Last time: Harmonic Functions

The Laplacian operator, written ∇^2 or Δ, acts on functions $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$\nabla^2 g = \nabla \cdot \nabla g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}$$

Definition

A function $u : \mathbb{R}^2 \to \mathbb{R}$ is *harmonic* if $\nabla^2 f = 0$.

Lemma

Let $f(z) = u(x, y) + iv(x, y)$ be analytic on a domain D. Then u and v are harmonic on D.

Can you go backwards?

Yes, if u defined on a *simply-connected* region.

When is u are the real part of analytic functions?

From the 2012-2013 exam

(iii) Find all the functions k analytic on $\mathbb{C}$ with $\operatorname{Re}(k(x+iy)) = 2x - \sinh x \sin y$, giving an explicit expression for $k(z)$ in terms of z. Show that you have found **all** the functions satisfying the above conditions. ***(6 marks)***

The real part of an analytic function is harmonic

First, check $\nabla^2 u = 0$. If not, the answer is no.

If it is, find f' using Cauchy-Riemann

$$f' = \frac{\partial}{\partial x}\Big(u(x,y) + iv(x,y)\Big) = u_x + iv_x = u_x - iu_y$$

This gives us f' in terms of x and y. We'd *like* to write f' in terms of z, and integrate to find f. But how?

Maybe we need a clever little trick....

Dr. Hart's "Clever little trick"

Given:

We know f' in terms of x and y, want it terms of z.

Guess:

Set $y = 0$; to get $f'(x)$ in terms of just x. Integrate to get $f(x)$.
Guess that this is actually formula for $f(z)$

Check:

Show that $\text{Re}(f) = u(x, y)$

To find *all* such k:

Lemma

Suppose that f and g are analytic on a region D and that $\text{Re}(f) = \text{Re}(g)$ on D. Then $f = g + ia$ for some $a \in \mathbb{R}$.

A last application of Cauchy-Riemann

- Cauchy-Riemann relates between $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$
- If we have more relations, then f is *very* constrained

Example from the notes:

The function f is analytic in $\mathbb{C}$ and its real and imaginary parts u and v satisfy

$$ue^{v} = 12$$

at all points in $\mathbb{C}$. Prove that f is constant.

Next up – Section 6: Power series

- Not much different than real power series, so largely review
- Spoiler: analytic functions have convergent power series