MAS439 Lecture 9
k-algebras

October 26th

Today's goal:

Understand the statement "$\mathbb{C}[x, y]$ is a finitely generated $\mathbb{C}$-algebra"

Why algebras?

$\mathbb{C}[x, y]$ is NOT a finitely generated ring!

But this is "just" because $\mathbb{C}$ is not finitely generated. We have made our peace with $\mathbb{C}$, and are no longer scared of it ($\mathbb{R}$, really). If we are willing to take $\mathbb{C}$ for granted, then to get $\mathbb{C}[x, y]$ we just need to add x and y. A primary purpose of introducing $\mathbb{C}$-algebras is to make this idea precise.

Never leave home without an algebraically closed field

We want to build in an (algebraically closed) field into our rings. $\mathbb{C}$-algebras do just that.

Formal definition of k-algebra

Let k be any commutative ring.

Definition

A k-algebra is a pair (R, ϕ), where R is a ring and $\phi : k \to S$ a morphism.

Definition

A *map of k-algebras* between $f : (R, \phi_1) \to (S, \phi_2)$ is a map of rings $f : R \to S$ such that $\phi_2 = f \circ \phi_1$, that is, the following diagram commutes:

$$\begin{tikzcd} & k \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ R \arrow[rr, "f"] & & S \end{tikzcd}$$

Important examples of k-algebras

- $k[x]$ is a k-algebra, with $\phi : k \to k[x]$ the inclusion of k as constant polynomials.
- $\mathbb{C}$ is an $\mathbb{R}$-algebra, with $\phi : \mathbb{R} \to \mathbb{C}$ the inclusion
- $\mathbb{C}$ is also a $\mathbb{C}$-algebra, with $\phi : \mathbb{C} \to \mathbb{C}$ the identity
- The ring $\mathrm{Fun}(X, R)$ of functions is an R-algebra, with $\phi : R \to \mathrm{Fun}(X, R)$ the inclusion of R as the set of constant functions
- As there is a unique homomorphism $\phi : \mathbb{Z} \to R$ to any ring R, we see that any ring R is a $\mathbb{Z}$-algebra in a unique way – that is, rings are the same thing as $\mathbb{Z}$ algebras.

Examples of maps of k-algebras

- Complex conjugation from $\mathbb{C}$ to itself is a map of $\mathbb{R}$-algebras but NOT a map of $\mathbb{C}$-algebras.
- If R is a k-algebra, and I an ideal, R/I is a k-algebra, and the quotient map $R \to R/I$ is a morphism of k-algebras
- A $\mathbb{Z}$-algebra map is just a ring homomorphism

Slogan: Algebras are rings that are vector spaces

We will usually take k to be a field. This has the following consequences:

- As maps from fields are injective, we have that $\phi : k \to R$ is injective, and so $k \subset R$ is a subring.
- The ring R becomes a vector space over k, with structure map $\lambda \cdot_{vs} r = \phi(\lambda) \cdot_R r$
- Multiplication is linear in each variable: if we fix s, then $r \mapsto r \cdot s$ and $r \mapsto s \cdot r$ are both linear maps.
- Going backwards, if V is a vector space over k, with a bilinear, associative multiplication law and a unit 1_V, then V is naturally a k-algebra, with structure map $\phi : k \to V$ defined by $\lambda \mapsto \lambda \cdot 1_V$

Finite-dimensional algebras

Definition

Let k be a field. We say a k-algebra R is *finite dimensional* if R is finite dimensional as a k-vector space.

Example

- $\mathbb{C}$ is a two dimensional $\mathbb{R}$-algebra
- $\mathbb{C}[x]/(x^n)$ is an n-dimensional $\mathbb{C}$-algebra
- $\mathbb{C}[x]$ is not a finite-dimensional $\mathbb{C}$ algebra

Toward finitely generated k-algebras

Being finite dimensional is too strong a condition to place on k-algebras for our purposes. We now define what it means to be finitely generated.

This is completely parallel to how we defined finitely generated for rings.

Subalgebras

Definition

Let (R, ϕ) be a k-algebra. A k-subalgebra is a subring S that contains $\mathrm{Im}(\phi)$.

- if S is a k-subalgebra, then in particular it is a k-algebra, where we can use the same structure map ϕ
- The inclusion map $S \hookrightarrow R$ is a k-algebra map
- To check if a subset $S \subset R$ is a subalgebra, we must check it is closed under addition and multiplication, and containts $\phi(k)$.

Generating subalgebras

Definition

Let R be a k-algebra, and $T \subset R$ a set. The subalgebra generated by T, denoted $k[T]$, is the smallest k-subalgebra of R containing T

Lemma

The elements of $k[T]$ are precisely the k-linear combinations of monomials in T; that is, elements of the form

$$\sum_{i=1}^{m} \lambda_i m_i$$

where $\lambda_i \in \phi(k)$ and m_i is a product of elements in T

Three ways of generating

Let $T \subset \mathbb{C}[x]$ be the single element x. Then

- The subring generated by x, written $\langle x \rangle$ is polynomials with integer coefficients: $\langle x \rangle = \mathbb{Z}[x] \subset \mathbb{C}[x]$.
- The ideal generated by x, written (x), are all polynomials with zero constant term
- The $\mathbb{C}$-subalgebra generated by T is the full ring $R = \mathbb{C}[x]$.

$\mathbb{C}[x, y]$ is a finitely generated $\mathbb{C}$-algebra

Definition

We say that a k-algebra R is finitely generated if we have $R = k[T]$ for some finite subset $T \subset R$.

Indeed, we have $\mathbb{C}[x, y]$ is generated as a $\mathbb{C}$ algebra by $\{x, y\}$.