

Chromatic Polynomial

Rather than just knowing whether we *can* colour a graph G with k colours, we can count the different colourings that are possible.

Definition

Let G be a simple (why?) graph, and let $k \geq 1$ be an integer. The *chromatic polynomial* $P_G(k)$ is the number of different ways to colour the vertices of G with k colours, so that adjacent vertices have different colours.

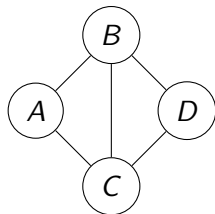
Remarks:

- ▶ It is *not* obvious from the definition that $P_G(k)$ is a polynomial
- ▶ $P_G(k) = 0 \iff 0 \leq k < \chi(G)$

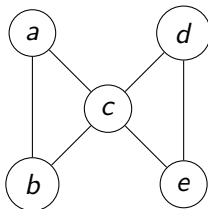
In easy examples, can colour vertex by vertex:

Colour vertex v_1 , then vertices adjacent to v_1 , then..

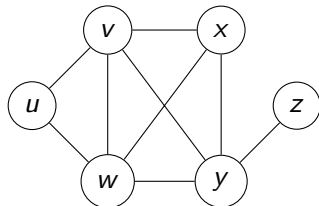
- ▶ $P_{E_n}(k) = k^n$
- ▶ $P_{K_n}(k) = k(k-1)(k-2)\cdots(k-n+1)$
- ▶ If T is a tree with n vertices, then $P_T(k) = k(k-1)^{n-1}$



G



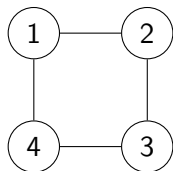
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What patterns do you notice?

A harder example: C_4



- ▶ k ways to colour vertex 1
- ▶ $k - 1$ ways to colour vertex 2
- ▶ $k - 1$ ways to colour vertex 3
- ▶ Don't know if v_1 and v_3 have same colour

Case 1: v_1 and v_3 have the same colour

- ▶ k choices for this colour
- ▶ Then $k - 1$ choices for each of v_2 and v_4

Case 2: v_1 and v_3 have the same colour

- ▶ k choices for v_1 , then $k - 1$ choices for v_3
- ▶ $k - 2$ choices for each of v_2 and v_4

Combining the cases, we see:

$$P_{C_4} = k(k-1)^2 + k(k-1)(k-2)^2 = k(k-1)(k^2 - 3k + 3)$$

Foreshadowing: the two cases are chromatic polynomials

The type of reasoning we used to find the chromatic polynomial of C_4 will work to find the chromatic polynomial of any graph; however, many cases might need to be considered, and the argument will get quite complicated.

It will help to repackage the reasoning

Case 1: v_1 and v_2 are the same colour

If they're the same colour, then we can make them same vertex...

Case 2: v_2 and v_3 are different colours

If there's an edge between two vertices, then they need to be different colours.

Generalizing the observation we just made

A lemma, with some definitions baked in

Suppose that G is a graph, and x and y are two vertices that aren't adjacent. Define:

- ▶ G_{+xy} to be the graph with the edge xy added
- ▶ $G_{x=y}$ to be the graph with x and y identified

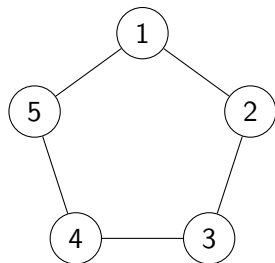
Then:

$$P_G(k) = P_{G_{+xy}}(k) + P_{G_{x=y}}(k)$$

Proof.

Consider a colouring of G ; in some of them x and y will have different colourings, and in others x and y will have the same colour. The colourings in the first case are exactly the colourings of G_{+xy} , the colourings in the second are the colourings of $G_{x=y}$. \square

The chromatic polynomial of C_5



Three cases, but two are the same:

- ▶ Case 1: 1, 2 and 3 all have different colours
- ▶ Case 2: 1 and 2 have the same colour
- ▶ Case 3: 1 and 3 have the same colour

By symmetry, Cases 2 and 3 are the same.

- ▶ Case 1 gives: $k(k-1)(k-2)^3$
- ▶ Cases 2 and 3 each give: $k(k-1)^2(k-2)$

$$\begin{aligned}P_{C_5}(k) &= k(k-1)(k-2)^3 + 2k(k-1)^2(k-2) \\&= k(k-1)(k-2) [k^2 - 4k + 4 + 2k - 2] \\&= k(k-1)(k-2)(k^2 - 2k + 2)\end{aligned}$$