BIG E
Saturday, October 28, 2000
9am-12 noon

1. Define
$$Q_k = \frac{1}{(k+2)!} + \frac{2}{(k+3)!} + \frac{3}{(k+4)!} + \cdots$$
Show that Q_0 is rational, but that Q_k is irrational for every positive integer k.

2. Let S be a set of points in the plane. A circle C is said to be *framed* by S if C has a diameter whose endpoints both lie in S. Find all sets S of four points in the plane such that, for any two circles C_1 and C_2 framed by S, the set $S \cap C_1 \cap C_2$ is nonempty.

3. Let f be a real-valued continuous function of a real variable with the property that
$$\lim_{x\to+\infty} f(f(x)) = +\infty \quad \text{and} \quad \lim_{x\to-\infty} f(f(x)) = -\infty$$
Prove that $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both exist and are infinite.

4. Let a and b be non-zero complex numbers which satisfy the equation
$$a\,2^{|a|} + b\,2^{|b|} = (a+b)\,2^{|a+b|}$$
Prove that $a^6 = b^6$.

5. Find the value of the infinite product
$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{a_n}\right)$$
where $a_1 = 1$, and $a_n = n\,(a_{n-1} + 1)$ for all $n \geq 2$.