Colouring Graphs

Started with 4 colour theorem:

Any map can be coloured with four colours so that adjacent countries have different colours.

- Posed by Guthrie 1852
- Proved by Appel and Haken 1976

Definition

The *chromatic number* $\chi(G)$ of a graph G is the least number of colours needed to colour the vertices so that adjacent vertices have different colours.

Theorem (The four colour theorem)

A planar graph G has $\chi(G) \leq 4$.

First examples of chromatic number

The empty graph E_n

The only graphs with $\chi(G) = 1$ are the empty graphs E_n.

The complete graph K_n

The complete graph has $\chi(K_n) = n$

Bipartite graphs

A graph G has $\chi(G) = 2$ if and only if G is bipartite.

The wheel graph W_n

$$\chi(W_n) = \begin{cases} 3 & n \text{ even} \\ 4 & n \text{ odd} \end{cases}$$

Why?

How to find $\chi(G)$: sandwich it!

Start by finding rough upper and lower bounds.

Upper bound: colour it

If you can colour the vertices of G with k colours so that adjacent vertices don't share a vertex, then $\chi(G) \leq k$.

Lower bound: often case by case

A few trivial tricks:

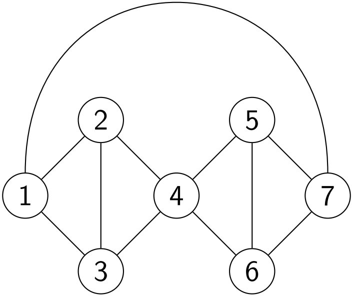
- If a vertex is adjacent to everything, as in W_n
- If H is a subgraph of G, then $\chi(G) \geq \chi(H)$.

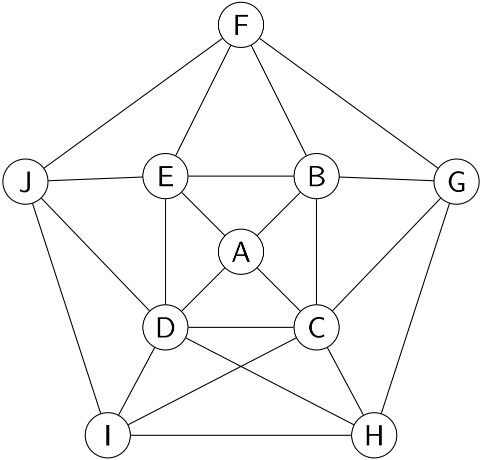
If lower bound isn't equal to upper bound, refine.

Finding the $\chi(G)$ is NP-hard

So no beautiful answer. But for small graphs not that bad.

Example: find $\chi(G)$ for the graphs shown below





General upper bounds

Definition

$\Delta(G)$ is the maximum degree of any vertex of G.

Lemma

$$\chi(G) \leq \Delta(G) + 1$$

Proof.

Colour the vertices one by one in any order. $\square$

The bound is tight, but for very few graphs:

- $\chi(K_n) = n = \Delta(K_n) + 1$
- For n odd, $\chi(C_n) = 3 = \Delta(C_n) + 1$

Theorem (Brooks)

If G isn't a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.

A story problem for $\chi(G)$ – not on exam this year

Suppose you have some things you want to separate into groups, but certain things can't be in the same group. How many groups do you need?

- Group vacation, several cottages, some people don't get on
- Storing chemicals, some react dangerously with each other

Make a graph G with:

- Vertices are the things
- Edges mean the vertices can't be in the same group

The groups are the colours.

$\chi(G)$ is the lowest feasible number of groups