Commutative Algebra MAS439
Lecture 2: Homomorphisms

Paul Johnson
paul.johnson@sheffield.ac.uk
Hicks J06b

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Slogan: (Homo)-morphisms preserve structure

Objects

Often in math we study things that are sets with some extra sturcture (Groups, rings, fields, vector spaces, metric spaces, topological spaces, measure space, . . .).

Morphisms

In these situations, usually there is a notion of *morphism* between these objects – i.e., set maps that “preserve the extra structure”

- Group homomorphisms preserve addition, units, inverses
- Vector space morphisms (linear maps) preserve addition and multiplication by scalars

This is the beginnings of *category theory*, which we’ll talk about later.

What do we mean by "preserve structure"?

More specifically, recall that a group G has:

- an identity e
- a multiplication map $G \times G \to G : (g, h) \mapsto g \cdot h$
- an inverse map $G \to G : g \mapsto g^{-1}$.

Definition

A group homomorphism $\varphi : G \to H$ is a map of sets so that

1. $\varphi(e_G) = e_H$
2. $\varphi(g^{-1}) = \varphi(g)^{-1}$
3. $\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2)$

Warning: For us, preserve will always be easy like this; but it can be more different.

What is a ring homomorphism?

Ring Homomorphisms

Definition

A ring homomorphism $\varphi : R \to S$ is a function so that

1. $\varphi(0_R) = 0_S$
2. $\varphi(1_R) = 1_S$
3. $\varphi(-r) = -\varphi(r)$
4. $\varphi(r + s) = \varphi(r) + \varphi(s)$
5. $\varphi(rs) = \varphi(r)\varphi(s)$

This is slightly more involved than the definition in the notes. *Why?*

A shortcut to ring homomorphisms

Some of the properties in the definition are implied by other properties.

- Since S has additive inverses and φ preserves addition, it automatically preserves 0's:

$$\varphi(0_R) = \varphi(0_R + 0_R) = \varphi(0_r) + \varphi(0_r)$$

  and add $-\varphi(0_r)$ to both sides.

- Similarly, φ must preserve additive inverses:

$$\varphi(r) + \varphi(-r) = \varphi(r - r) = \varphi(0_R) = 0_S$$

  and now add $-\varphi(r)$ to both sides.

So we only need to check that φ preserves addition, multiplication, and multiplicative units.

Examples of ring homomorphisms.

Non-examples

- $\det : M_{n \times n}(R) \to R$ is not a homomorphism: doesn't preserve addition
- The map $f : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ defined by $f([n]) = [4n]$ satisfies everything but doesn't preserve the identity
- The map zero map $R \to S$ sending everything to 0_S is only a homomorphism if S is the trivial ring; otherwise it doesn't preserve multiplicative identities

Is there a ring homomorphism $\varphi : \mathbb{Z} \to M_{3\times 3}(\mathbb{R})$?

How many such ring homomorphisms?

A useful lemma

Lemma

For any ring R, there is a unique ring homomorphism $f : \mathbb{Z} \to R$.

To prove the lemma, we need to write down a ring homomorphism $f : \mathbb{Z} \to R$ to show there *is* one.

Then, we need to prove that any other ring homomorphism *has* to be the same as f (uniqueness).

Isomorphisms

Informally, we think of things as being isomorphic if they are "the same". This is subtly and importantly different than being "equal".

Definition

A ring homomorphism $\varphi : R \to S$ is a *isomorphism* if there is another ring homomorphism $: S \to R$ with $\varphi \circ = \mathrm{Id}_S, \circ \varphi = \mathrm{Id}_R$.

A silly example

Let R be a copy of $\mathbb{Z}$ painted red. Let S be a copy of $\mathbb{Z}$ painted green.

Then R and S are isomorphic, but they aren't equal.

A more serious example

Let R be a commutative ring, and for a set X recall that $\text{Fun}(X, R)$, the set of functions from X to R, is a ring under pointwise addition and multiplication. Let $\{x\}$ be a one element set.

$\text{Fun}(\{x\}, R) \cong R$

To prove this, we define $\varphi : \text{Fun}(\{x\}, R) \to R$ by $\varphi(f) = f(x)$.

For $r \in R$ let $g_r \in \text{Fun}(\{x\}, R)$ be defined by $g_r(x) = r$. Then we define $\psi : R \to \text{Fun}(\{x\}, R)$ by $\psi(r) = g_r$.

Then ϕ and ψ are inverses to each other.

Similarly, $\text{Fun}(\{x, y\}, R) \cong R \times R$.

Another viewpoint on isomorphisms

Lemma

If $\varphi : R \to S$ is a bijective homomorphism, then φ is an isomorphism.

Proof.

Since φ is a bijection, we know from first year that there is an inverse map φ^{-1} of sets, we need to show that φ^{-1} is a ring homomorphism.

We need to check... (See board and/or notes)

$\square$

Nonisomorphic rings

Any *reasonable* property of rings (i.e., defined in terms of properties of the ring structure, and not in terms of something extraneous like being green or red) are invariant under isomorphism.

So, for example, if R and S are isomorphic, and R is an integral domain, than so is S.

To show two rings R and S are *not* isomorphic, it is usually easiest to find something true about one ring but not the other.

Lemma

None of the rings $\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$ are isomorphic to each other.

Sneak peek at next week

From a ring homomorphism $\varphi : R \to S$, we define the kernel $\ker(\phi)$ and the image Im() by

$$\operatorname{Im}(\varphi) = \{s \in S : s = \varphi(r) \text{ for some } r \in R\}$$

$$\ker(\varphi) = \{r \in R : \varphi(r) = 0_S\}$$

Though the kernel and the image are both subsets of a ring, it turns out they are very different types of subsets.

- The kernel is the prototypical (only!) example of an *ideal*
- The image is the prototypical (only!) example of a *subring*

A simple use of image and kernel

Lemma

Let $\varphi : R \to S$ a ring homomorphism. Then

1. *φ is surjective if and only if $Im(\varphi) = S$*
2. *φ is injective if and only if $\ker(\varphi) = \{0_R\}$*

Proof

? ? ?