Section 2.3 Worked examples in notes

Example 1

Find M such that

$$\left|\frac{e^z + \cos(z)}{z+6}\right| \leq M$$

for all z with $|z| = 1$.

Example 2

Find the zeroes of $\cos(z)$

Section 2.4: Complex Logarithm

Logs for real numbers

For a real number x, $e^x > 0$, and $\ln(x) : (0, \infty) \to \mathbb{R}$ is defined to be its inverse function, i.e., $\ln(x)$ is *defined* by

$$e^{\ln(x)} = x \qquad \ln(e^x) = x$$

Complex case: defining log

We have defined $e^{x+iy} = e^r [\cos(x) + i \sin(y)]$ and so the codomain of the complex exp is $\mathbb{C} \setminus \{0\}$. We want to define log to be the inverse function, but exp isn't one-to-one, so log won't be a function!

Example

$\log(1)$ could be $0, 2\pi i, 4\pi i, -2\pi, -56834756380\pi i \cdots$

Real and imaginary parts of log:

For $z = re^{i\theta}$, the real and imaginary parts of log are interesting.

$$\log(z) = \log(re^{i\theta}) = \ln(r) + i\theta$$

So $\mathrm{Re}(\log(z)) = \ln(r) = \ln(|z|)$ *is* well defined.
$\mathrm{Im}(\log(z)) = \theta + 2\pi n = \arg(z)$

Example

Find all values of $\log(-1 - i)$.

Section 3: Simple integrals of complex valued functions

In MAS211, you did line integrals of real functions $f(x, y)$ along a curve in the plane $\mathbb{R}^2$. Complex line integrals aren't any harder.

Parametric curves

IN MAS211, you parametrized curves in $\mathbb{R}^2$ by

$$x = x(t) \quad y = y(t) \quad a \leq t \leq b$$

In the complex plane, we write things slightly differently, since $z = x + iy$:

$$z = z(t) = x(t) + iy(t) \quad a \leq t \leq b$$

Example (Unit circle)

In $\mathbb{R}^2$, we can parameterize the unit circle by

$$x(t) = \cos(t) \quad y(t) = \sin(t) \quad 0 \leq t \leq 2\pi$$

In $\mathbb{C}$, we can write this more compactly with Euler's theorem:

$$z(t) = e^{it} = \cos(t) + i\sin(t) \quad 0 \leq t \leq 2\pi$$

Vocabulary: Types of curves

Definition (A curve)

A *curve* γ is a continuously differentiable complex-valued function z of a real variable t on $[a, b]$.

Definition (A path)

A *path* is a finite union of curves, joined successively at end points.

Definition (Contour)

A *contour* is a path whose final point is the same as the initial point. A contour is *simple* if it has no self-intersection.

Important examples of paths. How to parameterize?

- $C_r(a)$ – the circle of radius r around a
- The straight line segment from z_0 to z_1

Line integrals: Definition is chain/rule u-substitution

Definition (3.3 in Notes; need to use, not quote)

Let f be continuous on a region containing the path γ. Let γ be given by $z = z(t), a \leq t \leq b$. Then

$$\int_\gamma f(z)\mathrm{d}z := \int_a^b f(z(t))z'(t)\mathrm{d}t$$

What about the fact that it's complex?

Just use linearity of integrals to turn it into real integrals!
If $g = \operatorname{Re} f$ and $h = \operatorname{Im}(f)$ then

$$\int_a^b f(t)\mathrm{d}t = \int_a^b g(t) + ih(t)\mathrm{d}t = \int_a^b g(t)\mathrm{d}t + i\int_a^b g(t)\mathrm{d}t$$

Basic properties of line integrals

Basic properties – analogous to usual integrals

- $\int_{-\gamma} f(z)\mathrm{d}z = -\int_{\gamma} f(z)\mathrm{d}z$
- $\int_{\gamma}(af(z) + bf(z)\mathrm{d}z = a\int_{\gamma} f(z)\mathrm{d}z + b\int_{\gamma} g(z)\mathrm{d}z \quad a, b \in \mathbb{C}$
- $\int_{\gamma_1+\gamma_2} f(z)\mathrm{d}z = \int_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_2} f(z)\mathrm{d}z$

One other fact

The length of curve γ, parametrized by $z : [a, b] \to \mathbb{C}$ is calculated similarly as

$$\int_a^b |z'(t)|\mathrm{d}t$$