Housekeeping: Office Hours

Office hours mean I'm in my office for peopoe to drop in and chat. Take advantage of them!

Office Hours this week:

- Friday 2-3

Standard office hours starting next week:

- Monday: 1-2
- Wednesday: 10-11

These are subject to change; if you'd like to come a few times and those will never work, let me know, and maybe I'll change them! You can also email to try to set up another time to drop in.

Section 1.3: Inequalities

Covered 1.1 Yesterday, 1.2 is worked examples!

Lemma (Triangle Inequality)

For $z, w \in \mathbb{C}$ *we have*

$$\big||z| - |w|\big| \leq |z + w| \leq |z| + |w|$$
$$\big||z| - |w|\big| \leq |z - w| \leq |z| + |w|$$

Proof.

The two inequalities are equivalent: replace w with $-w$.

- Consider the triangle with vertices $0, z$ and $z + w$.
- The sum of any two sides lengths is $\geq$ the third side length
- Use this for all three sides and repackage

$\square$

Worked example of triangle inequality

Show that if $|z + 2i| \leq 1$, then we

$$\frac{1}{\sqrt{5}+1} \leq \left|\frac{1}{z+1}\right| \leq \frac{1}{\sqrt{5}-1}$$

Hints/ideas

- Treat denominator and numerator separately
- Rewrite what you want in terms of what you know
- Be careful of the direction of your inequalities
- Drawing a picture can help...

Clicker Session
Turning Point app or
ttpoll.eu

Z < W

|Z| < |W|

Section 2: Special functions

We won't sweat this, but defining $\exp(z) = e^z$ properly is tricky...

Definition

$$\exp(x) = 1 + z + z^2/2 + z^3/3! + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Binomial theorem means $\exp(z)$ behaves like an exponential:

$$\exp(z + w) = \exp(z) \exp(w)$$

And using Taylor series for $\sin(z)$ and $\cos(z)$ Euler's formula follows, and for $z = x + iy$

$$\exp(z) = e^x (\cos(y) + i \sin(y))$$

So $\exp(z)$ is periodic with period $2\pi i$.

Other functions

Using Euler's formula, for real z we have

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

The right hand sides make sense for complex z, however, and so we can use these as the definition for $\sin(z)$ and $\cos(z)$ for $z \in \mathbb{C}$.

- Still have $\cos^2(z) + \sin^2(z) = 1$
- No longer have $|\sin(z)| \leq 1$

Hyperbolic versions: leave out the i

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2} \qquad \sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

So $\sin(z)$ and $\cos(z)$ are periodic with period 2π.

Section 2.3 Worked examples

Example 1

Find M such that

$$\left|\frac{e^z + \cos(z)}{z+6}\right| \leq M$$

for all z with $|z| = 1$.

Example 2

Find the zeroes of $\cos(z)$