MAS439 Lecture 9
Polynomial rings

October 26th

Universal Property of $k[x]$

Lemma

Let (R, ϕ) be any k-algebra, and let $r \in R$ any element. Then there exists a unique homomorphism of k-algebras $f : k[x] \to R$ such that $f(x) = r$.

Proof.

- Since f is a k-algebra homomorphism, f must send k to k; i.e., if $a \in k \subset k[x]$, then we must have $f(a) = \phi(a)$.
- By assumption, we know $f(x) = r$.
- Since f preserves addition and multiplication, we have:

$$f(a_n x^n + a_{n_1} x^{n-2} + \cdots + a_0) = \phi(a_n) r^n + \phi(a_{n-1}) r^{n-2} + \cdots + \phi($$

So f just substitutes r for x, and there can be at most one such homomorphism; it's easy to check that f actually is a homomorphism. $\square$

Universal objects characterize properties

If X and Y have the same universal property, then typically it will follow, simply from the definition of the Universal Property, that X and Y are isomorphic.

Lemma

Suppose (X, x) and (Y, y) both satisfy the Universal Property of the polynomial algebra over k. Then there is a unique isomorphism $\varphi : X \to Y$ with $\varphi(x) = y$.

Proof that Universal Property characterizes $k[x]$

How do we get a map from X to Y?

The Universal Property: Since X satisfies the universal property, there is a unique k-algebra homomorphism φ with $\varphi(x) = y$.

How do we show φ is an isomorphism?

Construct an inverse $\psi : Y \to X$.

How do we construct ψ?

The Universal Property: If ψ is inverse to φ we must have $\psi(y) = x$. Since Y satisfies the UP, there is a unique $\psi : Y \to X$ with $\psi(y) = x$.

Universal property characterizes $k[x]$, continued

How do we prove that $\psi \circ \varphi = \mathrm{Id}_X$

The Universal Property Applying the Universal Property of (X, x) to itself, we see there is a unique morphism $f : X \to X$ with $f(x) = x$.
Clearly, the identity map Id_X is such a map.
On the other hand, $\psi \circ \varphi : X \to X$, and $\psi \circ \varphi(x) = \psi(y) = x$.
Since f was unique, we must have $\psi \circ \varphi = \mathrm{Id}_X$.

How do we prove that $\varphi \circ \psi = \mathrm{Id}_Y$

Exactly the same, i.e.: The Universal Property for Y.

Polynomial rings in more than one variable

Tom spends most of a page defining $k[x_1, \ldots, x_n]$. Are we comfortable with what this is?

k linear combinations of

Monomials: $x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$

Lemma

For any n, there is an isomorphism of k-algebras:

$$k[x_1, \ldots, x_{n-1}][x_n] \cong k[x_1, \ldots, x_n]$$

Proof.

To go from the left to the right we multiply out; to go from the right to the left we factor out x_n^k's. $\square$

Universal property for $k[x_1, \dots, x_n]$

Lemma

Let (R, ϕ) be a k-algebra, and let $r_1, dots, r_n$ be (not necessarily distinct) elements of R. Then there exists a unique k-algebra homomorphism $f : k[x_1, \dots, x_n]$ such that $f(x_i) = r_i$.

One proof is to just adapt the proof we gave in the one variable case. Being a k-algebra homomorphism and knowing $f(x_i) = r_i$ forces f to be the map that substitutes r_i in for x_i, and this is indeed a k-algebra homomorphism.

Another proof uses induction on n, and is perhaps instructive to work through.

Proof the multivariables UP via induction

Base Case: n=1

We already did.

Inductive step

Suppose we know the universal property for $k[x_1, \ldots, x_{n-1}]$. We now prove it for $k[x_1, \ldots, x_{n-1}, x_n]$. The key idea is to use the isomorphism $k[x_1, \ldots, x_n] \cong k[x_1, \ldots, x_{n-1}][x_n]$.

Main idea:

Use the Universal property of $k[x_1, \ldots, x_{n-1}][x_n]$ among $k[x_1, \ldots, x_{n-1}]$ algebras.

First problem:

To use the UP to get a map to R, we need R to be a $k[x_1, \ldots, x_{n-1}]$-algebra.

Solution:

From the $n-1$ variables UP for k algebras, we have a unique k-algebra homomorphism ϕ from $k[x_1, \ldots, x_{n-1}]$ to R sending x_i to r_i.
We now use this map ϕ as the structure map to view R as a $k[x_1, \ldots, x_{n-1}]$ algebra.

Then, using the 1-variable UP for $k[x_1, \ldots, x_{n-1}][x_n]$, there exists a unique $k[[x_1, \ldots, x_{n-1}]$-algebra homomorphism $f : k[x_1, \ldots, x_n] \to R$ with $f(x_n) = r_n$.

The map f is a k-algebra homomorphism and has $f(x_i) = r_i$.

Need to show f unique with these properties

But unraveling the definitions, being a k-algebra homomorphism sending x_i to r_i for $1 \leq i \leq n-1$ is exactly the same as being a $k[x_1, \ldots, x_{n-1}]$-algebra homomorphism. So, the map f is the unique k-algebra homomorphism because it is the unique $k[x_1, \ldots, x_{n-1}]$-algebra homomorphism.

An important corollary of the Universal Property

Lemma

Let R be a finitely generated k-algebra. Then R is the quotient of a polynomial ring: $R \cong k[x_1, \ldots, x_n]/I$ for some ideal I.

Proof.

- Since R is finitely generated, suppose it is generated by $r_1, \ldots, r_n$.

$\square$

An important corollary of the Universal Property

Lemma

Let R be a finitely generated k-algebra. Then R is the quotient of a polynomial ring: $R \cong k[x_1, \ldots, x_n]/I$ for some ideal I.

Proof.

- Since R is finitely generated, suppose it is generated by $r_1, \ldots, r_n$.
- By the Universal Property of polynomial algebras, there is a k-algebra homomorphism $f : k[x_1, \ldots, x_n]$ sending x_i to r_i.

□

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Proof.

- Since R is finitely generated, suppose it is generated by $r_1, \ldots, r_n$.
- By the Universal Property of polynomial algebras, there is a k-algebra homomorphism $f : k[x_1, \ldots, x_n]$ sending x_i to r_i.
- Since $\mathrm{Im}(f)$ is a subring containing k and the r_i, we have $R = k[r_1, \ldots, r_n] \subset \mathrm{Im}(f) \subset R$, and so $\mathrm{Im}(f) = R$.

$\square$

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- Then, by the first isomorphism theorem, we have $R = k[x_1, \ldots, x_n]/\ker(f)$.

$\square$