Last week: Laurent series, residues, singularities

A Laurent series is like a power series but we're allowed to have negative terms.

Theorem (Laurent's Theorem)

Suppose that f has an isolated singularity at α, so f analytic on $D' = \{z : 0 < |z - \alpha| < R\}$. Then f can be represented by a Laurent series around α that converges on D':

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - \alpha)^n$$

- a_1 is called the *residue of f at α*
- $\int_{C_r(\alpha)} f(z)dz = a_{-1}$ for small r
- f has a removable singularity/pole/essential singularity if it has no/finite / infinite $a_{-k} \neq 0$

What's left:

Today Residue Theorem

Tomorrow Applying Residue Theorem to Real Integrals

Next Monday Residue Theorem tricks:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Tuesday Application: Laplace Transform? Revision?

Week 12 Revision?

No office hours this week due to the strike.

At last!

Theorem (The Residue Theorem)

Let D be a simply connected region containing a simple positively oriented contour γ. Suppose f is analytic on D except for finitely many singularities $\beta_1, \ldots, \beta_n$, none of which like on γ. Then

$$\int_\gamma f(z)dz = 2\pi i \times (\textit{sum of the residues of } f \textit{ at the } \beta_i \textit{ inside } \gamma)$$

Proof.

The proof is really putting together things we've already done:

- Deform contour so one singularity in each piece
- Expand f in Laurent series
- Use formula for a_{-1} / our first important example

$\square$

Using the Residue Theorem

Show you understand and check hypotheses!

1. Find the isolated singularities (bad points) β_i of f
2. Draw picture showing γ and bad points to see which are inside
3. Find the residues of the singularities inside γ

Examples; let $c = 5e^{it} \quad (0 \leq t \leq 2\pi)$

1. $\int_c \frac{dz}{z^2(z-3)^3}$
2. $\int_c \frac{dz}{\tan(z)}$
3. $\int_c z^3 \cos(1/z) dz$

Finding the residues can be a lot of work.

Laurent Expansion often easiest. For simple poles use shortcut.