Colouring Graphs

Started with 4 colour theorem:
Any map can be coloured with four colours so that adjacent countries have different colours.

- Posed by Guthrie 1852
- Proved by Appel and Haken 1976

Definition

The *chromatic number* $\chi(G)$ of a graph G is the least number of colours needed to colour the vertices so that adjacent vertices have different colours.

Theorem (The four colour theorem)

A planar graph G has $\chi(G) \leq 4$.

First examples of chromatic number

The empty graph E_n

The only graphs with $\chi(G) = 1$ are the empty graphs E_n.

The complete graph K_n

The complete graph has $\chi(K_n) = n$

Bipartite graphs

A graph G has $\chi(G) = 2$ if and only if G is bipartite.

The wheel graph W_n

$$\chi(W_n) = \begin{cases} 3 & n \text{ even} \\ 4 & n \text{ odd} \end{cases}$$

Why?

How to find $\chi(G)$: sandwich it!

Start by finding rough upper and lower bounds.

Upper bound: colour it

If you can colour the vertices of G with k colours so that adjacent vertices don't share a vertex, then $\chi(G) \leq k$.

Lower bound: often case by case

A few trivial tricks:

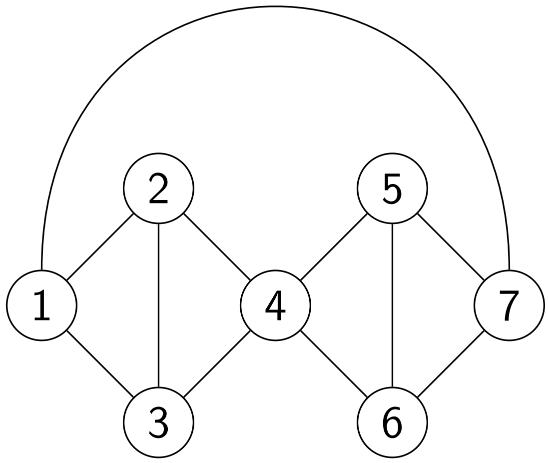
- If a vertex is adjacent to everything, as in W_n
- If H is a subgraph of G, then $\chi(G) \geq \chi(H)$.

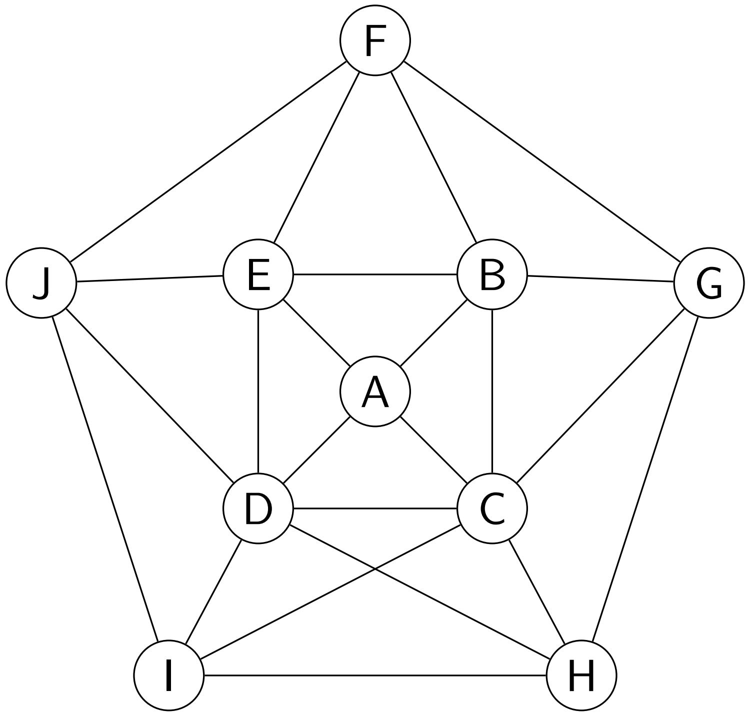
If lower bound isn't equal to upper bound, refine.

Finding the $\chi(G)$ is NP-hard

So no beautiful answer. But for small graphs not that bad.

Example: find $\chi(G)$ for the graphs shown below





General upper bounds

Definition

$\Delta(G)$ is the maximum degree of any vertex of G.

Lemma

$$\chi(G) \leq \Delta(G) + 1$$

Proof.

Colour the vertices one by one in any order. $\square$

The bound is tight, but for very few graphs:

- $\chi(K_n) = n = \Delta(K_n) + 1$
- For n odd, $\chi(C_n) = 3 = \Delta(C_n) + 1$

Theorem (Brooks)

If G isn't a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.

A story problem for $\chi(G)$

Suppose you have some things you want to separate into groups, but certain things can't be in the same group. How many groups do you need?

- Group vacation, several cottages, some people don't get on
- Storing chemicals, some react dangerously with each other

Make a graph G with:

- Vertices are the things
- Edges mean the vertices can't be in the same group

The groups are the colours.

$\chi(G)$ is the lowest feasible number of groups