Computational topology: Image classification using Vietoris-Rips complex

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Obtaining and preprocessing data

Listing 1 shows a Perl script.

Listing 1: Sample Perl Script With Highlighting

```perl
#!/usr/bin/perl

use strict;
use warnings;

for (1..99) { print $_." Luftballons\n"; }

# This is a commented line

my $string = "Hello World!";

print $string."\n\n";

$string =~ s/Hello/Goodbye/;

print $string."\n\n";

test();

exit;

sub test { print "All good.\n"; }
```

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The classification model

Definition 1 (Vietoris-Rips complex)**.** *Let X be a a set of m-dimensional points $X \in \mathbb{R}^m$ and let d be a metric. Pick a parameter $r > 0$. Construct a simplicial complex as follows:*

- *Add a 0-simplex for each point in X.*
- *For $x_1, x_2 \in X$ add a 1-simplex between x_1, x_2 if $d(x_1, x_2) \leq r$.*
- *For $x_1, x_2, x_3 \in X$ add a 2-simplex with vertices x_1, x_2, x_3 if $d(x_1, x_2), d(x_1, x_3), d(x_2, x_3) \leq r$.*
- *...*
- *For $x_1, x_2, ..., x_m \in X$, add a $(m-1)$-simplex with vertices $x_1, x_2, ..., x_m$ if $d(x_i, x_j) \leq r$ for $0 \leq i, j \leq m$; that is, if all the points are within a distance of r from each other.*

The classification model continued on next page...

The simplicial complex is called the Vietoris-Rips complex and is denoted $V_r(X)$.

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Relation to single linkage clustering algorithm

Our intuition tells us, that the model we build using the Vietoris-Rips complex to classify the images, produces the same results as the well known single linkage clustering algorithm. In this section we aim to prove or at least give a strong intuition that this is indeed the case.

Definition 2. *A connected component in a graph $G(V, E)$ (V is a set of vertices, E a set of edges) is a set of vertices C such that between every two points $a, b \in C$ there exists a path in the graph G between them.*

Both algorithms take a set X of n samples with m features as input. We can think of a sample x in the set X as a point in the m-dimensional space $x \in \mathbb{R}^m$. The algortihms then constructs a graph with points X as the vertices (V) and edges E. The k connected components in the constructed graph corespond to the classes of samples.

Single linkage algorithm. The algorithm starts with n connected components (no edges in the graph). n each step the algorithm chooses the two connected components that are closest to each according to some distance metric d (in our case the euclidean distance) and joins them into one by adding an edge between their closest two vertices.

Definition 3. *The distance D between two connected components A and B is defined as the distance of the pair of vertices (one from A and one from B) that are closest to each other. More formally*

$$D(A, B) = \min_{a \in A, b \in B} d(a, b).$$

The algorithm stops when there are only k connected components left.

Vietoris-Rips classification algorithm. The algorithm builds a 1-dimensional Vietoris-Rips complex $V_r(X)$ with parameter r. We choose the biggest r such that the Vietoris-Rips complex $V_r(X)$ has k connected components.

Testing

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References

[1] Leslie Lamport, *LaTeX: a document preparation system*, Addison Wesley, Massachusetts, 2nd edition, 1994.