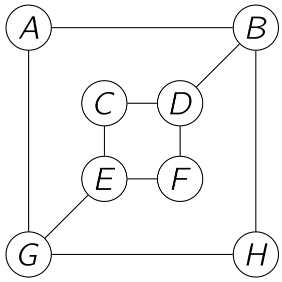
The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2017–2018

Graph Theory — 2 hours 30 minutes

*Attempt all the questions. The allocation of marks is shown in brackets.*

**1** The first two parts use the graph $\Gamma$ shown below.




(i) State what it means for a graph to be semi-Eulerian, and prove that $\Gamma$ is not semi-Eulerian. List *all* edges that could be added or deleted from $\Gamma$ to make it semi-Eulerian, while not creating any loops or multiple edges.

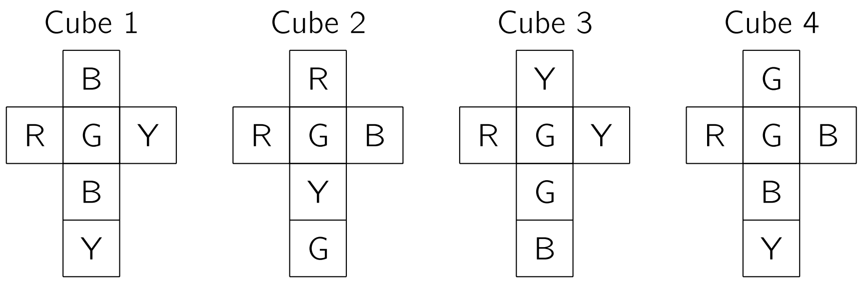
***(5 marks)***

(ii) State what it means for a graph to be Hamiltonian, and prove that $\Gamma$ is not Hamiltonian. List *all* edges that could be added or deleted from $\Gamma$ to make it Hamiltonian.

***(5 marks)***

(iii) Find a solution to the set of instant insanity cubes shown below. How many other solutions can you get by keeping the position of Cube 1 fixed, but rotating the remaining three cubes? Explain your answer.

***(9 marks)***




(iv) Recall that in biochemistry, atoms of carbon, nitrogen, oxygen and hydrogen have valency 4, 3, 2 and 1, respectively. Prove that any isotope of $C_2NOH_7$ is a tree. How many such isotopes are there? Draw them.

***(6 marks)***

2 The first four parts of this question use a weighted complete graph $H$ on seven vertices $A - G$. The weights of the edges are given below.

| | | | | | | |
|---|---|---|---|---|---|---|
| $A$ | | | | | | |
| 20 | $B$ | | | | | |
| 4 | 11 | $C$ | | | | |
| 6 | 14 | 5 | $D$ | | | |
| 19 | 10 | 13 | 15 | $E$ | | |
| 10 | 17 | 11 | 19 | 9 | $F$ | |
| 15 | 8 | 14 | 7 | 12 | 13 | $G$ |

(i) List the edges of a cheapest spanning tree of $H$ in the order they are added if the tree is built using Kruskal's algorithm.

***(2 marks)***

(ii) List the edges of a cheapest spanning tree of $H$ in the order they are added if the tree is built using Prim's algorithm beginning at vertex $B$.

***(2 marks)***

(iii) Suppose the vertices of $H$ are cities and the edge weights are costs to travel between cities on a direct flight. If journeys with multiple legs are allowed (for instance, getting to F by first going through B), which city is the most expensive to fly to from city $A$?

***(5 marks)***

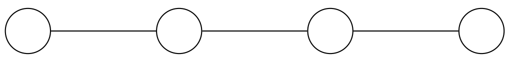
(iv) State the traveling salesperson problem. Give a solution to the traveling salesperson for $H$, proving that your solution is optimal.

***(6 marks)***

(v) Draw the tree with Prüfer code 331668.

***(4 marks)***

(vi) Recall that the path graph $P_n$ consists of $n$ vertices in a line. $P_4$ is shown below.



How can you tell from the Prüfer code whether or not the corresponding tree is the path graph $P_n$ with some labelling? Explain your answer.

***(6 marks)***

**3** (i) Define the chromatic number, $\chi(G)$, the chromatic index $\chi'(G)$, and the chromatic polynomial $\chi_G(k)$. Explain how to determine $\chi(G)$ from $\chi_G(k)$, and give an example of two graphs $G$ and $H$ whose chromatic polynomials are equal, but whose chromatic indexes are not equal.

***(6 marks)***

(ii) Prove that for $k \geq 1$ we have $\chi_G(k+1) \geq \chi_G(k)$. Give an example of a $G$ and a $k \geq 1$, with $\chi_G(k+1) = \chi_G(k)$.

***(4 marks)***

(iii) Prove that if every vertex of $G$ has degree at most $d$, then $\chi(G) \leq d+1$.

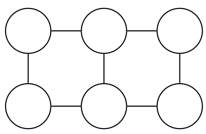
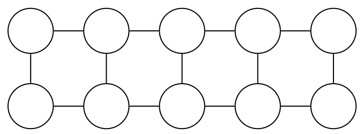
***(3 marks)***

(iv) Let $G$ and $\Gamma$ be two subgraphs of $H$, with the union of $G$ and $\Gamma$ being all of $H$, but the intersection $G \cap \Gamma$ consisting of two vertices $v$ and $w$ connected by an edge $e$. Prove that

$$k(k-1)\chi_H(k) = \chi_G(k)\chi_\Gamma(k)$$

***(4 marks)***

(v) Let $n \geq 0$. The "ladder graph" $L_n$ has $2n+2$ vertices arranged as $n$ squares in a line. For example, $L_1$ is isomorphic to the 4-cycle $C_4$, and $L_2$ and $L_4$ are shown below:

$L_2$  $L_4$ 

Determine the chromatic number, the chromatic index, and the chromatic polynomial of $L_n$. Justify your answers.

***(8 marks)***

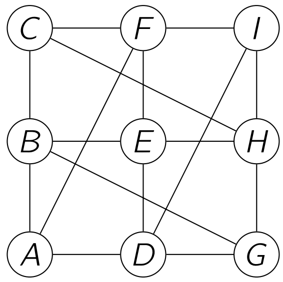
**4** (i) Describe the Planarity Algorithm for determining whether or not Hamiltonian graphs are planar, and explain why it works. Using the Planarity Algorithm, prove that $K_{3,3}$ and $K_5$ aren't planar.

***(5 marks)***

(ii) State Kuratowski's theorem and prove the "easy" direction, that gives a way to prove graphs aren't planar. You may use Part (i) even if you didn't complete it.

***(4 marks)***

Parts (iii) and (iv) of this question use the graph $\Gamma$ shown below.




(iii) Prove $\Gamma$ is not planar. Draw $\Gamma$ on the torus, with the vertices labeled.

***(5 marks)***

(iv) Prove that we can delete any edge $e$ of $\Gamma$ and the resulting graph $\Gamma \setminus e$ is still not planar. If $e$ is the edge $EH$, draw $\Gamma \setminus e$ on the Möbius band, with the vertices labeled.

***(5 marks)***

(v) State Euler's theorem for graphs drawn on the sphere. Use it to prove the following theorem: If a regular graph $G$ of degree 4 is drawn on the sphere so that every face is a triangle, quadrilateral, or pentagon (i.e., every face has 3,4 or 5 edges), then there are exactly 8 more triangles than pentagons.

***(6 marks)***

## End of Question Paper