Chapter 8

空间解析几何

8.1 向量与坐标系

1: Here only prove (1)

Proof

If $\mathbf{a} = \mathbf{0}$,or one of $\lambda, \mu, \lambda + \mu$ is zero, the equation is established.

[1] If $\lambda\mu > 0$,$(\lambda+\mu)\mathbf{a}$ and $\lambda\mathbf{a}+\mu\mathbf{b}$ have the same direction, and $|(\lambda+\mu)\mathbf{a}| = |\lambda+\mu||\mathbf{a}| = (|\lambda|+|\mu|)|\mathbf{a}| = |\lambda\mathbf{a}|+|\mu\mathbf{a}| = |\lambda\mathbf{a}+\mu\mathbf{a}|$ then $(\lambda+\mu)\mathbf{a} = \lambda\mathbf{a}+\mu\mathbf{a}$

[2] If $\lambda\mu < 0$, For convenience, we can set $\lambda > 0, \mu < 0$, we only discuss the case $\lambda + \mu > 0$, the case $\lambda + \mu < 0$ is similar. since $(\lambda+\mu)\mathbf{a} + (-\mu)\mathbf{a} = [(\lambda+\mu)+(-\mu)]\mathbf{a} = \lambda\mathbf{a}$, then $(\lambda+\mu)\mathbf{a} = \lambda\mathbf{a} - (-\mu)\mathbf{a} = \lambda\mathbf{a}+\mu\mathbf{a}$

2: 略

3: 解：(1)不成立。若$\vec{a}$,$\vec{b}$都不为$\vec{0}$，且有$\vec{a}\perp\vec{b}$,则有$\vec{a}\cdot\vec{b}$=0

(2)不成立。如$\vec{a}$,$\vec{b}$大小相等，但$\theta(\vec{a},\vec{b})$与$\theta(\vec{a},\vec{c})$互补

(3)不成立。$\vec{e_1}\cdot\vec{e_2}=|\vec{e_1}|\,|\vec{e_2}|\cos\theta(\vec{e_1},\vec{e_2})=\cos\theta(\vec{e_1},\vec{e_2})$,与两单位向量的夹角有关，大小$\in[-1,1]$

(4)不成立。$(\vec{a}\cdot\vec{b})\vec{c}$与$\vec{c}$共线，$\vec{a}(\vec{b}\cdot\vec{c})$与$\vec{a}$共线,当$\vec{a}$与$\vec{c}$ 不共线时，结论显然不成立。

(5)不成立。$\left|\vec{a}\cdot\vec{b}\right|^2=|\vec{a}|^2\left|\vec{b}\right|^2\cos^2\theta(\vec{a},\vec{b})$，当$\theta=0$或$\pi$ 时，即$\vec{a}$,$\vec{b}$不共线时，$\left|\vec{a}\cdot\vec{b}\right|^2\neq|\vec{a}|^2\left|\vec{b}\right|^2$

(6)不成立。由向量叉乘的分配律，$(\vec{a}+\vec{b})\times(\vec{a}+\vec{b})=(\vec{a}+\vec{b})\times\vec{a}+(\vec{a}+\vec{b})\times\vec{b}=\vec{a}\times\vec{a}+\vec{b}\times\vec{a}+\vec{a}\times\vec{b}+\vec{b}\times\vec{b}=\vec{0}$

4: 这三个式子的大小同为以$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$为棱的平行六面体体积, 且有序向量组$\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\},\{\boldsymbol{b},\boldsymbol{c},\boldsymbol{a}\},\{\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}\}$同为左手系或右手系, 从而有

$$\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}=\boldsymbol{b}\times\boldsymbol{c}\cdot\boldsymbol{a}=\boldsymbol{c}\times\boldsymbol{a}\cdot\boldsymbol{b}.$$

5:

$$\begin{aligned}
\vec{OM}&=\vec{OA}+\vec{AM}\\
&=\vec{OA}+\frac{1}{2}\vec{AB}\\
&=\vec{OA}+\frac{1}{2}(\vec{OB}-\vec{OA})\\
&=\frac{1}{2}(\vec{OA}+\vec{OB})
\end{aligned}$$

6: 解:

$$\because 2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = (\vec{a}+\vec{c})\cdot\vec{b}+(\vec{b}+\vec{c})\cdot\vec{a}+(\vec{a}+\vec{b})\cdot\vec{c}$$
$$= -\vec{b}\cdot\vec{b}-\vec{a}\cdot\vec{a}-\vec{c}\cdot\vec{c}$$
$$= -3$$
$$\therefore \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} = -\frac{3}{2}$$

7: 不妨设两者均为非零向量（零向量的情况结论平凡）

$(\overrightarrow{a}+3\overrightarrow{b})\cdot(7\overrightarrow{a}-5\overrightarrow{b}) = 7|\overrightarrow{a}|^2+16\overrightarrow{a}\cdot\overrightarrow{b}-15|\overrightarrow{b}|^2 = 0;$

$(\overrightarrow{a}-4\overrightarrow{b})\cdot(7\overrightarrow{a}-2\overrightarrow{b}) = 7|\overrightarrow{a}|^2-30\overrightarrow{a}\cdot\overrightarrow{b}+8|\overrightarrow{b}|^2 = 0;$

则$|\overrightarrow{a}|^2 = 2\overrightarrow{a}\cdot\overrightarrow{b}$，$|\overrightarrow{b}|^2 = 2\overrightarrow{a}\cdot\overrightarrow{b}$

于是$\overrightarrow{a}, \overrightarrow{b}$的夹角$\theta$满足$cos\theta = \frac{\overrightarrow{a}\cdot\overrightarrow{b}}{|\overrightarrow{a}|\cdot|\overrightarrow{b}|} = \frac{1}{2}$, 故$\theta$为$\frac{\pi}{3}$

8: (1)

$$|(\boldsymbol{a}+\boldsymbol{b})\times(\boldsymbol{a}-\boldsymbol{b})| = |\boldsymbol{b}\times\boldsymbol{a}-\boldsymbol{a}\times\boldsymbol{b}| = ||\boldsymbol{b}|\,|\boldsymbol{a}|-|\boldsymbol{a}|\,|\boldsymbol{b}|| = 0$$

(2)

$$|(3\boldsymbol{a}-\boldsymbol{b})\times(\boldsymbol{a}-2\boldsymbol{b})| = |-6\boldsymbol{a}\times\boldsymbol{b}-\boldsymbol{b}\times\boldsymbol{a}| = 7\,|\boldsymbol{a}|\,|\boldsymbol{b}| = 84$$

9: Using the operation law of "$\times$", we can get

(1)$|\mathbf{a}\times\mathbf{b}|^2 = (|\mathbf{a}|\cdot|\mathbf{b}|\cdot\sin\theta)^2 = 1\cdot 4\cdot\sin^2(\frac{2\pi}{3}) = 3$

(2)$|(\mathbf{a}+\mathbf{3b})\times(\mathbf{3a}-\mathbf{b})|^2 = |10\cdot(\mathbf{a}\times\mathbf{b})|^2 = 300$

10: $\vec{a}\times\vec{b}=\vec{a}\times(-\vec{a}-\vec{c})=-\vec{a}\times\vec{a}-\vec{a}\times\vec{c}=-\vec{a}\times\vec{c}=\vec{c}\times a$

11: 证明：由题意知，$\vec{a}$ﬁ$\vec{b}$ﬁ$\vec{c}$均不为零向量，$\therefore(\vec{a}+\vec{b}+\vec{c})\times\vec{a}=\vec{a}\times\vec{a}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}=\vec{b}\times\vec{a}+\vec{a}\times\vec{b}=\vec{0}$,同理有$(\vec{a}+\vec{b}+\vec{c})\times\vec{b}=(\vec{a}+\vec{b}+\vec{c})\times\vec{c}=\vec{0}$,故只有$\vec{a}+\vec{b}+\vec{c}=\vec{0}$

12: 将等式两边展开, 都是$(|\boldsymbol{a}||\boldsymbol{b}|\sin\theta)^2$.

13:

$$\begin{aligned}
V &= |\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}| \\
&= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -3 & 1 \\ 2 & 1 & -3 \\ 1 & 2 & 1 \end{vmatrix} \\
&= 25
\end{aligned}$$

14: 解:

$$\begin{aligned}
\left|\vec{a}-\vec{b}\right| &= |(4,-6,12)| \\
&= \sqrt{4^2+(-6)^2+12^2} \\
&= 14
\end{aligned}$$

方向余弦为:

$$(\cos\alpha, \cos\beta, \cos\gamma) = (\frac{2}{7}, -\frac{3}{7}, \frac{6}{7})$$

15: $|\overrightarrow{a}| = \sqrt{3^2+4^2+12^2} = 13$
故$\overrightarrow{a^0} = (\frac{3}{13}, \frac{4}{13}, -\frac{12}{13})$

16: 设x轴和y轴基向量为$\hat{i}$, $\hat{j}$, $\boldsymbol{a} = a_1\hat{i} + a_2\hat{j}$, a_1, a_2满足$a_1^2 + a_2^2 = 4$, 将$\boldsymbol{a}$与两基向量点乘

$$\boldsymbol{a}\cdot\hat{i} = a_1 + a_2\hat{i}\cdot\hat{j} = |\boldsymbol{a}|\cos\alpha = 1$$
$$\boldsymbol{a}\cdot\hat{j} = a_1\hat{i}\cdot\hat{j} + a_2 = |\boldsymbol{a}|\cos\beta = -1$$

上两式相加: $(a_1+a_2)(1+\hat{i}\cdot\hat{j}) = 0$ 因为$\hat{i}\cdot\hat{j} \neq -1$, 所以$a_0 + a_1 = 0$
则$a_1 = \pm\sqrt{2}, a_2 = \pm\mp\sqrt{2}$ 所以$\boldsymbol{a} = (\sqrt{2}, -\sqrt{2})$或$\boldsymbol{a} = (-\sqrt{2}, \sqrt{2})$

17: Using the coordinate operation law of vector, we can get
$\overrightarrow{AB}$=$\overrightarrow{OB} - \overrightarrow{OA}$= $(1, 3, 4)$,$\overrightarrow{AC}$=$\overrightarrow{OC} - \overrightarrow{OA}$= $(2, 6, 8)$
since$\overrightarrow{AB} \mathbin{//} \overrightarrow{AC}$ then A, B, C are collinear

18: (1)-6, (2)-61

19: 解：(1)$\vec{a}\cdot\vec{b}=24+6+8=38$

(2)$\sqrt{\vec{b}\cdot\vec{b}}=\left|\vec{b}\right|=\sqrt{36+9+4}=7$

(3)$(2\vec{a}-3\vec{b})\cdot(\vec{a}+2\vec{b})=2\left|\vec{a}\right|^2+\vec{a}\cdot\vec{b}-6\left|\vec{b}\right|^2=2\cdot(16+4+16)+38-6\cdot7^2=-184$

(4)$\vec{a}-\vec{b}=(-2,1,2),\left|\vec{a}-\vec{b}\right|^2=4+1+4=9$

20: $\cos\theta=\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}=\dfrac{5}{21}.$

21:

$$\boldsymbol{a}\cdot\boldsymbol{e_b}=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|}=\frac{18}{3}=6$$

22: 解：

(1)

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\3&-1&-2\\1&2&-1\end{vmatrix}$$
$$=5\vec{i}+\vec{j}+7\vec{k}$$

(2)

$$2\vec{a}-\vec{b}=(5,-4,-3)$$
$$2\vec{a}+\vec{b}=(7,0,-5)$$
$$\therefore(2\vec{a}-\vec{b})\times(2\vec{a}-\vec{b})=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\5&-4&-3\\7&0&-5\end{vmatrix}$$
$$=20\vec{i}+4\vec{j}+28\vec{k}$$

23: $\overrightarrow{AB}=(2,-2,-3)$，$\overrightarrow{AC}=(4,0,6)$

则$S=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=14$

24: 先求出

$$\boldsymbol{AB}=(3,6,3),\boldsymbol{AC}=(1,3,-2),\boldsymbol{AD}=(2,2,2)$$

四点构成的四面体体积为

$$V=|\boldsymbol{AB}\cdot(\boldsymbol{AC}\times\boldsymbol{AD})=18|$$

25: Using the coordinate operation law of the mixed product of vectors, we can get

(1) $\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}=\begin{vmatrix}2 & 3 & -1\\ 1 & -1 & 3\\ 1 & 9 & -11\end{vmatrix}=0$

since $\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}=0$ then $\mathbf{a},\mathbf{b},\mathbf{c}$ are coplanar

(2) $\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}=\begin{vmatrix}3 & -2 & 1\\ 2 & 1 & 2\\ 3 & -1 & -2\end{vmatrix}\neq 0$

since $\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}\neq 0$ then $\mathbf{a},\mathbf{b},\mathbf{c}$ are non coplanar

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26: 共面

27: (1)关于xOy平面的对称点：(a,b,-c)

关于xOz平面的对称点：(a,-b,c)

关于yOz平面的对称点：(-a,b,c)

(2)关于x轴对称点：(a,-b,-c)

关于y轴对称点：(-a,b,-c)

关于z轴对称点：(-a,-b,c)

28: 到原点距离$5\sqrt{2}$, 到x轴, y轴和z轴距离分别为$\sqrt{34}, \sqrt{41}, 5$.

29: 设所求点为$(0, y, z)$,由条件可列方程

$$9+(y-1)^2+(z-2)^2=16+(y+2)^2+(z+2)^2=(y-5)^2+(z-1)^2$$

解得$y=1, z=-2$,故所求点为$(0,1,-2)$

8.2 平面与直线

1: 略

2: $M_1\vec{M}_2 = (1,2,-1)$,平面的法向量$\vec{n} = M_1\vec{M}_2 \times \vec{v} = (7,-7,-7)$.不妨取$\vec{n_0} = (1,-1,-1)$,则$x-2+(-1)*(y+1)-(z-3)=0$,化简即为$x-y-z=0$

3: 设平面方程为a(x-5)+b(y+7)+c(z-4)=0

(1)若截距不为零，则由

$$\frac{a}{5a-7b+4c}x+\frac{b}{5a-7b+4c}y+\frac{c}{5a-7b+4c}z=1$$

可取$a=b=c=1$则方程为$x+y+z-2=0$

(2)若截距为0则存在无数多解，可用平面束方程。

4: 平行, 相交, 重合, 相交.

5: 两平面法向量分别为$\boldsymbol{n_1} = (2, 0, -1)$和$\boldsymbol{n_2} = (0, 1, 0)$,所求平面的法向量为

$$\begin{aligned} \boldsymbol{n} &= \boldsymbol{n_1} \times \boldsymbol{n_2} \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} \\ &= (1, 0, -2) \end{aligned}$$

又平面过$M(3, -1, 1)$,平面方程为$x + 2z - 5 = 0$

6: 解: ∵该平面平行于坐标面Oyz,故其一个法向量为$\vec{n} = (1, 0, 0)$
又∵该平面过点M,∴其平面方程为$x + 5 = 0$.

7: (1).两平面对应的法向量为$\overrightarrow{a} = (2, -1, 1)$与$\overrightarrow{b} = (1, 1, 2)$

则$cos\theta = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}||\overrightarrow{b}|} = \frac{1}{2}$

夹角为$\frac{\pi}{3}$

（2）两平面对应的法向量为$\overrightarrow{a} = (4, 2, 4)$与$\overrightarrow{b} = (3, -4, 0)$

则$cos\theta = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{a}||\overrightarrow{b}|} = \frac{2}{15}$

夹角为$arccos(\frac{2}{15})$

8: (1)

$$d=\frac{|16\times 2-12\times(-1)+15\times(-1)-4|}{\sqrt{16^2+12^2+15^2}}=1$$

(2)

$$d=\frac{|12\times 2-5\times(-2)+5|}{\sqrt{12^2+5^2}}=3$$

9: (1)

$$\frac{|14+7|}{\sqrt{9+36+4}}=3. \tag{8.1}$$

(2)

$$\frac{|18+21|}{\sqrt{16+4+16}}=\frac{13}{2} \tag{8.2}$$

10: (1)同侧, (2)异侧

11: 两平面平行，易知所求平面方程为$x+y-2z+1=0$

(1)$x^2+y^2-\frac{z^2}{4}=1$，为单叶双曲面

(2)$\sqrt{(y^2+z^2)}=sinx(0\leq x\leq\pi)$，名称未知

(3)$4x^2+9y^2+4z^2=36$，为椭球面

12: 两平面的法向量$\vec{v_1}=(2,-1,1)$和$\vec{v_2}=(1,1,2)$模长相等, 于是两个平分面的法向量为$\vec{v_3}=\vec{v_1}+\vec{v_2}=(3,0,3)$与$\vec{v_4}=\vec{v_1}-\vec{v_2}=(1,-2,-1)$. 现在选取两平面的交点$(6,5,0)$, 那么平分面的方程为

$$(x-6,y-5,z)\cdot\vec{v_3}=0,\quad (x-6,y-5,z)\cdot\vec{v_4}=0.$$

也就是$x+z-6=0$与$x-2y-z+4=0$.

13: 由于该点到三个坐标平面的距离相等,因此设该点为$P(x,x,x)$,现在计算该点到平面$x+y+z-1=0$的距离(用书上公式),

$$d=\frac{|ax+by+cz+d|}{\sqrt{a^2+b^2+c^2}}=\frac{|3x-1|}{\sqrt{3}}$$

14: 解:

(1)易知该平面过A,B两点所构成线段的中点$C(\frac{3}{2},\frac{1}{2},\frac{7}{2})$,且该平面的一个法向量$\vec{n}=\vec{AB}=(1,-3,1)$

$\therefore$该平面的平面方程为$x-\frac{3}{2}-3(y-\frac{1}{2})+z-\frac{7}{2}=0$,即

$$x-3y+z-\frac{7}{2}=0$$

(2)$\because$该平面与平面$6x+3y+2z+12=0$平行

$\therefore$可设平面方程为$6x+3y+2z+D=0$

由题意得,$\frac{|6-2+D|}{\sqrt{6^2+3^2+2^2}}=\frac{|6-2+12|}{\sqrt{6^2+3^2+2^2}}$

$\therefore D=-20$

$\therefore$平面方程为

$$6x+3y+2z-20=0$$

(3)通过x轴的平面方程可设为$By+z=0$

$\therefore \frac{|4B+13|}{\sqrt{B^2+1}}=8$,$\therefore B=-\frac{3}{4}$ 或 $\frac{35}{12}$,

$\therefore$平面方程为

$$-3y+4z=0 \quad 或 \quad 35y+12z=0$$

(4)由题知,可设平面方程为$\frac{x}{3}+\frac{y}{m}+\frac{z}{1}=1$,即$mx+3y+3mz-3m=0$

$\therefore$该平面的一个法向量为$\vec{n_1}=(m,3,3m)$,而Oxy平面的法向量为$\vec{n_2}=$

$(0,0,1)$

$\therefore \cos\frac{\pi}{3} = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$,即$\frac{3m}{\sqrt{m^2+(3m)^2+9}} = \frac{1}{2}$

$\therefore m = \pm\frac{3\sqrt{26}}{26}$

$\therefore$平面方程为

$$x+\sqrt{26}y+3z-3=0 \quad 或 \quad x-\sqrt{26}y+3z-3=0$$

15: (1).直线方向向量为$(1,0,2)\times(0,1,-3) = (-2,3,1)$ 从而直线方程为$\frac{x}{-2} = \frac{y-2}{3} = z-4$

(2).直线方向向量为$(1,1,-2)\times(1,2.,-1) = (3,-1,1)$ 从而直线方程为$\frac{x+1}{3} = -y+2 = z-1$

(3).$\frac{x-2}{2} = \frac{y+3}{-3} = \frac{z-4}{0}$

(4).先计算两条直线的方向向量，分别为$\overrightarrow{a} = (-3,1,10)$与$\overrightarrow{b} = (4,-1,2)$

则l具有方向向量$\overrightarrow{l} = \overrightarrow{a}\times\overrightarrow{b} = (12,46,-1)$

从而方程为$\frac{x+1}{12} = \frac{y+4}{46} = \frac{z-3}{-1}$

16: 两平面的方向量为$\boldsymbol{n}_1 = (2,3,-1), \boldsymbol{n}_2 = (3,-5,2)$,则直线的方向向量为$\boldsymbol{v} = \boldsymbol{n}_1\times\boldsymbol{n}_2 = (1,-7,-19)$.又有$(1,0,-2)$为直线上的一点，可求出此直线的参数方程为

$$\frac{x-1}{1} = \frac{y}{-7} = \frac{z+2}{-19}$$