MAS439 Lecture 14
Coordinate Ring

November 23rd

Where we are:

We've seen that the maps V and I are inverse to each other and set up 1:1 correspondences between certain types of ideals in $R = k[x_1, \ldots, x_n]$ and certain types of subsets of $\mathbb{A}_k^n$.

$$\begin{array}{c} \text{Radical Ideals} \\ \text{Prime Ideals} \\ \text{Maximal Ideals} \end{array} \underset{I}{\overset{V}{\rightleftarrows}} \begin{array}{c} \text{Algebraic Subsets} \\ \text{Irreducible subsets} \\ \text{Points} \end{array}$$

What's next?

The correspondence above holds for ideals of $k[x_1, \ldots, x_n]$, which is a very special example of a k-algebra. We'd like a geometric way to study the ideals of other rings.

We've seen that any finitely generated reduced k-algebra S, we have $S \cong k[x_1, \ldots, x_n]/I$ for some n and radical ideal I.

Today we will see that we can view $S = k[x_1, \ldots, x_n]/I$ as functions on the algebraic subset $V(I)$: we will call S the *coordinate ring of* $V(I)$, and that ideals of S will be related to the geometry of $V(I)$.

Tomorrow, we will study maps between different algebraic subsets, and how they relate to maps between their coordinate rings.

The Coordinate ring

Definition

Let $X \subset \mathbb{A}^n_k$ be an algebraic subset. The *coordinate ring* $k[X]$ of X is defined to be the quotient ring

$$k[X] = k[x_1, \ldots, x_n]/I(X)$$

Since $I(X)$ is always radical, the coordinate ring $k[X]$ is always reduced.

Since $k[x_1, \ldots, x_n]$ is a k-algebra, so is $k[X]$.

How is $k[X]$ related to X?

Polynomial Functions

We can view the polynomial ring $R = k[x_1, \ldots, x_n]$ as a subring of the space of all functions from $\mathbb{A}^n_k$ to k.

If $X \subset \mathbb{A}^n_k$, then we can also view a polynomial as a function on X.

Definition

Let $X \subset \mathbb{A}^n_k$ be algebraic. We call a function $f : X \to k$ *polynomial* if there is a polynomial $g \in R = k[x_1, \ldots, x_n]$ so that $f(x) = g(x)$ for all $x \in X$.

Let $\mathrm{Fun}_{\mathrm{poly}}(X)$ denote the set of all polynomial functions on X.

Claim:

$\mathrm{Fun}_{\mathrm{poly}}(X)$ is a k-algebra

The coordinate ring is the polynomial functions

Lemma

Let $X \subset \mathbb{A}^n_k$ algebraic. Then

$$k[X] := k[x_1, \ldots, x_n]/I(X) \cong Fun_{poly}(X)$$

Proof.

- The restriction map $\mathrm{Res} : R = k[x_1, \ldots, x_n] \to \mathrm{Fun}_{\mathrm{poly}}(X)$ is a homomorphism, and is surjective by definition.
- A polynomial $f \in R$ is in the kernel of Res is equivalent to $f(x) = 0$ for all $x \in X$; that is, that $f(x) \in I(X)$.
- The first isomorphism theorem then says that $\mathrm{Fun}_{\mathrm{poly}}(X) \cong R/I(X)$, as desired.

$\square$

Geometry and the Ideals of $K(X)$

Since $K(X) = R/I(X)$, the second isomorphism theorem says that ideals in $K(X)$ are in 1-1 correspondence with ideals of R containing $I(X)$.

Since I is inclusion reversing, we see that radical ideals of $K(X)$ are in 1-1 correspondence with algebraic subsets of $\mathbb{A}^n_k$ contained in X.

$$\begin{array}{ccc} K(X) & & X \\ \text{Radical Ideals} & \xrightarrow{V} & \text{Algebraic Subsets} \\ \text{Prime Ideals} & & \text{Irreducible subsets} \\ \text{Maximal Ideals} & \xleftarrow[I]{} & \text{Points} \end{array}$$

Example:

Find all the radical/prime/maximal ideals of
$R = k[x, y]/(x^2y + xy^2 - xy)$.

Technique:

We'd like to use the previous slide and relate the ideals to of R to the geometry of $V(x^2y + xy^2 - xy)$, but to do this, we must first check that $(x^2y + xy^2 - xy)$ is radical.

This is not hard as $(x^2y + xy^2 - xy)$ is principal and $k[x, y]$ is a unique factorisation domain...