# Last time: Basics of Laurent series

A Laurent series is like a power series but we're allowed to have negative terms:

$$\sum_{n=-\infty}^{\infty} a_n(z-a)^n$$

## Theorem (Laurent's Theorem)

*Suppose that $f$ has an isolated singularity at $\alpha$, so $f$ analytic on $D' = \{z : 0 < |z-\alpha| < R\}$. Then $f$ can be represented by a Laurent series around $\alpha$ that converges on $D'$:*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n$$

*Where:*

$$a_n = \frac{1}{2\pi i}\int_{C_r(\alpha)} \frac{f(w)}{(w-\alpha)^{n+1}}\,dw$$

# What's left to cover:

| | |
|---|---|
| 11.5+11.6 | Classification of singularities |
| 11.7+11.8 | Calculating residues at poles |
| 12+12.1 | Residue Theorem |
| 12.2 | Application of Residue Theorem to real integrals |

## Definition (Residue)

Let $f$ have an isolated singularity at $\alpha$, and Laurent expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-\alpha)^n$$

Then $a_{-1}$ is called *The residue of $f$ at $\alpha$*, and written $\mathrm{Res}\{f;\alpha\}$.

## Awkward ordering in notes:

The end of 11 is really about finding Residues, but we only care about these because of the Residue Theorem.

# To fix this, changing order of lectures

To motivate the material at the end of Section 11, going to cover the Residue Theorem first.

| | |
|---|---|
| Today | Residue Theorem: Proof + first examples |
| Thursday | Classifying Singularities + finding residues |
| Next Tuesday | Applying Residue Theorem to real integrals |
| Next Thursday | Revision; focus on last two weeks |

The material next lecture will make applying Residue Theorem easier in nice cases.

## Theorem (The Residue Theorem)

*Let $D$ be a simply connected region containing a simple positively oriented contour $\gamma$. Suppose $f$ is analytic on $D$ except for finitely many singularities $\beta_1, \ldots, \beta_n$, none of which like on $\gamma$. Then*

$$\int_\gamma f(z)dz = 2\pi i \times (\textit{sum of the residues of } f \textit{ at the } \beta_i \textit{ inside } \gamma)$$

## Proof.

The proof is really putting together things we've already done:

- Deform contour so one singularity in each piece
- Expand $f$ in Laurent series
- Use formula for $a_{-1}$ / our first important example

$\square$

~~Every mother on Christmas morning~~
Every lecturer when they prove the big theorem of the module

# Using the Residue Theorem

Show you understand and check hypotheses!

1. Find the bad points (isolated singularities) $\beta_i$ of $f$
2. Draw picture showing $\gamma$ and bad points to see which are inside
3. Find the residues at the bad points inside $\gamma$

Examples; let $c = 5e^{it} \quad (0 \leq t \leq 2\pi)$

1. $\int_c \frac{dz}{z^2(z-3)^3}$
2. $\int_c \frac{dz}{\tan(z)}$
3. $\int_c z^3 \cos(1/z) dz$

Take-away:

Using Residue Theorem from definition can be slightly painful; can we find residue without finding whole Laurent expansion?