Section 1.3: Inequalities

Covered 1.1 Yesterday. 1.2 is worked examples!

Lemma (Triangle Inequality)

For $z, w \in \mathbb{C}$ we have

$$\begin{aligned} \big||z| - |w|\big| &\le |z + w| \le |z| + |w| \\ \big||z| - |w|\big| &\le |z - w| \le |z| + |w| \end{aligned}$$

Proof.

The two inequalities are equivalent: replace w with $-w$.

- Consider the triangle with vertices $0, z$ and $z + w$.
- The sum of any two sides lengths is $\ge$ the third side length
- Use this for all three sides and repackage

$\square$

Worked example of triangle inequality

Show that for z with $|z + 2i| \leq 1$ we have

$$\frac{1}{\sqrt{5}+1} \leq \left|\frac{z}{z+1}\right| \leq \frac{3}{\sqrt{5}-1}$$

General Hints and ideas

- Treat denominator and numerator separately
- Rewrite what you want in terms of what you know
- Be careful of the direction of your inequalities
- Drawing a picture can help...

Section 2: Special functions

We won't sweat this, but defining $\exp(z) = e^z$ properly is tricky...

Definition

$$\exp(x) = 1 + z + z^2/2 + z^3/3! + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Binomial theorem means $\exp(z)$ behaves like an exponential:

$$\exp(z + w) = \exp(z)\exp(w)$$

And using Taylor series for $\sin(z)$ and $\cos(z)$ Euler's formula follows, and for $z = x + iy$

$$\exp(z) = e^x (\cos(y) + i\sin(y))$$

So $\exp(z)$ is periodic with period $2\pi i$: $\exp(z + k2\pi i) = \exp(z)$.

Other functions

Using Euler's formula, for real z we have

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

The right hand sides make sense for complex z, however, and so we can use these as the definition for $\sin(z)$ and $\cos(z)$ for $z \in \mathbb{C}$.

- Still have $\cos^2(z) + \sin^2(z) = 1$
- No longer have $|\sin(z)| \leq 1$!

Hyperbolic versions: leave out the i

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2} \qquad \sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

So $\sin(z)$ and $\cos(z)$ are periodic with period 2π.

Section 2.3 Worked examples

Example 1

Find M such that

$$\left|\frac{e^z + \cos(z)}{z+6}\right| \leq M$$

for all z with $|z| = 1$.

Example 2

Find the zeroes of $\cos(z)$