AN INTRODUCION TO FJRW THEORY

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1. Introduction

Our starting point is:

Theorem 1.1. Witten-Kontsevich Intersection theory of ψ classes on $\overline{\mathcal{M}}_{g,n}$ is governed by the KdV hierarchy

Immediately after making this conjecture, Witten gave two generalizations of $\overline{\mathcal{M}}_{g,n}$.

First, the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$, which are the main object of study for Gromov-Witten theory.

Second, the moduli space of r-spin curves $\overline{\mathcal{M}}_{g,n}^{1/r}$. A point of $\overline{\mathcal{M}}_{g,n}^{1/r}$ is a curvec $\mathcal{C}$ together with a line bundle $\mathcal{L}$ satisfying $\mathcal{L}^{\otimes r} \cong \mathcal{K}_{\log}$.

FJRW is a further generalization of r-spin curves, that develops in parallel to Gromov-Witten theory.

	GW-theory	FJRW
Input	X	$W \in \mathbb{C}[x_1,\ldots,x_N]$; G finite
State Space	$H^*(X)$	$\mathcal{A}^{W,G}$
Moduli space	$\overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$	$\overline{\mathcal{M}}_{g,n}^{W,G} \to \overline{\mathcal{M}}_{g,n}$
Virtual Class	Yes	Yes
CohFT	Λ^{GW}	Λ^{FJRW}

Our goal in the first day is to develop the second column of this table. Specifically, we describe the basic set-up of FJRW theory, and outline how to produce a CohFT from this set-up.

1.1. Larger picture.

FJRW theory is not just parallel to GW theory, but in many cases closely related by the Landau-Ginzburg correspondence. FJRW theory is the A-model for LG theory; and we have the following table:

	A	B
CY	GW	BK, BCOV, Costello-Li, etc.
LG	FJRW	Saito-Givental, Dubrovin-Zhang

2. Initial set-up

FJRW theory requires the following inputs:

- A $\mathbb{C}^*$ action on $\mathbb{C}^N$ with weights $c_1,\ldots,c_n$; that is:
$$\lambda \cdot (x_1,\ldots,x_n) = (\lambda^{c_1}x_1,\ldots,\lambda^{c_N}x_N)$$
- A polynomial $W \in \mathbb{C}[x_1,\ldots,x_N]$ that is quasihomogenous of weight d with respect to $\mathbb{C}^*$ action – that is: $W(\lambda\cdot x) = \lambda^d W(x)$.

- Quasihomogeneity forces W to have a singularity at the origin. We require this singularity to be isolated. This is equivalent to requiring $X_W = \{W = 0\} \subset WP(c_1, \ldots, c_N)$ to be smooth.
- We require that W should have no terms of the form $x_i x_j$. This implies that the weights $c_i/d \leq 1/2$ and weights are uniquely determined by W.

Example 2.1. $W = x^r$ is the A_{r-1} singularity.

Example 2.2. $W = x^5 + xy^2$ is the D_5 singularity; we have $d = 5, c_1 = 1, c_2 = 2$.

2.1. **Orbifolding.** It will be essential to work with a finite group G of symmetries of W.

The $\mathbb{C}^*$ action and W together give the exponential grading element J:

$$J = (\exp(2\pi i c_1/d), \ldots, \exp(2\pi i c_N/d))$$

It is clear that J fixes W.

We require G to be a finite abelian group acting diagonally on $\mathbb{C}^N$ that fixes W and contains J.

Let $G^{\max} = \{(\alpha_1, \ldots, \alpha_N) | W(\alpha_1 x_1, \ldots, \alpha_N x_N) = W(x_1, \ldots, x_N)\}$ denote the maximal diagonal symmetry group of W.

Example 2.3. We illustrate these concepts for $W = D_6$.

$$G^{\max}_{D_6} = \{(\zeta^{-2}, \zeta) | \zeta^1 0 = 1\}$$

and $J_{D_6} = (\exp(2\pi i/5), \exp(2pii2/5))$, so that $\langle J \rangle \neq G^{max}$.

For all $g \in G^{\max}$, we can writ(e $g = (\exp(2\pi i \theta_1), \ldots, \exp(2\pi i \theta_N))$, s.t. $\theta_i \in [0, 1) \cap \mathbb{Q}$. We call θ_i the *phases* of g.

$$\operatorname{Age}(g) = \sum \theta_i$$
$$N_g = \dim \operatorname{Fix}(g)$$

Note: $\operatorname{Age}(g) + \operatorname{Age}(g^{-1}) = N - N_g$.

3. Cohomological Field Theories

The idea is that, given a the data described above, FJRW theory will produce a CohFT. What is a CohFT? It is a collection of tautological classes that behave like you would expect them to if you've been trained in Gromov-Witten theory.

More specifically, there are three maps between moduli spaces of curves: two gluing maps $\rho_{\text{tree}}, \rho_{\text{loop}}$, and the forgetting marked point map τ. Roughly speaking, a cohomological field theory is a family of cohomology classes in $\overline{\mathcal{M}}_{g,n}$ that behave well with respect to these three maps.

A CohFT with flat identity consists of a state space $\mathcal{A}$, that has a pairing $\langle , \rangle : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$, and a family $\{\Lambda_{g,n}\}$, with $\Lambda_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}) \otimes (\mathcal{A}^*)^n$ and a vector $1 \in \mathcal{A}$ that behave well with respect to the gluing maps, that is:

$$\rho^*_{\text{tree}} \Lambda_{g_1+g_2, k_1+k_2}(\alpha_{k_1}, \ldots, \alpha_{k_1+k_2}) = \sum_{r,s} \Lambda_{g_1, k_1+1}(\alpha_1, \ldots, \alpha_{k_1}, r) \eta^{rs} \Lambda_{g_2, k_2+1}(s, \alpha_{k_1+1}, \ldots, \alpha_{k_1+k_2})$$

here r, s run over a basis of $\mathcal{A}$, and (η^{rs}) is the inverse of the metric $\langle , rangle$ in terms of a basis of $\mathcal{A}$.

We have analogous requirements for ρ^*_{loop}:

$$\rho^*_{\text{loop}}\Lambda_{g+1,k}(\alpha_1,\dots,\alpha_k) = \sum_{r,s}\Lambda_{g,k+2})(\alpha_1,\dots,\alpha_k,r,s)\eta^{rs}$$

We want $\Lambda_{g,k}$ to be symmetric; that is, S_k invariant, as an element of $H^*(\overline{\mathcal{M}}_{g,n})\otimes(\mathcal{A}^*)^k$.

The flat identity condition states that y

$$\tau^*(\Lambda_{g,k}(\alpha_1,\dots,\alpha_k) = \Lambda_{g,k+1}(\alpha_1,\dots,\alpha_k,1)$$

and $\Lambda_{0,3}(\alpha_1,\alpha_2,1) = \langle\alpha_1,\alpha_2\rangle$.

To construct the CohFT, we will use a moduli space (stack) consisting of curves with with extra structure.

The moduli space will be $\overline{\mathcal{M}}^{W,G}_{g,k}$ with maps e_i to $\mathcal{IB}G$, and a finite stabilization map st to $\overline{\mathcal{M}}_{g,k}$.

The moduli space $\overline{\mathcal{M}}^{W,G}_{g,k}$ is smooth and proper.

This will be based off

$$\mathcal{W}^{W,G}_{g,k} = \{\mathcal{C}, p_1,\dots,p_k,\mathcal{L}_1,\dots,\mathcal{L}_N,\phi_i\}$$

where

- $(\mathcal{C}, p_1,\dots,p_k)$ is a stable pointed orbicurve.
- $\mathcal{L}_i$ are orbifold line bundles on $\mathcal{C}$
- $W = \sum i = 1^s a_i W_i$, with w_i monomials in the x_i
- $\phi_i : W_i(\mathcal{L}_\infty,\dots,\mathcal{L}_N) \to \mathcal{K}_{\log}$
- The group action at each marked point and node factors through G.

Here $\mathcal{K}_{\log=\mathcal{K}\otimes\mathcal{O}(\sum p_i)}$ has sections locally given by dz/z around the marked points.

Example 3.1. *r-spin curves Let $W = x^r$, $J = \exp(2\pi i/r)$, $G^{\max} = \langle J\rangle$ and*

$$\overline{\mathcal{M}}^{x^r,G^{\max}}_{g,k} = (\mathcal{C}, p_1,\dots,p_k$$

Polischchuck-Vaintrob have an alternate definition.

Let $\Gamma = \mathbb{C}^*G \subset GL(N)$ MORE IN NOTES

4. State Space

First we will give the high-brow definition; then we will give a more down-to-earth version.

$W : \mathbb{C}^N \to \mathbb{C}$ is G-invariant, and so defines a map $W : [\mathbb{C}^N/G]\mathbb{C}$. - $W^\infty = \text{Re}W^{-1}\left((M,\infty)\right), M >> 0$; that is, we want to consider those points in $[\mathbb{C}^N/G]$ that map to something with very large real part under W. Then:

$$\mathcal{A} = H^{*+2\sum c_i/d}_{CR}([\mathbb{C}^N/G], W^\infty, \mathbb{C})$$

In another way, this is

$$\bigoplus_{\gamma\in G} H^*(\text{Fix}(\gamma); W^\infty_\gamma, \mathbb{C})^G$$

Note that $\text{Fix}(\gamma) = \mathbb{C}^{N_\gamma}$.

If $\gamma = J = (\exp(2\pi i c_1/d),\dots,\exp(2\pi i c_N/d))$, then $\text{Fix}(J) = \{0\}$, and $\mathcal{A}_J = H^*(\{0\},\mathbb{C}) = \mathbb{C}$, and $1\in\mathcal{A}$ is defined to be $1\in\mathcal{A}_J$.

Note that this is a bit surprising for those used to Chen-Ruan cohomology, as the identity lives in a twisted sector.

Another, more computational description:

Theorem 4.1. Wall, Orlik-Solomon, Sebastrani

$$H^*(\mathbb{C}^n, W^\infty; \mathbb{C}) \cong \Omega^N/(dW \wedge \Omega^{N-1}$$

as G-modules.

This is germs of N-forms on $\mathbb{C}^N$ at 0, modulo $dW = 0$.

Another way to put this is $\mathbb{C}[x_1, \ldots, x_N]dx_1 \wedge \cdots \wedge dx_N/(\frac{\partial W}{\partial x_1}dx_1 + \cdots + \frac{\partial W}{\partial x_N}dx_N) \wedge dx_1 \wedge \cdots dx_N$.

$\cong \mathbb{C}[x_1, \ldots, x_N]/(\frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_N})dx_1 \wedge \cdots \wedge dx_N$ THis is the Milnor ring, or local algebra, of W.

Fact: $\Omega^N/dW \wedge \Omega^{N-1}$ is finite dimneionsal ifa nd only if W has an isolated singularty at 0. The dimension is called the milnor number, and is denoted μ.

The Milnor ring is powerful: for quasihomogenous singularities, that is a natural class of equivalence of singularities, so that two singularities are equivalent if and only if their Milnor rings are isomorphic.

The pairing on $H^*(\mathbb{C}^N, W^\infty, \mathbb{C})$ matches the residue pairing on $\Omega^N/dW \wedge \Omega^{N-1}$.

Residue pairing:

$$\langle f dx_1 \wedge \cdots \wedge dx_N, g dx_1 \wedge \cdots \wedge dx_n \rangle = \frac{1}{(2\pi i)^N} \int \frac{fg dx_1 \wedge \cdots \wedge dx_n}{\frac{\partial W}{\partial x_1} \cdots \frac{\partial W}{\partial x_N}}$$

Fact: in $\mathbb{C}[x_1, \ldots, x_N]/(frac\partial W \partial x_1, \ldots, \frac{\partial W}{\partial x_n})$ there is an obvious gradiing. The Hessian of W spans the part of highest degree. If $\langle f, g \rangle = \alpha\mu$, then $fg = \alpha \mathrm{Hess} +$ lower order terms.

Fact:

$$H^{N-p,p}(\mathbb{C}^n, W^\infty.\mathbb{C})^{\langle J \rangle} \to \{\phi dx_1 \wedge \ldots dx_N \,|\, \deg \phi = p\}$$

Highest degree in Milnor ring is the degree of the Hessian, which is denoted $\hat{c} = \sum(1 - 2(c_i/d))$.

Then $\langle , \rangle : \mathcal{A}^{p,q} \otimes \mathcal{A}^{\hat{c}-p,\hat{c}-q} \to \mathbb{C}$.

So that $\mathcal{A}$ behaves like the cohomology of a manifold of dimension $\hat{c}$, which may be fractional $-$ $\mathcal{A}$ may have fractional grading anyway. Bu if $\hat{c} = N - 2$, then $X_W = \{W = 0\}$ is CY, and $\mathcal{A}$ agrees with the cohomology of X_W.

MORE ON BIGRADING LATER

We have a moduli space, and a state space, the idea is that we will construct a virtual fundamental class $[\overline{\mathcal{M}}^{W,G}_{g,k}]^{\in} H_*(\overline{\mathcal{M}}^{W,G}_{g,l}) \otimes (\mathcal{A}^*)^k$ such that the classes we get by pushing down the virtual classes to $\overline{\mathcal{M}}_{g,k}$ we get a CohFT.

More specifically, define:

$$\Lambda_{g,k} = \frac{PD\mathrm{st}_*[\overline{\mathcal{M}}^{W,G}_{g,k}]}{|G|^g \deg \mathrm{st}}$$

$\Lambda_{g,k}$ forms a CohFT

$\dim \Lambda_{g,k}(\alpha_1, \ldots, \alpha_k) = (\hat{c} - 3)(1 - g) + k - \sum \deg \alpha_i$ $\mathrm{G}^{\max} - invariant$.

Deformation invariance: If W_t is a 1-parameter family of polynomials, then $\Lambda^{W_t,G}_{g,k}$ is independent of t.

Decomposition: $\Lambda^{W_1+W_2,G_1\times G_2}_{g,k} = \Lambda^{W_1,G_1}_{g,k} \otimes \Lambda^{W_2,G_2}_{g,k}$.

Additional properties that facilitate computation.