MAS439 COMMUTATIVE ALGEBRA
PROBLEM SET 3

DUE FRIDAY, NOVEMBER 22ND

1. Ideals in product rings

Ideals in products are products of ideals. Prove that the ideals of $R \times S$ are *precisely* the sets of the form

$$I \times J\{(x, y) : x \in I, y \in J\}$$

where $I \subset R$ and $J \subset S$ are ideals.

Ideals of $\mathbb{Z}/2\times\mathbb{Z}/4$. What are all the ideals of $\mathbb{Z}/2\times\mathbb{Z}/4$? Which ones are radical? Prime? Maximal? Hint: to explain and figure out this last one, it might be easier to look at the quotient ring $(\mathbb{Z}/2 \times \mathbb{Z}/4)/I$.

2. A nonfinitely generated ideal

Later we will prove that if k is a field, then every ideal in $k[x_1, \ldots, x_n]$ is finitely generated. Before we do so, it may be beneficial to have see an example of an ideal that *isn't* finitely generated.

Let R be the ring of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the usual pointwise addition and multiplication. We say that a function $f \in R$ has bounded support if there is a real number $M > 0$ such that

$$|x| > M \implies f(x) = 0$$

(1) Prove that the subset $I \subset R$ of functions with bounded support is an ideal of R.
(2) Prove that I is radical but not prime.
(3) Prove that I is not finitely generated.

Question 1: Dimensions of $\mathbb{C}$-algebras

Compute the dimensions of the following $\mathbb{C}$-algebras, that are all quotients of $[x, y]$ by ideals generated by a few elements. Note that they might be infinite dimensional... In each case give a basis for the algebra as a complex vector space. Some explanation of your answer is required, but ironclad proofs are unnecessary.

(1) $R_1 = \mathbb{C}[x, y]/(x^3, xy, y^2)$
(2) $R_2 = \mathbb{C}[x, y]/(x^2y, x^3)$
(3) $R_3 = \mathbb{C}[x, y]/(x^2 + y^2, x - y^3)$