A forest is a bunch of trees

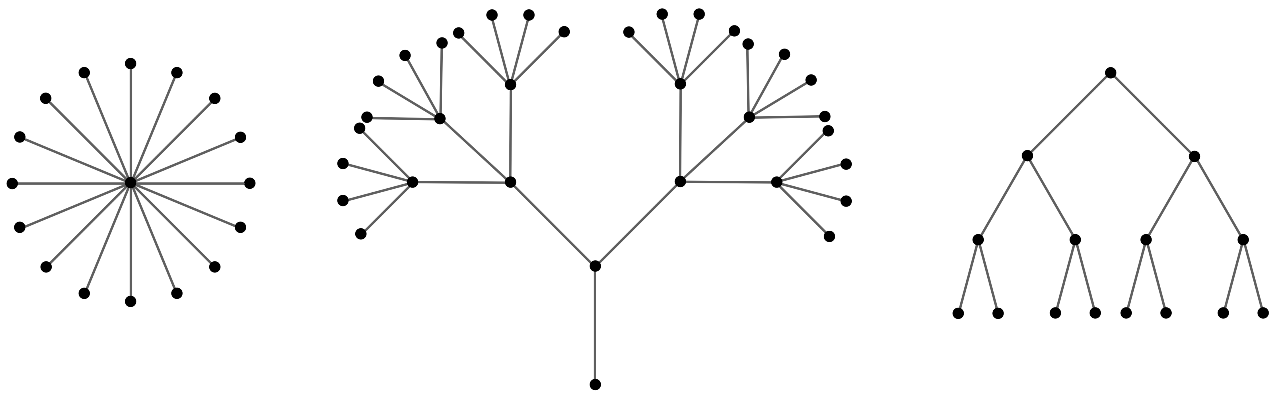


Figure: A forest of three trees

Definition

- A *forest* is a graph without cycles
- A *tree* is a connected graph without cycles

The Treachery of Definitions (After Magritte)

Figure: Ceci n'est pas un arbre (This is not a tree)

$\lfloor 13/2 \rfloor$ ways of looking a tree (After Wallace Stevens)

Proposition:

Let G be a graph with n vertices. The following are equivalent.

1. G is a tree
2. There is a unique path in G between any two vertices
3. G is connected and has $n-1$ edges
4. G has no cycles and has $n-1$ edges
5. G is connected, but removing any edge disconnects G
6. G has no cycles, but adding any edge creates a cycle

Informally: Trees are Goldilocks graphs

- Trees have enough edges: they're connected
- Trees don't have too many edges: they have no cycles

Make like a tree and get out of here (After Biff Tannen)

Definition (Tree)

Let T be a tree. A vertex $v \in T$ is a *leaf* if it has degree 1.

Lemma

Let T be a tree with $2 \leq n < \infty$ vertices. Then T has at least two leaves.

Proof 1: See title of slide.

Pick an edge, and try to "leave" – that is, walk as far as you can.

- No loops, so you'll never return to where you are
- Finitely many vertices, so it can't go on forever

Eventually you'll get stuck – that's a leaf.

Pruning Trees

Part of Proposition:

If T is a tree with n vertices, then T has $n-1$ edges.

Proof: Induct on n

- Base case: $n = 1$
- Now assume that all trees with $n-1$ vertices have $n-2$ edges
- If T is a tree with n vertices, it has a leaf v (by Lemma)
- Delete v and the edge next to it to get a new tree T'
- T' has $n-1$ vertices, so $n-2$ edges, so T has $n-1$ edges.

Another use of the handshaking lemma

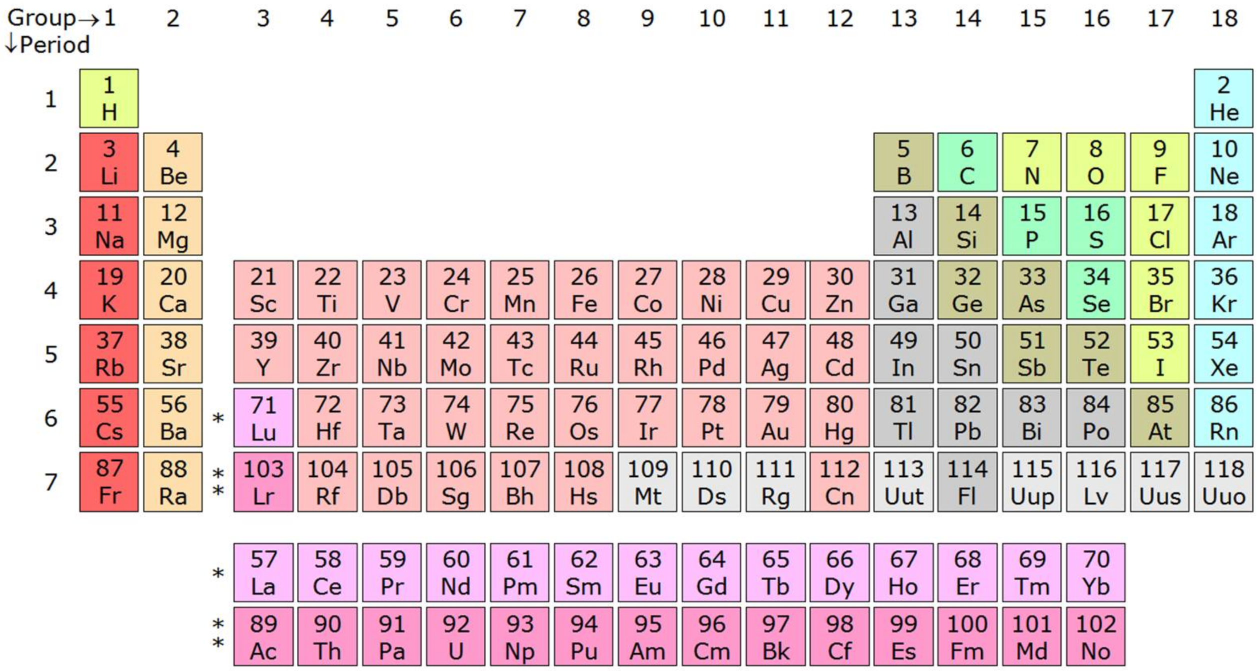
Part of Proposition:

If G is a connected graph with n vertices and $n-1$ edges, then G is a tree.

Proof: induct on n

- Base case: $n = 1$
- Assume proposition is true for all graphs with $n-1$ vertices
- Since G is connected, it has no vertices of degree 0
- Use handshaking to show G must have a vertex v of degree 1
- Delete v and the edge next to it to get a new graph G'
- G' is a tree, so G must have been as well

Chemical formulas encode degree sequences



Atom	C	N	O	H
Degree	4	3	2	1

Shortcuts around Carbon and Hydrogen

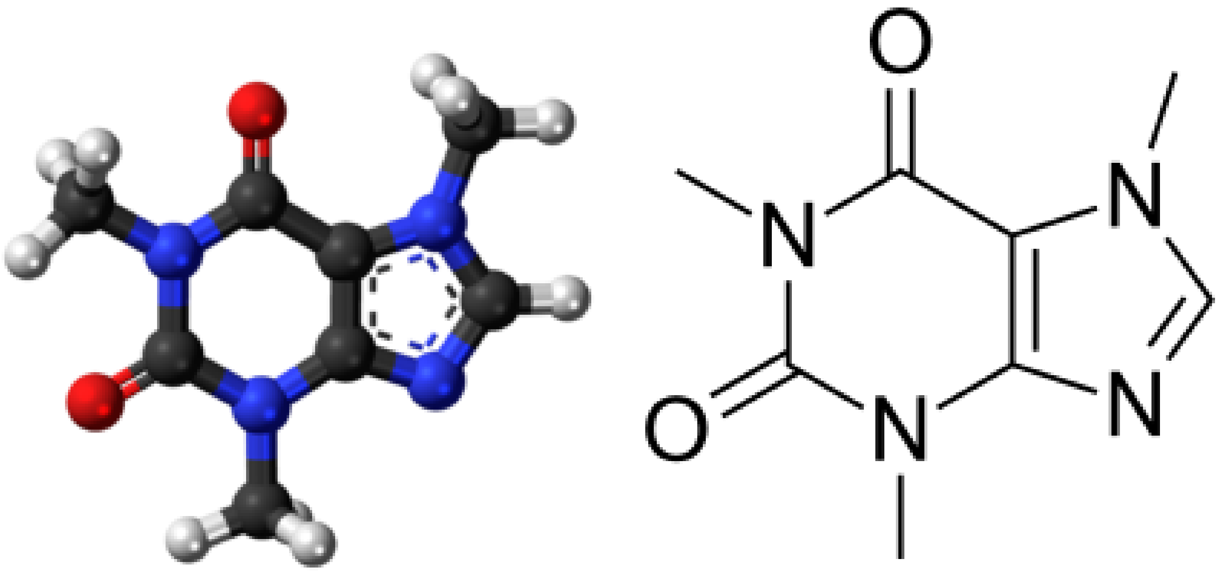


Figure: Two pictures of Caffeine

- Unlabelled vertices are Carbon
- Hydrogen not drawn; inferred to make degrees correct

Isomers are graphs with the same degree sequence

Definition

An *Alkane* is a molecule with formula $C_n H_{2n+2}$

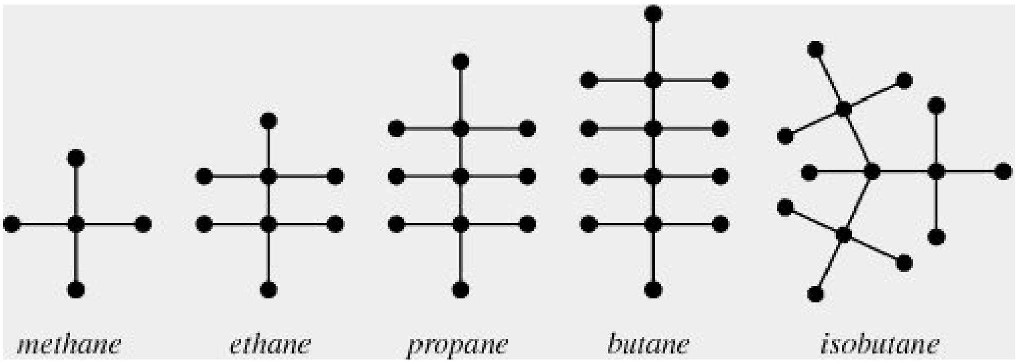


Figure: Alkanes. Butane and isobutane are isomers

Lemma: Any alkane is a tree.

Proof: Handshaking.

Question: How many isomers does $C_5 H_{12}$ have?