Exercises from Meeting Two

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1 Warm-up to an open question

Remember that for a cell $\in \lambda$ of a partition λ, we defined

2 Bijective proofs of Euler's Theorem

The first Theorem of partitions, and illustration of the power of generating functions, is Euler's theorem that the number of partitions of n into odd parts are equal to the number of partitions of n into distinct parts.

Though very elegant, Euler's Theorem could be improved in that it's not "bijective" – we prove that two finite sets have the same size, but we don't give a bijection between those.

I know of three different bijective proofs of Euler's Theorem (learned from Igor Pak's survey of partition bijections); the discussion of these bijections begins in Section 3.1 on Page 20. Read Pak's terse descriptions of them and decipher how they work.

1. Glaisher's bijection uses binary
2. Sylvester's bijection cuts up a partition of one type and reassembles it into another
3. The "Iterated Dyson's Map"

Dyson's map ψ_r is defined in Section 2.5.1 of Pak's notes, on page 15.

3 Glaisher's Theorem

Recall *Glaisher's Theorem*, a generelization of Euler's Theorem that "Odd=Distinct". Let $\mathcal{R}(k)$ (the $\mathcal{R}$ is for 'repeat') denote the set of partitions where no part occurs k times or more. Hence $\mathcal{R}(1)$ consists only of the empty partition, and $\mathcal{R}(2)$ consists of partitions with distinct parts.

Let $\mathcal{D}(k)$ (the $\mathcal{D}$ is for 'divisible') denote the set of partitions with no part divisible by k; hence $\mathcal{D}(1)$ again consists only of the empty partition, and $\mathcal{D}(2)$ consists of partitions with odd parts.

Then we have

Theorem 3.1 (Glaisher)**.**

$$\sum_{\lambda\in\mathcal{R}(k)} q^{|\lambda|} = \sum_{\lambda\in\mathcal{D}(k)} q^{|\lambda|}$$

and so the $k = 2$ case recovers Euler's Theorem that Odd=Distinct.

3.1 Easy, do me

Write out a careful proof of Glaisher's Theorem by adapting the generating function proof of Euler's Theorem. First find a product formula for the generating functions of partitions in $\mathcal{R}(k)$ and $\mathcal{D}(k)$, and show these different looking formulas are actually the same using

$$(1-q^m)(1+q^m+q^{2m}+\cdots+q^{(k-1)m}) = 1-q^{km}$$

3.2 A little harder

Now find a *bijective* proof of Glaisher's theorem, by adapting Glaisher's bijective proof of Euler's Theorem. Briefly, if $\lambda \in \mathcal{D}(k)$, and the part of size m occurs a times, then write a in base k:

$$a = a_\ell k^\ell + a_{\ell-1}k^{\ell-1} + a_1 k + a_0$$

with $0 \leq a_i < k$. Then to get a partition in $\mathcal{R}(k)$ we convert the a parts of size m into a_ℓ parts of size $k^\ell m$, $a_{\ell-1}$ parts of size $k^{\ell-1}m$, etc. until a_1 parts of size km and a_0 parts of size m.

4 The Jacobi Triple Product formula

The Jacobi triple product formula states that

$$\prod_{m\geq 1}(1-x^{2m})(1+x^{2m-1}y^2)(1+x^{2m-1}y^{-2}) = \sum_{n\in Z} x^{n^2}y^{2n}$$

It's called the triple product formula because the infinite product has three terms in it.

4.1 A specialization, easy

Show that if we specialize $x = q^{3/2}, y^2 = q^{1/2}$, the the Jacobi Triple Product formula becomes Euler's Pentagonal Number Theorem (covered at the end of Haiman's notes, for instance):

$$\prod_{m\geq 1}(1-q^m) = \sum_{n\in}(-1)^n q^{\frac{3n^2-n}{2}}$$

The original proofs of the Jacobi triple product formula used some analysis and depended on the Euler Pentagonal Number Theorem.

4.2 Borcherd's proof of the Jacobi Triple Product Formula

A very illuminating proof of the Jacobi Triple Product Formula uses Dirac's Electron Sea.

Let a state S consist of a finite number of electrons and a finite number of positrons, and let $\mathcal{S}$ denote the set of all possible states. (Recall electrons have charge -1, positrons have charge +1, their energy can be any number of the form $k - 1/2, k \in, k > 0$, and we can have at most one electron/positron of any given energy in a state).

The energy $e(S)$ and charge $c(S)$ are the sums of the energies and charges of the electrons and positrons in the state. So, for example, if the sate S consisted of three electrons with energies 11/2, 7/2, and 1/2, and one positron with energy 9/2, then it would have $c(S) = -2$, and $e(S) = 28/2 = 14$.

We will prove the Jacobi Triple product by computing the generating function

$$F(q,t) = \sum_{S \in \mathcal{S}} q^{e(s)} t^{c(S)}$$

in two different ways.

First, working "energy level by energy level", show

$$F(q,t) = \prod_{m \geq 1} (1 + q^{m-1/2} t)(1 + q^{m-1/2} t^{-1})$$

Second, we can work "charge by charge" – for any charge c, show:

- There is unique state vac_c (The "vacuum of charge c") with minimal energy among all states with charge c
- That energy is $e(\mathrm{vac}_c)) = c^2/2$
- There is a bijection between partitions and states of charge c, that sends a partition λ with $|\lambda| = n$ to a state S with energy $c^2/2 + n$

Deduce:

$$F(q,t) = \sum_{n \in Z} t^n q^{n^2/2} \prod_{m \geq 1} \frac{1}{1 - q^m}$$

Finally, prove the Jacobi Triple Product formula by setting the two expressions for $F(q,t)$ equal and doing some algebraic manipulations.