Warm-up: what's all this then?

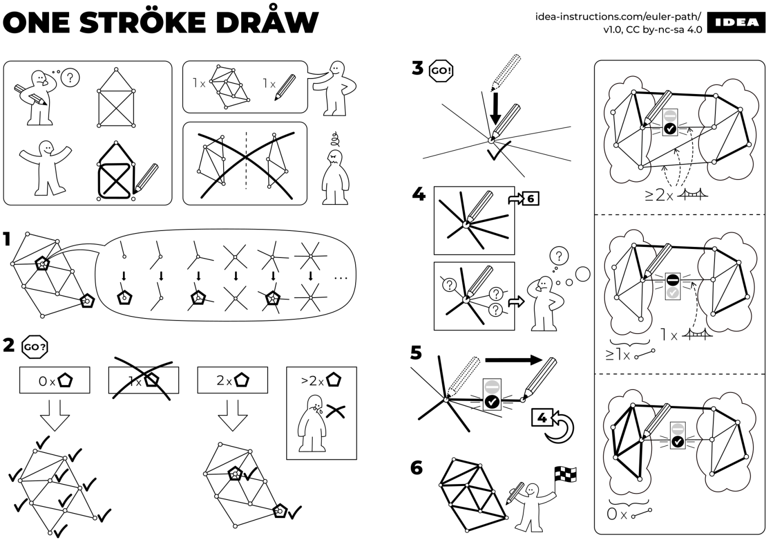


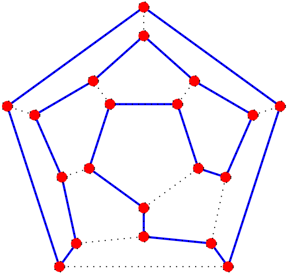
Figure: Fleury's algorithm for finding Hamiltonian paths/cycles

To prove a graph is Hamiltonian, find a cycle

Definition

A graph G is *Hamiltonian* if there is a closed walk that visits every vertex exactly once. G is *semi-Hamiltonian* if there is a not necessarily closed walk that visits every vertex exactly once.

Figure: The dedecahedron graph D_{20} is Hamiltonian



Proving a graph *isn't* Hamiltonian is hard

In *theory* it's easy:

Only finitely many possible paths; check them all.

But number of possible paths grows very quickly

We can't prove there's no easy way to check if a graph is Hamiltonian or not, but we've bet the world economy that there isn't.

Mathematical culture: NP-completeness

Determining whether or not a graph is Hamiltonian is "NP-complete" i.e., any problem in NP can be reduced to checking whether or not a certain graph is Hamiltonian.

If we found an easy algorithm, could break standard encryption.

Theorem

If we remove a vertex from D_{20}, it is not longer Hamiltonian.

Proof Sketch

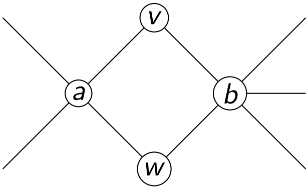
- Suppose D_{20} was Hamiltonian
- At every vertex, exactly two edges used in cycle
- Suppose certain edge in cycle, chase consequences:
  Other edges must be in/out of cycle
- Iterate

Useful observations: edges can't make a smaller cycle

This strategy will work in general, but may be very complicated.

Proving G isn't Hamiltonian: tricks

Can't contain certain configurations:



Lemma

If G is bipartite and Hamiltonian, then G has the same number of red and blue vertices.

Proof.

- The vertices in the Hamiltonian cycle alternate red and blue
- The Hamiltonian cycle contains all the vertices

□

Tool: assume G is Hamiltonian, consider “extra” edges

Theorem

The Petersen graph P isn't Hamiltonian

Proof.

Suppose P is Hamiltonian. The Hamiltonian cycle uses up two edges at each vertex, so we have one more edge meeting each vertex.

Analyze how to place these five edges.

- Can't go to next vertex in cycle: no multiple edges
- Can't “skip” one or two: P has no 3 or 4 cycles
- So extra edges “straight across” ± 1
- Rule out straight across
- Rule out all “skip 3”

$\square$

A sufficient condition to be Hamiltonian

If we have "enough" edges, should be Hamiltonian

If G is Hamiltonian and we add extra edges, the result is still Hamiltonian.

Theorem (Ore)

Let G be a simple graph with n vertices, so that for any two nonadjacent vertices v and w, we have $deg(v) + deg(w) \geq n$. Then G is Hamiltonian.

Not an "If and only If!" – won't prove G isn't Hamiltonian

Proof ingredients

- "Minimal Criminal": minimal/maximal counterexamples have extra structure
- "adding extra edges"
- Pigeonhole principal

Proof of Ore's Theorem

Minimal criminimal:

Assume G is a counterexample, but then adding any edge to G makes it Hamiltonian.
Then G must be semi-Hamiltonian.

Adding extra edges

Let $v_1 - v_2 - \cdots - v_n$ be the Hamiltonian path. v_1 and v_2 not adjacent so $deg(v_1) + deg(v_2) \geq n$.
So we need to add $n - 2$ more edges to v or w.

Pigeon hole principle

Let $i \in 3, \ldots, n-1$ with v_1 adjacent to v_i and v_n adjacent to v_{i-1}: then we have a Hamiltonian path!
Use pigeon-hole principle to find such an i.