LIOUVILLE'S THEOREM QUESTIONS AND SOLUTIONS

Questions

Exercise 6.6 for convenience. State Liouville's Theorem.

(i) The function f is analytic in the complex plane and $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. Show that f is constant.

**(ii) The function f is analytic in the complex plane and $\mathrm{Re}(f(z)) > 1$ in $\mathbb{C}$. Prove that f is constant in $\mathbb{C}$.

An extra Liouville's Theorem question. The function f is analytic in $\mathbb{C}$ and $|f(z)+1| < |f(z)|$ for all $z \in \mathbb{C}$. Show that f is constant.

Solutions

Solution to Exercise 6.6. Liouville's Theorem A function which is analytic and bounded in the complex plane is a constant.

(i) Since $|f(z)| \geq 1$ we see that $f(z) \neq 0$ in the complex plane. Write $g = \frac{1}{f}$. Then g is analytic in the complex plane since f is non-zero and analytic in $\mathbb{C}$ and $|g(z)| \leq 1$, i.e. g is analytic and bounded in $\mathbb{C}$. By Liouville's Theorem g is a constant. This constant is not zero as $f(z)$ is defined for all z. Hence $f = \frac{1}{g}$ is also constant.

(ii) We use the standard notation $f(z) = u(x,y) + iv(x,y)$ where u and v are real valued. Now we are given that $\mathrm{Re}f(z) \geq 1$ and so $|f(z)|^2 = u^2 + v^2 \geq 1$ i.e $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. It follows from part (i) that f is constant.

Solution to the extra question. Since

$$0 \leq |f(z)+1| < |f(z)|,$$

we see that $f(z)$ is never zero on $\mathbb{C}$.

Hence $\dfrac{f(z)+1}{f(z)}$ is analytic in $\mathbb{C}$ and $\left|\dfrac{f(z)+1}{f(z)}\right| < 1$ for all z.

By Liouville's Theorem $\dfrac{f(z)+1}{f(z)} = k$, where the constant $k < 1$.

Hence

$$f(z) = \frac{1}{1-k}.$$