Section 9: Cauchy's Integral Formula and Consequences

Theorem (Cauchy's integral formula)

Let γ be a simple contour described in the positive direction. Let w lie inside γ. Suppose that f is analytic on a simply connected region D containing γ and its interior. Then:

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} dz.$$

Uses of Cauchy's Integral Formula

- Calculate contour integrals
- Values of f inside a contour determined by values on boundary
- Proving f has convergent Taylor series!

High level overview of Cauchy's Integral Formula

Many proofs we see have essential no real ideas to them – they're just unpacking definitions. Cauchy's integral formula has one real idea (Point 1) and one sneaky trick (Point 2).

Sketch of proof

1. Theorem 9.1 "Deforming contours": replace γ with $w + \varepsilon e^{2\pi i t}$ and take $\varepsilon \to 0$
2. $$???? \quad \frac{f(z) - f(w)}{z - w} \quad ????$$
3. ~~Profit~~ Prove the Theorem

Theorem 9.1 depends on Cauchy's Theorem, and it and the ideas that go into it are independently useful.

Theorem 9.1 Deforming Contours

Theorem

Let γ be a simple contour described in the positive direction. Let z_0 be a point inside γ, and let C be another simple contour in positive direction, contained entirely inside γ. Suppose that f is analytic on a region D which contains γ, C and all points in between. Then

$$\int_\gamma f(z)dz = \int_C f(z)dz$$

- Crucially, D need not be simply connected
- Proof: join C and γ together by two paths, and rearrange contours to apply Cauchy.

Proving Cauchy's Integral Formula

Let $C_\varepsilon(w)$ be the circle of radius ε around w. By Deforming Contours, we have

$$\int_\gamma \frac{f(z)}{z-w}dz = \int_{C_\varepsilon(w)} \frac{f(z)}{z-w}dz$$

Using the trick, we have that this is

$$= \int_{C_\varepsilon(w)} \frac{f(z)-f(w)}{z-w}dz + \int_{C_\varepsilon(w)} \frac{f(w)}{z-w}dz$$

We already computed that the second integral is $2\pi i f(w)$, so we need to see the first integral tends toward zero.

Vanishing of the first integral

We use the ML estimate

- As $\varepsilon \to 0$, $z \to w$, and the integrand $\frac{f(z)-f(w)}{z-w} \to f'(w)$
- In particular, integrand is bounded by some M.
- The length of $C_\varepsilon(w) = 2\pi\varepsilon$.

Thus

$$\left| \int_{C_\varepsilon(w)} \frac{f(z) - f(w)}{z - w} dz \right| \leq M2\pi\varepsilon$$

And so the integral $\to 0$ as $\varepsilon \to 0$

Contour integrals via of Cauchy's Integral Formula

Example (Section 9.3)

Let γ be the simple, positively oriented triangular contour from 0 to $2-3i$ to $2+2i$ and back to zero. Evaluate

$$\int_\gamma \frac{e^z}{z-1}\,dz \quad \int_\gamma \frac{e^z}{z+1}\,dz \quad \int_\gamma \frac{1}{z^2-1}\,dz$$

$$\int_\gamma \frac{ze^{z^2}}{(z-1)(2z-1)}\,dz \quad \int_\gamma \frac{e^{z^2}}{z^2-1}\,dz$$

Tips and Tricks

- Draw Picture containing contour and "bad" points
- Avoid partial fractions – deform contour instead 😎