Lecture Notes on
Linear Algebra and Probability

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1 What is a vector?

1.1 Preamble

Before studying linear algebra, your conception of a vector may come from physics, in that a vector is an arrow in space with direction and magnitude. From a computer science perspective, a vector is a list of numbers that holds n numbers, which means it has n-dimensions. But in linear algebra, a vector in linear algebra is none of these definitions. In linear algebra, a vector is any object that *belongs in a vector space.*

1.2 Vector Spaces

But what is a vector space, you may wonder? Simply put, a vector space is a set of vectors satisfying the vector space properties. To make an analogy with computer science, think of vector space properties as an interface in object oriented programming that details the properties an object must have. For example, an interface for a car may be that it has four wheels and a steering wheel (two requirements). For vector spaces, there are six required properties:

Notation

1. $\vec{v} \in V$ means vector $\vec{v}$ is in the vector space V.
2. $\mathbb{R}$ means the set of all real numbers e.g. $a \in \mathbb{R}$ means a is a constant scalar.

1. **Commutativity:** $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ for all $\vec{u}, \vec{v} \in V$.
2. **Associativity:** $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ and $(cd)\vec{v} = c(d\vec{v})$ for all $\vec{u}, \vec{v}, \vec{w} \in V$ and $c, d \in \mathbb{R}$.
3. **Additive Identity:** There exists an element $\vec{0} \in V$ such that $\vec{0} + \vec{v} = \vec{v}$ for all $\vec{v} \in V$.
4. **Additive Inverse:** For every $\vec{v} \in V$, there exists an element $\vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{0}$. It can later be shown that $\vec{w} = -1\vec{w}$.
5. **Multiplicative Identity:** $1\vec{v} = \vec{v}$ for all $\vec{v} \in V$.
6. **Distributivity:** $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ and $(c + d)\vec{u} = c\vec{u} + d\vec{u}$ for all $\vec{u}, \vec{v} \in V$ and $c, d \in \mathbb{R}$.

It's also worth noting that these properties are dependent on *how you define vector addition and scalar multiplication.* Our conventional definition of these ideas can be summarized as below:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

$$a \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} ax_1 \\ ay_1 \end{bmatrix}$$

However, we could have defined vector addition in a different way, which could have led to different results. Below is an unconventional definition of vector addition:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ y_1 + y_2 \end{bmatrix}$$

So, when checking if something is a valid vector space, make sure to follow how vector addition and scalar multiplication is defined in your vector space world. If your candidate vector space satisfies all these properties, then it is a valid vector space. Objects from this vector space are then defined to be *vectors*.

1.3 Vectors as common mathematical objects

Some specific examples of vectors are column vectors, polynomials, and matrices. At first, this might sound counterintuitive since a vector up until this point has always been expressed as a column vector as shown below:

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

But it can be argued that polynomial spaces follow the vector space properties and hence imply that polynomials can be viewed as vectors. For example:

$$f(x) + g(x) = g(x) + f(x)$$

$$(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$$

$$0 + f(x) = f(x)$$

$$f(x) + (-f(x)) = 0$$

$$1 \cdot f(x) = f(x)$$

$$c(f(x) + g(x)) = cf(x) + cg(x)$$

Because polynomial spaces can satisfy vector space properties, this shows that there is a good argument to view functions as vectors. Likewise, for matrices similar reasons can be argued to view matrices as vectors. Proof is left as an exercise to the reader.

1.4 Vector Subspaces

A vector subspace is a vector space that has two more requirements.

1. **Closure under vector addition:** If $\vec{u}, \vec{v} \in V$, then $\vec{v} + \vec{w} \in V$
2. **Closure under scalar multiplication:** If $\vec{v} \in V$, $c \in \mathbb{R}$, then $c\vec{v} \in V$

You may be wondering, why would we want to impose these additional two requirements on vector spaces in the first place? The reason for this is because these two requirements are surprisingly useful, and the rest of linear algebra relies on these two properties.

Consider, for example, that you have two vectors $\vec{v_1}$ and $\vec{v_2}$ that come from V. These two vectors represent the building blocks of your vector space (this is a concept that we will revisit later). By building blocks, what I mean is that any vector $\vec{v}$ can be written as a *linear combination* of $\vec{v_1}$ and $\vec{v_2}$.

$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2}.$$

If this arbitrary vector is contained in V, then it implies that all the information that is stored within V can be expressed with just two vectors. There can be an infinite number of vectors in the subspace, but all of it can be captured by just two vectors. This is incredibly useful because if we want to study the the vector space V, we don't need to look at every single vector in V, we just need to study $\vec{v_1}$ and $\vec{v_2}$.

One example of a vector subspace is the Cartesian x-y plane. Any point on the plane can be described by two numbers: x and y, written as (x, y). To write (x, y) as the sum of two vectors, we can imagine that x and y are scalar multiples that scale two unique vectors that sum to any point. One such construction could be $x \cdot (1, 0) + y \cdot (0, 1) = (x, 0) + (0, y) = (x, y)$, where $(x, 0)$ and $(0, y)$ are our two vectors.