ORBIFOLD HILBERT SCHEMES AND A GENERALIZATION OF CORES AND QUOTIENTS

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1. Introduction

[1]

2. Hilbert schemes of points on surfaces

2.1. Hilbert schemes of points in the plane.

Theorem 2.1 (Ellingsrud and Strømme, 1987)**.**

$$\sum_{k,n\geq 0} b_k(\mathrm{Hilb}_n(\mathbb{C}^2))t^k q^n = \prod_{\ell=1}^{\infty} \frac{1}{1-t^{2\ell-2}q^{\ell}}$$

2.1.1. *Białynicki-Birula decomposition.* Proved using the Białynicki-Birula decomposition. The Białynicki-Birula decomosition should be understood in analogy with Morse theory.

The role of the Morse flow will be played by the flow $x \mapsto \varepsilon x$ as $\varepsilon \in \mathbb{C}^*$ tends toward zero. We assume that the $\mathbb{C}^*$ action on X is such that this limit point exists for all $x \in X$.

Let p be a fixed point of the $\mathbb{C}^*$ action. Then linearizing the $\mathbb{C}^*$ action on X gives a $\mathbb{C}^*$ action on T_pX, and so T_pX is not just a vector space but a $\mathbb{C}^*$ representation, and hence decomposes into a direct sum of irreducible representations. Let V_a denote the irreducible representation of $\mathbb{C}^*$ where $\varepsilon \in \mathbb{C}^*$ acts as ε^a.

Let T_p^+X (respectively T_p^-X) denote the subspace of T_pX where the $\mathbb{C}^*$ action acts with a positive (respectively negative) exponent. I.e., if

$$T_pX = \bigoplus_{n\in\mathbb{Z}} V_n^{e_n}$$

then

Let

$$\mathcal{S}_p = \{x \in X | \lim_{\varepsilon\to 0} \varepsilon x = p\}$$

Then clearly we have $X = \sqcup_p \mathcal{S}_p$; the point is that $\mathcal{S}_p$ is a subvariety isomorphic to $\mathbb{C}^{\iota(p)}$.

We first observe that V_0 cannot occur in T_pX, as

We saw in Section [REFER BACK TO] that if X is a variety with a $\mathbb{C}^*$ action with k isolated fixed points, then $\chi(X) = k$. If we know the weights of the $\mathbb{C}^*$ action on the tangent spaces of the fixed points, the Białynicki-Birula decomposition leverages this result to give the betti numbers of X, or even further the class of X in the Grothendieck ring of varieties.

2.1.2. *Tangent space to a monomial ideal.* To apply the Białynicki-Birula decomposition to $\mathrm{Hilb}_n(\mathbb{C}^2)$, we need to calculate the weights of the $\mathbb{C}^*$ action on $T_\lambda \mathrm{Hilb}_n(\mathbb{C}^2)$.

[Ellingsrud and Strømme, Cheah]

$$T_\lambda \mathrm{Hilb}_n(\mathbb{C}^2) = \sum_{\square \in \lambda} \left(x^{-\ell(\square)} y^{a(\square)+1} + x^{\ell(\square)+1} y^{-a(\square)} \right)$$

Proof. First, we use $T_\lambda \mathrm{Hilb}_n(\mathbb{C}^2) =_R (\mathcal{I}_\lambda, R/\mathcal{I}_\lambda)$.

Analogy with the Grassmannian makes this intuitively plausible; indeed, if $V \subset W$ is a k dimensional subspace, then $T_V Gr_k(W) = (V, V^\perp) = (V, W/V)$. Thus, we understand the deformations of $\mathcal{I}_\lambda$ as a vector space; for the deformation to remain an ideal it is plausible that we should require the deformation to be a map of R modules and not just vector spaces.

More formally, it is a general fact that first order deformations of objects are given by $^1(\mathcal{F}, \mathcal{F})$, and obstructions to these deformations are given by $^2(\mathcal{F}, \mathcal{F})$; starting from this fact and considering the long exact sequences by taking $(\mathcal{I})$ to

$$0 \to \mathcal{I} \to R \to R/\mathcal{I} \to 0$$

gives the result.

□

References

[1] S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández. A power structure over the Grothendieck ring of varieties. *Math. Res. Lett.*, 11(1):49–57, 2004.

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