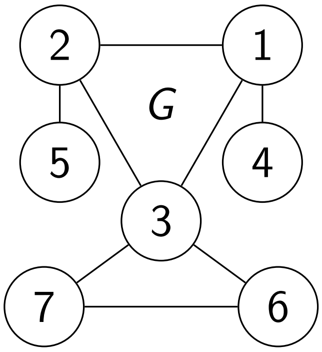
Where were we? Colouring graphs

- Chromatic number $\chi(G)$: colour vertices with fewest colours
- Chromatic index $\chi'(G)$: colour edges with fewest colours
- Chromatic polynomial $P_G(k)$: number of ways to colour vertices with k colours

Some graphs: colour vertex by vertex

$$P_G(k) = k(k-1)^4(k-2)^2$$



In general: need case-by-case

Examples we looked at: C_4, C_5.

Today: prove $P_G(k)$ is a polynomial

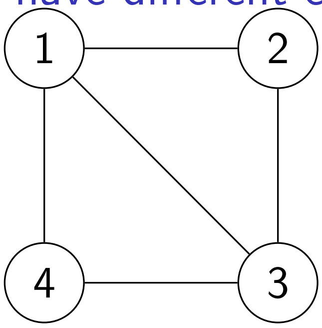
Review: Chromatic polynomial of C_4 and a Lemma

Lemma

Let x and y be two non-adjacent vertices in G. Then

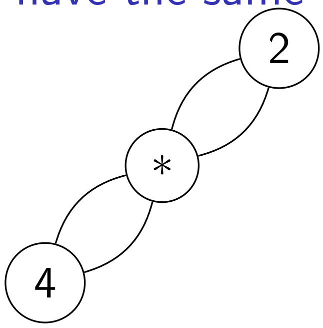
$$P_G(k) = P_{G_{+xy}}(k) + P_{G_{x=y}}(k)$$

1 and 3 have different colours



$k(k-1)(k-2)^2$

1 and 3 have the same colour



$k(k-1)^2$

$$P_{C_4}(k) = k(k-1)(k-2)^2 + k(k-1)^2 = k(k-1)(k^2-3k+3)$$

Deletion-Contraction: restructuring the lemma

The lemma as stated is useful for calculating, but awkward for induction:

- G_{+xy} has an extra edge; "bigger"
- $G_{x=y}$ has fewer vertices: "smaller"

Rewrite so $H \cong G_{+xy}$ is the "starting" graph.

Lemma (Deletion-Contraction)

Let H be a graph, and let e be an edge in H. Then

$$P_H(k) = P_{H \setminus e}(k) - P_{H/e}(k)$$

The edge e is between xy

$$H = G + xy, \qquad H \setminus e = G, \qquad H/e = G_{x=y}$$

Chromatic polynomial is a polynomial

Theorem

Let G be a simple graph. Then $P_G(k)$ is a polynomial in k. Moreover, if G has n vertices and m edges, then

$$P_G(k) = k^n - mk^{n-1} + \textit{lower order terms}$$

Proof idea:

Induct on the number of edges using deletion-contraction.

Base case: $m = 0$

If G has no edges and n vertices, then $G = E_n$ empty graph.
$P_{E_n} = k^n$ is a polynomial of the right form.

Inductive step

Assume that G has $m > 0$ edges and n vertices, and that for any graph H with $\ell < m$ edges and p vertices, we have $P_H(k) = k^p - \ell k^{p-1} + \cdots$.

Let $e \in G$ be any edge:

- $G \setminus e$ has n vertices and $m - 1$ edges
- G/e has $n - 1$ vertices and *at most* $m - 1$ edges

So by the inductive hypothesis, theorem holds for $G \setminus e$ and G/e

So applying Deletion-Contraction:

$$\begin{aligned} P_G(k) &= P_{G\setminus e}(k) - P_{G/e}(k) \\ &= \left(k^n - (m-1)k^{n-1} + \cdots\right) - \left(k^{n-1} - \cdots\right) \\ &= k^n - mk^{n-1} + \cdots \end{aligned}$$

Which is what we needed to show. $\square$

Odds and Ends

Deletion-Contraction as an algorithm

- Can always find $P_G(x)$ by iterating deletion-contraction
- In practise, often faster to add edges

Information in $P_G(k)$

- Number of vertices is the degree
- Number of edges is negative the coefficient of next highest term
- $\chi(G)$ is the lowest k with $P_G(k) \neq 0$.