MAS439 COMMUTATIVE ALGEBRA
PROBLEM SET 2

DUE FRIDAY, NOVEMBER 1ST

Question 1: A subring

For f polynomial, let f' be its derivative and f'' be its second derivative. Define

$$S = \{f \in \mathbb{Z}[x] : f'(1) = f''(1) = 0\}$$

Part 1. Prove that S is a subring of $\mathbb{Z}[x]$.

Part 2. Prove that S is finitely generated.

Question 2: Ideals?

Let R be a commutative ring.

Part 1. Let N be the union of $\{0\}$ and all the nilpotent elements of R. Prove that N is an ideal of R.

Part 2. Let D be the union of $\{0\}$ and all the zero divisors of R. Give an example to show that D need not be an ideal.

Question 3: Rings with four elements

We already know three rings with four elements: $\mathbb{Z}/4, \mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{F}_4$, the field with four elements, which we constructed in lecture and in the notes as the quotient ring $\mathbb{F}_4 = \mathbb{F}_2[x]/(x^2 + x + 1)$.

In addition to $x^2 + x + 1$, there are three more quadratic polynomials in $\mathbb{F}_2[x]$ (Recall that $\mathbb{F}_2$ is just what we write for $\mathbb{Z}/2$ when we want to emphasize it's a field):

$$f_1 = x^2, \quad f_2 = x^2 + x, \quad f_3 = x^2 + 1$$

Part one. Prove that each of the rings $R_i = \mathbb{F}_2[x]/(f_i)$ has four elements – do this for all three at one time.

Part two. Write down the multiplication tables for each of the rings R_i, and for each one determine:

(1) Is R_i reduced?
(2) Is R_i an integral domain?
(3) Is R_i a field?
(4) Is R_i isomorphic to one of three rings with four elements we already knew?

Part three. Finally, decide which of the R_i are isomorphic to each other, and which are not?