ORBIFOLD HILBERT SCHEMES AND A GENERALIZATION OF CORES AND QUOTIENTS

PAUL JOHNSON

1. Introduction

[2]

2. Hilbert schemes of points on surfaces

2.1. **Colored square counting.** The core construction helps us with the colored box square counting:

Consider the function

$$P_G(q_0, \dots, q_{r-1}) = \sum_{\lambda \in \mathcal{P}} \mathbf{q}^{|\lambda|^G}$$

where

$$\mathbf{q}^{|\lambda|_G} = \prod_{i=0}^{r-1} q_i^{|\lambda|_i^G}$$

that counts partitions according to their full colored square count as opposed to just their size; alternatively, the coefficient of $\mathbf{q}^v$ is the euler characteristic of $\mathrm{Hilb}_v([\mathbb{C}^2/G])$.

What can we say about P_G? First, we consider the case where $G \subset SL_2$. Let $Q = q_0 q_1 \cdots q_{r-1}$. The core construction gives

$$P_G = \prod_{i=1}^{\infty} \frac{1}{(1-Q^i)^r} \sum_w Q^{A(w)} \mathbf{q}^w$$

Thus, we see in this case P_G is a multivariable theta function.

In case $r = 2$, this has an infinite product expansion using the Jacobi identity. In fact, work of Boulet shows that actually the case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ has an infinite product expansion. Letting $q_{00}, q_{01}, q_{10}, q_{11}$ denote the variables, we have

$$P_{\mathbb{Z}_2 \times \mathbb{Z}_2} = \prod_{i=1}^{\infty} \frac{(1 + q_{00}^j q_{01}^{j-1} q_{10}^{j-1} q_{11}^{j-1})(1 + q_{00}^j q_{01}^j q_{10}^j q_{11}^{j-1})}{(1 - q_{00}^j q_{01}^j q_{10}^j q_{11}^j)(1 - q_{00}^j q_{01}^j q_{10}^{j-1} q_{11}^{j-1})(1 - q_{00}^j q_{01}^{j-1} q_{10}^j q_{11}^{j-1})}$$

However, even the one variable specialization $P_{\mathbb{Z}_3}(q, 1, 1)$ does not have a nice infinite product expression, as observed by Balázs Szendröi [3]; as it has a root at $-e^{\pi/\sqrt{3}}$ [1].

2.2. **Hilbert schemes of points in the plane.**

Theorem 2.1 (Ellingsrud and Strømme, 1987)**.**

$$\sum_{k,n\geq 0} b_k(\mathrm{Hilb}_n(\mathbb{C}^2))t^k q^n = \prod_{\ell=1}^{\infty} \frac{1}{1-t^{2\ell-2}q^\ell}$$

2.2.1. *Białynicki-Birula decomposition.* Proved using the Białynicki-Birula decomposition. The Białynicki-Birula decomosition should be understood in analogy with Morse theory.

The role of the Morse flow will be played by the flow $x \mapsto \varepsilon x$ as $\varepsilon \in \mathbb{C}^*$ tends toward zero. We assume that the $\mathbb{C}^*$ action on X is such that this limit point exists for all $x \in X$.

Let p be a fixed point of the $\mathbb{C}^*$ action. Then linearizing the $\mathbb{C}^*$ action on X gives a $\mathbb{C}^*$ action on T_pX, and so T_pX is not just a vector space but a $\mathbb{C}^*$ representation, and hence decomposes into a direct sum of irreducible representations. Let V_a denote the irreducible representation of $\mathbb{C}^*$ where $\varepsilon \in \mathbb{C}^*$ acts as ε^a.

Let T_p^+X (respectively T_p^-X) denote the subspace of T_pX where the $\mathbb{C}^*$ action acts with a positive (respectively negative) exponent. I.e., if

$$T_pX = \bigoplus_{n\in\mathbb{Z}} V_n^{e_n}$$

then

Let

$$\mathcal{S}_p = \{x \in X | \lim_{\varepsilon\to 0} \varepsilon x = p\}$$

Then clearly we have $X = \sqcup_p \mathcal{S}_p$; the point is that $\mathcal{S}_p$ is a subvariety isomorphic to $\mathbb{C}^{\iota(p)}$.

We first observe that V_0 cannot occur in T_pX, as

We saw in Section [REFER BACK TO] that if X is a variety with a $\mathbb{C}^*$ action with k isolated fixed points, then $\chi(X) = k$. If we know the weights of the $\mathbb{C}^*$ action on the tangent spaces of the fixed points, the Białynicki-Birula decomposition leverages this result to give the betti numbers of X, or even further the class of X in the Grothendieck ring of varieties.

2.2.2. *Tangent space to a monomial ideal.* To apply the Białynicki-Birula decomposition to $\mathrm{Hilb}_n(\mathbb{C}^2)$, we need to calculate the weights of the $\mathbb{C}^*$ action on $T_\lambda \mathrm{Hilb}_n(\mathbb{C}^2)$.

Lemma 2.2 (Ellingsrud and Strømme, Cheah)**.**

$$T_\lambda \mathrm{Hilb}_n(\mathbb{C}^2) = \sum_{\square\in\lambda} \left(x^{-\ell(\square)} y^{a(\square)+1} + x^{\ell(\square)+1} y^{-a(\square)} \right)$$

Proof. First, we use $T_\lambda \operatorname{Hilb}_n(\mathbb{C}^2) = \operatorname{Hom}_R(\mathcal{I}_\lambda, R/\mathcal{I}_\lambda)$.

Analogy with the Grassmannian makes this intuitively plausible; indeed, if $V \subset W$ is a k dimensional subspace, then $T_V Gr_k(W) = \operatorname{Hom}(V, V^\perp) = \operatorname{Hom}(V, W/V)$. Thus, we understand the deformations of $\mathcal{I}_\lambda$ as a vector space; for the deformation to remain an ideal it is plausible that we should require the deformation to be a map of R modules and not just vector spaces.

More formally, it is a general fact that first order deformations of objects are given by $\operatorname{Ext}^1(\mathcal{F}, \mathcal{F})$, and obstructions to these deformations are given by $\operatorname{Ext}^2(\mathcal{F}, \mathcal{F})$; starting from this fact and considering the long exact sequences by taking $\operatorname{Hom}(\mathcal{I})$ to

$$0 \to \mathcal{I} \to R \to R/\mathcal{I} \to 0$$

gives the result. $\square$

References

[1] J. M. Borwein and P. B. Borwein. A cubic counterpart of Jacobi's identity and the AGM. *Trans. Amer. Math. Soc.*, 323(2):691–701, 1991. http://www.cecm.sfu.ca/personal/pborwein/PAPERS/P50.pdf.

[2] S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández. A power structure over the Grothendieck ring of varieties. *Math. Res. Lett.*, 11(1):49–57, 2004.

[3] Balázs Szendrői. Math overflow answer. http://mathoverflow.net/a/202902/1102.

University of Sheffield

E-mail address: paul.johnson@sheffield.ac.uk