Classification of Singularities

By Laurent's Theorem, if $f(z)$ is analytic in a punctured disk around α, it has a convergent Laurent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - \alpha)^n$$

Three possibilities:

Removable singularity None of the a_n with $n < 0$ are nonzero

A pole Only finitely many a_n with $n < 0$ are nonzero

Essential singularity Infinitely many a_n with $n < 0$ are nonzero

Punchline first:

Only for essential singularities will you need to compute the Laurent series to find the pole!

Removable singularities: no negative powers

f has a removable singularity at α means it has no negative powers of $z - \alpha$ in its Laurent series.
But then it's Laurent series is really a Taylor series, and it makes sense to plug in α. Thus f extends to an analytic function around α.

Examples:

1. $\frac{z^2-1}{z-1}$
2. $\frac{\sin(z)}{z}$

Since the Laurent series has no negative powers, the residue of a removable singularity is always zero.

Poles: only finitely many negative terms

Definition

We say that $f(z)$ has a *pole of order k* at α if its Laurent series $f(z) = \sum_{n \in \mathbb{Z}} a_n (z-\alpha)^n$ has $a_{-k} \neq 0$, but $a_n = 0$ for $n < -k$.

In other words

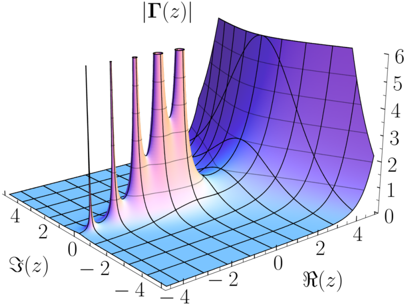
$$f(z) = \sum_{n \geq -k} a_n (z-\alpha)^n$$

A pole of order one is also called a *simple pole*.

Examples: poles of order 2

1. $\frac{e^z}{z^2}$
2. $\tan^2(z)$

Poles of the gamma function



$\Gamma(z)$ extends $(n-1)!$ to an analytic function, and has simple poles at the non-positive integers

Essential singularity: infinitely many negative powers

Definition

If the Laurent series of $f(z)$ around α has infinitely many $a_k \neq 0$ with $k < 0$, then we say α is an *essential singularity* of f

Examples

- $e^{1/z}$
- $\cosh(1/z)$

Theorem (Great Picard's Theorem – just for culture)

If $f(z)$ has an essential singularity at α, then on any punctured disk around α, $f(z)$ takes on all possible complex values, with at most one exception, infinitely often.

So $\lim_{z\to\alpha} f(z)$ is infinity for a pole, but horribly doesn't exist for an essential singularity.

Easy (and examinable!) theorems about poles

Theorem

f has a pole of order k at α if and only if

$$f(z) = \frac{g(z)}{(z-\alpha)^k}$$

where $g(z)$ analytic and nonzero in some disk around α.

Theorem

If f has a zero of order k at α, then $1/f$ has a pole of order k at α.

Corollary

If f has a zero of order m at α, and g has a zero of order n at α, then

- $\frac{f}{g}$ *has a pole of order $n - m$ if $n > m$*
- $\frac{f}{g}$ *has a removable singularity if $m \geq n$*

Easy way to find residues at poles

Theorem

Suppose that f has a pole of order k at α. Then

$$\operatorname{Res}\{f;\alpha\} = \frac{1}{(k-1)!}\lim_{z\to\alpha}\frac{d^{k-1}}{dz^{k-1}}(z-\alpha)^k f(z)$$

Proof.

Just compute the right hand side. $\square$

Corollary

If $f = g/h$, where g and h are analytic at α,
$g(\alpha) \neq 0, h(\alpha) = 0, h(\alpha) \neq 0$, then f has a simple pole at α and

$$\operatorname{Res}\{f;\alpha\} = \frac{g(\alpha)}{h'(\alpha)}$$