MAS439 Lecture 9
Noetherian Rings

October 27th

Noetherian Rings

Definition

We say that a ring R is *Noetherian* if it satisfies the *Ascending Chain Condition*: any increasing chain of ideals eventually stabilizes. That is, if

$$I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots \subset R$$

is a chain of ideals, then there exists some N so that $I_n = I_n$ for all $n > N$

Why care about Noetherian rings?

Lemma

A ring R is Noetherian if and only if every ideal I of R is finitely generated

Suppose that R has every ideal finitely generated

How to show R Noetherian?

Given an ascending chain $I_1 \subset I_2 \subset I_3 \subset \cdots \subset I_n \subset \cdots$, we must show the chain stabilizes.

Only thing we can do is use that some ideal I is finitely generated. Trying to use this on the I_k doesn't help. What other ideal could we use?

A lemma we've already used...

Lemma

Suppose $I_1 \subset I_2 \subset \cdots \subset I_n \subset \ldots$ *ideals. Then*

$$I =_n I_n$$

is an ideal

Proof.

- Nonempty as it contains 0
- Closed under addition: if $r, s \in I$, then $\exists n, m$ with $r \in I_n, s \in I_m$. Let $k = \max(n, m)$; then $r + s \in I_k \subset I$.
- Closed under multiplication by R: if $x \in I, r \in I$, then $x \in I_n$ for some n, so $r \cdot x \in I_n \subset I$

$\square$

Back to Noetherian rings

Assume every ideal of R is finitely generated, and $I_1 \subset I_2 \subset \cdots \subset I_n \subset \ldots$ a chain of ideal; need to show the ideal stabilizes.

But $I = I_n$ is finitely generated

Suppose that $I = (x_1, \ldots, x_k)$.

Then $x_i \in I$, so $x_i \in I_{n_i}$ for some n_i. Taking $N = \max(n_k)$, we see that $x_i \in I_N$. Hence:

$$I = (x_i) \subset I_N \subset I$$

and so the chain stabilizes at I_N to I.

Noetherian $\implies$ every ideal finitely generated

Now suppose that R Noetherian, and $I \subset R$ an ideal. We want to show that I is finitely generated.

Where can we find a chain of ideals to apply the ACC to?

Try to generate I at random