

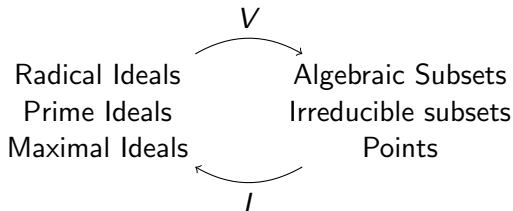
# MAS439 Lecture 14

## Coordinate Ring

November 23rd

## Where we are:

We've seen that the maps  $V$  and  $I$  are inverse to each other and set up 1:1 correspondences between certain types of ideals in  $R = k[x_1, \dots, x_n]$  and certain types of subsets of  $\mathbb{A}_k^n$ .



# What's next?

The correspondence above holds for ideals of  $k[x_1, \dots, x_n]$ , which is a very special example of a  $k$ -algebra. We'd like a geometric way to study the ideals of other rings.

We've seen that any finitely generated reduced  $k$ -algebra  $S$ , we have  $S \cong k[x_1, \dots, x_n]/I$  for some  $n$  and radical ideal  $I$ .

Today we will see that we can view  $S = k[x_1, \dots, x_n]/I$  as functions on the algebraic subset  $V(I)$ : we will call  $S$  the *coordinate ring of  $V(I)$* , and that ideals of  $S$  will be related to the geometry of  $V(I)$ .

Tomorrow, we will study maps between different algebraic subsets, and how they relate to maps between their coordinate rings.

# The Coordinate ring

## Definition

Let  $X \subset \mathbb{A}_k^n$  be an algebraic subset. The *coordinate ring*  $k[X]$  of  $X$  is defined to be the quotient ring

$$k[X] = k[x_1, \dots, x_n] / I(X)$$

Since  $I(X)$  is always radical, the coordinate ring  $k[X]$  is always reduced.

Since  $k[x_1, \dots, x_n]$  is a  $k$ -algebra, so is  $k[X]$ .

How is  $k[X]$  related to  $X$ ?

# Polynomial Functions

We can view the polynomial ring  $R = k[x_1, \dots, x_n]$  as a subring of the space of all functions from  $\mathbb{A}_k^n$  to  $k$ .

If  $X \subset \mathbb{A}_k^n$ , then we can also view a polynomial as a function on  $X$ .

## Definition

Let  $X \subset \mathbb{A}_k^n$  be algebraic. We call a function  $f : X \rightarrow k$  *polynomial* if there is a polynomial  $g \in R = k[x_1, \dots, x_n]$  so that  $f(x) = g(x)$  for all  $x \in X$ .

Let  $\text{Fun}_{\text{poly}}(X)$  denote the set of all polynomial functions on  $X$ .

## Claim:

$\text{Fun}_{\text{poly}}(X)$  is a  $k$ -algebra

# The coordinate ring is the polynomial functions

## Lemma

Let  $X \subset \mathbb{A}_k^n$  algebraic. Then

$$k[X] := k[x_1, \dots, x_n] / I(X) \cong \text{Fun}_{\text{poly}}(X)$$

## Proof.

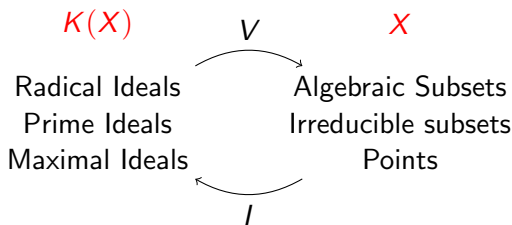
- ▶ The restriction map  $\text{Res} : R = k[x_1, \dots, x_n] \rightarrow \text{Fun}_{\text{poly}}(X)$  is a homomorphism, and is surjective by definition.
- ▶ A polynomial  $f \in R$  is in the kernel of  $\text{Res}$  is equivalent to  $f(x) = 0$  for all  $x \in X$ ; that is, that  $f(x) \in I(X)$ .
- ▶ The first isomorphism theorem then says that  $\text{Fun}_{\text{poly}}(X) \cong R / I(X)$ , as desired.



# Geometry and the Ideals of $K(X)$

Since  $K(X) = R/I(X)$ , the second isomorphism theorem says that ideals in  $K(X)$  are in 1-1 correspondence with ideals of  $R$  containing  $I(X)$ .

Since  $I$  is inclusion reversing, we see that radical ideals of  $K(X)$  are in 1-1 correspondence with algebraic subsets of  $\mathbb{A}_k^n$  contained in  $X$ .



## Example:

Find all the radical/prime/maximal ideals of  $R = k[x, y]/(x^2y + xy^2 - xy)$ .

### Technique:

We'd like to use the previous slide and relate the ideals of  $R$  to the geometry of  $V(x^2y + xy^2 - xy)$ , but to do this, we must first check that  $(x^2y + xy^2 - xy)$  is radical.

This is not hard as  $(x^2y + xy^2 - xy)$  is principal and  $k[x, y]$  is a unique factorisation domain...