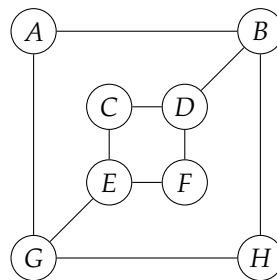


MAS341 Graph Theory 2018 Solutions

April 6, 2018

Question 1

Parts (i) and (ii) use the graph Γ show below.



Question 1, Part (i)

A graph G is semi-Eulerian if it has a (not necessarily closed) walk that uses every edge exactly once. A variation of Euler's theorem says that a graph is semi-Eulerian if and only if it has at most two vertices of odd degree, but Γ shown in the picture has 4, namely B, D, E and G . To make G semi-Eulerian, we must reduce the number of vertices with odd degree by 2. That is, we must either delete an edge connecting two vertices of odd degree (BD and EG) or add an edge connecting two vertices of odd degree that aren't already connected (namely, BE, BG, DE or DG).

Question 1, Part (ii)

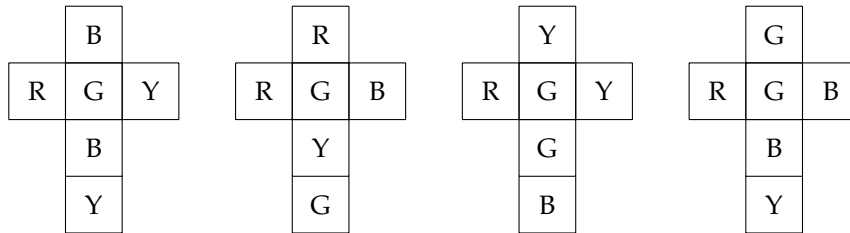
A graph is Hamiltonian if it has a closed walk that uses every vertex exactly once. To see that Γ is not Hamiltonian, note that since A has degree two, the path near A must be BAG or its reverse, and similarly since H has degree 2, the path near H must be BHG or its reverse, but now we have a closed four cycle $BAGH$ or its reverse, and so Γ isn't Hamiltonian.

A similar argument would hold using the inside four cycle $CDFE$, with vertices C and F having degree two. To make Γ Hamiltonian, we must break up both of these arguments, and hence connect either A or H to C or F , giving four possible edges to add. For instance, if we connect A to C , then we have $ACDFEGHBA$ as a Hamiltonian path.

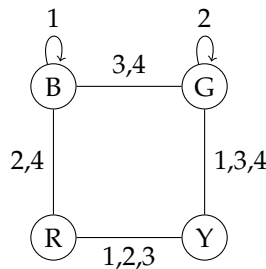
Removing an edge from a graph just makes it harder to be Hamiltonian, and so doesn't work.

Question 1, Part (iii)

We use the following set of instant insanity cubes:



We construct an edge-labeled graph on the vertices B, G, R, Y where there's an edge labeled i between two colors if they occur on opposite sides of cube i . For these cubes, this graph is the following:



Any solution gives rise to two disjoint subgraphs, each of which uses each edge label exactly once and has degree two at every vertex. We quickly see that neither the loop at B or G may be used in such a pair of subgraphs; the graph has no triangles so once one is used the other needs to be used, but then we need two edges between R and Y labeled 3 and 4, which don't exist. So any solution must use edges marked 1 and 2 between R and Y , the edge marked 2 between B and R , and the edge marked 1 between G and Y .

We have two cases: either the two edges used between R and Y are in the same subgraph, or there is one in each subgraph. If they're in the same subgraph, call it the first subgraph, then we must use the edges 3 and 4 between B

and G as the other two edges of the first subgraph. Then, there are no edges between B and G left to use in the second subgraph, and so the second subgraph cannot be a four cycle, and we quickly see the second one must use edges 2 and 4 between B and R , and edges 1 and 3 between G and Y . This gives rise to the following as one example solution:

	1	2	3	4
Front	G	B	Y	R
Back	Y	R	G	B
Top	R	Y	B	G
Bottom	Y	R	G	B

We may switch B and R on the front and back rows by only rotating cubes 2 and 4, and still have a solution, and similarly we may switch B and G between top and bottom by only rotating 3 and 4, and so once we fix the position of first cube. Thus, this pair of subgraphs actually corresponds to 4 separate solutions.

For the other case, suppose edge 1 between R and Y is in the first subgraph, and edge 2 between R and Y is in the second subgraph. We quickly see both of these subgraphs must be four cycles – if either subgraph also contained the edge 3 between R and Y , then we couldn't use the edge 3 between B and G in the same subgraph. If we follow the cycle in the order $BGYR$, the options for the edges used in the first subgraph are 3412 and 4312, and the second one must be 3124. This shares an edge with the solution 3412 for the first subgraph, but not with the second solution, so we have one more set of subgraphs. Choosing the same position for the first, this gives the following solution:

	1	2	3	4
Front	G	Y	B	R
Back	Y	R	G	B
Top	R	B	G	Y
Bottom	Y	R	B	G

We can't get any more solutions from this one without rotating cube 1 – i.e., since Y is on the back of cube 1, it must be on the front of cube 2. Then since R is on the back of cube 2, it must be on the front of cube 4. Then, since B is on the back of cube 4, it must on the front of cube 3. A similar argument fixes the top/bottoms of the cubes.

Hence, there are 5 solutions to the cubes with a given position of the first cube.

Question 1, Part (iv)

A graph with n vertices is a tree if and only if it is connected and has $n - 1$ edges. Molecules are automatically connected, and C_2NOH_7 has 11 vertices,

so we need to show any isotope has 10 edges. By the handshaking lemma, we have $2e = 2 \cdot 4 + 3 + 2 + 7 = 20$, giving the desired result.

To understand what the isotopes are, note that the hydrogen atoms will be the leaves, and it is enough to consider the layout of the other 4 vertices. If they're in a straight line, there are $4! = 24$ ways to arrange four things, but only half as many because two of the things are C, and half as many again because we can reverse the order of the chain, leaving six possibilities: CCNO, CNCO, COCN, CNOC, NCCO, CNOC.

The other tree on four vertices has one central element, and three elements coming off of it; the central element needs to have valence at least 2, and so can't be O, but could be C or N, giving us 2 more possibilities, and 8 possibilities in total.

Question 2

The next parts use the following weighted complete graph:

A						
20	B					
4	11	C				
6	14	5	D			
19	10	13	15	E		
10	17	11	19	9	F	
15	8	14	7	12	13	G

Question 2, Part (i)

Using Kruskal's algorithm to find a cheapest spanning trees, the edges added in order are AC, CD, DG, BG, FE, AF (or EB).

Question 2, Part (ii)

Using Prim's algorithm starting from vertex B, we have BG, GD, DC, CA, (AF or EB), EF.

Question 2, Part (iii)

We run Dijkstra's algorithm starting at A. Initially, the cheapest distances are AB: 20, AC: 4, AD: 6, AE: 19 AF: 10, AG: 15, so we make the direct route from C to 4 permanent, and look for new cheapest routes through C. Our updated list is:

$$ACB : 15 \quad AD : 6 \quad ACE : 17 \quad AF : 10 \quad AG : 15$$

We make AD permanent, and look for new cheapest routes through D. Our updated list:

$$AB : 15 \quad ACE : 17 \quad AF : 10, \quad ADG : 13$$

Making AF permanent and looking for new cheapest routes through F gives:

$$AB : 15 \quad ACE : 17 \quad ADG : 13$$

Now our most expensive path is within 4 of the new cheapest route, and the only edge with cost 4 or less isn't in the way, so we're not going to find any cheaper routes, and the city E has the most expensive shortest path, ACE costing 17.

Question 2, Part (iv)

The traveling salesperson problem (TSP) is to find the cheapest Hamiltonian cycle in a weighted graph; statements in terms of salespeople and costs are acceptable.

The TSP in general is quite difficult. In lecture we gave a method to get lower bounds by observing that deleting any edge from the Hamiltonian cycle would give a spanning tree, and more powerfully deleting a vertex v gives a spanning tree of $\Gamma \setminus v$.

We saw in the previous parts that Γ had two cheapest spanning trees, and that both of them were paths: ACDGBEF, or EFACDGB. Even more promisingly, their union is a Hamiltonian cycle: ACDGBEFA. We propose this is a solution to the TSP. As it's a Hamiltonian cycle it's definitely an upper bound; we have to show that it's also a lower bound. S

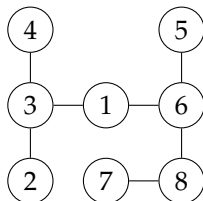
Suppose we took an arbitrary Hamiltonian cycle, call it S , which will have two edges, and deleted the more expensive of the two edge e out of vertex E in this cycle. The result would be a path, call it P . The path P is a spanning tree of Γ , and would have to be at least as expensive as the cheapest spanning tree, namely EFACDGB; that is, $w(P) \geq w(EFACDGB)$. The edge e we deleted would have to be at least as expensive as the second most expensive of ALL edges out of E, namely EB; that is $w(e) \geq w(EB)$.

Putting this together,

$$w(S) = w(P) + w(e) \geq w(EFACDGB) + w(EB) = w(ACDGBEFA)$$

and our solution is as cheap as any other cycle, and so is a solution to the TSP.

Question 2, Part v



Question 2, Part (vi)

For the graph to be a path graph, we need every vertex to have degree 1 or 2. When we write the $n - 1$ edges down in the proper order, vertex v will appear $d(v)$ times; and to create the Prüfer code we remove cells that count each number exactly once; hence, in the Prüfer code each number appears $d(v) - 1$ times. Thus, to be a path graph, we need that each number appears one or zero times in the Prüfer code; equivalently, that no number is repeated.

Question 3

Question 3, Part (i)

The chromatic number $\chi(G)$ is the minimum number of colours needed to colour the vertices of G so that no two adjacent vertices have the same colour, while the chromatic polynomial $\chi_G(k)$ counts the number of ways to colour the vertices of G with k colours, so that no two adjacent vertices have the same colour. From this, we see that $\chi(G)$ is the minimum positive number m with $\chi_G(m) \neq 0$.

The chromatic index $\chi'(G)$ is the number of colours needed to colour the edges of G so that no two edges that share a vertex have the same colour.

One way to find graphs G, H with $\chi_G(k) \neq \chi_H(k)$ but $\chi'(G) = \chi'(H)$ is to note that all trees T with n vertices have $\chi_T(k) = k(k-1)^{n-1}$, simply by starting at any vertex and then colouring adjacent vertices. On the other hand, colouring edge by edge one can see that for a tree T , $\chi'(T)$ will be the maximal degree of any vertex, and so the two nonisomorphic trees on four vertices will work, as one has a vertex of degree three, but the other doesn't.

Question 3, part (ii)

To see $\chi_G(k+1) \geq \chi_G(k)$, we work from the definition: any colouring of G using k colours is also a colouring with $k+1$ colours where one of the colours isn't used, so we immediately have the result.

As long as there are at least some colourings with $k + 1$ colours, we can forget different colours, and get a strict inequality $\chi_G(k + 1) > \chi_G(k)$. So to have $\chi_G(k + 1) = \chi_G(k)$, we need $\chi_G(k + 1) = 0$. As a concrete example, let $G = K_n$ and take $k + 1 < n$, then $\chi_G(k + 1) = \chi_G(k) = 0$.

Question 3, Part (iii)

Prove that if every vertex of G has degree at most d , then $\chi(G) \leq d + 1$.

We colour the vertices one by one in any order we choose. When we go to colour vertex v , we need to make sure it's not the same colour as any vertex w that's already been coloured and that is adjacent to v . But since v has degree at most d , there are at most d such vertices, and since we have $d + 1$ colours we'll have at least one colour available to colour v .

Question 3, Part (iv)

A colouring of H gives a colouring of G and a colouring of Γ by restriction, and the two vertices v and w that appear in $G \cap \Gamma$ must have the same colour in both G and Γ . On the other hand, given any colouring Γ and any colouring of G where these v is coloured the same in both colourings, and w has the same colour in both colourings, we will be able to take the union of these colourings and get a colouring of $G \cup \Gamma = H$.

Suppose we are given a k -colouring of G – then in particular, v is one of k colours, and w is one of $k - 1$ colours that v isn't, so there are $k(k - 1)$ colourings of v and w . Swapping colourings around, we that there are $\chi_\Gamma(k) / [k(k - 1)]$ colourings of Γ that have the given colours for v and w , and hence we have the result.

Question 3, Part (v)

All of the L_n have at least two vertices and are bipartite, so $\chi(L_n) = 2$.

Not that L_2 has exactly one edge, and so $\chi'(L_2) = 1$, and $L_2 = C_4$, and we can colour the edges alternating red-blue-red-blue, so $\chi'(L_2) = 2$.

Now, suppose $n \geq 2$, and consider L_n . Since L_n has vertices of degree 3, we have $\chi'(L_n) \geq 3$. We now show that the edges can be three coloured – there are n vertical edges that don't share any vertices, and they can be all be coloured green. The result is the disjoint union of two paths of length $n + 1$, and we can two colour the edges of these alternating red blue red blue, to get a three colouring of the entire graph.

To calculate $\chi_G(k)$, we use induction. First, as a base case, note that $\chi_{L_2}(k) = k(k - 1)$ since L_0 is just an edge, and the first vertex can be coloured in k ways and the second can be coloured any colour by the first, and hence in $k - 1$ ways.

Now, we can relate L_n to L_{n-1} by deletion-contraction applied to the edge e on the far right of L_n . If we delete the edge, we get L_{n-1} with two extra vertices hung on, each adjacent to one extra vertex. Hence a colouring of $L_n \setminus e$ consists of a colouring of L_{n-1} together with a colouring of these two vertices, each of which has $k - 1$ choices, and so $\chi_{L_n \setminus e}(k) = (k - 1)^2 \chi_{L_{n-1}}(k)$.

If we contract the edge e , we get L_{n-1} with one extra vertex glued on, adjacent to both the last two edges, which were themselves adjacent. Hence, a colouring of L_n / e is a colouring of L_{n-1} with one more vertex that has $(k - 2)$ choices of colouring it, and so $\chi_{L_n / e}(k) = (k - 2) \chi_{L_{n-1}}(k)$.

Using deletion-contraction $\chi_G(k) = \chi_{G \setminus e}(k) - \chi_{G / e}(k)$, we see

$$\begin{aligned} \chi_{L_n}(k) &= \chi_{L_n \setminus e}(k) - \chi_{L_n / e}(k) \\ &= (k - 1)^2 \chi_{L_{n-1}}(k) - (k - 2) \chi_{L_{n-1}}(k) \\ &= (k^2 - 3k + 3) \chi_{L_{n-1}}(k) \end{aligned}$$

Hence, using induction, we have $\chi_{L_n}(k) = (k^2 - 3k + 3)^n \chi_{L_0}(k) = k(k - 1)(k^2 - 3k + 3)^n$.

Alternatively, you can use deletion contraction to prove $\chi_{C_4}(k) = k(k - 1)(k^2 - 3k + 3)$, and then use induction and Part (iv) to derive the answer; be sure to handle the case $\chi_{L_1}(k)$ as well.

Question 4

Suppose that a graph Γ is drawn on the sphere, and let C_n be a Hamiltonian cycle in Γ . Then C_n will be drawn on the sphere as a simple closed curve, and hence have an inside and an outside (by the Jordan Curve Theorem). Each remaining edge must either be drawn entirely on the inside or the outside of the circle. Make a graph $\text{Cross}(\Gamma)$ with vertices the edges in Γ but not C_n , and an edge between e and f if e and f would cross. Then, since Γ is planar, we see that $\text{Cross}(\Gamma)$ is bipartite. In the other direction, if $\text{Cross}(\Gamma)$ is bipartite, that tells us how to draw the edges of Γ so that none cross.

For $K_{3,3}$ we see that after a Hamiltonian cycle is chosen, there are three more edges, which all would cross each other, and so $\text{Cross}(\Gamma)$ is the triangle C_3 which isn't bipartite.

For K_5 , we see that after a Hamiltonian cycle is chosen there are 5 more edges, each of which would cross with two other edges, and in fact $\text{Cross}(K_5)$ is C_5 , which isn't bipartite.

Question 4, Part (ii)

Kuratowski's theorem states that a graph G is planar if and only if it has no subgraph that is a subdivision of $K_{3,3}$ or K_5 .

The easy direction is the “only if” – taking the contrapositive, we need to show that if G has a subgraph H that is a subdivision of a $K_{3,3}$ or a K_5 , then G isn’t planar. If G were planar, then any drawing of it on the plane would include a drawing of H on the plane, and so H would be planar. But H being a subdivision of $K_{3,3}$ or K_5 means that H is one of these graphs with some extra vertices of degree two added to the edges. Hence, drawing H on the plane would give a drawing of $K_{3,3}$ or K_5 on the plane, but we saw in Part (i) that these graphs aren’t planar.

Question 4, Part (iii)

We use Kuratowski’s theorem. Γ has exactly 5 vertices of degree 4, so if Γ had a subgraph that was a subdivision of K_5 they’d have to be the 5 vertices, but one can check that they can’t be completed to a K_5 . So we must have a subgraph that’s a subdivision of a $K_{3,3}$. Γ nearly has a $K_{3,3}$ as a subgraph: taking, say, ACE to be red vertices and BDF to be blue vertices, we see that ABEF have all the connections made already, and all we need to do is to connect C to D, which can be done either through DGHC or DIHC.

Another proof adapts the Planarity Algorithm; Γ isn’t Hamiltonian (it is bipartite with different number of vertices in the two colours), but the cycle $ABCFIHGD$ uses every vertex by E . If Γ were drawn on the plane, this would have to be a circle, and E would have to be inside or outside (say, inside, as in the drawing); then every other edge would have to be outside, which isn’t possible.

To draw Γ on the torus we can just draw the 3 by 3 grid and horizontal and vertical edges as are, and wrap the four diagonal edges around the sides of the torus – AF and DI over the top/bottom, and BG, CH over the left/right edges.

Question 4, Part (iv)

We need to show that we can delete any edge of Γ and still have find a subgraph that’s a subdivision of $K_{3,3}$. Note that Γ has an order 4 cyclic symmetry by rotating by 90 degrees. Γ has 16 edges, and this edges divides them into 4 equivalence classes (the orbits), each with 4 elements: the diagonal edges are one orbit, the edges incident to E another, and the edges along the outside of the square are split into two orbits depending on whether they go clockwise or anticlockwise away from a corner (i.e., AB, CF, IH, GD are in one orbit, and AD, CB, IF, and GH are in the other).

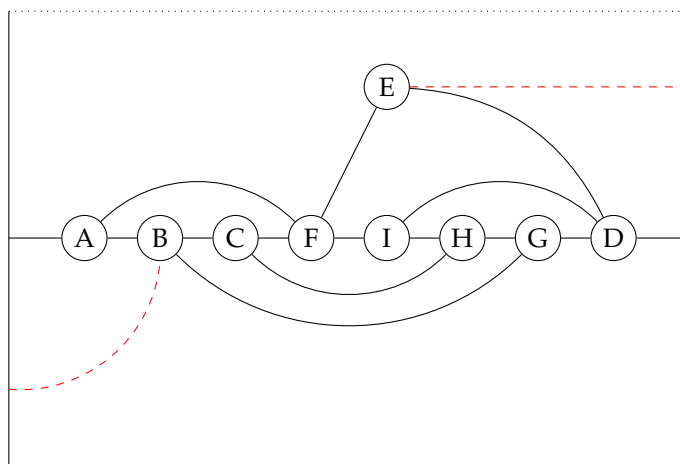
If we look at the subgraph we found in Part (iii) that connected D to C through DGHC, the edges *not* used in that subgraph are DI, IH, IF and EH, which conveniently is one edge from each equivalence class. Hence, whichever

edge we pick, some rotation of the graph from part iii will not contain that edge, and no matter what edge we delete, the graph will still not be planar.

There are many possible drawings of $\Gamma \setminus EH$ on the Mobius band. One way to find one is by trial and error; here's a more systematic way to get started.

The graph $\Gamma \setminus EH$ is not Hamiltonian (it is bipartite and has an odd number of vertices), but it is nearly so – take the cycle ABCFIHGDA around the outside. If that cycle is drawn bounding a disk on the Mobius band, either E would be outside the disk or inside the disk, and in either case arguments like the planarity algorithm would quickly show it isn't possible.

Hence, that cycle must not bound a disk, i.e., it must wrap around the Mobius band. Once we realize this, it's not too hard to add E and the remaining edges, for instance, as shown below where the dotted line is the one boundary circle of the Mobius band, the two solid edges of the Mobious band are glued together in the opposite directions, and the two edges AD and EB wrap around the glued edge.



Question 4, Part (v)

Euler's theorem states the following: suppose that a connected graph G is drawn on the sphere, and let v, e and f be the number of vertices, edges and faces, respectively. Then $v - e + f = 2$.

Let T, Q and P be the number of triangles, quadrilaterals, and pentagons, respectively; we want to show that $T - P = 8$.

The total number of faces is $T + Q + P$, and so Euler's theorem becomes $V - E + T + Q + P = 2$. On the other hand, handshaking between vertices and edges gives $4V = 2E$, and handshaking between faces and edges gives $2E = 3T + 4Q + 5P$. Taking the difference of these two equations and rearranging

gives $4V - 4E = -(3T + 4Q + 5P)$. Multiplying Euler's theorem by 4, and then substituting this in, gives

$$\begin{aligned} 8 &= (4V - 4E) + 4T + 4Q + 4P \\ &= -(33T + 4Q + 5P) + 4T + 4Q + 4P \\ &= T - P \end{aligned}$$

as desired.