Previously: Cauchy's Integral Formula and Consequences

Theorem (Cauchy's integral formula)

Let γ be a simple contour described in the positive direction. Let w lie inside γ. Suppose that f is analytic on a simply connected region D containing γ and its interior. Then:

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} dz.$$

Uses of Cauchy's Integral Formula

- Calculate contour integrals
- Values of f inside a contour determined by values on boundary
- Proving f has convergent Taylor series!

Today: Other applications

Analytic functions have all derivatives!

Theorem (Cauchy's Integral formula for the derivatives)

Let γ be a simple contour described in the positive direction. let w be any point inside γ. Suppose f analytic on a simply-connected region D containing γ. Then

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^{n+1}} dz$$

Proof.

Take $\frac{d}{dw}$ of both sides of CIF. Differentiate inside the integral. $\square$

Example

Let γ be the square with vertices $1, i, -1, -i$. Evaluate

$$\int_\gamma \frac{e^z}{z^n} dz$$

Another application of CIF: Liouville's Theorem

Theorem (Liouville's Theorem)

A function which is analytic and bounded in the complex plane is a constant.

Proof.

Let $a, b \in \mathbb{C}$.

1. Rewrite $f(a) - f(b)$ as an integral around $|z| = R$ using CIF.
2. Use ML to bound $|f(a) - f(b)|$
3. As $R \to \infty$, the bound goes to 0.

$\square$

Toy applications of Liouville's Theorem frequently on exam, and in problem sheets. More excitingly, Liouville's Theorem can prove the fundamental theorem of algebra.

A theorem you've long used...

Theorem (Fundamental Theorem of algebra)

Let $p(z)$ be a non-constant polynomial with complex coefficients. Then there is a point $w \in \mathbb{C}$ such that $p(w) = 0$.

Proof.

Suppose not, and $p(z)$ has no roots. Then show $1/p(z)$ is bounded and analytic on $\mathbb{C}$, and apply Liouville's Theorem. $\square$

Using induction, it follows that a polynomial of degree n has n roots, counted with multiplicity.

Note

The trick of dividing by a nonzero function appears frequently in applications.