Last time: trees, ending with applications to chemistry

Review: Alkanes are trees

- An Alkane is a molecue with formula C_nH_{2n+2}
- $3n + 2$ vertices
- Handshaking says: $(4n + 2n + 2)/2 = 3n + 1$ edges
- Connected, so a tree.

Question: How many isomers does C_5H_{12} have?

Case-by-case reasoning. Two possible ways to organize:

- Length of longest path in carbons
- Highest degree just looking at carbons

Bridges of Konigsberg: birth of graph theory

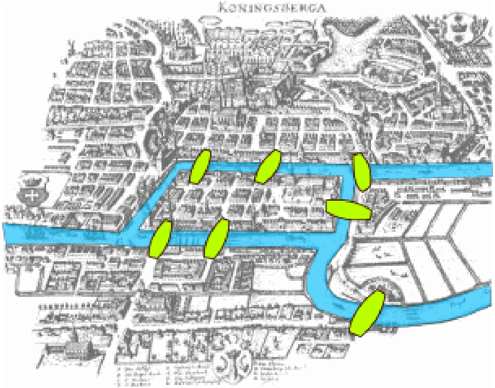
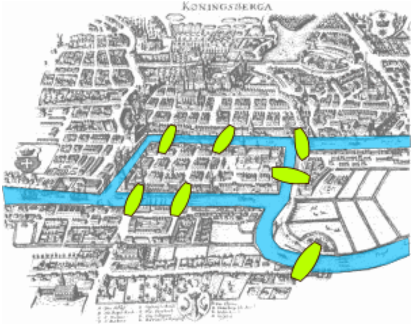


Figure: The city of Konigsberg

A puzzle:

Cross every bridge exactly once and return to where you started.

The puzzle has no solution



Suppose there was a walk:

- Stand on far bank
- Watch friend do walk
- Comes by one bridge
- Leaves by another bridge
- When they cross third bridge they're stuck with you

Generalizing with graph theory

Definition

- A walk is *closed* if it starts and ends at the same point.
- A graph is *Eulerian* if it has a closed walk that uses every edge exactly once

Lemma:

If G is Eulerian, then every vertex has even degree.

Proof.

Every time the walk visits v, pair the edge it arrives by with the edge it leaves by. $\square$

The first theorem in graph theory

Theorem (Euler)

A connected graph G is Eulerian if and only if every vertex of G has even degree.

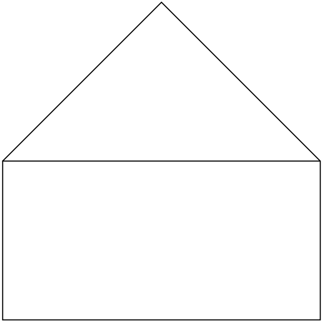
Proof.

We proved Eulerian $\implies$ even degree in last lemma.
For other direction:

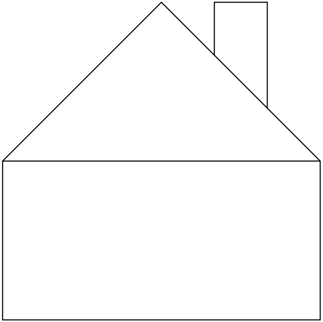
- Walking randomly will eventually get back to where we started (why?)
- Remove edges we used to get a smaller graph
- By induction, each connected piece is Eulerian
- Glue the cycles back together

$\square$

What can you draw without lifting your pen or retracing?



A house?



A house with a chimney?

Formalizing our observations

Definition

A graph G is *semi-Eulerian* if it has a (not necessarily closed) walk that uses every edge exactly once.

Theorem

A connected graph G is semi-Eulerian if and only if it has at most two vertices of odd degree

One proof: tweak the original proof

Easy direction: every point but start and end needs even degree.
Hard direction:

- Start walk at one odd degree point
- Walking randomly can only end at other odd degree point
- Delete this path, then use induction + gluing as before

A devious trick to avoid doing work

Mathematicians are lazy

- It's unsatisfying to "redo" the work of the proof
- Slicker to reduce it to a problem we've already solved

The tricky/easy proof:

- Let $u, v \in G$ be the two vertices of odd degree
- Add an edge e from u to v to get G'
  (this may make G non-simple, that's okay)
- In G', every vertex has even degree, so it has Eulerian cycle
- Delete e from the eulerian cycle to get an Eulerian walk from u to v

A preview of next lecture

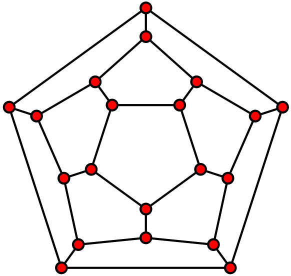
Definition

A graph G is *Hamiltonian* if there is a closed walk that visits every vertex exactly once. G is *semi-Hamiltonian* if there is a not necessarily closed walk that visits every vertex exactly once.

- It was Easy to tell if a graph was Eulerian (Edges)
- It's Hard to tell if a general graph is Hamiltonian (Vertices)

Question to lead into it:

- Is D_{20} Hamiltonian?
- Is the Petersen graph?
- If we remove a vertex from D_{20} is it Hamiltonian? How about Petersen?



Dodecahedron graph D_{20}