Notes, problems and exercises from the first session

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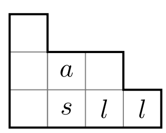
July 3, 2020

1 Arms, legs, and hooks

Definition 1.1. Let λ be a partition, and $\square \in \lambda$ a cell. The *arm* of a cell $\square \in \lambda$, written $a(\square)$, is the number of cells inside λ and above $\square$, the *leg* of a cell, written $l(\square)$ or $\ell(\square)$, is the number of cells in the partition and to the right of $\square$.

Example 1.2. The cell $(2,1)$ of $\lambda = 3+2+2+1$ is marked s; the cells in the leg and arm of s are labeled a and l, respectively.



$$a(s) = \#a = 1$$
$$l(s) = \#l = 2$$
$$h(s) = 4$$

Definition 1.3. The *hook length* of a cell $\square$, written $h(\square)$ is

$$h(\square) = a(\square) + \ell(\square) + 1$$

The *weighted hook length*

$$h_k(\square) = a(\square) + k(\ell(\square) + 1)$$

Definition 1.4. A partition is called an n-core if it has no squares $\square$ with $h(\square)$ divisible by n.

A partition is called a (k,n)-core if it has no squares $\square$ with $h_k(\square)$ divisible by n.

We will use $\mathcal{C}(k,n)$ to denote the set of all (k,n) core partitions.

So, an n-core partition is a $(1,n)$-core.

Exercise 1. Show that the 2-core partitions are exactly the staircase partitions $k + (k-1) + (k-2) + (k-3) + \cdots + 2 + 1$.

Exercise 2. Prove that the generating function for $(-1,3)$ cores is $1/(1-q)$

$$\sum_{\lambda \in \mathcal{C}(-1,3)} q^{|\lambda|} = \frac{1}{1-q}$$

More explicitly, prove that for every n there is exactly one partition (describe it!) of n that is a (-1,3) core partition.

Open Problem 1. Give a *combinatorial* proof that

$$\sum_{\lambda \in \mathcal{C}(2,5)} q^{|\lambda|} = \frac{1}{(1-q)(1-q^2)}$$

I have proven something much stronger than this, but the proof uses a lot of high-power algebraic geometry. We would like even a bijective proof. The right hand side is the generating function for partitions where every part is size 1 or size 2, so we'd like a bijection

$$f : \mathbb{N} \times \mathbb{N} \to \mathcal{C}(2,5) \qquad |f(x,y)| = x + 2y$$

Although this is an open problem, I think it should be very doable! I will be explaining some ideas and tools that might help with it, but it should be approachable even now, and thinking about is good practice!

Question 1. For a partition λ, for $k = 0, 1, 2$ define:

$$d_k(\lambda) = \#\{\Box \in \lambda : a(\Box) - \ell(\Box) = k \mod 3\}$$

and generating functions

$$\mathcal{G}_k(q,t) = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} t^{d_k(\lambda)}$$

- Write sage code to compute the generating functions
- Explain why $\mathcal{G}_1 = \mathcal{G}_2$.
- Conjecture a product formula expression for $\mathcal{G}_1$? (proving your conjecture is HARD)
- Can you say anything about $\mathcal{G}_0$?

I haven't really thought about the last bullet point, so I'd be curious what you can find!