MAS439 COMMUTATIVE ALGEBRA
PROBLEM SET 5

DUE BEFORE TERM 2

Question 1

Let k be an algebraic closed field, and let $X \subset \mathbb{A}_k^n$ and $Y \subset \mathbb{A}_k^m$ be algebraic sets. Let $\varphi : X \to Y$ be a polynomial map and let $\varphi^* : k[Y] \to k[X]$ be the corresponding map between coordinate rings.

(1) Prove that if φ is surjective, then φ^* is injective.
(2) Prove that if φ^* is surjective, then φ is injective.
(3) Show that the converses of parts 1 and 2 don't hold.

Hints.

(1) Two polynomials are the same if and only if they take on the same value at every point.
(2) Two points p and q are different if and only if there is a polynomial that takes on different values at p and q.
(3) Consider the map $V(xy-1) \to \mathbb{A}_k^1$ given by $(a,b) \mapsto a$

Question 2

The *nodal cubic* is the algebraic set $X = V(y^2 - x^3 - x^2)$.

(1) Show that the map $\varphi : \mathbb{A}_{\mathbb{C}}^1 \to \mathbb{A}_{\mathbb{C}}^2$ given by $\varphi(t) = (t^2-1, t^3-t)$ lands inside X, and hence gives a polynomial map $\varphi : \mathbb{A}_{\mathbb{C}}^1 \to X$
(2) Show that the map φ is surjective onto X. Where does φ fail to be injective?
(3) Prove that X is irreducible. Using φ or φ^* might help...
(4) Construct a rational function $r \in \mathbb{C}(X)$ that is an inverse to φ wherever it is defined.

Geometric explanation. There's nothing to prove here, and no information that's required to prove the parts above, but it may provide some intuition for the mysterious formula for φ.

Sketch the real part of X (maybe by considering how it's related to the graph of $y = x^3 + x^2$).

The map φ has the following geometric interpretation, which may be enlightening. Consider the line through the origin with slope t, i.e., $y = tx$. If we restrict $y^2 - x^3 - x^2$ to this line, it will be a cubic polynomial in one variable, and hence have three roots. The origin will always be a double root, and hence there will be one more root. Geometrically, this means the line $y = tx$ intersects X at the origin with multiplicity two, and at one *other* point. This other point is $\varphi(t)$.