MAS439 COMMUTATIVE ALGEBRA
PROBLEM SET 4

DUE TUESDAY, DECEMBER 10TH

1. Using Universal Properties

It is tedious to completely check that a given set map is a homomorphism by hand, but if we use the universal properties of polynomial rings and quotient rings we can construct maps we know are homomorphisms without doing the labor of checking it. Using both of these universal properties, carefully prove that:

(1) $\mathbb{C}[z]$ is isomorphic to $\mathbb{C}[x, y]/(y - 3)$
(2) For any ring k, $k[x]/(x^2 - x)$ is isomorphic to $k \times k$.

(The k-algebra structure on $k \times k$ is $\varphi(r) = (r, r)$)

In each case you should use the universal properties to construct a homomorphism of algebras from one algebra to the other, and then prove your homomorphism is an isomorphism.

2. "Bad" intersections

Let $I = (x^2 + y^2 - 2), J = (xy - 1) \subset \mathbb{R}[x, y]$.

(1) Describe and draw the algebraic subsets $X = V(I), Y = V(J) \subset \mathbb{A}^2_{\mathbb{R}}$.
(2) Find $X \cap Y$, and verify that $X \cap Y = V(I + J)$.
(3) Show that $x - y \in I(X \cap Y)$, but $x - y \notin I + J$. This verifies that that $I(X \cap Y) \neq I(X) + I(Y)$ in general.
(4) Finally, show that $(x - y)^2 \in I + J$.

3. Proving the Nullstellensatz in dimension one

Recall that r is a root of a polynomial f if $f(r) = 0$, and that a field k is algebraically closed if every nonconstant polynomial $f \in k[x]$ has a root $r \in k$.

(1) Prove that if k is algebraically closed, and $f \in k[x]$, then f factors completely into linear factors $f = a(x - r_1)(x - r_2) \cdots (x - r_n)$. (Hint: division algorithm and induction)
(2) Prove the Nullstellensatz in dimension 1: if k is algebraically closed and $I \subset k[x]$, then $I(V(I)) = \sqrt{I}$.