AN INTRODUCION TO FJRW THEORY IV

TYLER JARVIS

Today, Nathan Priddis is substituting in as Tyler is losing his voice.

The plan today is to talk about two things: LG mirror symmetry, and the LG/CY correspondence.

1. LG mirror symmetry

First, we will review what we have done

1.1. A-model review.

1.1.1. *Input.* Recall we started with:

A W-quasihomogeneous polynomial satisfying:

- weights $q_i = c_i/d$
- Isolated singularity
- No terms of the form $x_i x_j$

and a group of symmetries G^W:

$$G^W = \left\{(\alpha_1, \ldots, \alpha_N) \in (\mathbb{C}^*)^N | W(\alpha_1 x_1, \ldots, \alpha_N x_n) = W(x_1, \ldots, x_N)\right\}$$

With the grading element J_W.

G^W is what we were calling G^{max} before, but later we will have two maximal groups (one from the mirror).

1.1.2. *Output.* From this, we first constructed the state space

$$\begin{aligned} A^{W,G} &= H^*_{CR}([\mathbb{C}^N/G], W^\infty, \mathbb{C}) \\ &= \bigoplus_{g \in G} A_g \end{aligned}$$

The grading was

$$(p + \sum_{i=1}^N \Theta_i^g - \sum_{j=1}^N q_i, N_g - p + \sum_{i=1}^N \Theta_i^g - \sum_{j=1}^N q_i)$$

Here p is the internal weight, each piece was a Jacobi ring, and this was the weight coming from that. The Θ terms are the weights, similar to the age in Chen-RUan cohomology, and the q_i are an extra shifting, which is the age of J – recall, that our identity lived in A_J, and so the shifting of A_J should be trivial.

We had the moduli space of W structures, with a virtual fundamental class that we could push forward to $\overline{\mathcal{M}}_{g,n}$ to obtain a CohFT:

$$\Lambda_{g,n}(\phi_1, \dots, \phi_n) \in H^*(\overline{\mathcal{M}}_{g,n})$$

Integrating these classes over $\overline{\mathcal{M}}_{g,n}$, we get numbers, called correlators, which are the FJRW invariants.

1.2. **B-model.** We now turn to the B-model. The input to the B model is the same as the A-model; a quasihomogeneous polynomial W and a finite group of symmetries $G \subset G^W$.

We have already seen some of the ingredients that go into the B-model.

The main piece is $\Omega_W = \Omega^N / \partial W \wedge \Omega^{N-1} = \mathbb{C}[x_1, \dots, x_N]/(\partial_1 W, \dots, \partial_N W)$

Ω_W is called the Jacobi ring (or Milnor ring?)

The fact that W had an isolated singularity at the origin means that Ω_W is finite dimensional, with dimension

$$\mu = \prod_{j=1}^{N} (\frac{1}{q_j} - 1)$$

It has a top degree element called the Hessian

$$\mathrm{Hess}(W) = \det\left(\partial_i \partial_j W\right)$$

1.2.1. *The role of G.* The group G^W acts on Ω_W:

$$(\alpha_1, \dots, \alpha_n) \cdot \prod_{j=1}^{N} x_j^{m_j} dx_1 \wedge \dots \wedge dx_N = \prod_{j=1}^{N} \alpha_j^{m_j+1} \prod_{j=1}^{N} x_j^{m_j} dx_1 \wedge \dots \wedge dx_N$$

1.2.2. *Pairing.* The pairing is defined by taking the coefficient of the top dimensional part in fg, and normalizing by $\mu(W)$:

$$fg = \frac{\langle f, g \rangle}{\mu(W)} \cdot \text{hess(W)} + \text{ lower order}$$

From this, it is clear that

$$\langle fg, h \rangle = \langle f, gh \rangle$$

and so this structure gives us a Frobenius algebra.

1.2.3. *Groups.* On the A side, our group G had to be "admissible", which means it contains J. Dual to that on the B side, we have that G must be contained in $SL_W = SL_N \cap G^W$.

Our state space will be

$$\mathcal{Q}_{W,G} = \bigoplus_{g \in G} \left(\Omega_{W_g}\right)^G$$

Our grading is given by:

$$(p+\sum_{i=1}^{N}\Theta_i^g-\sum_{j=1}^{N}q_i, p+\sum_{i=1}^{N}\Theta_i^g-\sum_{j=1}^{N}q_i)$$

which is similar to the grading on the A model, but symmetric instead of antisymmetric.

The pairing on the orbifolded state space is constructed similarly to before:

$$\langle,\rangle:\Omega_{w_g}\otimes\Omega_{w_{g^{-1}}}\to\mathbb{C}$$

which gives us a Frobenius algebra.

There is an obvious isomorphism $A_{W,G}\cong\mathcal{Q}_{W,G}$, as vector spaces, but the gradings and pairings are different.

1.3. **Mirror Symmetry.**

Definition 1.1. A quasihomogeneous polynomial is *invertible* if it has the same number of monomials and variables, (and it satisfies the properties it had before)

So $W=\sum_{i=1}^{N}\prod_{j=1}^{N}x_j^{a_{ij}}$.

Theorem 1.2 (Kreuzer, Skarke)**.** Let W be an invertible polynomial satisfying our main conditions.

Then W is the disjoint sum of the following atomic types:

(1) Fermat: $x^a, a\geq 2$
(2) Chain: $x_1^{a_1}x_2+x_x^{a_2}x_3+\cdots x_{N-1}^{a_{N-1}}x_N+x_N^{a_N}$ with $a_i\geq 2$
(3) Loop: $x_1^{a_1}x_2+x_x^{a_2}x_3+\cdots x_{N-1}^{a_{N-1}}x_N+x_N^{a_N}x_1$ with $a_i\geq 2$

The exponent matrix is $E_W=(a_{ij})$.
The inverse matrix $E_w^{-1}=(a^{ij})$.
Define $\rho_j\in(\mathbb{C}^*)^N$

$$\rho_j=\big(\exp(2\pi i a^{1j}),\ldots,\exp(2\pi i a^{Nj})\big)$$

FACT 1: Then $\{\rho_j\}$ generated G^W. This is not at all a minimal generating set, but will be an important set of generators that will let us write the dual group.

We leave the proof as an exercise, but the hint it $E_W^{-1}E_W=\mathrm{Id}$.

FACT 2: $q_i=\sum_{j=1}^{N}a^{ij}$

This implies that $J=\rho_1\cdots\rho_N$.

Example 1.3 ($E_7=x^3+xy^3$)**.** We have

$$E_w=\begin{pmatrix}3&0\\1&3\end{pmatrix}$$

$$E_w^{-1}=\begin{pmatrix}\frac{1}{3}&0\\\frac{-1}{9}&\frac{1}{3}\end{pmatrix}$$

$$J = \left(e^{2\pi i/3}, e^{2\pi i 2/9}\right)$$

Definition 1.4 (Berglund, Hübsch, Krawitz, Henningson)**.**
(1) The dual polynomial to W, written W^T, is given by E_W^T.
(2) The dual grop G^T, is

$$\begin{aligned} G^T &= \left\{ \overline{\rho}_1^{b_1} \cdots \overline{\rho}_N^{b_N} \,|\, \prod_{j=1}^N x_j^{b_j} \text{ is } G\text{-invariant} \right\} \\ &= \left\{ \overline{\rho}_1^{b_1} \cdots \overline{\rho}_N^{b_N} \,|\, (b_1, \dots, b_n) E_w^{-1} (\ell_1, \dots, \ell_n)^T \in \mathbb{Z} \text{ for all } \rho_1^{\ell_1} \cdots \rho_N^{\ell_N} \in G \right\} \end{aligned}$$

Facts:
(1) $(G^T)^T = G$
(2)

$$\begin{aligned} \langle J_W \rangle^T &= SL_{W^T} \\ SL_W^T &= \langle J_{W^T} \rangle \end{aligned}$$

(3) $G_1 < G_2$ if and only if $G_2^T < G_1^T$
(4) $(G^W)^T = \{1\}$

Example 1.5 (E_7^T)**.** We had:

$$E_w = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}$$

So

$$E_w^T = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

and hence we see

$$E_y^T = x^3 y_y^3$$

and so E_7 is self-dual.

Conjecture 1.6. The LG A-model for (W, G) is mirror to the LG B-model for (W^T, G^T).

Theorem 1.7 (Krawitz)**.** There is an isomorphism

$$A_{w,G} \cong \mathcal{Q}_{W^T, G^T}$$

that preserves bi-degrees and the pairing.

This can be viewed as a state-space isomorphism. We do not know an isomorphism of Frobenius algebras for all (W, G) yet.

Furthermore, if W gave a CY in weighted projective space, we could look at the hodge diamonds of X_W/G_W and X_{W^T}/G_{W^T}, and they would be related by the familiar 90 degree rotation.

Proof. Look at the vector spaces

$$\bigoplus_{g\in G^W} \Omega_{W_g} \quad \text{and} \quad \bigoplus_{h\in G^{WT}} \Omega_{W_h^T}$$

this is like our state space, but we are summing over all of G^W, and not taking invariants.

There is an isomorphism between these spaces given by $(v,g)\mapsto(w,h)$ as:

$$\left(\prod_{j=1}^{N_g} x_{i_j}^{b_{i_j}}\, d\underline{x}, \prod_{r=1}^{N_h} \rho_{i_r}^{s_{i_r}+1}\right) \mapsto \left(\prod_{r=1}^{N_h} y_{i_r}^{s_{i_r}}\, d\underline{y}, \prod_{j=1}^{N_g} \overline{\rho}_{i_j}^{b_{i_j}+1}\right)$$

□

1.4. **Frobenius algebras.** Several cases of isomorphisms of Frobenius algebras have been proven, but not the general case. We have

- Krawitz showed we could take the maximal group:
$$A_{W,G^W} \cong \mathcal{Q}_{W^T,1}$$
- Francis, Jarvis, Johnson and Webb for all G if W is a fermat+loops, and W is a chain, for some groups.

Note: the restrictions on the groups is because even if the polynomial decomposes as a sum of basic pieces, its group of symmetries may or may not also decompose.

1.5. **Frobenius Manifolds.** In the first FJR paper, they proved an isomorphism of Frobenius manifolds for W an ADE case with $\hat{c}<1$. The A case had been proven earlier by Faber-Shadrin-Zvonkine.

Krawitz, Shen, Milanov proved an isomoprhism of Frobenius manifolds for simple elliptic singularities ($\hat{c}=1$) and maximal symmetry group on the A-side.

Li-Li-Saito-Shen proved the isomorphism of singularities for the 14 exceptional singularities in Arnold's classification and maximal symmetry group on the A-side.

He-Webb-Shen-Li: have the isomorphism of Frobenius manifolds for all W and maximal symmetry group on the A-side.

One reason they use maximal symmetry group on the A-side is that there is still some work to be done to understand orbifolding the B-side.

1.6. **J-functions and I-functions.** Another flavor of result is to show an equality of small J and I functions.

Chiodo-Ruan: $W=\sum_{i=1}^5 x_i^5, G=\langle J\rangle$

Priddis-Shoemaker: $W=\sum_{i=1}^5, G=SL_W$

Gruéré:W is a chain, $G=\langle J\rangle$

Most of these results are just in genus 0.

The first series of results actually hold for higher genus, because for G^{max} the B-model Frobenius manifold is generically semisimple, and so we can get the higher genus results using Teleman's result. There is still some difficulty as there is a shift that gives an infinite sum, and work is necessary to show that this infinite sum converges.

2. LG-CY correspondence

Not much time, but we'll give just a little bit of the flavor.

In the same Kahler moduli space that corresponds to FJRW (W, G), there will be a large-radius limit point (maximal unipotent monomdromy), that corresponds to the Gromov-Witten theory of $X_W/(G/J)$.

The J-function is function that lives on the Kahler moduli space.

Mirror to this, we have the I function on the B-model moduli space.

So, we use GW mirror symmetry, analytically continue the I-funciton on the B-model moduli space and apply a symplectic transformation to get the I-function at the orbifold point. Another mirror map should give the J-function of the FJRW theory of (W, G).

One of the main motivations for FJRW theory was to help compute GW-theory. The hope is that it is easier to compute, and then we can apply the network of mirror symmetry, analytic continuation and symplectic transformations to find the Gromov-Witten theory.