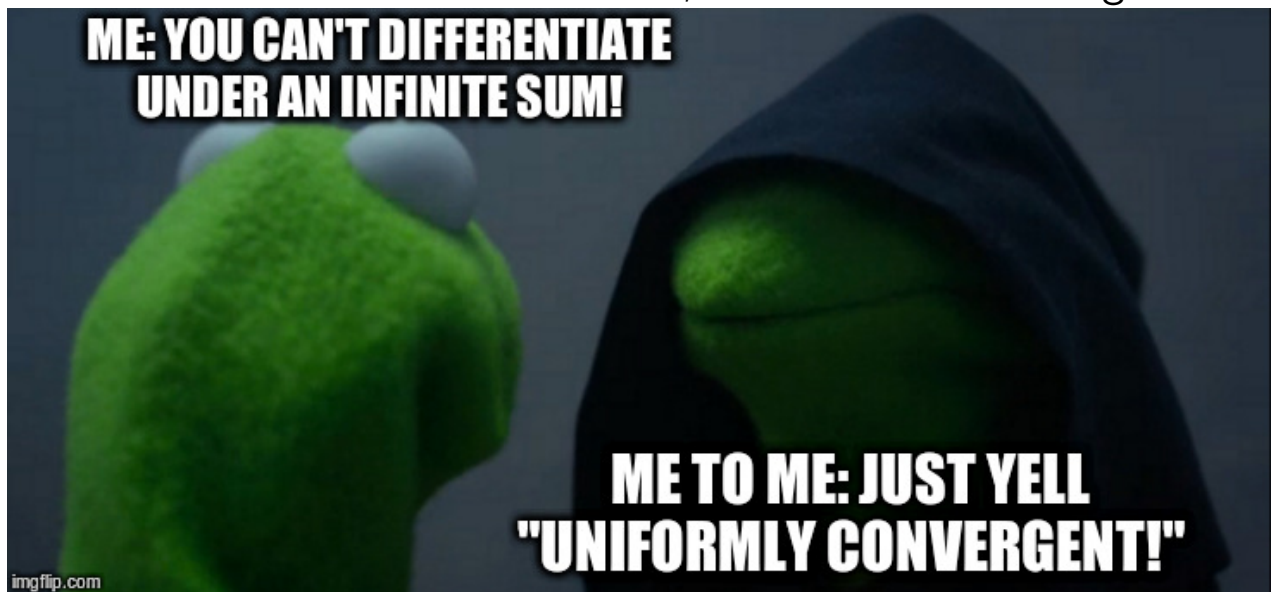


## On Switching Orders

“There are three big assumptions, all valid in this course with our situation, but which are false in general.”



## Sequences in $\mathbb{C}$ – just as in $\mathbb{R}$

### Definition

Let  $a_n$  be a sequence of complex numbers. Then we say the sequence converges to  $L$ , and write

$$\lim_{n \rightarrow \infty} a_n = L$$

if for all  $\varepsilon > 0$  there exists an  $N$  so for  $n > N$  we have  $|a_n - L| < \varepsilon$

### Lemma

$$\lim_{n \rightarrow \infty} a_n = L \iff \begin{array}{l} \lim_{n \rightarrow \infty} \operatorname{Re}(a_n) = \operatorname{Re}(L) \\ \lim_{n \rightarrow \infty} \operatorname{Im}(a_n) = \operatorname{Im}(L) \end{array}$$

### Proof.

$$\max(|\operatorname{Re}(z - L)|, |\operatorname{Im}(z - L)|) \leq |z - L| \leq |\operatorname{Im}(z - L)| + |\operatorname{Re}(z - L)|$$

□

## Series in $\mathbb{C}$ – just as in $\mathbb{R}$

### Definition

Let  $a_n$  be a sequence of complex numbers. Then we say the series  $\sum_{i=0}^{\infty} a_i$  converges to  $L$  if the sequence of partial sums  $S_n = \sum_{i=0}^n a_i$  converges to  $L$ .

### Definition

A series  $\sum a_n$  is *absolutely convergent* if  $\sum |a_n|$  converges.

### Two tools:

- ▶ Comparison test
- ▶ Geometric series

# We'll mostly be interested in power series

## Definition

Suppose that  $a_n, z_0 \in \mathbb{C}$ . A series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is called a *power series centred at  $z_0$* .

We will be interested in what values of  $z$  a given power series converges; for those values, we will have a function of  $z$ . E.g.

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \forall z \in \mathbb{C}$$

# Radius of convergence

## Theorem

Suppose  $w \neq 0$  and  $\sum a_n w^n$  converges. Then  $\sum a_n z^n$  is absolutely convergent for all  $z$  with  $|z| < |w|$

## Theorem (Abel)

For any power series, either:

1. The power series converges only at  $z = 0$ .
2. The power series is absolutely convergent for all  $z \in \mathbb{C}$
3. There is a real number  $R$  so that the power series is absolutely convergent if  $|z| < R$ , and divergent if  $|z| > R$

$R$  is called the radius of convergence. It is zero in case one, and infinite in case 2.

# The ratio test

The radius of convergence always exists (though it may be 0 or  $\infty$ ), but not clear how to find. We will mostly use:

## Theorem (Ratio Test)

If

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} = R$$

Then  $R$  is the radius of convergence.

Proof.

Idea: use comparison test to compare to geometric series



- ▶ **Can't** apply Ratio Test to “most” power series.
- ▶ *Can* apply Ratio Test to most power series we'll see!

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## Examples

Find the radius of convergence of the following functions.

$$\sum_{n=0}^{\infty} (\sinh(n)) z^n \quad (1)$$

$$\sum_{n=1}^{\infty} \frac{(2i)^n z^n}{n} \quad (2)$$

$$\sum_{n=1}^{\infty} \frac{(2i)^n z^{3n}}{n} \quad (3)$$

$$\sum_{n=0}^{\infty} \frac{(2n)! n!}{(3n)!} z^n \quad (4)$$



## Convergent Power series give analytic functions

Define  $f(z) = \sum a_n z^n$  inside the radius of convergence. Is  $f(z)$  analytic? We'd like to argue:

$$\begin{aligned} f'(z) &= \frac{d}{dz} \sum a_n z^n \\ &= \sum \frac{d}{dz} a_n z^n \\ &= \sum n a_n z^{n-1} \end{aligned}$$

We're being Evil Kermit

- ▶ Not clear we can move derivative inside sum
- ▶ Not clear final power series converges

Hence: Power series give analytic functions!

Will see later this gives *all* analytic functions!