16: (1)记 $\lim_{x\to x_0} f(x,y) = \phi(y)(y_1 \neq y_0)$. 由题设知，对任意$\epsilon > 0$, 存在$\delta > 0$, 只要 $|x-x_0| < \delta, |y_1 - y_0| < \delta, |y_2 - y_0| < \delta$, 便有

$$|f(x,y_1) - f(x,y_2)| < \epsilon$$

令 $x \to x_0$，则有$|\phi(y_1) - \phi(y_2)| < \epsilon$, 故 $\lim_{y\to y_0} \phi(y)$ 存在.

再证明 $\lim_{y\to y_0} \phi(y) = A$. 对上述的$\epsilon, \delta$, 当 $0 < |x-x_0|, |y_1 - y_0| < \delta$ 时，有

$$|f(x,y_1) - A| < \epsilon, \quad |f(x,y_1) - \phi(y_1)| < \epsilon$$

于是

$$|\phi(y_1) - A| = |\phi(y_1) - f(x,y_1) + f(x,y_1) - A| < 2\epsilon,$$

所以 $\lim_{y\to y_0} \phi(y) = A$, 命题得证. 同理可证(2).

17: (1)容易知道在$y \neq x$时整个函数是连续的。我们主要讨论$y = x$时的连续性。则当x和y趋于相等时。第一种情况x和y趋于相等且不等于0，那么分子不为零，分母趋于零，整个趋于无穷，所以在$x = y$的且不等于零的地方肯定不连续。下面再讨论在$(0,0)$处的连续性。分别取$y = 2x$和$y = x - x^2$所得极限不一样，所以在原点处也没有极限。因此整个函数在$y = x$处不连续，在其他地方连续。

(2).Consider the point on the line $y = 0$, we set it $(x_0, 0)$

[1].If $x_0 = 0$, that is to say $(0,0)$, then $f(0,0) = 0$.

since $|x \sin\frac{1}{y}| \leq |x|$, by the definition of limit we know $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} x\sin(\frac{1}{y}) = 0 = f(0,0)$ and along the y axis $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, then $f(x,y)$ is continuous at $(0,0)$.

[2].If $x_0 \neq 0$, we know $f(x_0, 0) = 0$

along the line $x = x_0$, $f(x, y) = x_0 \sin(\frac{1}{y})$ since the limit $\lim\limits_{(x,y)\to(x_0,0)} f(x, y) = \lim\limits_{(x,y)\to(x_0,0)} x_0 \sin(\frac{1}{y})$ is not exist, thus $\lim\limits_{(x,y)\to(x_0,0)} f(x, y)$ is not exist.
In short, $f(x, y)$ is continuous on the set $\{(x, y) \mid y \neq 0\} \cup \{(0, 0)\}$

(3). 令$x = \rho\cos\theta, y = \rho\sin\theta$，那么

$$|\frac{x^2y}{x^2 + y^2}| = |\rho\sin\theta\cos^2\theta| \leq |\rho| \to 0.$$

所以函数在整个平面是连续的。

(4). 函数在$x + y \neq 0$的地方肯定是连续的。下面我们讨论在$x + y = 0$地方的连续性。首先在$(0, 0)$处，令$y = kx$发现极限的结果与k的取值有关，所以在$(0, 0)$处不连续。在其他$x + y = 0$的地方，分母会趋于0，但是分子是有限非零数，这个时候整个函数会趋于无穷。所以在$x + y = 0$上是不连续的。

18: 略

19: 不妨设f(x,y)关于y是单调递增的

$\forall$取D中一点(x_0,y_0),因为f(x,y)关于y连续

$\therefore$ $\forall$ ϵ¿0,$\exists$ δ_1 当—y-y_0—$\leq$ δ_1时，—f(x_0,y)-f(x_0,y_0)—¡ϵ/2

对于点(x_0,y_0-δ_1),(x_0,y_0+δ_1)

$\because$f(x,y)关于x连续

$\therefore$对上述ϵ,$\exists\delta_2$¿0,当—x-x_0—¡δ_2时

—f(x,y_0-δ_1)-f(x_0,y_0-δ_1)—¡ϵ/2

—f(x,y_0+δ_1)-f(x_0,y_0+δ_1)—¡ϵ/2

令δ=min{δ_1,δ_2},则当—x-x_0—¡δ,—y-y_0—¡δ时

—f(x,y)-f(x_0,y_0)—≤max{—f(x,y_0-δ_1)-f(x_0,y_0)—,—f(x,y_0+δ_1)-f(x_0,y_0)—}

—f(x,y_0-δ_1)-f(x_0,y_0)—≤—f(x,y_0-δ_1)-f(x_0,y_0-δ_1)—+—f(x_0,y_0-δ_1)-f(x_0,y_0)—¡ϵ/2+ϵ/2=ϵ

—f(x,y_0+δ_1)-f(x_0,y_0)—≤—f(x,y_0+δ_1)-f(x_0,y_0+δ_1)—+—f(x_0,y_0+δ_1)-f(x_0,y_0)—¡ϵ/2+ϵ/2=ϵ

∴—f(x,y)-f(x_0,y_0)—¡ϵ,f(x,y)在点(x_0,y_0)处连续

由点(x_0,y_0)的任意性知，f(x,y)在D连续，证毕

20: 反例：$\{(x,y)|y \geq \frac{1}{x} > 0\}$.

21: 二元函数Cauchy收敛准则：设$f(x,y)$是定义在$D \in \mathbb{R}^2$上的二元函数，则$f(x,y)$在$M_0(x_0,y_0)$处收敛等价于$\forall\epsilon > 0, \exists\delta > 0, s.t.\forall M_1, M_2 \in B(M_0,\delta)\cap D$,有$|f(M_1) - f(M_2)| < \epsilon$

证明略.

22: 证：$f(x,y)$在(x_0,y_0)处连续

$\Rightarrow \forall\epsilon > 0\ \exists\delta > 0$, s.t. $\forall|x - x_0| < \delta, |y - y_0| < \delta$, 有$|f(x,y) - f(x_0,y_0)| < \epsilon$

$x(u,v), y(u,v)$在(u_0,v_0)处连续

$\Rightarrow \exists\delta' > 0$, s.t. $\forall|u-u_0| < \delta', |v-v_0| < \delta'$，有$|x(u,v)-x_0| < \delta, |y(u,v)-y_0| < \delta$

$\Rightarrow |f(x(u,v),y(u,v)) - f(x_0,y_0)| < \epsilon$, $\forall|u - u_0| < \delta', |v - v_0| < \delta'$

$\Rightarrow f(x(u,v),y(u,v))$在$(u_0,v_0)$处连续.

23: 证明：$f(x) = \frac{1}{1-xy}$在$[0,1]\times[0,1]\cap\{(x,y)\big|(x,y) \neq (1,1)\}$上由初等函数的四则运算产生，显然连续

下证其不一致连续:

取$\epsilon = \frac{1}{8}$

则对$\forall \delta > 0$(不妨设$\delta < 1$),

取$A = (x_1, y_1) = (1-\delta, 1-\delta), B = (x, y) = (1-\frac{\delta}{2}, 1-\frac{\delta}{2})$

则$|AB| < \delta$

且$|f(A) - f(B)| = |\frac{1}{1-x_1y_1} - \frac{1}{1-x_2y_2}| = \frac{4-3\delta}{\delta(4-\delta)(2-\delta)} > \frac{1}{8}$

从而f不一致连续

9.2 多变量函数的微分

1: (1)$f'_x(x,y) = 1 - \frac{x}{\sqrt{x^2+y^2}}, f'_x(3,4) = \frac{2}{5}$

(2)$f'_x(x,y) = 2xy\cos x^2y, f'_x(1,\pi) = -2\pi$

(3)$f'_x(x,y) = \frac{y^2+2xy+\frac{(xy^2+x^2y)(y^2+2xy)}{\sqrt{1+(xy^2+x^2y)^2}}}{xy^2+x^2y+\sqrt{1+(xy^2+x^2y)^2}}$

由对称性，$f'_y(x,y) = \frac{x^2+2xy+\frac{(xy^2+x^2y)(x^2+2xy)}{\sqrt{1+(xy^2+x^2y)^2}}}{xy^2+x^2y+\sqrt{1+(xy^2+x^2y)^2}}$

$f'_x(1,y) = \frac{y^2+2y+\frac{(y^2+y)(y^2+2y)}{\sqrt{1+(y^2+y)^2}}}{y^2+y+\sqrt{1+(y^2+y)^2}} = \frac{y^2+2y}{\sqrt{1+(y^2+y)^2}}$

$f'_y(1,y) = \frac{1+2y+\frac{(y^2+y)(1+2y)}{\sqrt{1+(y^2+y)^2}}}{y^2+y+\sqrt{1+(y^2+y)^2}} = \frac{1+2y}{\sqrt{1+(y^2+y)^2}}$

2: (1) $\frac{\partial z}{\partial x} = \frac{e^y}{y^2}$, $\frac{\partial z}{\partial y} = \frac{xe^y(y-2)}{y^3}$

(2) $\frac{\partial z}{\partial x} = \frac{ln3y3^{-\frac{y}{x}}}{x^2}$, $\frac{\partial z}{\partial y} = -\frac{3^{-\frac{y}{x}}ln3}{x}$

(3) $\frac{\partial z}{\partial x} = \frac{cos(\frac{x}{y})cos(\frac{y}{x})}{y} + \frac{ysin(\frac{x}{y})sin(\frac{y}{x})}{x^2}$, $\frac{\partial z}{\partial y} = -\frac{-xcos(\frac{x}{y})cos(\frac{y}{x})}{y^2} - \frac{sin(\frac{x}{y})sin(\frac{y}{x})}{x}$

(4) $\frac{\partial z}{\partial x} = \frac{1}{\sqrt{x^2+y^2}}$, $\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}(x+\sqrt{x^2+y^2})}$

(5) $\frac{\partial z}{\partial x} = -\frac{y}{x^2+y^2}$, $\frac{\partial z}{\partial y} = \frac{x}{x^2+y^2}$

(6) $\frac{\partial u}{\partial x} = (3x^2+y^2+z^2)e^{x(x^2+y^2+z^2)}$, $\frac{\partial u}{\partial y} = 2xye^{x(x^2+y^2+z^2)}$, $\frac{\partial u}{\partial x} = 2xze^{x(x^2+y^2+z^2)}$

(7) $\frac{\partial u}{\partial x} = x^{-1+yz}y^z$, $\frac{\partial u}{\partial y} = ln(x)zx^{y^z}y^{-1+z}$, $\frac{\partial u}{\partial z} = ln(x)ln(y)x^{y^z}y^z$

(8) $\frac{\partial u}{\partial x} = e^{-z} + \frac{1}{x+ln(y)}$, $\frac{\partial u}{\partial y} = \frac{1}{y(x+ln(y))}$, $\frac{\partial u}{\partial z} = 1 - xe^{-z}$

3:

$$\frac{\partial f}{\partial x} = \frac{\sin x^2y}{x^2y}2xy = \frac{2\sin x^2y}{x}, \frac{\partial f}{\partial y} = \frac{\sin x^2y}{x^2y}x^2 = \frac{\sin x^2y}{y}$$

4: 直接计算:

$$\frac{\partial f}{\partial x} = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = 0,$$

$$\frac{\partial f}{\partial y} = \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0}\frac{y\sin\frac{1}{y^2}}{y} = \lim_{y\to 0}\sin\frac{1}{y^2}\text{不存在.}$$

5: 点(x,y)与$(0,0)$的距离$\rho = \sqrt{x^2+y^2}$.只需令$\delta = \epsilon$,那么当$\rho < \delta$时,有$|z(x,y) - z(0,0)| = \rho < \delta = \epsilon$,即$z(x,y)$在$(0,0)$处连续.

由于

$$\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = \frac{|x|}{x}$$

极限不存在,故$z(x,y)$在$(0,0)$处对x偏导数不存在，对y偏导数同理

6: 解:

由题意可设切线的方向向量为$\vec{\tau} = (x_0, 0, z_0)$，且有$\frac{\partial z}{\partial x} = \frac{1}{2}x$，故在$(2,4,5)$点取方向向量为$\vec{\tau} = (1,0,1)$。而$Ox$轴正向单位方向向量为$\vec{n} = (1,0,0)$，则有:

$$\theta = \arccos\frac{\vec{\tau}\cdot\vec{n}}{|\vec{\tau}|\,|\vec{n}|} = \frac{\pi}{4}$$