故所成角为$\frac{\pi}{4}$。

7: $\frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2+1}}$

$\frac{\partial z}{\partial y}\Big|_{(x,y)=(1,1)}=\frac{\sqrt{3}}{3}$

则曲线在$(1,1,\sqrt{3})$处切线与x轴，y轴，z轴夹角分别为$\frac{\pi}{2},\frac{\pi}{6},\frac{\pi}{3}$

8: 考虑取对数 $\ln u=-\frac{1}{2}\ln t-\frac{x^2}{4t}$, 则有

$$\frac{\partial \ln u}{\partial t}=-\frac{1}{2t}+\frac{x^2}{4t^2}=\frac{1}{u}\frac{\partial u}{\partial t}$$

$$\frac{\partial \ln u}{\partial x}=-\frac{x}{2t}=\frac{1}{u}\frac{\partial u}{\partial x}$$

于是

$$\frac{\partial u}{\partial t}=u\left(-\frac{1}{2t}+\frac{x^2}{4t^2}\right),\frac{\partial u}{\partial x}=u\left(-\frac{x}{2t}\right),$$

因此

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial x}\left(-\frac{x}{2t}\right)+u(-\frac{1}{2t})=u\left(-\frac{1}{2t}+\frac{x}{2t^2}\right)=\frac{\partial u}{\partial t}.$$

9: $(1)z_{xx}=-\frac{4y}{(x+y)^3},z_{yy}=\frac{4x}{(x+y)^3},z_{xy}=\frac{2(x-y)}{(x+y)^3}$

$(2)z_{xx}=-\frac{2x}{(1+x^2)},z_{yy}=-\frac{2y}{(1+y^2)},z_{xy}=0$

$(3)z_{xx}=-\frac{x}{(x^2+y^2)^{3/2}},z_{xy}=-\frac{y}{(x^2+y^2)^{3/2}}$

$z_{yy}=\frac{x(x^2+y^2)-xy^2-2y^2\sqrt{x^2+y^2}}{(x\sqrt{x^2+y^2}+x^2+y^2)^2}$

$(4)z_{xx}=a^2\sin 4(ax+by),z_{yy}=b^2\sin 4(ax+by)\ ,z_{xy}=ab\sin 4(ax+by)$

(5)$z_{xx} = (-\frac{lny}{x^2} + (\frac{lny}{x})^2)e^{lnxlny}$

$z_{yy} = (-\frac{lnx}{y^2} + (\frac{lnx}{y})^2)e^{lnxlny}$

$z_{xy} = \frac{1+lnxlny}{xy}e^{lnxlny}$

(6)$z_{xx} = \frac{y^3x}{(1-x^2y^2)^{3/2}}, z_{yy} = \frac{x^3y}{(1-x^2y^2)^{3/2}}$

$z_{xy} = \frac{1}{(1-x^2y^2)^{3/2}}$

10: $\frac{\partial^3 u}{\partial x\partial y\partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$, $\frac{\partial^3 u}{\partial x\partial y^2} = (2+xyz)xz^2e^{xyz}$

11: (1)

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2+z^2}} \\
\frac{\partial^2 r}{\partial x^2} &= \frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\
\text{同理，}\frac{\partial^2 r}{\partial y^2} &= \frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\
\frac{\partial^2 r}{\partial z^2} &= \frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\
\text{故}\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} &= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{2}{r}
\end{aligned}$$

(2)

$$\begin{aligned}
\ln r &= \frac{1}{2}\ln(x^2+y^2+z^2) \\
\frac{\partial \ln r}{\partial x} &= \frac{x}{x^2+y^2+z^2} \\
\frac{\partial^2 \ln r}{\partial x^2} &= \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2} \\
\therefore \frac{\partial^2 \ln r}{\partial x^2} + \frac{\partial^2 \ln r}{\partial y^2} + \frac{\partial^2 \ln r}{\partial z^2} &= \frac{y^2+z^2+x^2}{(x^2+y^2+z^2)^2} = \frac{1}{r^2}
\end{aligned}$$

(3)

$$\begin{aligned}
\frac{1}{r} &= (x^2+y^2+z^2)^{-\frac{1}{2}} \\
\frac{\partial}{\partial x}\frac{1}{r} &= \frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\
\frac{\partial^2}{\partial x^2}\frac{1}{r} &= \frac{-(x^2+y^2+z^2)+3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} \\
\therefore \frac{\partial^2}{\partial x^2}\frac{1}{r}+\frac{\partial^2}{\partial y^2}\frac{1}{r}+\frac{\partial^2}{\partial z^2}\frac{1}{r} &= \frac{-3(x^2+y^2+z^2)+3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}} = 0
\end{aligned}$$

12: 函数的二阶偏导数

$$f''_{xx} = \begin{cases} \frac{-4x^3y^3+12xy^5}{(x^2+y^2)^3}, & x^2+y^2=0; \\ 0. & x^2+y^2\neq 0. \end{cases} \qquad f''_{xy} = \begin{cases} \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}, & x^2+y^2=0; \\ -1. & x^2+y^2\neq 0. \end{cases}$$

$$f''_{yy} = \begin{cases} \frac{4x^3y^3-12x^5y}{(x^2+y^2)^3}, & x^2+y^2=0; \\ 0. & x^2+y^2\neq 0. \end{cases} \qquad f''_{yx} = \begin{cases} \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}, & x^2+y^2=0; \\ 1. & x^2+y^2\neq 0. \end{cases}$$

沿着$y=kx$容易看出它们在$(0,0)$处都不连续, 且$f''_{xy}\neq f''_{yx}$.

13: (1)

$$\frac{\partial z}{\partial x}=\frac{2x}{x^2+y^2},\quad \frac{\partial z}{\partial y}=\frac{2y}{x^2+y^2},\quad dz=\frac{2}{x^2+y^2}(xdx+ydy)$$

(2)

$$\frac{\partial z}{\partial x}=\frac{y(y^2-x^2)}{(x^2+y^2)^2},\quad \frac{\partial z}{\partial y}=\frac{x(x^2-y^2)}{(x^2+y^2)^2},\quad dz=\frac{y^2-x^2}{(x^2+y^2)^2}(ydx-xdy)$$

(3)

$$\frac{\partial u}{\partial s} = \frac{-2t}{(s-t)^2}, \quad \frac{\partial u}{\partial t} = \frac{2s}{(s-t)^2}, \quad du = \frac{2}{(s-t)^2}(sdt - tds)$$

(4)

$$dz = \frac{1}{x^2+y^2}(xdy - ydx)$$

(5)

$$\frac{\partial z}{\partial x} = y\cos xy, \quad \frac{\partial z}{\partial y} = x\cos xy$$

故在$(0,0)$处,有

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$$

因此

$$dz = 0$$

(6)

$$\frac{\partial z}{\partial x} = 4x^3 - 8xy^2, \quad \frac{\partial z}{\partial y} = 4y^3 - 8x^2y$$

在$(0,0)$处,有

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$$

因此

$$dz = 0$$

在$(1,1)$处,有

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = -4$$

因此

$$dz = -4dx - 4dy$$

14: 证明:

由定理9.14知:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x}dy$$

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial x}dy$$

那么可以得到:

$$d\left(fg\right) = \frac{\partial fg}{\partial x}dx + \frac{\partial fg}{\partial x}dy = \left(\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial x}f\right)dx + \left(\frac{\partial f}{\partial y}g + \frac{\partial g}{\partial y}f\right)dy = fdg + gdf$$

证毕。

15: $f(x,y) = \sqrt{|xy|}$

若$f(x,y)$在(0,0)处可微

则在(0,0)的邻域有$f(x,y) - f(0,0) = ax + by + o(\rho)$

故$f(x,y) = ax + by + o(\rho)$

而$f(x,y) = f(x,-y) = f(-x,y)$

则必有$a = b = 0$

则$f(x,y) = o(\rho)$

而沿着直线y=kx

$$\lim_{x\to 0, y=kx} \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}} = \frac{\sqrt{|k|}}{\sqrt{1+k^2}} \neq 0$$

从而在(0,0)的邻域上,$f(x,y) \neq o(\rho)$

f在(0，0)处不可微

16: 对任意 $\epsilon > 0$, 取 $\delta = \epsilon$, 则当$0 < |x|, |y| < \delta$ 时有

$$|f(x,y)| = \frac{x^2|y|}{x^2+y^2} \le |y| < \epsilon$$

由定义知$f(x,y)$ 在点$(0,0)$ 处连续. 由定义易知f 在$(0,0)$ 处的偏导数均为0. 若f 在点$(0,0)$ 处可微，则

$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + o(\rho) \quad (\rho = \sqrt{x^2+y^2} \to 0)$$

因此 $f(x,y) = o(\rho)(\rho \to 0)$, 又若令$y = kx(k \neq 0)$, 则

$$\frac{f(x,y)}{\rho} = \frac{x^2 y}{(x^2+y^2)^{\frac{3}{2}}} = \frac{k}{(k^2+1)^{\frac{3}{2}}}$$

于是矛盾，故f 在$(0,0)$ 处不可微.

17: $\frac{\sin r}{r^2} \to 0, r \to \infty$故连续

$x^2+y^2 \neq 0, f_x = 2x\sin\frac{1}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}$

$f_y = 2y\sin\frac{1}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}$

$x = y = 0, f_x(0,0) = \lim_{x\to 0} x\sin\frac{1}{x} = 0, f_y(0,0) = 0$

$lim_{x\to 0} f_x(x,0) = lim_{x\to 0}\cos\frac{1}{x}$ *not exist*，偏导数不连续

$x, y \to 0, count, \frac{(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$

$= \frac{\sin r}{r} \to 0, r \to \infty$

18: $(1)\frac{\partial z}{\partial x} = 2xln(\frac{1}{1+y}),\ \frac{\partial z}{\partial y} = -\frac{x^2}{1+y}$

$\frac{\partial^2 z}{\partial x^2} = 2ln(\frac{1}{1+y}),\ \frac{\partial^2 z}{\partial x\partial y} = -\frac{2x}{1+y},\ \frac{\partial^2 z}{\partial x^2} = \frac{x^2}{(1+y)^2}$

$(2)\frac{\partial z}{\partial x} = -\frac{xy(1-2xy)}{(x-y)(1+(x-y-x^2y)^2)} + \frac{xyarctan(x-y-x^2y)}{(x-y)^2} - \frac{yarctan(x-y-x^2y)}{x-y}$

$\frac{\partial z}{\partial y} = -\frac{x(-1-x^2)y}{(x-y)(1+(x-y-x^2y)^2)} - \frac{xarctan(x-y-x^2y)}{x-y} - \frac{xyarctan(x-y-x^2y)}{(x-y)^2}$

$\frac{\partial^2 z}{\partial x^2} = \frac{2xy(1-2xy)^2(x-y-x^2y)}{(x-y)(1+(x-y-x^2y)^2)^2} + \frac{2xy^2}{(x-y)(1+(x-y-x^2y)^2)} + \frac{2xy(1-2xy)}{(x-y)^2(1+(x-y-x^2y)^2)} - \frac{2y(1-2xy)}{(x-y)(1+(x-y-x^2y)^2)} -$

$\frac{2xyarctan(x-y-x^2y)}{(x-y)^3}+\frac{2yarctan(x-y-x^2y)}{(x-y)^2}$

$\frac{\partial^2 z}{\partial x\partial y}=\frac{2x(-1-x^2)y(1-2xy)(x-y-x^2y)}{(x-y)(1+(x-y-x^2y)^2)^2}+\frac{x(-1-x^2)y}{(x-y)^2(1+(x-y-x^2y)^2)}+\frac{2x^2y}{(x-y)(1+(x-y-x^2y)^2)}-\frac{(-1-x^2)y}{(x-y)(1+(x-y-x^2y)^2)}-\frac{x(1-2xy)}{(x-y)(1+(x-y-x^2y)^2)}-\frac{xy(1-2xy)}{(x-y)^2(1+(x-y-x^2y)^2)}+\frac{xarctan(x-y-x^2y)}{(x-y)^2}-\frac{arctan(x-y-x^2y)}{x-y}+\frac{2xyarctan(x-y-x^2y)}{(x-y)^3}-\frac{yarctan(x-y-x^2y)}{(x-y)^2}$

$\frac{\partial^2 z}{\partial y^2}=\frac{2x(-1-x^2)^2y(x-y-x^2y)}{(x-y)(1+(x-y-x^2y)^2)^2}-\frac{2x(-1-x^2)}{(x-y)(1+(x-y-x^2y)^2)}-\frac{2x(-1-x^2)y}{(x-y)^2(1+(x-y-x^2y)^2)}-\frac{2xarctan(x-y-x^2y)}{(x-y)^2}-\frac{2xyarctan(x-y-x^2y)}{(x-y)^3}$

19: (1)

$$\begin{aligned}
\frac{\partial u}{\partial x}&=\frac{\partial u}{\partial t}\frac{\partial t}{\partial x}=e^t yx^{y-1}+\frac{2t}{1+t^2}yx^{y-1}=e^{x^y}yx^{y-1}+\frac{2yx^{2y-1}}{1+x^{2y}}\\
\frac{\partial u}{\partial y}&=\frac{\partial u}{\partial t}\frac{\partial t}{\partial y}=e^t x^y\ln x+\frac{2t}{1+t^2}x^y\ln x=e^{x^y}x^y\ln x+\frac{2\ln x x^{2y}}{1+x^{2y}}
\end{aligned}$$

(2)

$$\begin{aligned}
\frac{\partial u}{\partial r}&=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial r}=yze^{xyz}s+xze^{xyz}/s+xye^{xyz}sr^{s-1}\\
&=2(r^{s+1}e^{r^{s+2}})+r^{s+1}se^{r^{s+2}}\\
\frac{\partial u}{\partial s}&=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial s}=yze^{xyz}r+xze^{xyz}(-\frac{r}{s^2})+xye^{xyz}r^s\\
&=r^{s+2}e^{r^{s+2}}\ln r
\end{aligned}$$

(3)

$$\begin{aligned}
\frac{\partial u}{\partial r}&=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\frac{2x}{x^2+y^2}e^{t+s+r}+\frac{2y}{x^2+y^2}*0=\frac{2e^{2(t+s+r)}}{e^{2(t+s+r)}+16(s^2+t^2)^2}\\
\frac{\partial u}{\partial s}&=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}=\frac{2x}{x^2+y^2}e^{t+s+r}+\frac{2y}{x^2+y^2}8s=\frac{2e^{2(t+s+r)}+64s(s^2+t^2)}{e^{2(t+s+r)}+16(s^2+t^2)^2}\\
\frac{\partial u}{\partial t}&=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}=\frac{2x}{x^2+y^2}e^{t+s+r}+\frac{2y}{x^2+y^2}8t=\frac{2e^{2(t+s+r)}+64t(s^2+t^2)}{e^{2(t+s+r)}+16(s^2+t^2)^2}
\end{aligned}$$

(4)

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial u}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{e^{ax}}{a^2+1}a\cos x + \frac{e^{ax}}{a^2+1}\sin x = \frac{e^{ax}}{a^2+1}(\sin x + a\cos x)$$

20: (1)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 3t^2 f_1' + 4t f_2'$$

.

(2)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \cos t f_1' - \sin t f_2' + \mathrm{e}^t f_3'$$

.

(3)

$$\frac{\partial u}{\partial x} = 2x f_1' + y\mathrm{e}^{xy} f_2',$$
$$\frac{\partial^2 u}{\partial x \partial y} = -4xy f_{11}'' + (2x^2 - 2y^2)\mathrm{e}^{xy} f_{12}'' + xy\mathrm{e}^{2xy} f_{22}'' + (1+xy)\mathrm{e}^{xy} f_2'.$$

(4)

$$\frac{\partial u}{\partial x} = f_1' + 2x f_2', \qquad \frac{\partial^2 u}{\partial x^2} = f_{11}'' + 4x f_{12}'' + 2f_2' + 4x^2 f_{22}'',$$
$$\frac{\partial^2 u}{\partial x \partial y} = f_{11}'' + (2x+2y) f_{12}'' + 4xy f_{22}''.$$

21: 根据方向微商的计算公式

$$\frac{\partial u}{\partial \boldsymbol{l}} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) \cdot \frac{\boldsymbol{l}}{|\boldsymbol{l}|} = (-2,-1,2) \cdot \frac{(3,-1,1)}{\sqrt{11}} = -\frac{3\sqrt{11}}{11}$$

22: 解:

设$\vec{l}$为P点沿圆周逆时针方向的单位方向向量，易知:

$$\vec{l}=\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$$

由定理知:

$$grad\,(z)=\frac{\partial z}{\partial x}\vec{i}+\frac{\partial z}{\partial y}\vec{j}$$

且:

$$\frac{\partial z}{\partial x}=-\frac{y}{x^2+y^2},\frac{\partial z}{\partial y}=\frac{x}{x^2+y^2}$$

则在$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$点的方向微商为:

$$grad(z)\cdot\vec{l}_{x=\frac{1}{2},y=\frac{\sqrt{3}}{2}}=\frac{1}{2}$$

23: $u'_x=2x+y+3,u'_y=x+4y-2,u'_z=6z-2$

从而.$u'_x(1,1,-1)=6,u'_y(1,1,-1)=4,u'_z(1,1,-1)=-12$

$gradu\big|_{(1,1,-1)}=(6,3,-12)$

$(\frac{\partial f}{\partial\vec{e}})_{max}=|gradu\big|_{(1,1,-1)}|=3\sqrt{21}$

24: (1) 由$\frac{1}{r^2}=\frac{1}{x^2+y^2+z^2}$ 有

$$\begin{aligned}\frac{\partial\frac{1}{r^2}}{\partial x}&=-\frac{2x}{(x^2+y^2+z^2)^2}=-\frac{2x}{r^4},\\ \frac{\partial\frac{1}{r^2}}{\partial y}&=-\frac{2y}{(x^2+y^2+z^2)^2}=-\frac{2y}{r^4},\\ \frac{\partial\frac{1}{r^2}}{\partial z}&=-\frac{2z}{(x^2+y^2+z^2)^2}=-\frac{2z}{r^4},\end{aligned}\tag{9.1}$$

所以 $\mathbf{grad}\frac{1}{r^2}=-\frac{2}{r^4}\boldsymbol{r}$

(2) 由$\ln r=\frac{1}{2}\ln(x^2+y^2+z^2)$ 易有$\mathbf{grad}\ln r=\frac{1}{r^2}\boldsymbol{r}$.

25: $u_x=f'(\phi_1 y+\phi 2), u_y=f'(\phi_1 x+\phi_2)$

$u_{xy}=f''(\phi_1 x+\phi_2)(\phi_1 y+\phi_2)+f'(\phi_{11}xy+\phi_{12}y+\phi_{21}x+\phi_{22}+\phi_1)$

26: 略

27: 证明：可设中间变量$u=xy$,则$z=f(u)$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}=y\frac{\partial z}{\partial u}$$
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}=x\frac{\partial z}{\partial u}$$
$$\therefore x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=xy\frac{\partial z}{\partial u}-xy\frac{\partial z}{\partial u}=0$$

28:

$$\frac{\partial^2 u}{\partial t^2}=a^2\phi''(x-at)+a^2\psi''(x+at).$$

$$\frac{\partial^2 u}{\partial x^2}=\phi''(x-at)+\psi''(x+at).$$

从而$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$.

29: 证明：有

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\cos\varphi\frac{\partial u}{\partial x}+\sin\varphi\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial \varphi}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi}=-r\sin\varphi\frac{\partial u}{\partial x}+r\cos\varphi\frac{\partial u}{\partial y}$$

因此,有

$$\begin{aligned}
left&=\left(\cos\varphi\frac{\partial u}{\partial x}+\sin\varphi\frac{\partial u}{\partial y}\right)^2+\left(-\sin\varphi\frac{\partial u}{\partial x}+\cos\varphi\frac{\partial u}{\partial y}\right)^2\\
&=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\\
&=right
\end{aligned}$$

30: 证明:

由:

$$\begin{cases}\xi=x+y\\ \eta=3x-y\end{cases}$$

可知:

$$\begin{cases}x=\frac{1}{4}\left(\xi+\eta\right)\\ y=\frac{1}{4}\left(3\xi-\eta\right)\end{cases}$$

得到:

$$\frac{\partial x}{\partial \xi}=\frac{\partial x}{\partial \eta}=\frac{1}{3}\frac{\partial y}{\partial \xi}=-\frac{\partial y}{\partial \eta}=\frac{1}{4}$$

求得:

$$\frac{\partial u}{\partial \xi}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}=\frac{1}{4}\left(\frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial \eta\partial \xi}=\frac{1}{16}\left(\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+3\frac{\partial^2 u}{\partial y^2}\right)$$

代入得:

$$\frac{\partial^2 u}{\partial \eta\partial \xi}+\frac{1}{2}\frac{\partial u}{\partial \xi}=\frac{1}{16}\left(\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial \xi}+6\frac{\partial u}{\partial \xi}\right)=0$$

证毕。

31: $\xi = x - sinx + y, \eta = x + sinx - y$

则$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}(1 - cosx) + \frac{\partial u}{\partial \eta}(1 + cosx)$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}(1-cosx)^2 + \frac{\partial^2 u}{\partial \eta \partial \xi}(1-cos^2x) + \frac{\partial u}{\partial \xi} sinx + \frac{\partial^2 u}{\partial \eta^2}(1+cosx)^2 + \frac{\partial^2 u}{\partial \xi \partial \eta}(1-cos^2x) + \frac{\partial u}{\partial \eta}(-sinx)$

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2}(1 - cosx) + \frac{\partial^2 u}{\partial \eta \partial \xi}(1 + cosx) - \frac{\partial^2 u}{\partial \eta^2}(1 + cosx) - \frac{\partial^2 u}{\partial \xi \partial \eta}(1 - cosx)$

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta \partial \xi} - \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$

再由$\frac{\partial^2 u}{\partial \eta \partial \xi} = \frac{\partial^2 u}{\partial \xi \partial \eta}$

则$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} cosx - \frac{\partial^2 u}{\partial y^2} sin^2x - \frac{\partial u}{\partial y} sinx = 4\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$

故$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$

32: 由求导的链式法则有

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u} \tag{9.2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = -2\frac{\partial z}{\partial u} \tag{9.3}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u \partial v} \tag{9.4}$$

同理可有 $\frac{\partial^2 z}{\partial x \partial y} = a\frac{\partial^2 z}{\partial u \partial v}, \frac{\partial^2 z}{\partial y^2} = -2a\frac{\partial^2 z}{\partial u \partial v}$, 于是可得$a = 6$

33: $\int z_y dy = \int (x^2 + 2y)dy = yx^2 + y^2 + c(x)$

$z(x, x^2) = 1 = x^4 + x^4 + c(x) = 1$

$z(x, y) = yx^2 + y^2 + 1 - 2x^4$

34: $u(x, x^2) = 1$两边对x求导,得$u_x'(x, x^2) + 2xu_y'(x, x^2) = 0$

从而$u_y'(x, x^2) = -u_x'(x, x^2)/2x = -\frac{1}{2}$

35: 解：对$u(x,2x)=x$两边关于x求导可得：

$$u_x^{'}(x,2x)+2u_y^{'}(x,2x)=1,$$

再由已知

$$u_x^{'}(x,2x)=x^2,$$

则

$$u_y^{'}(x,2x)=\frac{1-x^2}{2},$$

以上两式关于x求导可得：

$$\begin{cases} u_{xx}^{''}(x,2x)+2u_{xy}^{''}(x,2x)=2x \\ u_{yx}^{''}(x,2x)+2u_{yy}^{''}(x,2x)=-x \end{cases}$$

由题设条件知

$$u_{xx}^{''}=u_{yy}^{''},u_{xy}^{''}=u_{yx}^{''}$$

联立解得

$$u_{xx}^{''}(x,2x)=u_{yy}^{''}(x,2x)=-\frac{4x}{3},u_{xy}^{''}(x,2x)=\frac{5x}{3}$$

36: (1)$\mathrm{d}u=f'\mathrm{d}x+f'\mathrm{d}y.$

(2)$\mathrm{d}u=(f_1'y+\frac{f_2'}{y})\mathrm{d}x+(f_1'x-\frac{xf_2'}{y^2})\mathrm{d}y.$

(3)$\mathrm{d}u=(f_1'+2tf_2'+3t^2f_3')\mathrm{d}t.$

(4)$\mathrm{d}u=(f_1'+2xf_2'+2xf_3')\mathrm{d}x+(2yf_2'+2yf_3')\mathrm{d}y+2zf_3'\mathrm{d}z.$

(5)$\mathrm{d}u=(2xf_1'+2xf_2'+2yf_3')\mathrm{d}x+(2yf_1'-2yf_2'+2xf_3')\mathrm{d}y.$

37: 球坐标下的Laplace方程的形式:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}=0$$

或:

$$\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}=0$$

38: 解:

由题意得:

$$\frac{\partial\left(x,y\right)}{\partial\left(r,\theta\right)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}$$

又知:

$$\frac{\partial x}{\partial r}=\cos\theta,\frac{\partial x}{\partial\theta}=-r\sin\theta,\frac{\partial y}{\partial r}=\sin\theta,\frac{\partial y}{\partial\theta}=r\cos\theta$$

代入得:

$$\frac{\partial\left(x,y\right)}{\partial\left(r,\theta\right)}=r\cos^2\theta+r\sin^2\theta=r$$

9.3 隐函数定理和逆映射定理

1: (1) Let $F(x,y)=x^2+xy+y^2-7$, then it's easy to check

[1] $F(x,y)\in C^1$

[2] $F(2,1)=2^2+2\cdot 1+1^2-7=0$

[3] $F_y(2,1)=2+2\cdot 1\neq 0$

By implicit function theorem, there exist $y=y(x)$ who is determined by $x^2+xy+y^2-7=0$ near point $(2,1)$.

derivation of x on both sides of the equation, we get $2x+y+xy'+2yy'=0$, then $y'(2,1)=-\frac{5}{4}$

derivation of x on both sides of the equation, $2+y'+y'+xy''+2y'y'+2yy''=0$, then $y''(2,1)=-\frac{21}{32}$

(2)Similar to (1). $y'(1,\frac{\pi}{2})=-\frac{\pi}{2}$, $y''(1,\frac{\pi}{2})=\pi$

2: (1)$\frac{dy}{dx}=\frac{2xy+ye^{xy}-ycos(xy)}{xcos(xy)-xe^{xy}-x^2}$

(2)$\frac{dy}{dx}=\frac{x+y}{x-y},\frac{d^2y}{dx^2}=\frac{2(x^2+y^2)}{(x-y)^3}$

(3)$\frac{dy}{dx}=\frac{ln(y)-\frac{y}{x}}{ln(x)-\frac{x}{y}}$

$\frac{d^2y}{dx^2}=\frac{2(\frac{1}{y}-\frac{1}{x})(ln(y)-\frac{y}{x})(ln(x)-\frac{x}{y})+\frac{y}{x^2}(ln(x)-\frac{x}{y})^2-\frac{x}{y^2}(ln(y)-\frac{y}{x})^2}{(ln(x)-\frac{x}{y})^3}$

(4)$\frac{\partial z}{\partial x}=\frac{ye^{-xy}}{e^z-2}$, $\frac{\partial z}{\partial y}=\frac{xe^{-xy}}{e^z-2}$

$\frac{\partial x}{\partial y}=-\frac{y}{x}$, $\frac{\partial^2 z}{\partial x^2}=\frac{-y^2e^{-xy}((e^z-2)^2+ez-xy)}{(e^z-2)^3}$

(5)$\frac{\partial z}{\partial x}=\frac{xz}{x^2+z^2}$, $\frac{\partial z}{\partial y}=\frac{z^3}{y(x^2+z^2)}$

(6)$\frac{\partial z}{\partial x}=-\frac{F_1^{'}+F_2^{'}+F_3^{'}}{F_3^{'}}$, $\frac{\partial z}{\partial y}=-\frac{F_2^{'}+F_3^{'}}{F_3^{'}}$

(7)$\frac{\partial z}{\partial x}=-\frac{zF_1^{'}}{xF_1^{'}+yF_2^{'}}$, $\frac{\partial z}{\partial y}=-\frac{zF_2^{'}}{xF_1^{'}+yF_2^{'}}$

3: 令$F(x,y)=x^2+xy+y^2-27$，易知道(3,3)和(-3,-3)为F的零点。对F求偏导有$F_x^{'}=2x+y,F_y^{'}=x+2y$，因此

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x'(x,y)}{F_y'(x,y)}=-\frac{2x+y}{x+2y}$$

令$\frac{\mathrm{d}y}{\mathrm{d}x}=0$有$y=-2x$，代入原方程得到$x^2+x(-2x)+(-2x)^2=0$，得到$x=3,y=-6$或$x=-3,y=6$。由于$F(x,y)$为椭圆，可知道y=6为极大值，y=-6为极小值

也可用二阶导确定是极大值还是极小值。对$y^{'}(x+2y)=-(2x+y)$两边对x求导有

$$y^{''}(x+2y)+y^{'}(1+2y^{'})=-2-y^{'}$$

化简有$y''=-2\frac{1+y^{'}+(y^{'})^2}{x+2y}=-6\frac{x^2+xy+y^2}{(x+2y)^3}$,$x=3,y=-6$时$y''=\frac{2}{9}$因此y=-6极小值，而$x=-3,y=6$时$y''=-\frac{2}{9}$因此y=6为极大值点。