Chromatic Number

Theorem

For G a planar graph $\chi(G) \leq 6$.

Step 1: Using Euler's Theorem

Definition

$\delta(G)$ denotes the *minimum* degree of all vertices in G.

Lemma

For G a simple planar graph $\delta(G) \leq 5$.

Step 2: Induction

We proved $\chi(G) \leq \Delta(G) + 1$ by colouring the vertices of G in any order. The Lemma bounds $\delta(G)$ and not $\Delta(G)$; need to be a little smarter.

Proof that simple planar graphs have $\delta(G) \leq 5$

Assume not, then every vertex has $d(v) \geq 6$.

The three ingredients:

- Euler's Theorem $V - E + F = 2$
- Face-Edge handshaking
  Simple, so $d(f) \geq 3$ for all faces. So $2E \geq 3F$
- Vertex-Edge handshaking
  By assumption, $d(v) \geq 6$, so $2E \geq 6V$

Proof of the Six Colour Theorem

Assume G is planar. We can assume that G is simple. Why?

Induct on n the number of vertices

Base case: $n \leq 6$

At most six vertices, so can give each vertex a different colour.

Inductive Step

Assume that G has n vertices, and every planar simple graph with less than n vertices can be coloured with six colours.

- By the Lemma, G has a vertex v with $d(v) \leq 5$
- The graph $G \setminus v$ has $n - 1$ vertices, so can be six coloured
- Now colour v

Chromatic index

Suppose six teams $A - F$ are in a soccer league, and each team will play three games:

$$
\begin{array}{cccccc}
A & & & & & \\
X & B & & & & \\
X & X & C & & & \\
X & & & D & & \\
 & X & & X & E & \\
 & & X & X & X & F
\end{array}
$$

If each team plays one game a week, can the tournament be run in three weeks? How about if we want AB and DE to play on different weeks?

Make it a graph

Chromatic index $\chi'(G)$

Definition

The *chromatic index* $\chi'(G)$ denotes the minimum number of colours needed to colour the *edges* of G so that any two edges that share a vertex have different colours.

In the application: the colours were the weeks?

Examples:

- $\chi'(K_4) = 3$
- $\chi'(K_5) = 5$

Lemma

$\chi'(G) \geq \Delta(G)$

Finding $\chi'(G)$

Theorem (Vizing)

For a simple graph $\chi'(G) = \Delta$ *or* $\Delta + 1$

We won't prove Vizing's Theorem, but will only implicitly use it.

One method to find $\chi'(G)$ with proof:

We know $\chi'(G) \geq \Delta(G)$. Try to colour it with $\Delta(G)$.

- If we can, we have shown $\chi'(G) = \Delta$
- If we can't, prove it: so we know $\chi'(G) \geq \Delta + 1$
- Find a colouring with $\Delta + 1$ colours

Another way to prove $\chi'(G) \geq \Delta + 1$

Example

K_n for n odd Suppose $n = 2k + 1$ is odd.

- Then $\Delta(K_n) = n - 1 = 2k$
- Note K_n has $k(2k + 1)$ edges. Why?
- We can have at most k edges of any given colour
- So using $2k$ colours, can only colour $2k^2 < k(2k + 1)$ edges