

Computing real integrals with the Residue Theorem

Basic idea is very flexible:

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{R, S \rightarrow \infty} \int_{-R}^S f(x) dx$$

- ▶ Include the finite integral as part of a contour integral
- ▶ Calculate the contour integral using residue theorem
- ▶ As $R, S \rightarrow \infty$ contributions of other parts of contour $\rightarrow 0$

ML Estimates often work for last point

Basic example / sanity check:

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx = \pi$$

Specific formulation appearing in notes on exam

Older exams wanted you to memorize this 😞

Theorem

Let $f(z) = \frac{p(z)}{q(z)} e^{i\lambda z}$ where $\lambda \in \mathbb{R}, \lambda > 0$, $p(z), q(z)$ are polynomials with $\deg q > \deg p$, no common zeroes, and q has no real roots. Then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{i\lambda x} dx = \sum_{i=1}^k \operatorname{Res}\{f; z_k\}$$

where z_1, z_2, \dots, z_k are the zeros of q in the upper half plane.

Proof.

Follow plan from last slide; if $\deg(q) + 1 < \deg(p)$ can just use *ML*-estimates, otherwise we need to sweat a bit more. □

Applying our specific formulation to compute real integrals

Take real and imaginary parts of our integrand

If p, q have real coefficients, and x is real, then

$$\operatorname{Re} \frac{p(x)}{q(x)} e^{i\lambda x} = \frac{p(x)}{q(x)} \cos(\lambda x) \quad \operatorname{Im} \frac{p(x)}{q(x)} e^{i\lambda x} = \frac{p(x)}{q(x)} \sin(\lambda x)$$

Examples from Section 12.3 of Notes

$$\int_{-\infty}^{\infty} \frac{x \sin(\pi x)}{x^2 + 2x + 5} dx = -\pi e^{-2\pi}$$

$$\int_0^{\infty} \frac{\cos(\pi x)}{(1+x^2)^2} dx = \frac{\pi(\pi+1)e^{-\pi}}{4}$$

$$\int_0^{\infty} \frac{\cos^2(x)}{1+x^2} dx = \frac{\pi(1+e^2)}{4e^2}$$