$$MPG(\theta \mid \alpha, \beta, \lambda) = \sum_{j=1}^{J} \lambda_j \, PG(\theta \mid \alpha_j, \beta_j)$$

$$= \int MPG(\theta, \gamma \mid \alpha, \beta) \, d\gamma$$

$$= \int \prod_{j=1}^{J} \left(\lambda_j \, PG(\theta \mid \alpha_j, \beta_j) \right)^{\gamma_j} d\gamma$$

$$= \int \prod_{j=1}^{J} \lambda_j^{\gamma_j} \, PG(\theta \mid \alpha_j, \beta_j)^{\gamma_j}$$

where $y' = \begin{pmatrix} \cos\theta_1, \\ \sin\theta_1 \cos\theta_2 \\ \vdots \\ \sin\theta_1 \cdots \sin\theta_{p-2} \cos\theta_{p-1} \\ \sin\theta_1 \cdots \sin\theta_{p-1} \end{pmatrix}$

$$= \int_{\gamma} \int_{r} \lambda_j^{\gamma_j} \prod_{k=1}^{K} Ga(r y' \mid \alpha_j, \beta_j)^{\gamma_j} \, |J| \, dr \, d\gamma$$

$$\Longrightarrow \pi(\alpha_{jk} \mid -) \propto \frac{\alpha_{jk}^{a_0 - 1} \prod (r_i y_{ijk}')^{\gamma_{ij} \alpha_{jk}}}{\Gamma(\alpha_{jk})^{\sum_i \gamma_{ij}}} \exp\{-b_0 \alpha_{jk}\} \frac{\Gamma(\alpha_{jk} \sum \gamma_{ij} + c_0)}{\left(\sum r_i y_{ijk}' + d_0 \right)^{\alpha_{jk} \sum \gamma_{ij} + c_0}}$$

$$\pi(\beta_{jk} \mid -) = Ga\left(\sum_{i=1}^{n} \gamma_{ij} \alpha_{jk} + c_0, \sum_{\{i : \gamma_{ij} = 1\}} r_i y_{ik}' + d_0 \right)$$

$$\pi(r_i \mid \gamma, \underline{\alpha}, \underline{\beta}) = Ga\left(\sum_{j=1}^{J} \sum_{k=1}^{k} \gamma_{ij} \alpha_{jk}, \sum_{k=1}^{K} \left[\sum_{j=1}^{J} \gamma_{ij} \beta_{jk} \right] y_{ik}' \right)$$

$$\pi(\gamma_{ij} \mid \lambda, \underline{\alpha}, \underline{\beta}) = Mult\left(1, \left\{ \frac{\lambda_j PG(\underline{y}_i' \mid \alpha_j, \beta_j)}{\sum_{\ell=1}^{J} \lambda_\ell PG(y_i' \mid \alpha_\ell, \beta_\ell)}, j = 1, \ldots, J \right\} \right)$$

$$\pi(\lambda \mid \gamma) = Dir\left(\left\{ \left(\sum_{i=1}^{n} \gamma_{ij} \right) + a_0 \quad \text{for } j = 1, \ldots J \right\} \right)$$