使用球极坐标换元，$0\le r\le t, 0\le\phi\le\pi, 0\le\theta\le 2\pi$

原式=$\int_0^{2\pi}\int_0^{\pi}\int_0^t f(r^2)r^2\sin\phi drd\phi d\theta$

易见只要知道$G(t)=\int_0^t f(r^2)r^2dr$的导数

一元变上限积分的导数利用莱布尼兹公式即得

10: $\int_D \rho dxdy=\frac{\pi}{2}ab$

11: 任取圆环上一圆周，圆周到圆心的距离为x

该圆周的面积ds=2πxdx,密度ρ=$\frac{1}{x}$

面质量dm=ρds=2πdx

质量m=$\int_L dm$=$\int_r^R 2\pi\, dx$=2π(R-r)

15: 不妨设$a,b,c>0$

考虑坐标变换$\begin{cases} x &= ar\cos\theta \\ y &= br\sin\theta \\ z &= z \end{cases}$

$\frac{\partial(x,y,z)}{\partial(r,\theta,z)}=abr\neq 0$

则$z_G=\frac{\iiint_V zdxdydx}{\iiint_V dxdydx}=\frac{\int_0^c dz\int_0^{2\pi}d\theta\int_0^{\frac{z}{c}} abrzdz}{\int_0^c dz\int_0^{2\pi}d\theta\int_0^{\frac{z}{c}} abrdz}=\frac{\frac{1}{4}\pi abc^2}{\frac{1}{3}\pi abc}=\frac{3}{4}\pi$

由对称性可知$x_G=y_G=0$

故重心为$(0,0,\frac{3}{4}\pi)$

17: 本题直接套用本章正文的质心计算公式不难算出

注意到对称性x，y不用去算，只要算z

两个计算区域可以分别用柱坐标换元和球坐标换元计算，然后再相加

18: (1)(a)$I=\int_0^{2\pi}\int_0^R \rho r^3 dr d\theta=\frac{1}{2}mR^2$

(b)$I=\int_{-R}^{R}dx\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\rho x^2 dxdy=\frac{1}{4}mR^2$

(2)$\frac{3}{5}mR^2$

19: 取球心为坐标原点，z轴过顶点

则顶点坐标为(0,0,l=$\frac{R}{tan\alpha}$)

由积分对称性可得$F_x=F_y=0$

$F_z=km\rho\iiint_V \frac{z-l}{r^3}\,dx\,dy\,dz$

$r=\sqrt{x^2+y^2+(z-l)^2}$,$V=V_1+V_2$

V_1为半球体 $x^2+y^2+z^2\leq R^2,(z\leq 0)$

V_2为锥体 $x^2+y^2=((l-z)\tan\alpha)^2\leq R^2,(0\leq z\leq l)$

将物体带入积分区域，仿照书本10.3节例10.3.13积分可得最终结果

10.4 n重积分

1: n重积分的计算转化为累次积分的计算

(1)

$$\int\cdots\int_{[0,1]^n} x_1^2+\cdots+x_n^2 \mathrm{dx_1}\cdots\mathrm{dx_n}=\int_0^1\mathrm{dx_n}\cdots\int_0^1\mathrm{dx_2}\int_0^1 x_1^2+\cdots+x_n^2\mathrm{dx_1} \tag{10.1}$$

$$=\int_0^1 x_n^2\mathrm{dx_n}+\cdots+\int_0^1 x_1^2\mathrm{dx_1} \tag{10.2}$$

$$=\frac{n}{3} \tag{10.3}$$

(2) 引理：对任意$0 \le i \le n$有

$$\int_0^1 (x_i + \cdots + x_n)^2 \mathrm{dx_i} = \frac{1}{12} + \left(\frac{1}{2} + x_{i+1} + \cdots + x_n\right)^2$$

证明：只需要关于x_i展开，积分后重新配方即可，易证

$$\int \cdots \int_{[0,1]^n} (x_1 + \cdots + x_n)^2 \mathrm{dx_1} \cdots \mathrm{dx_n} = \frac{1}{12} + \int \cdots \int_{[0,1]^{n-1}} (\frac{1}{2} + x_2 + \cdots + x_n)^2 \mathrm{dx_2} \cdots \mathrm{dx_n} \tag{10.4}$$

$$= \cdots = \frac{n}{12} + (\frac{n}{2})^2 \tag{10.5}$$

$$\tag{10.6}$$

(3)

$$\int_0^1 \mathrm{dx_1} \int_0^{x_1} \mathrm{dx_2} \cdots \int_0^{x_{n-1}} x_1 \cdots x_n \mathrm{dx_n} = \int_0^1 \mathrm{dx_1} \int_0^{x_1} \mathrm{dx_2} \cdots \int_0^{x_{n-2}} \frac{1}{2} x_1 \cdots x_{n-1}^3 \mathrm{dx_{n-1}} \tag{10.7}$$

$$= \cdots = \int_0^1 \frac{1}{2^{n-1}(n-1)!} x_1^{2n-1} \mathrm{dx_1} = \frac{1}{2^n n!} \tag{10.8}$$

$$\tag{10.9}$$

2: $\frac{\prod_{i=1}^n a_i}{n!}$

3: 证明：采用数学归纳法，当$n=1$时，结论显然成立。假设$n=k$时，结论成立，即

$$\int_0^a dx_1\int_0^{x_1}dx_2\cdots\int_0^{x_{k-1}}f(x_k)dx_k=\frac{1}{(k-1)!}\int_0^a f(t)(a-t)^{k-1}dt$$

那么，当$n=k+1$时，有

$$\int_0^a dx_1\int_0^{x_1}dx_2\cdots\int_0^{x_{k-1}}dx_k\int_0^{x_k}f(x_{k+1})dx_{k1}=\int_0^a dx_1\frac{1}{(k-1)!}\int_0^{x_1}f(t)(x_1-t)^{k-1}dt$$
$$=\int_0^a dt\frac{1}{(k-1)!}\int_t^a f(t)(x_1-t)^{k-1}dx_1=\int_0^a\frac{1}{k!}f(t)(a-t)^k dt$$

10.5 综合习题

2: I可以化简为$\int_{-1}^1\int_{-1}^1\frac{ArcTan(\frac{1}{\sqrt{1+x^2+y^2}})}{\sqrt{1+x^2+y^2}}dydx$
结果的数值近似是0.308425

3: (1)

$$I_1=\int_0^1 sin(ln\frac{1}{x})\cdot\frac{x^b-x^a}{lnx}dx=\int_0^1 sin(ln\frac{1}{x})\int_a^b x^y dydx=\int_a^b dy\int_0^1 sin(ln\frac{1}{x})x^y dx$$

计算

$$J_1=\int_0^1 sin(ln\frac{1}{x})x^y dx=\int_0^1\frac{sin(ln(\frac{1}{x}))}{y+1}dx^{y+1}=\frac{sin(ln(\frac{1}{x}))x^{y+1}}{y+1}\Big|_0^1+\int_0^1\frac{x^{y+1}}{y+1}\frac{cos(ln\frac{1}{x})}{x}dx$$
$$=\int_0^1\frac{cos(ln\frac{1}{x})}{(y+1)^2}dx^{y+1}=\frac{cos(ln\frac{1}{x})x^{y+1}}{(y+1)^2}\Big|_0^1-\int_0^1\frac{x^y}{(y+1)^2}sin(ln\frac{1}{x})dx=\frac{1}{(y+1)^2}-\frac{1}{(y+1)^2}J_1$$

解得

$$J_1=\frac{1}{(y+1)^2+1}$$

那么

$$I_1=\int_a^b J_1dy=\int_a^b \frac{1}{(y+1)^2+1}dy=\int_{a+1}^{b+1}\frac{1}{t^2+1}dt$$
$$=arctan(t)|_{a+1}^{b+1}=arctan(b+1)-arctan(a+1)=arctan\frac{b-a}{1+(a+1)(b+1)}$$

(2) 同理

$$I_2=\int_0^1 cos(ln\frac{1}{x})\cdot\frac{x^b-x^a}{lnx}dx=\int_0^1 cos(ln\frac{1}{x})\int_a^b x^ydydx=\int_a^b dy\int_0^1 cos(ln\frac{1}{x})x^ydx$$

计算

$$J_2=\int_0^1 cos(ln\frac{1}{x})x^ydx=\int_0^1\frac{cos(ln(\frac{1}{x}))}{y+1}dx^{y+1}=\frac{cos(ln(\frac{1}{x}))x^{y+1}}{y+1}\Big|_0^1-\int_0^1\frac{x^{y+1}}{y+1}\frac{sin(ln\frac{1}{x})}{x}dx$$
$$=\frac{1}{y+1}-\int_0^1\frac{sin(ln\frac{1}{x})}{(y+1)^2}dx^{y+1}=\frac{1}{y+1}-\frac{sin(ln\frac{1}{x})x^{y+1}}{(y+1)^2}\Big|_0^1-\int_0^1\frac{sin(ln\frac{1}{x})x^y}{(y+1)^2}dx$$
$$=\frac{1}{y+1}-\frac{1}{(y+1)^2}J_2$$

解得

$$J_2=\frac{y+1}{(y+1)^2+1}$$

那么

$$I_2=\int_a^b J_2dy=\int_a^b\frac{y+1}{(y+1)^2+1}dy\overset{t=y+1}{=}\int_{a+1}^{b+1}\frac{t}{t^2+1}dt\overset{u=t^2}{=}\int_{(a+1)^2}^{(b+1)^2}\frac{1}{2(u+1)}du$$
$$=\frac{1}{2}ln(u+1)\Big|_{(a+1)^2}^{(b+1)^2}=\frac{1}{2}ln\frac{(b+1)^2+1}{(a+1)^2+1}$$

6: 解：考虑到$y\geqslant 0$和曲面关于x,z对称，因此对于此体积分可以化归为第一象限内的体积分，也即：

$$\iiint_V dxdydz=4\iiint_{V'}dxdydz$$

其中V'是V在第一象限的部分。根据重积分的定理，可知：

$$\iiint_{V'} dxdydz = \int_0^{z_0} dz \iint_D dxdy$$

其中z_0可以根据柱坐标系代换得知：

$$z^4 = r\sin\theta - r^4 \geqslant r - r^4 \geqslant \frac{3}{4}\cdot 2^{-\frac{2}{3}}$$

因此：

$$z_0 = (\frac{3}{4}\cdot 2^{-\frac{2}{3}})^{\frac{1}{4}}$$

而对于内部的二重积分，则有r,θ满足：

$$z^4 = r\sin\theta - r^4$$

其中r,θ具体关系，涉及到四次方程的判别式因此该问题难以求得解析解。若使用数学软件Mathematica，亦无法获得解析解，但可以求得数值解，其解为：0.3702402451

7: 考虑坐标变换$\begin{cases} t &= xy \\ m &= x \end{cases}$

则$\begin{cases} x &= m \\ y &= \frac{m}{t} \end{cases}$

$\frac{\partial(x,y)}{\partial(m,t)} = \frac{1}{m} \neq 0$

记$D = \{(x,y)|0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$

则$D' = \{(m,t)|0 \leqslant m \leqslant 1, 0 \leqslant \frac{t}{m} \leqslant 1\} = \{(m,t)|0 \leqslant t \leqslant 1, t \leqslant m \leqslant 1\}$

而

$$\lim_{t\to 0^+} t^t = \lim_{t\to 0^+} e^{t\log t} = 1$$

则 $\iint_D (xy)^{xy} dxdy = \iint_{D'} t^t \frac{1}{m} dmdt$

$$
\begin{aligned}
&= \int_0^1 -t^t \log t dt \\
&= -t^t|_{t\to 0^+}^1 + \int_0^1 t^t dt \\
&= \int_0^1 t^t dt
\end{aligned}
$$

10:

$$
\begin{aligned}
\int_a^b dx \int_{\phi(x)}^{\psi(x)} f^2(x,y)dy &= \int_a^b dx \int_{\phi(x)}^{\psi(x)} (\int_{\phi(x)}^{y} f_y'(x,t)dt)^2 dy && (10.10)\\
&\le \int_a^b dx \int_{\phi(x)}^{\psi(x)} dy \int_{\phi(x)}^{y} (f'y(x,t)^2)dt \int_{\phi(x)}^{y} dt && (10.11)\\
&\le \int_a^b dx \int_{\phi(x)}^{\psi(x)} dy \int_{\phi(x)}^{\psi(x)} (f'y(x,t))^2 dt \int_{\phi(x)}^{\psi(x)} dt \quad (using\ Cauchy\ inequality) && (10.12)\\
&\le \int_a^b dx \int_{\phi(x)}^{\psi(x)} dy (f'(x,y))^2 \quad (define\ M := \max_{a\le x\le b} |\psi(x)-\phi(x)|) && (10.13)
\end{aligned}
$$

11: 做变元代换

$$x_i = at_i, i = 1, \cdots, n, \quad \frac{\partial(x_1, \cdots, x_n)}{\partial(t_1, \cdots, t_n)} = a^n$$

因此积分可以写成

$$I_n(a) = \int \cdots \int_{\Omega_n(a)} x_1 x_2 \cdots x_n dx_1 x_2 \cdots x_n = a^{2n} \int \cdots \int_{\Omega_n(1)} t_1 t_2 \cdots t_n dt_1 t_2 \cdots t_n = a^{2n} I_n(1)$$

区域也可写成

$$\Omega_n(1) : 0 \le x_n \le 1, 0 \le \sum_{i=1}^{n-1} x_i \le 1 - x_n, x_1, x_2, \cdots x_{n-1} \ge 0$$

因此积分可以写成

$$
\begin{aligned}
I_n(1) &= \int_0^1 tdt \int \cdots \int_{\Omega_{n-1}(1-t)} t_1 t_2 \cdots t_{n-1} dt_1 t_2 \cdots t_{n-1} = \int_0^1 tI_{n-1}(1-t)dt \\
&= \int_0^1 t(1-t)^{2n-2} I_{n-1}(1)dt = I_{n-1}(1) \int_0^1 t(1-t)^{2n-2} = \int_0^1 (1-t)^{2n-2}dt - \int_0^1 (1-t)^{2n-1} \\
&= \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{2n(2n-1)}
\end{aligned}
$$

又显然的$I_1(1) = \int_0^1 xdx = \frac{1}{2}$，所以

$$
I_n(1) = \frac{1}{(2n)!}, I_n(a) = \frac{a^{2n}}{(2n)!}
$$

Chapter 11

曲线积分和曲面积分

11.1 数量场在曲线上的积分

1: (1)

根据弧长公式

$$\int_0^{2\pi} \sqrt{[(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2 + e^{2t}]}dt = \sqrt{3}\int_0^{2\pi} e^t dt = \sqrt{3}(e^{2\pi}-1)$$

(2)

根据弧长公式有

$$\begin{aligned}&\int_0^1 \sqrt{9+(6t)^2+(6t^2)^2}\\&=\int_0^1 \sqrt{9+36t^2+36t^4}\\&=\int_0^1 6t^2+3dt=5\end{aligned}$$

(3)

由弧长公式得到

$$
\begin{aligned}
&\sqrt{(a\sin t)^2+(a\cos t)^2+\left[\frac{a\sin t}{\cos t}\right]^2}\\
&=\int_0^{\frac{\pi}{4}}\frac{a}{\cos t}dt\\
&=\int_0^{\frac{\pi}{4}}\frac{a\cot t}{\cos^2 t}dt=\int_0^{\frac{\sqrt{2}}{2}}\frac{a}{1-x^2}dx=\frac{a}{2}\ln(3+2\sqrt{2})
\end{aligned}
$$

(4)

可以解出曲线的参数方程为

$$
\begin{cases}
x &= \frac{z^2}{2a}\\
y &= \sqrt{\frac{8z^3}{9a}}\\
z &= z
\end{cases}
$$

所以由曲线的弧长公式

$$\int_0^{2a}\sqrt{(\frac{z}{a})^2+\frac{2z}{a}+1}dz=4|a|$$

(5)

令$y+z=0$，　那么可以解出参数方程为

$$
\begin{cases}
x &= \frac{t^2}{4a}\\
y &= \frac{1}{2}\left(t-\frac{t^3}{12a^2}\right)\\
z &= \frac{1}{2}\left(t+\frac{t^3}{12a^2}\right)
\end{cases}
$$

所以根据弧长公式

$$
\begin{aligned}
&\int\sqrt{\left(\frac{t}{2a}\right)^2+\frac{1}{4}(1-\frac{t^2}{4a^2})^2+\frac{1}{4}(1+\frac{t^2}{4a^2})^2}\\
&=\int\frac{t^2+4a^2}{4\sqrt{2}a^2}=\int_0^2\sqrt{2}zdz=\frac{\sqrt{2}}{2}z^2
\end{aligned}
$$

11.2 数量场在曲面上的积分

11.3 向量场在曲线上的积分

1: 第二型曲线积分基本计算方法是利用积分曲线的参数方程转化为第一型曲线积分

(1)两段分别为：$L_1 : y = x, L_2 : x + y = 2$，可以都选取$x$为参数

$$\int_{L_1} (x^2 + y^2)\mathrm{dx} + (x^2 - y^2)\mathrm{dy} = \int_0^1 2x^2\mathrm{dx} = \frac{2}{3}$$

$$\int_{L_2} (x^2 + y^2)\mathrm{dx} + (x^2 - y^2)\mathrm{dy} = \int_1^2 2(2 - x)^2\mathrm{dx} = \frac{2}{3}$$

故，原积分为$\int_L (x^2 + y^2)\mathrm{dx} + (x^2 - y^2)\mathrm{dy} = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$ 注：也可以用Green公式

(2)积分曲线为：$|x| + |y| = 1$，故在此曲线上，被积表达式为$\int_L \mathrm{dx} + \mathrm{dy}$

因为在直线AB, CD上，$\mathrm{dx} = -\mathrm{dy}$，积分为0

在直线BC, DA上，$\mathrm{dx} = \mathrm{dy}$，但是由于方向相反，在用参数方程表达时上下限相反，故求和为0

综上，此积分在正方形$ABCD$上的值为0

(3)思路同2，此积分表达式限制在积分曲线上可以表示为$\frac{1}{a^2} \int_L -x\mathrm{dx} + y\mathrm{dy}$

利用圆的参数方程$x(t) = a\cos t, y(t) = a\sin t$以及相应地$\mathrm{dx} = -a\sin t\mathrm{dt}, \mathrm{dy} = a\cos t\mathrm{dt},$

带入表达式有$\int_0^{2\pi} \sin 2t\mathrm{dt} = 0$

(4)$OA: 0 \leq x \leq 1, y = 0, z = 0$，此时对应表达式为0，相应的积分值为0

$AB: x = 1, 0 \leq y \leq 1, z = 0$，此时对应表达式为$\int_0^1 y\mathrm{dy} = \frac{1}{2}$ $BC: x = 1, y = 1, 0 \leq z \leq 1$，此时对应表达式为$\int_0^1 z\mathrm{dz} = \frac{1}{2}$

故积分值为1

(5)A对应$\phi = 0$，B对应$\phi = \frac{\pi}{2}$，故原积分表达式可化为

$$\int_0^{\frac{\pi}{2}} e^{\cos\phi + \sin\phi + \frac{\phi}{\pi}} \left(\cos\phi - \sin\phi + \frac{1}{\pi}\right) \mathrm{d}\phi = e^{\cos\phi + \sin\phi + \frac{\phi}{\pi}}|_0^{\frac{\pi}{2}} = e^{\frac{3}{2}} - e$$

(6)L的参数方程为$x = 1 - \cos t, y = 1 + \cos t, z = \sqrt{2}\sin t$，注意到原点看是顺时针，$0 \leq t \leq 2\pi$，带入被积表达式有$\int_0^{2\pi}(1 + \cos t)\sin t\mathrm{dt} = 0$

11.4 向量场在曲面上的积分

11.5 Gauss定理和Stokes定理

11.6 其它形式的曲线曲面积分*

1: 参照课本推导，结果为

$$\frac{1}{r}\frac{\partial rF_r}{\partial r} + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$