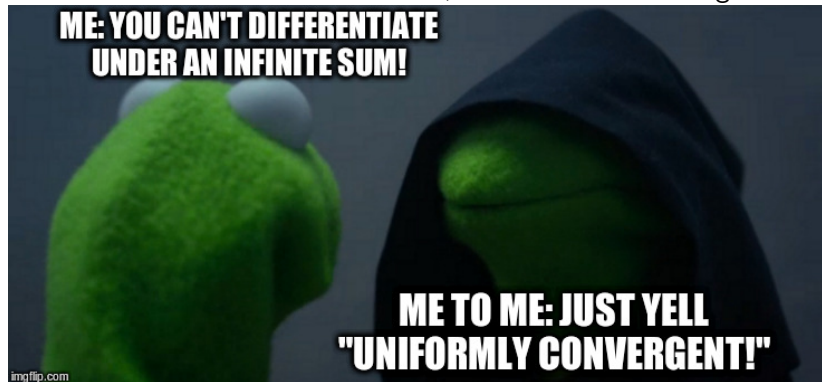


On Switching Orders

"There are three big assumptions, all valid in this course with our situation, but which are false in general."



Sequences in \mathbb{C} – just as in \mathbb{R}

Definition

Let a_n be a sequence of complex numbers. Then we say the sequence converges to L , and write

$$\lim_{n \rightarrow \infty} a_n = L$$

if for all $\varepsilon > 0$ there exists an N so for $n > N$ we have $|a_n - L| < \varepsilon$

Lemma

$$\lim_{n \rightarrow \infty} a_n = L \iff \begin{aligned} \lim_{n \rightarrow \infty} \operatorname{Re}(a_n) &= \operatorname{Re}(L) \\ \lim_{n \rightarrow \infty} \operatorname{Im}(a_n) &= \operatorname{Im}(L) \end{aligned}$$

Proof.

$$\max(|\operatorname{Re}(z - L)|, |\operatorname{Im}(z - L)|) \leq |z - L| \leq |\operatorname{Im}(z - L)| + |\operatorname{Re}(z - L)|$$



Series in \mathbb{C} – just as in \mathbb{R}

Definition

Let a_n be a sequence of complex numbers. Then we say the series $\sum_{i=0}^{\infty} a_i$ converges to L if the sequence of partial sums $S_n = \sum_{i=0}^n a_i$ converges to L .

Definition

A series $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ converges.

Two tools:

- ▶ Comparison test
- ▶ Geometric series

We'll mostly be interested in power series

Definition

Suppose that $a_n, z_0 \in \mathbb{C}$. A series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is called a *power series centred at z_0* .

We will be interested in what values of z a given power series converges; for those values, we will have a function of z . E.g.

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \forall z \in \mathbb{C}$$

Radius of convergence

Theorem

Suppose $w \neq 0$ and $\sum a_n w^n$ converges. Then $\sum a_n z^n$ is absolutely convergent for all z with $|z| < |w|$

Theorem (Abel)

For any power series, either:

- 1. The power series converges only at $z = 0$.*
- 2. The power series is absolutely convergent for all $z \in \mathbb{C}$*
- 3. There is a real number R so that the power series is absolutely convergent if $|z| < R$, and divergent if $|z| > R$*

R is called the radius of convergence. It is zero in case one, and infinite in case 2.

The ratio test

The radius of convergence always exists (though it may be 0 or ∞), but not clear how to find. We will mostly use:

Theorem (Ratio Test)

If

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} = R$$

Then R is the radius of convergence.

Proof.

Idea: use comparison test to compare to geometric series



- ▶ **Can't** apply Ratio Test to “most” power series.
- ▶ *Can* apply Ratio Test to most power series we'll see!

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Examples

Find the radius of convergence of the following functions.

$$\sum_{n=0}^{\infty} (\sinh(n)) z^n \quad (1)$$

$$\sum_{n=1}^{\infty} \frac{(2i)^n z^n}{n} \quad (2)$$

$$\sum_{n=1}^{\infty} \frac{(2i)^n z^{3n}}{n} \quad (3)$$

$$\sum_{n=0}^{\infty} \frac{(2n)! n!}{(3n)!} z^n \quad (4)$$

Convergent Power series give analytic functions

Define $f(z) = \sum a_n z^n$ inside the radius of convergence. Is $f(z)$ analytic? We'd like to argue:

$$\begin{aligned} f'(z) &= \frac{d}{dz} \sum a_n z^n \\ &= \sum \frac{d}{dz} a_n z^n \\ &= \sum n a_n z^{n-1} \end{aligned}$$

We're being Evil Kermit

- ▶ Not clear we can move derivative inside sum
- ▶ Not clear final power series converges

Hence: Power series give analytic functions!

Will see later this gives *all* analytic functions!