MAS341: Graph Theory

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2018 Edition

Work builds on notes from previous course, as well as sections of the book Applied Combinatorics by Keller and Trotter, and Discrete Math by Oscar Levin.

Preface

Course notes for MAS341: Graph Theory at the University of Sheffield.

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Chapter 1

Introduction

The first chapter is an introduction, including the formal definition of a graph and many terms we will use throughout, but more importantly, examples of these concepts and how you should think abotu them.

1.1 A first look a graphs

First and foremost, you should think of a graph as a certain type of picture, containing dots and lines connecting those dots, like so:

We will typically use the letters G, H, or Γ (capital Gamma) to denote a graph. The "dots" or the graph are called *vertices* or *nodes*, and the lines between the dots are called *edges*. Graphs occur frequently in the "real world", and typically how to show how something is connected, with the vertices representing the things and the edges showing connections.

- *Transit networks:* The London tube map is a graph, with the vertices representing the stations, and an edge between two stations if the tube goes directly between them. More generally, rail maps in general are graphs, with vertices stations and edges representing line, and road maps as well, with vertices being cities, and edges being roads.
- *Social networks:* The typical example would be Facebook, with the vertices being people, and edge between two people if they are friends on Facebook.
- *Molecules in Chemistry:* In organic chemistry, molecules are made up of different atoms, and are often represented as a graph, with the atoms being vertices, and edges representing covalent bonds between the vertices.

That is all rather informal, though, and to do mathematics we need very precise, formal definitions. The one we will begin with is the following.

Definition 1.1.1. A *graph* G consists of a set $V(G)$, called the *vertices* of G, and a set $E(G)$, called the *edges* of G, of the two element subsets of $V(G)$

Example 1.1.2. Consider the water molecule. It has three vertices, and so $V(G) = \{O, H1, H2\}$, and two edges $E(G) = \big\{\{O, H1\}, \{O, H2\}\big\}$

This formal definition has some perhaps unintended consequences about what a graph is. Because we have identified edges with the two things they connect, and have a set of edges, we can't have more than one edge between any two vertices. In many real world examples, this is not the case: for example, on the London Tube, the Circle, District and Picadilly lines all connect Gloucester Road with South Kensington, and so there should be multiple edges between those two vertices on the graph.

Another consequence is that we require each edge to be a two element subset of $V(G)$, and so we do not allow for the possibility of an edge between a vertex and itself, often called a *loop*.

Graphs without multiple edges or loops are sometimes called *simple graphs*. We will sometimes deal with graphs with multiple edges or loops, and will try to be explicit when we allow this. Our default assumption is that our graphs are simple.

Another consequence of the definition is that edges are symmetric, and work equally well in both directions. This is not always the case: in road systems, there are often one-way streets. If we were to model Twitter or Instragram as a graph, rather than the symmetric notion of friends we would have to work with "following". To capture these, we have the notion of a *directed graph*, where rather than just lines, we think of the edges as arrows, pointing from one vertex (the source) to another vertex (the target). To model twitter or instagram, we would have an ege from vertex a to vertex b if a followed b.

1.2 Degree and handhsaking

Intuitively, the *degree* of a vertex is the "number of edges coming out of it". If we think of a graph G as a picture, then to find the degree of a vertex $v \in V(G)$ we draw a very small circle around v, the number of times the G intersects that circle is the degree of v. Formally, we have:

Definition 1.2.1. Let G be a simple graph, and let $v \in V(G)$ be a vertex of G. Then the *degree of* v, written $d(v)$, is the number of edges $e \in E(G)$ with $v \in e$. Alternatively, $d(v)$ is the number of vertices v is adjacent to.

Example 1.2.2.

Note that in the definition we require G to be a simple graph. The notion of degree has a few pitfalls to be careful of G has loops or multiple edges. We still want to the degree $d(v)$ to match the intuitive notion of the "number of edges coming out of v" captured in the drawing with a small circle. The trap to beware is that this notion no longer agrees with "the number of vertices adjacent to v" or the "the number of edges incident to v"

Example 1.2.3.

Theorem 1.2.4. *(Euler's handshaking Lemma)*

$$\sum_{v \in V(G)} d(v) = 2|E(G)|$$

Proof. We count the "ends" of edges two different ways. On the one hand, every end occurs at a vertex, and at vertex v there are $d(v)$ ends, and so the total number of ends is the sum on the left hand side. On the other hand, every edge has exactly two ends, and so the number of ends is twice the number of edges, giving the right hand side. □