Section 4: Graphs on Surfaces

The Utilities Problem:

Connect three houses to three utilities without crossing the lines?

First, a definition

Definition

A graph G is *planar* if it can be drawn in the plane so that no edges cross.

This "definition" is vague, fixing it requires analysis/topology

- What do we mean "draw" an edge?
  - Injective continuous map $f : [0, 1] \to \mathbb{R}^2$

Not a course in topology; our intuition will be enough

The Utilities question becomes:

Is the complete bipartite graph $K_{3,3}$ planar?

Isn't possible, but how to organize proof?

Too many cases about how the edges could go...

Would be very lengthy, easy to miss a case.

Instead, assume it was:

- Any cycle $C_n \subset G$ would be drawn as a circle
- Any edge $e \in G, e \notin C_n$ would be inside or outside the circle
- Certain pairs of edges can't be on the same side...

Mathematical culture:

That a circle has two sides is surprisingly difficult topology.

Theorem (Jordan curve theorem)

A simple closed curve in the plane has an interior and an exterior

- Simple: doesn't cross itself
- Closed: starts where it ends

Theorem: $K_{3,3}$ isn't planar

Let A, B, C be blue vertices, X, Y, Z be red.

Suppose $K_{3,3}$ were planar:

- Then the Hamiltonian cycle $AXBYCZA$ would be a circle
- The three edges AY, BZ, CX still need to be drawn

Case 1: AY inside the cycle

- But then we need to draw BZ outside
- Two ways to do this, but either way can't draw CX

Case 2: AY outside the cycle

- Two ways to do this, but either way BZ needs to be inside
- Now we can't draw CX $\square$

The cases look awfully similar/redundant...

From the plane to the sphere

In the plane $\mathbb{R}^2$:

The outside and inside of a circle are different:

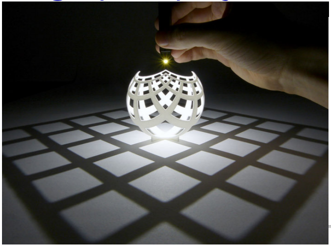
- The inside is bounded, the outside isn't
- One way to connect inside, two ways to connect outside

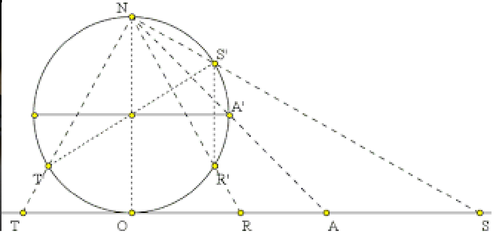
But on the sphere S^2 things are nicer:

- The ouside and inside of a circle look the same
- Only one way to connect outside

But we wanted to draw graphs on the plane, not the sphere

Stereographic projection: $S^2 = \mathbb{R}^2 \cup \{p\}$:





Corollary

G is planar if and only if G can be drawn on the sphere.

Proof.

If: Draw G on S^2; project from a point $p \in S^2 \setminus G$

Only if: Project from plane to sphere

$\square$

Upshot: don't need to treat inside/outside as separate cases

Wrapping up and moving to next time?

Other examples of this general method

Use or adapt this general method to prove that:

- K_5 isn't planar
- Petersen graph isn't planar
- Just draw a random graph and decide if it's planar

Does this work on any graph?

Next time:

- This method is *great* for Hamiltonian graphs
- Kuratowski's Theorem: when you're far from Hamiltonian