PARTITIONS AND BIJECTIONS

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1. Introduction

Partition theory is a subfield of Combinatorics and Number Theory. One of the first recorded appearances of the study of partitions came in 1674 when Gottfried Wilhelm Leibniz wrote to Johann Bernoulli asking about "divulsions of integers", which later became known as partitions. The first publication on integer partitions came from Leonhard Euler during a presentation at St. Petersburg Academy in 1741.

From here the study of partitions became more common and partition theory grew to become a widely studied area of mathematics. Since the work of Euler, a many famous mathematicians have worked in the field, including Goldbach, Hardy and Ramanujan.

Partition theory has been shown to have links to many different areas of mathematics and even physics, some of which we will explore. Our main aim throughout this paper shall be to understand some of the key theorems in partition theory. These theorems are normally proved via manipulation of generating functions. We however will aim to give bijective proofs using combinatorial methods, more specifically by constructing bijections between different types of partitions.

We shall start by introducing the concept of a partition, meaning that no prior knowlege of partition theory is needed. The main idea of the section will be counting partitions, a concept which naturally leads us to what is considered as the first major theorem in partition theory, Euler's Theorem. During the main bulk of Section 2, Euler's original proof using generating functions shall be explored.

Section 3 brings us to the first bijective proof of Euler's theorem which we will look at, Glaisher's bijections. We will understand how these bijections work before discussing the idea that they can lead to an extension of a theorem being discovered, one of the key reasons why bijective proofs are so important. We shall see that Glaisher's Theorem is a direct extension of Euler's Theorem. We will then continue to look at bijective proofs of Euler's Theorem by studying Sylvestre's bijection and it's extensions, followed shortly by a brief look at Pak's Iterated Dyson's map proof of Euler's Theorem.

Section 4 will concentrate one of the other major theorems in Partition theory, Euler's Pentagonal number theorem. Here a bijective proof of this theorem will be introduced before showing that the Pentagonal number theorem is a sub-case of the famous Jacobi Triple product. The section takes a swift digression into the world of Quantum Mechanics, which arms us with new definitions and bijections which we will be able to use to bijectively prove the Jacobi Triple Product.

At this stage it is expected that the reader will have a good understanding of the use of bijective proofs. Thus we shall begin a journey into the unknown, starting the search for a new bijective proof of a theorem proven by A. Buryak and B. L. Feigin. The theorem in question has strong links to partitions and is a major candidate for a bijective proof, as we shall see in Section 5.

Section 6 sees the start of the hunt. Here we will look at trying to understand a small sub-case of the full theorem, trying to describe the set of partitions that this sub-case corresponds to, using a variation on one of the bijections used to prove Euler's Theorem. The section is completely original and concludes with a new bijective proof that gives us a description of these partitions.

In section 7 we shall see if we can generalise these findings to a larger sub-case of Buryak-Feigin. Again the section concludes with a bijective proof which shows us how to define the partitions which correspond to this sub-case.

Before we can reach the heady heights of new bijective proofs which help us to understand Buryak-Feigin, we must first dive in with the basics of partition theory and what better place to start than the definition of a partition.

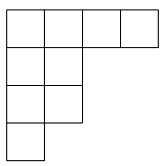
Acknowledgements. I would like to thank my supervisor Paul Johnson for all the help and insight he has provided throughout the project. The report exceeds the 40 page limit, totalling 49 pages due to a combination of factors. The main reasons are the 12 pages of diagrams contained within, which have been added to aid understanding, and the presence of original research in Sections 5-7, which contributes over 23 pages.

2. Partitions and Euler's Theorem

Definition 2.1. A *partition* λ of n is a sequence of non-increasing positive integers $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_m \geq 0$ that sum to n. We write $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ or $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_m$ and say that λ has m parts, with λ_i representing the ith part of the partition λ. We say that $|\lambda| = n$. We also define $\mathcal{P}$ to be the set of all partitions and $\mathcal{P}(n)$ to be the set of all partitions of n

Note 2.2. Let $\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}, \cdots, \lambda_s^{k_s}) \in \mathcal{P}$ be the partition with k_i parts of size λ_i for all i. Thus $(5^2, 3^4, 2, 1^3) = (5, 5, 3, 3, 3, 3, 2, 1, 1, 1)$. This definition will be used mostly from Section 5 onwards.

A partition can be represented as a set of dots or squares using a *Young Diagram*. In the young diagram of a partition, the nth row represents the nth part of the partition. Below is the young diagram for the partition $(4, 2, 2, 1)$.



We shall say that a young diagram of the above form is a young diagram in *standard notation*.

2.1. **Counting Partitions and Euler's Theorem.**

Now we actually know what a partition is, we can start to create different "sets" of partitions which we can show to have certain notable mathematical properties. We shall start by defining some of the most commonly studied collections of partitions.

Definition 2.3. A partition $\lambda \in \mathcal{P}(n)$ is *odd* if all the parts of λ are odd. We define $\mathcal{O}(n)$ to be the set of all odd partitions of n. We call a partition $\mu \in \mathcal{P}(n)$ *distinct*, if the size of each part is distinct and we let $\mathcal{D}(n)$ be the set of all distinct partitions of n.

One of the most natural first questions about partitions is, how many partition of a given number n are there? To answer this question, we define the following counting functions.

Definition 2.4. For any positive integer n,

$$\begin{aligned} p(n) &= \text{ the number of partitions of } n = |\mathcal{P}(n)| \\ d(n) &= \text{ the number of partitions of } n \text{ into distinct parts} = |\mathcal{D}(n)| \\ o(n) &= \text{ the number of partitions of } n \text{ into odd parts} = |\mathcal{O}(n)| \end{aligned}$$

Example 2.5. The partitions of 6 are: (6), $(5,1)$, $(4,2)$, $(4,1,1)$, $(3,3)$, $(3,2,1)$, $(3,1,1,1)$, $(2,2,2)$, $(2,2,1,1)$, $(2,1,1,1,1)$ and $(1,1,1,1,1,1)$. Thus $p(6) = 11$, $d(6) = 4$, and $o(6) = 4$.

At first it seems that $d(6) = o(6)$ may just be a coincidence. However upon further inspection we see this could be more than just a minor coincidence. The following table shows the values of $p(n)$, $o(n)$ and $d(n)$ for n up to 10.

n	1	2	3	4	5	6	7	8	9	10
$p(n)$	1	2	3	5	7	11	15	22	30	42
$o(n)$	1	1	2	2	3	4	5	6	8	10
$d(n)$	1	1	2	2	3	4	5	6	8	10

We can clearly see that for n up to 10, there are as many odd partitions as distinct but does this hold for all n?

Theorem 2.6 (Euler's Theorem)**.** *For each n, the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.*

In 1748 Euler published his proof of this remarkable theorem [4]. We now aim to recreate Euler's original proof of this theorem. To do this we must introduce the concept of generating functions for partitions, closely following Mark Haiman's notes [6] throughout.

Definition 2.7. Let $P(q) = \sum_n p(n)q^n$ be the generating function which counts the number of partitions of n, for all values of n, giving each partition of n a weight q^n. Analogously define $O(q)$ and $D(q)$ to be the generating functions for the odd and distinct partitions respectively.

Proposition 2.8.

1. *The generating function for all partitions is*

$$P(q) = \prod_{i=1}^{\infty} \frac{1}{1-q^i}.$$

2. *The generating function for partitions into odd parts is*

$$O(q) = \prod_{i \text{ odd}} \frac{1}{1-q^i}.$$

3. *The generating function for distinct partitions is*

$$D(q) = \prod_{i=1}^{\infty} (1+q^i).$$

Proof.

1. To describe a partition λ we need to choose for each positive integer i how many times to include i as a part of λ. Now as each repetition of i contributes i towards the total value of the partition, the generating function for the choice of repetitions of i is given by $1+q^i+q^{2i}+\cdots = 1/(1-q^i)$. Hence as all partitions are generated by making this choice for all integers i, we can multiply for all i and find that

$$P(q) = \prod_{i=1}^{\infty} \frac{1}{1-q^i}.$$

2. We know that for a partition to be odd, each part i must be odd. Hence to find $O(q)$ we can use the same generating function for parts of size i but then we restrict i to being the positive odd integers. Thus we find that

$$O(q) = \prod_{i \text{ odd}} \frac{1}{1-q^i}.$$

3. Now distinct partitions differ from general partitions in that each part i either occurs once or not at all. Thus the generating function for each i is simply $1+q^i$ and therefore by multiplying over all i, we find that the generating function for partitions of n into distinct parts is

$$D(q) = \prod_{i=1}^{\infty} (1+q^i)$$

□

We now have the necessary tools to recreate Euler's original proof of Theorem 2.6.

Proof of Euler's Theorem. The main idea of the proof is to take $O(q)$, our number of odd partitions, and express it as a product over all i, rather than just odd values of i. Now we know that

$$O(q) = \prod_{i \text{ odd}} \frac{1}{1-q^i} = \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots}.$$

Our first step is to multiply numerator and denominator by the terms corresponding to the even values of i, $(1-q^2)(1-q^4)\cdots$. Doing this we find that

$$\begin{aligned} O(q) =& \frac{(1-q^2)(1-q^4)(1-q^6)\cdots}{(1-q)(1-q^2)(1-q^3)\cdots} \\ =& \prod_{i=1}^{\infty} \frac{(1-q^{2i})}{(1-q^i).} \end{aligned}$$

Since we know that $\frac{(1-q^{2i})}{(1-q^i)} = (1+q^i)$, we find that

$$O(q) = \prod_{i=1}^{\infty}(1+q^i) = D(q).$$

Hence for any n, $o(n) = d(n)$. $\square$

Thanks to Euler's Theorem, we know that there are as many odd partitions of n as distinct partitions of n. Despite this, the proof feels rather disappointing as we don't have a way to construct an odd partition from a distinct partition, or vice versa. We'd like a bijective proof of Euler's Theorem and as it happens we are about to see three.

3. Bijective Proofs of Euler's Theorem

3.1. **Glaisher's Bijection.** The first bijective proof of Euler's Theorem we will study was devised in 1883 by James Whitbread Lee Glaisher. Glaisher defined a pair of maps, $\varphi : \mathcal{O}(n) \to \mathcal{D}(n)$ and $\phi : \mathcal{D}(n) \to \mathcal{O}(n)$, that are inverse to each other, thus proving Euler's Theorem by giving a method of constructing odd partitions from distinct partitions and vice versa.

We will start by defining φ.

Definition 3.1. Let λ be a partition in $\mathcal{O}(n)$. Now, let λ have m_i parts of size i. Each m_i has a unique binary expansion of the form $m_i = a_{i,0}+2a_{i,1}+4a_{i,2}+\cdots+2^k a_{i,k}$ for some sequence of $a_{i,j} \in \{0,1\}$. Define $\varphi : \mathcal{O}(n) \to \mathcal{D}(n)$ so that $\varphi(\lambda)$ is the partition which contains the part $i \cdot 2^j$ once, if the binary expansion of m_i is such that $a_{i,j} = 1$.

Remark 3.2. The size of a part k in $\varphi(\lambda)$ uniquely determines the values of i and j such that $k = i\dot{2}^j$. Thus it follows that k cannot be repeated in $\varphi(\lambda)$ and hence the image of φ is contained in $\mathcal{D}(n)$.

Example 3.3. Take the partition $\lambda = (9,7,7,5,3,3,3,1,1,1,1,1) \in \mathcal{O}(42)$. Then λ is such that $m_1 = 1+4$, $m_3 = 1+2$, $m_5 = 1$, $m_7 = 2$ and $m_9 = 1$. Then up to a rearrangement of parts we see that $\varphi(\lambda) = (1\cdot 1, 1\cdot 4, 3\cdot 1, 3\cdot 2, 5\cdot 1, 7\cdot 2, 9\cdot 1)$. We can now rearrange this into the standard form and find that $\varphi(\lambda) = (14,9,6,5,4,3,1)$.

Example 3.4. Let $n = 7$. Then the elements, λ, of $\mathcal{O}(7)$ are mapped to elements, $\varphi(\lambda) \in \mathcal{D}(7)$ in the following way:

λ	$\varphi(\lambda)$
(7)	(7)
(5,1,1)	(5,2)
(3,3,1)	(6,1)
(3,1,1,1,1)	(4,3)
(1,1,1,1,1,1,1)	(4,2,1)

Now we will define the inverse to φ.

Definition 3.5. Take a partition $\mu = (\mu_1, \mu_2, \cdots, \mu_l) \in \mathcal{D}(n)$. Each part is of the form $\mu_i = 2^{k_i} \cdot l_i$, for some positive odd integer l_i and $k_i \in \mathbb{Z}^+$. Then $\phi : \mathcal{D}(n) \to \mathcal{O}(n)$ is defined by mapping the partition μ to the partition with 2^{k_i} repetitions of the part l_i, for all i.

Example 3.6. Take the partition $\mu = (14, 9, 6, 5, 4, 3, 1) \in \mathcal{D}(42)$. Then $\mu_1 = 2 \cdot 7$, $\mu_2 = 9$, $\mu_3 = 2 \cdot 3$, $\mu_4 = 5$, $\mu_5 = 4 \cdot 1$, $\mu_6 = 3$ and $\mu_7 = 1$. Hence $\phi(\mu) = (9, 7, 7, 5, 3, 3, 3, 1, 1, 1, 1, 1) \in \mathcal{O}(42)$.

Example 3.7. Let $n = 8$. Then the elements, μ, of $\mathcal{D}(8)$ are mapped to elements, $\phi(\mu)$, of $\mathcal{O}(8)$ in the following way:

μ	$\phi(\mu)$
(8)	(1,1,1,1,1,1,1,1)
(7,1)	(7,1)
(6,2)	(3,3,1,1)
(5,3)	(5,3)
(5,2,1)	(5,1,1,1)
(4,3,1)	(3,1,1,1,1,1)

We have seen that for the partitions λ and μ from Examples 3.3 and 3.6 respectively, that φ and ϕ are inverse to each other. We will now prove that this holds in general.

Proposition 3.8. *φ and ϕ are inverse.*

Proof. First we show $\phi \circ \varphi$ is the identity map on $\mathcal{O}(n)$. Take $\lambda \in \mathcal{O}(n)$. Now when viewed in the sum notation for partitions, $\lambda = \sum_{i \text{ odd}} i \cdot m_i$, where m_i is the number of parts of size i in λ. Now we express the m_i's in terms of their binary expansions, $m_i = a_{i,0} + 2a_{i,1} + 4a_{i,2} + \cdots + 2^k a_{i,k}$. Thus

$$\varphi(\lambda) = \sum_{i \text{ odd}} \left(i \cdot (a_{i,0} + 2a_{i,1} + 4a_{i,2} + \cdots + 2^k a_{i,k}) \right) \in \mathcal{D}(n). \tag{1}$$

Therefore we see that

$$\phi(\varphi(\lambda)) = \phi\left(\sum_{i \text{ odd}} i \cdot (a_{i,0} + 2a_{i,1} + 4a_{i,2} + \cdots + 2^k a_{i,k}) \right) = \sum_i i \cdot m_i = \lambda,$$

since ϕ sends a distinct part of the form $i \cdot 2^j$ to 2^j parts of size i (and if we do this j we obtain m_i parts of size i). Hence $\phi \circ \varphi$ is the identity map on $\mathcal{O}(n)$.

We must now show that $\varphi \circ \phi$ is the identity map on $\mathcal{D}(n)$. Take any partition $\mu \in \mathcal{D}(n)$. Now we want to show that μ is of the form of $\varphi(\lambda)$ as in equation (1), for some partition $\lambda \in \mathcal{O}(n)$. If this is possible, then $\varphi(\phi(\mu)) = \varphi(\phi(\varphi(\lambda))) = \varphi(\lambda) = \mu$.

Since μ is an element of $\mathcal{D}(n)$, we can write each part of μ in the form $2^j \cdot i$, for a positive odd integer i and $j \in \mathbb{Z}^+$. We notice that each part has a unique combination of i and j. Now we can group together parts that have the same value i into one sum of the form $i \cdot (a_{i,0} + 2a_{i,1} + 4a_{i,2} + \cdots + 2^k a_{i,k})$, where $a_{i,j} = 1$ if $i \cdot 2^j$ is a part of μ and $a_{i,j} = 0$ otherwise. By doing this we can clearly write μ in the required form and so $\varphi \circ \phi$ is the identity map on $\mathcal{D}(n)$. $\square$

We can now use Glaisher's bijections to prove Euler's Theorem.

Glaisher's proof of Euler's Theorem. Take φ and ϕ as defined above in definitions 3.1 and 3.5 respectively. Now $\varphi : \mathcal{O}(n) \to \mathcal{D}(n)$ and $\phi : \mathcal{D}(n) \to \mathcal{O}(n)$ are bijections and hence we have that for any n, $|\mathcal{O}(n)| = |\mathcal{D}(n)|$. $\square$

3.1.1. *Glaisher's Theorem.* Having just found our first bijective proof, one may still be wondering how this gives us extra insight into Euler's theorem, specifically information which cannot obtain from the standard generating function proof. The answer to that question is a theorem, proven by Glaisher, which follows nearly immediately from the argument used here.

Theorem 3.9 (Glaisher's Theorem)**.** *For any $r \geq 2$, the number of partitions of n into parts not divisible by r is the same as the number of partitions of n so that no part is repeated more than $r - 1$ times.*

It is immediate that this is an extension of Euler's theorem, since Euler's theorem follows from Glaisher's theorem by setting $r = 2$. To prove Glaisher's theorem we would normally have to create a new generating function proof. Alternatively, thanks to our bijective proof, we can simply modify φ and ϕ so that they depend on r, considerably shortening the proof.

Definition 3.10. We define $\mathcal{A}_r(n)$ to be the set of partitions of n into parts not divisible by r. Equally we let $\mathcal{B}_r(n)$ be the set of partitions of n with no part repeated more than $r - 1$ times.

Definition 3.11. Take $\lambda \in \mathcal{A}_r(n)$ and say λ has m_i parts of size i. Instead of taking the binary expansion of m_i we take the base r expansion so that

$$m_i = a_{i,0} + ra_{i,1} + r^2 a_{i,2} + \cdots + r^k a_{i,k},$$

for some sequence of $a_{i,j} \in \{0, 1, \cdots, r-1\}$. As in the binary case, this is unique.

Define $\varphi_r : \mathcal{A}_r(n) \to \mathcal{B}_r(n)$ so that $\varphi_r(\lambda)$ is the partition which contains the part $i \cdot r^j$, $a_{i,j}$ times for each combination of i and j.

Equally we can define the inverse in a similar fashion to ϕ.

Definition 3.12. Take a partition $\mu \in \mathcal{B}_r(n)$ where each part part of μ is of the form $\mu_i = l_i \cdot r^{k_i}$ for some $l_i, k_i \in \mathbb{Z}_{\geq 0}$ with $l_i \nmid r$. Then we define $\phi_r : \mathcal{B}_r(n) \to \mathcal{A}_r(n)$ by mapping μ to the partition with r^{k_i} repetitions of the part l_i, for all i.

We can see that φ_r and ϕ_r are bijections, via an (omitted) argument which is very similar to that with φ and ϕ in the Euler's theorem case. Glaisher's theorem now follows trivially. $\square$

Thanks to the greater understanding that Glaisher's bijective proof of Euler's theorem has given us, we have been able to easily generalise the theorem. This is one of the major advocates for finding bijective proofs. As such we shall return to finding bijective proofs of Euler's theorem, to see what other insights it may give us.

3.2. **Sylvester's Bijections.** Our next bijective proof of Euler's Theorem was suggested by James Sylvester. He, like Glaisher, proved Euler's Theorem by finding a bijection between $\mathcal{O}(n)$ and $\mathcal{D}(n)$.There are three different, but correspondent, definitions of Slyvestre's map from odd partitions to distinct partition. We will expand on the process of defining one of them, and its inverse, as shown by David Lonoff and Daniel McDonald [10]. For more information on the correspondent definitions, see Igor Pak's survey on partition theory [11]. We start by finding the map from odd partitions to distinct partitions.

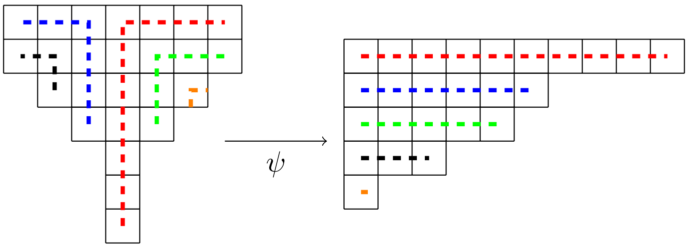
Definition 3.13. The young diagram of a partition $\lambda \in \mathcal{O}(n)$ is *centre-aligned* if it is symmetrical about a vertical line through the centre of the largest part. To convert the standard notation young diagram of a partition $\lambda \in \mathcal{O}(n)$, into the centre-aligned young diagram, move all parts so that the centre line of all parts line up.

Definition 3.14. Let $\psi : \mathcal{O}(n) \to \mathcal{D}(n)$ be the function as defined by the following iterative process. Take $\lambda \in \mathcal{O}(n)$, draw a centre-aligned young-tableaux for λ and then use the following algorithm to create $\psi(\lambda)$:

1) Draw one fishhook, of shape $\Rsh$, passing upwards along the centre line and through all the squares to the right of the centre line on the top row.
2) Draw a fishhook of shape $\neg$ along the remaining top row boxes and down the column to the left of the centre line.
3) If the partition is not covered by fishhooks, add a $\Rsh$ fishhook with corner one square south-west of the last $\Rsh$ fishhook.
4) If the partition is still not covered by fishhooks, add a $\neg$ fishhook with corner one square directly south-east of the last $\neg$ fishhook.
5) Repeat steps 3 and 4 alternately until the partition is completely covered by fishhooks.
6) Once λ has been covered by fishhooks, for each fishhook add a part of size equal to the number of squares in the fishhook to the new partition $\psi(\lambda)$.

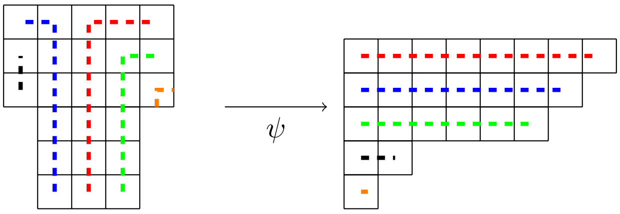
This process is best demonstrated via examples.

Example 3.15. Take $\lambda = (7, 7, 5, 3, 1, 1, 1) \in \mathcal{O}(25)$. Then we run our algorithm to find $\psi(\lambda)$, adding hooks in the following colour order: red, blue, green, black then orange.



Thus we find that $\psi(\lambda) = (10, 6, 5, 3, 1) \in \mathcal{D}(25)$.

Example 3.16. Take $\lambda = (5, 5, 5, 3, 3, 3) \in \mathcal{O}(24)$. We now run our algorithm, adding coloured hooks in the same order as in Example 3.15.

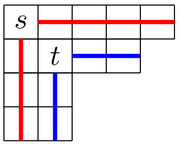


Hence $\psi(\lambda) = (8, 7, 6, 2, 1) \in \mathcal{D}(24)$.

In order to show that for any partition $\lambda \in \mathcal{O}(n)$, it's image $\psi(\lambda) \in \mathcal{D}(n)$, we make some new definitions.

Definition 3.17. Take any square $\square$ in the young diagram representation of a partition λ. The *arm* of $\square$, often written $a(\square)$, is the number of squares on $\square$'s row that are to the right of $\square$. Similarly the *leg* of $\square$, often written as $l(\square)$, is the number of squares in $\square$'s column that are below $\square$.

Example 3.18. Take the partition $\lambda = (5, 4, 2, 2)$. Consider the young diagram of λ below, specifically the squares marked s and t.



The square labelled s is such that $a(s) = 4$ and $l(s) = 3$. We can also see from the diagram above, that $a(t) = 2$ and $l(t) = 2$. The two diagrams below show the arm and leg for each square in λ, with the tableaux on the left showing the arm of each square and the tableaux on the right showing the leg.

4	3	2	1	0
3	2	1	0	
1	0			
1	0			

The *arm* of each $\square$.

3	3	1	1	0
2	2	0	0	
1	1			
0	0			

The *leg* of each $\square$.

Definition 3.19. Let λ be any partition. Then $a(\lambda)$ denotes the largest part of λ and $l(\lambda)$ denotes the number parts of lambda.

Note 3.20. It is important to note that $a(x)$ and $l(x)$ for x a partition, are different from $a(x)$ and $l(x)$ where x is a square in a partition. The reason for this notation is that if λ is a partition, then $a(\lambda)$ and $l(\lambda)$ correspond to the arm and leg of the origin of λ respectively.

To avoid confusion this author will state whenever which version is being used if it is deemed unclear, though as a general rule of thumb Greek letters will denote partitions and squares are denoted with either $\square$ or lower-case English letters.

Definition 3.21. Let the fishhooks of the shape $\Lsh$ be called right hooks and define r_i to be the size of the ith right hook. Equally let fishhooks of shape $\neg$ be called left hooks and define l_i to be the size of the ith left hook.

Note 3.22. Under this definition $\psi(\lambda) = (r_1, l_1, r_2, l_2, ...)$, with the last part of $\psi(\lambda)$ being the last hook (either left or right) with non-zero size.

Proposition 3.23. *For any partition $\lambda \in \mathcal{O}(n)$, when we apply the map ψ the lengths of the fishhooks produced are strictly decreasing so that $r_1 > l_1 > r_2 > l_2 > \cdots$.*

Proof. We will first show that $r_i > l_i$. Take the ith left hook. Its corner is in the same row as the ith right hook but is further away from the centre line. Thus the column containing the ith left hook's leg is no longer than the column containing the leg of the ith right hook. Therefore the legs are at most equal.

Next consider the arm of these two hooks. For the young diagram to be centre aligned with the centre line passing through the 1st right hook, we must have that the arm of the ith left hook is 1 smaller than the ith right hook. Hence $r_i > l_i$.

Next we show that $l_i > r_{i+1}$. Recall that $\lambda_i \geq \lambda_{i+1}$. As our young diagram is centre-aligned, this conditions means that the number of squares in the arm of the $(i+1)$th right hook is at most equal to the number in the ith left hook, so $a(i\text{th left hook}) \geq a((i+1)\text{th right hook})$.

Now since our diagram of λ is symmetric about the centre column, the columns containing the leg of the i left hook and the leg of the $(i+1)$th right hook must be the same length. However the corner of the i left hook is in the row above the corner of the $(i+1)$th right hook, so $l(i\text{th left hook}) > l((i+1)\text{th right hook})$. Hence $l_i > r_{i+1}$.

Therefore by combining our two inequalities we see that the size of the fishhooks are strictly decreasing so that $r_1 > l_1 > r_2 > l_2 > \cdots$. $\square$

Note 3.24. The chain $r_1 > l_1 > r_2 > l_2 > \cdots$ is finite and terminates with the last hook of non-zero size.

Corollary 3.25. *The image of ψ is contained in $\mathcal{D}(n)$*

Proof. From the proposition, we know that the chain $r_1 > l_1 > r_2 > l_2 > \cdots$ is strictly descending. Hence as $\psi(\lambda) = (r_1, l_1, r_2, l_2, ...)$, we see the parts of $\psi(\lambda)$ are strictly decreasing, so $\psi(\lambda) \in \mathcal{D}(n)$. $\square$

Example 3.26. Let $n = 7$. Then the elements, λ, of $\mathcal{O}(7)$ are mapped to following elements $\psi(\lambda) \in \mathcal{D}(7)$:

λ	$\psi(\lambda)$
(7)	(4,3)
(5,1,1)	(5,2)
(3,3,1)	(4,2,1)
(3,1,1,1,1)	(6,1)
(1,1,1,1,1,1,1)	(7)

From this we can see that Sylvester's bijection from $\mathcal{O}(n)$ to $\mathcal{D}(n)$ maps elements in a different way to Glaisher's bijection, since ψ sends (7) to $(4,3)$ rather than to itself, as φ (as in definition 3.1) does.

We now search for an inverse to ψ. We will use another iterative function to turn a distinct partition into the centre-aligned young diagram for a partition in $\mathcal{O}(n)$.

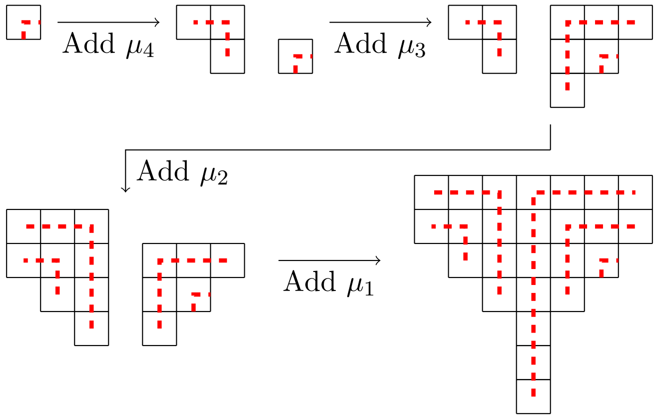
Definition 3.27. Let $\chi : \mathcal{D}(n) \to \mathcal{O}(n)$ be defined by the following iterative process. Take $\mu = (\mu_1, \mu_2, \cdots, \mu_s) \in \mathcal{D}(n)$.

1) If s is odd, place a column of μ_s squares on the right. If s is even, place a row of μ_s squares on the left.
2) If we placed μ_s on the right then add a fishhook of size μ_{s-1} with leg one greater then the μ_s fishhook on the left. Place this fishhook so that it has it's corner 1 row above μ_s. Else skip this step.
3) If we placed μ_s on the left then add a fishhook of size μ_{s-1} on the right with arm 1 greater than the μ_s fishhook. Place this fishhook so it's corner is on the same row as the μ_s fishhook. Else skip this step.
4) If we just inserted μ_{i+1} as a right hook $\Lsh$, then insert a left hook, $\neg$, of length μ_i with leg 1 greater than the preceding hook. Place it on the left, directly above the hook represented by μ_{i+2}, so that it creates a new top row.
5) If we just inserted μ_i as a left hook $\neg$, then insert a right hook, $\Lsh$, of length μ_{i-1} with arm 1 greater than the preceding hook. Place it on the right, directly above the hook represented by μ_{i+2}, so the top of this hook is on the same row as the top of the last $\neg$ hook.
6) Alternately repeat steps 4 and 5 until all s hooks have been placed.

This process creates a centre-aligned partition in $\mathcal{O}(n)$. Take the partition that this represents as $\chi(\mu)$.

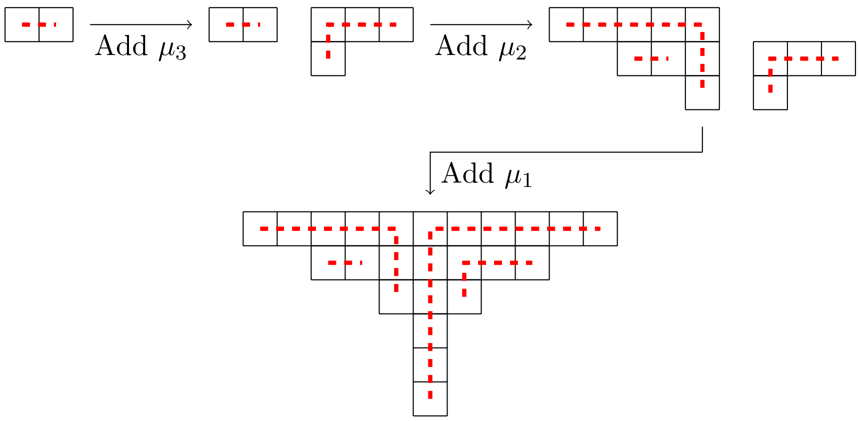
Again the process is best understood via examples.

Example 3.28. Let $\mu = (10, 6, 5, 3, 1) \in \mathcal{D}(25)$. Since μ has 5 parts, $\mu_5 = 1$ represents a right hook $\Lsh$. We now run our algorithm starting with a right hook.



Hence $\chi(\mu) = (7, 7, 5, 3, 1, 1, 1) \in \mathcal{O}(25)$.

Example 3.29. Let $\mu = (11, 7, 4, 2) \in \mathcal{D}(24)$. Now μ has 4 parts so $\mu_4 = 2$ represents a left hook $\neg\!\!\cdot$. We now run our algorithm starting with this left hook.



Therefore $\chi(\mu) = (11, 7, 3, 1, 1, 1) \in \mathcal{O}(24)$.

Proposition 3.30. *ψ and χ are inverse to each other*

Proof. It is clear that $\psi \circ \chi$ is the identity map on $\mathcal{D}(n)$. After completing the 4th step of χ, we have the centre-aligned diagram of $\chi(\mu)$ for any $\mu \in \mathcal{D}(n)$. The diagram here is clearly identical to the diagram of $\psi(\chi(\mu))$ after step 6 and so we recover μ.

Equally $\chi \circ \psi$ is also the identity map. To show we want to show that any partition $\lambda \in \mathcal{O}(n)$ is of the form $\chi(\mu)$ for some $\mu \in \mathcal{D}(n)$. However we can use the first 6 steps of the algorithm for ψ to find a centre-aligned diagram for λ with all its fishhooks on. We can then take μ to be the lengths of these fishhooks, so that $\chi(\mu) = \lambda$. Therefore $\chi(\psi(\lambda)) = \chi(\psi(\chi(\mu)) = \chi(\mu) = \lambda$. $\square$

We can now use Sylvester's bijections to prove Euler's Theorem.

Sylvester's proof of Euler's Theorem. Take ψ and χ as defined above in definitions 3.17 and 3.27 respectively. Now $\psi : \mathcal{O}(n) \to \mathcal{D}(n)$ is a bijection with inverse $\chi : \mathcal{D}(n) \to \mathcal{O}(n)$ and hence we have that for any n, $|\mathcal{O}(n)| = |\mathcal{D}(n)|$. $\square$

We have now proven Euler's theorem using Sylvester's bijection. However, Sylvester's original description of his bijection was slightly different than our own. For more information on the original bijection (as well as two other correspondent descriptions of the bijection) see Igor Pak's survey on partitions [11].

As with Glaisher's bijection, Sylvester's bijections lead to new extensions of Euler's theorem. We shall state these extensions for flavour, though in the interest of time, the proofs shall be omitted. They can be found in the book *Integer Partitions* by G. E. Andrews and K. Eriksson [1].

Theorem 3.31 (Sylvester's Theorem)**.** *The set of partitions in $\mathcal{O}(n)$ with exactly k distinct parts is in bijection with the set of partition in $\mathcal{D}(n)$ which have exactly k different sequences of consecutive integer parts.*

Theorem 3.32 (Fine's Theorem)**.** *The number of partitions in $\mu \in \mathcal{D}(n)$ with $a(\mu) = k$ is equal to the number of partitions $\lambda \in \mathcal{O}(n)$ with $a(\lambda) + 2l(\lambda) = 2k + 1$.*

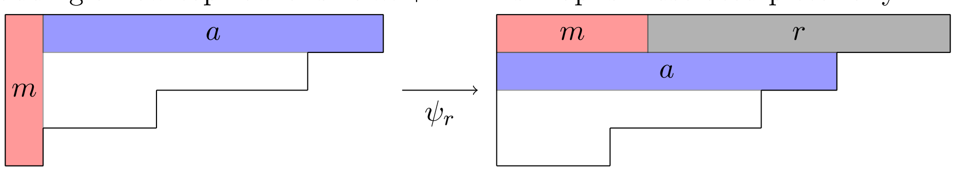
3.3. **The Iterated Dyson Map.** Our final bijective proof of Euler's Theorem was created by Igor Pak using a map first invented by Dyson. We will start by defining Dyson's map before showing Pak's proof. We will again closely follow work by David Lonoff and Daniel McDonald [10], using papers by W. Y. C. Chen and K. Q. Ji [3] and Pak[11] to elaborate on discussions where relevant.

Definition 3.33. The rank of λ, denoted $r(\lambda)$, is the difference between the largest part of λ and the number of parts of λ. So $r(\lambda) = a(\lambda) - l(\lambda)$. If $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ then $a(\lambda) = \lambda_1$ and $l(\lambda) = m$. Hence $r(\lambda) = \lambda_1 - m$.

Definition 3.34. We define the following sets of partitions.

$$\mathcal{P}(n, r) = \{\lambda \in \mathcal{P}(n) \mid r(\lambda) = n\}.$$
$$\mathcal{H}(n, r) = \{\lambda \in \mathcal{P}(n) \mid r(\lambda) \leq n\}.$$
$$\mathcal{G}(n, r) = \{\lambda \in \mathcal{P}(n) \mid r(\lambda) \geq n\}.$$

Definition 3.35. We now define Dyson's map $\psi_r : \mathcal{H}(n, r+1) \to \mathcal{G}(n+r, r-1)$ so that for $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathcal{H}(n, r+1)$, $\psi_r(\lambda)$ is the partition obtained from the young diagram of λ by removing the first column, of size m, and adding a new top row of size $m + r$. This map is illustrated pictorially:



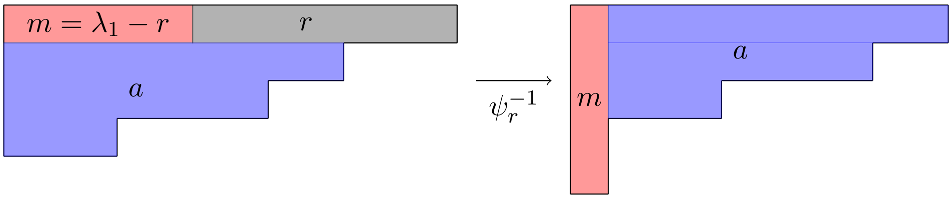
Note 3.36. The Dyson's map ψ_r is defined from $\mathcal{H}(n, r+1)$ to $\mathcal{G}(n+r, r-1)$ rather than just from $\mathcal{P}(n)$ to itself. This is because ψ_r must produce a valid partition. For this to happen we must have that the new first part in $\psi_r(\lambda)$ is greater than or equal to the first part of λ minus 1. i.e. in the diagram above, $m + r \geq a$. Thus $\lambda_1 - 1 \leq m + r$, so $r(\lambda) \leq r + 1$ and hence $\lambda \in \mathcal{H}(n, r + 1)$.

Consider the rank of the partition $\psi_r(\lambda)$. The leg of $\psi_r(\lambda)$ will be at most $m + 1$ since it can be at most one more than the leg of λ. The arm of $\psi_r(\lambda)$

will always be $m+r$. Hence $r(\psi_r(\lambda)) \geq (m+r)-(m+1) = r-1$ and so $\lambda \in \mathcal{G}(n+r, r-1)$.

This map can clearly be reversed.

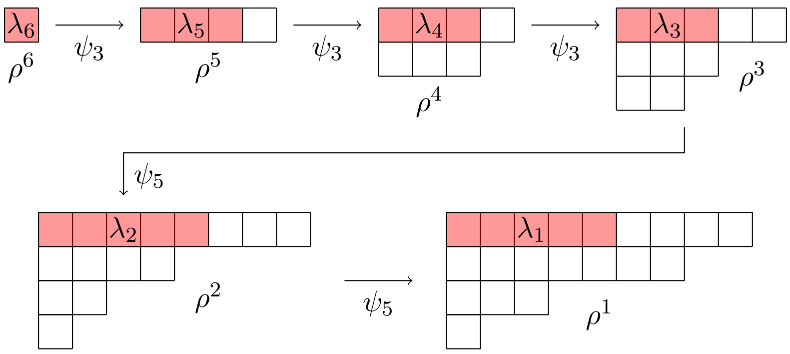
Definition 3.37. We define ψ_r^{-1} as the map which removes the top row of size a and adds a column on the left of length $a-r$. We illustrate the map pictorially below.



From these definitions we clearly see that the Dyson's map is a bijection. We may now define the iterated Dyson's map in the following way.

Definition 3.38. We define the iterated Dyson's map $\zeta : \mathcal{O}(n) \to \mathcal{D}(n)$ such that for $\lambda = (\lambda_1, \cdots, \lambda_s) \in \mathcal{O}(n)$, $\zeta(\lambda) = \rho^1$, where $\rho^1, \rho^2, ..., \rho^s$ are partitions defined by the conditions $\rho^s = (\lambda_s)$ and $\rho^i = \psi_{\lambda_i}(\rho^{i+1})$.

Example 3.39. Take the partition $\lambda = (5,5,3,3,3,1) \in \mathcal{O}(20)$. We shall find $\zeta(\lambda)$. The diagram below shows the sequence of partitions ρ, with shaded boxes in ρ^i representing λ_i.



Hence $\zeta(\lambda) = (9,7,3,1)$.

In order to define the inverse to the Iterated Dyson's map, we will need to check the following Lemma. We also notice that this lemma guarantees that if $\lambda \in \mathcal{O}(n)$ then $\zeta(\lambda) \in \mathcal{D}(n)$.

Lemma 3.40. *If $\lambda = (\lambda_1, \cdots, \lambda_s) \in \mathcal{O}(n)$, then $|\rho^1| = n$ and for all i, ρ^i is a partition into distinct parts with rank $r(\rho^i) = \lambda_i$ or $\lambda_i - 1$.*

Proof. First note that $|\rho^i| = \lambda_i + \cdots + \lambda_s$. Thus $|\rho^1| = |\lambda| = n$. We will now use strong induction to show that for all i, ρ^i is a partition into distinct parts with rank $r(\rho^i) = \lambda_i$ or $\lambda_i - 1$.

Base Case: ρ^s has one part of size λ_s. Thus ρ_s is a distinct partition with rank $\lambda_s - 1$.

Inductive Step: Suppose that ρ^k is a partition into distinct parts with $r(\rho^k) = \lambda_k$ or $\lambda_k - 1$, for all $i < k \leq s$. First we show ρ^i is a distinct partition. Let $\rho^j = (\rho_1^j, \rho_2^j, \cdots, \rho_{m_j}^j)$ for all $1 \leq j \leq s$. Now by the definition of the Dyson's map, we know that $\rho_1^i = m_{i+1} + \lambda_i$.

We also know from the supposition that the rank of ρ^{i+1} is either λ_{i+1} or $\lambda_{i+1}-1$. Thus we see that $m_{i+1}=\rho_1^{i+1}-\lambda_{i+1}$ or $m_{i+1}=\rho_1^{i+1}-\lambda_{i+1}+1$.

Finally we know from the definition of a partition that $\lambda_i \geq \lambda_{i+1}$. Using all the facts we can see that

$$\begin{aligned} p_1^i &= m_{i+1}+\lambda_i \\ &\geq (\rho_1^{i+1}-\lambda_{i+1})+\lambda_i \\ &\geq \rho_1^{i+1} \\ &> \rho_1^{i+1}-1. \end{aligned}$$

Now $\rho_2^i=\rho_1^{i+1}-1$ as it was the largest part in the ρ^{i+1} and the Dyson's map process causes it to lose 1 square from its first column (see the diagram in Definition 3.35). Hence $\rho_1^i > \rho_2^i$.

By considering ρ^{i+j} we can show using a similar argument that $\rho_j^{i+j} > \rho_{j+1}^{i+j}$. As Dyson's map removes 1 square from both ρ_j^{i+j} and ρ_{j+1}^{i+j} when creating ρ^{i+j-1}, these two parts remain distinct in ρ^i. Therefore we can see that $\rho_1^i > \rho_2^i > \cdots > \rho_k^i$, where ρ_k^i is the last part of ρ^i. Hence ρ^i is a distinct partition.

We will next show that $r(\rho^i)=\lambda_i$ or λ_i-1. Now

$$\begin{aligned} r(\rho^i) &= a(\rho^i)-l(\rho^i) \\ &= \big(l(\rho^{i+1})+\lambda_i\big)-l(\rho^i). \end{aligned}$$

Since the parts of ρ^{i+1} are distinct, there is at most one part of size 1 in ρ^{i+1}. This means that when we apply the Dyson's map, we remove no more than one row when we delete the first column. We then add a new top row, so in total we either have the same number of parts or one more. Hence $l(\rho^i)=l(\rho^{i+1})$ or $l(\rho^i)=l(\rho^{i+1})+1$. So $r(\rho^i)=\lambda_i$ or $r(\rho^i)=\lambda_i-1$ □

We will now be able to define the inverse to the Iterated Dyson's map in the following way.

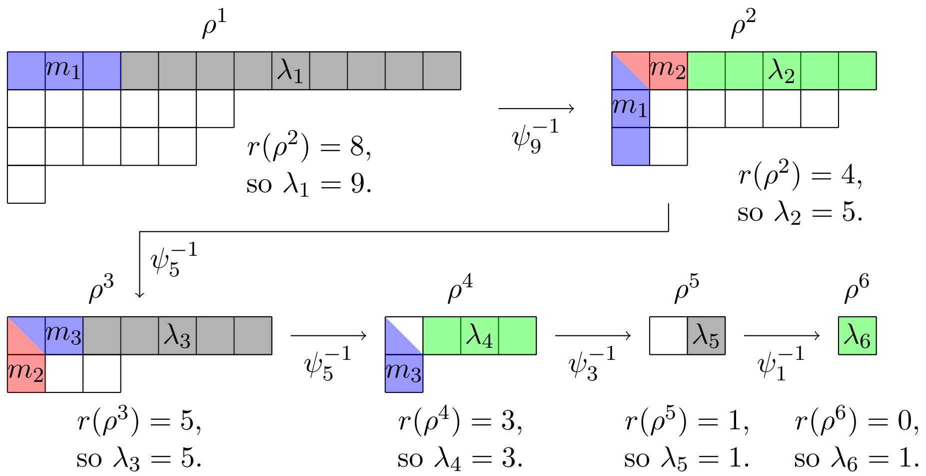
Definition 3.41. Let $\mu=(\mu_1,\mu_2,\cdots,\mu_l)\in\mathcal{D}(n)$. We first set $\rho^1=\mu$ and then use the following iterative process:

1) If $r(\rho^1)$ is odd, we set $\lambda_1=r(\rho^1)$. Else we set $\lambda_1=r(\rho^1)+1$.
2) Now take $\rho^2=\psi_{\lambda_1}^{-1}(\rho^1)$.
3) Repeat steps 1 and 2 iteratively with ρ^i and λ_i until $\rho^{s+1}=\emptyset$.

Set $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_s)$ and notice that each of the λ_i is odd, so $\lambda\in\mathcal{O}(n)$. We define $\zeta^{-1}:\mathcal{D}(n)\to\mathcal{O}(n)$ to be the function such that $\mu\mapsto\lambda$.

Remark 3.42. Despite a rather technical definition, the idea of ζ^{-1} is to set $\rho^1=\mu$ and then repeatedly apply the map ψ_r^{-1} for the largest possible odd r. We then take $\zeta^{-1}(\mu)$ to be the partition with parts of size equivalent to the values of r used.

Example 3.43. Take the partition $\mu=(12,6,5,1)\in\mathcal{D}(24)$. We look for $\zeta^{-1}(\mu)$.



Thus $\lambda = (9, 5, 5, 3, 1, 1)$.

Example 3.44. Take the partition $\mu = (9, 7, 3, 1) \in \mathcal{D}(20)$. We find $\zeta^{-1}(\mu)$. The following table shows the partition ρ^i, $r(\rho^i)$ and the corresponding value of λ_i.

i	ρ^i	$r(\rho^i)$	λ_i
1	(9,7,3,1)	5	5
2	(8,4,2,1)	4	5
3	(5,3,2)	2	3
4	(4,3)	2	3
5	(4)	3	3
6	(1)	1	1
7	$\emptyset$	-	-

Thus $\lambda = (5, 5, 3, 3, 3, 1)$.

Iterated Dyson's map proof of Euler's Theorem. We clearly see that ζ and ζ^{-1} are inverse to each other, since they use the same sequence of partitions ρ^i, with the maps between them uniquely determined. Thus ζ and ζ^{-1} are bijections and so $|\mathcal{O}(n)| = |\mathcal{D}(n)|$. $\square$

4. Euler's Pentagonal Number Theorem and the Jacobi Triple Product

The Jacobi triple product is a famous mathematical identity first discovered by Carl Gustav Jacob Jacobi in 1829. The product was presented in his work *Fundamenta nova theoriae functionum ellipticarum* [8] in the following form.

Theorem 4.1 (Jacobi Triple Product)**.** *Let $x, y \in \mathbb{C}$ with $|x| < 1$ and $y \neq 0$. Then*

$$\prod_{m=1}^{\infty} (1 - x^{2m})(1 + x^{2m-1}y^2)\left(1 + \frac{x^{2m-1}}{y^2}\right) = \sum_{n=-\infty}^{\infty} x^{n^2} y^{2n}.$$

There are many different proofs of this theorem. The aim of this section is to give a bijective proof using partitions. First we look at a special case of the Jacobi Triple Product, Euler's Pentagonal Number Theorem.

4.1. **Euler's Pentagonal Number Theorem.** On 6th April 1741, Euler gave a talk at the St. Petersburg Academy. The published notes from this talk [5] are widely regarded as the first publication on partitions. During the talk Euler stated, but did not prove, a partition identity which later became known as Euler's Pentagonal Number Theorem. Nine years later, Euler finally proved the identity in a letter which he sent to Christian Goldbach, eventually publishing the result in two papers.

We aim to prove Euler's Pentagonal Number Theorem considerably quicker than the 9 years Euler took to do so. Let us begin with the statement:

Theorem 4.2 (Euler's Pentagonal Number Theorem)**.**

$$\prod_{n=1}^{\infty}(1-q^n) = 1 + \sum_{k=1}^{\infty}(-1)^k \left(q^{k(3k+1)/2} + q^{k(3k-1)/2}\right)$$

Although seemingly daunting, the theorem tells us that

$$(1-q)(1-q^2)(1-q^3)\cdots = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots,$$

where the exponents in the sum are the generalized pentagonal numbers.

Note 4.3. The pentagonal numbers are the set of numbers that are obtained from generalising the concepts of triangle or square numbers to the pentagon. The nth pentagonal number has equation $p_n = \frac{n^2-n}{2}$ and the generalized pentagonal numbers are the values of p_n for $n \in \mathbb{Z}$ and are ordered $p_0, p_1, p_{-1}, p_2, p_{-2}, \cdots$.

Before we prove Euler's Pentagonal Number Theorem (EPNT) we shall prove our earlier claim that it is a special case of the Jacobi Triple Product (JTP). Placing the two identities next to each other, as we do so below, this seems to be a strange claim.

$$\text{JTP: } \prod_{m=1}^{\infty}(1-x^{2m})(1+x^{2m-1}y^2)\left(1+\frac{x^{2m-1}}{y^2}\right) = \sum_{n=-\infty}^{\infty} x^{n^2}y^{2n}.$$

$$\text{EPNT: } \prod_{n=1}^{\infty}(1-q^n) = 1 + \sum_{k=1}^{\infty}(-1)^k(q^{k(3k+1)/2} + q^{k(3k-1)/2}).$$

This is the case though, as we shall now see.

Claim 4.4. *Euler's Pentagonal Number Theorem is a special case of the Jacobi Triple Product*

Proof. Consider the substitution given by $x = q^{3/2}$ and $y^2 = -q^{1/2}$. Then we find that the Jacobi Triple Product reduces to

$$\prod_{m=1}^{\infty}(1-q^{3m})(1-q^{3m-1})(1-q^{3m-2}) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3n^2+n}{2}}.$$

From this we can see that

$$\prod_{m=1}^{\infty}(1-q^m) = 1 + \sum_{n=1}^{\infty}(-1)^n q^{\frac{3n^2+n}{2}} + \sum_{n=-\infty}^{-1}(-1)^n q^{\frac{3n^2+n}{2}},$$

and thus obtain that

$$\prod_{m=1}^{\infty}(1-q^m) = 1 + \sum_{n=1}^{\infty}(-1)^n(q^{\frac{3n^2+n}{2}} + q^{\frac{3n^2-n}{2}}).$$

□

We shall now give a bijective proof of Euler's Pentagonal Number Theorem originally given by Franklin in the late 19th century. We will be following Haiman's notes [6]. To understand this proof, we must first return to generating functions to find a new generating function for distinct partitions.

Definition 4.5. We define $\mathcal{D}(n,k)$ be the set of all distinct partitions of n which have exactly k parts. Let $d(n,k)$ be the number of partitions of n into k distinct parts. Thus $\mathcal{D}(n,k) = \{\lambda \in \mathcal{D}(n) : l(\lambda) = k\}$ and $d(n,y) = |\mathcal{D}(n,y)|$.

Lemma 4.6. *Let $D(q,t) = \sum_{n,k} d(n,k)t^k q^n$ be the two-variable generating function where we count each partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \in \mathcal{D}(n)$ with weight $t^k q^n$. Then*

$$D(q,t) = \prod_{i=1}^{\infty}(1+tq^i).$$

Proof. Fix i and consider a general partition $\lambda \in \mathcal{D}(n)$. Now we know that as $\lambda \in \mathcal{D}(n)$, the monomial giving us the contribution of parts of size i is $1+tq^i$ as every part of size i contributes 1 towards the total number of parts and i towards the total value of the partition. Hence multiplying by all i we find that

$$D(q,t) = \prod_{i=1}^{\infty}(1+tq^i).$$

□

Proof of Euler's Pentagonal Number Theorem. First, notice that

$$D(q,-1) = \prod_{n=1}^{\infty}(1-q^n).$$

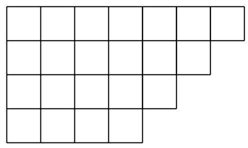
It remains to show that

$$D(q,-1) = 1 + \sum_{k=1}^{\infty}(-1)^k(q^{k(3k+1)/2} + q^{k(3k-1)/2}).$$

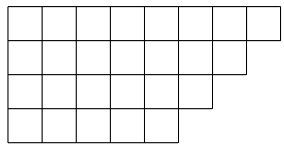
Now consider a distinct partition λ. The contribution of λ to $D(q,-1)$ will be $(-1)^{l(\lambda)}q^n$, so is q^n if $l(\lambda)$ is even, and $-q^n$ if $l(\lambda)$ is odd. Thus we notice that the coefficient of q^n in $D(q,-1)$ is $|\{\lambda \in \mathcal{D}(n) : l(\lambda) \text{ is even}\}| - |\{\lambda \in \mathcal{D}(n) : l(\lambda) \text{ is odd}\}|$. Thus it suffices to show that this difference, which we shall call $\Theta(n)$, is such that

$$\Theta(n) = \begin{cases} 1 & \text{if } n = (3k^2+k)/2 \text{ or } (3k^2-k)/2, \text{ for even } k \\ -1 & \text{if } n = (3k^2+k)/2 \text{ or } (3k^2-k)/2, \text{ for odd } k \\ 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Suppose $k > 0$ and consider the collection of partitions of the form $A_k = (2k-1, 2k-2, \cdots, k)$ and $B_k = (2k, 2k-1, \cdots, k+1)$, for example see A_4 and B_4 below. A_k and B_k are partition of $(3k^2 - k)/2$ and $(3k^2 + k)/2$ respectively.



$A_4 \in P(22)$



$B_4 \in P(26)$

Let $W = \{A_k, B_k \mid k \in \mathbb{Z}^+\} \cup \{(0)\}$. Now take $\alpha, \beta \in W$. It is clear that if $\alpha \neq \beta$ then $|\alpha| \neq |\beta|$, else $(3k^2+k)/2 = (3k^2-k)/2$ and then $k = 0$. Thus if $n = 0$ or $n = 3k^2 \pm k$ there is one, and only one, partition in W with size n, else there are none.

If we can construct a bijection

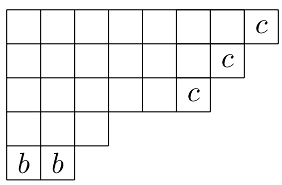
$$\{\lambda \in \mathcal{D}(n) : \lambda \notin W,\, l(\lambda) \text{ is even}\} \leftrightarrow \{\lambda \in \mathcal{D}(n) : \lambda \notin W,\, l(\lambda) \text{ is odd}\},$$

then only partitions in W contribute to the value of $\Theta(n)$, since these two sets above are the distinct partitions of n, not in W, which contribute positively and negatively to $\Theta(n)$ respectively.

Notice that the elements of W clearly contribute to $\Theta(n)$ in the desired way. This is because $l(A_k) = l(B_k) = k$, so partitions in W contribute positively if k is even and negatively if k is odd. Hence constructing this bijection completes the proof. We shall do this via an involution.

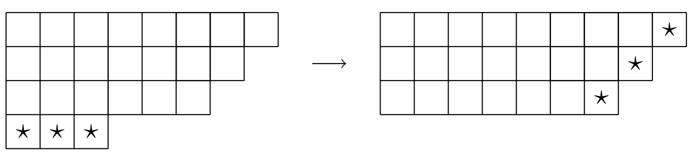
Let our involution be S and let $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathcal{D}(n) \setminus W$. Define $b(\lambda)$ to be the smallest part of λ, in this case λ_m. Also set $c(\lambda)$ to be the number of consecutive parts, starting with λ_1, which differ by 1, i.e. the length of the "staircase" that starts at λ_1.

For example, if $\lambda = (8, 7, 6, 3, 2)$, then $b(\lambda) = 2$ and $c(\lambda) = 3$.

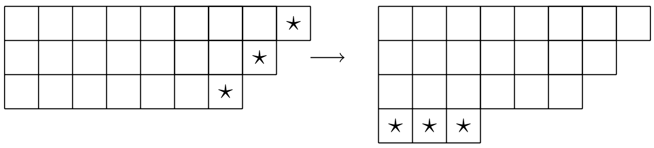


Now there are two cases to investigate.

Case I: $b(\lambda) \leq c(\lambda)$. Take the young diagram of λ and remove the final part. Now extend the first $b(\lambda)$ rows by 1 square. This is well-defined provided that the row to be removed isn't also needing to be extended. If this was not the case then $b(\lambda) \geq c(\lambda)$, so $b(\lambda) = c(\lambda)$ and then $\lambda \in W$ (as it would be a partition of the A_k type). Thus the process is well-defined. Note that $c(S(\lambda)) = b(\lambda)$ and that $b(S(\lambda)) > b(\lambda)$.



Case II: $b(\lambda) > c(\lambda)$. Remove 1 cell from the first $c(\lambda)$ rows of the young diagram of λ and then add these cells as a new smallest part of size $c(\lambda)$. This process is viable provided that the new row we're adding is smaller than the old smallest part. The old smallest part has size $b(\lambda)$, but we may have to remove a square from it, so the smallest possible size of this row in $S(\lambda)$ is $b(\lambda)-1$. But $b(\lambda) > c(\lambda)$ so this is only a problem if $b(\lambda) = c(\lambda)+1$. This would mean that $\lambda \in W$ (as it would be partition of type B_k) and thus has already been ruled out. Again notice that $b(S(\lambda)) = c(\lambda)$ and that the first $c(\lambda)$ rows will remain consecutive after applying S so $c(S(\lambda)) \geq c(\lambda)$.



Obviously in either case the number of parts changes by one. Now if λ was in Case I, then as we have that $b(S(\lambda)) > b(\lambda) = c(S(\lambda))$, we see that $S(\lambda)$ is in Case II. Equally if λ was in Case II, then since $c(S(\lambda)) > c(\lambda) = b(S(\lambda))$, $S(\lambda)$ is in Case I. Since the operations in both cases are the opposite to each other we conclude that S is an involution. Therefore

$$\prod_{n=1}^{\infty}(1-q^n) = D(q,-1) = 1 + \sum_{k=1}^{\infty}(-1)^k(q^{k(3k+1)/2} + q^{k(3k-1)/2}).$$

□

This link between the Jacobi Triple Product and Euler's Pentagonal Number Theorem, a theorem which has a bijective proof, hints that the Jacobi Triple Product may too have a bijective proof. Our search to find such a proof is about to take us somewhere few would expect, into to the wonderful world of Quantum Mechanics.

4.2. **A Quantum Mechanics Excursion.** This author imagines that you are currently questioning why we are about to discuss quantum mechanics and how it could possibly relate to partitions and more specifically, the Jacobi Triple Product. Rest assured that this section will lead to us having a new method of describing partitions, a method from which our desired combinatorial proof will follow with relative ease. This chapter will follow a paper authored by Paul Johnson [9].

We shall start with a physics based story, which is told in the cited paper (although is not original to this paper). It is asked that this is not misinterpreted as an attempt at an accurate account of the physics involved. This tale is included only as motivation for the definitions that will shortly follow.

4.2.1. *A Physics Fairy Tale.* In quantum mechanics, it is a well known that the possible energy levels of electrons are half integers, i.e. elements of $\mathbb{Z}_{1/2} = \{a + 1/2 : a \in \mathbb{Z}\}$. It does not make sense for an electron to have negative energy but basic quantum mechanics predicts this. A method of solving this problem was suggested in 1930 by British physicist Paul Dirac.

The Pauli exclusion principle tells us that each possible energy state can only have at most one electron in it. This means that we can view any set of electrons as a subset $S \subset \mathbb{Z}_{1/2}$. Dirac's solution was to take the vacuum state vac to be the negative half integers i.e. $vac = \mathbb{Z}^{-}_{1/2} = \{a+1/2 : a \in \mathbb{Z}, a < 0\}$. Then by Pauli's exclusion principle we can only add electrons with positive energy, solving our negative energy electron problem.

Dirac's sea also predicts the positron, a particle which has the same energy levels as electrons but positive charge (whereas an electron has negative charge). These positrons corresponds to a negative energy level not filled with an electron. Removing a negative energy electron results in adding positive charge and energy to the system and so can be interpreted as adding a positron.

4.2.2. *Returning to mathematics.* Armed with our quantum mechanics setting, we are now ready to introduce the definitions and arguments that will allow us to prove the Jacobi Triple Product.

Definition 4.7. Let $\mathbb{Z}^{\pm}_{1/2}$ be the set of all positive/negative half integers, respectively. Then we define the vacuum state to be $vac = \mathbb{Z}^{-}_{1/2}$. A state $S \subset \mathbb{Z}_{1/2}$ is a finite collection of electrons and a finite collection of positrons, which we shall denote S^+ and S^- respectively.

Definition 4.8. The charge $c(S)$ of a state S is the number of positrons minus the number of electrons. So $c(S) = |S^-| - |S^+|$.

Definition 4.9. The energy $e(S)$ of a state S is the sum of the energy of all electrons and positrons. Thus

$$e(S) = \sum_{k \in S^+} k + \sum_{k \in S^-} -k$$

.

Definition 4.10. The *Maya diagram* of a state S, is an infinite sequence of circles on the x-axis, each centred at an element of $\mathbb{Z}_{1/2}$, with positive circles extending to the left and negative circles to the right. We shade the circle centred at $k \in \mathbb{Z}_{1/2}$ black if and only if $k \in S$

Example 4.11. The Maya diagram of the vacuum state vac is given by:

$$\begin{array}{cccccc|cccccc} \cdots & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots \\ & \frac{9}{2} & \frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-3}{2} & \frac{-5}{2} & \frac{-7}{2} & \frac{-9}{2} & \end{array}$$

Example 4.12. Let S be the state given by having electrons of energy level $7/2$ and $1/2$ and positrons of energy levels $3/2$ and $7/2$. Then the Maya diagram of S is:

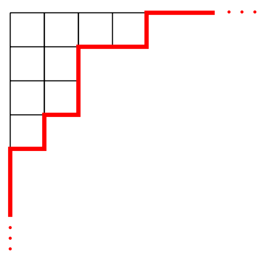
$$\begin{array}{cccccc|cccccc} \cdots & \bigcirc & \bullet & \bigcirc & \bigcirc & \bullet & \bullet & \bigcirc & \bullet & \bigcirc & \bullet & \cdots \\ & \frac{9}{2} & \frac{7}{2} & \frac{5}{2} & \frac{3}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-3}{2} & \frac{-5}{2} & \frac{-7}{2} & \frac{-9}{2} & \end{array}$$

4.2.3. *The states of zero charge bijection.* Our aim now is to show that there is a bijection between partitions and states with zero charge, which sends a partition $\lambda \in \mathcal{P}(n)$ to a state S_λ with $c(S_\lambda) = 0$ and $e(S_\lambda) = n$. Before we can show this, we will have to introduce a new way to draw the young diagram of a partition.

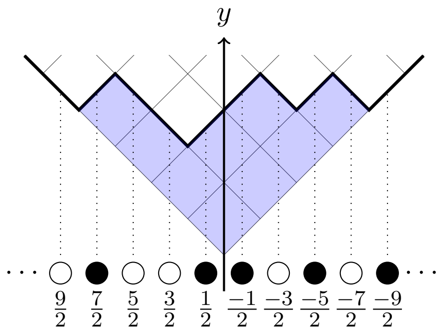
Definition 4.13. To draw a young diagram in Russian notation, we take a young diagram in standard notation, rotated it $3\pi/4$ radians counter-clockwise and scale up by a factor of $\sqrt{2}$. This means that the segments of the boundary path of the tableaux are centred at the half integers.

We can now draw a Maya diagram directly under the young diagram, so that the bottom corner of the partition is at the line $y = 0$ (i.e. is directly between the circles $1/2$ and $-1/2$). We shade the circle k black if the segment directly above k is sloping upwards. If the segment is sloping downwards then we leave k unshaded.

Example 4.14. Consider the partition $\lambda = (4, 2, 2, 1) \in P(9)$. Then the standard notation young diagram for λ is directly below. The boundary path has been marked in red.



The Russian notation young diagram for λ is below. In the diagram below, λ has been coloured blue.



As we can see in the above diagram, the y-axis of the Russian notation young diagram, will split the partition into two parts, as in the diagram above. The left hand side of the partition will have a rows, where a is the number of electrons in the corresponding Maya diagram. The length of the ith row in this half of the partition corresponds to the energy level of the ith electron (where the electrons are ordered in decreasing energy level).

Similarly for the right hand side of the partition, the number of rows corresponds to the number of positrons and the ith row has length equivalent to the energy level of the ith positron (where the positrons are again ordered in decreasing energy levels).

Proposition 4.15. *The Russian notation young diagram gives us a bijection from partitions of n to states with zero charge and energy n.*

Proof. Take a partition $\lambda \in \mathcal{P}(n)$. Then we find that the Russian notation young diagram, as defined in Definition 4.13, will give us a state S_λ with zero charge. This state will also have energy n, because we know that the energy is the sum of the e_i and p_i, and this is the number of squares under the boundary line in the young diagram, which is n.

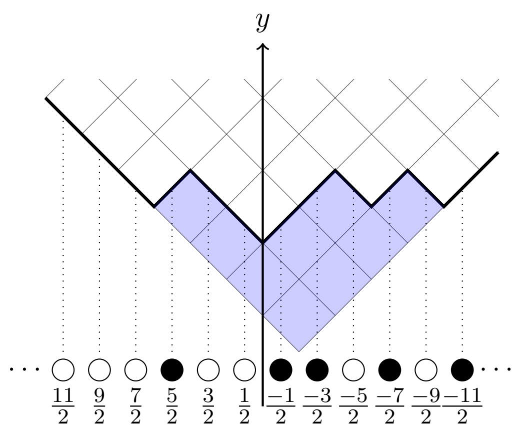
If we are given a zero charge state S with energy n then we can clearly construct a partition $\lambda \in \mathcal{P}(n)$ by using the Maya diagram of S to create the boundary of the partition. This will be a partition of n as the energy of the system corresponds to the number of squares under the boundary path of the partition. Hence partitions of n and states with zero charge and energy n are in bijection. $\square$

This bijection can be modified to give us a bijection between partitions and states of charge $c \in \mathbb{Z}$, simply by shifting the partition so that the bottom corner of the partition (in Russian notation) is on the line $y = -c$ rather than at $y = 0$.

Example 4.16. Consider the state S with one electron at energy level $5/2$ and two positrons at energy levels $5/2$ and $9/2$. This state clearly has charge of 1 and an energy of $19/2$. The Maya diagram is given below.

$\cdots$	○	○	○	●	○	○ |	●	●	○	●	○	●	$\cdots$
	$\frac{11}{2}$	$\frac{9}{2}$	$\frac{7}{2}$	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{-3}{2}$	$\frac{-5}{2}$	$\frac{-7}{2}$	$\frac{-9}{2}$	$\frac{-11}{2}$	

This corresponds to the following partition in Russian notation using our bijection.



This gives us the partition $(4, 2, 2, 1)$ again, as in Example 4.14, though this time the partition is centred on the line $y = -1$, as we claimed it would be for a state with charge 1.

We can view a Maya diagram as an abacus, rather than a fixed diagram, so that we can slide all the "beads" simultaneous either left or right along the "rung", preserving the pattern. Doing this we will find that this "sliding" motion will not change the partition acquired, though the state will have a different charge. For example, no matter where we place the origin in this sequence of "stones",

Ex4.16

$$\cdots \ \bigcirc\ \bigcirc\ \bigcirc\ \bullet\ \bigcirc\ \bigcirc \,|\, \bullet \,|\, \bullet\ \bigcirc\ \bullet\ \bigcirc\ \bullet\ \bullet\ \bullet \cdots,$$

Ex4.14

we will always find that the Maya diagram will correspond to the partition $(4, 2, 2, 1)$, though the state will have a different charge and energy for all the possible states. The origins for examples 4.14 and 4.16, have been placed on the above diagram in red and blue respectively.

We will now prove this bijection holds.

Proposition 4.17. *There is a bijection between partitions and states with charge c for any $c \in \mathbb{Z}$.*

Proof. Start with a partition λ. We can use the bijection from Proposition 4.15 to obtain a state with charge zero. Now we have a state with charge zero we can take its Maya diagram and slide all beads c places to the right. This sliding motion is clearly a bijection. Thus we have a unique state of charge c. The process is clearly reversible since both the actions applied to the partition are bijections. Hence the whole process is a bijection. □

Before we move on to prove the Jacobi triple product we shall prove one last lemma which we shall need.

Lemma 4.18. *The generating function for the vacuum state vac at charge $n \in \mathbb{Z}$ is given by the sum*

$$\sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} t^n,$$

where the exponent of t is the charge of the system and the exponent of q is the energy.

Proof. To obtain the vacuum state at charge $n \in \mathbb{Z}$ we take the Maya diagram for vac and slide all the beads n spaces to the right. If $n = 0$, we remain as the vacuum state and thus the charge and energy are both 0. If n is positive it is equivalent to creating n new positrons. If n is negative we are creating n new electrons. In any case the charge will be n and the energy will be

$$\sum_{i=1}^{n}\left(i - \frac{1}{2}\right) = \left(\sum_{i=1}^{n} i\right) - \frac{n}{2} = \frac{n(n+1)}{2} - \frac{n}{2} = \frac{n^2}{2}.$$

Hence the sum

$$\sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} t^n,$$

represents all possible charge levels of the vacuum state. □

4.3. **Proving the Jacobi Triple Product.** Thanks to the bijection between states and partitions that we have just discovered, we can now prove the Jacobi triple product. We shall start by once again stating the theorem.

Theorem 4.19 (Jacobi Triple Product)**.**

$$\prod_{m=1}^{\infty}(1-x^{2m})(1+x^{2m-1}y^2)\left(1+\frac{x^{2m-1}}{y^2}\right)=\sum_{n=-\infty}^{\infty}x^{n^2}y^{2n}.$$

Proof. First take the change of variables given by $t=y^2$ and $q=x^2$. Then the formula is reduced to

$$\prod_{m=1}^{\infty}(1-q^m)(1+q^{m-\frac{1}{2}}t)(1+q^{m-\frac{1}{2}}t^{-1})=\sum_{n=-\infty}^{\infty}q^{\frac{n^2}{2}}t^n.$$

This can then be rearranged to give the formula

$$\prod_{m=1}^{\infty}(1+q^{m-\frac{1}{2}}t)(1+q^{m-\frac{1}{2}}t^{-1})=\prod_{m=1}^{\infty}\frac{1}{1-q^m}\sum_{n=-\infty}^{\infty}q^{\frac{n^2}{2}}t^n. \qquad (2)$$

We claim that both sides represent the generating function for all possible states. By the bijection given by Proposition 4.17 we can create any state by choosing a partition and a charge level to set it at. We know from Lemma 4.18 that the sum on the right hand side represents all possible charge levels for the vacuum state. We also know that the product on the right hand side is the generating function for all partitions. Thus the right hand side takes the vacuum state, applies a charge to it and then "excites" it with a partition. By making the correct choice of charge and partition we can obtain any state and thus the right hand side is the generating function for all states.

It remains to show that the left hand side also represents the generating function for all states. As a positron has charge 1 and an energy level of $p-\frac{1}{2}$, for some $p\in\mathbb{Z}$, this positron contributes $q^{p-\frac{1}{2}}t$ to the total charge and energy of a state. Equally an electron of energy level $e-\frac{1}{2}$, for some $e\in\mathbb{Z}$, contributes $q^{e-\frac{1}{2}}t^{-1}$ to the total charge and energy of a state.

Suppose that positrons and electrons can only have an energy level of $m-\frac{1}{2}$ for some $m\in\mathbb{Z}^+$. Now consider the generating function for all states given that we can only use the energy level $m-\frac{1}{2}$. To create such a state, we must choose whether to include the electron of energy $m-\frac{1}{2}$ and also whether to include the positron of energy $m-\frac{1}{2}$. Thus if we can only use one fixed energy level of $m-\frac{1}{2}$, for some $m\in\mathbb{Z}^+$, the generating function for states would be $\left(1+q^{p-\frac{1}{2}}t\right)\left(1+q^{p-\frac{1}{2}}t^{-1}\right)$.

Now to obtain any state (with no restriction on energy levels), we must make a decision as to whether to include the electron of energy $m-\frac{1}{2}$ and whether to include the positron of energy $m-\frac{1}{2}$, for all $m\in\mathbb{Z}^+$. Thus the generating function for all states is the previous generating function multiplied through by all values of $m\in\mathbb{Z}^+$, i.e.

$$\prod_{m=1}^{\infty}(1+q^{m-\frac{1}{2}}t)(1+q^{m-\frac{1}{2}}t^{-1}).$$

This is the left hand side of equation (2).

Therefore the left and right hand side of equation (2) represent the generating function for all possible states and so are equal. Hence by reversing our original change of variables,

$$\prod_{m=1}^{\infty}(1-x^{2m})(1+x^{2m-1}y^2)\left(1+\frac{x^{2m-1}}{y^2}\right)=\sum_{n=-\infty}^{\infty}x^{n^2}y^{2n}.$$

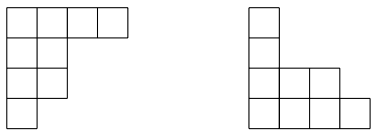
□

5. A New Bijection?

In a 2012, A. Buryak and B. Feigin jointly published a paper [2] which contains a combinatorial identity proven using methods from algebraic geometry. This identity can be viewed from a partition theory perspective and seems to be a plausible candidate for a bijective proof. The aim for the rest of the paper is to explain why it is such an ideal candidate before taking steps towards finding one for specific special cases of the theorem, although is full proof is as yet unknown. Unless otherwise stated, from henceforth all work is original research, as far as this author is aware.

Before the Theorem is stated, we will have to introduce some new notation as well as new partition functions.

Definition 5.1. The vertical notation for young diagram of a partition $\lambda \in \mathcal{P}$ is the young diagram of λ in standard notation rotated $\frac{\pi}{2}$ radians counter-clockwise. For example the standard notation and vertical notation young diagram of $\lambda = (4, 2^2, 1)$ are displayed below.

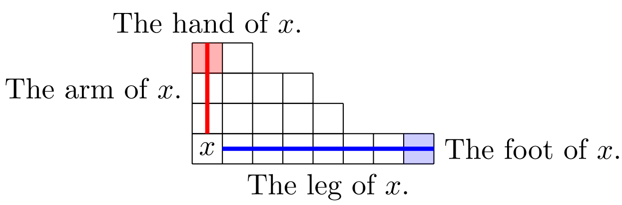


Standard Notation Vertical Notation

Note 5.2. From henceforth, unless explicitly stated otherwise, we shall draw all young diagram in vertical notation.

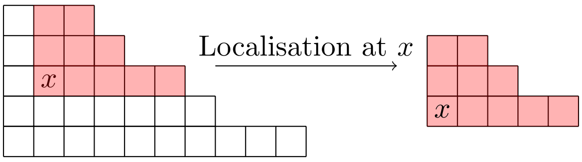
Remark 5.3. Since the vertical notation is obtained by rotating the standard notation $\frac{\pi}{2}$ radians counter-clockwise, we must also rotate the arm and leg. This means that the arm of a square □ is now all the squares directly above □ and the leg is all the squares directly to the right of □. We will also call the last square in the arm of x, the hand of x and similarly, the last square in the leg of x shall be the foot of x.

For example, in the diagram below, $a(x) = 3$ and $l(x) = 7$. The hand of x is the red square and the foot of x is the blue square.



Definition 5.4. The localisation of a vertical notation young diagram λ to a square x, is the vertical notation young diagram λ but with all columns to the left of x and all rows below x removed.

Example 5.5. Consider the partition $(5,4,3^2,2,1^3)$ localised at the square x in the 2nd column and 3rd row as below.



Definition 5.6. We define the *(α,β)-dimension* of a square $\square$ in a partition λ to be $d_{\alpha,\beta} = \alpha(a(\square)+1) - \beta(l(\square))$.

Definition 5.7. For any partition λ, the *(α,β)-dimension* of λ is defined to be

$$\begin{aligned} dim_{\alpha,\beta}(\lambda) &= |\{\square \in \lambda \mid d_{\alpha,\beta}(\square) = 0\}| \\ &= |\{\square \in \lambda \mid \alpha(a(\square)+1) - \beta l(\square) = 0\}|, \end{aligned}$$

i.e. the number of squares in λ with an (α,β)-dimension of 0.

Example 5.8. Take the partition $\lambda = (5^3, 4^4, 2^2, 1)$. We shall work out $dim_{1,1}(\lambda)$, $dim_{2,1}(\lambda)$ and $dim_{2,3}(\lambda)$ by computing $d_{1,1}$, $d_{2,1}$ and $d_{2,3}$ for all squares in λ and shading boxes red if the relevant statistic is 0.

$d_{1,1}$ of each square of λ:

-1	0	1							
-5	-4	-3	-2	-1	0	1			
-4	-3	-2	-1	0	1	2			
-5	-4	-3	-2	-1	0	1	0	1	
-5	-4	-3	-2	-1	0	1	0	1	1

So $dim_{1,1}(\lambda) = 7$

$d_{2,1}$ of each square of λ:

0	1	2							
-2	-1	0	-1	0	1	2			
0	1	2	1	2	3	4			
0	1	2	1	2	3	4	1	2	
1	2	3	2	3	4	5	2	3	2

So $dim_{2,1}(\lambda) = 5$

$d_{2,3}$ of each square of λ:

-4	-1	2							
-14	-11	-8	-7	-4	-1	2			
-12	-9	-6	-5	-2	1	4			
-16	-13	-10	-9	-6	-3	0	-1	2	
-17	-14	-11	-10	-7	-4	-1	-2	1	2

So $dim_{3,2}(\lambda) = 1$

We are now ready to introduce Buryak and Feigin's theorem.

Theorem 5.9 (Buryak-Feigin). *Let α and β be arbitrary positive coprime integers. Then*

$$\sum_{\lambda \in \mathcal{P}} t^{dim_{\alpha,\beta}(\lambda)} q^{|\lambda|} = \prod_{\substack{m \geq 1 \\ (\alpha+\beta) \nmid m}} \frac{1}{1-q^m} \prod_{m \geq 1} \frac{1}{1-tq^{(\alpha+\beta)m}}.$$

Upon inspection, we see that the right hand side of Buryak-Feigin is very similar to the generating function for partitions, though some partitions have an added variable t. The power of this added variable t counts the number of parts which are divisible by $\alpha + \beta$. If we set $t = 1$ in Buryak-Feigin, we recover the generating function for partitions, $P(q)$. These observations leads us to believe that both sides of the identity can be interpreted in terms of partitions, motivating the following conjecture.

Conjecture 5.10. *There is a bijection between sets of partitions which proves Buryak-Feigin.*

The major argument for this conjecture occurs when looking at subcases of Buryak-Feigin, specifically when $t = 0$ and we start investigating different values for α and β. When doing this we find that Buryak-Feigin is an extension of Euler's Theorem.

Proposition 5.11. *Euler's theorem is a special case of Buryak-Feigin.*

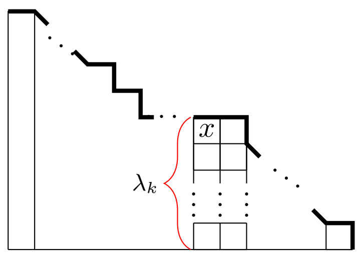
Proof. Consider Buryak-Feigin (B-F). Let $\alpha = \beta = 1$ and $t = 0$. Then we find that B-F becomes

$$\sum_{\lambda \in \mathcal{P}} 0^{dim_{1,1}(\lambda)} q^{|\lambda|} = \prod_{\substack{m \geq 1 \\ 2 \nmid m}} \frac{1}{1 - q^m} = O(q). \tag{3}$$

Now the terms on the left hand side are zero unless $dim_{1,1}(\lambda) = 0$. Thus it remains to show that $dim_{1,1}(\lambda) = 0$ if and only if $\lambda \in \mathcal{D}(n)$.

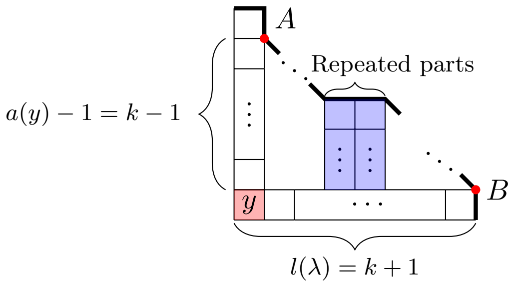
($\Rightarrow$) Let $\lambda \in \mathcal{P}(n)$ with $dim_{1,1}(\lambda) = 0$. Suppose that λ has a repeated part λ_k and so $\lambda \notin \mathcal{D}(n)$. Then draw the young diagram of λ (in vertical notation) and consider the square x which is at the top of the penultimate part of size λ_k. Then $a(x) = 0$ and $l(x) = 1$, so $d_{1,1}(x) = 0$ (see picture below) and hence $dim_{1,1}(\lambda) > 0$. This is a contradiction and therefore $\lambda \in \mathcal{D}(n)$.

$\lambda =$



($\Leftarrow$) Let $\lambda \in \mathcal{D}(n)$. Suppose $dim_{1,1}(\lambda) > 0$. Then there exists a square $y \in \lambda$ with $d_{1,1}(y) = 0$ and so $l(y) = a(y) + 1$. If $a(y) = 0$ then $l(y) = 1$ and so we have a repeated part, as y would be equivalent to the x in the above diagram. Thus assume $a(y) = k > 0$ and then $l(y) = k + 1$. We shall use the pigeon hole principle to show that λ must have a repeated part in this case.

Consider the young diagram of λ localised at y. Set the bottom right corner of the hand of y as A and the top right corner of the foot of y as B. Then the vertical difference between A and B is $a(y) - 1 = k - 1$. The horizontal difference between A and B is the the leg of $y = k + 1$. Thus the boundary of the partition must have $k - 1$ down steps and $k + 1$ right steps and so by the pigeon hole principle, there must be 2 right steps next to each other. Hence there is a repeated part in λ, namely the parts directly under the consecutive right steps on the boundary. This is a contradiction and so $dim_{1,1}(\lambda) = 0$.



Therefore $dim_{1,1}(\lambda) = 0$ if and only if $\lambda \in \mathcal{D}(n)$ and we are done. □

The fact that Buryak-Feigin is an extension of Euler's Theorem is remarkable in it's own right. It actually turns out that using an analogous argument, we can prove the following Proposition.

Proposition 5.12. *Glaisher's theorem is a special case of Buryak-Feigin*

Proof. Set $\beta = 1$ and $t = 0$ in B-F. We use a similar argument as in Proposition 5.11, except when showing that having no part repeated more than a times implies $dim_{\alpha,1}(\lambda) = 0$, we look for $a + 1$ consecutive right steps using the pigeon hole principle, rather than 2. □

Now we have seen links between bijectively provable theorems and Buryak-Feigin, Conjecture 5.10 seems very reasonable. The remainder of this paper will see us attempt to make steps towards a proof of this conjecture. We will not produce a complete proof of the conjecture as such a proof is currently unknown. We shall however use a bijection to construct the set of partitions which have the maximum possible $(\alpha, 1)$-dimension. First we investigate the $(1, 1)$ case.

6. First Steps

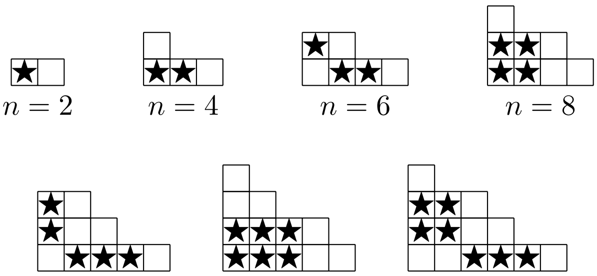
In this section we shall consider the $dim_{1,1}$ case of Buryak-Feigin. We have already seen in this case that the partitions of n with $dim_{1,1} = 0$ are in bijection with the distinct partitions of n. This leads us to speculated about what the partitions of n with the maximum possible $dim_{1,1}$ look like.

We shall start by re-stating Buryak-Feigin in the dimension $(1, 1)$ case.

$$\sum_{\lambda \in \mathcal{P}} t^{dim_{1,1}(\lambda)} q^{|\lambda|} = \prod_{m \text{ odd}} \frac{1}{1 - q^m} \prod_{m \geq 1} \frac{1}{1 - tq^{2m}}. \tag{4}$$

From this equation we see that if $n = 2k + r$ for $r \in \{0, 1\}$ and $k \in \mathbb{Z}_{\geq 0}$ then the maximum possible value of $dim_{1,1}$ will be k and that the partition with $dim_{1,1} = k$ will be unique. This is since a partition of n with maximum $dim_{1,1}$ corresponds to taking k terms of the form $\frac{1}{1-tq^2}$ and r terms of the form $\frac{1}{1-q}$ in the product on the right hand side of equation 4.

We now know that these maximal dimension partitions are unique for each value of n. The next step is to calculate the $dim_{1,1}$ of all partitions of a small n to find them and see if they share a similar trait. We shall start by doing this for small, even n since the required $\frac{1}{1-q}$ term for odd n may change the pattern. Below are the unique maximal partitions for the first 7 even values of n, with the $\frac{n}{2}$ squares which have $d_{1,1} = 0$, marked with a star.

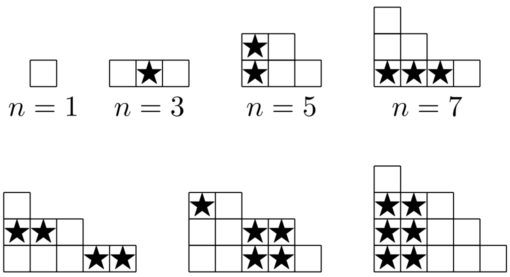


$n = 2$ $n = 4$ $n = 6$ $n = 8$

$n = 10$ $n = 12$ $n = 14$

Remarkably, there is a pattern to be seen here. Each successive partition above is acquired by applying the inverse Dyson's map of order -2 (as in Section 3.3) to the previous partition, starting with the empty partition. Indeed it seems that the unique partition of $2k$ with $dim_{1,1} = k$ is $\psi_{-2}^{-k}(0)$.

A similar pattern can be seen below with the first few maximal partitions for odd values of n. These partitions will have $\frac{n-1}{2}$ squares with $d_{1,1} = 0$, which again have been marked with a star.



$n = 1$ $n = 3$ $n = 5$ $n = 7$

$n = 9$ $n = 11$ $n = 13$

These partitions have again been obtained from the last by applying the inverse Dyson's map of order -2. This implies that the maximal $dim_{1,1}$ partition of $2k + 1$ seems to be $\psi_{-2}^{-k}(1)$.

So far we have identified a possible pattern and possible description of the set of partitions with maximum possible $dim_{1,1}$. It turns out that this pattern does hold.

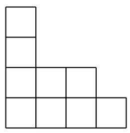
Theorem 6.1. *Let $n = 2k + r$ for $k \in |z_{\geq 0}$ and $r \in \{0, 1\}$. Then the unique partition of n, with $dim_{1,1} = k$ is given by $\psi_{-2}^{-k}(r)$.*

Proving this theorem will take some work. We start by introducing a few new definitions so we can prove that the map ψ_{-2}^{-1} increases the $dim_{1,1}$ of a partition by 1. Next we show that we can always apply this map to the partitions (0) and (1). Combined these will allow us to prove Theorem 6.1.

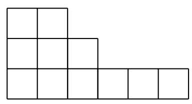
Remark 6.2. If $\lambda = (\lambda_1, \cdots, \lambda_m)$ is a partition of n, then $\psi_{-2}^{-1}(\lambda)$ is a valid partition of $n + 2$ if only if $\lambda \in \mathcal{G}(n, -3)$. This is since the newly added bottom row of $\psi_{-2}^{-1}(\lambda)$ must be at least as long as the row above it. This means that $\lambda_1 + 2$, the size of the added new row, must be longer than the bottom row of λ minus 1, $m-1$. Thus $r(\lambda) = \lambda_1 - m \geq -3$ and so $\lambda \in \mathcal{G}(n, -3)$.

Definition 6.3. The young diagram of a partition $\lambda \in \mathcal{G}(n, -3)$ is said to be in ψ_{-2}^{-1}*-notation* if the vertical notation young diagram of λ is placed with its corner on the lattice point with coordinates $(-1, 0)$ and the vertical notation young diagram of $\psi_{-2}^{-1}(\lambda)$ is placed with its corner at the lattice point with coordinates $(0, -1)$. This is so that the corner of λ is one square north west of the corner of $\psi_{-2}^{-1}(\lambda)$.

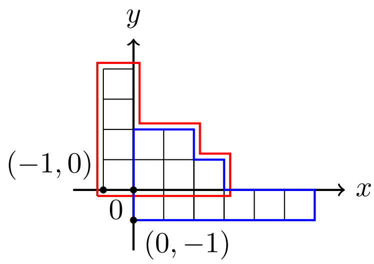
Example 6.4. Let $\lambda = (4, 2, 2, 1)$. Below we show the the vertical notation young tableau of λ and $\psi_{-2}^{-1}(\lambda)$ as well as the ψ_{-2}^{-1}*-notation* young diagram of λ. In the ψ_{-2}^{-1}*-notation* young diagram, λ is all the squares inside the red section and the boundary of $\psi_{-2}^{-1}(\lambda)$ has been coloured blue.



Vertical notation λ



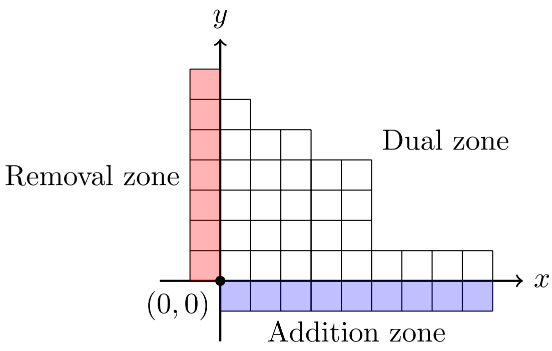
Vertical notation $\psi_{-2}^{-1}(\lambda)$



ψ_{-2}^{-1}-notation λ

Definition 6.5. Take the ψ_{-2}^{-1}-notation young diagram of a partition λ. The section of this young diagram which is in λ but not $\psi_{-2}^{-1}(\lambda)$ is called the removal zone. Similarly the section of this young diagram which is in $\psi_{-2}^{-1}(\lambda)$ but not λ, is called the addition zone. These correspond to the first part of λ and the bottom row of $\psi_{-2}^{-1}(\lambda)$ respectively. Any square in the young diagram which is not in either the removal or addition zone is in both λ and $\psi_{-2}^{-1}(\lambda)$. We call this section the dual zone.

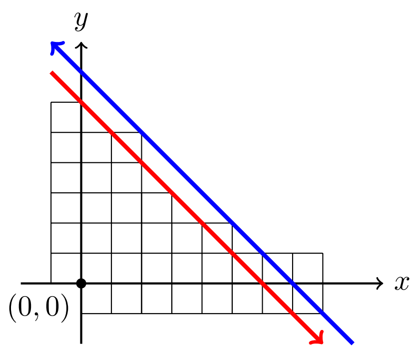
Example 6.6. Consider the partition $(7, 6, 5^2, 4^2, 1^4)$. We draw the ψ_{-2}^{-1}-notation young diagram below. In this diagram, the removal zone has been shaded red, the removal addition zone blue and the dual zone left unshaded.



Definition 6.7. The *hand line* of λ is the line on the ψ_{-2}^{-1}-notation young-tableaux of λ with gradient -1 which passes through the top right hand corner of the first part of λ, i.e. the line with equation $y = -x + \lambda_1$. We orient the line so that it has a south-east direction.

Definition 6.8. We define the *foot line* of λ to be the line on the ψ_{-2}^{-1}-notation young-tableaux of λ with gradient -1 which passes through the bottom right hand corner of the bottom row of $\psi_{-2}^{-1}(\lambda)$. Since this point has coordinates $(\lambda_1 + 2, -1)$ this line also passes through the point $(0, \lambda_1 + 1)$ so has equation $y = -x + \lambda_1 + 1$. We orient the line so that it has a north-west direction

Example 6.9. Take the ψ_{-2}^{-1}-notation young-tableaux of $\lambda = (6, 5^2, 3, 2^2, 1^3)$. In the diagram below we will add the hand line in red and the foot line in blue.



Lemma 6.10. *A square x in the removal zone with $d_{1,1}(x) = 0$ corresponds to the hand line leaving the partition at the bottom right hand corner of the foot of x. We call these points the hand line leaving points.*

Similarly a square y in the addition zone with $d_{1,1}(y) = 0$ corresponds to the foot line leaving the partition at the top right hand corner of the hand of y. We call these points the foot line leaving points.

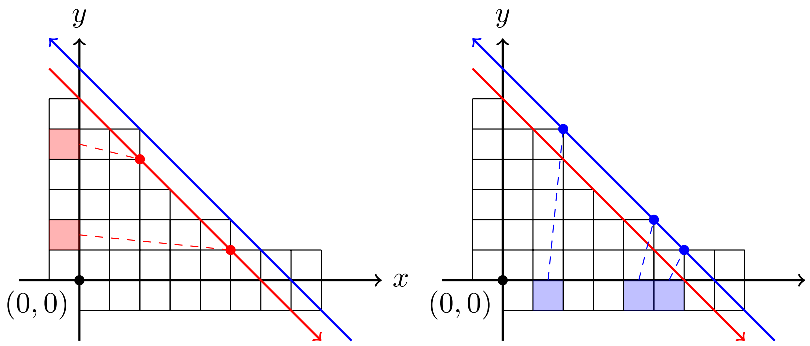
Proof. For the hand line leaving points, consider the the square x at the top of the first part. This square has no squares in it's arm so $d_{1,1}(x) = 0$ if the square y, directly to the right of x, is the last square on that row. The hand line will always intersect y, (whether it is in the partition of not) at it's bottom right corner.

Now the required increase in leg for the square s which is r rows directly below x will be precisely r. Thus we need the square $r + 1$ to the right of

s to be the last square on it's row. But since the gradient of the hand line is -1, if this square is the last on it's row then the hand line intersects it at it's bottom right corner.

We can use a similar argument for the foot line leaving points. $\square$

Example 6.11. Take the partition $\lambda = (6, 5^2, 3, 2^2, 1^3)$. We shall demonstrate how this correspondence works.



Theorem 6.12. *If* $\lambda \in \mathcal{G}(n, -3)$*, then* $dim_{1,1}(\psi_{-2}^{-1}(\lambda)) = dim_{1,1}(\lambda) + 1$.

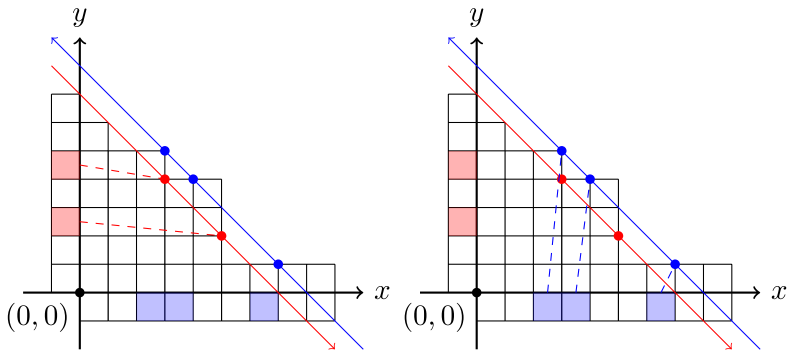
Note 6.13. The diagrams will show how to apply the method used in the proof to the partition $(7, 6, 5^2, 4^2, 1^4)$. The proof will detail why this process works in general.

Proof. Take a partition $\lambda \in \mathcal{G}(n, -3)$. We know that $\psi_{-2}^{-1}(\lambda)$ is a valid partition so draw the ψ_{-2}^{-1}-notation young diagram of λ. We want to show that there is one more square $\square$ in $\psi_{-2}^{-1}(\lambda)$ with $d_{1,1}(\square) = 0$ than in λ.

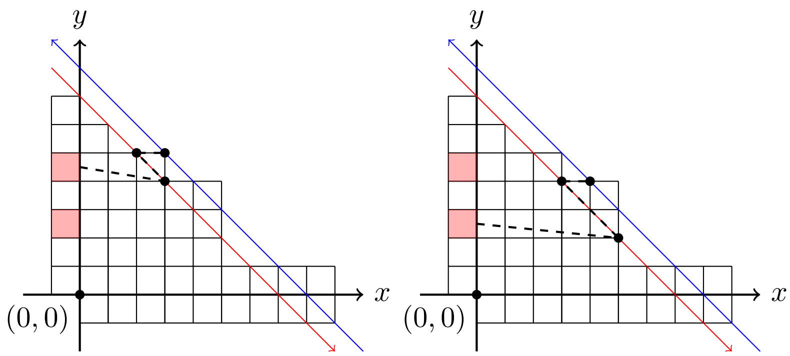
Localization to any square in a partition clearly doesn't affect the value of $d_{1,1}$ for a square in the localisation. More specifically if $d_{1,1}(x) = a$ in a localisation of λ containing the square x, then $d_{1,1}(x) = a$ in λ. Thus, as all squares in the dual zone are in both λ and $\psi_{-2}^{-1}(\lambda)$, we can discount these squares. This means that we only need to prove there is one more square $\square$ with $d_{1,1}(\square) = 0$ in the addition zone than in the removal zone.

To show this we will construct an injective map ϕ which sends a $d_{1,1} = 0$ square in the removal zone to a $d_{1,1} = 0$ square in the addition zone. We shall then prove that there will be one, and only one, $d_{1,1} = 0$ square in the addition zone that ϕ does not map to. We will use the hand and foot lines to do this.

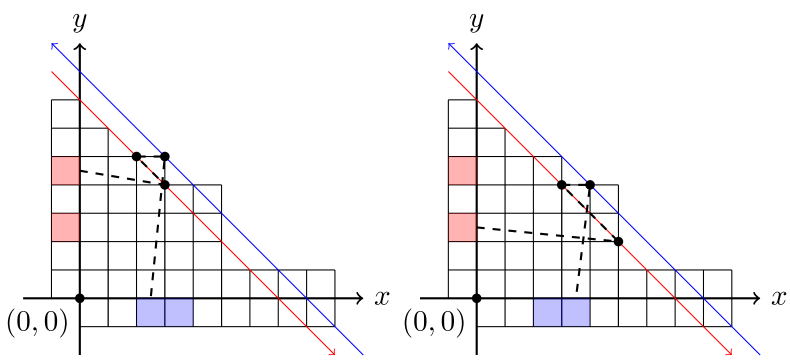
Take the ψ_{-2}^{-1}-notation young diagram of λ and add the hand and foot lines. By Lemma 6.10 we know that the $d_{1,1} = 0$ squares in the removal zone correspond to a unique leaving point of the hand line and that $d_{1,1} = 0$ squares in the addition zone correspond to a unique leaving points of the foot line.



Now consider these hand line leaving points. They must be preceeded by a unique entering point. We first map the leaving points back up the hand line to this unique proceeding entering point. This will be at the top left hand corner of a square of the partition. We now map this entering point horizontally across to the foot line.

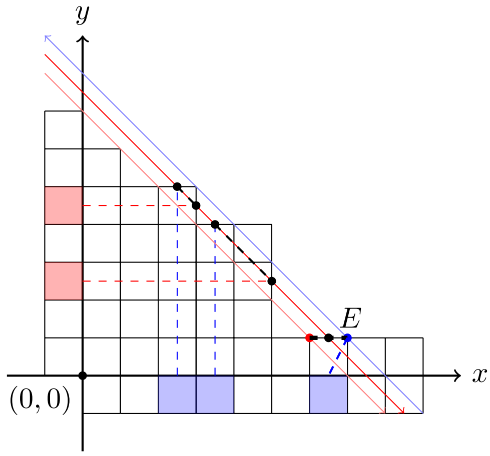


This point on the foot line must be an leaving point of the foot line. If it were not then the hand line "entering point" would be on the apex of a ⌝ shaped corner, which cannot be a hand line entering point (since it must then also be the leaving point, which contradicts the definition of a leaving point). We can now map this foot line leaving point to its correspondent removal zone $d_{1,1} = 0$ square. At each step the point acquired was uniquely determined and thus ϕ is injective.



It remains to show that there is precisely one $d_{1,1} = 0$ square in the addition zone that we haven't associated to a $d_{1,1} = 0$ square in the removal zone. Consider a line that runs directly between the hand and foot lines, with south-east orientation. This line starts outside the partition and finishes inside. Thus it has one more entering point than leaving point. Call the last entering point E.

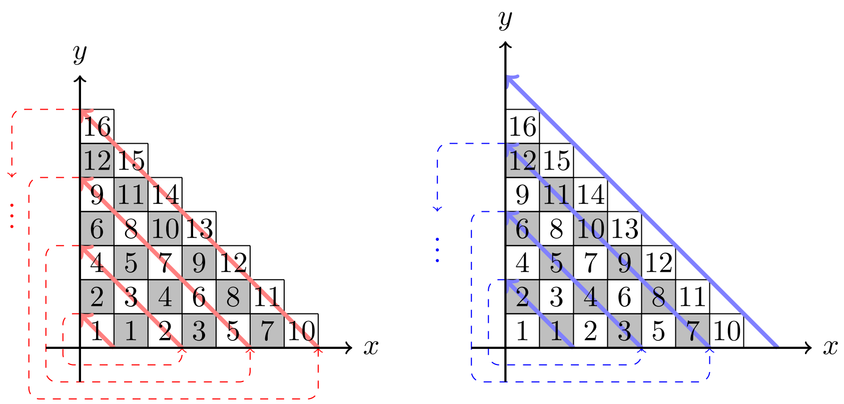
The hand line entering point to the left of E will not correspond to a $d_{1,1} = 0$ square in the removal zone (as there is no hand line leaving point after it), but the foot line leaving point to the right of E will correspond to a $d_{1,1} = 0$ square in the addition zone. All entering points other than E are directly between a hand line entering and foot line leaving point which are "joined" together by ϕ.



Hence there is exactly one more $d_{1,1} = 0$ square in the addition zone than in the removal zone and so $dim_{1,1}(\psi_{-2}^{-1}(\lambda)) = dim_{1,1}(\lambda) + 1$. □

Now we have proven that ψ_{-2}^{-1} increases the $dim_{1,1}$ of a partition, we have almost finished. It remains to show that we are able to repeatedly apply this map to the partitions (0) and (1). To show this we will have to introduce a new way to look at the Dyson's map.

Definition 6.14. The numbered checkerboad grid is a checkerboard grid (i.e. a grid shaded alternatively black and white) with boxes of each colour numbered by following diagonal north-west directed lines that start at the x-axis as demonstrated below.



Definition 6.15. Take a partition λ and colour and number it according to this checkerboard grid. Let φ be the map which sends each square of λ

to the next numbered square of that colour in the checkerboard numbering (i.e. a square numbered x gets mapped to the square numbered $x+1$ of the same colour) and then adds 2 new squares to the partition at the boxes numbered 1 in the checkerboard grid numbering.

Proposition 6.16. $\varphi = \psi_{-2}^{-1}$

Proof. It is clear from the diagrams in Definition 6.14 that φ will remove the first column of the partition, slide every other square 1 box diagonally north-west, and then add a new bottom row consisting of the removed squares plus an additional two. Thus the two maps are identical. □

We can now prove the following lemma.

Lemma 6.17. *$\psi_{-2}^{-k}(0)$ is the partition made up of all the black and white squares numbered up to k. Similarly $\psi_{-2}^{-k}(1)$ will be the partition with the first k black squares and the first $k+1$ white squares.*

Proof. Follows immediately from the φ description of ψ_{-2}^{-1}. □

Corollary 6.18. *$\psi_{-2}^{-k}(0) \in \mathcal{G}(2k,-3)$. Similarly $\psi_{-2}^{-k}(1) \in \mathcal{G}(2k+1,-3)$.*

Proof. Consider the diagonal lines which we use to number the checkerboard grid. Since both partitions correspond to filling the lines in order up to a fixed value (for each k), we can have at most one incomplete line in each colour, specifically the furthest diagonals from the origin. We can't have two incomplete lines of the same colour since the partition would not then consist of the first k (or $k+1$ white squares in $\psi_{-2}^{-k}(1)$). Thus the bottom row of the partition can be at most 2 longer than the first part and so the rank must be greater than or equal to -2. Therefore $\psi_{-2}^{-k}(0) \in \mathcal{G}(2k,-3)$. □

The proof of Theorem 6.1 now follows directly from Corollary 6.18 and Theorem 6.12. Therefore we have now found that the unique partition of $n = 2k+r$, for $k \in \mathbb{Z}_{\geq 0}$ and $r \in \{0,1\}$, is $\psi_{-2}^{-k}(r)$.

7. Generalisation to the $(\alpha, 1)$ case

Earlier we mentioned how it was possible to generalise our understanding of minimal dimension partitions from the $dim_{1,1}$ case to the $dim_{\alpha,1}$, seeing that partitions $\lambda \in \mathcal{P}(n)$ with $dim_{\alpha,1}(\lambda) = 0$ are in bijection with partitions where no part is repeated more than α times. We now know that the maximal dimension partitions in the 1,1 case, are in bijection with the set of partitions $\mathcal{B}_{1,1} = \{\psi_{-2}^{-k}(r) \mid k \in \mathbb{Z}_{\geq 0}, r \in \{0,1\}\}$. We wish to generalise this to a higher dimensional case.

We start much the same as in the $dim_{1,1}$ case. In the $(\alpha,1)$ case Buryak-Feigin reduces to the following identity:

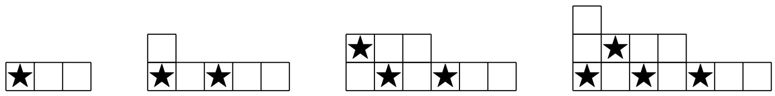
$$\sum_{\lambda \in \mathcal{P}} t^{dim_{\alpha,1}(\lambda)} q^{|\lambda|} = \prod_{\substack{m \geq 1 \\ (\alpha+1) \nmid m}} \frac{1}{1-q^m} \prod_{m \geq 1} \frac{1}{1-tq^{(\alpha+1)m}} \tag{5}$$

From this we see that, in much the same way as in the $(1,1)$ case, if $n = (\alpha+1)k + r$, for $k \in \mathbb{Z}_{\geq 0}$ and $r \in \{0,1,\cdots,\alpha\}$, then the maximal possible value of $dim_{\alpha,1}$ is k. However it is no longer necessarily the case that a partition $\lambda \in \mathcal{P}(n)$ with $dim_{\alpha,1} = k$ is unique. In fact it is clear to see that for our setup of n, the number of maximal partitions will be the number of partitions of r, $p(r)$.

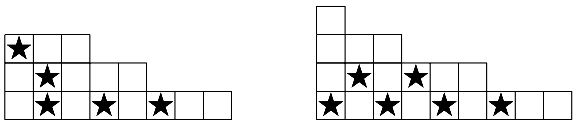
We however will look at just the case where $r = 0$ and so $n = k(\alpha + 1)$. This will continue to give us a unique partition with $dim_{\alpha,1} = k$. Given what we know about the $dim_{1,1}$ case, it may seem reasonable to conjecture that these partitions are of the form $\psi_{\alpha+1}^{-k}(0)$, however this is not the case. Something slightly more subtle is afoot, as we shall see by taking a closer look at the $dim_{2,1}$ and $dim_{3,1}$ cases.

Example 7.1 (Maximal $dim_{2,1}$ partitions)**.** First recall that a square $\square$ in a partition λ has $d_{2,1}(\square) = 0$ if $2(a(\square)+1) - l(\square) = 0$ and that $dim_{2,1}(\lambda) = |\{\square \in \lambda \mid d_{2,1}(\square) = 0\}|$.

We will only consider the case where $r = 0$ and so will look at partitions of $n = 3k$. The unique partition of $3k$ with maximum possible $dim_{2,1}$ will have k squares with $d_{2,1} = 0$. Thus, for small k, we can find these partitions by inspection. The unique partition of $3k$ with $dim_{2,1} = k$ for $k = 1$ to $k = 6$ are displayed below, with the squares with $d_{2,1} = 0$ containing a star.



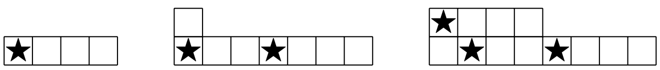
$n = 3$ $n = 6$ $n = 9$ $n = 12$



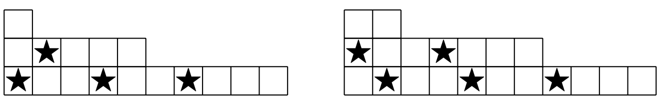
$n = 15$ $n = 18$

From these partitions we immediately see that we cannot use the Dyson's map to acquire the maximal partition of $3(k+1)$ from the partition of $3k$. There does however, seem to be a Dyson-like map which does allow us to easily find the next maximal partition from the previous. This map sees us removing the first two columns of the partition before adding a new bottom row consisting of the removed squares as well as an additional 3 squares.

Example 7.2 (Maximal $dim_{3,1}$ partitions)**.** The unique partition of $n = 4k$ with maximum possible value of $\mathrm{dim}_{3,1}$, will have k squares with $d_{3,1} = 0$. The partitions for $k = 1$ to $k = 5$ can be seen below.



$n = 4$ $n = 8$ $n = 12$



$n = 16$ $n = 20$

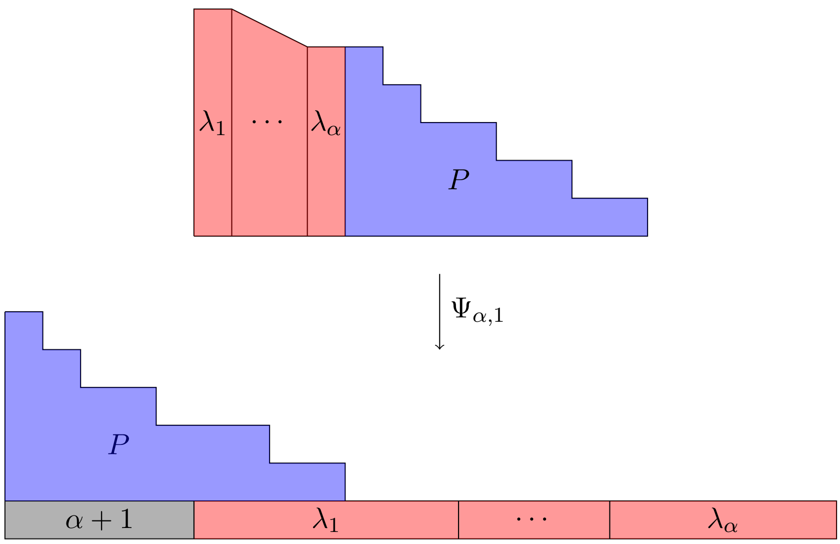
Again we can see that these partitions can be obtained by applying another Dyson-like map to the partition that proceeded it in the sequence. This time the map removes the first 3 columns and adds a single new row at the bottom composed of the removed squares plus 4 additional squares.

We have now seen that in both the $dim_{2,1}$ and $dim_{3,1}$ cases, we seem to obtain the maximal partitions from a similar Dyson-like map. The aim for the rest of the section is to formally define this map and then prove that it can be used to obtain the maximal $dim_{\alpha,1}$ partitions of $n = (\alpha+1)k$, much in the same way as we did in the previous section with the $dim_{1,1}$ case.

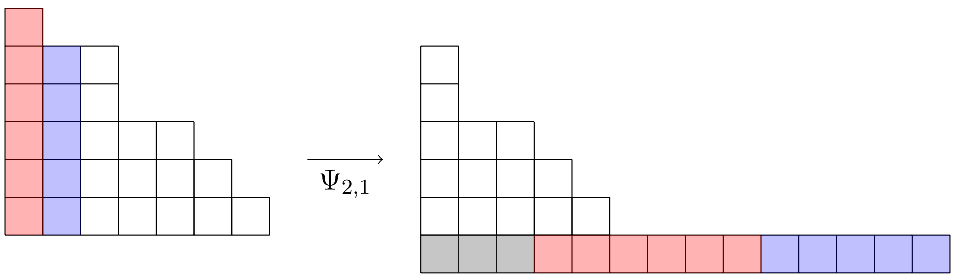
Definition 7.3. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m) \in \mathcal{P}(n)$. We define the $(\alpha, 1)$-*rank* of λ to be $r_{\alpha,1}(\lambda) = \lambda_1 + \lambda_2 + \cdots + \lambda_\alpha - m$.

Definition 7.4. We define $\mathcal{H}_{\alpha,1}(n, r)$ to be the set of partitions of n with an $(\alpha, 1)$-rank of at most r. Similarly we define $\mathcal{G}_{\alpha,1}(n, r)$ to be the set of partitions with an $(\alpha, 1)$-rank of at least r.

Definition 7.5. We shall define the modified $(\alpha, 1)$-Dyson's map $\Psi_{\alpha,1}$: $\mathcal{G}_{\alpha,1}(n, -2\alpha - 1) \to \mathcal{H}_{\alpha,1}(n + \alpha + 1, -1)$ to be the Dyson-like map which takes a partition, removes the first α parts and adds a new bottom row consisting of the removed squares plus an additional $\alpha + 1$ squares, i.e. of size $\lambda_1 + \cdots + \lambda_\alpha + \alpha + 1$. This process can be seen pictorially in the following diagram.



Example 7.6. Consider the partition $\lambda = (6, 5^2, 3^2, 2, 1) \in \mathcal{G}_{2,1}(23, -5)$. We shall find $\Psi_{2,1}(\lambda)$.



So $\Psi_{2,1}(\lambda) = (6, 4^2, 3, 2, 1^9) \in \mathcal{H}_{2,1}(26, -1)$.

Remark 7.7. If $\lambda = (\lambda_1, \cdots, \lambda_m)$ is a partition, then for $\Psi_{\alpha,1}(\lambda)$ to be a valid partition, we must have that $\lambda \in \mathcal{G}_{\alpha,1}(n, -2\alpha - 1)$. This is since the new bottom row must be longer than the row directly above it, i.e. $\lambda_1 + \cdots + \lambda_\alpha + \alpha + 1 \geq m - \alpha$ and so $r_{\alpha,1}(\lambda) \geq -2\alpha - 1$. It is also easy to show that $\Psi_{\alpha,1} \in \mathcal{H}_{\alpha,1}(n + a + 1, -1)$, since

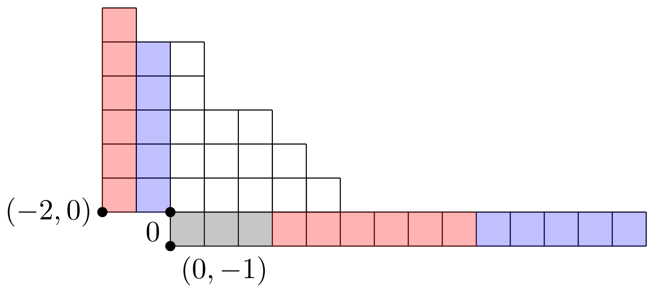
$$\begin{aligned} r_{\alpha,1}(\Psi_{\alpha,1}(\lambda)) &= (\lambda_{\alpha+1} + \cdots + \lambda_{2\alpha} + \alpha) - (\lambda_1 + \cdots + \lambda_\alpha + \alpha + 1) \\ &= ((\lambda_{\alpha+1} + \cdots + \lambda_{2\alpha}) - (\lambda_1 + \cdots + \lambda_\alpha) - 1. \end{aligned}$$

Now the $(\alpha, 1)$-rank of $\Psi_{\alpha,1}(\lambda)$ is maximal if $\lambda_{2\alpha} = \lambda_1$, which gives us that for $1 \leq i \leq 2\alpha$, $\lambda_i = \lambda_1$. Hence $r_{\alpha,1}(\Psi_{\alpha,1}(\lambda)) \leq (\alpha \cdot \lambda_1) - (\alpha \cdot \lambda_1) - 1 = -1$ and so $\Psi_{\alpha,1} \in \mathcal{H}_{\alpha,1}(n + \alpha + 1, -1)$.

Recall the ψ_{-2}^{-1}-notation for young diagram. We wish to create a new analogous version of this notation for the map $\Psi_{\alpha,1}$.

Definition 7.8. Let the $\Psi_{\alpha,1}$-notation young diagram of a partition λ be created by first placing the standard notation young diagram of λ so that it's corner is at the lattice point $(-\alpha, 0)$, then superimposing the standard notation young diagram of $\Psi_{\alpha,1}$ so that it's corner is at the lattice point $(0, -1)$.

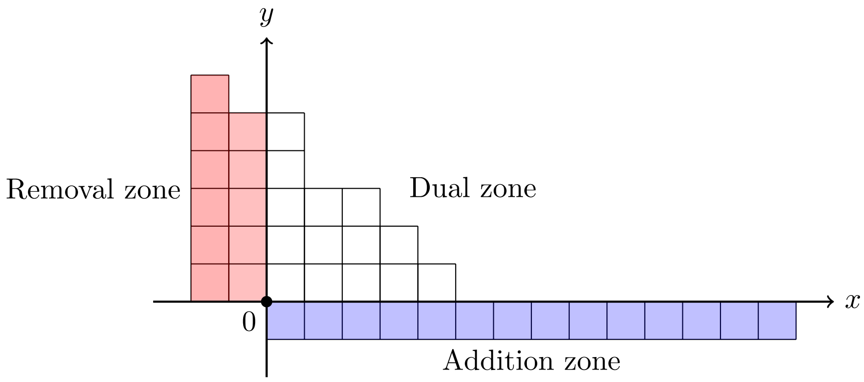
Example 7.9. The $\Psi_{2,1}$-notation young diagram of $\lambda = (6, 5^2, 3^2, 2, 1)$ is given by the young diagram below.



Definition 7.10. Take the $\Psi_{\alpha,1}$-notation young diagram of a partition λ. In an analogous way to as in the ψ_{-2}^{-1}-notation young diagram, the section of this young diagram which is in λ but not $\Psi_{\alpha,1}(\lambda)$ is called the removal zone and the section of this young diagram which is in $\Psi_{\alpha,1}(\lambda)$ but not λ, is called the addition zone.

These correspond to the first α parts of λ and the bottom row of $\Psi_{\alpha,1}(\lambda)$ respectively. Any square in the young diagram which is not in either the removal or addition zone is in both λ and $\Psi_{\alpha,1}(\lambda)$ and we call this section the dual zone.

Example 7.11. Consider the partition $(6, 5^2, 3^2, 2, 1)$. We draw the $\Psi_{\alpha,1}$-notation young diagram below. In this diagram, the removal zone has been shaded red, the addition zone blue and the dual zone left unshaded.



Definition 7.12. Take $1 \leq i \leq \alpha$. The *ith hand line* of λ is the line on the $\Psi_{\alpha,1}$-notation young-tableaux of λ with gradient $-\frac{1}{\alpha}$ which passes through the top right hand corner of the ith part of λ. Since this line passes through the point $(-\alpha+i, \lambda_i)$, it passes through the y-axis at the point $(0, \lambda_i + \frac{i-\alpha}{\alpha})$. Thus this is the line with equation

$$y_i = -\frac{1}{\alpha}x_i + (\lambda_i + \frac{i-\alpha}{\alpha}).$$

We orient the line so that it has a south-east direction.

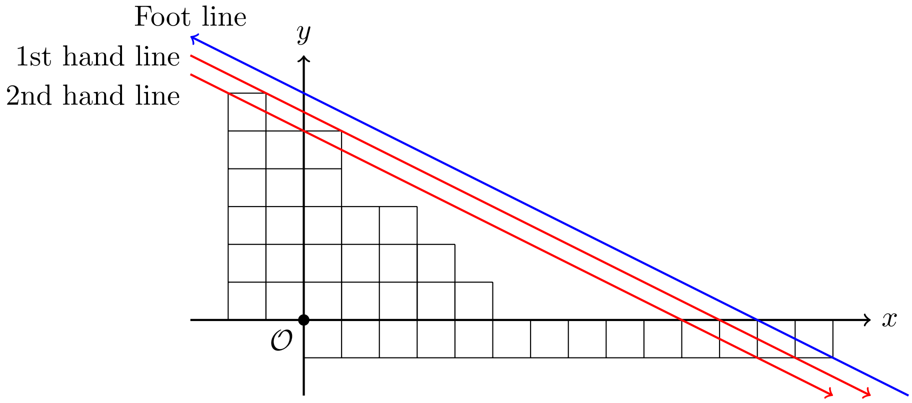
Definition 7.13. Similarly we define the *foot line* of λ to be the line on the $\Psi_{\alpha,1}$-notation young-tableaux of λ with gradient $-\frac{1}{\alpha}$ which passes through the bottom right hand corner of the bottom row of $\Psi_{\alpha,1}(\lambda)$. Since this point has coordinates $(\lambda_1 + \cdots + \lambda_\alpha + \alpha + 1, -1)$, this line also passes through the point $(0, \frac{\lambda_1+\cdots+\lambda_\alpha+1}{\alpha})$ so has equation

$$y = -\frac{1}{\alpha}x + \frac{\lambda_1 + \cdots + \lambda_\alpha + 1}{\alpha}.$$

We orient the line so that it has a north-west direction

Note 7.14. The ith hand line can be below the $(i+1)$th hand line. The number of the line relates to which part it passes through the top right corner of, not the order they appear. For example we could have that the 2nd hand line is the highest, the 1st next, then the 4th and finally the 3rd being the lowest. Due to the fixed gradient of $-\frac{1}{\alpha}$ which all the lines have, we can't have a hand line which passes through the top right corner of more than one part in the removal zone, i.e. this numbering system is well-defined as the ith hand line can't also be the jth hand line, unless $i = j$.

Example 7.15. Consider the partition $(6, 5^2, 3^2, 2, 1)$. Below is the $\Psi_{\alpha,1}$-notation young-tableaux with all hand and foot lines added. The hand lines are in red and the foot line in blue.



Lemma 7.16. *A square x in the ith part of the partition (the ith part in the removal zone) with $d_{\alpha,1}(x) = 0$ corresponds to the ith hand line leaving the partition at the bottom right hand corner of the hand of x. We call these points the ith hand line leaving points.*

Similarly square y in the addition zone with $d_{\alpha,1}(y) = 0$ corresponds to the foot line leaving the partition at the top right hand corner of the foot of y. We call these points the foot line leaving points.

Proof. Analogous to the proof of Lemma 6.10 □

Lemma 7.17. *The ith hand line is below the foot line if and only if*

$$\lambda_i < \frac{\lambda_1 + \cdots + \lambda_\alpha + \alpha + 1 - i}{\alpha}.$$

Proof. This is clear from definitions 7.12 and 7.13 that the ith hand line is below the foot line if and only if

$$\lambda_i + \frac{i - \alpha}{\alpha} < \frac{\lambda_1 + \cdots + \lambda_\alpha + 1}{\alpha}$$

and thus the result follows. □

We are getting closer to being able to prove that we can construct the maximal $dim_{\alpha,1}$ partitions from our modified Dyson's map $\Psi_{\alpha,1}$. Before we prove this, we first need to prove the proposition, which will become a vital condition for the proof to work.

Proposition 7.18. *All the hand lines are below the foot line if and only if $\lambda_1 - \lambda_\alpha \leq 1$.*

Proof. ($\Rightarrow$) Let all the hand lines be below the foot line. Now the highest hand line will clearly be the last i such that $\lambda_i = \lambda_1$. If this is λ_α then we are done, since $\lambda_1 - \lambda_\alpha = 0 \leq 1$. Thus let this drop happen after the kth part, for some $1 \leq k < \alpha$. Then $\lambda_1 = \lambda_2 = \cdots = \lambda_k$ and $\lambda_i = \lambda_1 - t_{i-k}$, where $t_{i-k} \in \mathbb{Z}_{>0}$, for $k < i \leq \alpha$,. Thus

$$\begin{aligned}\lambda_1 = \lambda_k &< \frac{\lambda_1 + \cdots + \lambda_\alpha + \alpha + 1 + k}{\alpha} \\ &= \frac{\alpha\lambda_1 - (t_1 + \cdots + t_{\alpha-k}) + \alpha + 1 - k}{\alpha}.\end{aligned}$$

From this we see that

$$\frac{(t_1 + \cdots + t_{\alpha-k})}{\alpha} < \frac{\alpha + 1 - k}{\alpha}$$

and thus that $t_1 + \cdots + t_{\alpha-k} - (\alpha - k) < 1$. Now since each $t_j \geq 1$,

$$\begin{aligned} 0 < t_{\alpha-k} &\leq t_{\alpha-k} + (t_1 - 1) + (t_2 - 1) + \cdots + (t_{\alpha-k-1} - 1) \\ &= t_1 + \cdots + t_{\alpha-k} - (\alpha - k - 1) \\ &< 2. \end{aligned}$$

But $t_{\alpha-k} \in \mathbb{Z}_{>0}$ so $t_{\alpha-k} = 1$ and thus $\lambda_1 - \lambda_\alpha = t_{\alpha-k} = 1$.
($\Leftarrow$) Let $\lambda_1 - \lambda_\alpha \leq 1$. Since $\lambda_\alpha \leq \lambda_1$ we have that $0 \leq \lambda_1 - \lambda_\alpha \leq 1$. Thus either $\lambda_1 = \lambda_\alpha$ or $\lambda_1 = \lambda_\alpha + 1$.

Case I: $\lambda_\alpha = \lambda_1$. In this case, the maximum hand line is clearly the αth hand line. Now

$$\lambda_\alpha = \frac{\alpha\lambda_\alpha}{\alpha} = \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_\alpha}{\alpha} < \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_\alpha + 1}{\alpha},$$

so the highest hand line is below the foot line and thus they all are.

Case II: $\lambda_\alpha = \lambda_1 - 1$. The highest hand line is going to be the last part j such that $\lambda_j = \lambda_1$. Let this be λ_k. Then $\lambda_i = \lambda_1$ if $1 \leq i \leq k$ and $\lambda_i = \lambda_1 - 1$ for $k < i \leq \alpha$. Then

$$\begin{aligned} \lambda_k = \frac{\alpha\lambda_k}{\alpha} &= \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_k + (\lambda_{k+1} + 1) + (\lambda_{k+2} + 1) + \cdots + (\lambda_\alpha + 1)}{\alpha} \\ &= \frac{\lambda_1 + \cdots + \lambda_\alpha + \alpha - k}{\alpha} < \frac{\lambda_1 + \cdots + \lambda_\alpha + \alpha + 1 - k}{\alpha}. \end{aligned}$$

Hence the highest hand line is below the foot line and so all the hand lines are. This completes the proof. ☐

Remark 7.19. If all the hand lines are below the foot line we can see that all the lines are equally spaced from each other. We order the $\alpha + 1$ lines by height, $l_1, \cdots, l_\alpha, l_{\alpha+1}$. If l_1 passes through the y-axis at the point $(0, y)$, then l_i passes through the y-axis at the point $(0, y + \frac{i-1}{a})$.

This ordering is governed by the last part of λ to be equal to λ_1. If this is λ_k, for some $1 \leq k \leq \alpha$, then it is easy to check that the ordering of the lines would be such that

$$l_j = \begin{cases} \text{the footline} & \text{if } j = \alpha + 1 \\ \text{the } (k+j)\text{th hand line} & \text{if } 1 \leq j \leq \alpha - k \\ \text{the } (k+j-\alpha)\text{th hand line} & \text{if } \alpha - k < j \leq \alpha \end{cases}.$$

Theorem 7.20. *Let $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathcal{G}_{\alpha,1}(n, -2\alpha - 1)$ with $\lambda_1 - \lambda_\alpha \leq 1$. Then $dim_{\alpha,1}(\Psi_{\alpha,1}(\lambda)) = dim_{\alpha,1}(\lambda) + 1$.*

This theorem is the $dim_{\alpha,1}$ generalisation of Theorem 6.12. The major difference is the addition of $\alpha - 1$ more hand lines. These complicate the proof. If we can show however that we can "merge" all α hand line to one "super hand line", uniquely mapping the relevant entering and leaving points of all α hand lines to entering and leaving points on the "super hand line", then the proof of Theorem 7.20 will be analogous to the proof of Theorem 6.12.

Definition 7.21. Let $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathcal{G}_{\alpha,1}(n, -2\alpha-1)$ with $\lambda_1 - \lambda_\alpha \leq 1$. The *heart line* of λ is the line on the $\Psi_{\alpha,1}$-notation young diagram of λ which is parallel to and directly between the foot line and the highest hand line. We orient the line in the same direction as the hand lines, south-east.

Remark 7.22. By Remark 7.19, the hand and foot lines are equally spaced with distance $\frac{1}{\alpha}$ between l_i and l_{i+1}. Thus from the definition of the heart line and the definition of the foot line, we see that the heart line will have equation

$$y = -\frac{1}{\alpha}x + \frac{\lambda_1 + \cdots + \lambda_\alpha + \frac{1}{2}}{\alpha}.$$

This means that the heart line will not enter or leave the partition at lattice points as the hand and foot lines do. Thus the following definition is necessary.

Definition 7.23. We define entering points of the heart line to be any point on a horizontal edge of the boundary of the partition, at which the heart line enters the partition. Similarly, leaving points of the heart line are defined as any point on a vertical edge of the boundary of a partition, where the heart line leaves the partition.

Note 7.24. The condition that the leaving points must be on the vertical edge of the boundary of the partition, is needed to exclude the times where the heart line would continue to be in the partition if a new bottom row, of equal length to the previous bottom row, was added to the partition. This is explicitly designed to stop there being a heart line leaving point on the bottom edge of $\Psi_{\alpha,1}(\lambda)$ in the $\Psi_{\alpha,1}$-notation young diagram of λ.

Lemma 7.25. *The height difference between any hand line and the heart line is less than 1.*

Proof. This follows immediately from Remark 7.19 since the heart line is below the foot line and the greatest distance between a hand line and the foot line is 1, namely between the lines l_1 and $l_{\alpha+1}$. □

Lemma 7.26. *If (x, y) is a hand line leaving point, for some $x, y \in \mathbb{Z}_{\geq 0}$, then the heart line has a leaving point at (x, y_h) where $y < y_h < y + 1$.*

Proof. Consider the hand line leaving point. It is at the bottom right hand corner of a square in the partition, so the line from (x, y) to $(x, y + 1)$ is on the boundary of the partition. But by Lemma 7.25 the maximum distance between the hand line and the heart line is less than one and thus (x, y_h) is in the partition. It will clearly be a point where the heart line leaves the partition. □

Lemma 7.27. *If (x, y_h) is a heart line leaving point, then one of the hand lines below it has a leaving point, namely the point (x, y) for the only $y \in \mathbb{Z}$ such that $y_h - 1 < y < y_h$.*

Proof. The heart line doesn't leave the partition at lattice points. Thus as the height difference between any hand line and the heart line is less than 1 and all the lines are equally spaced, one of the hand lines intersects a lattice point of the form (x, y). This point will clearly be a hand line leaving point. □

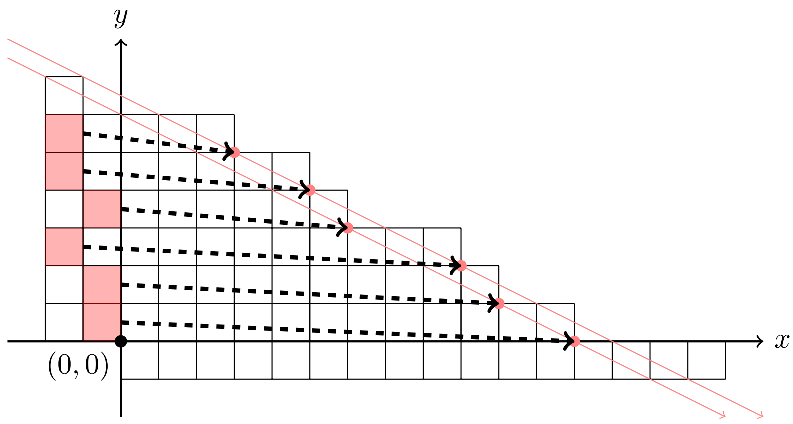
Proposition 7.28. *The set of leaving points of all hand lines is in bijection with the leaving points of the heart line.*

Proof. Follows immediately from Lemmas 7.26 and 7.27. □

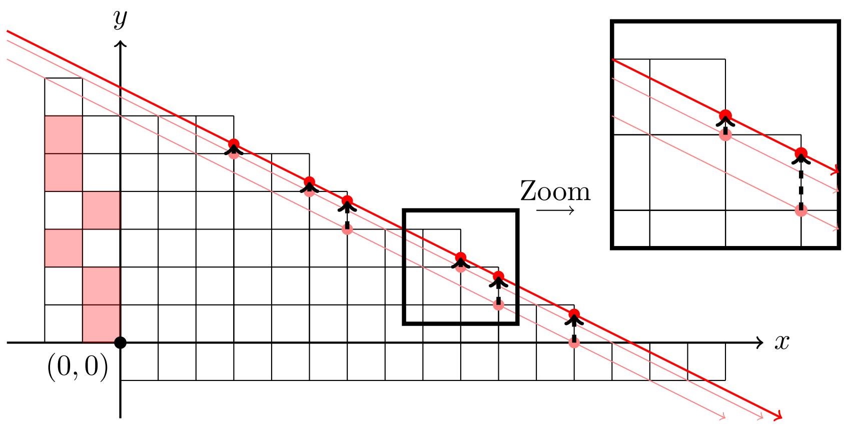
Proof of Theorem 7.20. Take the $\Psi_{\alpha,1}$-notation young diagram of λ and add the α hand lines as well as the foot and heart lines. By a similar argument as to in the proof of Theorem 6.12, we only need to show that there is one more $d_{\alpha,1} = 0$ square in the addition zone than in the removal zone.

Take a $d_{\alpha,1} = 0$ square □ in the removal zone and consider the map defined by the following algorithm. We shall show each step of this map working for the partition $\lambda = (7, 6^4, 5^2, 4, 3^3, 2, 1^2)$ in the $\dim_{2,1}$ case.

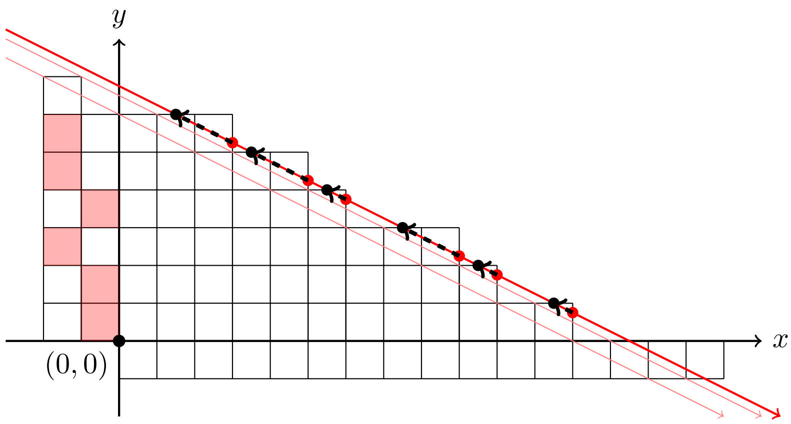
1) First map □ to it's hand line leaving point h. This is unique by Lemma 7.16.



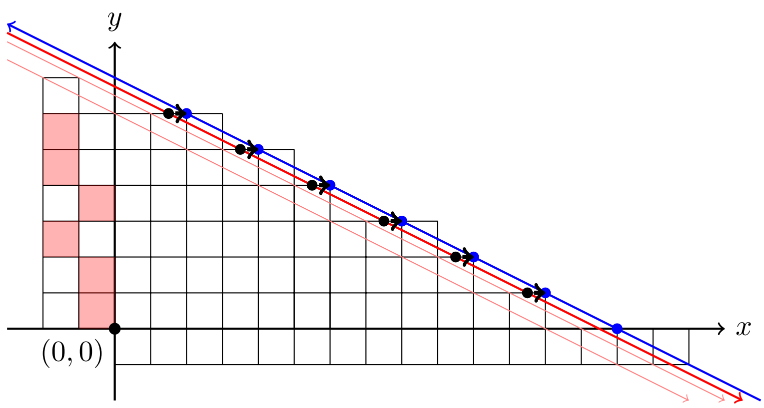
2) Next map the hand line leaving point h to it's unique heart line leaving point l. This is unique by Proposition 7.28.



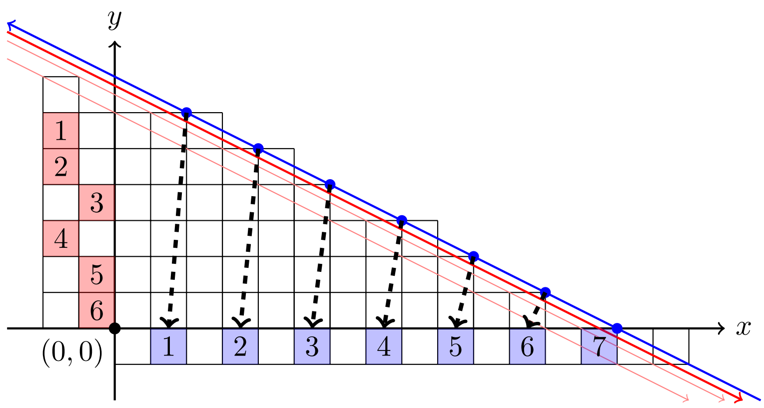
3) Now map the heart line leaving point l to it's proceeding heart line entering point e. This is clearly unique by the definition of entering and leaving points of the heart line.



4) Our next step is to map the heart line entering point e directly across horizontally to the foot line leaving point f. As e is directly in the middle of the top edge of a square on the boundary of λ, f is on the boundary of the partition. Hence f will be a foot line entering point. f is clearly uniquely determined by e.



5) Finally map f to it's corresponding $d_{\alpha,1} = 0$ square in the addition zone. This is unique by Lemma 7.16.



Thus we have uniquely mapped every $d_{\alpha,1} = 0$ square in the removal zone to a $d_{\alpha,1} = 0$ square in the addition zone.

It remains to prove that there is one, and only one, $d_{\alpha,1} = 0$ square in the addition zone which we have not mapped a removal zone $d_{\alpha,1} = 0$ square to. This is follows immediately from noticing that the heart line starts outside the partition and ends inside and thus has one more entering point than leaving (in an analogous way to Theorem 6.12's proof). □

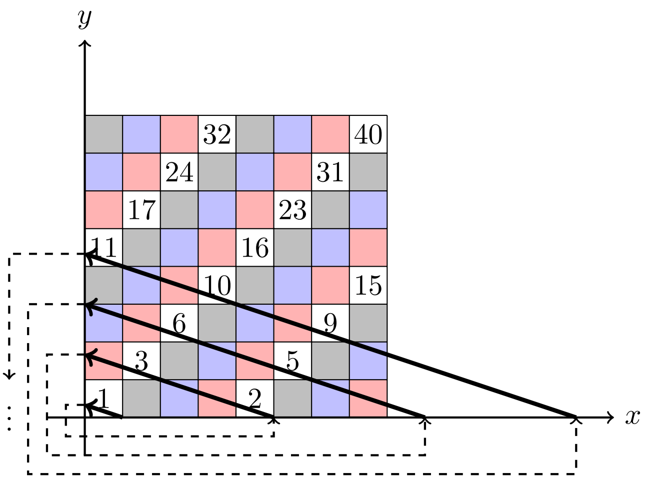
Now we know that the map $\Psi_{\alpha,1}$ will increase the $dim_{\alpha,1}$ of a partition by one under specific conditions. Now to confirm that the maximal $dim_{\alpha,1}$ partitions of $3k$ are the partitions $\Psi^k_{\alpha,1}(0)$ we simply need to check that the conditions of Theorem 7.20 hold after applying the map an undetermined amount of applications. To do this we will generalise the checkerboard style grid to the $dim_{\alpha,1}$ case.

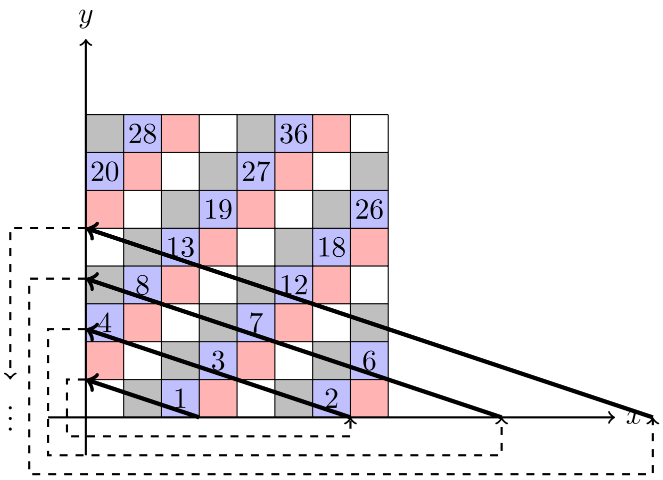
Definition 7.29. The $(\alpha, 1)$-checkerboard is the square grid which has been coloured alternately using $\alpha + 1$ colours. We do this by ordering the colours $C_0, C_1, \cdots, C_\alpha$ and then colour the square with bottom left corner at (x, y) in the colour numbered $x - y \pmod{\alpha + 1}$.

To number the squares we draw lines of gradient $-\frac{1}{\alpha}$ which pass through the points $(x, 0)$ for $x \in \mathbb{Z}$. We give these lines north-west direction and colour these lines according to the value of $x-1 \pmod{\alpha+1}$, i.e. if $x-1 \equiv 1$ colour it in using C_0, if $x - 1 \equiv 2$, use C_1 and so on. We then travel along them numbering squares in an analogous method to in the $dim_{1,1}$ case.

Below is an example of the numbered $(\alpha, 1)$-checkerboard in the $(3, 1)$ case where the colours are set as white, black, blue then red.(with the lines for the white and blue squares also shown).

26	28	30	32	34	36	38	40
20	22	24	25	27	29	31	33
15	17	18	19	21	23	24	26
11	12	13	14	16	17	18	20
7	8	9	10	11	12	13	15
4	5	6	6	7	8	9	10
2	3	3	3	4	5	5	6
1	1	1	1	2	2	2	3





As in the $dim_{1,1}$ case, the map we are interested in, $\Psi_{\alpha,1}$, has an equivalent definition. We take a partition λ and number and colour the squares as in the the numbered $(\alpha,1)$-checkerboard. Then the map $\Psi_{\alpha,1}$ will move a square numbered s to the square of the same colour but with number $s+1$ and then add one new square of each colour, namely the square of each colour numbered 1.

This will slide the square in a part after λ_α, with bottom left corner at lattice point (x,y), to the square with bottom left corner at $(x-\alpha,y+1)$. The squares in the first α parts will be mapped to the new bottom row. This is since they must have been the last square of their colour on their specific line, so they move to be the first square on the next line of that colour (i.e. to being on the bottom row).

From this new viewpoint on $\Psi_{\alpha,1}$ we can prove the following proposition.

Proposition 7.30. *$\Psi^k_{\alpha,1}(0)$, for $k \in \mathbb{Z}_{\geq 0}$, will be the partition consisting of the first k squares of each colour in the numbered $(\alpha,1)$-checkerboard grid.*

Proof. Easy to verify using Induction on k. □

Thus we can quickly verify the following two lemmas.

Lemma 7.31. *Let $k \in \mathbb{Z}_{\geq 0}$, then $\Psi^k_{\alpha,1}(0) \in \mathcal{G}_{\alpha,1}((\alpha+1)k, -2\alpha-1)$.*

Proof. Consider the diagonal lines which are used to colour and number $\Psi^k_{\alpha,1}(0)$. We call a line complete in λ if every square in the line is contained in the partition λ. We call a line non-empty in λ if there is one square which the line colours and numbers that is in the partition λ

Now every non-empty line in $\Psi^k_{\alpha,1}(0)$, will contribute 1 to the bottom row of $\Psi^k_{\alpha,1}(0)$, namely the first square of the line. This is since the partition $\Psi^k_{\alpha,1}(0)$ is filled up to the number k for each colour and if the line was non-empty in $\Psi^k_{\alpha,1}(0)$ but did not include it's first square then $\Psi^k_{\alpha,1}(0)$ would not be filled to k (since the first square, which would have number less than k, would have been omitted).

Every line only colours and numbers 1 square in the first α parts of a partition because the lines have gradient $\frac{1}{\alpha}$ so can't reach more than 1 square in this region. Since this square will be the last on it's respective line, we see that the line will contribute 1 to the sum $\lambda_1 + \cdots + \lambda_\alpha$ if it is complete and 0 otherwise.

Hence the contribution of a complete line to the $(\alpha, 1)$-rank of $\Psi^k_{\alpha,1}(0)$ will be 0, and the contribution of a non-empty, incomplete line, to the $(\alpha, 1)$-rank of $\Psi^k_{\alpha,1}(0)$ will be -1.

Because $\Psi^k_{\alpha,1}(0)$ corresponds to the partition with the squares numbered up to k for each colour, we know there can be at most α lines which are both non-empty and not complete in $\Psi^k_{\alpha,1}(0)$. Therefore the $(\alpha, 1) - rank$ of $\Psi^k_{\alpha,1}(0)$ will be at least, $-\alpha$ and so $\Psi^k_{\alpha,1}(0) \in \mathcal{G}_{\alpha,1}((\alpha+1)k, -2\alpha-1)$. □

Lemma 7.32. *Let $\Psi^k_{\alpha,1}(0) = \lambda$ for some $k \in \mathbb{Z}_{\geq 0}$. Then $\lambda_1 - \lambda_\alpha \leq 1$.*

Proof. Take the partition and colour and number it in accordance with the $(\alpha, 1)$-checkerboard grid. Call the top square in λ_1, R. Place a rectangle with width $\alpha+1$ and height 2 so that it's top left square is S. This rectangle will be of the form given below.

λ_1	λ_2	λ_3		$\lambda_{\alpha-1}$	λ_α	$\lambda_{\alpha+1}$
R			$\cdots$			
			$\cdots$		T	S

Now the square S is on the same coloured line as R but will have number 1 less than R. Thus as R is in the partition and $\Psi^k_{\alpha,1}(0)$ is the partition consisting of the first k squares for each colour, S is also in the partition. Thus T is in the partition and so $\lambda_1 - \lambda_\alpha = 1$ or $\lambda_1 - \lambda_\alpha = 0$. □

Combining these two Lemmas and Theorem 7.20, we can finally prove the following theorem.

Theorem 7.33. *The unique partition λ of $n = (\alpha+1)k$ with $dim_{\alpha,1}(\lambda) = k$ is given as $\lambda = \Psi^k_{\alpha,1}(0)$.*

Proof. We prove this by induction on k.

Base Case: $k = 0$, then the unique partition of 0 with $dim_{\alpha,1} = 0$ is the only partition of 0, $(0) = \Psi^0_{\alpha,1}(0)$.

Induction Step: Assume that the unique partition of $n = (\alpha+1)t$ with $dim_{\alpha,1}(\lambda) = t$ is $\lambda = \Psi^t_{\alpha,1}(0)$. Then by the Lemmas 7.31 and 7.32, $\lambda \in \mathcal{G}_{\alpha,1}((\alpha+1)t, -2\alpha-1)$ and $\lambda_1 - \lambda_\alpha \leq 1$ so by Theorem 7.20,

$$dim_{\alpha,1}(\Psi^{t+1}_{\alpha,1}(0)) = dim_{\alpha,1}(\Psi_{\alpha,1}(\lambda)) = dim_{\alpha,1}(\lambda) + 1 = t + 1.$$

Hence the induction step holds and we complete the proof. □

We have now found, using bijections, that the maximal $dim_{\alpha,1}$ partitions for $n = (\alpha+1)k$ is $\Psi^{-k}_{\alpha,1}$. This author believes that we can generalise this method even further to the full (α, β) case. To do this one would have to modify the map Ψ so that it splits the added squares between β new rows. This is believed to be possible by jumping straight in with a checkerboard grid definition of the map, rather than the Dyson-like style of definition.

Once the map has been defined, Theorem 7.20 should be clear to generalise. This author has not yet verified the conditions required for this generalised version of the Theorem to work, though with some work, these conditions should be discoverable. It is expected that these conditions would continue to hold after multiple applications of the map. This serves as a starting point for future research into proving Conjecture 5.10.

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