

# Computational topology: Image classification using Vietoris-Rips complex

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## Project description

As the title implies, the idea behind the project is to use the Vietoris-Rips (in future abbreviated VR-cx) complex for image classification. We took pictures of three generic objects and preprocessed them so we obtained vector representations of pictures. The vector representations of images can then be looked upon as points in some  $n$ -dimensional space  $\mathbb{R}^n$ . Over this points we build the VR-cx for two different parameters. We use the complexes to build a classification model for the images and classify them. We also perform several interesting tests on the generated VR-cx.

## Obtaining and preprocessing data

TODO: Describe our dataset (and show them)

TODO: Describe the process of generating matrix  $X$

TODO: Describe the preprocessing steps and the idea behind it

## The classification model

**Definition 1** (Vietoris-Rips complex). *Let  $X$  be a set of  $m$ -dimensional points  $X \in \mathbb{R}^m$  and let  $d$  be a metric. Pick a parameter  $r > 0$ . Construct a simplicial complex as follows:*

- *Add a 0-simplex for each point in  $X$ .*
- *For  $x_1, x_2 \in X$  add a 1-simplex between  $x_1, x_2$  if  $d(x_1, x_2) \leq r$ .*
- *For  $x_1, x_2, x_3 \in X$  add a 2-simplex with vertices  $x_1, x_2, x_3$  if  $d(x_1, x_2), d(x_1, x_3), d(x_2, x_3) \leq r$ .*
- *...*
- *For  $x_1, x_2, \dots, x_m \in X$ , add a  $(m-1)$ -simplex with vertices  $x_1, x_2, \dots, x_m$  if  $d(x_i, x_j) \leq r$  for  $0 \leq i, j \leq m$ ; that is, if all the points are within a distance of  $r$  from each other.*

*The simplicial complex is called the Vietoris-Rips complex and is denoted  $VR_r(X)$ .*

**Definition 2.** *We say that disjoint subsets  $A_1 \dots A_k$  of vertices  $V$  in graph  $G(V, E)$  are  $k$  connected components of the graph  $G$  if the following is true:*

1. *The vertices inside  $A_i$  are connected i.e. there exists a path between arbitrary two vertices  $a, b \in A_i$ , for every  $i \in 1 \dots k$ .*
2. *The sets of vertices  $A_1, \dots, A_k$  are disconnected i.e. there isn't an edges  $e(a, b)$  between a pair of two points  $(a, b)$  such that  $a \in A_i, b \in A_j, i \neq j$ .*

Given the input of  $n$   $m$ -dimensional points  $X \in \mathbb{R}^m$  and a metric  $d$  (in our case the euclidian distance) we build a VR-cx for parameter  $r = r_2$ , where  $r_2$  is the biggest  $r$  such that the VR-cx has three connected components. We find the parameter  $r_2$  using a binary search on a set of all possible values of  $r$ . We then perform the classification, simply by saying that all the objects in some connected component belong to the same class. Seeing as we have three connected components in our VR-cx we will get three distinct classes.

It does not take much thought to see, that our classification model will produce the exact same results if we only use simplices of dimension 1, as it would, if we use all the simplices up to dimension  $h$ , where  $h$  is an arbitrary number in the range  $h \in 1 \dots m$ . This follows from Definition 1.

## Relation to single linkage clustering algorithm

Our intuition tells us, that the model we build using the VR-cx to classify the images, produces the same results as the well known single linkage clustering algorithm. In this section we aim to prove or at least give a strong intuition that this is indeed the case.

Both algorithms take a set  $X$  of  $n$  samples with  $m$  features as input. We can think of a sample  $x$  in the set  $X$  as a point in the  $m$ -dimensional space  $x \in \mathbb{R}^m$ . The algorithm then constructs a graph with points  $X$  as the vertices ( $V$ ) and edges  $E$ . The  $k$  connected components in the constructed graph correspond to the classes of samples.

**Single linkage algorithm.** The algorithm starts with  $n$  connected components (no edges in the graph). In each step the algorithm chooses the two connected components that are closest to each according to some distance metric  $d$  (in our case the euclidean distance) and joins them into one by adding an edge between their closest two vertices.

**Definition 3.** The distance  $D$  between two connected components  $A$  and  $B$  is defined as the distance of the pair of vertices (one from  $A$  and one from  $B$ ) that are closest to each other. More formally

$$D(A, B) = \min_{a \in A, b \in B} d(a, b).$$

The algorithm stops when there are only  $k$  connected components left.

**Vietoris-Rips classification algorithm.** The algorithm builds a (1-dimensional) Vietoris-Rips complex  $V_r(X)$  with parameter  $r$ . We choose the biggest  $r$  such that the Vietoris-Rips complex  $V_r(X)$  has  $k$  connected components.

To prove that the two algorithms indeed produce the same connected components we will first prove the next claim.

**Claim 1.** Let  $G_{sl}(V, E_{sl})$  be a graph produced by the single linkage algorithm for finding  $k$  clusters and let  $d_{max}$  denote the distance between vertices in graph  $G_{sl}$  that were connected in the last iteration of the algorithm. Graph  $G_{sl}$  has connected components  $A_1 \dots A_k$ . The graph  $G_{vr}(V, E_{vr})$  induced by the Vietoris-Rips complex  $VR_{d_{max}}(V)$  has the same connected components  $A_1 \dots A_k$ .

*Proof.* We prove the Claim 1 by induction on the steps in the single linkage algorithm. We start with a set of vertices  $V$ . Let  $j$  denote the step (iteration) of the algorithm,  $e_j$  the edge added in  $j$ -th step and  $d_j$  its length. We claim that at each  $j$  the graph  $G_{sl}^j$  constructed by the algorithm up to that point, has the exact same connected components as  $G_{vr}^j$ , that is the graph induced by the Vietoris-Rips complex  $VR_{d_j}(V)$ .

**Base case.** For  $j = 0$  this is obvious, since this is the initial state of the algorithm. Both graphs  $G_{sl}^0$  and  $G_{vr}^0$  consist only of vertices  $V$ . For  $j = 1$  the algorithm adds the smallest edge  $e_1$  out of all possible candidates and builds a graph  $G_{sl}^1$ . Edge  $e_1$  has length  $d_1$ . It is obvious that  $VR_{d_1}(V)$  will induce a graph  $G_{vr}^1$  that will also only contain edge  $e_1$ , since no other pairwise distance between vertices  $V$  is smaller.

**Induction step.** Here we show that if for some  $j$  our claim holds, it will also hold after another iteration of the algorithm i.e. for  $j + 1$ . In  $(j + 1)$ -th iteration, the algorithm finds the edge  $e_{j+1}$  with length  $d_{j+1}$  and adds it to the graph. By the definition of the algorithm  $e_{j+1}$  is the smallest such edge that connects (joins) two separate connected components. This means that every other edge  $e'$  with length  $d' < d_{j+1}$  would not join connected components, but would instead just connect two vertices, that are both already in the same connected component. From the definition of the Vietoris-Rips complex we can see that in the graph  $G_{vr}^{j+1}$  there will only be one new edge that will join two separate connected components, and that will be exactly

edge  $e_{j+1}$ . All the other extra edges that will be added in  $G_{vr}^{j+1}$ , but do not appear in  $G_{sl}^{j+1}$  have length less than  $d_{j+1}$  and will therefore only connect vertices inside of already existing connected components of the graph  $G_{vr}^j$ . Since by our induction hypothesis graphs  $G_{sl}^j$  and  $G_{vr}^j$  had the same connected components and we joined two of the same connected components in both graphs, this means that the graphs  $G_{sl}^{j+1}$  and  $G_{vr}^{j+1}$  also have the same connected components.

We have proven that the graph  $G_{sl}^j$  constructed in  $j$ -th iteration of the single linkage algorithm indeed contains the same connected components as the graph  $G_{vr}^j$  induced by  $VR_{d_j}(V)$  for an arbitrary  $j$ . This also proves Claim 1.  $\square$

Using Claim 1 we see that the connected components in  $G_{vr}$  and  $G_{sl}$  are indeed the same. We need to take into account that the Vietoris-Rips algorithm takes the biggest such  $r$ , so that the graph has  $k$  connected components, so  $r > d_{max}$ . But we can quickly see that the extra edges in the graph induced by  $V_r(V_{sl})$  will not change the connected components. After all we already have  $k$  connected components in  $G_{vr}$ . To join any two together would mean a violation of a fundamental rule of the algorithm.

## Computational complexity

Let us now consider the computational complexity of our model. Since we are only interested in Vietoris-Rips complexes  $VR_r$  with simplices of dimension 1, the simplest approach to construct such complex requires us to check the distance between every pair of vertices  $(x_1, x_2) \in X \times X$ , adding such pair to the final complex if the distance  $d(x_1, x_2) \leq r$ . In worst case the algorithm would have to return all distinct pairs of vertices, meaning construction of  $VR_r(X)$  requires  $O(n^2)$  time and consumes  $O(n^2)$  space, where  $n$  is the number of vertices in  $X$ .

The problem is that we don't know the appropriate value for the parameter  $r$ . Recall that we are interested in finding biggest  $r$ , such that the Vietoris-Rips complex  $VR_r$  has as many connected components as there are distinct classes of images. Let  $r_{max}$  denote the largest distance between two vertices from  $X$ . Note that  $r$  we are looking for will always be bounded on the interval  $[0, r_{max}]$ , and it will furthermore be exactly one of the distances between some pair of vertices. Thus we only have  $n^2$  different possible values of  $r$  to check, and if we sort them by size and use binary search to find the right one, we can do it in  $O(n \log n)$  time and  $O(n^2)$  space. To count the number of connected components obtained with each of different  $VR$  complexes, we can use a union-find algorithm, which roughly adds a  $O(n^2)$  time to each run.

With this the final time complexity of our approach is  $O((n^2 + n^2) \log n^2) = O(n^2 \log n)$  using  $O(n^2)$  space.

**TODO: Double check the time complexity of union-find**

Contrast this with the computational complexity of single-linkage clustering, which with a clever implementation can in optimal case produce solution in  $O(n^2)$  time and  $O(n)$  space [1].

## Results

**TODO: Explain we get same results as with S-L clustering, and with that the same problems as S-L clustering.**

**TODO: Present MDS graph of our dataset after applying preprocessing, and emphasize different classes/recognized clusters.**

**TODO: Explain that simplices of larger dimension are just about useless**

**TODO: Explain what datasets our solution actually works with (e.g. determining position of objects on image)**

**TODO: Extra tests: images from midpoints of edges, barycenters of simplices, ...**

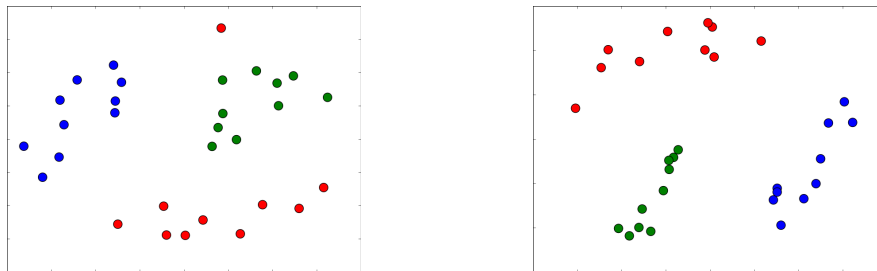


Figure 1: Caption place holder

## Summary

TODO: Brief summary

TODO: Further work (if there even is any), what we didn't explore.

## References

- [1] John C Gower and GJS Ross. Minimum spanning trees and single linkage cluster analysis. *Applied statistics*, pages 54–64, 1969.