

## Last time: trees, ending with applications to chemistry

### Review: Alkanes are trees

- ▶ An Alkane is a molecule with formula  $C_nH_{2n+2}$
- ▶  $3n + 2$  vertices
- ▶ Handshaking says:  $(4n + 2n + 2)/2 = 3n + 1$  edges
- ▶ Connected, so a tree.

### Question: How many isomers does $C_5H_{12}$ have?

Case-by-case reasoning. Two possible ways to organize:

- ▶ Length of longest path in carbons
- ▶ Highest degree just looking at carbons

# Bridges of Konigsberg: birth of graph theory

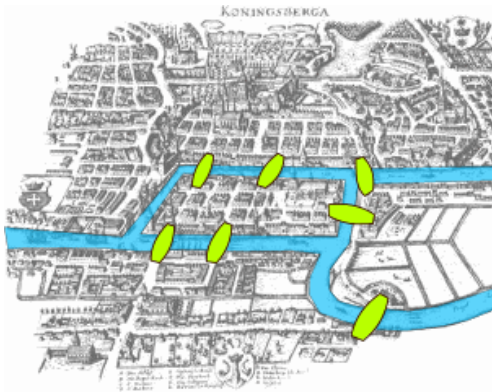
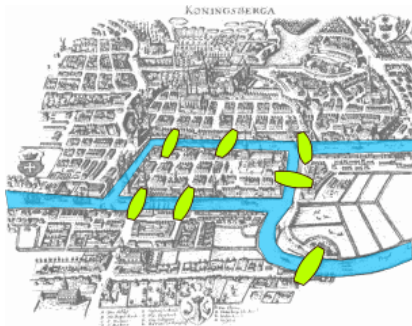


Figure: The city of Königsberg

A puzzle:

Cross every bridge exactly once and return to where you started.

# The puzzle has no solution



Suppose there was a walk:

- ▶ Stand on far bank
- ▶ Watch friend do walk
- ▶ Comes by one bridge
- ▶ Leaves by another bridge
- ▶ When they cross third bridge they're stuck with you

# Generalizing with graph theory

## Definition

- ▶ A walk is *closed* if it starts and ends at the same point.
- ▶ A graph is *Eulerian* if it has a closed walk that uses every edge exactly once

## Lemma:

If  $G$  is Eulerian, then every vertex has even degree.

## Proof.

Every time the walk visits  $v$ , pair the edge it arrives by with the edge it leaves by. □

# The first theorem in graph theory

## Theorem (Euler)

*A connected graph  $G$  is Eulerian if and only if every vertex of  $G$  has even degree.*

## Proof.

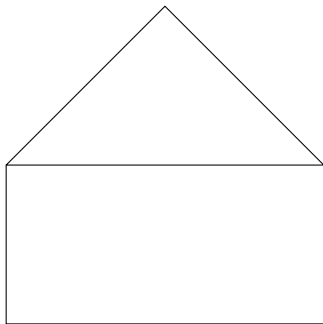
We proved Eulerian  $\implies$  even degree in last lemma.

For other direction:

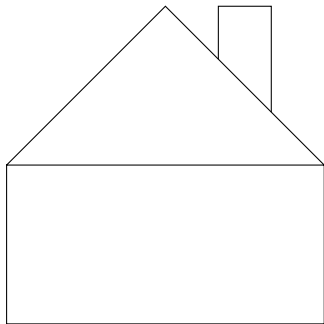
- ▶ Walking randomly will eventually get back to where we started (why?)
- ▶ Remove edges we used to get a smaller graph
- ▶ By induction, each connected piece is Eulerian
- ▶ Glue the cycles back together



A variation: what can you draw without lifting your pen?



A house?



A house with a chimney?

# Formalizing our observations

## Definition

A graph  $G$  is *semi-Eulerian* if it has a (not necessarily closed) walk that uses every edge exactly once.

## Theorem

*A connected graph  $G$  is semi-Eulerian if and only if it has at most two vertices of odd degree*

## One proof: tweak the original proof

Easy direction: every point but start and end needs even degree.

Hard direction:

- ▶ Start walk at one odd degree point
- ▶ Walking randomly can only end at other odd degree point
- ▶ Delete this path, then use induction + gluing as before

# A devious trick to avoid doing work

## Mathematicians are lazy

- ▶ It's unsatisfying to “redo” the work of the proof
- ▶ Slicker to reduce it to a problem we've already solved

## The tricky/easy proof:

- ▶ Let  $u, v \in G$  be the two vertices of odd degree
- ▶ Add an edge  $e$  from  $u$  to  $v$  to get  $G'$   
(this may make  $G$  non-simple, that's okay)
- ▶ In  $G'$ , every vertex has even degree, so it has Eulerian cycle
- ▶ Delete  $e$  from the eulerian cycle to get an Eulerian walk from  $u$  to  $v$



# A preview of next lecture

## Definition

A graph  $G$  is *Hamiltonian* if there is a closed walk that visits every vertex exactly once.  $G$  is *semi-Hamiltonian* if there is a not necessarily closed walk that visits every vertex exactly once.

- ▶ It was Easy to tell if a graph was Eulerian (Edges)
- ▶ It's Hard to tell if a general graph is Hamiltonian (Vertices)

## Question for this afternoon:

- ▶ Is  $D_{20}$  Hamiltonian?
- ▶ If we remove a vertex from  $D_{20}$  is it Hamiltonian?

