The topology of Hilbert schemes of points on orbifolds

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Outline

Goal:
Understand Betti numbers of $\mathrm{Hilb}^n([\mathbb{C}^2/G])$

1. Betti numbers of $\mathrm{Hilb}^n(\mathbb{C}^2)$
   - Ellingsrud and Strømme
2. Betti numbers of $\mathrm{Hilb}^n([\mathbb{C}^2/G])$ with $G \subset SL_2$
   - Gusein-Zade, Luengo, Melle-Hernández
3. Betti numbers of general case
   - Me (some conjectures, some theorems)

Basics of the Hilbert scheme of points on $\mathbb{C}^2$

Let $R = \mathbb{C}[x, y]$. Then:

$$\mathrm{Hilb}^n(\mathbb{C}^2) := \{\text{ideals } \mathcal{I} \subset R \mid \dim R/\mathcal{I} = n\}$$

- $\mathrm{Hilb}^n(\mathbb{C}^2)$ is smooth and connected of dimension $2n$.
- Generically, $\mathcal{I}$ will be the ideal sheaf of n distinct points in $\mathbb{C}^2$, so $\dim \mathrm{Hilb}^n(\mathbb{C}^2) = 2n$.
- When two or more points collide they become a "fat point" that remembers some information about how they collided.

Warm-up: Euler-characteristic of $\mathrm{Hilb}^n(\mathbb{C}^2)$

Before we find the Betti numbers let's find $\chi(\mathrm{Hilb}^n(\mathbb{C}^2))$:

- The action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^2$ induces a $(\mathbb{C}^*)^2$ action on $\mathrm{Hilb}^n(\mathbb{C}^2)$
- The fixed points of the $(\mathbb{C}^*)^2$ action are the monomial ideals
- Since $\chi(\mathbb{C}^*) = \chi((\mathbb{C}^*)^2) = 0$, the non-fixed orbits contribute nothing to the euler characteristic

So $\chi(\mathrm{Hilb}^n(\mathbb{C}^2))$ is the number ofmonomial ideals of length n.

How many monomial ideals of length n are there?

Bijection between monomial ideals and partitions

Monomials not in $\mathcal{I}$ are the cells of the partition. Exterior corners of the partition are the generators of the monomial ideal.

x^0y^3	x^1y^3	x^2y^3	x^3y^3	x^4y^3
x^0y^2	x^1y^2	x^2y^2	x^3y^2	x^4y^2
x^0y^1	x^1y^1	x^2y^1	x^3y^1	x^4y^1
x^0y^0	x^1y^0	x^2y^0	x^3y^0	x^4y^0

$\mathcal{I}$	λ
(x^3, xy, y^2)	$(2,1,1)$

So $\chi(\mathrm{Hilb}^n(\mathbb{C}^2)) = p(n)$.

Main motivating theorem

Packaged into generating functions:

Theorem (Warm-up)

$$\sum_{n\geq 0} \chi(Hilb^n(\mathbb{C}^2))q^n = \sum_{n\geq 0} p(n)q^n = \prod_{\ell\geq 1} \frac{1}{1-q^\ell}$$

Theorem (Ellingsrud and Strømme, 1987)

$$\sum_{k,n\geq 0} b_k(Hilb^n(\mathbb{C}^2))t^k q^n = \prod_{\ell=1}^{\infty} \frac{1}{1-t^{2\ell-2}q^\ell}$$

Proof

Main tool is the Białynicki-Birula decomposition

Białynicki-Birula decomposition $\approx$ Morse theory

Suppose X has a $\mathbb{C}^*$ action so that

1. $\lim_{\lambda \to 0} \lambda x$ exists for all $x \in X$
2. There are isolated fixed points

Then we can compute the homology of X by thinking of $x \mapsto \lambda x$ as $\lambda \to 0$ as a Morse flow, with the fixed points p acting as the critical points.

What's the Morse index of a fixed point p?

Morse index $= 2\dim T_p^- X$

At each fixed point p, T_pX is a $\mathbb{C}^*$ representation, and so splits into eigenspaces where $\lambda v = \lambda^a v$

$a = 0$ Can't occur since fixed points are isolated

$a > 0$ Flowing toward p

$a < 0$ Flowing away from p

$T_p^- X$ is the subspace where $a < 0$.

Theorem

Białynicki-Birula

$$P_t(X) = \sum_{p \text{ fixed}} t^{index(p)}$$

Proof.

The differential is zero since all fixed points have even index. $\square$

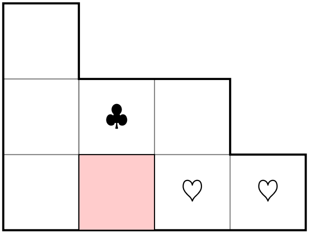
Proof of Ellingsrud and Strømme

Tangent spaces at fixed points

Lemma (Ellingsrud and Strømme, Cheah)

$$T_\lambda Hilb^n(\mathbb{C}^2) = \sum_{\square \in \lambda} \left(x^{-\ell(\square)} y^{a(\square)+1} + x^{\ell(\square)+1} y^{-a(\square)} \right)$$

Here $a(\square)$ and $\ell(\square)$ are the arm and leg of the square:



$$a(\square) = \#\clubsuit = 1$$

$$\ell(\square) = \#\heartsuit = 2$$

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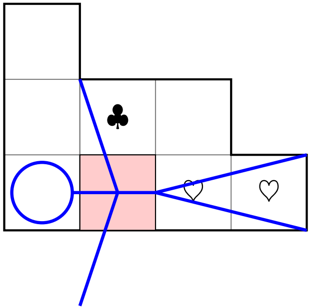

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Proof of Ellingsrud and Strømme

Putting everything together

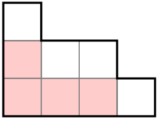
Pick a $\mathbb{C}^* \subset (\mathbb{C}^*)^2$

Use the $\mathbb{C}^*$ acting by

$$\lambda \cdot (x, y) = (\lambda^{\epsilon} x, \lambda y)$$

With $0 < \epsilon << 1$.

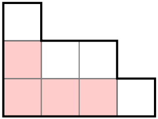
- $x^{-\ell(\square)} y^{a(\square)+1} \mapsto \lambda^{1+a(\square)-\epsilon\ell(\square)}$ is always positive
- $x^{\ell(\square)+1} y^{-a(\square)} \mapsto \lambda^{-a(\square)+\epsilon(1+\ell(\square))}$ negative when $a(\square) > 0$.



Morse index $= 2$ $\#$ red boxes

Proof of Ellingsrud and Strømme

Putting everything together



Morse index $= 2 \#$ red boxes

A column of height h contributes $q^h t^{2h-2}$

$$\sum_{k,n\geq 0} b_k(\mathrm{Hilb}^n(\mathbb{C}^2))t^k q^n = \prod_{\ell=1}^{\infty} \frac{1}{1-t^{2\ell-2}q^{\ell}} \qquad \square$$

Building on Ellingsrud-Strømme

Theorem (Göttsche, 1990)

Let S be a smooth quasi-projective surface with Betti numbers b_i. Let $S^{(n)} = \mathcal{H}ilb^n(S)$. Then

$$\sum b_k(S^{(n)}) t^k q^n = \prod_{\ell \geq 1} \frac{(1 + t^{2\ell-1} q^\ell)^{b_1} (1 + t^{2\ell+1} q^\ell)^{b_3}}{(1 - t^{2\ell-2} q^\ell)^{b_0} (1 - t^{2\ell} q^\ell)^{b_2} (1 - t^{2\ell+2} q^\ell)^{b_4}}$$

Proof.

Ellingsrud and Strømme + Weil Conjectures $\square$

Theorem (Nakajima, Grojnowski)

$\bigoplus H_k(\mathcal{H}ilb^n(S))$ is a highest weight representation for a Heisenberg algebra generated by $H^*(S)$.

Orbifold Hilbert Schemes are fixed point sets

$$\begin{aligned} \mathrm{Hilb}^n([\mathbb{C}^2/G]) &:= \{G\text{-equivariant ideals } \mathcal{I} \subset R\} \\ &= \mathrm{Hilb}^n(\mathbb{C}^2)^G \subset \mathrm{Hilb}^n(\mathbb{C}^2) \end{aligned}$$

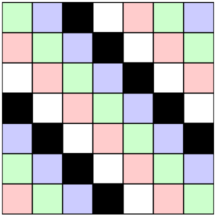
- As a fixed point set in a smooth variety, we see $\mathrm{Hilb}^n([\mathbb{C}^2/G])$ is smooth.
- Not connected. One discrete invariant: $R/\mathcal{I}$ isn't just a vector space, it's a representation of G.

For $v \in K_0(G)$, let Hilb_G^v denote the component where $R/\mathcal{I} = v$.

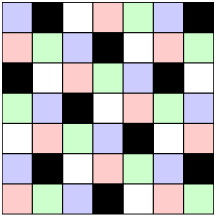
Colo(u)red boxes

Restrict to $G = \mathbb{Z}/r\mathbb{Z}$, with action $(\exp(2\pi i/r), \exp(2\pi i m/r))$. For a monomial ideal, keeping track of $K_0(G)$ class is counting colored boxes:

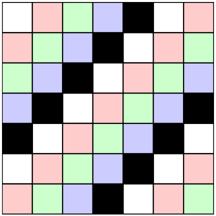
(1/5,1/5)



(1/5,2/5)



(1/5,-1/5)



Special McKay Correspondence

When S is smooth, $\mathrm{Hilb}^1(S) = S$, but $\mathrm{Hilb}^1([\mathbb{C}^2/G]) = \mathrm{point}$. The ideal sheaf of a smooth point on $[\mathbb{C}^2/G]$ corresponds to the regular representation of G.

Theorem

$Hilb_G^G([\mathbb{C}^2/G])$ is the minimal resolution of $\mathbb{C}^2/G$.

- The minimal resolution of $\mathbb{C}^2/G$ is a chain of c rational curves
- When $G \subset SL_2, c = |G| - 1$, and so $\chi(\mathrm{Hilb}_G^G([\mathbb{C}^2/G]) = |G|$
- Otherwise, $c < |G| - 1$, and $\mathrm{Hilb}_G^G([\mathbb{C}^2/G])$ only sees some of $K_0(G)$

Generating series for orbifold Hilbert schemes

Restrict to $G = \mathbb{Z}/r\mathbb{Z}$, with action $(\exp(2\pi i/r), \exp(2\pi im/r))$.

Disconnected generating series

$$\mathcal{DH}_{m/r} := \sum_{n,k\geq 0} b_k(\mathrm{Hilb}^n([\mathbb{C}^2/G]))t^k q^n$$

Call an element $\delta \in K_0(G)$ small if Hilb^{δ}_G is nonempty but compact; equivalently, if it is nonempty but $\mathrm{Hilb}^{\delta-G}_G$ is empty.

Connected generating series

For $\delta \in K_0(G)$ small, define

$$\mathcal{CH}^{\delta}_{m/r} := \sum_{n,k\geq 0} b_k(\mathrm{Hilb}^{\delta+nG}([\mathbb{C}^2/G]))t^k q^n$$

How to calculate these Betti numbers?

Follow proof of Ellingsrud-Strømme, but the index of each partition will change:

Lemma (Ellingsrud and Strømme, Cheah)

$$T_\lambda Hilb^n(\mathbb{C}^2) = \sum_{\square \in \lambda} \left(x^{-\ell(\square)} y^{a(\square)+1} + x^{\ell(\square)+1} y^{-a(\square)} \right)$$

A tangent direction only contributes to $T_\lambda \mathrm{Hilb}^n([\mathbb{C}^2/G])$ if it is G-invariant.

Example (Balanced $\mathbb{Z}_r$ action)

A generator acts as $(-1/r, 1/r)$, so we need $\ell(\square) + a(\square) + 1$ to be divisible by r

Theorem (Gusein-Zade, Luengo, Melle-Hernández)

$$\mathcal{H}^0_{-1/r} = \prod_{\ell \geq 1} \frac{1}{1 - t^{2\ell-2} q^{\ell}} \frac{1}{(1 - t^{2\ell} q^{\ell})^{r-1}}$$

$$\mathcal{H}_{-1/r} = \prod_{\ell \geq 1} \frac{(1 - q^{r\ell})^r}{1 - q^{\ell}} \frac{1}{1 - t^{2\ell-2} q^{r\ell}} \frac{1}{(1 - t^{2\ell} q^{r\ell})^{r-1}}$$

Proof.
Cores and quotients of partitions. $\square$

$\mathrm{Hilb}^{\delta+nG}_{-1/r}$ are Nakajima quiver varieties, and are all diffeomorphic to $\mathrm{Hilb}^{nG}_{-1/r}$.

What about the unbalanced case?

Start of unbalanced case

Recall $(a;x)_\infty := \prod_{\ell \geq 0}(1 - ax^\ell)$.

Example (Göttsche)

$$\sum_{n\geq 0} b_k(\mathrm{Hilb}^n(S))t^k q^n = \frac{1}{(q;qt^2)_\infty^{b_0}} \frac{1}{(qt^2;qt^2)_\infty^{b_2}} \frac{1}{(qt^4;qt^2)_\infty^{b_4}}$$

Conjecture (Gusein-Zade, Luengo, Melle-Hernández)]

$$\mathcal{H}_{1/3} = \frac{1}{(q;t^2q^3)_\infty} \frac{1}{(q^2t^2;t^2q^3)_\infty} \frac{1}{(q^3;t^2q^3)_\infty}$$

Why stop there?

Main conjecture on disconnected series

It seems if $G \cap SL_2 = \emptyset$ then

$$\mathcal{H}_G = \prod_{h=1}^{r} \frac{1}{(q^h t^{\epsilon(h)}; q^r t^2)_\infty}$$

with $\epsilon(h)$ either 2 or 0.

Conjecture (Johnson)

If $g \in G$ acts on $\mathbb{C}^2$ as $(\exp(2\pi ia/r), \exp(2\pi ib/r))$, let $AF(g)$ and $AI(g)$ denote the fractional and integral parts of $a/r + b/r$. Let $k = |G \cap SL_2|$.

$$\mathcal{H}_G(q,t) = \frac{(q^k; q^k)_\infty^k}{(q,q)_\infty} \prod_{g \in G} \frac{1}{q^{r(1-AF(g))} t^{2AI(g)}, q^r t^2)_\infty}$$

Results on connected Hilbert schemes

Theorem (Johnson)

$P_t(Hilb_G^{\delta+nG})$ stabilizes to $1/(t,t)_\infty^{|G|}$

Note that the right hand side is independent of m and δ.

Conjecture (Johnson)

Let c be the number of rational curves in the minimal resolution of $\mathbb{C}^2/G$. Then

$$\mathcal{H}_G^\delta \cdot (q, qt^2)_\infty \cdot (qt^2, qt^2)_\infty^c$$

has positive coefficients.

Though total homology sees all of $K_0(G)$, only get a Heisenberg action for the minimal resolution.