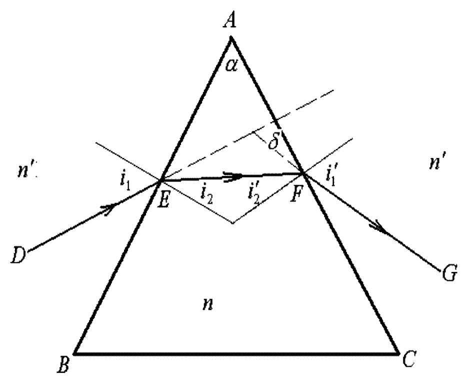
1. 使用费马原理推导光的反射定律

   (a) 入射光线、反射光线、法线在同一平面内

   (b) 入射光线、反射光线位于法线两侧

   (c) 入射角等于反射角

2. 证明棱镜折射率n与最小偏向角δ_{min}的关系 $n = \frac{\sin\frac{\alpha+\delta_{min}}{2}}{\sin\frac{\alpha}{2}}$



由几何关系

$$\alpha = i_2 + i_2' \tag{1}$$

$$\begin{aligned}\delta &= i_1 + i_1' - i_2 - i_2' \\ &= i_1 + i_1'(i_1) - \alpha \end{aligned} \tag{2}$$

当δ取最小值δ_{min}时

$$\frac{\mathrm{d}\delta_{min}}{\mathrm{d}i_1} = 0 \Rightarrow \mathrm{d}i_1 = -\mathrm{d}i_1' \tag{3}$$

由折射定律

$$\sin i_1 = n\sin i_2 \Rightarrow \cos i_1\ \mathrm{d}i_1 = n\cos i_2\ \mathrm{d}i_2$$
$$\sin i_1' = n\sin i_2' \Rightarrow \cos i_1'\ \mathrm{d}i_1' = n\cos i_2'\ \mathrm{d}i_2'$$

$$\Rightarrow \frac{\cos i_1\ \mathrm{d}i_1}{\cos i_1'\ \mathrm{d}i_1'} = \frac{\cos i_2\ \mathrm{d}i_2}{\cos i_2'\ \mathrm{d}i_2'} \tag{4}$$

(1), (3) 带入 (4), 并用折射定律消去 i_1', i_2' 得

$$\frac{\cos i_1}{\sqrt{n^2 - \sin^2 i_1}} = \frac{\cos i_2}{\sqrt{n^2 - \sin^2 i_2}} \tag{5}$$

因此δ_{min}对应于$i_1 = i_2$, 此时由(1)和(2)得$i_1 = \frac{\alpha+\delta_{min}}{2}$, $i_1' = \frac{\alpha}{2}$, 代入折射定律即可得

$$n = \frac{\sin\frac{\alpha+\delta_{min}}{2}}{\sin\frac{\alpha}{2}} \tag{6}$$

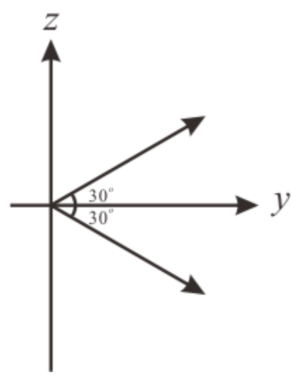
3. 代数法, 复数法, 振幅矢量法计算光波的叠加

4. 连续多个振幅矢量的叠加

$$E_l = Ae^{il\phi} \tag{7}$$

$$E = \sum_{l=1}^{n} E_l = A\sum_{l=1}^{n} e^{il\phi} = A\frac{e^{i\phi}(1-e^{in\phi})}{1-e^{i\phi}} = A\frac{\sin\frac{n}{2}\phi}{\sin\frac{1}{2}\phi}e^{i\frac{n+1}{2}\phi} \tag{8}$$

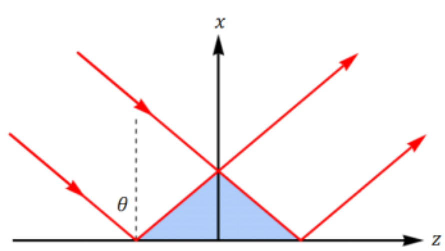
5. 写出在$y-z$平面内, 沿着$y-$轴夹角为30^o的方向传播的平面波函数



$$E = Ae^{ik(\frac{\sqrt{3}}{2}y+\frac{1}{2}z)}$$

$$E = Ae^{ik(\frac{\sqrt{3}}{2}y-\frac{1}{2}z)} \tag{9}$$

6. 如图,一列波矢量在$x-z$平面的平面波,入射后在分界面$x=0$处发生反射. 求反射波和入射波重叠区光矢量的复振幅



入射波复振幅

$$E_i = Ae^{ik(-x\cos\theta+z\sin\theta)} \tag{10}$$

反射波复振幅

$$E_r = Ae^{ik(x\cos\theta+z\sin\theta)} \tag{11}$$

叠加区域复振幅

$$E = E_i + E_r = Ae^{ik(-x\cos\theta+z\sin\theta)} + Ae^{ik(x\cos\theta+z\sin\theta)} = 2Ae^{ikz\sin\theta}\cos(kx\cos\theta) \tag{12}$$

7. 产生干涉的相干光, 必须来自同一发光原子, 同一次发射的波列, 解释其理由

 (a) 频率相同

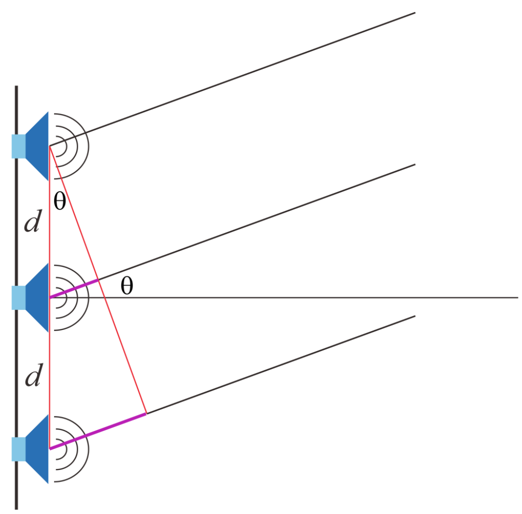
 (b) 初始相位稳定

 (c) 振动方向相同

8. 用很薄的云母片覆盖在双缝实验的一条缝上,看到干涉条纹移动了9个条纹间距,求云母片的厚度. 已知云母片折射率为1.58,光源波长550nm.

$$\Delta l = (n-1)d = 9\lambda \ \Rightarrow \ d = 8.53\mu m \tag{13}$$

9. 三个扬声器排成直线, 相距为d, 播放单频信号$s_j(t) = A\cos(\omega t + \phi_j), j = 1,2,3$. 远处一个麦克风在夹角为$\theta$的方向接收声音. 欲使麦克风处消音, 三个初相位$\phi_1, \phi_2, \phi_3$应该满足什么关系



使用复振幅形式

$$s_j(t) = Ae^{i(\omega t + \phi_j)} \tag{14}$$

麦克风处

$$s_j(t) = Ae^{i(\omega t + \phi_j - kx_i)} \tag{15}$$

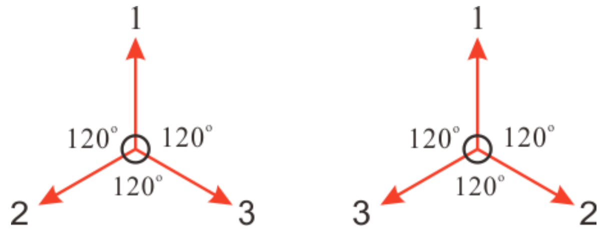
由于麦克风在无穷远处接收信号, 光程差为

$$\begin{cases} \Delta d_{21} = x_2 - x_1 = d\sin\theta \\ \Delta d_{32} = x_3 - x_2 = d\sin\theta \end{cases} \tag{16}$$

麦克风处三个扬声器信号的叠加为

$$S(t) = Ae^{i(\omega t - kx_1)}(e^{\phi_1} + e^{(\phi_2 - kd\sin\theta)} + e^{(\phi_3 - 2kd\sin\theta)}) \tag{17}$$

消音 $\Rightarrow$ $S(t)=0$

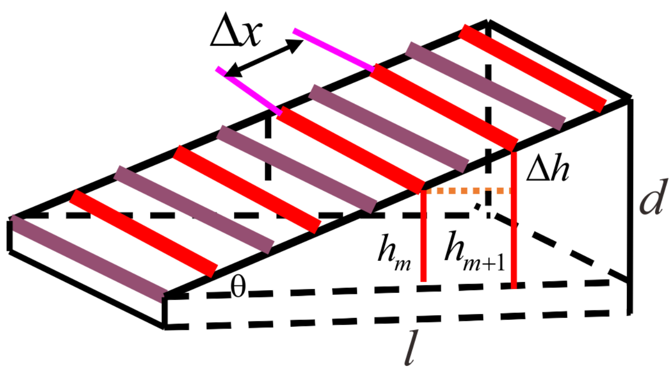


$$\begin{cases}\phi_2 - kd\sin\theta - \phi_1 = \dfrac{2}{3}\pi + 2n\pi \\ \phi_3 - 2kd\sin\theta - (\phi_2 - kd\sin\theta) = \dfrac{2}{3}\pi + 2m\pi\end{cases} \tag{18}$$

或

$$\begin{cases}\phi_2 - kd\sin\theta - \phi_1 = \dfrac{4}{3}\pi + 2n\pi \\ \phi_3 - 2kd\sin\theta - (\phi_2 - kd\sin\theta) = -\dfrac{2}{3}\pi + 2m\pi\end{cases} \tag{19}$$

10. **两块平板玻璃叠合在一起, 一端接触, 在离接触线$12.5cm$处用金属丝垫在两板之间. 用$546nm$的单色光垂直入射, 测得条纹间距为$1.5mm$. 求细丝直径**



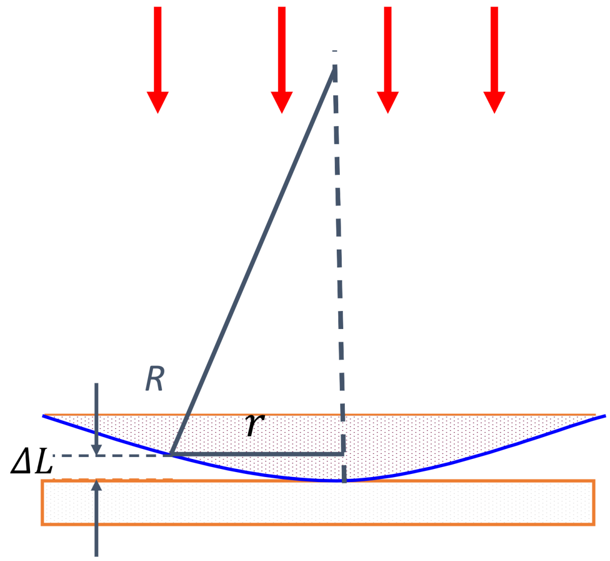
光程差每变化一个波长, 干涉条纹移动一级, 则Δx对应的光程差为

$$2\Delta x\sin\theta = \lambda \tag{20}$$

细丝直径$d = l\sin\theta$, 将(20) 代入即得

$$d = \frac{l\lambda}{2\Delta x} = 22.75\mu m \tag{21}$$

11. **牛顿环从中间数第5暗环和第15暗环直径分别是d_1和$d_2 (d_1 < d_2)$. 设入射单色光的波长为λ.**



(1) **求透镜凸面的曲率半径.**

环半径为r处光程差为

$$\Delta L = 2(R - \sqrt{R^2 - r^2}) = \frac{r^2}{R} \tag{22}$$

因此第5暗环与第15暗环对应的光程差为

$$\begin{cases} \Delta L_5 = \dfrac{d_1^2}{4R} = 5\lambda \\ \Delta L_{15} = \dfrac{d_2^2}{4R} = 15\lambda \end{cases} \tag{23}$$

$$\Rightarrow \quad R = \frac{d_2^2 - d_1^2}{40\lambda} \tag{24}$$

(2) **若牛顿环间隙充满折射率为n的介质, 这两个暗环的直径变为多大?**

在折射率为n的介质中, 光程差变为

$$\Delta L = n\frac{r^2}{R} \tag{25}$$

暗环对应于

$$\Delta L = n\frac{r^2}{R} = m\lambda \;\Rightarrow\; r = \sqrt{\frac{m\lambda R}{n}} \tag{26}$$

所以两暗环直径d_1'和d_2'会变为原来的$\frac{1}{\sqrt{n}}$倍

$$d_1' = \frac{d_1}{\sqrt{n}}, \quad d_2' = \frac{d_2}{\sqrt{n}} \tag{27}$$

12. **在折射率为1.5的玻璃表面, 镀上一层折射率为1.30的透明薄膜. 对于$550nm$的黄绿光垂直入社的情形, 为了增强透射光束强度, 应使反射光干涉相消. 求膜的厚度.**

$$2nd = \frac{2m+1}{2}\lambda \;\Rightarrow\; m = 0, \; d = \frac{1}{4n}\lambda = 105.77nm \tag{28}$$

13. 用波长为$589.3nm$的钠黄光作为夫琅禾费单缝衍射的光源, 测得第二极小到干涉图样中心的距离为$0.30cm$. 改用未知波长的单色光源, 测得第三极小到中心的距离为$0.42cm$. 求位置波长.

夫琅禾费单缝衍射光强

$$I = I_0(\frac{\sin\alpha}{\alpha}), \quad \alpha = \frac{\pi a \sin\theta}{\lambda} = \frac{\pi a d}{\lambda f} \tag{29}$$

极小对应于

$$\alpha = m\pi \quad m \neq 0 \tag{30}$$

根据题意, $\alpha_1 = \pi$, $\alpha_2 = 3\pi$, 即

$$\frac{\alpha_1}{\alpha_2} = \frac{d_1\lambda_2}{d_2\lambda_1} = \frac{2}{3} \ \Rightarrow \ \lambda_2 = \frac{2d_2}{3d_1}\lambda_1 = 550nm \tag{31}$$

14. 评估你的手机像素数目是否超过了镜头的光学衍射极限. 估算所需的参数, 如手机摄像头模组的光圈系数, 像素, CMOS图像传感器的尺寸等, 请自行在网络上搜索.

像素1200万, COMS大小为$44mm^2$, 可得单个像素尺寸约为

$$l = \sqrt{\frac{44mm^2}{1200 \times 10^5}} \approx 3.7\mu m \tag{32}$$

光圈值$f/2.8$, 即$f/D = 2.8$. 若取λ为$500nm$, 根据衍射极限可得线分辨率Δl最小为

$$\Delta l = \Delta\theta f = 1.22\frac{\lambda}{D}f \approx 1.7\mu m \tag{33}$$

$l > \Delta l$, 所以没有超过衍射极限

15. 天空的两颗星相对于望远镜的角距离为$4.8 \times 10^{-6}rad$, 都发出$550nm$的光. 望远镜的口径至少多大, 才能分辨这两颗星?

刚好能分辨时

$$\Delta\theta = 1.22\frac{\lambda}{D} \ \Rightarrow \ D = 1.22\frac{\lambda}{\Delta\theta} = 0.14m \tag{34}$$

16. 四个偏振片依次前后排列. 每个偏振片的投射方向, 均相对于前一偏振片沿顺时针方向转过30^o角. 不考虑吸收, 反射等光能损失, 则自然光透过此偏振片系统的光强是入射光强的多少倍

自然光经过第一个波片变为线偏光

$$I_1 = \frac{1}{2}I_0 \tag{35}$$

线偏光经过偏振片后的光强由马吕斯定律给出

$$I = (\cos^2(30^o))^3 I_1 = \frac{27}{128}I_0 \tag{36}$$

17. 有一空气-玻璃界面, 光从空气一侧射入时, 布儒斯特角是58^o, 求光从玻璃一侧入射时的布儒斯特角.

$$\theta_B = \arctan\frac{n_a}{n_g} = 90^o - \arctan\frac{n_g}{n_a} = 32^o \tag{37}$$

18. 用偏振器件分析, 检验光的偏振态.

(a) I不变 $\Rightarrow$ 自然光或圆偏光

(b) I变, 有消光 $\Rightarrow$ 线偏光

(c) I变, 无消光 $\Rightarrow$ 部分偏振光或椭圆偏振光

19. 热核爆炸中火球的温度可达10^7K,

(1) 求辐射最强的波长;

根据维恩位移定律

$$\lambda_{max}T = b \;\Rightarrow\; \lambda_{max} = \frac{b}{T} \tag{38}$$

其中b为维恩常量, $b \approx 2.898 \times 10^6 nm \cdot K$, 代入(38)可得

$$\lambda_{max} = 0.290\ nm \tag{39}$$

(2) 这种波长的光子能量是多少?

$$E = h\nu = \frac{hc}{\lambda_{max}} = 4.29 \times 10^3\ eV \tag{40}$$

20. 铝的脱出功是$4.2eV$, 用波长为$200nm$的光照射铝表面,

(1) 求铝的截止波长.

入射光子能量仅能克服脱出功

$$\lambda_c = \frac{hc}{W_e} = 296\ nm \tag{41}$$

(2) 光电子的最大初动能.

$$E_{max} = h\nu - W_e = 2.01\ eV \tag{42}$$

(3) 求截止电压.

$$V_c = \frac{E_{max}}{e} = 2.01\ V \tag{43}$$

(4) 如果入射光强是$2.0W/m^2$，阴极面积是$1m^2$, 光束垂直照射阴极, 那么饱和电流最大是多少?

单位时间入射光子的数目为

$$n = \frac{PS\lambda}{hc} = 2.01 \times 10^{18}\ s^{-1} \tag{44}$$

每个光子都打出一个光电子, 并且所有光电子都到达阳极, 此时饱和电流为

$$I = ne = 0.322\ A \tag{45}$$

21. **能量为$0.41MeV$的X射线光子与静止的自由电子碰撞, 反冲电子的速度为$0.6c$，求散射光的波长和散射角.**

根据能量守恒定律

$$E_p + E_e = E_{p'} + E_e' \tag{46}$$

$$E_{p'} = E_p + m_e c^2 - \frac{m_e c^2}{\sqrt{1-(\frac{v_e}{c})^2}} \tag{47}$$

则散射光的波长可表示为

$$\lambda' = \frac{hc}{E_{p'}} = 4.40 \times 10^{-12}\ m \tag{48}$$

散射角可由$Compton$散射公式得出

$$\theta = \arccos\,(1 - \frac{\lambda' - \lambda}{\lambda_e}) = 64.2^o \tag{49}$$

22. **已知氢原子的电离能为$13.6\ eV$, 求B^{4+}离子从$n=2$能级跃迁到基态的辐射能量, 波长.**

B^{4+}离子质子数$Z=5$, 则其核外电子基态能量与氢原子基态能量的关系为

$$E_0 = Z^2 E_H = -340\ eV \tag{50}$$

根据类氢离子能级关系, 可求得B^{4+}离子从$n=2$能级跃迁到基态的辐射能量为

$$\Delta E = \frac{1}{n^2}E_0 - E_0 = 255\ eV \tag{51}$$

对应的波长为

$$\lambda = \frac{hc}{\Delta E} = 4.87\ nm \tag{52}$$

23. **某种类氢离子的光谱中, 已知属于同一线系的三条谱线波长为$99.2nm$, $108.5nm$和$121.5nm$. 可以预言还有哪些光谱线?**

根据题意可知此线系是由He^{+1}离子核外电子从高能级($n=4,5,7$)向低能级($m=2$)跃迁所产生

$$\frac{1}{\lambda} = 2^2 R(\frac{1}{2^2} - \frac{1}{n^2}) \tag{53}$$

当$n \geq 3$, $n \neq 4,5,7$时, 代入(53)可预言此线系其他谱线的波长.

24. **气体放电管用$12.2eV$的电子轰击氢原子, 确定此时氢所发出的谱线波长.**

氢原子各能级分别为

$$E_0 = -13.6\ eV \quad E_1 = -3.4\ eV \quad E_2 = -1.5\ eV \quad E_3 = -0.85\ eV \tag{54}$$

$$E_2 - E_0 < 12.2\ eV \quad E_3 - E_0 > 12.2\ eV \tag{55}$$

$12.2\ eV$的电子轰击氢原子最多将其激发至第三能级, 从此能级向下跃迁共可发出三条谱线

$$3\ \rightarrow 2 \quad \Rightarrow \quad \lambda_{32} = \frac{hc}{E_2 - E_1} = 653.9nm \tag{56}$$

$$2\ \rightarrow 1 \quad \Rightarrow \quad \lambda_{32} = \frac{hc}{E_1 - E_0} = 121.8nm \tag{57}$$

$$3\ \rightarrow 1 \quad \Rightarrow \quad \lambda_{32} = \frac{hc}{E_2 - E_0} = 102.6nm \tag{58}$$

25. 要使处于基态的氢原子受激发后, 能发射莱曼系最长波长的谱线, 则至少需向氢原子提供多少能量?

莱曼系对应于氢原子从高能级跃迁至基态

$$\frac{1}{\lambda} = R(1 - \frac{1}{n^2}) \tag{59}$$

波长最长的谱线对应于氢原子从第一激发态$n = 2$跃迁至基态所产生, 因此所需最少的能量为

$$\Delta E = E_1 - E_0 = 10.2\ eV \tag{60}$$

26. 当电子的德布罗意波长与可见光波长($\lambda = 550nm$)相同时, 求它的动能是多少电子伏.

$$E_k = \frac{p^2}{2m} = \frac{h^2}{2\lambda^2 m} = 4.98 \times 10^{-6}\ eV \tag{61}$$

27. 显微镜可以分辨的最小尺寸, 约为光波的波长. 电子显微镜的电子束能量为$50keV$, 计算这种电子显微镜的最高分辨本领.

电子显微镜的最高分辨本领约为电子的波长, 将电子能量代入(61)

$$\lambda_e = \sqrt{\frac{h^2}{2Em}} = 5.49\ pm \tag{62}$$

28. 求单粒子的不确定关系$\Delta L_x \Delta L_y \geq ?$ 设此粒子的量子态满足$\langle L_x \rangle = \langle L_y \rangle = 0$, $\langle L_z \rangle = 2\hbar$.

$$\Delta L_x \Delta L_y \geq \left| \frac{\langle [L_x, L_y] \rangle}{2} \right| = \hbar^2 \tag{63}$$

29. 下列非相对论粒子被限制在宽为L的盒子中. 利用海森堡不确定关系估算它们的动能最小值:

(1) 电子关在$L = 1\dot{A}$的盒子.

$$\langle p^2 \rangle = \Delta p^2 \geq \frac{\hbar^2}{4\Delta x^2} \tag{64}$$

$$E_k = \frac{\langle p^2 \rangle}{2m} \geq 0.95\ eV \tag{65}$$

(2) 中子(质量$940MeV \cdot c^{-2}$)限制在$L = 10fm$(原子核尺寸)的盒子中.

$$\langle p^2 \rangle = \Delta p^2 \geq \frac{\hbar^2}{4\Delta x^2} \tag{66}$$

$$E_k = \frac{\langle p^2 \rangle}{2m} \geq 5.20 \times 10^4\ eV \tag{67}$$

(3) 质量$10^{-6}g$的灰尘被关在$L = 1\mu m$的盒中.

$$\langle p^2 \rangle = \Delta p^2 \geq \frac{\hbar^2}{4\Delta x^2} \tag{68}$$

$$E_k = \frac{\langle p^2 \rangle}{2m} \geq 8.69 \times 10^{-30}\ eV \tag{69}$$

30. **设一个二维量子系统在$t=0$时的波函数为$\psi(x,y,t=0)=(x+iy)e^{-(x^2+y^2)}$, $x,y\in\mathbb{R}$求几率密度.**

首先将波函数归一化

$$A=\int|\psi(x,y)|^2\,\mathrm{d}x=\int_{-\infty}^{+\infty}(x^2+y^2)e^{-2(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y=2\pi\int_0^{+\infty}r^2e^{-2r^2}\,r\mathrm{d}r=\pi/4 \tag{70}$$

$$\psi(x,y)=\frac{2}{\sqrt{\pi}}(x+iy)e^{-(x^2+y^2)} \tag{71}$$

概率密度为

$$|\psi|^2=\frac{4(x^2+y^2)}{\pi}e^{-2(x^2+y^2)} \tag{72}$$

31. **一维空间中粒子的波函数若为$\psi(x)=e^{-|x|}$, $x\in\mathbb{R}$**

(1) **求归一化的波函数.**

归一化系数为

$$A=\int|\psi(x)|^2\,\mathrm{d}x=\int_{-\infty}^{+\infty}e^{-2|x|}\,\mathrm{d}x=2\int_0^{+\infty}e^{-2x}\,\mathrm{d}x=1 \tag{73}$$

所以归一化波函数即为$\psi(x)$

(2) **求动量表象的波函数$\phi(p)$.**

$$\begin{aligned}\phi(p)&=\frac{1}{\sqrt{2\pi\hbar}}\int\psi(x)e^{-i\frac{p}{\hbar}x}\,\mathrm{d}x\\&=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^0e^{x(1-i\frac{p}{\hbar})}\,\mathrm{d}x+\frac{1}{\sqrt{2\pi\hbar}}\int_0^\infty e^{-x(1+i\frac{p}{\hbar})}\,\mathrm{d}x\\&=\frac{1}{\sqrt{2\pi\hbar}}(\frac{1}{1-i\frac{p}{\hbar}}+\frac{1}{1+i\frac{p}{\hbar}})\\&=\frac{1}{\sqrt{2\pi\hbar}}\frac{2\hbar^2}{\hbar^2+p^2}\end{aligned} \tag{74}$$

32. **归一化高斯型波包 $\psi(x)=e^{-\frac{x^2}{\sigma^2}}$, 求坐标算符$\hat{x}$, 动量算符$\hat{p}$和动能算符$\hat{T}=\frac{1}{2m}\hat{p}^2$的期望值.**

首先将波函数归一化

$$A=\int|\psi(x)|^2\,\mathrm{d}x=\int_{-\infty}^{+\infty}e^{-2\frac{x^2}{\sigma^2}}\,\mathrm{d}x=\sqrt{\frac{\pi\sigma^2}{2}} \tag{75}$$

$$\psi(x)=(\frac{2}{\pi\sigma^2})^{\frac{1}{4}}e^{-\frac{x^2}{\sigma^2}} \tag{76}$$

使用归一化波函数$\psi(x)$计算坐标算符$\hat{x}$, 动量算符$\hat{p}$和动能算符$\hat{T}=\frac{1}{2m}\hat{p}^2$的期望值

$$\langle\hat{x}\rangle=\int\psi^*(x)\hat{x}\psi(x)\,\mathrm{d}x=\sqrt{\frac{2}{\pi\sigma^2}}\int_{-\infty}^{+\infty}xe^{-\frac{2x^2}{\sigma^2}}\,\mathrm{d}x=0 \tag{77}$$

$$\langle\hat{p}\rangle=\int\psi^*(x)\hat{p}\psi(x)\,\mathrm{d}x=-i\hbar\sqrt{\frac{2}{\pi\sigma^2}}\int_{-\infty}^{+\infty}e^{-\frac{x^2}{\sigma^2}}\frac{\partial}{\partial x}e^{-\frac{x^2}{\sigma^2}}\,\mathrm{d}x=i\hbar\sqrt{\frac{8}{\pi\sigma^6}}\int_{-\infty}^{+\infty}e^{-\frac{2x^2}{\sigma^2}}x\,\mathrm{d}x=0 \tag{78}$$

$$\begin{aligned}\langle\hat{T}\rangle&=\int\psi^*(x)\frac{\hat{p}^2}{2m}\psi(x)\,\mathrm{d}x\\&=-\frac{\hbar^2}{2m}\sqrt{\frac{2}{\pi\sigma^2}}\int_{-\infty}^{+\infty}e^{-\frac{x^2}{\sigma^2}}\frac{\partial^2}{\partial x^2}e^{-\frac{x^2}{\sigma^2}}\,\mathrm{d}x\\&=-\frac{\hbar^2}{2m}\sqrt{\frac{2}{\pi\sigma^2}}(\frac{4}{\sigma^4}\int_{-\infty}^{+\infty}e^{-\frac{2x^2}{\sigma^2}}x^2\,\mathrm{d}x-\frac{2}{\sigma^2}\int_{-\infty}^{+\infty}e^{-\frac{2x^2}{\sigma^2}}\,\mathrm{d}x)\\&=-\frac{\hbar^2}{2m}\sqrt{\frac{2}{\pi\sigma^2}}(\sqrt{\frac{\pi}{2\sigma^2}}-\sqrt{\frac{2\pi}{\sigma^2}})\\&=\frac{\hbar^2}{2m}\frac{1}{\sigma^2}\end{aligned} \tag{79}$$

33. **证明定态薛定谔方程的本征值必然不小于势能函数的最小值.**

定态薛定谔方程可写为, 其中ψ为哈密顿量能量本征值为E的本征波函数

$$\begin{aligned}&\hat{H}\psi(\vec{r})=E\psi(\vec{r})\\&[\frac{\hat{p}^2}{2m}+\hat{V}(\vec{r})]\psi(\vec{r})=E\psi(\vec{r})\end{aligned} \tag{80}$$

$\psi(\vec{r})$与(80)做内积, 且动量算符$\hat{p}$为厄米算符, 则有

$$\begin{aligned}E=\int\psi^*(\vec{r})\hat{H}\psi(\vec{r})\,\mathrm{d}\vec{r}&=\frac{1}{2m}\int\psi^*(\vec{r})\hat{p}^2\psi(\vec{r})\,\mathrm{d}\vec{r}+\int\psi^*(\vec{r})\hat{V}(\vec{r})\psi(\vec{r})\,\mathrm{d}\vec{r}\\&=\frac{1}{2m}\int|\hat{p}\psi(\vec{r})|^2\,\mathrm{d}\vec{r}+\int\psi^*(\vec{r})\hat{V}(\vec{r})\psi(\vec{r})\,\mathrm{d}\vec{r}\\&\geq\langle V\rangle\geq V_{min}\end{aligned} \tag{81}$$

34. **证明对于无限深势阱, 定态能量E一定大于零.**

35. **对一维无限深势阱中的第n个定态, 求解对应的算符平均值$\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$，计算算符的涨落$\Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}$, $\Delta p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}$, 验证不确定关系$\Delta x\Delta p\geq\hbar/2$, 并指出哪个状态最接近上述不等式极限.**

一维无限深势阱中电子的定态波函数为

$$\psi(x)=\begin{cases}\sqrt{\frac{2}{a}}\cos\frac{n\pi x}{a}, & n\text{为奇数}\\\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}, & n\text{为偶数}\end{cases} \tag{82}$$

$\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$ 分别可计算为

$$\langle x\rangle = \int \psi^*(x)x\psi(x)\,\mathrm{d}x = \begin{cases} \dfrac{2}{a}\displaystyle\int_{-\frac{a}{2}}^{\frac{a}{2}} x\cos^2(\frac{n\pi x}{a})\,\mathrm{d}x = 0, & n\text{为奇数} \\ \dfrac{2}{a}\displaystyle\int_{-\frac{a}{2}}^{\frac{a}{2}} x\sin^2(\frac{n\pi x}{a})\,\mathrm{d}x = 0, & n\text{为偶数} \end{cases} \tag{83}$$

$$\begin{aligned} \langle x^2\rangle = \int \psi^*(x)x^2\psi(x)\,\mathrm{d}x &= \frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2\cos^2(\frac{n\pi x}{a})\,\mathrm{d}x \\ &= \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2(1+\cos(\frac{2n\pi x}{a}))\,\mathrm{d}x \\ &= \frac{a^2}{12} + \frac{1}{2n\pi}\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2\,\mathrm{d}\sin(\frac{2n\pi}{a}x) \\ &= \frac{a^2}{12} + \frac{1}{2n\pi}x^2\sin(\frac{2n\pi}{a}x)\Big|_{-\frac{a}{2}}^{\frac{a}{2}} - \frac{1}{n\pi}\int_{-\frac{a}{2}}^{\frac{a}{2}} x\sin(\frac{2n\pi}{a}x)\,\mathrm{d}x \\ &= \frac{a^2}{12} + \frac{a}{2n^2\pi^2}\int_{-\frac{a}{2}}^{\frac{a}{2}} x\,\mathrm{d}\cos(\frac{2n\pi}{a}x) \\ &= \frac{a^2}{12} + \frac{a}{2n^2\pi^2}x\cos(\frac{2n\pi}{a}x)\Big|_{-\frac{a}{2}}^{\frac{a}{2}} - \frac{a}{2n^2\pi^2}\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos(\frac{2n\pi}{a}x)\,\mathrm{d}x \\ &= \frac{a^2}{12} - \frac{a^2}{2n^2\pi^2} \qquad n\text{为奇数} \end{aligned} \tag{84}$$

$$\begin{aligned} \langle x^2\rangle = \int \psi^*(x)x^2\psi(x)\,\mathrm{d}x &= \frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2\sin^2(\frac{n\pi x}{a})\,\mathrm{d}x \\ &= \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2(1-\cos(\frac{2n\pi x}{a}))\,\mathrm{d}x \\ &= \frac{a^2}{12} - \frac{1}{2n\pi}\int_{-\frac{a}{2}}^{\frac{a}{2}} x^2\,\mathrm{d}\sin(\frac{2n\pi}{a}x) \\ &= \frac{a^2}{12} - \frac{1}{2n\pi}x^2\sin(\frac{2n\pi}{a}x)\Big|_{-\frac{a}{2}}^{\frac{a}{2}} + \frac{1}{n\pi}\int_{-\frac{a}{2}}^{\frac{a}{2}} x\sin(\frac{2n\pi}{a}x)\,\mathrm{d}x \\ &= \frac{a^2}{12} - \frac{a}{2n^2\pi^2}\int_{-\frac{a}{2}}^{\frac{a}{2}} x\,\mathrm{d}\cos(\frac{2n\pi}{a}x) \\ &= \frac{a^2}{12} - \frac{a}{2n^2\pi^2}x\cos(\frac{2n\pi}{a}x)\Big|_{-\frac{a}{2}}^{\frac{a}{2}} + \frac{a}{2n^2\pi^2}\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos(\frac{2n\pi}{a}x)\,\mathrm{d}x \\ &= \frac{a^2}{12} - \frac{a^2}{2n^2\pi^2} \qquad n\text{为偶数} \end{aligned} \tag{85}$$

$$\langle p\rangle = \int \psi^*(x)(-i\hbar\frac{\partial}{\partial x})\psi(x)\,\mathrm{d}x = \begin{cases} \dfrac{i\hbar n\pi}{a^2}\displaystyle\int_{-\frac{a}{2}}^{\frac{a}{2}} \sin(\frac{2n\pi x}{a})\,\mathrm{d}x = 0, & n\text{为奇数} \\ \dfrac{-i\hbar n\pi}{a^2}\displaystyle\int_{-\frac{a}{2}}^{\frac{a}{2}} \sin(\frac{2n\pi x}{a})\,\mathrm{d}x = 0, & n\text{为偶数} \end{cases} \tag{86}$$

$$\langle p^2\rangle = \int \psi^*(x)(-\hbar^2\frac{\partial^2}{\partial x^2})\psi(x)\,\mathrm{d}x = \begin{cases} \frac{2n^2\hbar^2\pi^2}{a^3}\int_{-\frac{a}{2}}^{\frac{a}{2}}\cos^2(\frac{n\pi x}{a})\,\mathrm{d}x = \frac{2n^2\hbar^2\pi^2}{a^3}\int_{-\frac{a}{2}}^{\frac{a}{2}}\frac{1+\cos(\frac{2n\pi x}{a})}{2}\,\mathrm{d}x \\ \qquad = \frac{n^2\hbar^2\pi^2}{a^2},\ n\text{为奇数} \\ \frac{2n^2\hbar^2\pi^2}{a^3}\int_{-\frac{a}{2}}^{\frac{a}{2}}\sin^2(\frac{n\pi x}{a})\,\mathrm{d}x = \frac{2n^2\hbar^2\pi^2}{a^3}\int_{-\frac{a}{2}}^{\frac{a}{2}}\frac{1-\cos(\frac{2n\pi x}{a})}{2}\,\mathrm{d}x \\ \qquad = \frac{n^2\hbar^2\pi^2}{a^2},\ n\text{为偶数} \end{cases}$$

可以看出$\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$与n的奇偶性无关

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{a^2}{12} - \frac{a^2}{2n^2\pi^2}} \tag{87}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{n\hbar\pi}{a} \tag{88}$$

$$\Delta x\Delta p = \hbar\pi\sqrt{\frac{n^2}{12} - \frac{1}{2\pi^2}} \geq \frac{\hbar}{2}\sqrt{\frac{\pi^2}{3} - 2} \approx 1.136\frac{\hbar}{2} \tag{89}$$

36. **一维无限深势阱中, 若假定势能函数取值范围为**$[0,a]$**, 试证明基态波函数为**

$$\phi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a}) \qquad .$$

若此时粒子的初始波函数形式为$\psi(x,0) = A\sin^3(\pi x/a)$**, 试求**$A$**及**$\psi(x,t)$**, 并计算平均值**$\langle x\rangle$**,** $\langle p\rangle$**.**

定态薛定谔方程可写为

$$[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)]\phi(x) = E\phi(x) \tag{90}$$

$x \notin [0,a]$时, $\phi(x) = 0$; $x \in [0,a]$时, 令$k^2 = \frac{2mE}{\hbar^2}$, 方程变为

$$\frac{\partial^2}{\partial x^2}\phi(x) + \frac{2mE}{\hbar^2}\phi(x) = 0 \quad \Rightarrow \quad \phi(x) = \alpha\sin(kx) + \beta\cos(kx) \tag{91}$$

利用边界条件$\phi(0) = 0$, $\phi(a) = 0$, 可得

$$\begin{cases} \phi(0) = \beta = 0 \\ \phi(a) = \alpha\sin(ka) + \beta\cos(ka) = 0 \quad \Rightarrow \quad k = \frac{n\pi}{a}, \quad n \in \mathbb{N}^+ \end{cases} \tag{92}$$

本征能量可写为

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n \in \mathbb{N}^+ \tag{93}$$

计算归一化常数可确定α为

$$\int |\phi_n(x)|^2\,\mathrm{d}x = \int_0^a \alpha^2\sin^2(\frac{n\pi}{a})\,\mathrm{d}x = \frac{\alpha^2 a}{2} = 1 \quad \Rightarrow \quad \alpha = \sqrt{\frac{2}{a}} \tag{94}$$

即一维无限深势阱中粒子的能量本征函数可写作

$$\phi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a}), \quad n \in \mathbb{N}^+ \tag{95}$$

若此时粒子的初始波函数形式为$\psi(x,0)=A\sin^3(\pi x/a)$, 将其用本征波函数展开

$$\begin{aligned}\psi(x,0)&=A\sin^3(\frac{\pi x}{a})\\&=\frac{A}{2}\sin(\frac{\pi x}{a})[1-\cos(\frac{2\pi x}{a})]\\&=\frac{A}{2}\{\sin(\frac{\pi x}{a})-\frac{1}{2}[\sin(\frac{3\pi x}{a})-\sin(\frac{\pi x}{a})]\}\\&=\frac{A}{4}[3\sin(\frac{\pi x}{a})-\sin(\frac{3\pi x}{a})]\\&=\frac{A}{4}\sqrt{\frac{a}{2}}[3\phi_1(x)-\phi_3(x)]\end{aligned}\tag{96}$$

归一化(96)可得

$$\int|\psi(x,0)|^2\,\mathrm{d}x=1\quad\Rightarrow\quad A=\frac{4}{\sqrt{5a}}\tag{97}$$

综上粒子初态波函数为

$$\psi(x,0)=\frac{3}{\sqrt{10}}\phi_1(x)-\frac{1}{\sqrt{10}}\phi_3(x)\tag{98}$$

含时波函数则为

$$\psi(x,t)=\frac{3}{\sqrt{10}}\phi_1(x)e^{-i\frac{E_1}{\hbar}t}-\frac{1}{\sqrt{10}}\phi_3(x)e^{-i\frac{E_3}{\hbar}t}\tag{99}$$

坐标, 动量平均值为

$$\begin{aligned}\langle x\rangle&=\int\psi^*(x)x\psi(x)\,\mathrm{d}x\\&=\frac{9}{10}\int x\,|\phi_1(x)|^2\,\mathrm{d}x+\frac{1}{10}\int x\,|\phi_3(x)|^2\,\mathrm{d}x+\frac{3}{10}\int[\phi_1^*(x)x\phi_3(x)e^{-i\frac{E_3-E_1}{\hbar}t}+\phi_3^*(x)x\phi_1(x)e^{-i\frac{E_1-E_3}{\hbar}t}]\,\mathrm{d}x\\&=\frac{a}{2}+\frac{3}{5}\cos(\frac{E_3-E_1}{\hbar}t)\int_0^a\sin(\frac{\pi x}{a})\sin(\frac{3\pi x}{a})x\,\mathrm{d}x\\&=\frac{a}{2}\end{aligned}\tag{100}$$

$$\begin{aligned}\langle p\rangle&=\int\psi^*(x)\hat{p}\psi(x)\,\mathrm{d}x\\&=\int(\frac{3}{\sqrt{10}}\phi_1(x)e^{i\frac{E_1}{\hbar}t}-\frac{1}{\sqrt{10}}\phi_3(x)e^{i\frac{E_3}{\hbar}t})(-i\hbar\frac{\partial}{\partial x})(\frac{3}{\sqrt{10}}\phi_1(x)e^{-i\frac{E_1}{\hbar}t}-\frac{1}{\sqrt{10}}\phi_3(x)e^{-i\frac{E_3}{\hbar}t})\,\mathrm{d}x\\&=0\end{aligned}\tag{101}$$

37. **一维无限深势阱中粒子的初始波函数为两个定态的叠加**$\psi(x,0)\sim\phi_1(x)+i\phi_2(x)$.

(1) **归一化上述波函数**,

不同能态的波函数时正交, 归一的, 因此可得归一化的波函数为

$$\psi(x,0)=\frac{1}{\sqrt{2}}[\phi_1(x)+i\phi_2(x)]\tag{102}$$

(2) **求时间演化状态**$\psi(x,t)$**，并计算**$|\psi(x,t)|^2$,

$$\psi(x,t) = \frac{1}{\sqrt{2}}[\phi_1(x)e^{-i\frac{E_1}{\hbar}t} + i\phi_2(x)e^{-i\frac{E_2}{\hbar}t}] \tag{103}$$

其中E_1为$n=1$能态的能量, 其中E_2为$n=2$能态的能量.

$$\begin{aligned}|\psi(x,t)|^2 &= \frac{1}{2}(|\phi_1(x)|^2 + |\phi_2(x)|^2)\\ &+ \frac{i}{\sqrt{2}}\phi_1^*(x)\phi_2(x)e^{-i\frac{E_2-E_1}{\hbar}t} - \frac{i}{\sqrt{2}}\phi_1(x)\phi_2^*(x)e^{-i\frac{E_1-E_2}{\hbar}t}\\ &= \frac{1}{2}(|\phi_1(x)|^2 + |\phi_2(x)|^2) + \phi_1(x)\phi_2(x)\sin(\frac{E_2-E_1}{\hbar}t)\end{aligned} \tag{104}$$

(3) **计算算符平均值**$\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$,

取势能函数为偶函数, 且宽度为a, 粒子波函数则如式(82)

$$\begin{aligned}\langle x\rangle &= \int \psi^*(x,t)x\psi(x,t)\,\mathrm{d}x\\ &= \frac{1}{2}\int x\,|\phi_1(x)|^2\,\mathrm{d}x + \frac{1}{2}\int x\,|\phi_2(x)|^2\,\mathrm{d}x\\ &+ \frac{1}{2}\int [\phi_1^*(x)x\phi_2(x)e^{-i\frac{E_2-E_1}{\hbar}t} + \phi_2^*(x)x\phi_1(x)e^{-i\frac{E_1-E_2}{\hbar}t}]\,\mathrm{d}x\\ &= \frac{1}{2}\langle x\rangle_1 + \frac{1}{2}\langle x\rangle_2 + \sin(\frac{E_2-E_1}{\hbar}t)\int \phi_1(x)\phi_2(x)x\,\mathrm{d}x\\ &= \sin(\frac{E_2-E_1}{\hbar}t)\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos\frac{\pi x}{a}\sin\frac{2\pi x}{a}x\,\mathrm{d}x\\ &= \sin(\frac{E_2-E_1}{\hbar}t)\frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} (\sin\frac{3\pi x}{a} + \sin\frac{\pi x}{a})x\,\mathrm{d}x\\ &= \frac{16a}{9\pi^2}\sin(\frac{E_2-E_1}{\hbar}t)\end{aligned} \tag{105}$$

$$\tag{106}$$

$$\begin{aligned}\langle x^2\rangle &= \int \psi^*(x,t)x^2\psi(x,t)\,\mathrm{d}x\\ &= \frac{1}{2}\int x^2\,|\phi_1(x)|^2\,\mathrm{d}x + \frac{1}{2}\int x^2\,|\phi_2(x)|^2\,\mathrm{d}x\\ &+ \frac{1}{2}\int [\phi_1^*(x)x^2\phi_2(x)e^{-i\frac{E_2-E_1}{\hbar}t} + \phi_2^*(x)x^2\phi_1(x)e^{-i\frac{E_1-E_2}{\hbar}t}]\,\mathrm{d}x\\ &= \frac{1}{2}\langle x^2\rangle_1 + \frac{1}{2}\langle x^2\rangle_2 + \sin(\frac{E_2-E_1}{\hbar}t)\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos\frac{\pi x}{a}\sin\frac{2\pi x}{a}x^2\,\mathrm{d}x\\ &= \frac{a^2}{12} - \frac{5a^2}{16\pi^2}\end{aligned} \tag{107}$$

$$
\begin{aligned}
\langle p\rangle &= \int \psi^*(x,t)\hat{p}\psi(x,t)\,\mathrm{d}x \\
&= \frac{1}{2}\int \phi_1^*(x,t)\hat{p}\phi_1(x,t)\,\mathrm{d}x + \frac{1}{2}\int \phi_2^*(x,t)\hat{p}\phi_2(x,t)\,\mathrm{d}x \\
&+ \frac{1}{2}\int [\phi_1^*(x)\hat{p}\phi_2(x)e^{-i\frac{E_2-E_1}{\hbar}t} + \phi_2^*(x)\hat{p}\phi_1(x)e^{-i\frac{E_1-E_2}{\hbar}t}]\,\mathrm{d}x \\
&= \frac{1}{2}\langle\hat{p}\rangle_1 + \frac{1}{2}\langle\hat{p}\rangle_2 + \frac{2\pi\hbar}{a^2}e^{i(\frac{E_2-E_1}{\hbar}t)}\int_{-\frac{a}{2}}^{\frac{a}{2}}\cos\frac{\pi x}{a}\cos\frac{2\pi x}{a}\,\mathrm{d}x + \frac{\pi\hbar}{a^2}e^{i(\frac{E_2-E_1}{\hbar}t)}\int_{-\frac{a}{2}}^{\frac{a}{2}}\sin\frac{\pi x}{a}\sin\frac{2\pi x}{a}\,\mathrm{d}x \\
&= \frac{\pi\hbar}{a^2}e^{i(\frac{E_2-E_1}{\hbar}t)}\int_{-\frac{a}{2}}^{\frac{a}{2}}(\cos\frac{\pi x}{a} + \cos\frac{3\pi x}{a})\,\mathrm{d}x + \frac{\pi\hbar}{2a^2}e^{i(\frac{E_2-E_1}{\hbar}t)}\int_{-\frac{a}{2}}^{\frac{a}{2}}(\cos\frac{\pi x}{a} - \cos\frac{3\pi x}{a})\,\mathrm{d}x \\
&= \frac{4\hbar}{3a}e^{i(\frac{E_2-E_1}{\hbar}t)} + \frac{4\hbar}{3a}e^{i(\frac{E_2-E_1}{\hbar}t)} \\
&= \frac{8\hbar}{3a}\cos(\frac{E_2-E_1}{\hbar}t)
\end{aligned}
\tag{108}
$$

$$
\begin{aligned}
\langle p^2\rangle &= \int \psi^*(x,t)\hat{p}^2\psi(x,t)\,\mathrm{d}x \\
&= \frac{1}{2}\int \phi_1^*(x,t)\hat{p}^2\phi_1(x,t)\,\mathrm{d}x + \frac{1}{2}\int \phi_2^*(x,t)\hat{p}^2\phi_2(x,t)\,\mathrm{d}x \\
&+ \frac{1}{2}\int [\phi_1^*(x)\hat{p}^2\phi_2(x)e^{-i\frac{E_2-E_1}{\hbar}t} + \phi_2^*(x)\hat{p}^2\phi_1(x)e^{-i\frac{E_1-E_2}{\hbar}t}]\,\mathrm{d}x \\
&= \frac{1}{2}\langle p^2\rangle_1 + \frac{1}{2}\langle p^2\rangle_2 + \frac{4\pi^2\hbar^2}{a^3}e^{i(\frac{E_2-E_1}{\hbar}t)}\int_{-\frac{a}{2}}^{\frac{a}{2}}\cos\frac{\pi x}{a}\sin\frac{2\pi x}{a}\,\mathrm{d}x + \frac{\pi^2\hbar^2}{a^3}e^{i(\frac{E_2-E_1}{\hbar}t)}\int_{-\frac{a}{2}}^{\frac{a}{2}}\cos\frac{\pi x}{a}\sin\frac{2\pi x}{a}\,\mathrm{d}x \\
&= \frac{5\pi^2\hbar^2}{2a^2}
\end{aligned}
\tag{109}
$$

(4) **计算系统能量的平均值. 如果测量粒子的能量, 则得到什么结果, 相应的几率是多少?**

$$
\langle E\rangle = \int \psi^*(x)\hat{H}\psi(x)\,\mathrm{d}x = \frac{1}{2}E_1 + \frac{1}{2}E_2 = \frac{5\pi^2\hbar^2}{4ma^2}
\tag{110}
$$

若测量粒子的能量, 有一半的几率系统坍缩到$n=1$的能态, 测得能量为$\frac{\pi^2\hbar^2}{2ma^2}$;
另一半的几率系统坍缩到$n=2$的能态, 测得能量为$\frac{4\pi^2\hbar^2}{2ma^2}$.

38. **假定粒子所处量子态波函数为**

$$
\psi(x,y,z) = \frac{\alpha^{\frac{5}{2}}}{\sqrt{2}}ze^{-\alpha(x^2+y^2+z^2)},
$$

证明粒子处在角动量的本征态上, 并给出相应的本征值(L^2, L_z).
在球坐标系下$r=\sqrt{x^2+y^2+z^2}$, $z=r\cos\theta$, 重写波函数为

$$
\psi(r,\theta,\phi) = \frac{\alpha^{\frac{5}{2}}}{\sqrt{2}}r\cos\theta e^{-\alpha r^2}
\tag{111}
$$

将球坐标系下角动量算符作用于波函数上

$$
\begin{aligned}
\hat{L}^2\psi(r,\theta,\phi) &= -\hbar^2[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}][\frac{\alpha^{\frac{5}{2}}}{\sqrt{2}}r\cos\theta e^{-\alpha r^2}] \\
&= -\hbar^2[\frac{1}{\sin\theta}(\cos\theta\frac{\partial}{\partial\theta}+\sin\theta\frac{\partial^2}{\partial\theta^2})+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}][\frac{\alpha^{\frac{5}{2}}}{\sqrt{2}}r\cos\theta e^{-\alpha r^2}] \\
&= -\hbar^2[-\cos\theta-\cos\theta]\frac{\alpha^{\frac{5}{2}}}{\sqrt{2}}re^{-\alpha r^2} \\
&= 2\hbar^2\frac{\alpha^{\frac{5}{2}}}{\sqrt{2}}r\cos\theta e^{-\alpha r^2} = 2\hbar^2\psi(r,\theta,\phi)
\end{aligned}
\tag{112}
$$

$$
\hat{L}_z\psi(r,\theta,\phi) = -i\hbar\frac{\partial}{\partial\phi}[\frac{\alpha^{\frac{5}{2}}}{\sqrt{2}}r\cos\theta e^{-\alpha r^2}] = 0 \quad\Rightarrow\quad l=1, \quad m=0 \tag{113}
$$

39. **证明L_z的本征态下, $\langle L_x\rangle = \langle L_y\rangle = 0$**

考虑角动量的对易关系$[\hat{L}_x,\hat{L}_z] = -i\hbar\hat{L}_y$, $[\hat{L}_y,\hat{L}_z] = i\hbar\hat{L}_x$.

$$
\begin{aligned}
\langle L_y\rangle &= -\frac{1}{i\hbar}\langle[\hat{L}_x,\hat{L}_z]\rangle \\
&= -\frac{1}{i\hbar}\int\psi_m^*[\hat{L}_x,\hat{L}_z]\psi_m\,\mathrm{d}\Omega \\
&= -\frac{1}{i\hbar}\int\psi_m^*(\hat{L}_x\hat{L}_z-\hat{L}_z\hat{L}_x)\psi_m\,\mathrm{d}\Omega \\
&= 0
\end{aligned}
\tag{114}
$$

$$
\begin{aligned}
\langle L_x\rangle &= \frac{1}{i\hbar}\langle[\hat{L}_y,\hat{L}_z]\rangle \\
&= \frac{1}{i\hbar}\int\psi_m^*[\hat{L}_y,\hat{L}_z]\psi_m\,\mathrm{d}\Omega \\
&= \frac{1}{i\hbar}\int\psi_m^*(\hat{L}_y\hat{L}_z-\hat{L}_z\hat{L}_y)\psi_m\,\mathrm{d}\Omega \\
&= 0
\end{aligned}
\tag{115}
$$

40. **证明L_z的本征态下, 角动量沿与z方向成θ角方向上分量的平均值为$m\hbar\cos\theta$.**

考虑角动量$\hat{\vec{L}}$朝着与z方向成θ角方向上的投影

$$
\hat{L}_\theta = \hat{\vec{L}}\cdot\vec{e}_\theta = \hat{L}_x\vec{e}_x\cdot\vec{e}_\theta+\hat{L}_y\vec{e}_y\cdot\vec{e}_\theta+\hat{L}_z\vec{e}_z\cdot\vec{e}_\theta \tag{116}
$$

$$
\langle\hat{L}_\theta\rangle = \langle\hat{L}_x\rangle\vec{e}_x\cdot\vec{e}_\theta+\langle\hat{L}_y\rangle\vec{e}_y\cdot\vec{e}_\theta+\langle\hat{L}_z\rangle\vec{e}_z\cdot\vec{e}_\theta \tag{117}
$$

在角动量$\hat{L}_z$的本征态下, $\langle\hat{L}_x\rangle$ 与 $\langle\hat{L}_y\rangle$ 都为0, 因此

$$
\langle\hat{L}_\theta\rangle = \langle\hat{L}_z\rangle\vec{e}_z\cdot\vec{e}_\theta = m\hbar\cos\theta \tag{118}
$$

41. 计算角动量算符与动能算符的对易子$[\hat{L}_j, \hat{T}]$.

$$\begin{aligned}[\hat{L}_j, \hat{p}_k^2] &= \sum_{kk'l'} \varepsilon_{jk'l'}[\hat{r}_{k'}\hat{p}_{l'}, \hat{p}_k^2] \\ &= \sum_{kk'l'} \varepsilon_{jk'l'}[\hat{r}_{k'}, \hat{p}_k^2]\hat{p}_{l'} \\ &= 2i\hbar \sum_{kk'l'} \varepsilon_{jk'l'}\delta_{k,k'}\hat{p}_{k'}\hat{p}_{l'} \\ &= 2i\hbar \sum_{kl'} \varepsilon_{jkl'}\hat{p}_k\hat{p}_{l'} \\ &= 2i\hbar(\hat{\vec{p}} \times \hat{\vec{p}})_j = 0 \end{aligned} \tag{119}$$

$$[\hat{L}_j, \hat{T}] = \frac{1}{2m}\sum_k [\hat{L}_j, \hat{p}_k^2] = 0 \tag{120}$$

42. 计算对易子$[\hat{L}^2, \hat{L}_z]$.

$$[\hat{L}^2, \hat{L}_z] = [\hat{L}_x^2, \hat{L}_z] + [\hat{L}_y^2, \hat{L}_z] + [\hat{L}_z^2, \hat{L}_z] = -i\hbar(\hat{L}_y\hat{L}_x + \hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x - \hat{L}_x\hat{L}_y) = 0 \tag{121}$$

43. 设系统的正交归一化基矢为$\{|0\rangle\}$, $\{|1\rangle\}$, $\{|2\rangle\}$, 定义右矢为

$$|\alpha\rangle = |0\rangle - 2|1\rangle + 2i|2\rangle, \quad |\beta\rangle = i|0\rangle - 3|2\rangle,$$

(1) 试给出对偶矢量$\langle\alpha|$, $\langle\alpha|$. (用左矢形式$\{\langle 0|\}$, $\{\langle 1|\}$, $\{\langle 2|\}$)

$$\langle\alpha| = \langle 0| - 2\langle 1| - 2i\langle 2| \tag{122}$$

$$\langle\beta| = -i\langle 0| - 3\langle 2| \tag{123}$$

(2) 求出$\langle\alpha|\beta\rangle$和$\langle\beta|\alpha\rangle$, 并验证$\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$.

$$\langle\alpha|\beta\rangle = (\langle 0| - 2\langle 1| - 2i\langle 2|)(i|0\rangle - 3|2\rangle) = 7i \tag{124}$$

$$\langle\beta|\alpha\rangle = (-i\langle 0| - 3\langle 2|)(|0\rangle - 2|1\rangle + 2i|2\rangle) = -7i \tag{125}$$

$$\Rightarrow \quad \langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^* \tag{126}$$

(3) 写出算符$|\alpha\rangle\langle\beta|$和$|\beta\rangle\langle\alpha|$对应的矩阵形式, 并验证它们互为伴随算子.

$$|\alpha\rangle\langle\beta| = (|0\rangle - 2|1\rangle + 2i|2\rangle)(-i\langle 0| - 3\langle 2|) = \begin{pmatrix} -i & 0 & -3 \\ 2i & 0 & 6 \\ 2 & 0 & -6i \end{pmatrix} \tag{127}$$

$$|\beta\rangle\langle\alpha| = (|\beta\rangle = i|0\rangle - 3|2\rangle)(\langle 0| - 2\langle 1| - 2i\langle 2|) = \begin{pmatrix} i & -2i & 2 \\ 0 & 0 & 0 \\ -3 & 6 & 6i \end{pmatrix} \tag{128}$$

$$\Rightarrow \quad |\alpha\rangle\langle\beta| = (|\beta\rangle\langle\alpha|)^\dagger \tag{129}$$

44. **计算以下对易关系**

(1) $\left[\alpha\hat{A}+\beta\hat{B},\hat{C}\right]$

$$\begin{aligned}\left[\alpha\hat{A}+\beta\hat{B},\hat{C}\right] &= (\alpha\hat{A}+\beta\hat{B})\hat{C}-\hat{C}(\alpha\hat{A}+\beta\hat{B})\\ &= \alpha(\hat{A}\hat{C}-\hat{C}\hat{A})+\beta(\hat{B}\hat{C}-\hat{C}\hat{B})\\ &= \alpha\left[\hat{A},\hat{C}\right]+\alpha\left[\hat{B},\hat{C}\right]\end{aligned} \tag{130}$$

(2) $\left[\hat{A},\left[\hat{B},\hat{C}\right]\right]$

$$\begin{aligned}[\hat{A},[\hat{B},\hat{C}]] &= \hat{A}[\hat{B},\hat{C}]-[\hat{B},\hat{C}]\hat{A}\\ &= \hat{A}\hat{B}\hat{C}-\hat{A}\hat{C}\hat{B}-\hat{B}\hat{C}\hat{A}+\hat{C}\hat{B}\hat{A}\\ &= \hat{A}\hat{B}\hat{C}-\hat{B}\hat{A}\hat{C}+\hat{B}\hat{A}\hat{C}-\hat{B}\hat{C}\hat{A}+\hat{C}\hat{A}\hat{B}-\hat{A}\hat{C}\hat{B}-\hat{C}\hat{A}\hat{B}+\hat{C}\hat{B}\hat{A}\\ &= [\hat{A},\hat{B}]\hat{C}+\hat{B}[\hat{A},\hat{C}]-[\hat{A},\hat{C}]\hat{B}-\hat{C}[\hat{A},\hat{B}]\\ &= [[\hat{A},\hat{B}],\hat{C}]+[\hat{B},[\hat{A},\hat{C}]]\end{aligned} \tag{131}$$

(3) $\left[\hat{A},\hat{B}\hat{C}\right]$ **与** $\left[\hat{A}\hat{B},\hat{C}\right]$

$$\begin{aligned}[\hat{A},\hat{B}\hat{C}] &= \hat{A}\hat{B}\hat{C}-\hat{B}\hat{C}\hat{A}\\ &= \hat{A}\hat{B}\hat{C}-\hat{B}\hat{A}\hat{C}+\hat{B}\hat{A}\hat{C}-\hat{B}\hat{C}\hat{A}\\ &= [\hat{A},\hat{B}]\hat{C}+\hat{B}[\hat{A},\hat{C}]\end{aligned} \tag{132}$$

$$\begin{aligned}[\hat{A}\hat{B},\hat{C}] &= \hat{A}\hat{B}\hat{C}-\hat{C}\hat{A}\hat{B}\\ &= \hat{A}\hat{B}\hat{C}-\hat{A}\hat{C}\hat{B}+\hat{A}\hat{C}\hat{B}-\hat{C}\hat{A}\hat{B}\\ &= \hat{A}[\hat{B},\hat{C}]+[\hat{A},\hat{C}]\hat{B}\end{aligned} \tag{133}$$

(4) $\left[\left[\hat{A},\hat{B}\right],\hat{C}\right]+\left[\left[\hat{B},\hat{C}\right],\hat{A}\right]+\left[\left[\hat{C},\hat{A}\right],\hat{B}\right]$

$$\begin{aligned}[[\hat{A},\hat{B}],\hat{C}]+[[\hat{B},\hat{C}],\hat{A}]+[[\hat{C},\hat{A}],\hat{B}] &= -[[\hat{C},\hat{A}],\hat{B}]-[\hat{A},[\hat{C},\hat{B}]]+[[\hat{B},\hat{C}],\hat{A}]+[[\hat{C},\hat{A}],\hat{B}]\\ &= 0\end{aligned} \tag{134}$$

45. **矩阵力学应用至一维无限深势阱(第五章ppt)**

46. **满足$U^\dagger U=UU^\dagger=I$, 且$\mathrm{Det}(U)=1$的$n\times n$的矩阵称为特殊幺正矩阵(SU_n), 试写出SU_2的一般形式.**

对任意一个二维矩阵U, 可写为

$$U=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \tag{135}$$

其中a, b, c, d均为复数, 所以U共存在8个参数. 考虑以下约束

$$U^\dagger U=I,\ \mathrm{Det}(U)=1 \quad\Rightarrow\quad \begin{cases}|a|^2+|c|^2=1\\ |b|^2+|d|^2=1\\ ab^*+cd^*=0\\ ad-bc=1\end{cases} \tag{136}$$

可得$a=d^*$, $c=-b^*$. 因此U可写为

$$U=\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad |a|^2+|b|^2=1 \tag{137}$$

47. **证明等式**

$$e^{-i\frac{\alpha}{2}\sigma_x}\sigma_y e^{i\frac{\alpha}{2}\sigma_x}=\sigma_y\cos\alpha+\sigma_z\sin\alpha\ ,$$
$$e^{-i\frac{\alpha}{2}\sigma_x}\sigma_z e^{i\frac{\alpha}{2}\sigma_x}=\sigma_z\cos\alpha+\sigma_y\sin\alpha\ .$$

$$e^{-i\frac{\alpha}{2}\sigma_x}=\cos\frac{\alpha}{2}I-i\sin\frac{\alpha}{2}\sigma_x \tag{138}$$

将其代入即可

$$\begin{aligned} e^{-i\frac{\alpha}{2}\sigma_x}\sigma_y e^{i\frac{\alpha}{2}\sigma_x} &= (\cos\frac{\alpha}{2}I-i\sin\frac{\alpha}{2}\sigma_x)\sigma_y(\cos\frac{\alpha}{2}I+i\sin\frac{\alpha}{2}\sigma_x) \\ &= \cos^2\frac{\alpha}{2}\sigma_y+i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sigma_y\sigma_x-i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sigma_x\sigma_y+\sin^2\frac{\alpha}{2}\sigma_x\sigma_y\sigma_x \\ &= \cos^2\frac{\alpha}{2}\sigma_y+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sigma_z-\sin^2\frac{\alpha}{2}\sigma_y \\ &= \cos\alpha\ \sigma_y+\sin\alpha\ \sigma_z \end{aligned} \tag{139}$$

$$\begin{aligned} e^{-i\frac{\alpha}{2}\sigma_x}\sigma_z e^{i\frac{\alpha}{2}\sigma_x} &= (\cos\frac{\alpha}{2}I-i\sin\frac{\alpha}{2}\sigma_x)\sigma_z(\cos\frac{\alpha}{2}I+i\sin\frac{\alpha}{2}\sigma_x) \\ &= \cos^2\frac{\alpha}{2}\sigma_z+i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sigma_z\sigma_x-i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sigma_x\sigma_z+\sin^2\frac{\alpha}{2}\sigma_x\sigma_z\sigma_x \\ &= \cos^2\frac{\alpha}{2}\sigma_z+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sigma_y-\sin^2\frac{\alpha}{2}\sigma_z \\ &= \cos\alpha\ \sigma_z+\sin\alpha\ \sigma_y \end{aligned} \tag{140}$$

48. **对于电子自旋**$|\uparrow\rangle=\binom{1}{0}$, $S_z|\uparrow\rangle=\frac{\hbar}{2}|\uparrow\rangle$, **求其在算符**$S_x$, S_y**对应的涨落**ΔS_x^2,ΔS_y^2
下面以此计算$\langle S_x\rangle$, $\langle S_x^2\rangle$, ΔS_x以及$\langle S_y\rangle$, $\langle S_y^2\rangle$, ΔS_y

$$\langle S_x\rangle=\langle\uparrow|S_x|\uparrow\rangle=\frac{\hbar}{2}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=0 \tag{141}$$

$$\left\langle S_x^2\right\rangle=\langle\uparrow|S_x^2|\uparrow\rangle=\frac{\hbar^2}{4}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\frac{\hbar^2}{4} \tag{142}$$

$$\Delta S_x=\sqrt{\langle\uparrow|S_x^2|\uparrow\rangle-\langle\uparrow|S_x|\uparrow\rangle^2}=\frac{\hbar}{2} \tag{143}$$

$$\langle S_y\rangle=\langle\uparrow|S_y|\uparrow\rangle=\frac{\hbar}{2}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=0 \tag{144}$$

$$\left\langle S_y^2\right\rangle=\langle\uparrow|S_y^2|\uparrow\rangle=\frac{\hbar^2}{4}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\frac{\hbar^2}{4} \tag{145}$$

$$\Delta S_y=\sqrt{\langle\uparrow|S_y^2|\uparrow\rangle-\langle\uparrow|S_y|\uparrow\rangle^2}=\frac{\hbar}{2} \tag{146}$$

49. 设电子处在自旋态$|\uparrow\rangle = \binom{1}{0}$, $\sigma_z |\uparrow\rangle = |\uparrow\rangle$, 若选择自选测量投影的方向为$\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, 求可能的测量值和相应的概率.
在$\vec{n}$方向上的自旋算符可表示为$S_{\vec{n}} = \vec{n}\cdot\hbar\vec{\sigma}/2$, 因此我们可以求出在自旋$\vec{n}$方向上的本征态

$$S_{\vec{n}} = \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta\end{pmatrix} \quad\Rightarrow\quad \begin{cases}|\vec{n}_+\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\varphi}\sin\frac{\theta}{2}|\downarrow\rangle\,,\ S_{\vec{n}}|\vec{n}_+\rangle = \frac{\hbar}{2}|\vec{n}_+\rangle \\ |\vec{n}_-\rangle = e^{-i\varphi}\sin\frac{\theta}{2}|\uparrow\rangle - \cos\frac{\theta}{2}|\downarrow\rangle\,,\ S_{\vec{n}}|\vec{n}_-\rangle = -\frac{\hbar}{2}|\vec{n}_-\rangle\end{cases} \tag{147}$$

将$|\uparrow\rangle$用$|\vec{n}_+\rangle$, $|\vec{n}_-\rangle$展开可得$|\uparrow\rangle = \cos\frac{\theta}{2}|\vec{n}_+\rangle + e^{i\varphi}\sin\frac{\theta}{2}|\vec{n}_-\rangle$, 因此有$\cos^2\frac{\theta}{2}$的几率测得电子处于$|\vec{n}_+\rangle$, 相应的自旋大小为$\frac{\hbar}{2}$; 有$\sin^2\frac{\theta}{2}$的几率测得电子处于$|\vec{n}_-\rangle$, 相应的自旋大小为$-\frac{\hbar}{2}$.

50. 非耦合表象与耦合表象之间变换, 第六章ppt

51. 自旋交换算符P_{12}可以对两自旋系统的自旋状态实现交换操作, 例如在非耦合表象$\{S_1^2, S_2^2, S_{1z}, S_{2z}\}$对应的基矢量下, 算符$P_{12}$的作用表示成

$$P_{12}|\uparrow\rangle|\downarrow\rangle = |\downarrow\rangle|\uparrow\rangle\,, \quad P_{12}|\downarrow\rangle|\uparrow\rangle = |\uparrow\rangle|\downarrow\rangle$$

(a) 试在非耦合表象中写出算符P_{12}的矩阵形式.

$$P_{12} = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix} \tag{148}$$

(b) 证明$P_{12}^2 = I$; 用泡利算符表示P_{12}.

$$P_{12}^2 = \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix} = I \tag{149}$$

$$P_{12} = \frac{1}{2}(I + \sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z) \tag{150}$$

52. 证明: $e^{i\theta\sigma_z} = I\cos\theta + i\sin\theta\sigma_z$
在σ_z本征表象下

$$e^{i\theta\sigma_z} = \begin{pmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix} = I\cos\theta + i\sin\theta\sigma_z \tag{151}$$

53. 两个态$|\psi_1\rangle$和$|\psi_2\rangle$的保真度(相似程度)定义为$F = |\langle\psi_1|\psi_2\rangle|^2$, 计算如下两个态之间的保真度: $|\psi_1\rangle = \cos\frac{\theta_1}{2}|0\rangle + \sin\frac{\theta_1}{2}|1\rangle$, $|\psi_2\rangle = \cos\frac{\theta_2}{2}|0\rangle + \sin\frac{\theta_2}{2}|1\rangle$.

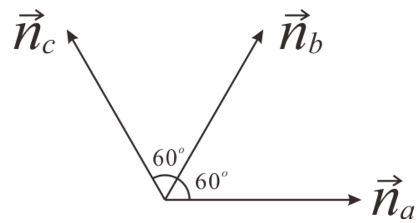
$$F = |\langle\psi_1|\psi_2\rangle|^2 = (\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2})^2 = \cos^2\frac{\theta_1-\theta_2}{2} \tag{152}$$

54. Bell's inequality

考虑反对称最大纠缠态$|\psi^-\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle)$, Alice与Bob分别测量$\vec{n}_a$与$\vec{n}_b$方向上的自旋, 得到关联函数$P(\vec{n}_a, \vec{n}_b)$

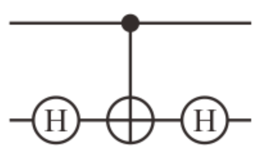
$$P(\vec{n}_a, \vec{n}_b) = \langle\psi^-|(\vec{n}_a \cdot \vec{\sigma}) \otimes (\vec{n}_b \cdot \vec{\sigma})|\psi^-\rangle = -\cos(\vec{n}_a, \vec{n}_b) \tag{153}$$

将其代入贝尔不等式$1+P(\vec{n}_b, \vec{n}_c) \geq |P(\vec{n}_a, \vec{n}_b) - P(\vec{n}_a, \vec{n}_c)|$得到$1-\cos(\vec{n}_b, \vec{n}_c) \geq |\cos(\vec{n}_a, \vec{n}_b) - \cos(\vec{n}_a, \vec{n}_c)|$.



考虑上图所示三个矢量夹角, $\cos(\vec{n}_a, \vec{n}_b) = 1/2$, $\cos(\vec{n}_a, \vec{n}_c) = -1/2$, $\cos(\vec{n}_b, \vec{n}_c) = 1/2$, 显然违背了上述不等式.

55. **基于下图线路, 证明受控相位门和受控非门(CNOT)等价.**



$$|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \sigma_z = |0\rangle\langle 0| \otimes HIH + |1\rangle\langle 1| \otimes H\sigma_x H \tag{154}$$

因此受控相位门只是在受控非门中的目标比特上增加了两个单比特$Hadamard$门.

56. $\langle\sigma_1\sigma_2\rangle = \langle\psi_{1,2}|\sigma_1\sigma_2|\psi_{1,2}\rangle$**是两粒子体系的一个关联,** σ_1**和**σ_2**是分别作用于粒子**1**和粒子**2**的算符. 如果**$|\psi_{1,2}\rangle = 1/\sqrt{2}(|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2)$, **求**$\langle\sigma_{x1}\sigma_{x2}\rangle$, $\langle\sigma_{y1}\sigma_{y2}\rangle$, $\langle\sigma_{z1}\sigma_{z2}\rangle$. **其中**$\sigma_x$, σ_y, σ_z**是泡利算符.**

$$\begin{aligned}
\langle\sigma_{x1}\sigma_{x2}\rangle &= \langle\psi_{1,2}|\sigma_{x1} \otimes \sigma_{x2}|\psi_{1,2}\rangle = 1 \\
\langle\sigma_{y1}\sigma_{y2}\rangle &= \langle\psi_{1,2}|\sigma_{y1} \otimes \sigma_{y2}|\psi_{1,2}\rangle = -1 \\
\langle\sigma_{z1}\sigma_{z2}\rangle &= \langle\psi_{1,2}|\sigma_{z1} \otimes \sigma_{z2}|\psi_{1,2}\rangle = 1
\end{aligned} \tag{155}$$

57. **纠缠交换中, 初始**1, 2**粒子和**3, 4**粒子分别处于态**$|\psi_{1,2}\rangle = 1/\sqrt{2}(|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2)$**及**$|\psi_{3,4}\rangle =$

$1/\sqrt{2}(|0\rangle_3|0\rangle_4 - |1\rangle_3|1\rangle_4)$. **试推导**2, 3**粒子做**Bell**测量后**, 1, 4**粒子所处的状态**.

$$
\begin{aligned}
|\psi_{1,2}\rangle|\psi_{3,4}\rangle = \frac{1}{2}\Big[&\frac{1}{\sqrt{2}}(|0\rangle_2|0\rangle_3 + |1\rangle_2|1\rangle_3)\frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_4 - |1\rangle_1|1\rangle_4) \\
&+\frac{1}{\sqrt{2}}(|0\rangle_2|0\rangle_3 - |1\rangle_2|1\rangle_3)\frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_4 + |1\rangle_1|1\rangle_4) \\
&-\frac{1}{\sqrt{2}}(|0\rangle_2|1\rangle_3 + |1\rangle_2|0\rangle_3)\frac{1}{\sqrt{2}}(|0\rangle_1|1\rangle_4 - |1\rangle_1|0\rangle_4) \\
&-\frac{1}{\sqrt{2}}(|0\rangle_2|1\rangle_3 - |1\rangle_2|0\rangle_3)\frac{1}{\sqrt{2}}(|0\rangle_1|1\rangle_4 + |1\rangle_1|0\rangle_4)\Big]
\end{aligned}
\tag{156}
$$

所以当2, 3粒子投影至$\frac{1}{\sqrt{2}}(|0\rangle_2|0\rangle_3 + |1\rangle_2|1\rangle_3)$时, 1, 4粒子处于$\frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_4 - |1\rangle_1|1\rangle_4)$;
所以当2, 3粒子投影至$\frac{1}{\sqrt{2}}(|0\rangle_2|0\rangle_3 - |1\rangle_2|1\rangle_3)$时, 1, 4粒子处于$\frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_4 + |1\rangle_1|1\rangle_4)$;
所以当2, 3粒子投影至$\frac{1}{\sqrt{2}}(|0\rangle_2|1\rangle_3 + |1\rangle_2|0\rangle_3)$时, 1, 4粒子处于$\frac{1}{\sqrt{2}}(|0\rangle_1|1\rangle_4 - |1\rangle_1|0\rangle_4)$;
所以当2, 3粒子投影至$\frac{1}{\sqrt{2}}(|0\rangle_2|1\rangle_3 - |1\rangle_2|0\rangle_3)$时, 1, 4粒子处于$\frac{1}{\sqrt{2}}(|0\rangle_1|1\rangle_4 + |1\rangle_1|0\rangle_4)$.