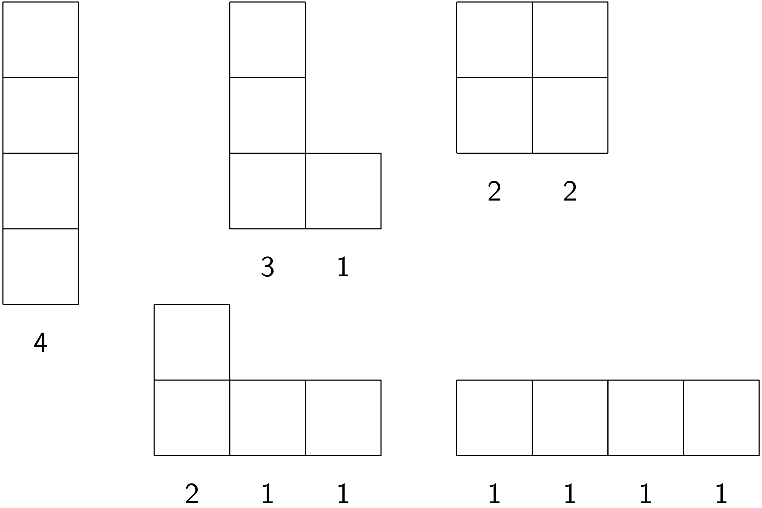
How I think of partitions: Young Diagrams



Partitions: lists of numbers

A *partition* λ is a nondecreasing sequence of positive integers

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$$

My favoured notation

- Preferred variables: λ, μ, ν
- $|\lambda| = \sum \lambda_i$ is the *size*
- The λ_i are the *parts*
- $\ell(\lambda)$ the number of parts is the *length*
- $\mathcal{P}$ will denote the set of all partitions
- $\mathcal{P}_n$ will denote the partitions of size n

First theorems in Partitions

Theorem (Euler Product)

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} t^{\ell(\lambda)} = \prod_{m \geq 1} \frac{1}{1 - q^m t}$$

Proof.

Expand RHS as geometric series; $(q^m t)^k$ means k parts of size m. □

Theorem

Euler's Odd=Distinct Let $\mathcal{O}$ be the set of partitions into odd parts Let $\mathcal{D}$ be the set of partitions into distinct parts Then $|\mathcal{O}_n| = |\mathcal{D}_n$

Proof.

$$\sum_{\lambda \in \mathcal{D}} q^{|\lambda|} = \prod_{m \geq 1} (1 + q^m) = \prod_{m \geq 1} \frac{1}{1 - q^{2m-1}} = \sum_{\lambda \in \mathcal{O}} q^{|\lambda|}$$

□

Algebra vs. Bijections

Euler's theorem is interesting in that it works by algebraic manipulation of power series. It tells us the two sets $\mathcal{O}_n$ and $\mathcal{D}_n$ have the same number of elements, but it doesn't give us a *bijection* between those two sets.

Exercise

In Igor Pak's survey on partition bijections, he gives three different bijective proofs of Euler's Odd=Distinct:

- Sylvester's bijection
- Glaisher's bijection
- Iterated Dyson maps (This will be important one for us!)

Read his notes to understand how all three work; find more in depth sources for hints / to check yourself.

Glaisher's theorem

- $\mathcal{R}(k)$ denotes partitions with no part repeated k times or more
- $\mathcal{D}(k)$ is the set of partitions with no part divisible by k

Theorem (Glaisher)

$$\sum_{\lambda \in \mathcal{R}(k)} q^{|\lambda|} = \sum_{\lambda \in \mathcal{D}(k)} q^{|\lambda|}$$

Proof.

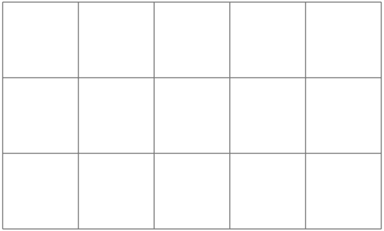
$$(1 + q^m + q^{2m} + \cdots q^{(k-1)m})(1 - q^m) = 1 - q^{km}$$

$\square$

- When $k = 2$ this is Euler's theorem that "Odd=Distinct"
- But only Glaisher's bijection known to adapt?

Start of "Fermionic" P.O.V.: Partitions in a box

How many partitions fit inside an $a \times b$ box?



Partitions $\leftrightarrow$ lattice paths $\leftrightarrow$ binary sequences $\leftrightarrow$ binary sequences

Now keep track of their size...

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Setting $q = 1$ gives back all the usual formulas.

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$$

$$[n]_q! = \prod_{k=1}^{n} [k]_q$$

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

Theorem (Exercise!)

Let $\mathcal{P}(a \times b)$ denote the set of partitions inside an $a \times b$ box.

$$\sum_{\lambda \in \mathcal{P}(a \times b)} q^{|\lambda|} = \binom{a+b}{a}_q$$

Proof.

Both sides satisfy a q-analog of Pascal's triangle. Induct. $\square$

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