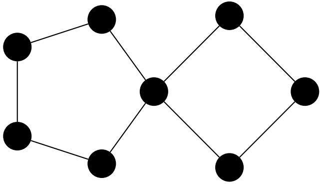
# Remaining bits

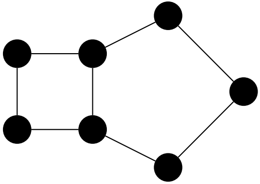
## Most important: Chromatic Polynomial is a polynomial

Nice induction proof, using a lemma called *Deletion Contraction*.

## Other topics:

- "Gluing formulas"
- Induction proof using chromatic polynomial: e.g. $C_n$





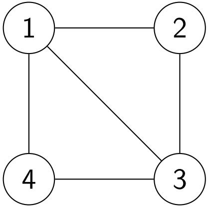
# Review: Chromatic polynomial of $C_4$ and a Lemma

**Lemma**

*Let $x$ and $y$ be two non-adjacent vertices in $G$. Then*

$$P_G(k) = P_{G_{+xy}}(k) + P_{G_{x=y}}(k)$$

1 and 3 have different colours

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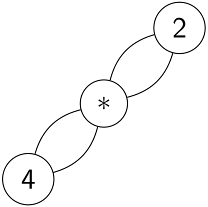

# Deletion-Contraction: restructuring the lemma

The lemma as stated is useful for calculating, but awkward for induction:

- $G_{+xy}$ has an extra edge; "bigger"
- $G_{x=y}$ has fewer vertices: "smaller"

Rewrite so $H \cong G_{+xy}$ is the "starting" graph.

**Lemma (Deletion-Contraction)**

*Let $H$ be a graph, and let $e$ be an edge in $H$. Then*

$$P_H(k) = P_{H\setminus e}(k) - P_{H/e}(k)$$

The edge $e$ is between $xy$

$$H = G + xy, \qquad H \setminus e = G, \qquad H/e = G_{x=y}$$

# Chromatic polynomial is a polynomial

**Theorem**

*Let $G$ be a simple graph. Then $P_G(k)$ is a polynomial in $k$. Moreover, if $G$ has $n$ vertices and $m$ edges, then*

$$P_G(k) = k^n - mk^{n-1} + \textit{lower order terms}$$

**Proof idea:**

Induct on the number of edges using deletion-contraction.

**Base case: $m = 0$**

If $G$ has no edges and $n$ vertices, then $G = E_n$ empty graph.
$P_{E_n} = k^n$ is a polynomial of the right form.

# Inductive step

Assume that $G$ has $m > 0$ edges and $n$ vertices, and that for any graph $H$ with $\ell < m$ edges and $p$ vertices, we have $P_H(k) = k^p - \ell k^{p-1} + \cdots$.

Let $e \in G$ be any edge:

- $G \setminus e$ has $n$ vertices and $m - 1$ edges
- $G/e$ has $n - 1$ vertices and *at most* $m - 1$ edges

So by the inductive hypothesis, theorem holds for $G \setminus e$ and $G/e$

So applying Deletion-Contraction:

$$\begin{aligned} P_G(k) &= P_{G\setminus e}(k) - P_{G/e}(k) \\ &= \left(k^n - (m-1)k^{n-1} + \cdots\right) - \left(k^{n-1} - \cdots\right) \\ &= k^n - mk^{n-1} + \cdots \end{aligned}$$

Which is what we needed to show. $\square$

# Odds and Ends

## Deletion-Contraction as an algorithm

- Can always find $P_G(x)$ by iterating deletion-contraction
- In practise, often faster to use cases

## Information in $P_G(k)$

- Number of vertices is the degree
- Number of edges is negative the coefficient of next highest term
- $\chi(G)$ is the lowest $k$ with $P_G(k) \neq 0$.

# Gluing formulas: intro

**Lemma**

*If $G$ is the disjoint union of $G_1$ and $G_2$, then*

$$P_G(k) = P_{G_1}(k)P_{G_2}(k)$$

**Proof.**

Colouring $G$ is exactly the same as colouring $G_1$ and $G_2$ independently. □

**Gluing formulas: when $G$ isn't *quite* a disjoint union**

Idea: Colour $G_1$, then extend to a colouring of $G_2$.

# Gluing formulas: statements

**Lemma**

*If $G$ is made by gluing $G_1$ and $G_2$ along a vertex $v$, then:*

$$P_G(k) = \frac{1}{k} P_{G_1}(k) P_{G_2}(k)$$

**Proof.**

First colour $G_1$ in any of the $P_{G_1}(k)$ ways. Now, vertex $v$ of $G_2$ is already coloured, but none of the rest. Since the colours are interchangable, exactly $1/k$ of the ways of colouring $G_2$ will have the right colour at $v$. $\square$

**Lemma**

*If $G$ is made by gluing $G_1$ and $G_2$ along an edge $e$, then*

$$P_G(k) = \frac{1}{k(k-1)} P_{G_1}(k) P_{G_2}(k)$$

# Calculating the chromatic polynomial of $C_n$

Let $e$ be any edge of $C_n$. Then:

- $C_n/e \cong C_{n-1}$
- $C_n \setminus e = P_n$, a tree, so $P_{P_n}(k) = k(k-1)^{n-1}$

So we should be able to find $P_{C_n}(k)$ using induction, but need to "guess" the formula first.

$$
\begin{aligned}
P_4(k) =& k^4 - 4k^3 + 6k^2 - 3k \\
P_5(k) =& k^5 - 5k^4 + 10k^3 - 10k^2 + 4k \\
P_6(k) =& k^6 - 6k^5 + 15k^4 - 20k^3 + 15k^3 - 5k \\
P_7(k) =& k^7 - 7k^6 + 21k^5 - 35k^4 + 35k^3 - 21k^2 + 6k
\end{aligned}
$$

Looks like:

$$P_n(k) = (k-1)^n + (-1)^n(k-1)$$

# Inductive proof that $P_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$

**Base case: $n = 3$**

Plug in $n = 3$, get $k(k-1)(k-2) = P_{C_3}(k)$.

**Inductive step:**

Get to assume: $P_{C_{n-1}}(k) = (k-1)^{n-1} + (-1)^{n-1}(k-1)$

- $C_n \setminus e = P_n$, a tree, so $P_{C_{n-1} \setminus e} = k(k-1)^{n-1}$
- $C_n / e = C_{n-1}$, so $P_{C_n/e}(k) = (k-1)^{n-1} + (-1)^n(k-1)$.

**Plugging into deletion-contraction:**

$$\begin{aligned}
P_{C_n}(k) &= P_{C_n \setminus e}(k) - P_{C_n/e}(k) \\
&= k(k-1)^{n-1} - \left[(k-1)^{n-1} + (-1)^{n-1}(k-1)\right] \\
&= k(k-1)^{n-1} - (k-1)^{n-1} - (-1)^{n-1}(k-1) \\
&= (k-1)^{n-1}[k-1] + (-1)^n(k-1) \\
&= (k-1)^n + (-1)^n(k-1)
\end{aligned}$$