Commutative Algebra MAS439
Lecture 1

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Assessment is entirely via problem sets

- Problems are due *every* Wednesday at the *beginning* of class
- Each semester, the two lowest scores (out of ten) can be dropped
- You are encouraged, but not required, to write your solutions in LaTeX
- You are encouraged, but not required, to work together in groups of 2 or 3

Wait, groupwork?! How does that work?

- Each group member writes up and hands in their own solution
- If you do work in groups, please write who you worked with on every assignment

What is/isn't allowed:

- You should NOT be writing up identical solutions, or even writing up your solutions sitting together.
- Rather, in the group digest what the problem is actually asking, come up with an informal / pseudo-formal solution
- LATER, on your own, write up the full, rigorous solution

Rigour and intuition, proof and understanding

- Mathematics is all in our heads. Giving formal definitions and rigorous proofs make sure we're not just making up nonsense
- However, humans don't think very well in this rigorous structure. We have our own intuitive pictures
- Most of the work of doing mathematics is translating back and forth between rigorous and intuitive modes.

The *Oral tradition* in mathematics

Mathematics is written down in full rigor, but informal discussion of "how to think about this" or "what's really going on" aren't written down

- Terry Tao, There's more to mathematics than rigour and proofs
- William Thurston, On proof and progress in mathematics

Lectures and Notes

- Primary text: Tom Bridgeland's notes (rigor)
- I won't provide lecture notes (intuition)

Weekly webpage

- Terse description of what was covered in lecture
- Slides
- Problem set
- Feedback on problem set?
- Comment section

Please *Please* read the notes

I will be assuming you are

The first 3-4 weeks should be somewhat review

MAS220 Syllabus from 2014 MAS439 Table of Contents

- You've forogtten a lot of this not having used it for two years
- We do everything more in depth and sophisticated

I AM DEPENDING ON YOU TO LET ME KNOW IF I'M GOING TOO FAST (or too slow)

A Quiz

Answer on the paper provided

1. What's the formal definition of a ring homomorphism? Give an example.
2. What's the formal definition of an ideal? What's the *point* of this definition?
3. Give at least 5 examples of rings.

Up to you whether you put your name on them or not; I'm just using these as a quick gauge of our background.

Definition of a ring, ugly version

A *ring* is a set R with two binary operations $+, \cdot$ satisfying:

1. $\forall x, y, z \in R, (x + y) + z = x + (y + z)$
2. $\exists 0_R \in R$ such that $\forall x \in R, 0_R + x = x + 0_R = x$
3. $\forall x \in R, \exists -x \in R$ such that $x + (-x) = (-x) + x = 0_R$
4. $\forall x, y \in R, x + y = y + x$
5. $\forall x, y, z \in R, (x \cdot y) \cdot z = x \cdot (y \cdot z)$
6. $\exists 1_R \in R$ such that $\forall x \in R, 1_r \cdot x = x \cdot 1_R = x$
7. $\forall x, y, z \in R$:

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

$$(y + z) \cdot x = y \cdot x + y \cdot z$$

What are the names of the axioms?

Definition of a ring, take two

A *ring* is a set R with two binary operations $+, \cdot$ satisfying:

1. $(R, +)$ is an abelian group
2. $(R, \cdot)$ is a monad
3. Multiplication $(\cdot)$ distributes over addition $(+)$

A *monad* satisfies all the axioms of a group except perhaps the existence of inverses.

Back to the Quiz:
Let's list examples of rings

How'd we do?

1. The trivial ring has one element
2. The integers $\mathbb{Z}$
3. Any field $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_2, \cdots$
4. "clock arithmetic" $\mathbb{Z}/12\mathbb{Z}$ and more generally $\mathbb{Z}/n\mathbb{Z}$
5. Polynomial rings $\mathbb{R}[x], \mathbb{Z}[y, z]$
6. The set $M_n(\mathbb{R})$ of $n \times n$ matrices with real coefficients
7. The quaternions $\mathbb{H}$
8. The Gaussian integers $\mathbb{Z}[i] = \{z = a + bi \in \mathbb{C} | a, b \in \mathbb{Z}\}$
9. The set $\mathrm{Fun}(\mathbb{R}, \mathbb{R})$ of all functions from $\mathbb{R}$ to itself, under pointwise addition and multiplication (e.g., $(f \cdot g)(x) = f(x) \cdot g(x)$)
10. The set $C(\mathbb{R})$ of all *continuous* functions from $\mathbb{R}$ to itself

Commutative algebra is the study of commutative rings

Definition

A ring R is *commutative* if multiplication is commutative, i.e. $x \cdot y = y \cdot x$

Convention:

Unless otherwise specified, all rings R will be assumed to be commutative.

Types of elements

Definition

We say $r \in R$ is a *unit* if there exists an element $s \in R$ with $rs = 1_R$

Definition

We say that $r \in R$ is a *zero divisor* if there exists $s \in R, s \neq 0_R$ with $rs = 0_R$

Definition

We say that $r \in R$ is *nilpotent* if there exists some $n \in \mathbb{N}$ with $r^n = 0_R$

Examples?

Types of rings

Definition
We say R is *field* if every nonzero element is a unit.

By convention, the trivial ring is not a field.

Definition
We say R is an *integral domain* if it has no zero divisors.

Definition
We say that R is *reduced* if it has no nilpotent elements.

Examples?