## QUESTION 1: ELEMENTS OF $\mathbb{Z} \times \mathbb{Z}$

**What are the units in $\mathbb{Z} \times \mathbb{Z}$.** In general $(a, b) \in R \times S$ is a unit if and only if $a$ is a unit in $R$ and $b$ is a unit in $S$. This is because $(a, b)$ is a unit if and only if there is a $(c, d)$ with $(a, b) \cdot (c, d) = 1_{R \times S}$. But this is if and only if $a \cdot c = 1_R$ and $b \cdot d = 1_S$, i.e. if $a$ is a unit in $R$ and $b$ is a unit in $S$.

Since the only units in $\mathbb{Z}$ are $\pm 1$, we see that there are four units in $\mathbb{Z} \times \mathbb{Z}$ : $(1, 1), (1, -1), (-1, 1), (-1, 1)$.

**What are the nilpotent elements of $\mathbb{Z} \times \mathbb{Z}$.** In general, an element $(a, b) \neq (0, 0) \in R \times S$ is nilpotent if and only if both $a$ and $b$ are nilpotent or zero. This is because $(a, b)$ is nilpotent if and only if $(a, b)^n = (0, 0)$ if and only if $a^n = 0$ and $b^n = 0$, i.e., if and only if both $a$ and $b$ are nilpotent or zero.

Thus, since $\mathbb{Z}$ has no nilpotent elements, $\mathbb{Z} \times \mathbb{Z}$ has no nilpotent elements.

**What are the zero divisors in $\mathbb{Z} \times \mathbb{Z}$.** Suppose that $(a, b) \neq (0, 0) \in \mathbb{Z} \times \mathbb{Z}$ is a zero divisor. This happens if and only if there is a $(c, d) \neq (0, 0) \in \mathbb{Z} \times \mathbb{Z}$ so that $(a, b) \cdot (c, d) = 0$, i.e. $ac = 0$ and $bd = 0$. Since $\mathbb{Z}$ has no zero divisors, $ac = 0$ if and only at least one of $a, c = 0$. Similarly, at least one of $b, d = 0$. Since by assumption, both of $a, b$ cannot be zero, and both of $c, d$ cannot be zero, we see there are two possibilities: $a = 0$ and $d = 0$, or $b = 0$ and $c = 0$. Thus, the zero divisors of $\mathbb{Z} \times \mathbb{Z}$ are those elements of the form $(a, 0)$ or $(0, b)$.

## QUESTION 2: ZERO DIVISORS IN $R[x]$

Let $R$ be a nontrivial ring, and let $f = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in R[x]$. Prove that if $a_n$ is not a zero divisor in $R$, then $f$ is not a zero divisor in $R[x]$.

**Proof:** Suppose that $f = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in R[x]$ were a zero divsior; we must show that $a_n$ is a zero divisor. Then there exists a $g = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0 \in R[x]$ with $fg = 0$. Note that we can take $b_m \neq 0$. But expanding the product $fg$ out, we have

$$a_n b_m x^{n+m} + (a_n b_{m-1} + a_{n-1} b_m) x^{n+m-1} + \cdots + (a_1 b_0 + a_0 b_1) x + a_0 b_0 = 0$$

For this to be true, we need the coefficient of every $x^k$ to be equal to zero in $R$; in particular, the coefficient of $x^{n+m}$, namely $a_n b_m$, must be equal to 0. Since $b_m \neq 0$, this means $a_n$ is a zero divisor.

## QUESTION 3: COUNTING HOMOMORPHISMS

How many different homomorphism $\varphi : R \to S$ are there when:

**$R = \mathbb{Z}$ and $S = \mathbb{Z}[x]$.** Lemma 4.4 in the notes states that for any ring $S$ there is a unique homomorphism $\varphi : \mathbb{Z} \to S$. Hence, there is one homomorphism from $\mathbb{Z}$ to $\mathbb{Z}[x]$.

**$R = \mathbb{Z}/7\mathbb{Z}$ and $S = \mathbb{Z}/49\mathbb{Z}$.** Following the proof of Lemma 4.4, we see that since $\mathbb{Z}/7\mathbb{Z}$ is generated by 1, there can be at most one homomorphism: $\varphi(1) = 1$ and the fact that $\varphi$ preserves addition quickly gives $\varphi([n]_7) = [n]_{49}$. But this is not a homomorphism, since $[0]_7 = [3]_7 + [4]_7$ but

$$\varphi([3]_7) + \varphi([4]_7) = [3]_{49} + [4]_{49} = [7]_{49} \neq [0]_{49} = \varphi([0]_7).$$

So there are no homomorphisms from $\mathbb{Z}/7\mathbb{Z}$ to $\mathbb{Z}/49\mathbb{Z}$.

**$R = \mathbb{Z}/14\mathbb{Z}$ and $S = \mathbb{Z}/7\mathbb{Z}$.** The same argument as the previous part shows there can be at most one homomorphism, specicially $\varphi([n]_{14}) = [n]_7$. We claim this actually is a homormorphism. There's a lot of additions and multiplications to check, so it's easier to shift points of view and think of $\mathbb{Z}/n\mathbb{Z}$ to be the set of cosets of multiples of $n$; that is, elements of $\mathbb{Z}/n\mathbb{Z}$ are the subsets of the form:

$$r + n\mathbb{Z} = \{x \in \mathbb{Z} : x = r + nk, \text{ for some } k \in \mathbb{Z}\}$$

The homomorphism $\varphi : \mathbb{Z}/14\mathbb{Z} \to \mathbb{Z}/7\mathbb{Z}$ then can be written $\varphi(k + 14\mathbb{Z}) = k + 7\mathbb{Z}$. But one then sees:

$$\varphi(k + 14\mathbb{Z}) + \varphi(\ell + 14\mathbb{Z}) = k + 7\mathbb{Z} + \ell + 7\mathbb{Z} = k + \ell + 7\mathbb{Z} = \varphi(k + \ell + 14\mathbb{Z})$$

$$\begin{aligned}
\varphi(k + 14\mathbb{Z}) \cdot \varphi(\ell + 14\mathbb{Z}) &= (k + 7\mathbb{Z}) \cdot (\ell + 7\mathbb{Z}) \\
&= k\ell + 7k\mathbb{Z} + 7\ell\mathbb{Z} + 49\mathbb{Z} \\
&= k\ell + 7\mathbb{Z} \\
&= \varphi((k + 14\mathbb{Z}) \cdot (\ell + 14\mathbb{Z}))
\end{aligned}$$

**$R = \mathbb{Z} \times \mathbb{Z}$ and $S = \mathbb{Z}/12\mathbb{Z}$.** This question is more difficult because knowing that $\varphi(1) = 1$ and that $\varphi$ preserves addition no longer determines $\varphi$. Let $\varphi((1,0)) = a$, $\varphi((0,1)) = b$.

Suppose that $\varphi$ is a homomorphism. We see that asking $\varphi$ to be an additive group homomorphism is equivalent to asking $\varphi(x,y) = xa + by$, and so $a$ and $b$ determine $\varphi$. But not all choices of $a, b \in \mathbb{Z}/12\mathbb{Z}$ will result in $\varphi$ being a ring homomorphism. To ask that $\varphi$ preserves the identity, is asking $\varphi(1,1) = a + b = 1$.

Now suppose that $\varphi$ preserves multiplication. Since $(1,0)^2 = (1,0), (0,1)^2 = (0,1)$, and $(1,0) \cdot (0,1) = 0$, we see we must have $a^2 = a, b^2 = b, ab = 0$. We claim that if $a, b$ satisfiy these three equations, then $\varphi$ preserves multiplication:

$$\begin{aligned}
\varphi((x,y)) \cdot \varphi((v,w)) &= (ax + by) \cdot (av + bw) \\
&= a^2xy + ab(xw + vy) + b^2yw \\
&= axv + byw = \varphi(xv, yw) = \varphi((x,y) \cdot (v,w))
\end{aligned}$$

The only elements $a \in \mathbb{Z}/12\mathbb{Z}$ with $a^2 = a$ are $0, 1, 4, 9$, and this together with $a + b = 1, ab = 0$ means there are four homomorphisms $\varphi_i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$:

$$\varphi_1(x,y) = x \quad \varphi_2(x,y) = y \quad \varphi_3(x,y) = 4x + 9y \quad \varphi_4(x,y) = 9x + 4y$$