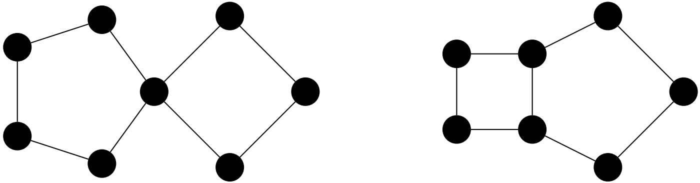
More techniques for computing chromatic polynomials

Main goal: two new bits

- Use induction to calculate $P_G(k)$ for a family of graphs. We'll do C_n
- Gluing formulas for graphs that are nearly disconnected



The graphs above glued from C_4 and C_5.

If we have time:

Compute $P_G(k)$ for a relatively complicated graph G.

Calculating the chromatic polynomial of C_n

Let e be any edge of C_n. Then:

- $C_n/e \cong C_{n-1}$
- $C_n \setminus e = P_n$, a tree, so $P_{P_n}(k) = k(k-1)^{n-1}$

So we should be able to find $P_{C_n}(k)$ using induction, but need to "guess" the formula first.

$$\begin{aligned}
P_4(k) =& k^4 - 4k^3 + 6k^2 - 3k \\
P_5(k) =& k^5 - 5k^4 + 10k^3 - 10k^2 + 4k \\
P_6(k) =& k^6 - 6k^5 + 15k^4 - 20k^3 + 15k^3 - 5k \\
P_7(k) =& k^7 - 7k^6 + 21k^5 - 35k^4 + 35k^3 - 21k^2 + 6k
\end{aligned}$$

Looks like:

$$P_n(k) = (k-1)^n + (-1)^n(k-1)$$

Inductive proof that $P_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$

Base case: $n = 3$

Plug in $n = 3$, get $k(k-1)(k-2) = P_{C_3}(k)$.

Inductive step:

Get to assume: $P_{C_{n-1}}(k) = (k-1)^{n-1} + (-1)^{n-1}(k-1)$

- $C_n \setminus e = P_n$, a tree, so $P_{C_{n-1} \setminus e} = k(k-1)^{n-1}$
- $C_n / e = C_{n-1}$, so $P_{C_n/e}(k) = (k-1)^{n-1} + (-1)^n(k-1)$.

Plugging into deletion-contraction:

$$
\begin{aligned}
P_{C_n}(k) &= P_{C_n \setminus e}(k) - P_{C_n/e}(k) \\
&= k(k-1)^{n-1} - \left[(k-1)^{n-1} + (-1)^{n-1}(k-1)\right] \\
&= k(k-1)^{n-1} - (k-1)^{n-1} - (-1)^{n-1}(k-1) \\
&= (k-1)^{n-1}\,[k-1] + (-1)^n(k-1) \\
&= (k-1)^n + (-1)^n(k-1)
\end{aligned}
$$

Gluing formulas: intro

Lemma

If G is the disjoint union of G_1 and G_2, then

$$P_G(k) = P_{G_1}(k)P_{G_2}(k)$$

Proof.

Colouring G is exactly the same as colouring G_1 and G_2 independently. □

Gluing formulas: when G isn't *quite* a disjoint union

Idea: Colour G_1, then extend to a colouring of G_2.

Gluing formulas: statements

Lemma

If G is made by gluing G_1 and G_2 along a vertex v, then:

$$P_G(k) = \frac{1}{k} P_{G_1}(k) P_{G_2}(k)$$

Proof.

First colour G_1 in any of the $P_{G_1}(k)$ ways. Now, vertex v of G_2 is already coloured, but none of the rest. Since the colours are interchangable, exactly $1/k$ of the ways of colouring G_2 will have the right colour at v. □

Lemma

If G is made by gluing G_1 and G_2 along an edge e, then

$$P_G(k) = \frac{1}{k(k-1)} P_{G_1}(k) P_{G_2}(k)$$

Find $P_G(k)$ for the following graphs