3) for runs R1D–R4D. The tail of the PDF for  $j$  extends further than the tail of that for  $\epsilon_u$  for all except the smallest values of  $Pr_M$  (figures 21(a.1), (a.2) and (a.3) for runs R1, R1C and R1D, respectively), so large values of  $j$  are more likely than large values of  $\omega$ . Thus, given that these PDFs have long tails, it is reasonable to expect that, except at the smallest values of  $Pr_M$  we have used, intermittency for the magnetic field might be larger than that for the velocity field. Moreover, the tail of the PDF of  $\omega$  is drawn in towards small values of  $\omega$  as we increase  $Pr_M$  (figures 21(a.1)–(e.1) for runs R1–R5, respectively) while holding  $\eta$  and the initial energy fixed; but if, while increasing  $\nu$ , we also increase the initial energy so that  $k_{\max} \eta_d^u$  and  $k_{\max} \eta_d^b$  are  $\simeq 1$ , we see that the tails of the PDFs of  $j$  and  $\omega$  get stretched out as we increase  $Pr_M$ , e.g. in figures 21(f.1)–(h.1) for runs R3B–R5B, respectively.



**Figure 21.** Semilog (base 10) plots of PDFs of the moduli of the local vorticity (blue dashed lines) and the current density (red full lines),  $\omega$  and  $j$ , respectively, with the arguments of the PDFs scaled by their standard deviations, for (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), (a.2)  $Pr_M = 0.01$  (R1C), (b.2)  $Pr_M = 0.1$  (R2C), (c.2)  $Pr_M = 1.0$  (R3C) and (d.2)  $Pr_M = 10.0$  (R4C) for decaying MHD turbulence; and for statistically steady MHD turbulence (a.3)  $Pr_M = 0.01$  (R1D), (b.3)  $Pr_M = 0.1$  (R2D), (c.3)  $Pr_M = 1.0$  (R3D), and (d.3)  $Pr_M = 10.0$  (R4D).

The values of the mean  $\mu_\omega$ , standard deviation  $\sigma_\omega$ , skewness  $\gamma_{3,\omega}$  and kurtosis  $\gamma_{4,\omega}$  of the PDFs of the modulus of the local vorticity  $\omega$  for all of our runs and their counterparts for  $j$  are given in table 4. From these values, we see that the right tails of these distributions fall much more slowly than the tail of a Gaussian distribution.

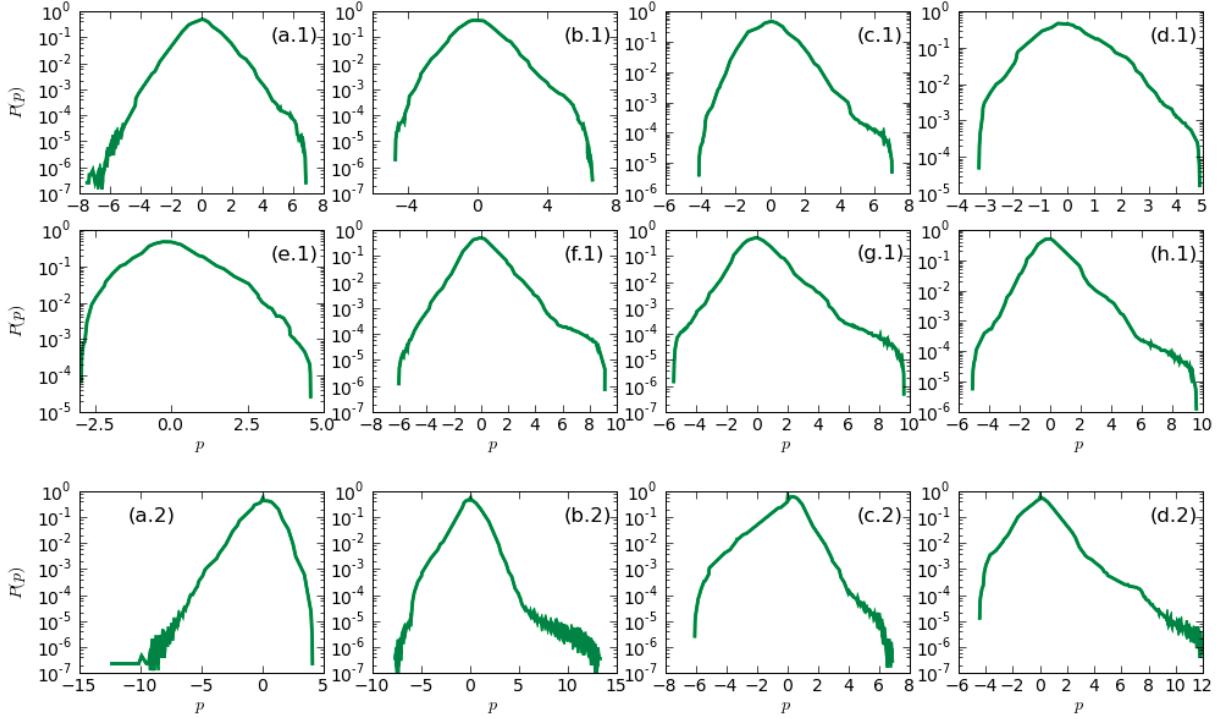
We move now to PDFs of the local effective pressure (green full lines), which are shown at  $t_c$  for runs R1–R5 in figures 22(a.1)–(e.1) and runs R3B–R5B in figures 22(f.1)–(h.1) for decaying MHD turbulence; for statistically steady MHD turbulence, they are shown in figures 22(a.2)–(d.2) for runs R1D–R4D. The values of the mean  $\mu_p$ , standard deviation  $\sigma_p$ ,

**Table 4.** The mean  $\mu_\omega$ , standard deviation  $\sigma_\omega$ , skewness  $\gamma_{3,\omega}$  and kurtosis  $\gamma_{4,\omega}$  of the PDFs of the modulus of the local vorticity  $\omega$ , and their analogues for  $j$ , for runs R1–R5, R3B–R5B, R1C–R4C and R1D–R4D.

Run	$\mu_\omega$	$\sigma_\omega$	$\gamma_{3,\omega}$	$\gamma_{4,\omega}$	$\mu_j$	$\sigma_j$	$\gamma_{3,j}$	$\gamma_{4,j}$
R1	3.817	3.138	2.174	10.25	3.109	2.340	2.206	11.42
R2	2.680	1.938	1.961	9.235	2.794	2.222	2.525	13.84
R3	2.189	1.512	1.871	8.598	2.606	2.215	2.873	17.16
R4	1.312	0.766	1.534	6.859	2.121	1.870	2.903	15.74
R5	1.022	0.567	1.363	6.098	1.906	1.695	2.789	13.95
R3B	11.50	8.706	1.949	8.908	13.27	12.06	2.791	15.61
R4B	21.30	14.98	1.828	8.246	32.58	34.11	3.243	19.05
R5B	26.93	18.23	1.770	8.049	47.08	51.90	3.360	19.79
R1C	25.31	23.10	2.265	11.17	21.00	18.47	2.456	14.56
R2C	15.70	13.07	2.062	9.831	18.35	17.95	2.899	17.01
R3C	21.68	15.49	1.746	7.910	38.85	45.50	3.563	23.00
R4C	2.945	2.661	3.027	26.37	1.446	0.861	1.451	6.816
R1D	12.14	8.700	2.297	14.16	5.336	3.456	1.702	7.683
R2D	14.26	9.802	1.745	7.821	12.58	9.544	2.336	12.64
R3D	10.07	6.544	1.644	7.463	13.88	11.24	2.320	11.36
R4D	4.119	2.349	1.388	6.526	11.92	10.47	2.429	12.46

skewness  $\gamma_{3,p}$  and kurtosis  $\gamma_{4,p}$  of the PDFs of the local effective pressure  $p$  are given for these runs in table 5. These have mean  $\mu_p = 0$  but are distinctly non-Gaussian as can be seen from the values of  $\gamma_{3,p}$  and  $\gamma_{4,p}$ . Pressure PDFs are negatively skewed in pure fluid turbulence, as we have mentioned above; however, for MHD turbulence, we find that the PDFs of the effective pressure  $\bar{p}$  can be positively skewed, as in runs R1–R5, R3B–R5B and run R4D, or negatively skewed, as in runs R1D–R3D; negative skewness seems to arise at low values of  $Pr_M$ .

The scale dependence of PDFs of velocity increments provides important clues to the nature of intermittency in fluid turbulence. To explore similar intermittency in MHD turbulence [73], we present data for the scale dependence of PDFs of velocity and magnetic-field increments. As mentioned above, these increments are of the form  $\delta a_{\parallel}(\mathbf{x}, l) \equiv \mathbf{a}(\mathbf{x} + \mathbf{l}, t) - \mathbf{a}(\mathbf{x}, t) \cdot \frac{\mathbf{l}}{l}$ , with  $\mathbf{a}$  being either  $\mathbf{u}$  or  $\mathbf{b}$ ,  $l = |\mathbf{l}|$  the length scale and  $\mathbf{x}$  an origin over which we can average to determine the dependence of the PDFs of  $\delta a_{\parallel}$  on the scale  $l$ ; for notational convenience, such velocity and magnetic-field increments are denoted by  $\delta u(l)$  and  $\delta b(l)$  in our plots. These PDFs are obtained at  $t_c$  for runs R1–R5 in figures 23(a.1)–(e.1), runs R3B–R5B in figures 23(f.1)–(h.1) and runs R1C–R4C in figures 23(a.2)–(d.2) for decaying MHD turbulence; and for statistically steady MHD turbulence, they are shown in figures 23(a.3)–(d.3) for runs R1D–R4D. The PDFs of velocity increments are shown for separations  $l = 2\delta x$  (red dashed thin line),  $l = 10\delta x$  (green dot-dashed thin line) and  $l = 100\delta x$  (blue full thin line), where  $\delta x$  is our real-space lattice spacing; for PDFs of magnetic-field increments we also use the separations  $l = 2\delta x$  (black dashed line),  $l = 10\delta x$  (cyan dot-dashed line) and  $l = 100\delta x$  (magenta full line); the arguments of these PDFs are scaled by their standard deviations. As in fluid turbulence, we see that these PDFs are nearly Gaussian if the length scale  $l$  is large. As  $l$  decreases, the PDFs develop long, non-Gaussian tails, a clear signature of intermittency. Furthermore, a comparison



**Figure 22.** Semilog (base 10) plots of PDFs of local effective pressure fluctuations (green full lines), with the arguments of the PDFs scaled by their standard deviations, for (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), for decaying MHD turbulence; and for statistically steady MHD turbulence (a.2)  $Pr_M = 0.01$  (R1D), (b.2)  $Pr_M = 0.1$  (R2D), (c.2)  $Pr_M = 1.0$  (R3D) and (d.2)  $Pr_M = 10.0$  (R4D).

of the red and black dashed lines in these plots indicates that the PDFs of the magnetic-field increments are broader than their velocity counterparts in most of our runs; this suggests, as we had surmised from the PDFs of energy-dissipation rates given above, that the magnetic field displays stronger intermittency than the velocity field at all but the smallest values of  $Pr_M$  (figures 23(a.1), (a.2) and (a.3) for runs R1, R1C and R1D); the general trend that we note from these figures is that the magnetic-field intermittency is stronger than that of the velocity field at large magnetic Prandtl numbers but the difference between these intermittencies decreases as  $Pr_M$  is lowered. We will try to quantify this when we present structure functions in section 3.5.

### 3.5. Structure functions

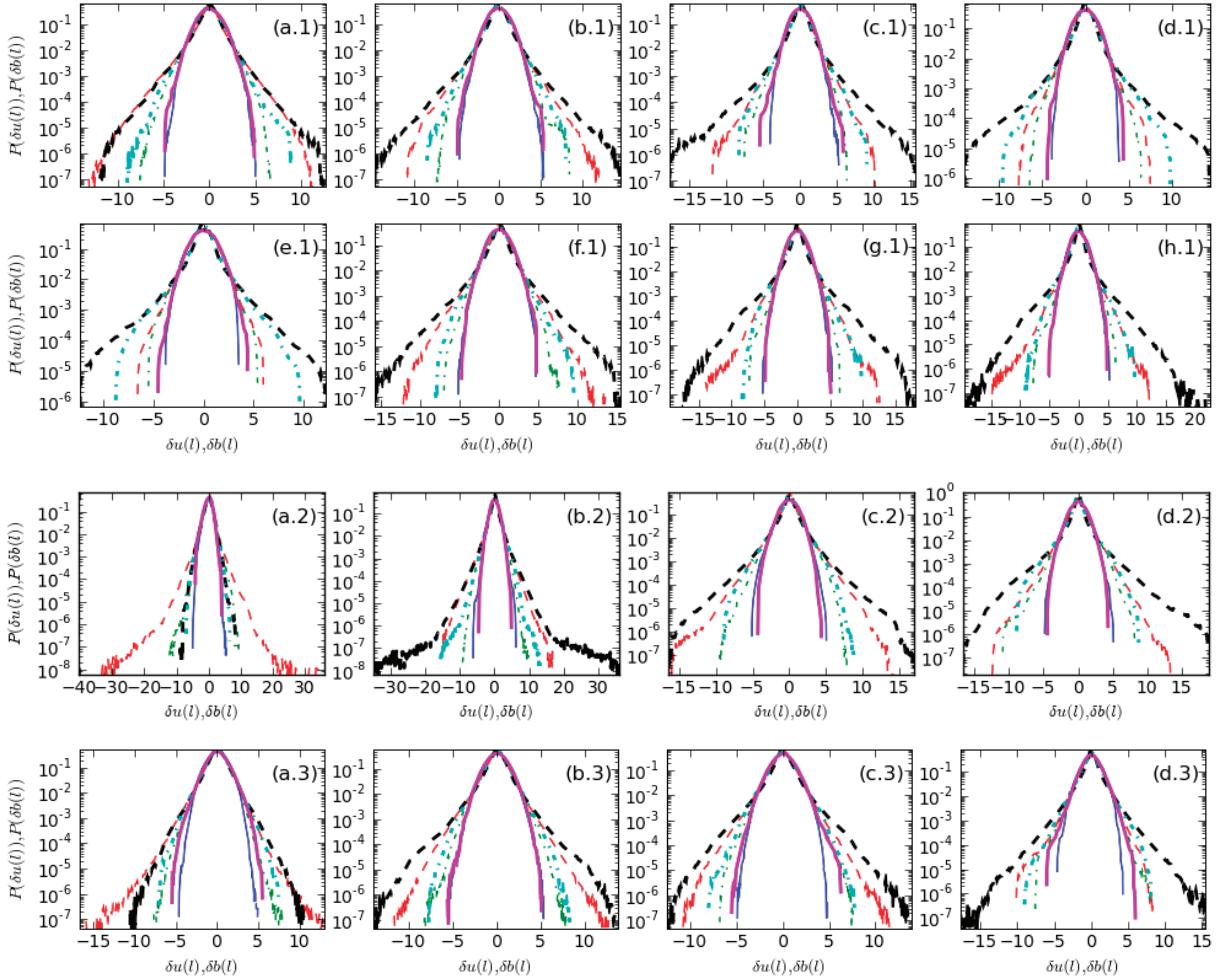
We continue our elucidation of intermittency in MHD turbulence by studying the scale dependence of order- $p$  equal-time, velocity and magnetic-field longitudinal structure functions  $S_p^u(l) \equiv \langle |\delta u_{\parallel}(\mathbf{x}, l)|^p \rangle$  and magnetic-field longitudinal structure functions  $S_p^b(l) \equiv \langle |\delta b_{\parallel}(\mathbf{x}, l)|^p \rangle$ , respectively, where  $\delta u_{\parallel}(\mathbf{x}, l) \equiv [\mathbf{u}(\mathbf{x} + \mathbf{l}, t) - \mathbf{u}(\mathbf{x}, t)] \cdot \frac{\mathbf{l}}{l}$  and  $\delta b_{\parallel}(\mathbf{x}, l) \equiv [\mathbf{b}(\mathbf{x} + \mathbf{l}, t) - \mathbf{b}(\mathbf{x}, t)] \cdot \frac{\mathbf{l}}{l}$ . From these structure functions, we also obtain the hyperflatnesses  $F_6^u(r) = S_6^u(r)/[S_2^u(r)]^3$  and  $F_6^b(r) = S_6^b(r)/[S_2^b(r)]^3$ . For the inertial range  $\eta_d^u, \eta_d^b \ll l \ll L$ , we expect  $S_p^u(l) \sim l^{\zeta_p^u}$

**Table 5.** The mean  $\mu_p$ , standard deviation  $\sigma_p$ , skewness  $\gamma_{3,p}$  and kurtosis  $\gamma_{4,p}$  of the PDFs of the local effective pressure  $\bar{p}$  for runs R1–R5, R3B–R5B and R1D–R4D.

Run	$\mu_p$	$\sigma_p$	$\gamma_{3,p}$	$\gamma_{4,p}$
R1	0.00	0.055	0.224	4.152
R2	0.00	0.057	0.256	4.052
R3	0.00	0.061	0.315	3.722
R4	0.00	0.060	0.397	3.493
R5	0.00	0.059	0.433	3.527
R3B	0.00	0.609	0.526	5.283
R4B	0.00	3.184	0.660	5.645
R5B	0.00	6.397	0.719	5.776
R1D	0.00	0.738	-0.533	3.882
R2D	0.00	0.313	-0.153	4.697
R3D	0.00	0.589	-1.066	5.338
R4D	0.00	0.363	0.221	5.560

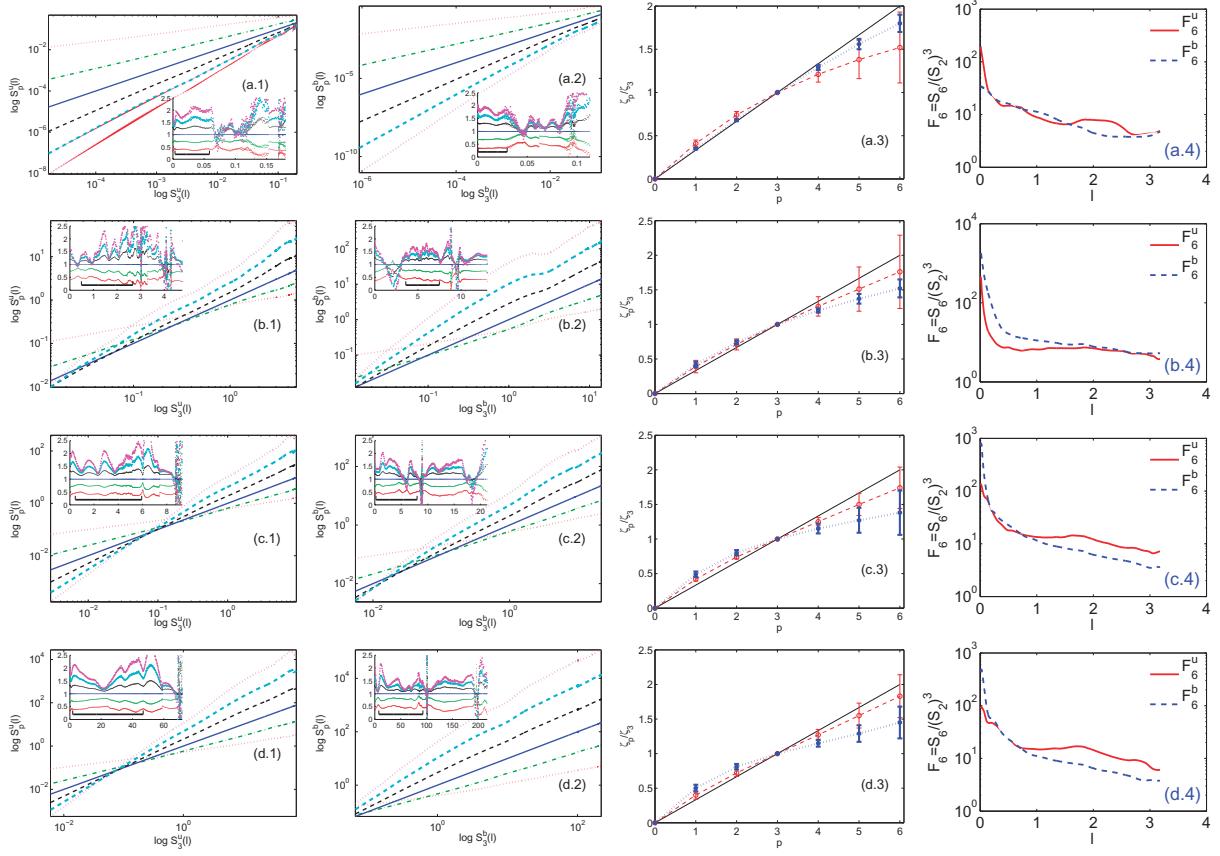
and  $S_p^b(l) \sim l^{\zeta_p^b}$ , where  $\zeta_p^u$  and  $\zeta_p^b$  are inertial-range multiscaling exponents for velocity and magnetic fields, respectively; if these fields show multiscaling, we expect significant deviations from the K41 result  $\zeta_p^{uK41} = \zeta_p^{bK41} = p/3$ . (Note that we do not expect any Iroshnikov–Kraichnan [74] scaling because we have no mean magnetic field in our simulations.) Given large inertial ranges, the multiscaling exponents can be extracted from slopes of log–log plots of structure functions versus  $l$ . However, in practical calculations, inertial ranges are limited, so we use the ESS procedure [40, 41] in which we determine the multiscaling exponent ratios  $\zeta_p^u/\zeta_3^u$  and  $\zeta_p^b/\zeta_3^b$ , respectively, from slopes of log–log plots of (a)  $S_p^u$  versus  $S_3^u$  and (b)  $S_p^b$  versus  $S_3^b$ ; we refer to these as ESS plots. Our data for structure functions are averaged over 51 and 400 origins, respectively, for simulations with  $512^3$  and  $1024^3$  collocation points.

We begin with data from our decaying-MHD-turbulence runs R1C–R4C, which use  $1024^3$  collocation points and span the  $Pr_M$  range 0.01–10. Figures 24(a.1)–(d.1) show ESS plots for  $S_p^u(r)$  for runs R1C–R4C, respectively, for  $p = 1$  (red small-dotted line),  $p = 2$  (green dot-dashed line),  $p = 3$  (blue line),  $p = 4$  (black thin-dashed line),  $p = 5$  (cyan thick-dashed line) and  $p = 6$  (magenta large-dotted line); their analogues for  $S_p^b(r)$  are given in figures 24(a.2)–(d.2); the local slopes of these ESS curves are shown in the insets of these figures. Flat portions in these plots of local slopes help us to identify the inertial ranges. The regions that we have chosen for our fits are indicated by black horizontal lines with vertical ticks at their ends. In such a region, the mean value and the standard deviation of the local slope of the ESS plot for  $S_p^u(r)$  (or  $S_p^b(r)$ ) yield, respectively, our estimates for the exponent ratio  $\zeta_p^u/\zeta_3^u$  (or  $\zeta_p^b/\zeta_3^b$ ) and its error bars. Figures 24(a.3)–(d.3) show plots of these exponent ratios versus  $p$  for the velocity field (blue dotted line with thick error bars) and the magnetic field (red dashed line with thin error bars); the black solid line shows the K41 result for comparison. Although earlier studies [25, 39] have obtained such exponents from DNS studies, they have done so, to the best of our knowledge, only for  $Pr_M = 1$ ; furthermore, they have not reported error bars. Although our (conservative) error bars are large, our plots of exponent ratios suggest the following: (a) deviations from the K41 result are significant, especially for  $p > 3$ , as in fluid turbulence;



**Figure 23.** Semilog (base 10) plots of PDFs of velocity increments  $\delta u(l)$ , for separations  $l = 2\delta x$  (red dashed thin line),  $10\delta x$  (green dot-dashed thin line) and  $100\delta x$  (blue full thin line), and of magnetic-field increments  $\delta b(l)$ , for separations  $l = 2\delta x$  (black dashed line),  $10\delta x$  (cyan dot-dashed line) and  $100\delta x$  (magenta full line), with the arguments of the PDFs scaled by their standard deviations, for (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), (a.2)  $Pr_M = 0.01$  (R1C), (b.2)  $Pr_M = 0.1$  (R2C), (c.2)  $Pr_M = 1.0$  (R3C) and (d.2)  $Pr_M = 10.0$  (R4C) for decaying MHD turbulence; and for statistically steady MHD turbulence (a.3)  $Pr_M = 0.01$  (R1D), (b.3)  $Pr_M = 0.1$  (R2D), (c.3)  $Pr_M = 1.0$  (R3D) and (d.3)  $Pr_M = 10.0$  (R4D).

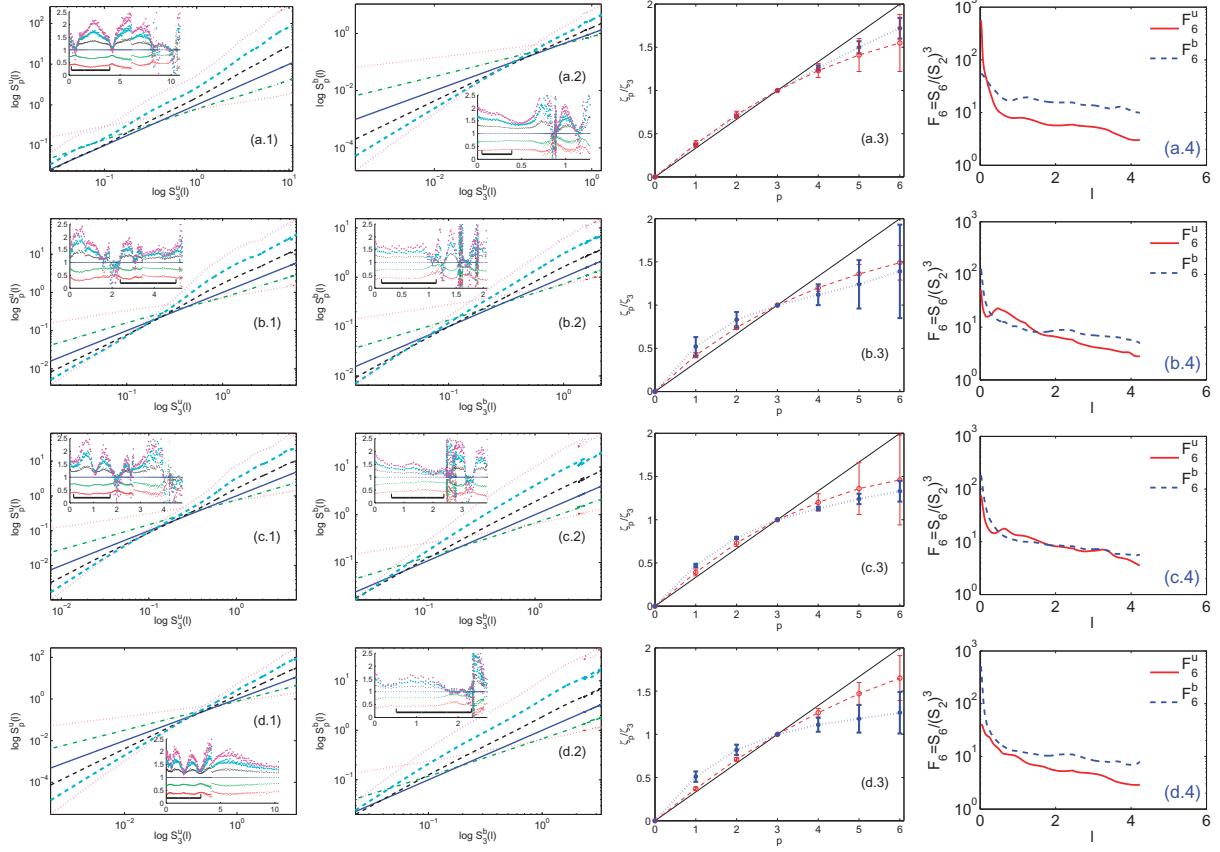
(b) at large values of  $Pr_M$  the magnetic field is more intermittent than the velocity field, insofar as the deviations of  $\zeta_p^b/\zeta_3^b$  from the K41 result  $p/3$  are larger than those of  $\zeta_p^u/\zeta_3^u$ ; (c) as we reduce  $Pr_M$  this difference in intermittency reduces until, at  $Pr_M = 0.01$ , the velocity field shows signs of becoming more intermittent than the magnetic field. This trend in intermittency is corroborated by plots versus  $l$  of the hyperflatnesses  $F_6^u(l) = \frac{S_6^u(l)}{S_2^u(l)^3}$  (red line) and  $F_6^b(l) = \frac{S_6^b(l)}{S_2^b(l)^3}$



**Figure 24.** Log–log (base 10) ESS plots of order- $p$  structure functions of the velocity  $S_p^u(l)$  ((a.1)–(d.1)) and magnetic-field  $S_p^b(l)$  ((a.2)–(d.2)) versus  $S_3^u(l)$  and  $S_3^b(l)$ , respectively; plots of the local slopes of these curves are shown in the inset. The black horizontal lines, with vertical ticks at their ends, show the inertial range over which we have averaged the exponent ratios  $\zeta_p^u/\zeta_3^u$  and  $\zeta_p^b/\zeta_3^b$ ; plots are shown for  $p = 1$  (red small-dotted line),  $p = 2$  (green dot-dashed line),  $p = 3$  (blue line),  $p = 4$  (black thin-dashed line),  $p = 5$  (cyan thick-dashed line) and  $p = 6$  (magenta large-dotted line). Subplots (a.3)–(d.3) show the exponent ratios  $\zeta_p/\zeta_3$  versus  $p$  for the velocity (red dashed line with thin error bars) and magnetic fields (blue dotted line with thick error bars); the black solid line shows the K41 result  $\zeta_p^{K41} = p/3$ . The semilog (base 10) plots (a.4)–(d.4) show the hyperflatnesses  $F_6^u(l)$  (red line) and  $F_6^b(l)$  (blue dashed line) versus  $l$ . Subplots in panels (a), (b), (c), and (d) are from our decaying-turbulence runs R1C, R2C, R3C and R4C, respectively, with  $Pr_M = 0.01, 0.1, 1$  and 10.

(blue dashed line) in figures 24(a.4)–(d.4) for runs R1C–R4C, respectively: As  $l$  decreases,  $F_6^b(l)$  rises more rapidly than  $F_6^u(l)$  except at  $Pr_M = 0.01$ .

Similar results follow from our studies of statistically steady MHD turbulence in runs R1D–R4D, which use  $512^3$  collocation points and span the  $Pr_M$  range 0.01–10. Figures 25(a.1)–(d.1) show ESS plots for  $S_p^u(r)$  for runs R1D–R4D, respectively, for  $p = 1$  (red small-dotted line),  $p = 2$  (green dot-dashed line),  $p = 3$  (blue line),  $p = 4$  (black thin-dashed



**Figure 25.** Log–log (base 10) ESS plots of order- $p$  structure functions of the velocity  $S_p^u(l)$  ((a.1)–(d.1)) and magnetic field  $S_p^b(l)$  ((a.2)–(d.2)) versus  $S_3^u(l)$  and  $S_3^b(l)$ , respectively; plots of the local slopes of these curves are shown in the inset. The black horizontal lines, with vertical ticks at their ends, show the inertial range over which we have averaged the exponent ratios  $\zeta_p^u/\zeta_3^u$  and  $\zeta_p^b/\zeta_3^b$ ; plots are shown for  $p = 1$  (red small-dotted line),  $p = 2$  (green dot-dashed line),  $p = 3$  (blue line),  $p = 4$  (black thin-dashed line),  $p = 5$  (cyan thick-dashed line), and  $p = 6$  (magenta large-dotted line). Subplots (a.3)–(d.3) show the exponent ratios  $\zeta_p/\zeta_3$  versus  $p$  for the velocity (red dashed line with thin error bars) and magnetic fields (blue dotted line with thick error bars); the black solid line shows the K41 result  $\zeta_p^{K41} = p/3$ . The semilog (base 10) plots (a.4)–(d.4) show the hyperflatnesses  $F_6^u(l)$  (red line) and  $F_6^b(l)$  (blue dashed line) versus  $l$ . Subplots in panels (a), (b), (c) and (d) are from our statistically steady MHD-turbulence runs R1D, R2D, R3D and R4D, respectively, with  $Pr_M = 0.01, 0.1, 1$  and 10.

line),  $p = 5$  (cyan thick-dashed line) and  $p = 6$  (magenta large-dotted line); their analogues for  $S_p^b(r)$  are given in figures 25(a.2)–(d.2); the local slopes of these ESS curves are shown in the insets of these figures. We obtain estimates for the exponent ratio  $\zeta_p^u/\zeta_3^u$  and  $\zeta_p^b/\zeta_3^b$  and their error bars, as in figure 24. Figures 25(a.3)–(d.3) show plots of these exponent ratios versus  $p$  for the velocity field (blue dotted line with thick error bars) and the magnetic field (red dashed line with thin error bars); the black solid line shows the K41 result for comparison. Plots versus  $l$  of the hyperflatnesses  $F_6^u(l)$  (red line) and  $F_6^b(l)$  (blue dashed line) are presented

**Table 6.** Multiscaling exponent ratios  $\zeta_p^u/\zeta_3^u$  and  $\zeta_p^b/\zeta_3^b$  from our decaying-MHD-turbulence runs R1C–R4C.

$p$	$\zeta_p^u/\zeta_3^u; \zeta_p^b/\zeta_3^b(Pr_M = 0.01)$	$\zeta_p^u/\zeta_3^u; \zeta_p^b/\zeta_3^b(Pr_M = 0.1)$
1	$0.41 \pm 0.04; 0.35 \pm 0.01$	$0.39 \pm 0.09; 0.42 \pm 0.04$
2	$0.74 \pm 0.04; 0.68 \pm 0.01$	$0.71 \pm 0.08; 0.74 \pm 0.03$
3	$1.00 \pm 0.00; 1.00 \pm 0.00$	$1.00 \pm 0.00; 1.00 \pm 0.00$
4	$1.21 \pm 0.09; 1.29 \pm 0.03$	$1.26 \pm 0.14; 1.20 \pm 0.03$
5	$1.38 \pm 0.22; 1.56 \pm 0.06$	$1.51 \pm 0.32; 1.37 \pm 0.07$
6	$1.52 \pm 0.41; 1.80 \pm 0.10$	$1.76 \pm 0.53; 1.52 \pm 0.13$
$p$	$\zeta_p^u/\zeta_3^u; \zeta_p^b/\zeta_3^b(Pr_M = 1)$	$\zeta_p^u/\zeta_3^u; \zeta_p^b/\zeta_3^b(Pr_M = 10)$
1	$0.42 \pm 0.03; 0.49 \pm 0.04$	$0.40 \pm 0.06; 0.50 \pm 0.05$
2	$0.74 \pm 0.03; 0.80 \pm 0.04$	$0.72 \pm 0.05; 0.81 \pm 0.04$
3	$1.00 \pm 0.00; 1.00 \pm 0.00$	$1.00 \pm 0.00; 1.00 \pm 0.00$
4	$1.25 \pm 0.06; 1.15 \pm 0.07$	$1.27 \pm 0.08; 1.15 \pm 0.05$
5	$1.50 \pm 0.16; 1.27 \pm 0.18$	$1.55 \pm 0.18; 1.29 \pm 0.12$
6	$1.74 \pm 0.30; 1.38 \pm 0.32$	$1.83 \pm 0.31; 1.45 \pm 0.23$

**Table 7.** Multiscaling exponent ratios  $\zeta_p^u/\zeta_3^u$  and  $\zeta_p^b/\zeta_3^b$  from our statistically steady MHD-turbulence runs R1D–R4D.

$p$	$\zeta_p^u/\zeta_3^u; \zeta_p^b/\zeta_3^b(Pr_M = 0.01)$	$\zeta_p^u/\zeta_3^u; \zeta_p^b/\zeta_3^b(Pr_M = 0.1)$
1	$0.38 \pm 0.04; 0.37 \pm 0.01$	$0.42 \pm 0.03; 0.52 \pm 0.11$
2	$0.72 \pm 0.04; 0.70 \pm 0.01$	$0.74 \pm 0.02; 0.83 \pm 0.09$
3	$1.00 \pm 0.00; 1.00 \pm 0.00$	$1.00 \pm 0.00; 1.00 \pm 0.00$
4	$1.23 \pm 0.08; 1.26 \pm 0.03$	$1.20 \pm 0.04; 1.12 \pm 0.12$
5	$1.41 \pm 0.19; 1.50 \pm 0.07$	$1.36 \pm 0.11; 1.24 \pm 0.28$
6	$1.55 \pm 0.33; 1.72 \pm 0.12$	$1.49 \pm 0.20; 1.39 \pm 0.54$
$p$	$\zeta_p^u/\zeta_3^u; \zeta_p^b/\zeta_3^b(Pr_M = 1)$	$\zeta_p^u/\zeta_3^u; \zeta_p^b/\zeta_3^b(Pr_M = 10)$
1	$0.39 \pm 0.04; 0.47 \pm 0.02$	$0.37 \pm 0.02; 0.51 \pm 0.06$
2	$0.73 \pm 0.04; 0.79 \pm 0.01$	$0.71 \pm 0.02; 0.82 \pm 0.06$
3	$1.00 \pm 0.00; 1.00 \pm 0.00$	$1.00 \pm 0.00; 1.00 \pm 0.00$
4	$1.20 \pm 0.10; 1.13 \pm 0.02$	$1.25 \pm 0.05; 1.11 \pm 0.08$
5	$1.36 \pm 0.30; 1.24 \pm 0.06$	$1.47 \pm 0.13; 1.18 \pm 0.16$
6	$1.46 \pm 0.52; 1.33 \pm 0.12$	$1.65 \pm 0.26; 1.25 \pm 0.24$

in figures 25(a.4)–(d.4) for runs R1D–R4D, respectively. All of the trends here are exactly as in the decaying-MHD-turbulence plots in figure 24.

Tables 6 and 7 summarize, respectively, our results for multiscaling exponent ratios for our decaying-MHD-turbulence runs R1C–R4C and our statistically steady MHD-turbulence runs R1D–R4D. The trends of these ratios with  $Pr_M$  have been discussed above. By comparing corresponding entries in the columns and rows of these tables, we see that exponent ratios from decaying and statistically steady MHD turbulence agree, given our (conservative) error

bars. Thus, at least at this level of resolution and accuracy, we have strong universality of these exponent ratios, for a given value of  $Pr_M$ , in as much as the ratios from decaying-MHD turbulence agree with those from the statistically steady case. The dependence on  $Pr_M$  will be examined in section 4. We have tried to fit our data for the multiscaling exponent ratios to the generalized She–Leveque formula used in [25, 75] to extract the codimensions of dissipative structures; these fits are not very good; they yield values for these codimensions that are close to or slightly lower than 1 and are closer to the estimates of [25] than those of [75]. (See [75] for a discussion of possible reasons for the discrepancies between these estimates.) Given the large error bars in multiscaling exponent ratios, we suggest that such fits are fraught with considerable uncertainty.

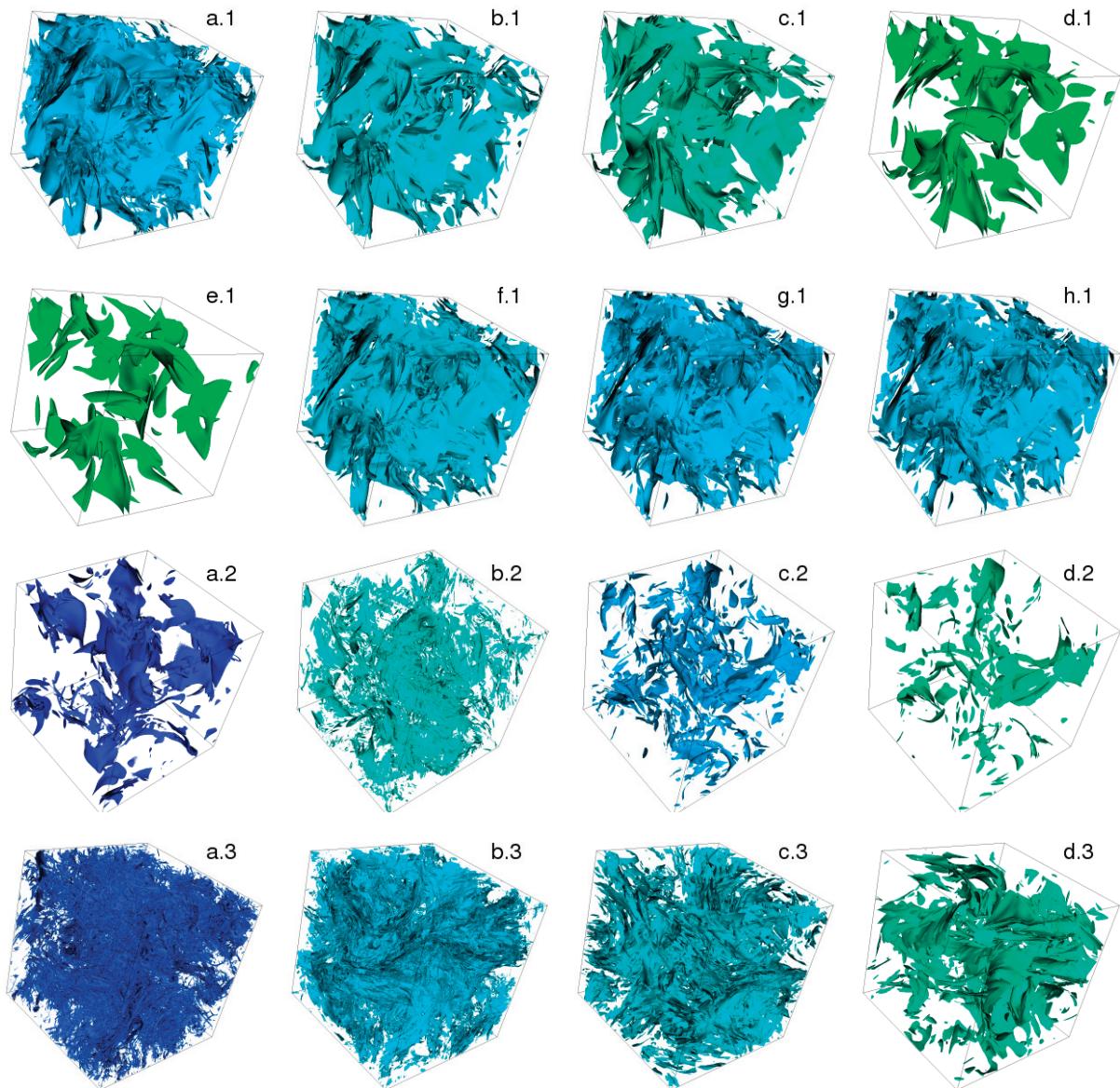
### 3.6. Isosurfaces

As we have mentioned in our discussion of fluid turbulence, isosurface plots of quantities such as  $\omega$ , the modulus of the vorticity, give us a visual appreciation of small-scale structures in a turbulent flow; in fluid turbulence, iso- $\omega$  surfaces are slender tubes if  $\omega$  is chosen to be well above its mean value [66, 72]. For the case of MHD turbulence, it is natural to consider isosurface plots [76] of  $\omega$ , the modulus  $j$  of the current density, energy dissipation rates and the effective pressure.

Isosurfaces of  $\omega$  are shown at  $t_c$  for runs R1–R5 in figures 26(a.1)–(e.1), runs R3B–R5B in figures 26(f.1)–(h.1) and runs R1C–R4C in figures 26(a.2)–(d.2) for decaying MHD turbulence; and for statistically steady MHD turbulence they are shown in figures 26(a.3)–(d.3) for runs R1D–R4D; these isosurfaces go through points at which the value of  $\omega$  is two standard deviations above its mean value (for any given plot). For  $Pr_M = 1$ , it has been noted in several DNS studies that such isosurfaces are sheets [5, 25, 76, 77] and that there is a general tendency for such sheet formation in MHD turbulence; our results show that this tendency persists even when  $Pr_M \neq 1$ . The number of high-intensity isosurfaces of  $\omega$  shrinks as we increase  $Pr_M$  (figures 26(a.1)–(e.1) for runs R1–R5, respectively), by increasing  $\nu$  while holding the initial energy fixed. However, if we compensate for the increase in  $\nu$  by increasing the energy in the initial condition such that  $k_{\max} \eta_d^u$  and  $k_{\max} \eta_d^b$  are both  $\simeq 1$ , we see that high- $\omega$  sheets reappear (figures 26(f.1)–(h.1) for runs R3B–R5B, respectively). These trends are also visible in our high-resolution, decaying-MHD-turbulence runs R1C–R4C (figures 26(a.2)–(d.2)) and the statistically steady ones, namely R1D–R4D (figures 26(a.3)–(d.3)). One interesting point that has not been noted before is that some tube-type structures appear along with the sheets at small values of  $Pr_M$ , as can be seen by enlarging figure 26(a.3) for run R1D.

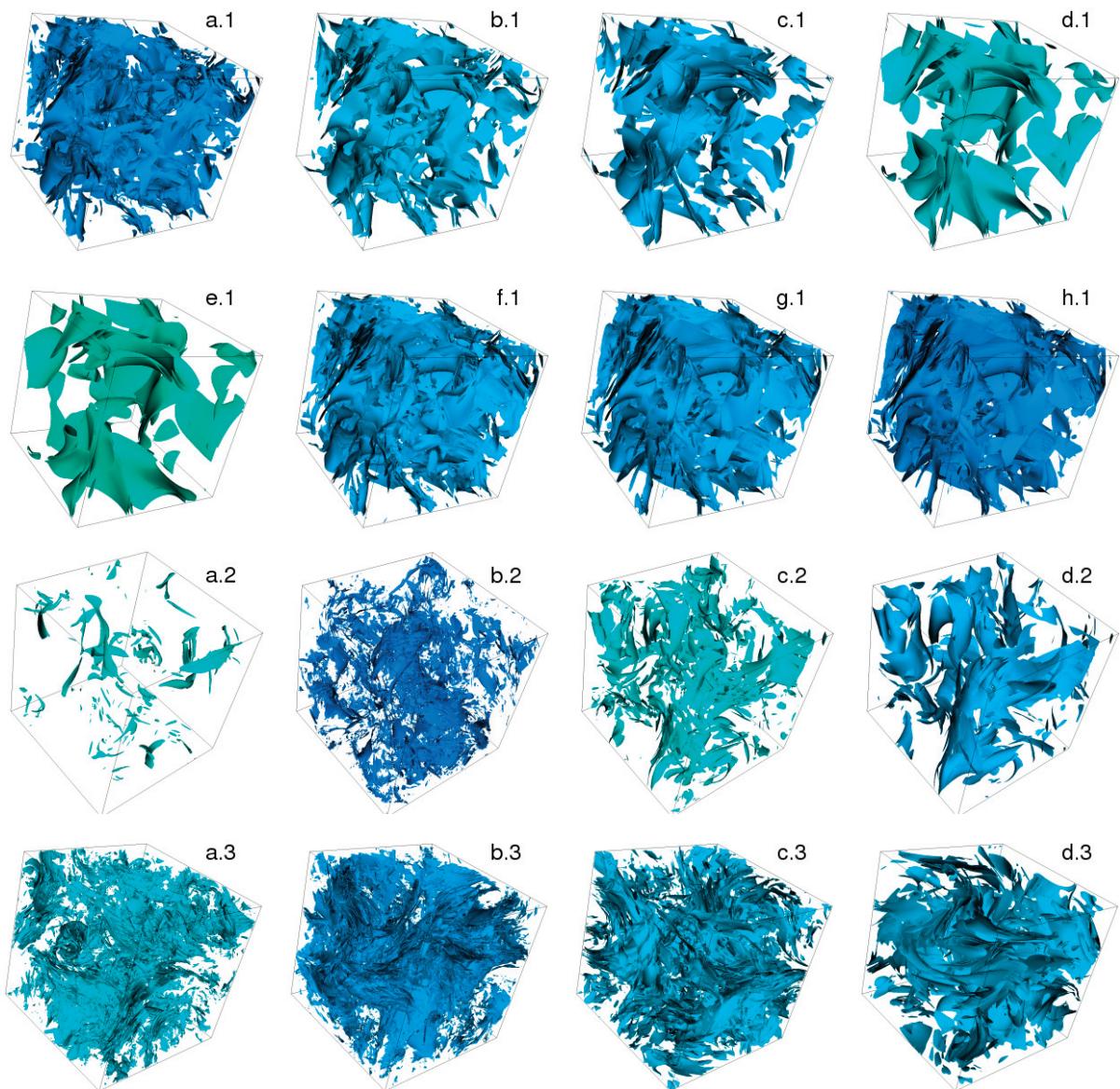
Similar features and trends appear in isosurfaces of  $j$  that are shown at  $t_c$  for runs R1–R5 in figures 27(a.1)–(e.1), runs R3B–R5B in figures 27(f.1)–(h.1) and runs R1C–R4C in figures 27(a.2)–(d.2) for decaying MHD turbulence; and for statistically steady MHD turbulence they are shown in figures 27(a.3)–(d.3) for runs R1D–R4D; these isosurfaces go through points at which the value of  $j$  is two standard deviations above its mean value (for any given plot). Again, the dominant features in these isosurface plots are sheets; their number goes down as  $Pr_M$  increases with  $\nu$  while the initial energy is held constant; but if this energy is increased, the number of high-intensity sheets increases.

Isosurfaces of  $\epsilon_u$  are shown at  $t_c$  for runs R1–R5 in figures 28(a.1)–(e.1), runs R3B–R5B in figures 28(f.1)–(h.1) and runs R1C–R4C in figures 28(a.2)–(d.2) for decaying MHD turbulence; and for statistically steady MHD turbulence they are shown in figures 28(a.3)–(d.3) for runs R1D–R4D; the isosurfaces go through points at which the value of  $\epsilon_u$  is two standard deviations



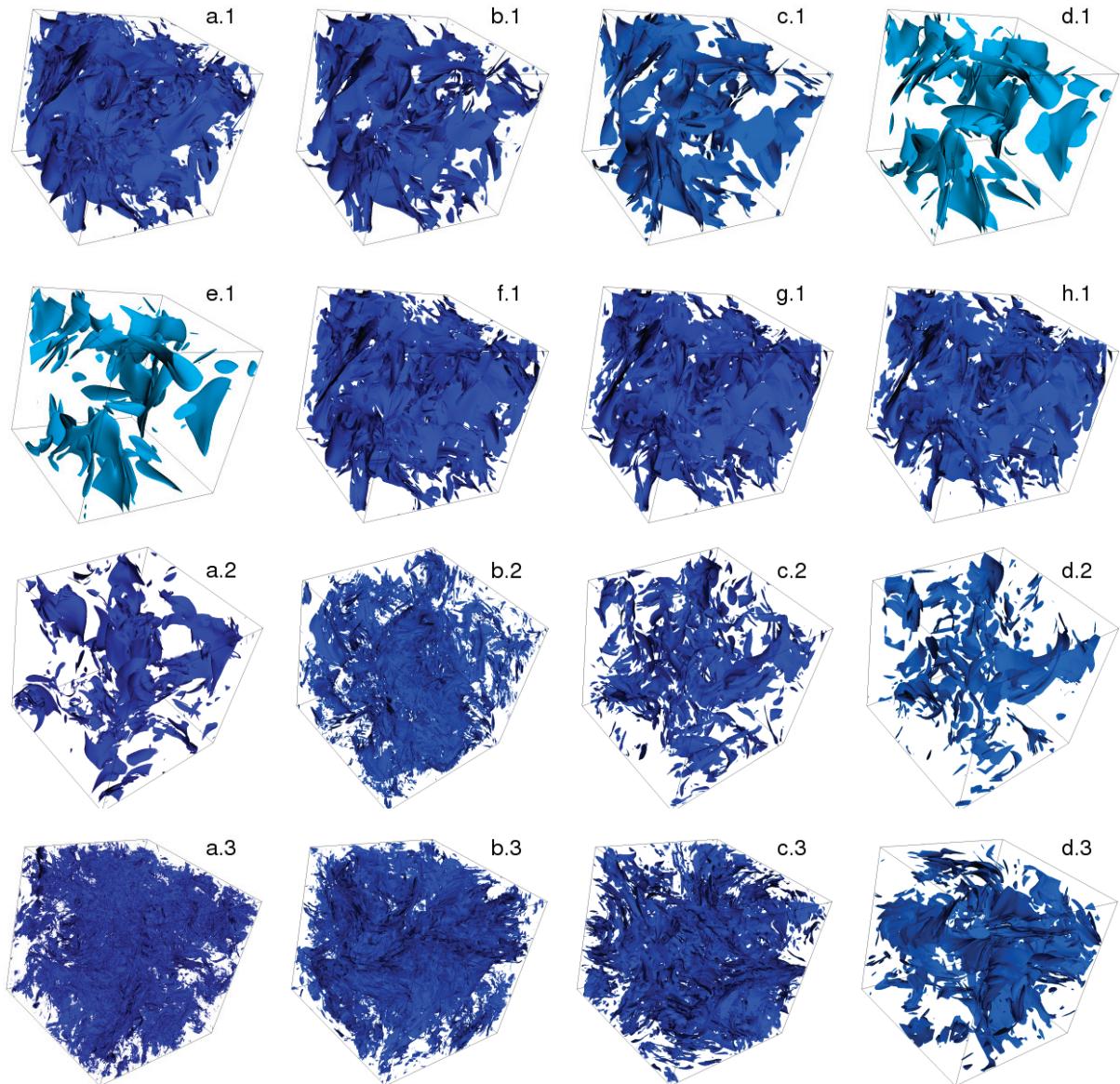
**Figure 26.** Isosurfaces of the modulus  $\omega$  of the vorticity: (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), (a.2)  $Pr_M = 0.01$  (R1C), (b.2)  $Pr_M = 0.1$  (R2C), (c.2)  $Pr_M = 1.0$  (R3C) and (d.2)  $Pr_M = 10.0$  (R4C) for decaying MHD turbulence; and for statistically steady MHD turbulence (a.3)  $Pr_M = 0.01$  (R1D), (b.3)  $Pr_M = 0.1$  (R2D), (c.3)  $Pr_M = 1.0$  (R3D) and (d.3)  $Pr_M = 10.0$  (R4D); these isosurfaces go through points at which the value of  $\omega$  is two standard deviations above its mean value (for any given plot).

above its mean value (for any given plot). Similar isosurfaces of  $\epsilon_b$  are shown at  $t_c$  for runs R1–R5 in figures 29(a.1)–(e.1), runs R3B–R5B in figures 29(f.1)–(h.1) and runs R1C–R4C in figures 29(a.2)–(d.2) for decaying MHD turbulence; and for statistically steady MHD turbulence



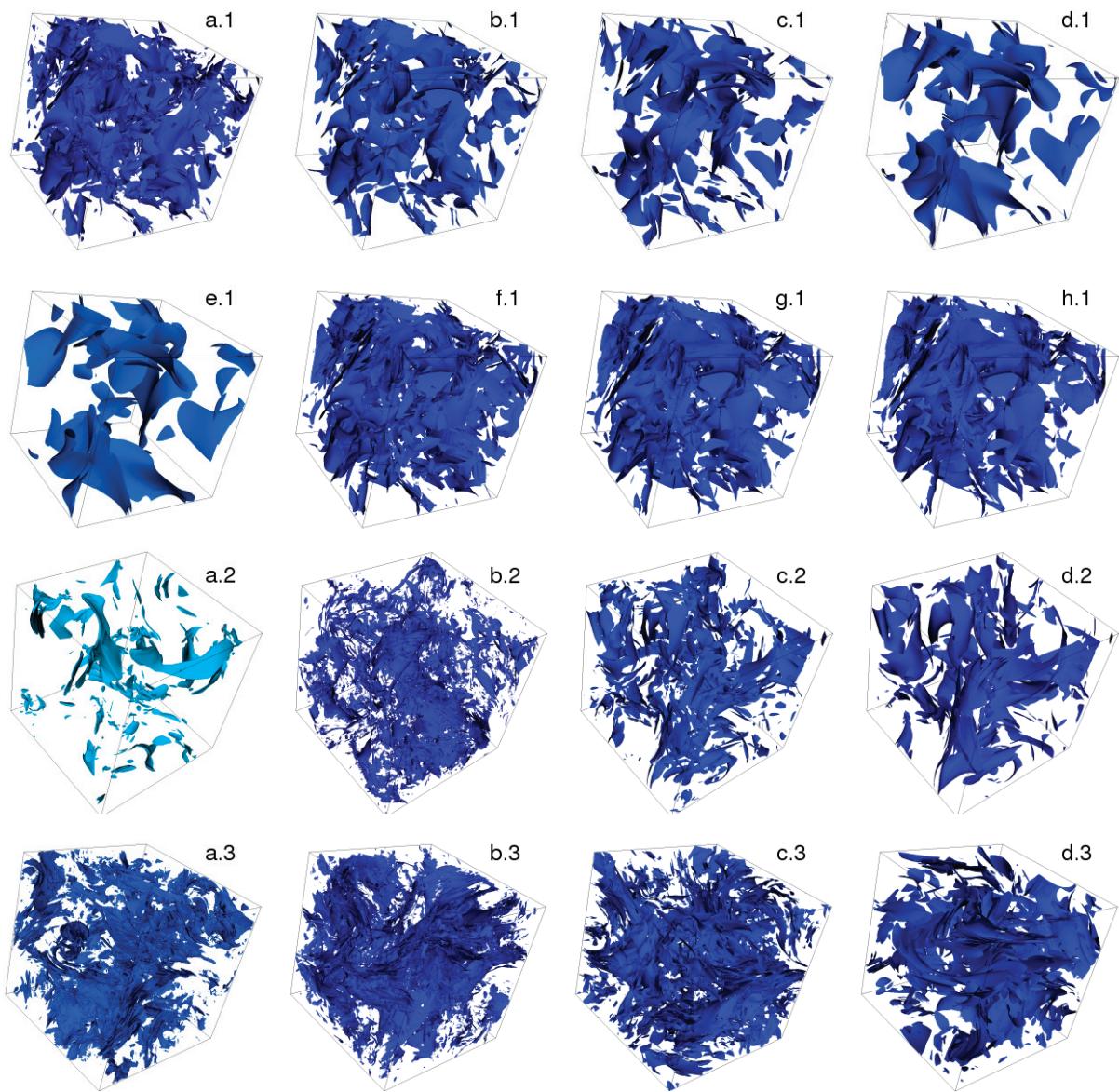
**Figure 27.** Isosurfaces of the modulus  $j$  of the current density: (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), (a.2)  $Pr_M = 0.01$  (R1C), (b.2)  $Pr_M = 0.1$  (R2C), (c.2)  $Pr_M = 1.0$  (R3C) and (d.2)  $Pr_M = 10.0$  (R4C) for decaying MHD turbulence; and for statistically steady MHD turbulence (a.3)  $Pr_M = 0.01$  (R1D), (b.3)  $Pr_M = 0.1$  (R2D), (c.3)  $Pr_M = 1.0$  (R3D) and (d.3)  $Pr_M = 10.0$  (R4D); these isosurfaces go through points at which the value of  $j$  is two standard deviations above its mean value (for any given plot).

they are shown in figures 29(a.3)–(d.3) for runs R1D–R4D; the isosurfaces go through points at which the value of  $\epsilon_b$  is two standard deviations above its mean value (for any given plot). Here too, the isosurfaces are sheets; they lie close to, but are not coincident with, isosurfaces of  $\omega$  and  $j$ ; changes in  $Pr_M$  affect these isosurfaces much as they affect isosurfaces of  $\omega$  and  $j$ .



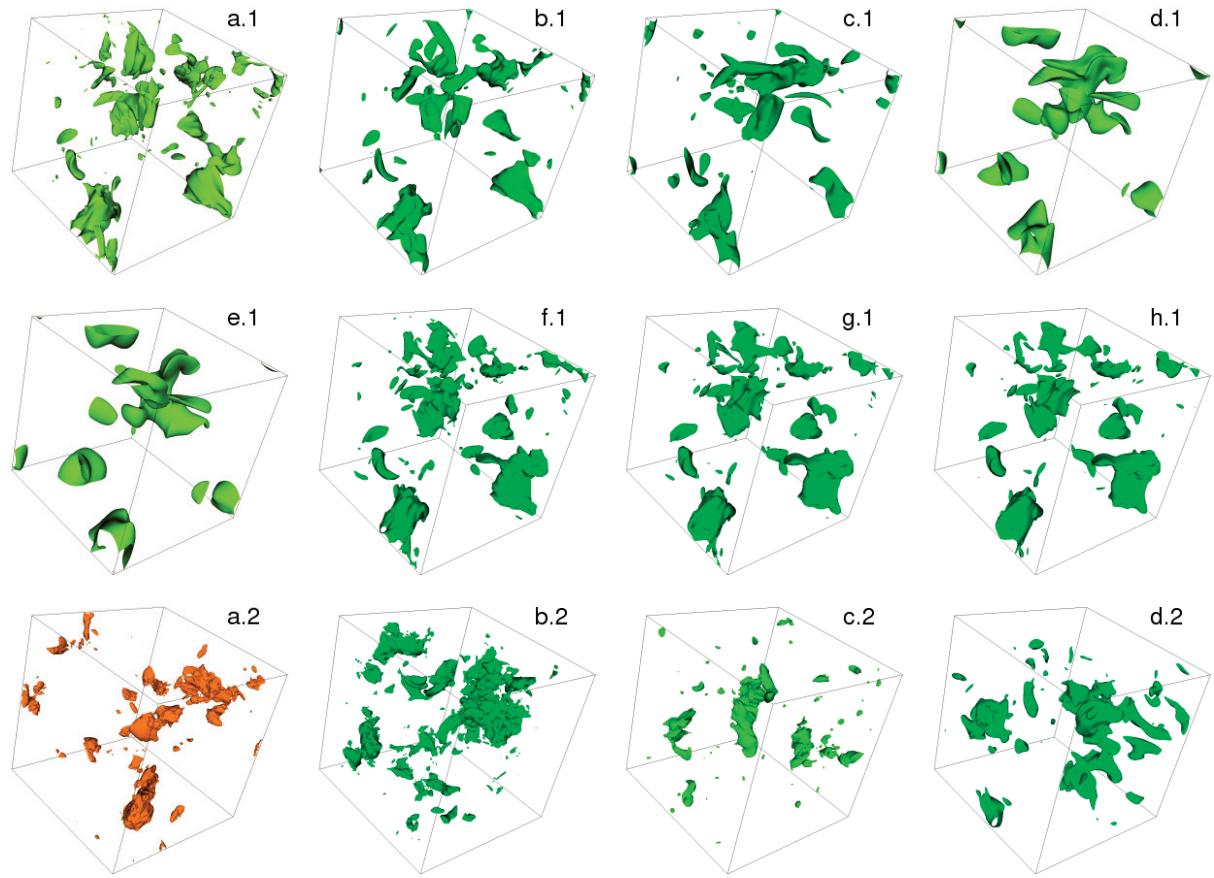
**Figure 28.** Isosurfaces of the local fluid energy dissipation rate  $\epsilon_u$ : (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), (a.2)  $Pr_M = 0.01$  (R1C), (b.2)  $Pr_M = 0.1$  (R2C), (c.2)  $Pr_M = 1.0$  (R3C) and (d.2)  $Pr_M = 10.0$  (R4C) for decaying MHD turbulence; and for statistically steady MHD turbulence (a.3)  $Pr_M = 0.01$  (R1D), (b.3)  $Pr_M = 0.1$  (R2D), (c.3)  $Pr_M = 1.0$  (R3D) and (d.3)  $Pr_M = 10.0$  (R4D); these isosurfaces go through points at which the value of  $\epsilon_u$  is two standard deviations above its mean value (for any given plot).

Isosurfaces of  $\bar{p}$  are shown at  $t_c$  for runs R1–R5 in figures 30(a.1)–(e.1) and runs R3B–R5B in figures 30(f.1)–(h.1) for decaying MHD turbulence; and for statistically steady MHD turbulence they are shown in figures 30(a.3)–(d.3) for runs R1D–R4D; the isosurfaces



**Figure 29.** Isosurfaces of the local magnetic-energy dissipation rate  $\epsilon_b$ : (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B) and (h.1)  $Pr_M = 10.0$  (R5B). (a.2)  $Pr_M = 0.01$  (R1C), (b.2)  $Pr_M = 0.1$  (R2C), (c.2)  $Pr_M = 1.0$  (R3C) and (d.2)  $Pr_M = 10.0$  (R4C) for decaying MHD turbulence; and for statistically steady MHD turbulence (a.3)  $Pr_M = 0.01$  (R1D), (b.3)  $Pr_M = 0.1$  (R2D), (c.3)  $Pr_M = 1.0$  (R3D) and (d.3)  $Pr_M = 10.0$  (R4D); these isosurfaces go through points at which the value of  $\epsilon_b$  is two standard deviations above its mean value (for any given plot).

go through points at which the value of  $\bar{p}$  is two standard deviations above its mean value (for any given plot). The general form of these isosurfaces is cloud-type, to borrow the term that has been used for isosurfaces of the pressure in fluid turbulence [61]. Here also changes in  $Pr_M$



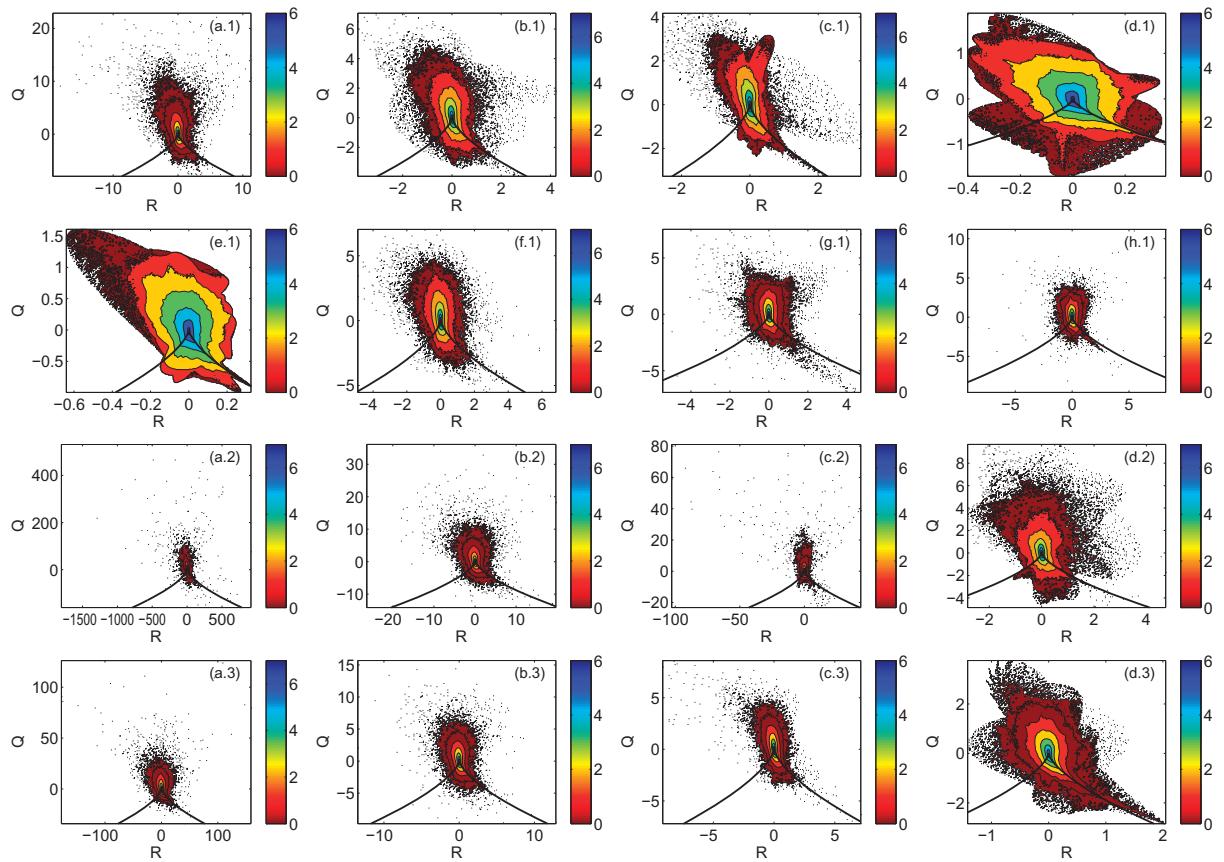
**Figure 30.** Isosurfaces of the local effective pressure  $\bar{p}$ : (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B) and (h.1)  $Pr_M = 10.0$  (R5B) for decaying MHD turbulence; and for statistically steady MHD turbulencs (a.2)  $Pr_M = 0.01$  (R1D), (b.2)  $Pr_M = 0.1$  (R2D), (c.2)  $Pr_M = 1.0$  (R3D) and (d.2)  $Pr_M = 10.0$  (R4D); these isosurfaces go through points at which the value of  $\bar{p}$  is two standard deviations above its mean value (for any given plot).

affect these isosurfaces much as they affect isosurfaces of  $\omega$  and  $j$ , in as much as high-intensity isosurfaces are suppressed as  $Pr_M$  increases via an increase in  $\nu$ , unless this is compensated for by an increase in the initial energy (in the case of decaying MHD turbulence) or  $Re_\lambda$ .

### 3.7. Joint probability distribution functions (PDFs)

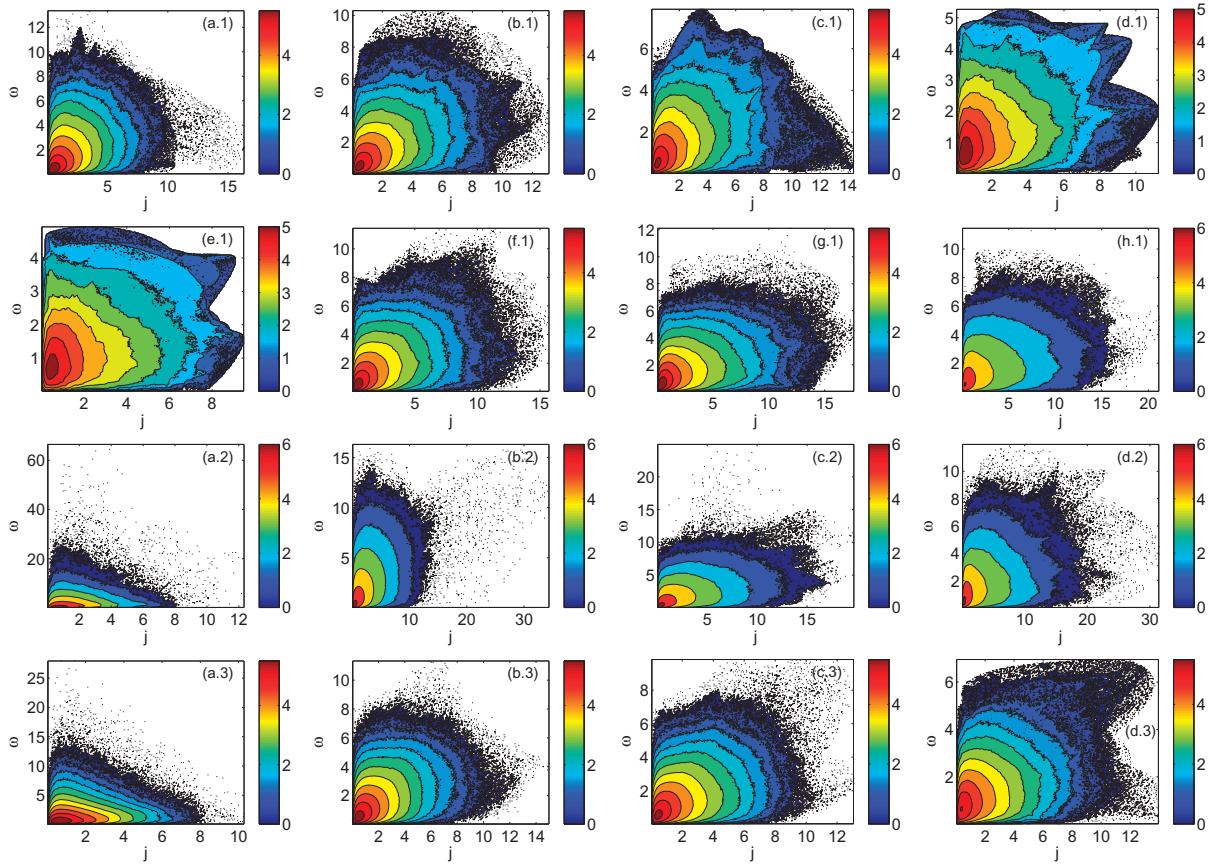
In this subsection, we present three sets of joint PDFs that have, to the best of our knowledge, not been used to characterize MHD turbulence previously. The first of these is a  $QR$  plot that is often used in studies of fluid turbulence, as we have discussed in sections 2.2 and 3.1; the next is a joint PDF of  $\omega$  and  $j$ ; and the last is a joint PDF of  $\epsilon_u$  and  $\epsilon_b$ .

We show  $QR$  plots, i.e. joint PDFs of  $Q$  and  $R$ , via filled contour plots; these are obtained at  $t_c$  for runs R1–R5 in figures 31(a.1)–(e.1), runs R3B–R5B in figures 31(f.1)–(h.1) and runs R1C–R4C in figures 31(a.2)–(d.2) for decaying MHD turbulence; and for statistically steady



**Figure 31.**  $QR$  plots, i.e. joint PDFs of  $Q$  and  $R$ , shown as filled contour plots on a logarithmic scale for (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), (a.2)  $Pr_M = 0.01$  (R1C), (b.2)  $Pr_M = 0.1$  (R2C), (c.2)  $Pr_M = 1.0$  (R3C) and (d.2)  $Pr_M = 10.0$  (R4C) for decaying MHD turbulence; and for statistically steady MHD turbulence (a.3)  $Pr_M = 0.01$  (R1D), (b.3)  $Pr_M = 0.1$  (R2D), (c.3)  $Pr_M = 1.0$  (R3D) and (d.3)  $Pr_M = 10.0$  (R4D). The arguments  $Q$  and  $R$  of the  $QR$  plots are normalised by  $\langle \omega^2 \rangle$  and  $\langle \omega^2 \rangle^{3/2}$ , respectively. The black curve is the zero-discriminant line  $D \equiv \frac{27}{4}R^2 + Q^3 = 0$ .

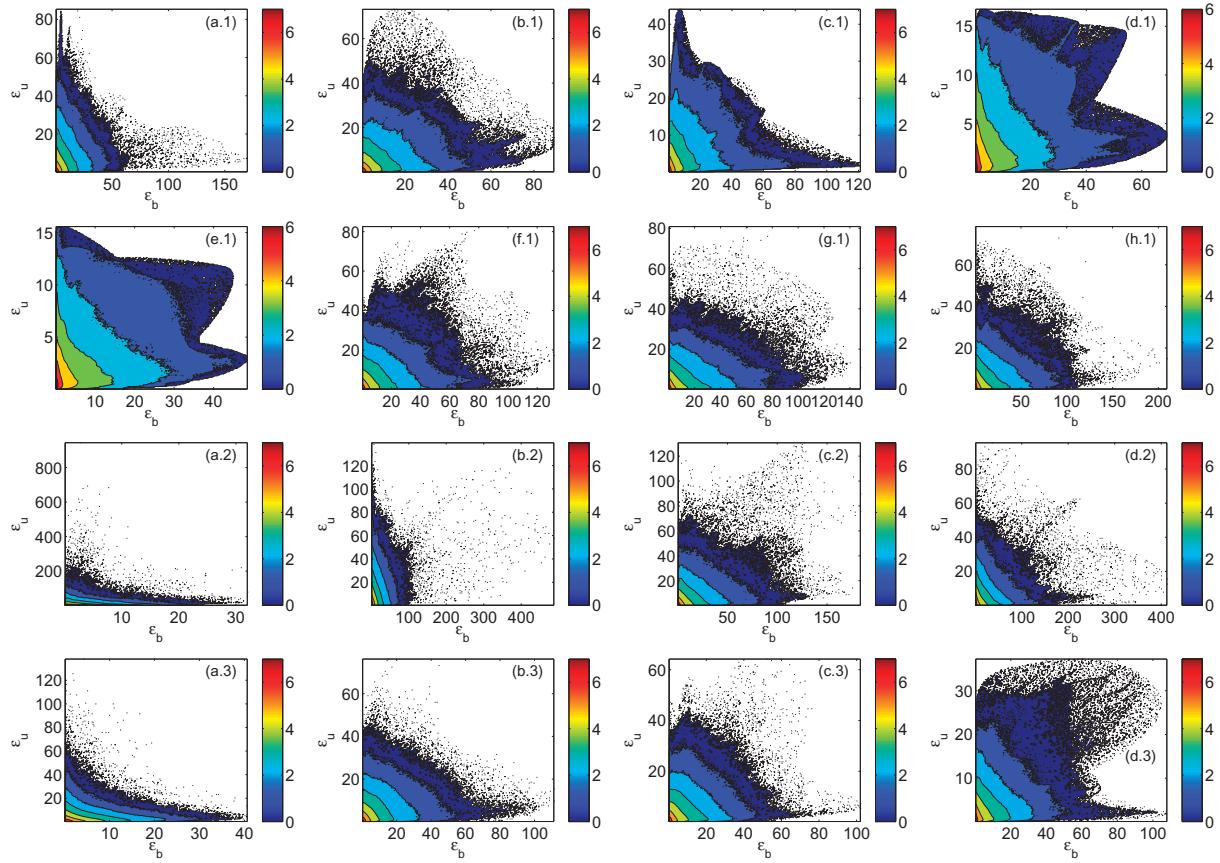
MHD turbulence they are shown in figures 31(a.3)–(d.3) for runs R1D–R4D; the black curve in these plots is the zero-discriminant line  $D \equiv \frac{27}{4}R^2 + Q^3 = 0$ . These  $QR$  plots retain overall, aside from some distortions, the characteristic tear-drop structure familiar from fluid turbulence (see section 3.1 and figure 5). If we recall our discussion of  $QR$  plots in section 2.2 and we note that, as we increase  $Pr_M$  (figures 31(a.1)–(e.1) for runs R1–R5, respectively) while holding  $\eta$  and the initial energy fixed, there is a general decrease in the probability of having large values of  $Q$  and  $R$ , i.e. regions of large strain or vorticity are suppressed; this corroborates what we have found from the PDFs and isosurfaces discussed above. However, if we compensate for the increase in  $\nu$  by increasing the initial energy, or  $Re_\lambda$ , so that  $k_{\max}\eta_d^u$  and  $k_{\max}\eta_d^b$  are both  $\approx 1$ , we see that  $Q$  and  $R$  can increase again. Note that when  $Pr_M$  is very small, as in run



**Figure 32.** Joint PDFs of  $\omega$  and  $j$  shown as filled contour plots on a logarithmic scale for (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), (a.2)  $Pr_M = 0.01$  (R1C), (b.2)  $Pr_M = 0.1$  (R2C), (c.2)  $Pr_M = 1.0$  (R3C) and (d.2)  $Pr_M = 10.0$  (R4C) for decaying MHD turbulence; and for statistically steady MHD turbulence (a.3)  $Pr_M = 0.01$  (R1D), (b.3)  $Pr_M = 0.1$  (R2D), (c.3)  $Pr_M = 1.0$  (R3D) and (d.3)  $Pr_M = 10.0$  (R4D). The arguments of the joint PDFs are normalized by their standard deviations.

R1D (figure 31(a.3)), the tear-drop structure is very much like its fluid-turbulence counterpart figure 5, which might well correlate with the appearance of some tube-type structures in the  $\omega$  isosurface in enlarged versions of figure 26(a.3).

We now consider joint PDFs of  $\omega$  and  $j$  that are obtained at  $t_c$  for runs R1–R5 in figures 32(a.1)–(e.1), runs R3B–R5B in figures 32(f.1)–(h.1) and runs R1C–R4C in figures 32(a.2)–(d.2) for decaying MHD turbulence; and for statistically steady MHD turbulence they are shown in figures 32(a.3)–(d.3) for runs R1D–R4D. All of these joint PDFs have long tails; as we move away from  $Pr_M = 1$ , they become more and more asymmetrical. Furthermore, as we expect, the tails of these PDFs are drawn in towards small values of  $\omega$  and  $j$  as we increase  $Pr_M$  (figures 32(a.1)–(e.1) for runs R1–R5, respectively) while holding  $\eta$  and the initial energy fixed. However, if we compensate for the increase in  $\nu$  by increasing the initial energy or  $Re_\lambda$ , so that  $k_{\max} \eta_d^u$  and  $k_{\max} \eta_d^b$  are both  $\simeq 1$ , we see that the tails of the PDFs become elongated again.



**Figure 33.** Joint PDFs of  $\epsilon_u$  and  $\epsilon_b$  shown as filled contour plots on a logarithmic scale for (a.1)  $Pr_M = 0.1$  (R1), (b.1)  $Pr_M = 0.5$  (R2), (c.1)  $Pr_M = 1.0$  (R3), (d.1)  $Pr_M = 5.0$  (R4), (e.1)  $Pr_M = 10.0$  (R5), (f.1)  $Pr_M = 1.0$  (R3B), (g.1)  $Pr_M = 5.0$  (R4B), (h.1)  $Pr_M = 10.0$  (R5B), (a.2)  $Pr_M = 0.01$  (R1C), (b.2)  $Pr_M = 0.1$  (R2C), (c.2)  $Pr_M = 1.0$  (R3C) and (d.2)  $Pr_M = 10.0$  (R4C) for decaying MHD turbulence; and for statistically steady MHD turbulence (a.3)  $Pr_M = 0.01$  (R1D), (b.3)  $Pr_M = 0.1$  (R2D), (c.3)  $Pr_M = 1.0$  (R3D) and (d.3)  $Pr_M = 10.0$  (R4D). The arguments of the joint PDFs are normalized by their standard deviations.

In the end, we consider joint PDFs of  $\epsilon_u$  and  $\epsilon_b$  that are obtained at  $t_c$  for runs R1–R5 in figures 33(a.1)–(e.1), runs R3B–R5B in figures 33(f.1)–(h.1) and runs R1C–R4C in figures 33(a.2)–(d.2) for decaying MHD turbulence; and for statistically steady MHD turbulence they are shown in figures 33(a.3)–(d.3) for runs R1D–R4D. The trends here are similar to the ones discussed in the previous paragraph. In particular, these joint PDFs have long tails; as we move away from  $Pr_M = 1$ , they become more and more asymmetrical; and the tails of these PDFs are drawn in towards small values of  $\epsilon_u$  and  $\epsilon_b$  as we increase  $Pr_M$  (figures 33(a.1)–(e.1) for runs R1–R5, respectively) while holding  $\eta$  and the initial energy fixed. But if we make up for the increase in  $\nu$  by increasing the initial energy or  $Re_\lambda$  so that  $k_{\max} \eta_d^u$  and  $k_{\max} \eta_d^b$  are both  $\approx 1$ , we see that the tails of the PDFs become elongated again.

#### 4. Discussions and conclusion

We have carried out an extensive study of the statistical properties of both decaying and statistically steady homogeneous, isotropic MHD turbulence. Our study, which has been designed specifically to study the systematics of the dependence of these properties on the magnetic Prandtl number  $Pr_M$ , uses a large number of statistical measures to characterize the statistical properties of both decaying and statistically steady MHD turbulence. Our study is restricted to incompressible MHD turbulence; we do not include a mean magnetic field as, e.g., in [33]; furthermore, we do not study Lagrangian properties considered, e.g., in [34]. In our studies, we obtain (a) various PDFs, such as those of the moduli of the vorticity and current density, the energy dissipation rates, of cosines of angles between various vectors and scale-dependent velocity and magnetic-field increments, (b) spectra, e.g. those of the energy and the effective pressure, (c) velocity and magnetic-field structure functions that can be used to characterize intermittency, (d) isosurfaces of quantities such as the moduli of the vorticity and current, and (e) joint PDFs, such as  $QR$  plots. The evolution of these properties with  $Pr_M$  has been described in detail in the previous section.

To the best of our knowledge, such a comprehensive study of the  $Pr_M$  dependence of incompressible, homogeneous, isotropic MHD turbulence, both decaying and statistically steady, has not been attempted before. Studies that draw their inspiration from astrophysics often consider anisotropic flows [78]–[85], flows that are compressible [35, 86] or flows that include a mean magnetic field [33, 83, 87, 88]. Yet other studies concentrate on the alignment between various vectors such as  $\mathbf{u}$  and  $\mathbf{b}$  as, e.g., in [28, 35, 53]; some of these include a few, but not all, of the PDFs we have studied; and, typically, these studies are restricted to the case  $Pr_M = 1$ . Some of the spectra we study have been obtained in earlier DNS studies but, typically, only for the case  $Pr_M = 1$ ; a notable exception is [38], which examines the  $Pr_M$  dependence of energy spectra but with a relatively low resolution. The papers [6, 32, 89] have also considered some  $Pr_M$  dependence but not for low  $Pr_M$ . Isosurfaces of the moduli of vorticity and current density have been obtained earlier [25, 39, 76] for the case  $Pr_M = 1$ . The  $Pr_M$  dependence of these and other isosurfaces is presented here for the first time. The joint PDFs we have shown above have also not been investigated previously.

Here we wish to highlight, and examine in detail, the implications of our study for intermittency. Some earlier DNS studies, such as [30], noted that, for the case  $Pr_M = 1$ , the magnetic field is more intermittent than the velocity field. This is why we have concentrated on velocity and magnetic-field structure functions. Our study confirms this finding, for the case  $Pr_M = 1$ . This can be clearly seen from the comparison of our exponent ratios, for  $Pr_M = 1$ , with those of the recent DNS of decaying-MHD turbulence in [30] in table 8; the error bars that we quote for our exponent ratios have been calculated as described in the previous section; we have obtained exponent ratios of the paper [30] by digitizing<sup>5</sup> the data in their plot (figure 3 of [30]) of multiscaling exponents versus the order  $p$  (error bars are not given in their plot). Thus, at least given our error bars, there is agreement between our exponent ratios, both for decaying and statistically steady MHD turbulence, and those of [30] for  $Pr_M = 1$ . We note in passing that the latter DNS is one of decaying MHD turbulence but with a very special initial condition, which allows an effective resolution greater than that we have obtained; however, the initial condition we use in our decaying-MHD-turbulence DNS is more generic than that of [30]. It is

<sup>5</sup> For digitizing we use G3DATA software (<http://www.frantz.fi/software/g3data.php>) available in the Ubuntu repositories.

**Table 8.** A comparison of multiscaling exponent ratios  $\zeta_p^u/\zeta_3^u$  and  $\zeta_p^b/\zeta_3^b$  from our statistically steady and decaying MHD simulations and from decaying MHD simulations by Mininni and Pouquet ([30]), for  $Pr_M = 1$ .

$p$	$\zeta_p^u$ [30]	$\zeta_p^u/\zeta_3^u$ [30]	$\zeta_p^u/\zeta_3^u$ (R3D)	$\zeta_p^u/\zeta_3^u$ (R3C)
1	0.30	0.40	$0.39 \pm 0.04$	$0.42 \pm 0.03$
2	0.55	0.74	$0.73 \pm 0.04$	$0.74 \pm 0.03$
3	0.74	1.00	$1.00 \pm 0.00$	$1.00 \pm 0.00$
4	0.91	1.22	$1.20 \pm 0.10$	$1.25 \pm 0.06$
5	1.04	1.39	$1.36 \pm 0.30$	$1.50 \pm 0.16$
6	1.17	1.56	$1.46 \pm 0.52$	$1.74 \pm 0.30$
$p$	$\zeta_p^b$ [30]	$\zeta_p^b/\zeta_3^b$ [30]	$\zeta_p^b/\zeta_3^b$ (R3D)	$\zeta_p^b/\zeta_3^b$ (R3C)
1	0.36	0.43	$0.47 \pm 0.02$	$0.49 \pm 0.04$
2	0.63	0.76	$0.79 \pm 0.01$	$0.80 \pm 0.04$
3	0.83	1.00	$1.00 \pm 0.00$	$1.00 \pm 0.00$
4	0.97	1.16	$1.13 \pm 0.02$	$1.15 \pm 0.07$
5	1.07	1.28	$1.24 \pm 0.06$	$1.27 \pm 0.18$
6	1.14	1.36	$1.33 \pm 0.12$	$1.38 \pm 0.32$

our expectation that non-universal effects, associated with different initial conditions [26, 50], might not affect multiscaling exponent ratios, except if we use non-generic, power-law initial conditions [26] in which  $E(k)$  grows with  $k$  (at least until some large- $k$  cut-off).

Direct numerical simulations of decaying-MHD turbulence, e.g. those of [25, 30], often average data obtained from field configurations at different times that are close to the time at which the peak appears in plots of the energy-dissipation rate. This is a reasonable procedure, for  $Pr_M = 1$ , because the temporal evolution of the system is slow in the vicinity of this peak. We have not adopted this procedure here because, as we move away from  $Pr_M = 1$ , the cascade-completion peaks occur at different times in plots of  $\epsilon_u$  and  $\epsilon_b$ , as we have discussed in detail in earlier sections of this paper.

Let us now turn to the  $Pr_M$  dependence of the multiscaling exponent ratios shown in tables 6 and 7 and in figures 24(a.3)–(d.3) and 25(a.3)–(d.3). Even though our error bars are large, given the conservative, local-slope error analysis we have described in the previous section, a trend emerges: at large values of  $Pr_M$ , the magnetic field is clearly more intermittent than the velocity field, in as much as the deviations of  $\zeta_p^b$  from the simple-scaling prediction are stronger than their counterparts for  $\zeta_p^u$ . However, the velocity field becomes more intermittent than the magnetic field as we lower  $Pr_M$ . Could this result, namely the dependence of our multiscaling exponent ratios on  $Pr_M$ , be an artifact? We believe not. As we have discussed above, dissipation ranges in our spectra are adequately resolved; furthermore, we have determined exponent ratios from a rather stringent local-slope analysis, which is rarely presented in earlier DNS studies of MHD turbulence. Ultimately, of course, this  $Pr_M$  dependence of multiscaling exponents in MHD turbulence must be tested in detail in very-high-resolution DNS studies of MHD turbulence; such studies should become possible with the next generation of supercomputers. In particular, such supercomputers should allow us to achieve high Reynolds numbers along with higher values  $k_{\max} \eta_d^u$  and  $k_{\max} \eta_d^b$  than we have been able to obtain in runs R1D and R2D so that we have both well-resolved inertial and dissipation ranges.

For a *fixed value* of  $Pr_M$ , we conjecture that, as in fluid turbulence, the energy-spectrum exponents and multiscaling exponent ratios do not depend on the  $Re_\lambda$  if the latter is large enough to ensure that we have fully developed, homogeneous, isotropic MHD turbulence with a substantial inertial range. Our results for the spectral exponents (see e.g. figures 8(c.1), (g.1), (c.2) and (c.3) for runs R3 ( $Re_\lambda = 121$ ), R3B ( $Re_\lambda = 210$ ), R3C ( $Re_\lambda = 172$ ) and R3D ( $Re_\lambda = 239$ ), respectively) are consistent with this conjecture, as are the multiscaling exponent ratios given in table 8 for  $Pr_M = 1$ .

It is useful to note at this stage that a recent experimental study of MHD turbulence in the solar wind [57] provides evidence for velocity fields that are more strongly intermittent than the magnetic field; this study does not give the value of  $Pr_M$ . However, its data for multiscaling exponents are qualitatively similar to those we obtain at low values of  $Pr_M$ . Furthermore, PDFs of  $H_C$  have also been obtained from solar-wind data [56]; these are similar to the PDFs we obtain for  $H_C$ . Of course, we must exercise caution in comparing results from DNS studies of homogeneous, isotropic, incompressible MHD turbulence with measurements on the solar wind, where anisotropy and compressibility can be significant; and, for the solar wind, we might also have to consider kinetic effects that are not captured by the MHD equations.

The last point we wish to address is the issue of strong universality of exponent ratios. In the fluid-turbulence context, such strong universality [42, 43] implies the equality of exponents (and, therefore, their ratios) determined from decaying-turbulence studies (say at the cascade-completion time) or from studies of statistically steady turbulence. Does such strong universality have an analogue in MHD turbulence? Our data, for any fixed value of  $Pr_M$  in tables 6 and 7, are consistent with such strong universality of multiscaling exponent ratios in MHD turbulence; but, of course, our large error bars imply that a definitive confirmation of such strong universality in MHD turbulence must await DNS studies that might become possible in the next generation of high-performance computing facilities.

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