

## Section 4: Graphs on Surfaces

### The Utilities Problem:

Connect three houses to three utilities without crossing the lines?



# First, a definition

## Definition

A graph  $G$  is *planar* if it can be drawn in the plane so that no edges cross.

This “definition” is vague, fixing it requires analysis/topology

- ▶ What do we mean “draw” an edge?
  - ▶ Injective continuous map  $f : [0, 1] \rightarrow \mathbb{R}^2$

Not a course in topology; our intuition will be enough

The Utilities question becomes:

Is the complete bipartite graph  $K_{3,3}$  planar?

# Isn't possible, but how to organize proof?

Too many cases about how the edges could go...

Would be very lengthy, easy to miss a case.

Instead, assume it was:

- ▶ Any cycle  $C_n \subset G$  would be drawn as a circle
- ▶ Any edge  $e \in G, e \notin C_n$  would be inside or outside the circle
- ▶ Certain pairs of edges can't be on the same side...

Mathematical culture:

That a circle has two sides is surprisingly difficult topology.

Theorem (Jordan curve theorem)

*A simple closed curve in the plane has an interior and an exterior*

- ▶ Simple: doesn't cross itself
- ▶ Closed: starts where it ends

## Theorem: $K_{3,3}$ isn't planar

Let  $A, B, C$  be blue vertices,  $X, Y, Z$  be red.

Suppose  $K_{3,3}$  were planar:

- ▶ Then the Hamiltonian cycle  $AXBYCZA$  would be a circle
- ▶ The three edges  $AY, BZ, CX$  still need to be drawn

Case 1:  $AY$  inside the cycle

- ▶ But then we need to draw  $BZ$  outside
- ▶ Two ways to do this, but either way can't draw  $CX$

Case 2:  $AY$  outside the cycle

- ▶ Two ways to do this, but either way  $BZ$  needs to be inside
- ▶ Now we can't draw  $CX$       $\square$

The cases look awfully similar/redundant...

# From the plane to the sphere

In the plane  $\mathbb{R}^2$ :

The outside and inside of a circle are different:

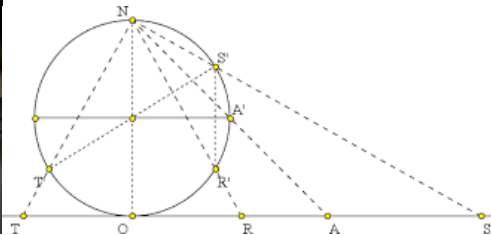
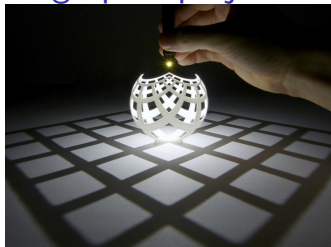
- ▶ The inside is bounded, the outside isn't
- ▶ One way to connect inside, two ways to connect outside

But on the sphere  $S^2$  things are nicer:

- ▶ The outside and inside of a circle look the same
- ▶ Only one way to connect outside

But we wanted to draw graphs on the plane, not the sphere

Stereographic projection:  $S^2 = \mathbb{R}^2 \cup \{p\}$ :



Corollary

*$G$  is planar if and only if  $G$  can be drawn on the sphere.*

Proof.

If: Draw  $G$  on  $S^2$ ; project from a point  $p \in S^2 \setminus G$

Only if: Project from plane to sphere



Upshot: don't need to treat inside/outside as separate cases

# Wrapping up and moving to next time?

## Other examples of this general method

Use or adapt this general method to prove that:

- ▶  $K_5$  isn't planar
- ▶ Petersen graph isn't planar
- ▶ Just draw a random graph and decide if it's planar

Does this work on any graph?

Next time:

- ▶ This method is *great* for Hamiltonian graphs
- ▶ Kuratowski's Theorem: when you're far from Hamiltonian