“ML-bounds”: M=Maximum, L=Length

Theorem

Suppose that f is continuous on a path γ of length L, and that $|f(z)| \leq M$ on γ. Then $|\int_\gamma f dz| \leq ML$

Proof.

$$\begin{aligned}\left|\int_\gamma f(z)dz\right| &= \left|\int_a^b f(z(t))z'(t)dt\right| \\ &\leq \int_a^b |f(z(t))||z'(t)|dt \\ &\leq \int_a^b M|z'(t)|dt = ML\end{aligned}$$

$\square$

The second line is triangle inequality for integrals.

Applications of the ML-inequality

Why do ML-inequalities?

The ML-inequality will help prove two of our big theorems.
We will take limits of paths where M or L are going to zero, and conclude the integral goes to zero.

"Toy" uses appear on exam

Example (From notes)

Let γ be a line segment lying with $D = \{z \in \mathbb{C} : |z| < 1$. Give an upper bound for

$$\int_\gamma \left(\frac{\operatorname{Re} z + z^2}{3 + \overline{z}} \right) dz$$

Section 8: Cauchy's Theorem

Theorem (Cauchy's Theorem)

Suppose the function f is analytic on a simply connected region D. Then $\int_\gamma f dz = 0$ for all contours γ in D.

Example

$$\int_{C_{0,1/2}} \frac{\sin(e^{3z}\cos(z))}{1+z^3}dz = 0$$

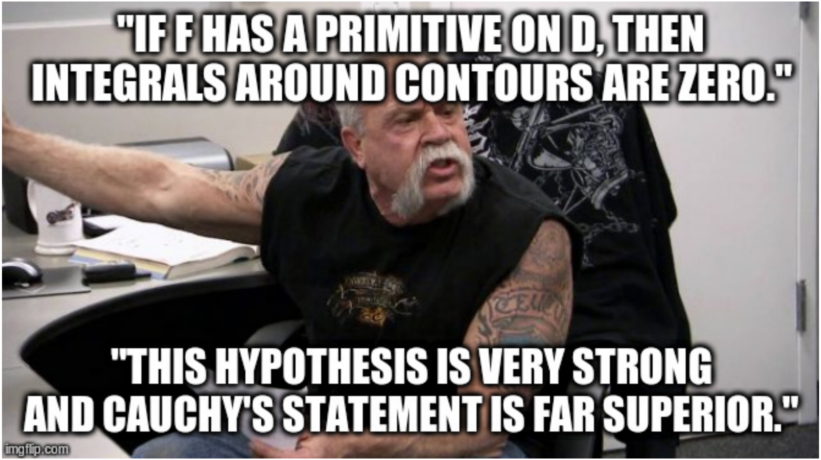
Example (Necessary to ask that D is simply connected)

The function $1/z$ is analytic on $\mathbb{C} \setminus \{0\}$. Letting $\gamma = e^{it}, 0 \leq t \leq 2\pi$ we have

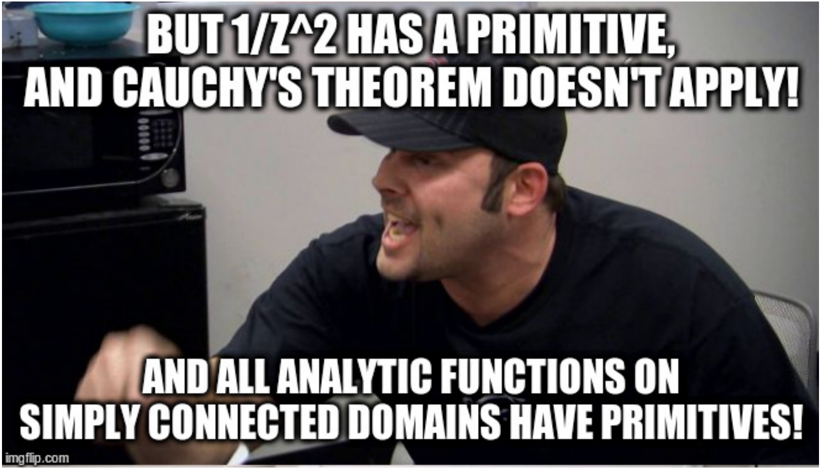
$$\int_\gamma \frac{1}{z}dz = 2\pi i$$

No Cauchy, because $\mathbb{C} \setminus \{0\}$ isn't simply connected.

I didn't agree with these quotes in the notes

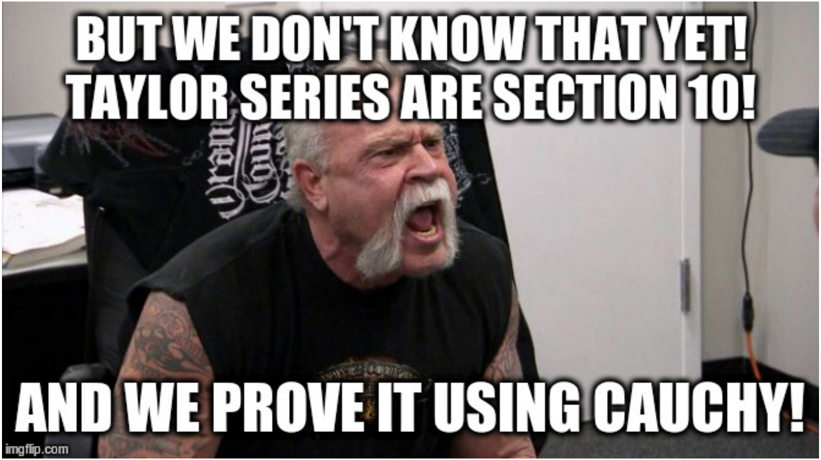


Here's why:



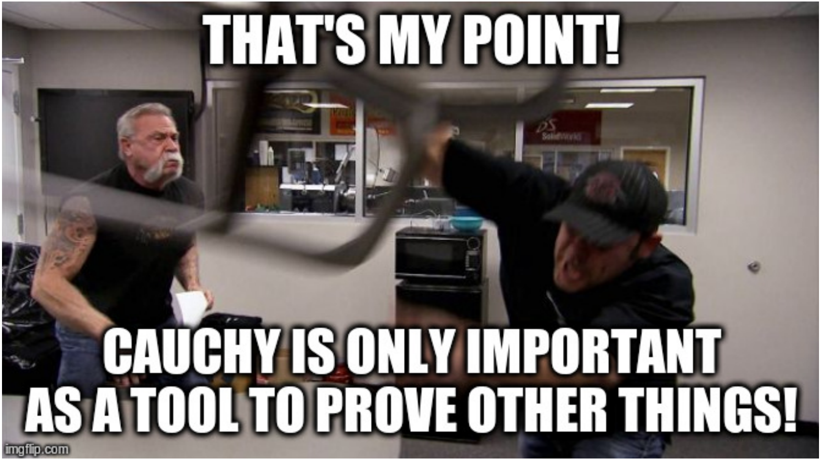
- Can prove $\int_{C_1(0)} \frac{1}{z^2} dz = 0$ using primitives, not Cauchy
- Hypotheses of Cauchy will tell us f *has* a primitive

Mary's rebuttal



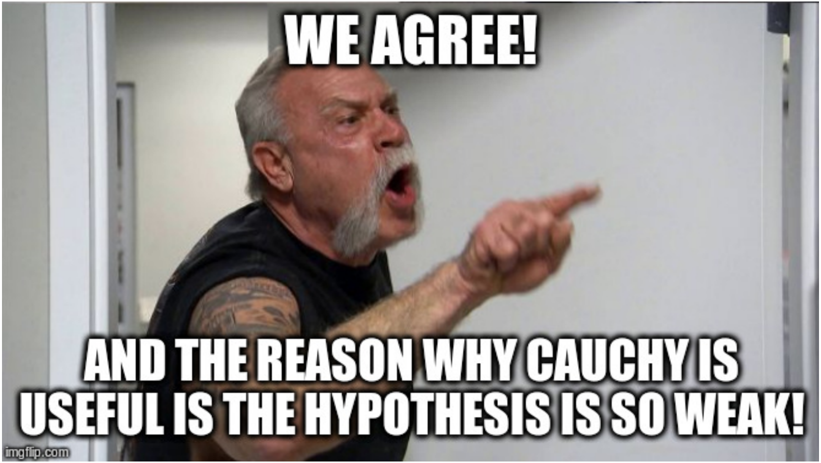
To prove that f analytic, D simply connected $\implies$ f has primitive comes much later in notes, and *depends* on Cauchy's theorem.

Well, sure, but...



The pay-off for Cauchy's theorem doesn't come until later

We're in agreement after all...



Even if I disagree about Cauchy being superior, I agree the point is that the hypotheses of Cauchy's Theorem are very weak

The proof of Cauchy's Theorem

The proof of Cauchy's Theorem is hard

Because the hypotheses are so weak!

- We won't prove it
- Notes: Green's Theorem + Cauchy-Riemann would prove it
- But Green's theorem requires *continuous* partial derivatives; we only have that f has a derivative!

Proof *is* in the recommended textbooks...

Theorem (Green)

Let Δ be the region bounded by an anticlockwise contuous γ, and let f and g have continous partial derivatives. Then:

$$\int_\gamma f\,dx + g\,dy = \iint_\Delta \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\,dx\,dy$$

A non-continuous derivative

Here's an example that shows that for *real* functions, the derivative of a function can be non-continuous:

Example

Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

exists at 0 but is not continuous there.