AN INTRODUCION TO FJRW THEORY

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.Our starting point is

Theorem 1. *Intersection theory of ψ classes on $\overline{\mathcal{M}}_{g,n}$ is governed by the KdV hierarchy*

Immediately after this, Witten gave two generalizations of $\overline{\mathcal{M}}_{g,n}$. The first, are the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

The second is $\overline{\mathcal{M}}_{g,n}^{1/r}$, the moduli space of r-spin curves, that is, curves $\mathcal{C}$ together with a line bundle $\mathcal{L}$ so that $\mathcal{L}^{\otimes r} \cong \mathcal{K}_{\log}$.

FJRW is a further generalization of r-spin curves, parallel to Gromov-Witten theory.

	GW-theory	FJRW
Input	X	$W \in \mathbb{C}[x_1,\dots,x_N]$; G finite
State Space	$H^*(X)$	$\mathcal{A}^{W,G}$
Moduli space	$\overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$	$\overline{\mathcal{M}}_{g,n}^{W,G} \to \overline{\mathcal{M}}_{g,n}$
Virtual Class	X	X
CohFT	Λ^{GW}	Λ^{FJRW}

Larger picture

	A	B
CY	GW	BK, BCOV, Costello-Li, etc.
LG	FJRW	Saito-Givental, Dubrovin-Zhang

FJRW theory requires the following inputs:

- $\mathbb{C}^*$ action on $\mathbb{C}^N$ with weights $c_1,\dots,c_n$; that is $\lambda\cdot(x_1,\dots,x_n) = (\lambda^{c_1}x_1,\dots,\lambda^{c_N}x_N)$.
- Polynomial $W \in \mathbb{C}[x_1,\dots,x_N]$ that is quasihomogenous of weight d with respect to $\mathbb{C}^*$ action – that is $W(\lambda\cdot x) = \lambda^d W(x)$.
- Quasihomogeneity forces W to have a singularity at the origin, we require this singularity to be isolated; that is, $X_W = \{W = 0\} \subset WP(c_1,\dots,c_N)$ is smooth.
- W should have no terms of form x_ix_j, which implies that the weights $c_i/d \leq 1/2$ and weights are uniquely determined by W.

Example 1. *$W = x^r$ is the A_{r-1} singularity.*

Example 2. *$W = x^5 + x^{y2}$ is the D_5 singularity; we have $d = 5, c_1 = 1, c_2 = 2$.*

The $\mathbb{C}^*$ action and W together give the exponential grading element

$$J = (\exp(2\pi i c_1/d),\dots,\exp(2\pi i c_N/d))$$

with J fixing W.

G is a finite abelian group acting diagonally on $\mathbb{C}^N$ that fixes W and contains J.

Let $G^{max} = \{(\alpha_1,\dots,\alpha_N) | W(\alpha_1x_1,\dots,\alpha_Nx_N) = W(x_1,\dots,x_N)\}$ denote the maximal diagonal symmetry group of W.

Example 3.
$$G^{max}_{D_6} = \{\zeta^{-2}, \zeta)|\zeta^10 = 1\}$$
and $J_{D_6} = (\exp(2\pi i/5), \exp(2pii2/5))$*, so that* $\langle J\rangle G^{max}$.

For all $g \in G^{max}$, we can writ(e $g = (\exp(2\pi i\theta_1), \ldots, \exp(2\pi i\theta_N))$, s.t. $\theta_i \in [0, 1) \cap \mathbb{Q}$. We call θ_i the *phases* of g.

$$\mathrm{Age}(g) = \sum \theta_i$$
$$N_g = \dim \mathrm{Fix}(g)$$

Note: $\mathrm{Age}(g) + \mathrm{Age}(g^{-1}) = N - N_g$.

The idea is that, given a the data described above, FJRW theory will produce a CohFT. What is a CohFT? It is a collection of tautological classes that behave like you would expect them to if you've been trained in Gromov-Witten theory.

More specifically, we have two gluing maps $\rho_{\text{tree}}, \rho_{\text{loop}}$, and the forgetting marked point map τ.

A CohFT with flat identity consists of a state space $\mathcal{A}$, that has a pairing $\langle,\rangle : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$, and a family $\{\Lambda_{g,n}\}$, with $\Lambda_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}) \otimes (\mathcal{A}^*)^n$ and a vector $1 \in \mathcal{A}$ that behave well with respect to the gluing maps, that is:

$$\rho^*_{\text{tree}}\Lambda_{g_1+g_2,k_1+k_2}(\alpha_{k_1}, \ldots, \alpha_{k_1+k_2}) = \sum_{r,s} \Lambda_{g_1,k_1+1}(\alpha_1, \ldots, \alpha_{k_1}, r)\eta^{rs}\Lambda_{g_2,k_2+1}(s, \alpha_{k_1+1}, \ldots, \alpha_{k_1+k_2})$$

here r, s run over a basis of $\mathcal{A}$, and (η^{rs}) is the inverse of the metric $\langle, rangle$ in terms of a basis of $\mathcal{A}$.

We have analogous requirements for ρ^*_{loop}:

$$\rho^*_{\text{loop}}\Lambda_{g+1,k}(\alpha_1, \ldots, \alpha_k) = \sum_{r,s} \Lambda_{g,k+2})(\alpha_1, \ldots, \alpha_k, r, s)\eta^{rs}$$

We want $\Lambda_{g,k}$ to be symmetric; that is, S_k invariant, as an element of $H^*(\overline{\mathcal{M}}_{g,n}) \otimes (\mathcal{A}^*)^k$.

The flat identity condition states that y

$$\tau^*(\Lambda_{g,k}(\alpha_1, \ldots, \alpha_k) = \Lambda_{g,k+1}(\alpha_1, \ldots, \alpha_k, 1)$$

and $\Lambda_{0,3}(\alpha_1, \alpha_2, 1) = \langle \alpha_1, \alpha_2 \rangle$.

To construct the CohFT, we will use a moduli space (stack) consisting of curves with with extra structure.

The moduli space will be $\overline{\mathcal{M}}^{W,G}_{g,k}$ with maps e_i to $\mathcal{I}\mathcal{B}G$, and a finite stabilization map st to $\overline{\mathcal{M}}_{g,k}$.

The moduli space $\overline{\mathcal{M}}^{W,G}_{g,k}$ is smooth and proper.

This will be based off

$$\mathcal{W}^{W,G}_{g,k} = \{\mathcal{C}, p_1, \ldots, p_k, \mathcal{L}_1, \ldots, \mathcal{L}_N, \phi_i\}$$

where

- $(\mathcal{C}, p_1, \ldots, p_k)$ is a stable pointed orbicurve.
- $\mathcal{L}_i$ are orbifold line bundles on $\mathcal{C}$
- $W = \sum i = 1^s a_i W_i$, with w_i monomials in the x_i
- $\phi_i : W_i(\mathcal{L}_\infty, \ldots, \mathcal{L}_N) \to \mathcal{K}_{\log}$
- The group action at each marked point and node factors through G.

Here $\mathcal{K}_{\log=\mathcal{K}\otimes\mathcal{O}(\sum p_i)}$ has sections locally given by dz/z around the marked points.

Example 4. *r-spin curves Let* $W = x^r, J = \exp(2\pi i/r), G^{max} = \langle J\rangle$ *and*

$$\overline{\mathcal{M}}_{g,k}^{x^r,G^{max}} = (\mathcal{C}, p_1, \ldots, p_k$$

Polischchuck-Vaintrob have an alternate definition.

Let $\Gamma = \mathbb{C}^* G \subset GL(N)$ *MORE IN NOTES*

1. State Space

First we will give the high-brow definition; then we will give a more down-to-earth version.

$W : \mathbb{C}^N \to \mathbb{C}$ *is G-invariant, and so defines a map* $W : [\mathbb{C}^N/G]\mathbb{C}$. - $W^\infty = ReW^{-1}((M,\infty)), M >> 0$*; that is, we want to consider those points in* $[\mathbb{C}^N/G]$ *that map to something with very large real part under* W*. Then:*

$$\mathcal{A} = H_{CR}^{*+2\sum c_i/d}([\mathbb{C}^N/G], W^\infty, \mathbb{C})$$

In another way, this is

$$\bigoplus_{\gamma\in G} H^*(Fix(\gamma); W_\gamma^\infty, \mathbb{C})^G$$

Note that $Fix(\gamma) = \mathbb{C}^{N_\gamma}$.

If $\gamma = J = (\exp(2\pi i c_1/d), \ldots, \exp(2\pi i c_N/d))$*, then* $Fix(J) = \{0\}$*, and* $\mathcal{A}_J = H^*(\{0\}, \mathbb{C}) = \mathbb{C}$*, and* $1 \in \mathcal{A}$ *is defined to be* $1 \in \mathcal{A}_J$.

Note that this is a bit surprising for those used to Chen-Ruan cohomology, as the identity lives in a twisted sector.

Another, more computational description:

Theorem 2. *Wall, Orlik-Solomon, Sebastrani*

$$H^*(\mathbb{C}^n, W^\infty; \mathbb{C}) \cong \Omega^N/(dW \wedge \Omega^{N-1}$$

as G-modules.

This is germs of N*-forms on* $\mathbb{C}^N$ *at 0, modulo* $dW = 0$.

Another way to put this is $\mathbb{C}[x_1, \ldots, x_N]dx_1\wedge\cdots\wedge dx_N/(\frac{\partial W}{\partial x_1}dx_1 + \cdots + \frac{\partial W}{\partial x_N}dx_N)\wedge dx_1 \wedge \cdots dx_N$.

$\cong \mathbb{C}[x_1, \ldots, x_N]/(\frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_N})dx_1 \wedge \cdots \wedge dx_N$ *THis is the Milnor ring, or local algebra, of* W.

Fact: $\Omega^N/dW \wedge \Omega^{N-1}$ *is finite dimneionsal ifa nd only if* W *has an isolated singularty at 0. The dimension is called the milnor number, and is denoted* μ.

The Milnor ring is powerful: for quasihomogenous singularities, that is a natural class of equivalence of singularities, so that two singularities are equivalent if and only if their Milnor rings are isomorphic.

The pairing on $H^*(\mathbb{C}^N, W^\infty, \mathbb{C})$ *matches the residue pairing on* $\Omega^N/dW\wedge\Omega^{N-1}$.

Residue pairing:

$$\langle f dx_1 \wedge \cdots \wedge dx_N, g dx_1 \wedge \cdots \wedge dx_n\rangle = \frac{1}{(2\pi i)^N}\int \frac{fg dx_1 \wedge \cdots \wedge dx_n}{\frac{\partial W}{\partial x_1}\cdots\frac{\partial W}{\partial x_N}}$$

Fact: in $\mathbb{C}[x_1, \ldots, x_N]/(frac\partial W\partial x_1, \ldots, \frac{\partial W}{\partial x_n})$ *there is an obvious gradiing. The Hessian of* W *spans the part of highest degree. If* $\langle f, g\rangle = \alpha\mu$*, then* $fg = \alpha Hess +$ *lower order terms.*

Fact:

$$H^{N-p,p}(\mathbb{C}^n, W^{\infty}.\mathbb{C})^{\langle J\rangle} \to \{\phi dx_1 \wedge \dots dx_N \,|\, \deg \phi = p\}$$

Highest degree in Milnor ring is the degree of the Hessian, which is denoted $\hat{c} = \sum(1 - 2(c_i/d))$.

Then $\langle , \rangle : \mathcal{A}^{p,q} \otimes \mathcal{A}^{\hat{c}-p,\hat{c}-q} \to \mathbb{C}$.

So that $\mathcal{A}$ *behaves like the cohomology of a manifold of dimension* $\hat{c}$*, which may be fractional –* $\mathcal{A}$ *may have fractional grading anyway. Bu if* $\hat{c} = N - 2$*, then* $X_W = \{W = 0\}$ *is CY, and* $\mathcal{A}$ *agrees with the cohomology of* X_W*.*

MORE ON BIGRADING LATER

We have a moduli space, and a state space, the idea is that we will construct a virtual fundamental class $[\overline{\mathcal{M}}^{W,G}_{g,k}]^{\in} H_*(\overline{\mathcal{M}}^{W,G}_{g,l}) \otimes (\mathcal{A}^*)^k$ *such that the classes we get by pushing down the virtual classes to* $\overline{\mathcal{M}}_{g,k}$ *we get a CohFT.*

More specifically, define:

$$\Lambda_{g,k} = \frac{PDst_*[\overline{\mathcal{M}}^{W,G}_{g,k}]}{|G|^g \deg st}$$

$\Lambda_{g,k}$ *forms a CohFT*

$\dim \Lambda_{g,k}(\alpha_1, \dots, \alpha_k) = (\hat{c} - 3)(1 - g) + k - \sum \deg \alpha_i$ $G^{max} - invariant$.

Deformation invariance: If W_t *is a 1-parameter family of polynomials, then* $\Lambda^{W_t,G}_{g,k}$ *is independent of* t.

Decomposition: $\Lambda^{W_1+W_2, G_1\times G_2}_{g,k} = \Lambda^{W_1,G_1}_{g,k} \otimes \Lambda^{W_2,G_2}_{g,k}$.

Additional properties that facilitate computation.