Revision: Classification of singularities + finding residues

Theorem

Suppose that f has a pole of order k at α. Then

$$\operatorname{Res}\{f;\alpha\} = \frac{1}{(k-1)!}\lim_{z\to\alpha}\frac{d^{k-1}}{dz^{k-1}}(z-\alpha)^k f(z)$$

Corollary

If $f = g/h$, where g and h are analytic at α,
$g(\alpha) \neq 0, h(\alpha) = 0, h(\alpha) \neq 0$, then f has a simple pole at α and

$$\operatorname{Res}\{f;\alpha\} = \frac{g(\alpha)}{h'(\alpha)}$$

Examples: Classify singularities; find residues

$$(1+z)^2 e^{i/z} \qquad \frac{z^2}{\sinh(\pi z)} \qquad \frac{1}{(1+z+z^2)^4}$$

Computing $\int_{-\infty}^{\infty} f(x)dx$ with the Residue Theorem

Basic idea is very flexible:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R,S\to\infty} \int_{-R}^{S} f(x)dx$$

- Include the finite integral as part of a contour integral
- Calculate the contour integral using residue theorem
- As $R, S \to \infty$ contributions of other parts of contour $\to 0$

ML Estimates *sometimes* work for last point

Basic example / sanity check:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \pi$$

Specific formulation appearing in notes on exam

I don't really like that you just need to memorize this, but

Theorem

Let $f(z) = \frac{p(z)}{q(z)} e^{i\lambda z}$ *where* $\lambda \in \mathbb{R}, \lambda > 0$, $p(z), q(z)$ *are polynomials with* $\deg q > \deg p$, *no common zeroes, and* q *has no real roots. Then*

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{i\lambda x} dx = \sum_{i=1}^{k} \operatorname{Res}\{f; z_k\}$$

where $z_1, z_2, \ldots, z_k$ *are the zeros of* q *in the upper half plane.*

Proof.

Follow plan from last slide; if $\deg(q) + 1 < \deg(p)$ can just use *ML*-estimates, otherwise we need to sweat a bit more. $\square$

Applying our specific formulation to compute real integrals

Take real and imaginary parts of our integrand

If p, q have real coefficients, and x is real, then

$$\operatorname{Re} \frac{p(x)}{q(x)} e^{i\lambda x} = \frac{p(x)}{q(x)} \cos(\lambda x) \quad \operatorname{Im} \frac{p(x)}{q(x)} e^{i\lambda x} = \frac{p(x)}{q(x)} \sin(\lambda x)$$

Examples from Section 12.3 of Notes

$$\int_{-\infty}^{\infty} \frac{x \sin(\pi x)}{x^2 + 2x + 5} dx = -\pi e^{-2\pi}$$

$$\int_{0}^{\infty} \frac{\cos(\pi x)}{(1 + x^2)^2} dx = \frac{\pi(\pi + 1)e^{-\pi}}{4}$$

$$\int_{0}^{\infty} \frac{\cos^2(x)}{1 + x^2} dx = \frac{\pi(1 + e^2)}{4e^2}$$

Thank you! Thursday will be revision