The method we used to show $K_{3,3}$ isn't planar generalises:

Take any cycle C in a graph G

- If the G is planar, C will be drawn as a "circle"
- Any other vertex or edge must lie inside or outside circle
- Handle possibilities case by case

Stereographic projection:

Don't have to treat inside/outside the circle as separate cases.

But for a complicated graph, could still be a lot of cases...

Best case: graph is Hamiltonian.

- Don't have to put vertices inside/outside circle

Another example: K_5 isn't planar

Use our method with Hamiltonian cycle $ABCDEA$:

- WLOG, edge AC drawn inside...
- Then B cut off from DE inside, so BD, BE outside
- BD cuts off C from E on outside, so CE inside

Only have AD left, but:

- CE cuts it off inside
- BE cuts it off outside

Therefore, K_5 isn't planar

The crossing graph packages the case by case analysis

"Edge e_1 is in, so edge e_2 out, so edge e_3 in, so ..." gets tiresome.

For this slide: G a graph with Hamiltonian cycle C

- Some pairs of edges in $G \setminus C$ cross if drawn inside C
- Some pairs of edges can be drawn on the same side

Definition

The *crossing graph* $\text{Cross}(G, C)$ has:

Vertices: the edges in $G \setminus C$

Edges e and f are adjacent if they cross inside C

Theorem (Planarity Algorithm for Hamiltonian graphs)

G is planar if and only if $Cross(G, C)$ is bipartite

Example of planarity algorithm:

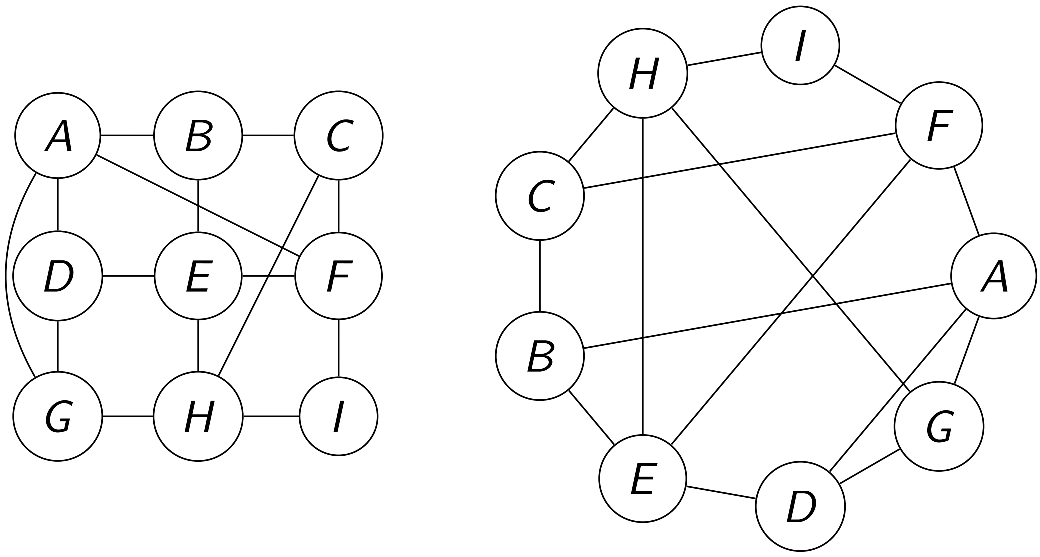


Figure: A graph Γ, then redrawn with Hamiltonian cycle outside

What is Cross(Γ, $AFIHCBEDGA$)?

Vertices = edges in middle

Edges = crossings in middle

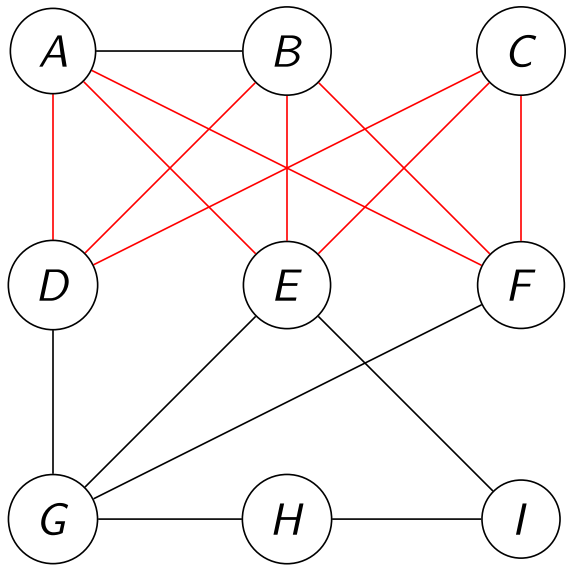
What if G isn't Hamiltonian? Two lemma suffice.

Lemma

If G is a subgraph of H, and G is nonplanar, then H is nonplanar.

Proof.

To draw H, we're drawing G and then adding some things. □



Another tool for showing G isn't planar

Definition

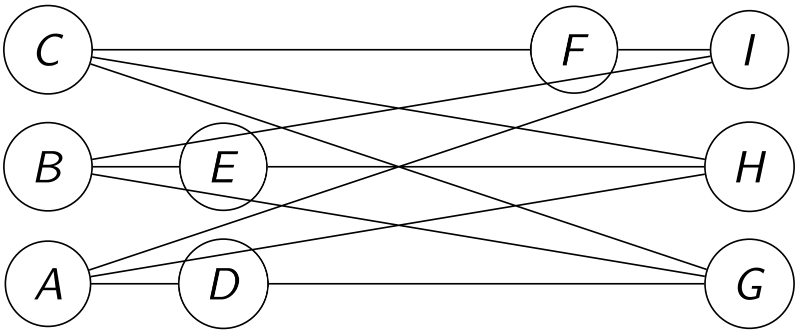
A graph H is a subdivision of G if it can be created from G by adding some vertices of degree two in the middle of edges.

Lemma

If H is a subdivision of G and, and G isn't planar, then H isn't planar.

Proof.

To draw H, we're drawing G and then adding some dots on the edges. □



Kuratowski's theorem – proves a general G not planar

Theorem

A graph G is not planar if and only if it has a subgraph that is a subdivision of $K_{3,3}$ or K_5

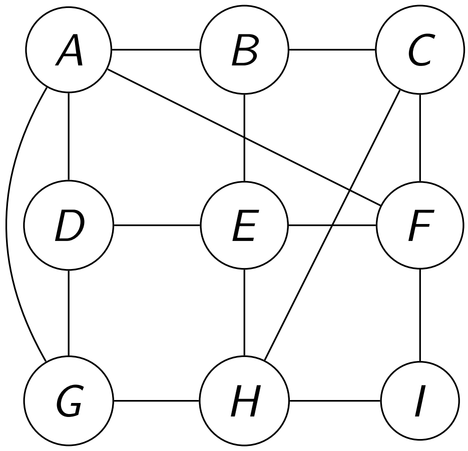
Proof of the "if" direction:

- $K_{3,3}$ and K_5 aren't planar
- So subdivisions of $K_{3,3}$ or K_5 aren't planar
- So graphs having subdivisions a $K_{3,3}$ or K_5

Remarks on the "only if" direction

- Harder to prove and we won't even sketch
- Won't *explicitly* use – if G is planar, prove it by drawing it!
- Will use *implicitly* – if G isn't planar, we know we can find a $K_{3,3}$ or K_5 hidden in it

Example of using Kuratowski's theorem



Give another proof that Γ isn't planar