Basic Generating Function Exercises

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1 Powers and basics

1.1 Summation of Geometric series

Let $G = 1 + x + x^2 + x^3 + \cdots$. Multiplying by x and subtracting prove that $G(x) = \frac{1}{1-x}$. What's the generating function for $1, 2, 4, 8, 16, \ldots$?

1.2 $1, 2, 3, 4, 5, \ldots$

Playing a similar game to previous exercise, find a closed form for the generating function for $1, 2, 3, \cdots$. Give another derivation for this formula by taking the derivatives of both sides of the previous part.

1.3 Triangles and squares

Similar to the previous problem, derive closed formulas for the generating function of the triangular numbers $1, 3, 6, 10, 15, 21, \ldots$ and squares $1, 4, 9, 16, 25, \ldots$ in multiple ways.

2 Fibonacci numbers

2.1 Generating Function

Let $F_0 = F_1 = 1$ and define $F_{n+1} = F_n + F_{n-1}$. Proving by using the recursive formula and tricks as in the previous section that

$$F(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{1}{1 - x - x^2}$$

2.2 Explicit Formula

Using the partial fractions and the previous problem, derive Binet's explicit formula for the Fibonacci numbers in terms of $\varphi = (1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$

2.3 Binomial Coefficients

Expanding the generating function for Fibonacci numbers out as a geometric series with ratio $x + x^2$ and using the binomial theorem, prove the following appearance of Fibonacci numbers in Pascal's triangle: