Which graphs are/aren't isomorphic? Prove it.
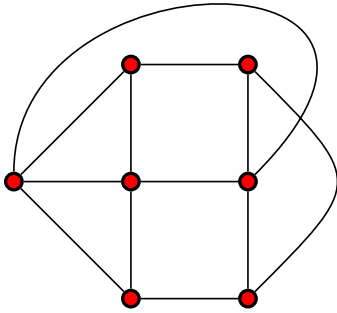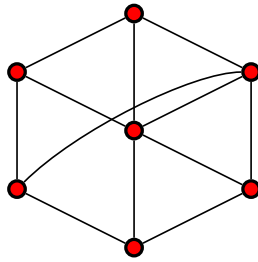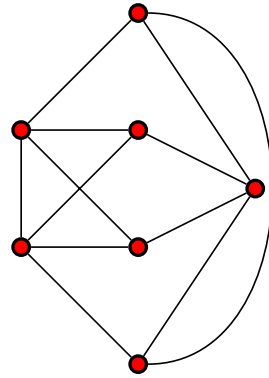
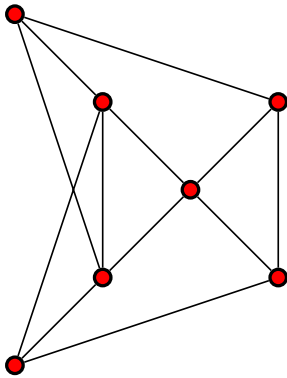# One solution to warm-up
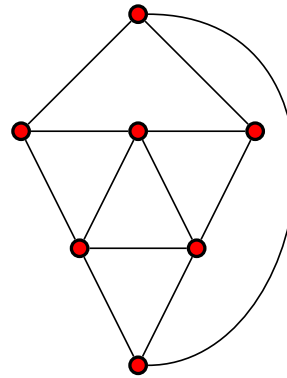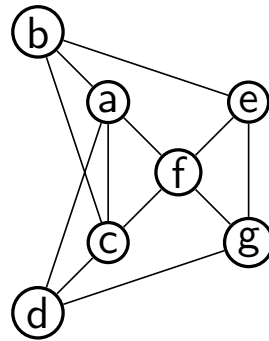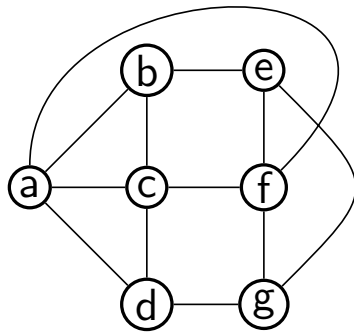
▶ Graph B has a vertex of degree 5; others have degree sequence $[4, 4, 4, 3, 3, 3, 3]$, so none are isomorphic to B.

▶ In A, D, E, the three vertices of degree 4 all touch, but not in C, so none are isomorphic to C.

▶ In A, D, every vertex is adjacent to a vertex of degree 4, but not in E, so none are isomorphic to E.

▶ But we see below A is isomorphic to D:

Basic graphs and concepts

- The *empty graph* E_n has n vertices and no edges
- The *complete graph* K_n has n vertices, and each vertex is connected to every other.
- The *path graph* P_n has n vertices $\{1, \ldots, n\}$ with an edge between i and $i+1$
- The *cycle graph* C_n has n vertices $\{1, \ldots, n\}$ with an edge between i and $i+1$ and between n and 1.

Definition

Let G be a simple graph with vertex set V. Its *complement* G^c is another graph with vertex set V, where two vertices $v, w \in V$ are adjacent in G^c if and only if they are not adjacent in G.

Obligatory Petersen graph
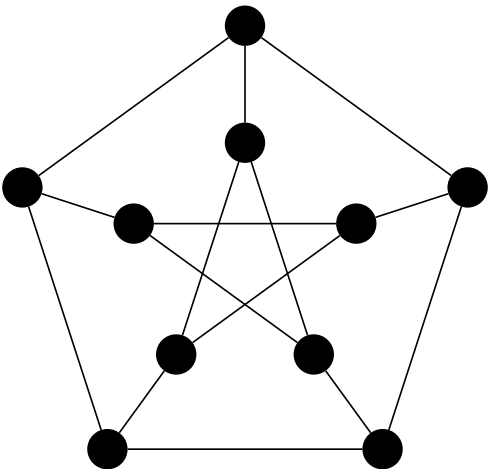
What does it mean for a graph to be connected?

Connected means we can "get from any vertex to another"

Definition (Walk)

Let G be a simple graph. A *walk* in G is a sequence of vertices $v_1, v_2, \ldots, v_n$ so that v_i is adjacent to v_{i+1}. We we say the walk goes from v_1 to v_n.

Definition (Connected)

A graph G is *connected* if there is a walk between any two vertices v and w in G.

Definitions I won't use without explaining

- A *trail* is a walk that doesn't repeat any edges
- A *path* is a walk that doesn't repeat any vertices

Bipartite graphs

Definition (Bipartite graphs)

A graph G is *bipartite* if we can colour every vertex either blue or red so that every edge goes between a blue vertex and a red vertex.

Definition (Complete bipartite graphs)

The *complete bipartite graph* $K_{m,n}$ consists of $m + n$ vertices, m coloured red, n coloured blue, and an edge between any red vertex and and any blue vertex.

Examples

Another way to characterise bipartite graphs

Lemma
A graph G is bipartite if and only if it doesn't have any cycles of odd length (i.e., subgraphs of the form C_{2k+1}).

Bipartite $\implies$ no odd cycles:
Subgraphs of bipartite graphs are bipartite

No odd cycles $\implies$ Bipartite:
Try to colour G by distance from v

Definition (Distance)
Let G be connected, and let v, w be two vertices. The *distance from v to w* is the least number of edges in any walk from v to w.