What we did last time, what we still have to do

Last time we stated:

Theorem (Euler's formula for graphs on the sphere)

Let G be a connected graph drawn on the sphere without edges crossing. Let V and E be the number of edges and vertices of G, respectively, and let F be the number of faces of the drawing. Then

$$V - E + F = 2$$

And explained how together with vertex-edge and face-edge handshaking can be used to prove:

- Every football has 12 pentagons
- K_5 isn't planar

Today we'll prove Euler's formula, and illustrate more applications.

Proof(s) of Euler's Theorem

Basic proof idea: induction

What happens if we delete an edge?

- Number of edges goes down by 1
- Number of faces goes down by 1?

Hence, $V - E + F$ remains unchanged.

Why the question mark?

First(?) proof of Euler's Theorem

We induct on the number of faces.

Base case: G has only one face

- Then G has no cycles (Jordan curve theorem)
- Assumed G connected, so it's a tree
- Therefore $E = V - 1$

Inductive step:

Assume G has $F > 1$ and theorem is true for all graphs with fewer than F faces.

- Then G has an edge separating two faces (why?)
- Deleting such an e doesn't disconnected G (why?)
- $G \setminus \{e\}$ has one less face, so theorem holds there

Back to videogames

Recall that the standard overhead view of a planet in video game produces not the sphere but the torus.

Definition

A *video game graph* is a graph drawn on a surface so that

- Every vertex has degree 4
- Every face has degree 4

Theorem

A video-game graph can never be the sphere. In fact, a video-game graph will always be the torus or the Klein bottle.

So the video-game designers didn't "mess up".

Proof: collect the standard three ingredients

Ingredient 1: Euler's theorem

Suppose that G was a video game graph drawn on the sphere:

$$V - E + F = 2$$

Ingredient 2: Vertex-edge handshaking

Since every vertex has degree 4, we have

$$2E = 4V$$

Ingredient 3: Vertex-face handshaking

Since every face has degree 4, we have

$$2E = 4F$$

Mix well to finish proof...

Duality

We noticed:

- The cube has $(V, E, F) = (8, 12, 6)$
- The octahedron has $(V, E, F) = (6, 12, 8)$

Definition

Let G be a planar connected graph. The *dual graph* G^* of G has

- One vertex for each face of G, placed in the middle
- One edge for each edge of G, drawn perpendicular
- One face for each vertex of G

Explains the pattern we saw in V, E, F!

Face-edge handshaking for G is vertex-edge handshaking for G^*, and vice versa.

Mathematical culture: points for discussion

- Euler's Theorem for *other* surfaces
- "Dualizing" our proof of Euler's Theorem: edge contraction
- Interlacing tree proof of Euler's Theorem
- Euler's theorem and curvature