Topology and combinatorics of Hilbert schemes of points on orbifolds

Paul Johnson

Colorado State University
www.math.colostate.edu/~johnson

March 11, 2013 UC Davis

Goal:

Topology of $\mathrm{Hilb}_n([\mathbb{C}^2/G])$

Warm-up:

Topology of $\mathrm{Hilb}_n(\mathbb{C}^2)$

The Hilbert scheme of points in the plane

Let $R = \mathbb{C}[x, y]$. Then:

$$\mathrm{Hilb}_n(\mathbb{C}^2) := \{\text{ideals } \mathcal{I} \subset R \,|\, \dim R/\mathcal{I} = n\}$$

- Generically $\mathcal{I}$ will be the ideal sheaf of n distinct points in $\mathbb{C}^2$
- When two or more points collide they become a "fat point" that "remembers" how they collided
- $\mathrm{Hilb}_n(\mathbb{C}^2)$ is connected and smooth of dimension $2n$

Question: What are the Betti numbers of $\mathrm{Hilb}_n(\mathbb{C}^2)$?

Warm-up to the warm-up: Euler-characteristic

Before we find the Betti numbers let's find $\chi(\mathrm{Hilb}_n(\mathbb{C}^2))$:

- The action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^2$ induces a $(\mathbb{C}^*)^2$ action on $\mathrm{Hilb}_n(\mathbb{C}^2)$
- The fixed points of the $(\mathbb{C}^*)^2$ action are the monomial ideals
- Since $\chi(\mathbb{C}^*) = \chi((\mathbb{C}^*)^2) = 0$, the non-fixed orbits contribute nothing to the Euler characteristic

So $\chi(\mathrm{Hilb}_n(\mathbb{C}^2))$ is the number of monomial ideals of length n.

How many monomial ideals of length n are there?

Bijection between monomial ideals and partitions

Monomials not in $\mathcal{I}$ are the cells of the partition. Exterior corners of the partition are the generators of the monomial ideal.

x^0y^3	x^1y^3	x^2y^3	x^3y^3	x^4y^3
x^0y^2	x^1y^2	x^2y^2	x^3y^2	x^4y^2
x^0y^1	x^1y^1	x^2y^1	x^3y^1	x^4y^1
x^0y^0	x^1y^0	x^2y^0	x^3y^0	x^4y^0

$\mathcal{I}$	λ
(x^3, xy, y^2)	$(2, 1, 1)$

So $\chi(\mathrm{Hilb}_n(\mathbb{C}^2)) = p(n)$.

Betti numbers of $\mathrm{Hilb}_n(\mathbb{C}^2)$

Important idea: it helps to consider $\mathrm{Hilb}_n(S)$ for all n at once':

Theorem (Warm-up)

$$\sum_{n\geq 0} \chi(\mathrm{Hilb}_n(\mathbb{C}^2))q^n = \sum_{n\geq 0} p(n)q^n = \prod_{\ell\geq 1} \frac{1}{1-q^\ell}$$

Theorem (Ellingsrud and Strømme, 1987)

$$\sum_{k,n\geq 0} b_k(\mathrm{Hilb}_n(\mathbb{C}^2))t^k q^n = \prod_{\ell=1}^{\infty} \frac{1}{1-t^{2\ell-2}q^\ell}$$

Main tool is the Białynicki-Birula decomposition

Białynicki-Birula decomposition $\approx$ Morse theory

Suppose X has a $\mathbb{C}^*$ action so that

1. $\lim_{\lambda \to 0} \lambda x$ exists for all $x \in X$
2. There are isolated fixed points

Then we can compute the homology of X by "Morse theory"

1. $x \mapsto \lambda x$ is the gradient flow
2. Fixed points are critical points

What's the Morse index of a fixed point p?

Morse index $= 2 \dim T_p^- X$

At each fixed point p, $T_p X$ is a $\mathbb{C}^*$ representation, and so splits into eigenspaces where $\lambda v = \lambda^a v$

- $a = 0$ Can't occur since fixed points are isolated
- $a > 0$ Flowing toward p
- $a < 0$ Flowing away from p

$T_p^- X$ is the subspace where $a < 0$.

Theorem

Białynicki-Birula

$$P_t(X) = \sum_{p \text{ fixed}} t^{\mathit{index}(p)}$$

Proof.

The differential is zero since all fixed points have even index. $\square$

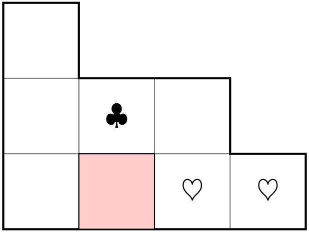
Proof of Ellingsrud and Strømme

Tangent spaces at fixed points

Lemma (Ellingsrud and Strømme, Cheah)

$$T_\lambda \operatorname{Hilb}_n(\mathbb{C}^2) = \sum_{\Box \in \lambda} \left(x^{-\ell(\Box)} y^{a(\Box)+1} + x^{\ell(\Box)+1} y^{-a(\Box)} \right)$$

Here $a(\Box)$ and $\ell(\Box)$ are the arm and leg of the square:



$$a(\Box) = \#\clubsuit = 1$$

$$\ell(\Box) = \#\heartsuit = 2$$

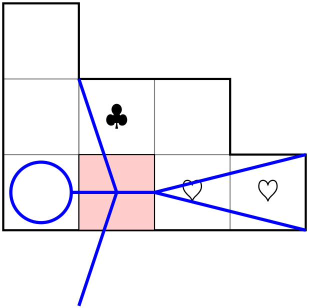
Proof of Ellingsrud and Strømme

Tangent spaces at fixed points

Lemma (Ellingsrud and Strømme, Cheah)

$$T_\lambda \operatorname{Hilb}_n(\mathbb{C}^2) = \sum_{\square \in \lambda} \left(x^{-\ell(\square)} y^{a(\square)+1} + x^{\ell(\square)+1} y^{-a(\square)} \right)$$

Here $a(\square)$ and $\ell(\square)$ are the arm and leg of the square:



$a(\square) = \#\clubsuit = 1$

$\ell(\square) = \#\heartsuit = 2$

Proof of Ellingsrud and Strømme

Putting everything together

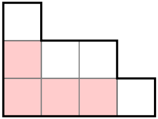
Pick a $\mathbb{C}^* \subset (\mathbb{C}^*)^2$

Use the $\mathbb{C}^*$ acting by

$$\lambda \cdot (x, y) = (\lambda^{\epsilon} x, \lambda y)$$

With $0 < \epsilon << 1$.

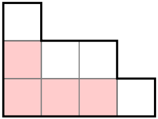
- $x^{-\ell(\square)} y^{a(\square)+1} \mapsto \lambda^{1+a(\square)-\epsilon\ell(\square)}$ is always positive
- $x^{\ell(\square)+1} y^{-a(\square)} \mapsto \lambda^{-a(\square)+\epsilon(1+\ell(\square))}$ negative when $a(\square) > 0$.



Morse index $= 2$ $\#$ red boxes

Proof of Ellingsrud and Strømme

Putting everything together



Morse index $= 2\ \#$ red boxes

A column of height h contributes $q^h t^{2h-2}$

$$\sum_{k,n\geq 0} b_k(\mathrm{Hilb}_n(\mathbb{C}^2))t^k q^n = \prod_{\ell=1}^{\infty} \frac{1}{1 - t^{2\ell-2}q^\ell} \qquad \square$$

Göttsche's formula: $\mathbb{C}^2 \rightsquigarrow S$:

Three Theorems

1. Product Formula
2. Homological Stability
3. Heisenberg Action

Theorem 1: Product Formula

Let S be a smooth quasi-projective surface with Betti numbers b_i.
Let $S^{[n]} = \mathrm{Hilb}_n(S)$

Theorem (Göttsche, 1990)

$$\sum_{k,n} b_k(S^{[n]}) t^k q^n = \prod_{\ell \geq 1} \frac{(1+t^{2\ell-1}q^\ell)^{b_1}(1+t^{2\ell+1}q^\ell)^{b_3}}{(1-t^{2\ell-2}q^\ell)^{b_0}(1-t^{2\ell}q^\ell)^{b_2}(1-t^{2\ell+2}q^\ell)^{b_4}}$$

Proof.
Reduce to case $S = \mathbb{C}^2$ using Weil conjectures $\square$

Theorem 2: Homological Stability

Theorem (Göttsche, 1990)

$$\sum_{k,n} b_k(S^{[n]})t^k q^n = \prod_{\ell \geq 1} \frac{(1+t^{2\ell-1}q^\ell)^{b_1}(1+t^{2\ell+1}q^\ell)^{b_3}}{(1-t^{2\ell-2}q^\ell)^{b_0}(1-t^{2\ell}q^\ell)^{b_2}(1-t^{2\ell+2}q^\ell)^{b_4}}$$

Corollary

Suppose S is connected. Then for fixed k and large n, $b_k(S^{[n]})$ stabilizes

Proof.

Exactly one factor with just q's and no t's:

$$\frac{1}{1-q}$$

□

Theorem 3: Heisenberg Action

Theorem (Göttsche, 1990)

$$\sum_{k,n} b_k(S^{[n]})t^k q^n = \prod_{\ell \geq 1} \frac{(1+t^{2\ell-1}q^\ell)^{b_1}(1+t^{2\ell+1}q^\ell)^{b_3}}{(1-t^{2\ell-2}q^\ell)^{b_0}(1-t^{2\ell}q^\ell)^{b_2}(1-t^{2\ell+2}q^\ell)^{b_4}}$$

Theorem (Nakajima, Grojnowski)

$\bigoplus H_k(\mathrm{Hilb}_n(S))$ *is a highest weight representation for a Heisenberg algebra modeled on* $H^*(S)$.

This reproves and categorifies Göttsche

Our Results: $S \rightsquigarrow [\mathbb{C}^2/\mathbb{Z}_r]$

1. Conjecture analogs of all 3
2. No obvious implications
3. Prove Homological stability.

$G \subset SL_2(\mathbb{C})$ already well studied

- $[\mathbb{C}^2/G]$, its minimal resolution, S_G, and any $\mathrm{Hilb}_n([\mathbb{C}^2/G])$ are all holomorphic symplectic
- McKay correspondence: ADE classification of G; exceptional divisor in S_G is the corresponding Dynkin diagram
- Every component of any $\mathrm{Hilb}_n([\mathbb{C}^2/G])$ is diffeomorphic to some $\mathrm{Hilb}_m(S_G)$; all connected by wall crossing
- Heisenberg action of Nakajima-Grojnowski is part of an action of the corresponding quantum group
- In the A_n case, these are also related to a construction in the combinatorics of partitions known as cores and quotients

When $G \not\subseteq SL_2(\mathbb{C})$, much less is known

When G is abelian, localization still works, and a modification of Ellingsrud-Strømme computes $b_k([\mathbb{C}^2/G])$ as a (q,t) count of partitions. A few lines in Sage give a vast amount of data to analyze.

Guesein-Zade, Luengo, Melle-Hernández

Conjectured product formula for $G = \mathbb{Z}_3, \mathbb{Z}_4$

Back to Earth:

Understanding $\mathrm{Hilb}_n([\mathbb{C}^2/G])$

Orbifold Hilbert Schemes are fixed point sets

$$\begin{aligned}\mathsf{Hilb}_n([\mathbb{C}^2/G]) &:= \{G\text{-equivariant ideals } \mathcal{I} \subset R\} \\ &= \mathsf{Hilb}_n(\mathbb{C}^2)^G \subset \mathsf{Hilb}_n(\mathbb{C}^2)\end{aligned}$$

- $\mathsf{Hilb}_n([\mathbb{C}^2/G])$ is smooth: it's a fixed point set in something smooth
- $\mathsf{Hilb}_n([\mathbb{C}^2/G]$ is not connected. One discrete invariant: $R/\mathcal{I}$ isn't just a vector space, it's a representation of G
- This is the only discrete invariant

For $\kappa \in K_0(G)$, let Hilb^κ_G denote the component where $R/\mathcal{I} = \kappa$. Then Hilb^κ_G is connected.

How to calculate Betti numbers?

Follow proof of Ellingsrud-Strømme, but the index of each partition will change:

Lemma (Ellingsrud and Strømme, Cheah)

$$T_\lambda \operatorname{Hilb}_n(\mathbb{C}^2) = \sum_{\square \in \lambda} \left(x^{-\ell(\square)} y^{a(\square)+1} + x^{\ell(\square)+1} y^{-a(\square)} \right)$$

A tangent direction only contributes to $T_\lambda \operatorname{Hilb}_n([\mathbb{C}^2/G])$ if it is G-invariant.

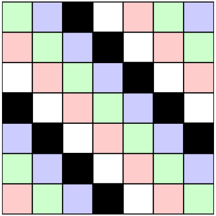
Example (Balanced $\mathbb{Z}_r$ action)

A generator acts as $(-1/r, 1/r)$, so we need $\ell(\square) + a(\square) + 1$ to be divisible by r

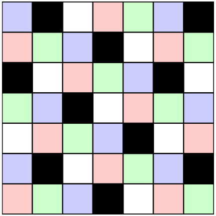
Colored boxes

Restrict to $G = \mathbb{Z}/r\mathbb{Z}$, with action $(\exp(2\pi i/r), \exp(2\pi i m/r))$. For a monomial ideal, keeping track of $K_0(G)$ class is counting colored boxes:

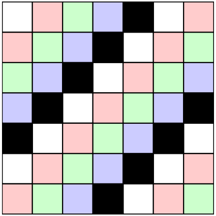
(1/5,1/5)



(1/5,2/5)



(1/5,-1/5)



Example: $\mathrm{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_3])$

Let $\mathbb{Z}_3$ act on $\mathbb{C}^2$ diagonally: $g \cdot (x, y) = (\omega x, \omega y)$.

- $\mathrm{Hilb}_1([\mathbb{C}^2/Z_3]) = \{(0,0)\}$
- $\mathrm{Hilb}_2([\mathbb{C}^2/Z_3]) = \mathbb{P}^1$
  Let v be a tangent direction at the origin:

$$\mathcal{I}_v = \{f \in R | f(0) = \partial_v f(0) = 0\}$$

- $\mathrm{Hilb}_3([\mathbb{C}^2/Z_3])$ has two components. One component is just an isolated point $\mathfrak{m}_0^2 = (x^2, xy, y^2)$

What's $R/\mathfrak{m}_0^2$ as a $\mathbb{Z}_3$ representation?

$\mathbb{Z}_3$ acts on 1 trivially

Acts as the same nontrivial representation on x and y

The other component is the minimal resolution

Let $p \neq (0,0) \in \mathbb{C}^2$. Its orbit consists of 3 points; let $\mathcal{I}$ be the ideal sheaf of these three points. Then $R/\mathcal{I}$ has the regular representation of G.

Over the origin, there are a $\mathbb{P}^1$ worth of ideals that give the regular representation:

$$\mathcal{I}_v^2 = \{f \in R | f(0) = \partial_v f(0) = \partial_v^2 f(0) = 0\}$$

This component is $\mathcal{O}(-3) \to \mathbb{P}^1$, the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_3$.

Hilb_G^G (often called $G\,\mathrm{Hilb}$) always gives the minimal resolution

Special McKay Correspondence

When S is smooth, $\mathrm{Hilb}^1(S) = S$, but $\mathrm{Hilb}^1([\mathbb{C}^2/G]) = \mathrm{point}$. The ideal sheaf of a smooth point on $[\mathbb{C}^2/G]$ corresponds to the regular representation of G.

Theorem

Hilb^G_G *is the minimal resolution of* $\mathbb{C}^2/G$.

- The minimal resolution of $\mathbb{C}^2/G$ is a tree of c rational curves
- When $G \subset SL_2, c = |G| - 1$, and so $\chi(\mathrm{Hilb}^G_G) = |G|$
- Otherwise, $c < |G| - 1$, and Hilb^G_G only sees a subset of the irreducible representations of G

Generating series for orbifold Hilbert schemes

Restrict to $G = \mathbb{Z}/r\mathbb{Z}$, with action $(\exp(2\pi i/r), \exp(2\pi i m/r))$.

Disconnected generating series

$$\mathcal{DH}_{m/r} := \sum_{n,k \geq 0} b_k(\mathrm{Hilb}_n([\mathbb{C}^2/G]))t^k q^n$$

Call an element $\delta \in K_0(G)$ small if Hilb_G^δ is nonempty but compact; equivalently, if Hilb_G^δ is nonempty but $\mathrm{Hilb}_G^{\delta - G}$ is empty.

Connected generating series

For $\delta \in K_0(G)$ small, define

$$\mathcal{CH}_{m/r}^\delta := \sum_{n,k \geq 0} b_k(\mathrm{Hilb}_G^{\delta + nG})t^k q^n$$

First Conjectural Product formula

Recall $(a;x)_\infty := \prod_{\ell \geq 0}(1 - ax^\ell)$.

Example (Göttsche)

$$\sum_{n\geq 0} b_k(\mathrm{Hilb}_n(S))t^k q^n = \frac{1}{(q;qt^2)_\infty^{b_0}} \frac{1}{(qt^2;qt^2)_\infty^{b_2}} \frac{1}{(qt^4;qt^2)_\infty^{b_4}}$$

Conjecture (Gusein-Zade, Luengo, Melle-Hernández)

$$\mathcal{DH}_{1/3} = \frac{1}{(q;t^2q^3)_\infty} \frac{1}{(q^2t^2;t^2q^3)_\infty} \frac{1}{(q^3;t^2q^3)_\infty}$$

Why stop there?

Intuition for conjectural product formula

It seems if $G \cap SL_2 = \{1\}$ then

$$\mathcal{DH}_G = \prod_{h=1}^{r} \frac{1}{(q^h t^{\epsilon(h)}; q^r t^2)_\infty}$$

with $\epsilon(h)$ either 2 or 0.

Question: what's $\epsilon(h)$?

In Göttsche's formula, $\epsilon(h) = 0$ corresponds to b_0, and $\epsilon(h) = 2$ corresponds to b_2.

Chen-Ruan cohomology

The Chen-Ruan cohomology of $[\mathbb{C}^2/G]$ is rationally graded, with d with $0 \leq d < 4$.

Idea: Round down the degree in Chen-Ruan cohomology to either 0 or 2

Chen-Ruan cohomology of $[\mathbb{C}^2/G]$

For G abelian:

- Basis given by the elements of G
- If g acts as $(\exp(2\pi ia/r), \exp(2\pi ib/r))$, the age of g is $\iota(g) = a/r + b/r$
- The degree of g is twice the age.

Formal statement of conjectural product formula

Let $F(g)$ and $I(g)$ denote the fractional and integral parts of $\iota(g)$. If $G \cap SL_2 = \{1\}$, then $F(G)$ gives a bijection between G and $\{0, 1/r, \ldots, (r-1)/r\}$.

Conjecture (Johnson)

Let G be cyclic, and define $k = |G \cap SL_2|$

$$\mathcal{H}_G(q,t) = \frac{(q^k;q^k)_\infty^k}{(q,q)_\infty} \prod_{g \in G} \frac{1}{(q^{r(1-F(g))}t^{2I(g)}, q^r t^2)_\infty}$$

Analog of Theorem 2: Homological Stability

The analogs of stabilization and geometric representation theory work on the level of connected Hilbert scheme.

Theorem (Johnson)

$P_t(\mathrm{Hilb}_G^{\delta+nG})$ stabilizes to $1/(t,t)_\infty^{|G|}$

Note that the right hand side is independent of m and δ.

Proof.

Combinatorics – a generalization of cores and quotients of partitions □

Conjecture (Johnson)

The stable cohomology of $\mathrm{Hilb}^{\delta+nG}$ is freely generated by the Chern classes of the $|G|$ tautological bundles.

Analog of Theorem 3: Heisenberg Action

Conjecture (Johnson)

Let $\delta \in K_0(G)$ be small, and G cyclic. Then

$$\bigoplus_{k \geq 0} H_*(\mathrm{Hilb}_G^{\delta + kG})$$

admits the action of a Heisenberg algebra based on the cohomology of the minimal resolution of $\mathbb{C}^2/G$.

Evidence:

Let c be the number of rational curves in the minimal resolution of $\mathbb{C}^2/G$. Then

$$\mathcal{CH}_G^{\delta} \cdot (q, qt^2)_\infty \cdot (qt^2, qt^2)_\infty^c$$

has positive coefficients; but higher powers start giving negative coefficients.

Thank you