

Complex Analysis: What's all this, then?

Up until now: derivatives and integrals of functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Derivatives and integrals of $f : \mathbb{C} \rightarrow \mathbb{C}$

- ▶ Easier than real case!
- ▶ Analysis not a prerequisite – not focused on *doing* proofs
- ▶ *Using* theorems correctly, building on each other

Theorem (Fundamental Theorem of Algebra)

Let $p(z) \in \mathbb{C}[x]$ be a nonconstant polynomial. Then there is a $w \in \mathbb{C}$ with $p(w) = 0$.

The Residue Theorem has bonkers applications:

$$\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{(1+x^2)^2} dx = \frac{\pi}{2} e^{-\pi} (\pi + 1)$$

Calculating Fourier and Laplace transforms. Number Theory.

Written Resources

Main resource: Dr. Hart's notes

- ▶ Fine-tuned toward the exam
- ▶ Many worked examples

Recommended texts: extra material + viewpoints, more rigour

- ▶ Beck et al (free and legal online!)
- ▶ Priestley (pdf through library)
- ▶ Stewart and Tall (old edition is cheap online)

Slides

- ▶ Posted online before lecture; space to write notes
- ▶ Roughly follow Dr. Hart's notes

Assessment and feedback

100% of marks from final exam

- ▶ Exam questions generally very similar to past years
- ▶ All questions mandatory, probably 4
- ▶ Forthcoming exam page will have more detail

Exercises and feedback

- ▶ Exercise and hints sheets on Blackboard **DO DURING TERM**
- ▶ Will collect a few to mark for feedback
- ▶ Worked solutions to all exercises will go up shortly after that

Interactive “clicker questions” in lecture

- ▶ Using TurningPoint app/webpage
- ▶ Everyone gets instant feedback on how things are going

Review of complex numbers

The complex plane $\mathbb{C} = \{z : z = x + iy; x, y \in \mathbb{R}\}$.

- ▶ $x = \operatorname{Re}(z)$ is called the *real part*
- ▶ $y = \operatorname{Im}(z)$ is called the *imaginary part*
- ▶ $\bar{z} = x - iy$ is called the *complex conjugate*
- ▶ $|z| = \sqrt{x^2 + y^2}$ is called the *modulus* of z

Complex numbers form a field

Can add, multiple, subtract by any complex number, and divide by nonzero complex numbers.

A few tricks

$$\mathbb{C} \cong \mathbb{R}^2$$

An equation $z = w$ of complex numbers can be viewed as a pair of equations between real numbers:

$$z = w \iff \operatorname{Re}(z) = \operatorname{Re}(w) \text{ and } \operatorname{Im}(z) = \operatorname{Im}(w)$$

Why care about \bar{z} ?

$$|z|^2 = z \cdot \bar{z}$$

This lets us find z^{-1} .

Example

What's $\frac{1-i}{3+4i}$?

Geometry of complex numbers

Theorem (Euler)

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

Polar form / “modulus-argument” form

If we write $z = x + iy$ in polar coordinates:

$$z = r \cos(\theta) + ir \sin(\theta) = re^{i\theta}$$

Multiplying and dividing easier in polar form!

$$(re^{i\theta})(se^{i\phi}) = (rs)e^{i(\theta+\phi)}$$

$$\frac{re^{i\theta}}{se^{i\phi}} = (r/s)e^{i(\theta-\phi)}$$

Powers and roots

Theorem (De Moivre)

Let $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

Proof.

Apply Euler's theorem to both sides of $(e^{i\theta})^n = e^{in\theta}$, or prove directly for $n \in \mathbb{N}$ using induction and trig formulas. □

Theorem

A complex number $z = re^{i\theta}$ has precisely n n th roots:

$$\sqrt[n]{r}e^{i(\theta+2k\pi)/n} \quad k \in \{0, 1, \dots, n-1\}$$