Examples

Find the radius of convergence of the following functions.

$$\sum_{n=0}^{\infty} (\sinh(n)) z^n \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{(2i)^n z^n}{n} \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{(2i)^n z^{3n}}{n} \tag{3}$$

$$\sum_{n=0}^{\infty} \frac{(2n)!n!}{(3n)!} z^n \tag{4}$$

Convergent Power series give analytic functions

Define $f(z) = \sum a_n z^n$ inside the radius of convergence. Is $f(z)$ analytic? We'd like to argue:

$$\begin{aligned} f'(z) &= \frac{d}{dz} \sum a_n z^n \\ &= \sum \frac{d}{dz} a_n z^n \\ &= \sum n a_n z^{n-1} \end{aligned}$$

We're being Evil Kermit

- Not clear we can move derivative inside sum
- Not clear final power series converges

Power series give analytic functions inside disk of convergence!

Will see later this gives *all* analytic functions!

If f has an antiderivative, integration is easy

Definition

Let f be defined on a region D. A *primitive* of f is an analytic function g on D with $g' = f$ at all points in D.

Note:

D does not need to be simply connected!

Lemma

If g is a primitive for f on D, and γ is any path from p to q in D, then $\int_\gamma f(z)dz = g(q) - g(p)$.

Corollary

If γ is a contour (i.e., $p = q$), and f has a primitive on D, then $\int_\gamma f(z)dz = 0$.

Examples of using primitives

1. Evaluate $\int_\gamma z dz$ where γ is the line segment from 0 to 1 followed by the line segment from 1 to $1+i$
2. Evaluate $\int_\gamma z \exp(z^2) dz$ where γ is the contour $z = e^{it}$
3. Homework: Evaluating $\int_\gamma (1+z) dz$ is easy!

What does the lemma say about our important example:

$$\int_{C_r(a)} \frac{1}{(z-a)^n} \mathrm{d}z = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$