Chapter 10

多变量函数的重积分

10.1 二重积分

1: 通过积分的边界确定积分区域，然后换序，一定要画图！

(1)积分区域是一个半圆，边界方程是$x^2+y^2=1$

$$\int_0^1 \mathrm{dy} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\mathrm{dx}$$

(2)积分区域是三角形，边界为$x=0, y=2x, x+y=6$

$$\int_0^4 \mathrm{dy} \int_0^{\frac{y}{2}} f(x,y)\mathrm{dx} + \int_4^6 \mathrm{dy} \int_0^{6-y} f(x,y)\mathrm{dx}$$

(3)积分区域是半圆，边界是$(x-a)^2+y^2=a^2$

$$\int_0^{2a} \mathrm{dx} \int_0^{\sqrt{a^2-(x-a)^2}} f(x,y)\mathrm{dy}$$

(4)积分区域是三角形，边界为$x=b, y=a, y=x$

$$\int_a^b \mathrm{dy} \int_a^x f(x,y)\mathrm{dy}$$

(5)积分区域是三角形，边界为$y=0, y=x, x+y=2$

$$\int_0^1 \mathrm{dy} \int_y^{2-y} f(x,y)\mathrm{dx}$$

(6)积分区域是曲边三角形，边界为$x=\frac{1}{2}, x=1, xy=1$

$$\int_{\frac{1}{2}}^1 \mathrm{dx} \int_0^{\frac{1}{x}} f(x,y)\mathrm{dy}$$

2: (1)$\iint_D \frac{y}{(1+x^2+y^2)^{\frac{3}{2}}}dxdy = \int_0^1 dx \int_0^1 \frac{y}{(1+x^2+y^2)^{\frac{3}{2}}}dy = \int_0^1 (\frac{-1}{\sqrt{x^2+2}} + \frac{1}{\sqrt{x^2+1}})dx = ln(\frac{2+\sqrt{2}}{\sqrt{3}+1})$

(2)$\iint_D sin(x+y)dxdy = \int_0^\pi dx \int_0^\pi sin(x+y)dy = \int_0^\pi 2cosxdx = 0$

(3)$\iint_D cos(x+y)dxdy = \int_0^\pi dx \int_x^\pi cos(x+y)dy = \int_0^\pi (-sinx - sin(2x))dx = -2$

(4)$\iint_D (x+y)dxdy = \int_0^a dx \int_0^{\sqrt{a^2-x^2}} (x+y)dy = \int_0^a (x\sqrt{a^2-x^2} + \frac{1}{2}(a^2 - x^2))dx = \frac{2}{3}a^3$

(5)$\iint_D (x+y-1)dxdy = \int_a^{3a} dy \int_{y-a}^y (x+y-1)dx = \int_a^{3a} (2ay - \frac{a^2}{2} - a)dy = 7a^3 - 2a^2$

(6)$\iint_D \frac{siny}{y}dxdy = \int_0^1 dy \int_{y^2}^y \frac{siny}{y}dx = \int_0^1 (siny - ysiny)dx = 1 - sin1$

(7)$\iint_D \frac{x^2}{y^2}dxdy = \int_1^2 dx \int_{\frac{1}{x}}^x \frac{x^2}{y^2}dy = \int_1^2 (x^3 - x)dx = \frac{9}{4}$

(8)$\iint_D |cos(x+y)|dxdy = \int_0^{\frac{\pi}{4}} dy \int_y^{\frac{\pi}{2}-y} cos(x+y)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} dx \int_{\frac{\pi}{2}-x}^{x} -cos(x+y)dy = \int_0^{\frac{\pi}{4}}(1-sin2y)dy + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(1-sin2x)dx = \frac{\pi}{2}-1$

3: 解:

(1)由积分区域的对称性和被积函数对x,y都是偶函数，可得:

$$\iint_D (x^2+y^2)dxdy = 4\int_0^1 dx \int_0^1 (x^2+y^2)dy = \frac{8}{3}$$

(2)由积分区域的对称性和被积函数对x,y都是奇函数，可得:

$$\iint_D (\sin x + \sin y)dxdy = 0$$

6: 解:

$\because f(x)$有二阶连续偏导数,$\therefore \frac{\partial^2 f(x,y)}{\partial x \partial y}$在$D$上可积,且

$$\begin{aligned}
\iint_D \frac{\partial^2 f(x,y)}{\partial x \partial y} dxdy &= \int_a^b dx \int_c^d \frac{\partial^2 f(x,y)}{\partial x \partial y} dy \\
&= \int_a^b dx \int_c^d \frac{\partial^2 f(x,y)}{\partial y \partial x} \\
&= \int_a^b \frac{\partial f(x,y)}{\partial x}|_{y=c}^{y=d} dx \\
&= \int_a^b (\frac{\partial f(x,d)}{\partial x} - \frac{\partial f(x,c)}{\partial x})dx \\
&= [f(x,d)-f(x,c)]|_{x=a}^{x=b} \\
&= f(b,d)+f(a,c)-f(b,c)-f(a,d)
\end{aligned}$$

7: 由积分中值定理

$\exists (x_0,y_0) \in D = \{(x,y)|x^2+y^2 \leqslant r^2\} s.t. \iint_{x^2+y^2 \leqslant r^2} f(x,y)dxdy = f(x_0,y_0)\pi r^2$

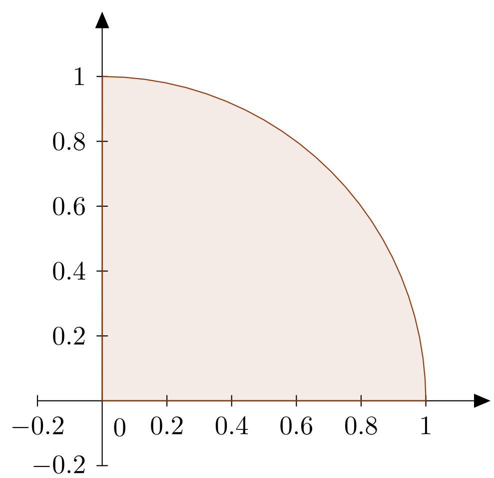
故

$$\lim_{r\to 0}\frac{1}{\pi r^2}\iint_{x^2+y^2\leqslant r^2} f(x,y)dxdy=\lim_{r\to 0} f(x_0,y_0)=f(0,0)$$

10.2 二重积分的换元

1: (1)

做极坐标换元，令$x = r\cos\theta$，$y = r\sin\theta$，其中$r\in[0,R]$，$\theta\in[0,\frac{\pi}{2}]$。那么则有



$$\begin{aligned}
&\int_0^R dx\int_0^{\sqrt{R^2-x^2}}\ln(1+x^2+y^2)dy\\
&=\int_0^{\frac{\pi}{2}} d\theta\int_0^R \ln(1+r^2)rdr\\
&=\frac{\pi}{2}\times\frac{1}{2}\left[(1+R^2)\ln(1+R^2)-(1+R^2)+1\right]\\
&=\frac{\pi}{4}\left[(1+R^2)\ln(1+R^2)-R^2\right]
\end{aligned}$$

(2)

$$\begin{aligned}
&\int_0^a dx\int_0^b xy(x^2-y^2)dy\\
&\int_0^a \frac{1}{2}b^2x^3-\frac{1}{4}b^4xdx\\
&=\frac{1}{8}b^2a^4-\frac{1}{8}b^4a^2
\end{aligned}$$

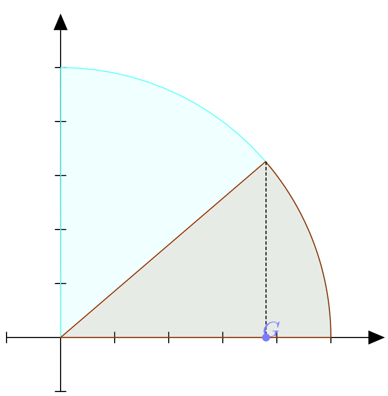
(3)

$$\begin{aligned}&\int_0^{\pi}\int_0^{\pi}\cos(x+y)dxdy\\&=\int_0^{\pi}\sin(x+\pi)-\sin(x)dx\\&=-2\int_0^{\pi}\sin xdx=-4\end{aligned}$$

(4)

$$\begin{aligned}&\int_0^{\frac{1}{\sqrt{2}}}dx\int_x^{\sqrt{1-x^2}}xy(1+y)dy\\&=\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{2}x^2(1-2x^2)+\frac{1}{3}x[(1-x^2)^{\frac{3}{2}}-x^3]dx\\&=\frac{1}{6}(\frac{1}{\sqrt{2}})^3-\frac{1}{5}(\frac{1}{\sqrt{2}})^5-\frac{1}{15}(\frac{1}{\sqrt{2}})^5+\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{3}x(1-x^2)^{\frac{3}{2}}dx=\frac{1}{15}\end{aligned}$$

(5)



$$\begin{aligned}&\int_0^{\frac{R^2}{\sqrt{1+R^2}}}dy\int_{\frac{R}{y}}^{\sqrt{R^2-y^2}}(1+\frac{y^2}{x^2})dx\\&=\int_0^{\frac{R^2}{\sqrt{1+R^2}}}\sqrt{R^2-y^2}-\frac{y}{R}+Ry-\frac{y^2}{\sqrt{R^2-y^2}}dy\\&=\frac{1}{2}\frac{R^3(R^2-1)}{R^2+1}+\int_0^{\frac{R^2}{\sqrt{1+R^2}}}\frac{R^2-2y^2}{\sqrt{R^2-y^2}}dy\end{aligned}$$

使用极坐标换元$x=r\cos\theta$，$y=r\sin\theta$，此时$\theta\in[0,\arcsin\frac{R}{\sqrt{1+R^2}}]$。不妨

设$\theta_0 = \arcsin \frac{R}{\sqrt{1+R^2}}$。那么则有

$$\begin{aligned}\frac{R^2-2y^2}{\sqrt{R^2-y^2}}dy &= \int_0^{\arcsin \frac{R}{\sqrt{1+R^2}}} R^2(1-2\sin^2\theta)d\theta \\ &= R^2 \sin\theta_0 \cos\theta_0 \\ &= \frac{R^3}{1+R^2}\end{aligned}$$

所以原结果即为

$$\frac{1}{2}\frac{R^3(R^2-1)}{1+R^2} + \frac{R^3}{1+R^2} = \frac{R^3(1+R^2)}{2(1+R^2)} = \frac{1}{2}R^3.$$

2: (1)做极坐标变换$\begin{cases} x = rcos\theta \\ y = rsin\theta \end{cases}$

则$D = \{(x,y)|x^2+y^2 < x+y\} = \{(r,\theta)|0 \le r \le sin\theta + cos\theta, -\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\}$

故$\iint_D \sqrt{x^2+y^2}dxdy = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \frac{0}{sin\theta+cos\theta} r^2 d\theta = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{3}(sin\theta+cos\theta)^3 d\theta = \frac{8\sqrt{2}}{9}$

(2)做类极坐标变换$\begin{cases} x = arcos\theta \\ y = brsin\theta \end{cases}$

则$D = \{(x,y)|\dots\} = \{(r,\theta)|a \le r \le 2, 0 \le \theta \le arctan\frac{a}{b}\}$

故$\iint_D \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}dxdy = ab\int_0^{arctan\frac{a}{b}} d\theta \int_0^2 r^2 dr = \frac{8ab}{3}\int_0^{arctan\frac{a}{b}} d\theta = \frac{8ab}{3}arctan\frac{a}{b}$

(3)做如下变换$\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases}$

则$D = \{(x,y)|\dots\} = \{(u,v)|1 \le u \le 2, 1 \le v \le 2\}$

故$\iint_D (x^2+y^2)dxdy = \int_1^2 dv \int_1^2 (\frac{u}{v}+uv)\frac{1}{2v}du = \frac{3}{4}\int_1^2 (1+\frac{1}{v^2})dv = \frac{9}{8}$

(4)做如下变换$\begin{cases} u = \frac{y^2}{x} \\ v = \frac{x^2}{y} \end{cases}$

则$D = \{(x,y)|\dots\} = \{(u,v)|b \le u \le a, n \le v \le m\}$

故$\iint_D dxdy = \int_a^b du \int_n^m \frac{1}{3} dv = \frac{1}{3}(a-b)(m-n)$

(5)做如下变换$\begin{cases} u = xy \\ v = \frac{y^2}{x} \end{cases}$

则$D = \{(x,y)| \ldots\} = \{(u,v)|a \le u \le b, c \le v \le d\}$

故$\iint_D dxdy = \int_a^b du \int_c^d \frac{u}{3v} dv = \frac{1}{3} ln\frac{d}{c} \int_a^b udu = \frac{1}{6} ln\frac{d}{c}(b^2 - a^2)$

(6)做极坐标变换$\begin{cases} x = \sqrt{rcos\theta} \\ y = \sqrt{rsin\theta} \end{cases}$

则$D = \{(x,y)|x^4 + y^4 < 1, x \ge 0, y \ge 0\} = \{(r,\theta)|0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}\}$

故$\iint_D 4xydxdy = \int_0^{\pi/2} d\theta \int_0^1 rdr = \frac{\pi}{4}$

(7)做如下变换$\begin{cases} u = x + y \\ v = x - y \end{cases}$

则$D = \{(x,y)||x| + |y| \le 1\} = \{(u,v)| -1 \le u \le 1, -1 \le v \le 1\}$

故$\iint_D \frac{x^2 - y^2}{\sqrt{x+y+3}} dxdy = \int_{-1}^1 du \int_{-1}^1 \frac{uv}{2\sqrt{u+3}} dv = 0$

(8)做如下变换$\begin{cases} u = x + y \\ v = y \end{cases}$

则$D = \{(x,y)| \ldots\} = \{(u,v)|0 \le u \le 1, 0 \le v \le u\}$

故$\iint_D sin\frac{y}{x+y} dxdy = \int_0^1 du \int_0^u sin\frac{v}{u} dv = (1 - cos1) \int_0^1 udu = \frac{1}{2}(1 - cos1)$

(9)做极坐标变换$\begin{cases} x = rcos\theta \\ y = rsin\theta \end{cases}$

则$D = \{(x,y)|x^2 + y^2 < a^2\} = \{(r,\theta)|0 \le r \le a, 0 \le \theta \le 2\pi\}$

故$\iint_D |xy| dxdy = \int_0^{2\pi} d\theta \int_0^a |sin\theta cos\theta| r^3 dr = 4 \int_0^{\frac{\pi}{2}} d\theta \int_0^a |sin\theta cos\theta| r^3 dr = \frac{a^4}{2}$

3: (1)所求的区域D是由关于原点对称的两部分组成，为第一象限D1面积的两倍。对于第一象限做变量代换$x = rcos\theta, y = \frac{1}{\sqrt{2}} rsin\theta$，由所围成的区

域可表示为

$$\begin{cases} 0 \leq r^2 \leq 3 \\ \dfrac{1}{\sqrt{2}} r^2 cos\theta sin\theta \geq 1 \end{cases}$$

$$\Rightarrow \begin{cases} 0 \leq r^2 \leq 3 \\ \dfrac{2\sqrt{2}}{r^2} \leq sin2\theta \leq 1 \end{cases}$$

$$\Rightarrow \begin{cases} 2\sqrt{2} \leq r^2 \leq 3 \\ \dfrac{arcsin(\frac{2\sqrt{2}}{r^2})}{2} \leq \theta \leq \dfrac{\pi}{2} - \dfrac{arcsin(\frac{2\sqrt{2}}{r^2})}{2} \end{cases}$$

并且

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} cos\theta & \frac{1}{\sqrt{2}} sin\theta \\ -rsin\theta & \frac{1}{\sqrt{2}} rcos\theta \end{vmatrix} = \frac{1}{\sqrt{2}} r$$

因此我们有

$$\begin{aligned} \iint_D 1 dxdy &= 2\int_{\sqrt{2\sqrt{2}}}^{\sqrt{3}} \int_{\frac{arcsin(\frac{2\sqrt{2}}{r^2})}{2}}^{\frac{\pi}{2} - \frac{arcsin(\frac{2\sqrt{2}}{r^2})}{2}} \frac{1}{\sqrt{2}} r d\theta dr = \sqrt{2} \int_{\sqrt{2\sqrt{2}}}^{\sqrt{3}} (\frac{\pi}{2} - arcsin(\frac{2\sqrt{2}}{r^2})) rdr \\ &\overset{t=\frac{r^2}{2\sqrt{2}}}{=} \frac{(3\sqrt{2}-4)\pi}{4} - 2\int_1^{\frac{3}{2\sqrt{2}}} arcsin(\frac{1}{t}) dt \end{aligned}$$

计算有

$$\begin{aligned} \int arcsin(\frac{1}{t}) dt &= tarcsin(\frac{1}{t}) + \int \frac{1}{t\sqrt{1-t^2}} dt \\ &= \ln(\sqrt{t^2-1} + t) + tarcsin\left(\frac{1}{t}\right) + \text{ constant} \end{aligned}$$

所以

$$
\begin{aligned}
\iint_D 1dxdy &= \frac{(3\sqrt{2}-4)\pi}{4} - 2(\ln(\sqrt{t^2-1}+t)+tarcsin\left(\frac{1}{t}\right)))\Big|_1^{\frac{3}{2\sqrt{2}}} \\
&= \frac{(3\sqrt{2}-4)\pi}{4} - 2(\ln\sqrt{2} + \frac{3}{2\sqrt{2}}arcsin(\frac{2\sqrt{2}}{3}) - arcsin(1)) \\
&= -\ln 2 + \frac{3\sqrt{2}}{2}(\frac{\pi}{2} - arcsin(\frac{2\sqrt{2}}{3})) \\
&= -\ln 2 + \frac{3\sqrt{2}}{2}(arcsin\frac{1}{3}))
\end{aligned}
$$

(2) 做变量代换$x-y=rcos\theta, x=rsin\theta$ 就把区域D' : $0 \le r \le a, 0 \le \theta \le 2\pi$映成$D:(x-y)^2+x^2 \le a^2$ 可知

$$
\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} sin\theta & sin\theta - cos\theta \\ rcos\theta & r(cos\theta + sin\theta) \end{vmatrix} = r
$$

所以

$$
\iint_{(x-y)^2+x^2\le a^2} 1dxdy = \int_0^{2\pi} d\theta \int_0^a rdr = \pi a^2
$$

(3) 变量代换$x+y=u, y=vx$就把$O'uv$平面上的区域D' : $a \le u \le b, k \le v \le m$映成$Oxy$平面上的区域$D$，解出

$$
x = \frac{u}{1+v}, y = \frac{uv}{1+v}
$$

可知

$$
\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{1}{1+v} & \frac{v}{1+v} \\ -\frac{u}{(1+v)^2} & \frac{u}{(1+v)^2} \end{vmatrix} = \frac{u}{(1+v)^2}
$$

故

$$
\iint_D 1dxdy = \int_k^m \frac{1}{(1+v)^2}dv \int_a^b udu = (\frac{1}{1+k} - \frac{1}{1+m})\frac{b^2-a^2}{2}
$$

6: 证明:

因为区域$D: |x|+|y| \le 1$关于原点对称，所以有

$$\begin{aligned}
\iint_{|x|+|y|\le 1} e^{f(x+y)}dxdy &= \iint_{D_1\cup D_2} e^{f(x+y)}dxdy \\
&= \iint_{D_1} e^{f(x+y)}dxdy + \iint_{D_2} e^{f(x+y)}dxdy \\
&= \iint_{D_1} e^{f(x+y)}dxdy + \iint_{D_1} e^{f(-x-y)}dxdy \\
&= \iint_{D_1} [e^{f(x+y)} + e^{f(-x-y)}]dxdy \\
&= \iint_{D_1} [e^{f(x+y)} + e^{-f(x+y)}]dxdy \\
&\ge \iint_{D_1} 2dxdy \\
&= 2\times\frac{1}{2}\times 2\times 1 \\
&= 2
\end{aligned}$$

其中,$D_1: |x|+|y|\le 1, x\ge 0$; $D_2: |x|+|y|\le 1, x\le 0$.

7: 令t=x-y,m=x

则$\iint_D f(x-y)dxdy = \iint_{D'} f(t)dtdm = \iint_{D'} f(x)dxdy$

其中$D' = \{(t,m)||t-m| < \frac{A}{2}, |m| < \frac{A}{2}\}$

$$= \{(x,y)|0\le x< A, x-\frac{A}{2}\le y<\frac{A}{2}\} \cup \{(x,y)|-A<x\le 0, -\frac{A}{2}\le y< x+\frac{A}{2}\}$$

故 $$\begin{aligned}
\iint_{D'} f(x)dxdy &= \int_0^A f(x)dx\int_{x-\frac{A}{2}}^{\frac{A}{2}} dy + \int_{-A}^0 f(x)dx\int_{-\frac{A}{2}}^{x+\frac{A}{2}} dy \\
&= \int_0^A (A-x)f(x)dx + \int_{-A}^0 (A+x)f(x)dx \\
&= \int_{-A}^A (A-|x|)f(x)dx
\end{aligned}$$

10.3 三重积分

1: 此类题目被积函数不是关键之处，重要的是确定积分区域

把题目给出的积分区域转成$\int_?^? \int_?^? \int_?^?$的形式

(1)直接了当，x，y，z之间没有纠缠关系

$\int_0^{1/2} \int_{-2}^1 \int_1^2 xydxdydz$

(2)先粗略画图，投影在xy平面上是一个三角区域，如果是投影在z轴相关平面z=xy的投影不好观察出

$\int_0^1 \int_0^x \int_0^{xy} (xy^2z^3)dzdydx$

(3)观察图像，和式子，先定x，再由x定y和z

$\int_0^{\pi/2} \int_0^{\pi/2-x} \int_0^{\sqrt{x}} y\cos(x+z)dydzdx$

(4)画图注意到，如果先定x的话，xy平面的图像是两部分，要拆成两部分去算

所以这里我们先定y，可以进一步定出x和z

$\int_0^a \int_{(a-y)/2}^{a-y} \int_0^{a-y} (a-y)dzdxdy$

2: (1)柱坐标变换，$\frac{16}{9}$

(2)$\frac{4\pi R^5}{15}$

(3)π

(4)$\frac{2}{5}(2^{3/2}-1)\pi$

3: (1) $\iiint_V (x^2+y^2)\,dx\,dy\,dz$=$\iint_{x^2+y^2\leq 4}\,dx\,dy\int_{\frac{x^2+y^2}{2}}^{2}\,x^2+y^2\,dz$

使用参数变换x=rcosθ y=rsinθ

0≤r≤2 0≤θ≤2π dxdy=rdrdθ

原式=$\int_0^{2\pi}\,d\theta\int_0^2 2r^3-\frac{r^5}{2}\,dr$=$\frac{16\pi}{3}$ (2) $\iiint_V \sqrt{x^2+y^2}\,dx\,dy\,dz$=$\iint_{x^2+y^2\leq 1}\,dx\,dy$

$\int_{\sqrt{x^2+y^2}}^{1} \sqrt{x^2+y^2}\, dz$
使用参数变换x=rcosθ y=rsinθ
0≤r≤1 0≤θ≤2π dxdy=rdrdθ
原式=$\int_0^{2\pi} d\theta \int_0^1 r^2 - r^3\, dr = \frac{\pi}{6}$ (3) $\iiint_V z\, dx\, dy\, dz = \iint_{x^2+y^2\le 3} dx\, dy \int_{\frac{x^2+y^2}{3}}^{\sqrt{4-x^2-y^2}} z\, dz$
使用参数变换x=rcosθ y=rsinθ
0≤r≤$\sqrt{3}$ 0≤θ≤2π dxdy=rdrdθ
原式=$\int_0^{2\pi} d\theta \int_0^{\sqrt{3}} 2r - \frac{r^3}{2} - \frac{r^5}{18}\, dr = \frac{13\pi}{4}$ (4) $\iiint_V xyz\, dx\, dy\, dz = \iint_{x^2+y^2\le 1} dx\, dy \int_0^{\sqrt{1-x^2-y^2}} xyz\, dz$
使用参数变换x=rcosθ y=rsinθ
0≤r≤1 0≤θ≤$\pi/2$ dxdy=rdrdθ
原式=$\int_0^{\frac{\pi}{2}} sin\theta cos\theta\, d\theta \int_0^1 \frac{r^3}{2} - \frac{r^5}{2}\, dr = \frac{1}{48}$ (5) V关于z轴的截面是由y=$\sqrt{z}$,y=$\frac{\sqrt{z}}{2}$,x=z,x=z/2围成
先xy后z的累次积分是
$\int_0^1 dz \int_{\frac{\sqrt{z}}{2}}^{\sqrt{z}} dy \int_{z/2}^{z} x^2\, dx = \frac{7}{216}$ (6) 作球坐标变换
原式=$\int_0^{2\pi} d\varphi \int_0^{\pi} sin\theta\, d\theta (\int_0^1 r^2 - r^4\, dr + \int_1^2 r^4 - r^2\, dr) = 16\pi$ (7）由对称性,先xy后z得
原式=$2\int_0^1 dz \iint_{D_z} e^z\, dx\, dy = 2\int_0^1 e^z \pi(1-z^2)\, dz = 2\pi$ (8) 由对称性得
原式=$\iiint_V |x| e^{-(x^2+y^2+z^2)}\, dx\, dy\, dz$,作球坐标变换
$=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos\varphi\, d\varphi \int_0^{\pi} sin^2\theta\, d\theta \int_1^2 r^3 e^{-r^2}\, dr = \pi(\frac{2}{e}$-$\frac{5}{e^4})$

7:

$$\begin{aligned} F(t) &= \iiint_{x^2+y^2+z^2\leqslant t^2} f(x^2+y^2+z^2) dxdydz \\ &= \iiint_{r^2\leqslant t^2} f(r^2) r^2 \sin\theta dr d\theta d\varphi \\ &= 4\pi \int_0^{|t|} f(r^2) r^2 dr \end{aligned}$$

故$F'(t) = 4\pi t^2 f(t^2) Sgn(t), \forall t \neq 0$,其中Sgn为符号函数

9: 这显然是一个用球极坐标换元的题目