

# MAS439 Lecture 9

## Polynomial rings

October 26th

# Universal Property of $k[x]$

## Lemma

*Let  $(R, \phi)$  be any  $k$ -algebra, and let  $r \in R$  any element. Then there exists a unique homomorphism of  $k$ -algebras  $f : k[x] \rightarrow R$  such that  $f(x) = r$ .*

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- ▶ Since  $f$  preserves addition and multiplication, we have:

$$\begin{aligned} f(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0) = \\ \phi(a_n) r^n + \phi(a_{n-1}) r^{n-1} + \cdots + \phi(a_0) \end{aligned}$$

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So  $f$  just substitutes  $r$  for  $x$ , and there can be at most one such homomorphism; it's easy to check that  $f$  actually is a

# Universal objects characterize properties

If  $X$  and  $Y$  have the same universal property, then typically it will follow, simply from the definition of the Universal Property, that  $X$  and  $Y$  are isomorphic.

## Lemma

*Suppose  $(X, x)$  and  $(Y, y)$  both satisfy the Universal Property of the polynomial algebra over  $k$ . Then there is a unique isomorphism  $\varphi : X \rightarrow Y$  with  $\varphi(x) = y$ .*



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How do we construct  $\psi$ ?

**The Universal Property:** If  $\psi$  is inverse to  $\varphi$  we must have  $\psi(y) = x$ . Since  $Y$  satisfies the UP, there is a unique  $\psi : Y \rightarrow X$  with  $\psi(y) = x$ .



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**The Universal Property** Applying the Universal Property of  $(X, x)$  to itself, we see there is a **unique** morphism  $f : X \rightarrow X$  with  $f(x) = x$ .

Clearly, the identity map  $\text{Id}_X$  is such a map.

On the other hand,  $\psi \circ \varphi : X \rightarrow X$ , and  $\psi \circ \varphi(x) = \psi(y) = x$ .

Since  $f$  was unique, we must have  $\psi \circ \varphi = \text{Id}_X$ .

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How do we prove that  $\varphi \circ \psi = \text{Id}_Y$

Exactly the same, i.e.: **The Universal Property** for  $Y$ .

# Polynomial rings in more than one variable

Tom spends most of a page defining  $k[x_1, \dots, x_n]$ . Are we comfortable with what this is?

$k$  linear combinations of

Monomials:  $x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$

## Lemma

*For any  $n$ , there is an isomorphism of  $k$ -algebras:*

$$k[x_1, \dots, x_{n-1}][x_n] \cong k[x_1, \dots, x_n]$$

## Proof.

To go from the left to the right we multiply out; to go from the right to the left we factor out  $x_n^k$ 's. □

# Universal property for $k[x_1, \dots, x_n]$

## Lemma

*Let  $(R, \phi)$  be a  $k$ -algebra, and let  $r_1, \dots, r_n$  be (not necessarily distinct) elements of  $R$ . Then there exists a unique  $k$ -algebra homomorphism  $f : k[x_1, \dots, x_n]$  such that  $f(x_i) = r_i$ .*

One proof is to just adapt the proof we gave in the one variable case. Being a  $k$ -algebra homomorphism and knowing  $f(x_i) = r_i$  forces  $f$  to be the map that substitutes  $r_i$  in for  $x_i$ , and this is indeed a  $k$ -algebra homomorphism.

Another proof uses induction on  $n$ , and is perhaps instructive to work through.

# Proof the multivariables UP via induction

## Base Case: $n=1$

We already did.

## Inductive step

Suppose we know the universal property for  $k[x_1, \dots, x_{n-1}]$ . We now prove it for  $k[x_1, \dots, x_{n-1}, x_n]$ . The key idea is to use the isomorphism  $k[x_1, \dots, x_n] \cong k[x_1, \dots, x_{n-1}][x_n]$ .



### Main idea:

Use the Universal property of  $k[x_1, \dots, x_{n-1}][x_n]$  among  $k[x_1, \dots, x_{n-1}]$  algebras.

### First problem:

To use the UP to get a map to  $R$ , we need  $R$  to be a  $k[x_1, \dots, x_{n-1}]$ -algebra.

### Solution:

From the  $n - 1$  variables UP for  $k$  algebras, we have a unique  $k$ -algebra homomorphism  $\phi$  from  $k[x_1, \dots, x_{n-1}]$  to  $R$  sending  $x_i$  to  $r_i$ .

We now use this map  $\phi$  as the structure map to view  $R$  as a  $k[x_1, \dots, x_{n-1}]$  algebra.

Then, using the 1-variable UP for  $k[x_1, \dots, x_{n-1}][x_n]$ , there exists a unique  $k[[x_1, \dots, x_{n-1}]]$ -algebra homomorphism  $f : k[x_1, \dots, x_n] \rightarrow R$  with  $f(x_n) = r_n$ .

The map  $f$  is a  $k$ -algebra homomorphism and has  $f(x_i) = r_i$ .

Need to show  $f$  unique with these properties

But unraveling the definitions, being a  $k$ -algebra homomorphism sending  $x_i$  to  $r_i$  for  $1 \leq i \leq n-1$  is exactly the same as being a  $k[x_1, \dots, x_{n-1}]$ -algebra homomorphism. So, the map  $f$  is the unique  $k$ -algebra homomorphism because it is the unique  $k[x_1, \dots, x_{n-1}]$ -algebra homomorphism.

# An important corollary of the Universal Property

## Lemma

*Let  $R$  be a finitely generated  $k$ -algebra. Then  $R$  is the quotient of a polynomial ring:  $R \cong k[x_1, \dots, x_n]/I$  for some ideal  $I$ .*

## Proof.

- ▶ Since  $R$  is finitely generated, suppose it is generated by  $r_1, \dots, r_n$ .



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## Proof.

- ▶ Since  $R$  is finitely generated, suppose it is generated by  $r_1, \dots, r_n$ .
- ▶ By the Universal Property of polynomial algebras, there is a  $k$ -algebra homomorphism  $f : k[x_1, \dots, x_n]$  sending  $x_i$  to  $r_i$ .



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- ▶ Since  $\text{Im}(f)$  is a subring containing  $k$  and the  $r_i$ , we have  $R = k[r_1, \dots, r_n] \subset \text{Im}(f) \subset R$ , and so  $\text{Im}(f) = R$ .



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- ▶ Then, by the first isomorphism theorem, we have  $R = k[x_1, \dots, x_n]/\ker(f)$ .

