MAS439 Lecture 11
Algebraic Subsets

November 3rd

Starting Geometry

We've been studying polynomial rings algebraically, from the viewpoint of their universal property. We now begin to study them from another angle, as rings of functions on affine space.

Definition

Let k be a field. Define affine space

$$\mathbb{A}_k^n = \{(a_1, \ldots, a_n) \in k^n\}$$

Note that as a set, this is just k^n. The notation highlights the fact that we are not going to view $\mathbb{A}_k^n$ as a ring or a vector space, but as a geometric space that has functions on it.

Vanishing loci

Definition

Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Define:

$$V(I) = \{p \in \mathbb{A}_k^n : f(p) = 0 \forall f \in I\}$$

Note that $V(X)$ makes sense for any subset $X \subset k[x_1, \ldots, x_n]$. But note that $V(X)$ agrees with $V(I)$, where $I = (X)$ is the ideal generated by elements of X. So we can work with only ideals instead.

In the other direction, since $k[x_1, \ldots, x_n]$ is Noetherian, any ideal I is generated by finitely many elements $f_1, \ldots, f_k$, and so $V(I)$ is the locus where all the f_i vanish simultaneously.

Ideals of functions that vanish

We now look at the other direction:

Definition

Given any subset $X \subset \mathbb{A}_k^n$, define

$$I(X) = \{f \in k[x_1, \ldots, x_n] : f(p) = 0 \quad \forall p \in V\}$$

Note that as the name suggests, $I(V)$ is an ideal. Why?

V and I are order reversing

Suppose that $I \subset J$, then $V(J) \subset V(I)$. (Why?)

Similarly, suppose that $X \subset Y \subset \mathbb{A}^n_k$. Then $I(Y) \subset I(X)$.

Algebraic subsets

Definition

A subset $X \subset \mathbb{A}^n_k$ is *algebraic* if it is of the form $V(I)$ for some ideal $I \subset k[x_1, \ldots, x_n]$

Now we'll look at a lot of examples (see the notes)

Unions and intersections

Tom's notes state this backwards!

Finite unions and arbitrary intersections of algebraic subsets are algebraic, and we can describe their ideals:

Lemma

Given an arbitrary collection of ideals I_s indexed by a set S, we have:

$$V\left(\sum_{s\in S} I_s\right) = \bigcap_{s\in S} V(I_s)$$

Given two ideals I, J, we have

$$V(I \cap J) = V(I) \cup V(J)$$

Examples

Let $I = (x, y) \in \mathbb{C}[x, y, z]$, and $J = (z)$.

Let's check the lemma in this case: What's $V(I)$, $V(J)$, $I + J, I \cap J$

Reminder on point-set topology

Definition

Let X be a space. A *topology* $\mathcal{T}$ on X is a collection of subsets of X (called *open* sets such that:

1. X and $\emptyset$ are open
2. Finite intersections of open sets are open:

$$U, V \in \mathcal{T} \implies U \cap V \in \mathcal{T}$$

3. Arbitrary unions of open sets are open:

$$U_s \in \mathcal{T} \forall s \in S \implies \bigcup_{s \in S} U_s \in \mathcal{T}$$

Motivating example

Let (X, d) be a metric space. (e.g., $\mathbb{R}^n$ with its usual distance function). Define a subset U to be *open* if for all $x \in X$, there exists an $r > 0$ such that

$$B_r(x) = \{y \in X : d(x, y) < r\} \subset U$$

Weirder examples of topologies

Let X be any set.

Discrete topology

Define any subset of X to be open.

Concrete topology

Define only X and $\emptyset$ to be open.

One more

Define $U \subset X$ to be open if and only the complement of U is finite

Algebraic subsets are closed sets

Definition

If X is a topological space, we call a subset $U \subset X$ *closed* if its complement U^c is open.

Warning: A set can be both open and closed (clopen). (There's a *downfall* parody video about this.

Since open sets are closed under finite intersection and arbitrary union, closed sets are closed under arbitrary union and finite intersection.

Thus, our lemma shows that the algebraic subsets form the closed sets of a topology on $\mathbb{A}^n_k$, called the *Zariski* topology.