4: (1) 两边求微分有

$$-2\cos x\sin x\mathrm{d}x - 2\cos y\sin y\mathrm{d}y - 2\cos z\sin z\mathrm{d}z = 0.$$

于是$\mathrm{d}z = -\dfrac{\cos x\sin x\mathrm{d}x + \cos y\sin y\mathrm{d}y}{\cos z\sin z}$.
(2) 两边求微分有

$$yz\mathrm{d}x + xz\mathrm{d}y + xy\mathrm{d}z = \mathrm{d}x + \mathrm{d}y + \mathrm{d}z.$$

于是$\mathrm{d}z = -\dfrac{(yz-1)\mathrm{d}x + (xz-1)\mathrm{d}y}{xy-1}$.
(3) 两边求微分

$$3u^2\mathrm{d}u - (3\mathrm{d}x + 3\mathrm{d}y)u^2 - 6(x+y)u\mathrm{d}u + 3z^2\mathrm{d}z = 0.$$

于是$\mathrm{d}u = \dfrac{u^2\mathrm{d}x + u^2\mathrm{d}y - z^2\mathrm{d}z}{u^2 - 2(x+y)u}$.
(4) 两边求微分

$$F_1'(\mathrm{d}x - \mathrm{d}y) + F_2'(\mathrm{d}y - \mathrm{d}z) + F_3'(\mathrm{d}z - \mathrm{d}x) = 0.$$

于是$\mathrm{d}z = \dfrac{(F_1' - F_2')\mathrm{d}y + (F_3' - F_1')\mathrm{d}x}{F_3' - F_2'}$.

5: 等式$1 + xy = k(x-y)$两边同时求微分，有

$$xdy + ydx = k(dx - dy)$$

两边同除dx,得

$$\frac{dy}{dx} = \frac{k-y}{k-x}$$

把$k=\frac{1+xy}{x-y}$代入上式,消去k,得到

$$\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$$

得证.

6: 证明:

令

$$F\left(x,y,z\right)=2\sin\left(x+2y-3z\right)-x-2y+3z$$

可得:

$$F_x^{'}=2\cos\left(x+2y-3z\right)-1$$

$$F_y^{'}=4\cos\left(x+2y-3z\right)-2$$

$$F_z^{'}=-6\cos\left(x+2y-3z\right)+3$$

则有

$$\frac{\partial z}{\partial x}=-\frac{F_x^{'}}{F_z^{'}}=3$$

$$\frac{\partial z}{\partial y}=-\frac{F_y^{'}}{F_z^{'}}=-2$$

代入得

$$\frac{\partial z}{\partial y}+\frac{\partial z}{\partial x}=1$$

证毕。

7: $\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z} = \frac{c\varphi_1}{a\varphi_1+b\varphi_2}$

$\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y}\Big/\frac{\partial F}{\partial z} = \frac{c\varphi_2}{a\varphi_1+b\varphi_2}$

从而$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = c$

8: 由$\mathrm{d}(x^2-xy-y^2) = 2x\mathrm{d}x - y\mathrm{d}x - x\mathrm{d}y + 2y\mathrm{d}y = 0$有$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x-y}{x-2y}(x \neq 2y)$，于是

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 2x + 2y\frac{\mathrm{d}y}{dx} = 2x + y\frac{2x-y}{x-2y}, \quad x \neq 2y$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = 2 + 2\left(\frac{\mathrm{d}y}{dx}\right)^2 + 2y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2 + \left(\frac{2x-y}{x-2y}\right)^2 + 6y\frac{x-y}{(x-2y)^2}$$

9: 对下面两个式子同时做全微分

$$\begin{cases} y = f(x+t) \\ y + g(x,t) = 0 \end{cases}$$

得到

$$\begin{cases} dy = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial t}dt = 0 \\ dy = f'dx + f'dt \end{cases}$$

联立两个方程，消去dt得到

$$\frac{dy}{dx} = \frac{1 - \frac{\partial g}{\partial x}}{\frac{\frac{\partial g}{\partial t}}{f'} + 1}$$

10: 在两个方程两端对z求导得到$\begin{cases}\frac{dx}{dz}+\frac{dy}{dz}+1=0\\ 2\frac{dx}{dz}x+2\frac{dy}{dz}y+2z=0\end{cases}$ 从而解得$\begin{cases}\frac{dx}{dz}=\frac{y-z}{x-y}\\ \frac{dy}{dz}=\frac{x-z}{y-x}\end{cases}$

12: (1)求微分得到

$$\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix}=\begin{pmatrix}f_1' & f_2'\\ g_1' & g_2'\end{pmatrix}\begin{pmatrix}\mathrm{d}u\\ \mathrm{d}v\end{pmatrix}.$$

从而

$$\begin{pmatrix}\mathrm{d}u\\ \mathrm{d}v\end{pmatrix}=\begin{pmatrix}f_1' & f_2'\\ g_1' & g_2'\end{pmatrix}^{-1}\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix}=\frac{1}{f_1'g_2'-f_2'g_1'}\begin{pmatrix}g_2' & -f_2'\\ -g_1' & f_1'\end{pmatrix}\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix}.$$

也就是

$$\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}=\frac{1}{f_1'g_2'-f_2'g_1'}\begin{pmatrix}g_2' & -f_2'\\ -g_1' & f_1'\end{pmatrix}.$$

(2) 取$f(u,v)=\mathrm{e}^u+u\sin v, g(u,v)=\mathrm{e}^u-u\cos v$代入(1)得到

$$\begin{aligned}\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}&=\frac{1}{f_1'g_2'-f_2'g_1'}\begin{pmatrix}g_2' & -f_2'\\ -g_1' & f_1'\end{pmatrix}.\\ &=\frac{1}{u\mathrm{e}^u(\sin v-\cos v)+u}\begin{pmatrix}u\sin v & -u\cos v\\ -\mathrm{e}^u+\cos v & \mathrm{e}^u+\sin v\end{pmatrix}.\end{aligned}$$

13: 方程$\varphi\left(x^2,e^y,z\right)=0$两边同时对$x$求导，得

$$2x\varphi_1'+\cos xe^{sinx}\varphi_2'+\frac{dz}{dx}\varphi_3'=0$$

从中解出

$$\frac{dz}{dx} = -\frac{2x\varphi_1' + \cos x e^{sinx}\varphi_2'}{\varphi_3'}$$

再将方程$u = f(x, y, z)$两边同时对x求导,有

$$\frac{du}{dx} = f_1' + f_2'\frac{dy}{dx} + f_3'\frac{dz}{dx} = f_1' + f_2'\cos x + f_3'\frac{dz}{dx}$$

将$\frac{dz}{dx}$的表达式代入上式,得

$$\frac{du}{dx} = f_1' + f_2'\cos x - f_3'\frac{2x\varphi_1' + \cos x e^{sinx}\varphi_2'}{\varphi_3'}$$

14: 解：令

$$G(x, y, z) = xf(x + y) - z$$

则有:

$$G_x^{'} = f(x + y) + xf^{'}(x + y)$$

$$G_y^{'} = xf^{'}(x + y)$$

$$G_z^{'} = -1$$

由隐函数定理可知:

$$\frac{dz}{dx} = -\frac{F_x^{'}G_y^{'} - F_y^{'}G_x^{'}}{F_y^{'}G_z^{'} - F_z^{'}G_y^{'}}$$

代入得:

$$\frac{dz}{dx} = \frac{F_x^{'}f^{'}(x + y)x - F_y^{'}f(x + y) - F_y^{'}f^{'}(x + y)x}{F_y^{'} + F_z^{'}f^{'}(x + y)x}$$

15: $F(x, y, u, v) = 0, G(x, y, u, v) = 0, u = u(x, y), v = v(x, y)$

$$\text{则}\begin{cases} \frac{\partial F}{\partial x} = F_1' + F_3'u_x' + F_4'v_x' = 0 \\ \frac{\partial F}{\partial y} = F_2' + F_3'u_y' + F_4'v_y' = 0 \\ \frac{\partial G}{\partial x} = G_1' + G_3'u_x' + G_4'v_x' = 0 \\ \frac{\partial G}{\partial y} = G_2' + G_3'u_y' + G_4'v_y' = 0 \end{cases}$$

从而$u'_x=-\frac{\partial(F,G)}{\partial(x,v)}\Big/\frac{\partial(F,G)}{\partial(u,v)}$

$u'_y=-\frac{\partial(F,G)}{\partial(y,v)}\Big/\frac{\partial(F,G)}{\partial(u,v)}$

$v'_x=-\frac{\partial(F,G)}{\partial(u,x)}\Big/\frac{\partial(F,G)}{\partial(u,v)}$

$v'_y=-\frac{\partial(F,G)}{\partial(u,y)}\Big/\frac{\partial(F,G)}{\partial(u,v)}$

故$du=u'_xdx+u'_ydy$

$dv=v'_xdx+v'_ydy$

将u'_x,u'_y,v'_x,v'_x代入即可

16: 由$u(x,y)=f(x,y,z,t)$知$z=z(x,y),t=t(x,y)$. 由方程 $g(y,z,t)=0,h(z,t)=0$ 有

$$g_zz_y+g_tt_y=-g_y$$

$$h_zz_y+h_tt_y=0$$

联立解得

$$\begin{pmatrix} z_y \\ t_y \end{pmatrix}=\begin{pmatrix} g_z & g_t \\ h_z & h_t \end{pmatrix}^{-1}\begin{pmatrix} -g_y \\ 0 \end{pmatrix} \tag{9.5}$$

$$u_y=f_y+f_zz_y+f_tt_y=f_y+(f_z,f_t)\begin{pmatrix} z_y \\ t_y \end{pmatrix} \tag{9.6}$$

$$=f_y+(f_z,f_t)\begin{pmatrix} g_z & g_t \\ h_z & h_t \end{pmatrix}^{-1}\begin{pmatrix} -g_y \\ 0 \end{pmatrix} \tag{9.7}$$

类似的有

$$g_zz_x+g_tt_x=0 \tag{9.8}$$

$$h_zz_x+h_tt_x=0 \tag{9.9}$$

易得$z_x=t_x=0$. 于是 $u_x=f_x+f_zz_x+f_tt_x=f_x$.

9.4 空间曲线与曲面

1:

$$\vec{r}' = (a\cos t, a\sin t, 2bt)$$
$$\vec{r}'' = (-a\sin t, a\cos t, 2b)$$

2: 设$\vec{r}(t) = (r_1(t), \ldots, r_n(t))$,则$\vec{r'}(t) = (r_1'(t), \ldots, r_n'(t))$.
由$r_1^2(t)+\cdots+r_n^2(t) = 1$两边对t求导得到$2(r_1(t)r_1'(t)+\cdots+r_n(t)r_n'(t)) = 0$,即得结论.
几何意义:长度不变的向量函数在其上每一点与其切向量正交.

3: 由题意可得曲线的切向量
$r'(t)$=(-asint,acost,b)
z轴的方向向量是k=(0,0,1)
所以切线与z轴的夹角余弦为
$\cos\theta=\frac{r'\cdot k}{|r'|\cdot|k|}=\frac{b}{\sqrt{a^2+b^2}}$为常数
$\therefore$曲线的切线与Oz轴夹角为常值

4: 是简单曲线也是光滑曲线. $\boldsymbol{r}'(t) = (\frac{1}{(1+t)^2}, -\frac{1}{t^2}, 2t)$,
将$t = 1$代入得切线的方向向量$\vec{v} = (\frac{1}{4}, -1, 2)$, 又$\boldsymbol{r}(1) = (\frac{1}{2}, 2, 1)$.
从而切线方程: $\dfrac{4x-2}{1} = \dfrac{y-2}{-1} = \dfrac{z-1}{2}$.
法平面方程: $\dfrac{1}{4}x - y + 2z - \dfrac{1}{8} = 0$.

5: (1)曲线切向量为$\left(2a\sin t\cos t, -b\sin^2 t + b\cos^2 t, -2c\sin t\cos t\right)$

在$t_0 = \pi/4$处切向量为$(a, 0, -c)$,且$t_0 = \pi/4$对应曲线上点$\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$

故切线方程为

$$\frac{x - \frac{a}{2}}{a} = \frac{y - \frac{b}{2}}{0} = -\frac{z - \frac{c}{2}}{c}$$

法平面方程为

$$a\left(x - \frac{a}{2}\right) - c\left(z - \frac{c}{2}\right) = 0$$

(2)曲线切向量为$(1 + \sin t, 2\sin t\cos t, -3\sin 3t)$

在$t_0 = \pi/2$处切向量为$(2, 0, 3)$,且$t_0 = \pi/2$对应曲线上点$\left(\frac{\pi}{2}, 4, 1\right)$

故切线方程为

$$\frac{x - \frac{\pi}{2}}{2} = \frac{y - 4}{0} = \frac{z - 1}{3}$$

法平面方程为

$$2\left(x - \frac{\pi}{2}\right) + 3(z - 1) = 0$$

6: 解：(1)

$$x_u^{'} = \cos v, y_u^{'} = \sin v, z_u^{'} = 0$$
$$x_v^{'} = -u\sin v, y_v^{'} = u\cos v, z_v^{'} = a$$

所以法向量为$\vec{n} = (x_u^{'}, y_u^{'}, z_u^{'}) \times (x_v^{'}, y_v^{'}, z_v^{'}) = (a\sin v, -a\cos v, u)$,则在$(u_0, v_0)$处的切平面方程为:

$$a\sin v_0(x - u_0\cos v_0) - a\cos v_0(y - u_0\sin v_0) + u_0(z - av_0) = 0$$

法线方程为:

$$\frac{x - u_0\cos v_0}{a\sin v_0} = \frac{y - u_0\sin v_0}{-a\cos v_0} = \frac{z - av_0}{u_0}$$

(2)

$$x_\theta^{'} = a\cos\theta\cos\varphi, y_\theta^{'} = b\cos\theta\sin\varphi, z_\theta^{'} = -c\sin\theta$$
$$x_\varphi^{'} = -a\sin\theta\sin\varphi, y_\varphi^{'} = b\sin\theta\cos\varphi, z_\varphi^{'} = 0$$

所以法向量为$\vec{n} = (x_\theta^{'}, y_\theta^{'}, z_\theta^{'})\times(x_\varphi^{'}, y_\varphi^{'}, z_\varphi^{'}) = (bc\sin^2\theta\cos\varphi, ac\sin^2\theta\sin\varphi, ab\sin\theta\cos\theta)$,则在$(u_0, v_0)$处的切平面方程为:

$$bc\sin^2\theta_0\cos\varphi_0(x - a\sin\theta_0\cos\varphi_0) + ac\sin^2\theta_0\sin\varphi_0(y - b\sin\theta_0\sin\varphi_0) +$$
$$ab\sin\theta_0\cos\theta_0(z - c\cos\theta_0) = 0$$

法线方程为:

$$\frac{x - a\sin\theta_0\cos\varphi_0}{bc\sin^2\theta_0\cos\varphi_0} = \frac{y - b\sin\theta_0\sin\varphi_0}{ac\sin^2\theta_0\sin\varphi_0} = \frac{z - c\cos\theta_0}{ab\sin\theta_0\cos\theta_0}$$

7: $F(x,y) = F(x, F(x)) = 0$

$\frac{dF}{dx} = F_1' + F_2' f'(x) = 0$

则$f'(x) = -\frac{F_1'}{F_2'}, f'(x_0) = -\frac{F_1'(x_0,y_0)}{F_2'(x_0,y_0)}$

故$\overrightarrow{n_1} = (1, -\frac{F_1'(x_0,y_0)}{F_2'(x_0,y_0)})$

同理$\overrightarrow{n_2} = (1, -\frac{G_1'(x_0,y_0)}{G_2'(x_0,y_0)})$

$< \overrightarrow{n_1}, \overrightarrow{n_2} >= arccos\frac{\overrightarrow{n_1}\cdot\overrightarrow{n_1}}{|\overrightarrow{n_1}|\cdot|\overrightarrow{n_2}|} = arccos\frac{F_1'(x_0,y_0)G_1'(x_0,y_0)+F_2'(x_0,y_0)G_2'(x_0,y_0)}{\sqrt{(F_1'(x_0,y_0))^2+(F_2'(x_0,y_0))^2)^2}\cdot\sqrt{(G_1'(x_0,y_0))^2+(G_2'(x_0,y_0))^2)^2}}$

8: (1) $\boldsymbol{n} = (17, 11, 5)$, $\boldsymbol{\pi} : 17x + 11y + 5z - 60 = 0$

(2) $\boldsymbol{n} = (1, -1, 2)$, $\boldsymbol{\pi} : x - y + 2z - \frac{\pi}{2} = 0$

(3) $\boldsymbol{n} = (1, 2, 0)$, $\boldsymbol{\pi} : x + 2y - 4 = 0$

(4) $\boldsymbol{n} = (5, 4, 1)$, $\boldsymbol{\pi} : 5x + 4y + z - 28 = 0$

9: 椭球面在(x_0, y_0, z_0)处的切平面为

$$xx_0 + 2yy_0 + zz_0 = 1$$
$$\frac{x_0}{1} = \frac{2y_0}{-1} = \frac{z_0}{2}$$
$$x_0^2 + 2y_0^2 + z_0^2 = 1$$

解得$(\frac{\sqrt{22}}{2}, -\frac{\sqrt{22}}{4}, \sqrt{22})$和$(-\frac{\sqrt{22}}{2}, \frac{\sqrt{22}}{4}, -\sqrt{22})$，再根据法向量即可得到平面方程。

10: 显然平面$x+3y+z = 0$的法向量为$\vec{n} = (1, 3, 1)$,而曲面上(x, y, z)点处的法向量为$\vec{n}_s = (z'_x, z'_y, -1) = (y, x, -1)$,由两法向量平行即可解出$\begin{cases} x = -3 \\ y = -1 \end{cases}$,故所求曲面上的点为$(-3, -1, 3)$,法线方程为$\frac{x+3}{1} = \frac{y+1}{3} = \frac{z-3}{1}$

11: 记点M的坐标为(x_0,y_0,z_0)

椭球面在M点的梯度gradF(M)=(x_0,2y_0,3z_0)

∴过点M的切平面的法向量为(x_0,2y_0,3z_0)

直线的方向向量为(2,1,-1)，过点(6,3,1/2),联立可得如下方程

$2x_0+2y_0-3z_0=0$

$x_0(6-x_0)+2y_0(3-y_0)+3z_0(1/2-z_0)=0$

${x_0}^2+2{y_0}^2+3{z_0}^2=21$

求解可得$x_0=1,y_0=2,z_0=2$或者$x_0=3,y_0=0,z_0=2$

当M为(1,2,2)时，切平面方程为x+4y+6z-21=0

当M为(3,0,2)时，切平面方程为3x+6z-21=0

12: 曲面于点$(1,-2,5)$处的法向量为$(2,-4,-1)$,
因此平面π: $2x-4y-z-5=0$. 任意选取直线上两点代入平面π, 得$a=-5, b=-2$.

13: 两个曲面在(x,y,z)处的法向量分别为

$$\boldsymbol{n_1}=(2x-a,2y,2z),\boldsymbol{n_2}=(2x,2y-b,2z)$$

$$\begin{aligned}\boldsymbol{n_1}\cdot\boldsymbol{n_2}&=4\left(x^2+y^2+z^2\right)-2ax-2by\\&=2\left(x^2+y^2+z^2-ax\right)+2\left(x^2+y^2+z^2-by\right)\\&=0\end{aligned}$$

因此两曲面正交.

14: 解：曲面$x+2y-\ln z+4=0$在点(x,y,z)处的法向量为$(1,2,-\frac{1}{z})$,曲面$x^2-xy-8x+z+5=0$在点(x,y,z)处的法向量为$(2x-y-8,-x,1)$,所以将点$(2,-3,1)$分别代入上面的两个法向量，得到$\vec{n_1}=(1,2,-1),\vec{n_2}=(-1,-2,1)$,即$\vec{n_1}\parallel\vec{n_2}$,则两曲面在该点有公共的切平面：

$$(x-2)+2(y+3)-(z-1)=0$$

15: 在$z = xe^{x/y}$上任取一点$(x_0, y_0, x_0e^{x_0/y_0})$

$\frac{\partial z}{\partial x}\Big|_{(x_0,y_0)} = (\frac{x_0}{y_0}+1)e^{x_0/y_0}$

$\frac{\partial z}{\partial y}\Big|_{(x_0,y_0)} = -\frac{x_0^2}{y_0^2}e^{x_0/y_0}$

则$\overrightarrow{n_1} = (1, 0, (\frac{x_0}{y_0}+1)e^{x_0/y_0})$

$\overrightarrow{n_2} = (0, 1, -\frac{x_0^2}{y_0^2}e^{x_0/y_0})$

$\overrightarrow{n_1} \times \overrightarrow{n_2} = (-(\frac{x_0}{y_0}+1)e^{x_0/y_0}, \frac{x_0^2}{y_0^2}e^{x_0/y_0}, 1)$

在该点的切平面为$-(\frac{x_0}{y_0}+1)e^{x_0/y_0}(x - x_0) + \frac{x_0^2}{y_0^2}e^{x_0/y_0}(y - y_0) + (z - x_0e^{x_0/y_0}) = 0$

将(x, y, z) =(0,0,0)代入得

$(\frac{x_0}{y_0}+1)e^{x_0/y_0}x_0 - \frac{x_0^2}{y_0^2}e^{x_0/y_0}y_0 - x_0e^{x_0/y_0} = 0$

该式恒成立，从而命题得证

16: (1) $l : x + y - 2 = 0, \quad l_n : x - y = 0$

(2) $l : x + 2y - 1 = 0, \quad l_n : 2x - y - 2 = 0$

17: (1)Let $F_1(x, y, z) = y^2 + z^2 - 25, F_2(x, y, z) = x^2 + y^2 - 10$, then for $F_1(x, y, z) = 0$ the point $(1, 3, 4)$ follows the normal vector is $\mathbf{n}_1 = (0, 6, 8)$, for $F_2(x, y, z) = 0$ the point $(1, 3, 4)$ follows the normal vector is $\mathbf{n}_2 = (2, 6, 0)$ then the tangent direction is $\mathbf{n}_1 \times \mathbf{n}_2 = (-48, 16, -12)$ it can be instead of $(-12, 4, -3)$

the equation of tangent line is $\frac{x-1}{-12} = \frac{y-3}{4} = \frac{z-4}{-3}$,

the equation of normal plane is $-12(x - 1) + 4(y - 3) - 3(z - 4) = 0$

(2)Similar to (1).the tangent direction is $(27, 28, 4)$, the equation of tangent line is $\frac{x+2}{27} = \frac{y-1}{28} = \frac{z-6}{4}$

the equation of normal plane is $27(x + 2) + 28(y - 1) + 4(z - 6) = 0$