Line integrals: an important example

Recall: $C_r(a)$ is the anti-clockwise circle of radius r around a.

A mysterious computation:

Let $n \in \mathbb{Z}$

$$\int_{C_r(a)} \frac{1}{(z-a)^n} \mathrm{d}z = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

Independent of a and r, works for n negative, too!

Coming attractions – conceptual explanation!

- *Antiderivatives* explain why the answer is zero unless $n = 1$
- *Cauchy's theorem* explains why it's independent of r
- *Residue theorem* reduces *any* integral to this computation!

Derivatives review

Definition

Let f be defined on some open subset $U \subset \mathbb{C}$. f is *differentiable* at $z_0 \in U$ if

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Looks like normal derivative, but...

- Numerator and denominator will be complex numbers
- Have to get the same limit no matter how we approach z_0

It will turn out that a complex function being differentiable is much strong than a real function being differentiable.

Cauchy-Riemann equations are first example of that

Cauchy-Riemann Equations

Theorem (Cauchy-Riemann Equations)

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at z_0. Then at

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{at } z_0$$

Proof.

Compute $f'(z_0)$ in two different ways:

- Keeping x constant
- Keeping y constant

$\square$

More on Cauchy-Riemann

Complex formulation:

Sometimes convenient to write both Cauchy-Riemann equations as one complex equation:

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

Extension (non-examinable): Analytic functions are conformal

In MAS211 you looked at the derivative of a map $f : \mathbb{R}^n \to \mathbb{R}^m$ as a linear map $Df : \mathbb{R}^n \to \mathbb{R}^m$, and hence as a matrix. If $f : \mathbb{C} \to \mathbb{C}$ is differentiable at z_0, this linear map corresponds to multiplication by a complex number $z = a + bi$, and in matrix form this is:

$$Df(z_0) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Hence the derivative is a rotation + a scaling, and *preserves angles*

Motivation for an "application": PDEs

The Laplacian operator, written ∇^2 or Δ, acts on functions $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$\nabla^2 g = \nabla \cdot \nabla g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}$$

and occurs in many PDEs important in applied math.

Examples

Let $f(x, y, t)$ be a function of two space variables and one time variable.

- The heat equation $\frac{\partial f}{\partial t} = \nabla^2 f$
- The wave equation $\frac{\partial^2 f}{\partial t^2} = \nabla^2 f$

A steady state solution to either of these equations would be $\nabla^2 f = 0$.

Harmonic Functions

Definition

A function $u : \mathbb{R}^2 \to \mathbb{R}$ is *harmonic* if $\nabla^2 f = 0$

Lemma

Let $f(z) = u(x, y) + iv(x, y)$ be analytic on a domain D. Then u and v are harmonic on D

Proof.

Cauchy-Riemann equations + mixed partials are equal. $\square$

This gives us lots of harmonic functions.

Does this give us *all* harmonic functions?

Given a harmonic function $u(x, y)$ on a domain, is it the real part of an analytic function $f(z)$?

Complete answer next time!