$(0,0,1)$

$\therefore \cos\frac{\pi}{3} = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$,即$\frac{3m}{\sqrt{m^2+(3m)^2+9}} = \frac{1}{2}$

$\therefore m = \pm\frac{3\sqrt{26}}{26}$

$\therefore$平面方程为

$$x + \sqrt{26}y + 3z - 3 = 0 \quad 或 \quad x - \sqrt{26}y + 3z - 3 = 0$$

15: (1).直线方向向量为$(1,0,2) \times (0,1,-3) = (-2,3,1)$ 从而直线方程为$\frac{x}{-2} = \frac{y-2}{3} = z - 4$

(2).直线方向向量为$(1,1,-2) \times (1,2.,-1) = (3,-1,1)$ 从而直线方程为$\frac{x+1}{3} = -y + 2 = z - 1$

(3).$\frac{x-2}{2} = \frac{y+3}{-3} = \frac{z-4}{0}$

(4).先计算两条直线的方向向量，分别为$\overrightarrow{a} = (-3,1,10)$与$\overrightarrow{b} = (4,-1,2)$

则l具有方向向量$\overrightarrow{l} = \overrightarrow{a} \times \overrightarrow{b} = (12,46,-1)$

从而方程为$\frac{x+1}{12} = \frac{y+4}{46} = \frac{z-3}{-1}$

16: 两平面的方向量为$\boldsymbol{n}_1 = (2,3,-1), \boldsymbol{n}_2 = (3,-5,2)$,则直线的方向向量为$\boldsymbol{v} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = (1,-7,-19)$.又有$(1,0,-2)$为直线上的一点，可求出此直线的参数方程为

$$\frac{x-1}{1} = \frac{y}{-7} = \frac{z+2}{-19}$$

17: (1)令

$$\frac{x-1}{1}+\frac{y+1}{-2}+\frac{z}{6}=t \tag{8.3}$$

代入得

$$2(t+1)+3(-2t-1)+6t-1=0, \quad \Rightarrow t=1. \tag{8.4}$$

因此$x=2, y=-3, z=6$。

(2)令

$$\frac{x+2}{-2}+\frac{y-1}{3}+\frac{z-3}{2}=t \tag{8.5}$$

代入得

$$-2t-2+2(3t+1)-2(2t+3)+6=0. \tag{8.6}$$

得知t任意都成立，所以直线在平面上。

18: (1)$\frac{\pi}{2}$, (2)$\frac{2\pi}{3}$

19: (1):

l_1=n_1×n_2=(-6,4,-2)

直线l_2=(3,-2,1),两向量平行，所以两直线平行

在两直线上分别取一点A=(-2,1,0),B=(1,-2,-1)

距离d=$\frac{|\vec{AB}\times l_2|}{|l_2|}$=$\sqrt{5}$

(2):

同理可得两直线平行

在两直线上分别取一点A=(-7,5,9),B=(0,-4,-18)

距离d=$\frac{|\vec{AB}\times l_2|}{|l_2|}$=25

20: (1)两直线点向式方程分别为$-x+1=2y=z+1$和$x=y+1=2z+3$. 方向向量$\vec{v_1}=(-1,\frac{1}{2},1)$与$\vec{v_2}=(1,1,\frac{1}{2})$垂直, 交点$(1,0,-1)$.

(2)两直线的方向向量$\vec{v_1}=(1,-4,0)$与$\vec{v_2}=(0,0,1)$垂直, 交点$(-2,-1,5)$.

21: (1)直线的点向式方程为

$$\frac{x}{2}=\frac{y+12}{3}=\frac{z-4}{6}$$

所以该直线的方向向量为$\boldsymbol{l}=(2,3,6)$,平面法向量为$\boldsymbol{n}=(6,15,-10)$,用书上公式,平面与直线的夹角$\phi$满足

$$sin\phi=\frac{|\boldsymbol{n}\cdot\boldsymbol{l}|}{|\boldsymbol{n}|\,|\boldsymbol{l}|}=\frac{3}{133}$$

(2)直线的点向式方程为

$$\frac{x}{-2}=\frac{y}{-4}=\frac{z}{2}$$

所以该直线的方向向量为$\boldsymbol{l}=(-1,-2,1)$,平面法向量为$\boldsymbol{n}=(1,-1,-1)$,用书上公式,平面与直线的夹角ϕ满足

$$sin\phi=\frac{|\boldsymbol{n}\cdot\boldsymbol{l}|}{|\boldsymbol{n}|\,|\boldsymbol{l}|}=0$$

故直线与平面平行

22: 解:

(1)直线过点$P_1(0,1,0)$,方向向量为$\vec{u}=(1,-2,1)$

$\therefore P_1\vec{P}_0=(1,-1,-1)$

$\therefore$距离$d=\frac{\left|P_1\vec{P}_0\times\vec{u}\right|}{|\vec{u}|}=\sqrt{3}$

(2)直线过点$M_1(1,1,1)$,方向向量为$\vec{u}=(1,1,1)\times(2,0,1)=(1,-3,-2)$

$\therefore M_1\vec{M}_0 = (0,1,2)$
$\therefore$距离$d = \frac{\left|M_1\vec{M}_0 \times \vec{u}\right|}{|\vec{u}|} = \frac{\sqrt{6}}{2}$

23: 记P，Q分别为两直线上的一点，$\vec{m}, \vec{n}$分别两直线的方向向量，则两直线的距离$d = \frac{|\vec{m} \times \vec{n} \cdot \overrightarrow{PQ}|}{|\vec{m} \times \vec{n}|}$

(1).(9,-2,0)，(0,-7,2)分别为两直线上的一点，$(4,-3,1),(-2,9,2)$分别两直线的方向向量，则两直线的距离$d = \frac{|(4,-3,1)\times(-2,9,2)\cdot(-9,-5,2)|}{|(4,-3,1)\times(-2,9,2)|} = 7$

(2).(1,0,0)，(0,0,-2)分别为两直线上的一点，$(1,1,-1) \times (2,1,-1) = (0,-1,-1), (1,2,-1) \times (1,2,2) = (6,-3,0)$分别两直线的方向向量，则两直线的距离$d = \frac{|(0,-1,-1)\times(6,-3,0)\cdot(-1,0,-2)|}{|(0,1,-1)\times(6,-3,0)|} = 1$

24: 直线方向向量

$$\boldsymbol{v} = (3,2,-1) \times (2,-3,2) = (1,-8,-13)$$

因为平面通过直线，所以$\boldsymbol{n} \perp \boldsymbol{n}$.又因为与法向量为$n_1 = (1,2,3)$的平面垂直，则$\boldsymbol{n} \perp \boldsymbol{n}_1$所以

$$\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{n}_1 = (2,-16,10)$$

取直线上一点$(0,0,-1)$，则平面也过此点。

设平面方程为$2x - 16y + 10z + d = 0$.代入点$(0,0,-1)$，得$d = 10$.整理后得待求直线方程为

$$x - 8y + 5z + 5 = 0$$

25: 直线$x=3t+1, y=2t+3, z=-t-2$，方向向量为$\vec{n_1}=(3,2,-1)$，其平行于平面。直线

$$\begin{cases} 2x-y+z-3 &= 0 \\ x+2y-z-5 &= 0 \end{cases} \tag{8.7}$$

方向向量为$\vec{n_2}=(-1,3,5)$，其也平行于平面。所以平面的法向量$\vec{n_1}\times\vec{n_2}=(13,-14,-11)$。令$t=0$，则平面经过点(1,3,-2)。所以平面方程为$13x-14y-11z+7=0$。

26: $2+6x+8y-2z=0$,$-x+2y+5z-3=0$

27: 投影：过点(-1,2,0)且与平面垂直的直线与平面的交点
直线的方向向量l=n=(1,2,-1)
将直线的参数方程带入平面方程，可得交点坐标为(-5/3,2/3,2/3)

28: 直线的方向向量$\vec{v}=(1,2,3)$, 设点$A(2,3,1)$, 在直线上的投影$P(t-7,2t-2,3t-2)$, 则由$\overrightarrow{PA}\cdot\vec{v}=0$得$P(-5,2,4)$.

29: 该平面的法向量为$(6,2,-9)$,设对称点为(x,y,z),则有

$$(x,y,z)-(0,0,0)=t(6,2,-9)$$

由于$(0,0,0)$到该平面的距离为

$$d=\frac{|ax+by+cz+d|}{\sqrt{a^2+b^2+c^2}}=\frac{121}{11}=11$$

所求对称点$(6t,2t,-9t)$到该平面的距离也应为11，故

$$d=\frac{|36t+4t+81t+121|}{11}=11$$

解得$t=-2$(舍0),故对称点为$(-12,-4,18)$

30: 解：过点$(1,2,3)$且与直线垂直的平面方程为$(x-1)-3(y-2)-2(z-3)=0$,即$x-3y-2z+11=0$

联立直线、平面方程得$x=\frac{1}{2},y=\frac{5}{2},z=2,$

$\therefore$根据中点坐标公式，知对称点坐标为$(0,3,1)$

31: 两直线上的点可分别设为$(1+\lambda u,-4+5u,3-3u)$和$(-3+3v,9-4v,-14+7v)$

令$(1+\lambda u,-4+5u,3-3u)=(-3+3v,9-4v,-14+7v)$

则可解得$\begin{cases}\lambda=2\\ u=1\\ v=2\end{cases}$

从而$\lambda=2$，交点为(3,1,0)

两直线方向向量分别为(2,5,-3)和(3,-4,7)

则平面法向量为$(2,5,-3)\times(3,-4,7)=(23,-23,-23)$

从而平面方程为x-y-z-2=0

32: 设题中给出的两条直线为l_1,l_2，取l_1上一点$A(0,5,-3)$，取l_2上一点$B(0,-7,10)$

l_1方向向量：$\boldsymbol{v}_1=(3,-1,0)\times(2,0,-1)=(1,3,2)$

l_2方向向量：$\boldsymbol{v}_2=(4,-1,0)\times(5,0,-1)=(1,4,5)$

因此l_1上的点可表示为

$$(0,5,-3)+t_1\boldsymbol{v}_1=(t_1,5+3t_1,-3+2t_1)$$

l_2上的点可表示为

$$(0,-7,10)+t_2\boldsymbol{v}_2=(t_2,-7+4t_2,10+5t_2)$$

待求直线l与直线l_1,l_2相交，则t_1,t_2应满足$(0,5,-3)+t_1\boldsymbol{v}_1=(t_1,5+3t_1,-3+2t_1)$,$(0,-7,10)+t_2\boldsymbol{v}_2=(t_2,-7+4t_2,10+5t_2)$,$(-3,5,-9)$三点共线，则

$$\frac{t_1+3}{t_2+3}=\frac{5+3t_1-5}{-7+4t_2-5}=\frac{-3+2t_1-(-9)}{10+5t_2-(-9)}$$

即

$$\frac{t_1+3}{t_2+3}=\frac{3t_1}{4t_2-12}=\frac{2t_1+6}{5t_2+19}$$

得到两组解

$$t_1=-3,t_2=-3$$

或

$$t_1=-\frac{66}{19},t_2=-\frac{13}{3}$$

对于第一组解，直线方程为

$$\begin{cases}x=-3\\y=-9\end{cases}$$

对于第二组解，直线方程为

$$\frac{x+3}{1}=\frac{y-5}{22}=\frac{z+9}{2}$$

33: 先求直线与平面的交点坐标Q，联立方程

$$\begin{cases} x+y-z-1 &= 0 \\ x-y+z+1 &= 0 \\ x+y+z &= 0 \end{cases} \tag{8.8}$$

求得$(0, \frac{1}{2}, -\frac{1}{2})$。任取直线上任意一点$A(0,0,-1)$。设在平面上的射影$B(u,v,w)$。由于$AB$垂直于平面，而平面法向量为(1,1,1)，所以$AB$直线方程为

$$\frac{x}{1} = \frac{y}{1} = \frac{z+1}{1} \tag{8.9}$$

那么(u,v,w)满足下面方程

$$u+v+w = 0 \tag{8.10}$$

$$\frac{u}{1} = \frac{v}{1} = \frac{w+1}{1} \tag{8.11}$$

所以解得$u = \frac{1}{4}, v = \frac{1}{4}, w = -\frac{1}{2}$。所以求得的直线方程为

$$\frac{x}{\frac{1}{4}} = \frac{y - \frac{1}{2}}{-\frac{1}{4}} = \frac{z + \frac{1}{2}}{0} \tag{8.12}$$

34: 平面5x-y+3z-2=0与Oxy平面的交线l:5x-y-2=0,z=0.平面的法向量$\vec{n_1} = (5,-1,3)$,直线的方向向量$\vec{n_2} = (\frac{1}{5},1,0)$,所求平面法向量为$\vec{n} = \vec{n_1} \times \vec{n_2} = (-15,3,26)$.因为平面过点(0,2,0),所以方程为$-15x+3y+26z+6=0$

35: 按照题28的思路可得点到直线的垂足的坐标为($\frac{33}{29}$,-$\frac{26}{29}$,$\frac{27}{29}$)
所以垂线方程为$\frac{x}{33}$=$\frac{y}{-26}$=$\frac{z}{27}$

8.3 二次曲面

1: (1)椭圆$\frac{x^2}{4}+\frac{r^2}{9}=1$绕x轴旋转，椭球面

(2)圆绕x,y,z轴其中一轴旋转，是球面

(4)双曲线$r^2-\frac{y^2}{4}=1$绕y轴旋转，单叶双曲面

(6)双曲线$x^2-r^2=1$绕x轴旋转，双叶双曲面

(7)抛物线$r^2=4z$绕z轴旋转，抛物面

2: (1)都是直线(2)都是直线(3)Oxy坐标系为圆，Oxyz坐标系为圆柱(4)Oxy坐标系为双曲线，Oxyz坐标系为双曲柱面(5)Oxy坐标系为抛物线，Oxyz坐标系为抛物柱面(6)Oxy坐标系为两个点，Oxyz坐标系为两条直线(7)Oxy坐标系为一个点，Oxyz坐标系为一条直线(8)Oxy坐标系为一个点，Oxyz坐标系为一条直线

3: (1)$x^2+y^2-\frac{z^2}{4}=1$，为单叶双曲面

(2)$\sqrt{(y^2+z^2)}=sinx(0\le x\le\pi)$，名称未知

(3)$4x^2+9y^2+4z^2=36$，为椭球面

4: Oyz平面上的直线$y=z$的方向向量为$\vec{v}=(0,1,1)$, 与这条直线垂直的平面为$y+z-t=0,t\in\mathbb{R}$. 平面与Oyz平面上的直线$y-2z+1=0$的交点为$A(0,\frac{2t-1}{3},\frac{t+1}{3})$, 平面上的点$P(x,y,z)$是旋转面上的点意味着$|\overrightarrow{OA}\times\vec{v}|=|\overrightarrow{OP}\times\vec{v}|$, 即$x^2+4y^2+4z^2-10yz+2y+2z-2=0$.

5: (1)平面π的法向量为$\boldsymbol{n}=(1,-1,2)$,直线L的方向向量为$\boldsymbol{l}=(1,1,-1)$,因此过L且与π垂直的平面的法向量$\boldsymbol{n'}=\boldsymbol{n}\times\boldsymbol{l}=(1,-3,-2)$,
投影直线L_0的方向向量$\boldsymbol{l_0}=\boldsymbol{n'}\times\boldsymbol{n}=(-4,-2,1)$,联立$L$与$\pi$的方程,可解得交点为$M(2,1,0)$,$M$在$L_0$上,故$L_0$方程为

$$\frac{x-2}{-4}=\frac{y-1}{-2}=\frac{z}{1}$$

(2)L_0绕y轴旋转,所得曲面上点$P(x,y,z)$对应子午线上$(2y,y,\frac{1-y}{2})$,由距离关系

$$x^2+y^2+z^2=(2y)^2+y^2+\left(\frac{1-y}{2}\right)^2$$

整理得曲面方程

$$4x^2-17y^2+4z^2+2y-1=0$$

6: 略

7: (1).将z=0代入得$x^2-y^2=9$

从而截痕为双曲线

(2).将x=0代入得$y^2+z^2=-9$

从而无截痕

(3).将y=0代入得$x^2-z^2=9$

从而截痕为双曲线

(4).将x=5代入得$y^2+z^2=16$

从而截痕为圆

8: 设动点坐标为(x,y,z)，由题意有

$$\sqrt{x^2+y^2+z^2}=|z-4|$$

两边平方，整理得

$$x^2 + y^2 + 8z - 16 = 0$$

可看出是抛物面

9: 联立两曲面方程，消去z即得母线平行于z轴的柱面方程$5x^2 - 3y^2 = 1$

10: 设球心$(0, 0, z_0)$,半径R。构造方程：$3^2 + (z_0 - 1)^2 = z_0^2 + 16$得$z_0 = -3, R = 5$。球的方程为$x^2 + y^2 + (z + 3)^2 = 25$

11: 利用两个方程消去z有$\frac{x^2}{16} + \frac{y^2}{4} - \frac{(x+3)^2}{20} = 1$,即$\frac{(x-12)^2}{260} + \frac{y^2}{13} = 1$

12: 在平面中$xy = h$在$h > 0$时表示位于一, 三象限的双曲线; 在$h < 0$时表示位于二, 四象限的双曲线, 在$h = 0$时表示两条坐标轴. 在空间坐标系中$xy = z$是马鞍面, 用平面$z = h$去截这个曲面, 在$h > 0$和$h < 0$时得到不同方向的双曲线，在$h = 0$时得到x, y坐标轴.

8.4 坐标变换和其他常用坐标系

1: $(x - 1)^2 - (y - 1)^2 - (z - 1/2)^2 = 3/4$，双叶双曲面

2: 平移、旋转变换可得$z = \frac{y^2}{2} - \frac{x^2}{2}$，双曲抛物面。

3: 解：此题需要对二次曲面进行化简，参考文献见链接。为消去yz项，令$\cot 2\theta = \frac{-3-3}{8} = \frac{-3}{4}$,取$\tan\theta = 2$,且$\theta$为锐角,做变换

$$\begin{cases} x = x' \\ y = \frac{1}{\sqrt{5}}y' - \frac{2}{\sqrt{5}}z' \\ z = \frac{2}{\sqrt{5}}y' + \frac{1}{\sqrt{5}}z' \end{cases}$$

代入原方程并化简得：$x'^2 + y'^2 - z'^2 = 1$，表示一个单叶双曲面。

4: (1)$r^2\cos 2\theta = 25, r^2\sin^2\theta\cos 2\phi = 25$;

(2)$r^2 + 4z^2 = 10, r^2\cos^2\theta = 3$;

(3)$\sin^2\theta = 2\cos^2\theta$;

(4)$r^2\sin^2\theta\cos 2\phi - r^2\cos^2\theta = 1$;

(5)$r^2\cos^2\theta = 3$;

(6)$r^2 + z^2 = 2z$;

(7)$r(\cos\theta + \sin\theta) = 4$;

(8)$r(\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta) = 1$;

(9)$x^2 + y^2 = 2y$;

(10)$x^2 - y^2 = z$;

(11$x^2 + y^2 = 1$.

5: 旋转后的方程为$z = 2(x^2 + y^2)$,柱面坐标系中$x^2 + y^2 = r^2$,故方程为$z = 2r^2$

6: 设曲面为S, 双曲线为Γ, $\forall(r,\theta,z)\in S$, $\exists(r,0,z)\in\Gamma \Leftrightarrow 2r^2-z^2=2$. 故曲面方程为$S: 2r^2-z^2=2$

7: $\begin{cases} x = x_0 + a\cos t \\ y = y_0 + b\sin t \end{cases} \quad t\in[0,2\pi)$

8:

$$\begin{cases} x = x_0 + a\sin u\cos v \\ y = y_0 + \sin u\sin v \\ z = z_0 + a\cos u \end{cases}$$

9: 类比球面的参数表示

$x = a\sin\theta\cos\phi; y = b\sin\theta\sin\phi; z = c\cos\theta$

$0\le\theta\le\pi; 0\le\phi\le 2\pi$

10: $\begin{cases} x = a\cosh t \\ y = b\sinh t \end{cases}$

11: 解:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

$$\begin{cases} x = a\cosh\theta\cos\varphi \\ y = b\cosh\theta\sin\varphi, \theta\in(-\infty,+\infty), \varphi\in[0,2\pi) \\ z = c\sinh\theta \end{cases}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$

$$\begin{cases} x = a\sinh\theta\cos\varphi \\ y = b\sinh\theta\sin\varphi, \theta \in [0, +\infty), \varphi \in [0, 2\pi) \\ z = c \pm \cosh\theta \end{cases}$$

用三角函数也可$\sec^2\theta - \tan^2\theta = 1$,双曲函数则是$\cosh^2\theta - \sinh^2\theta = 1$

8.5 综合习题

1: It's easy to check "M on a plane ABC" $\Leftrightarrow \overrightarrow{AM} = \mu_1 \overrightarrow{AB} + \mu_2 \overrightarrow{AC} \Leftrightarrow$ The conclusion

2: 略

3: 任取三维空间中的四个向量

A=(a1,a2,a3) B=(b1,b2,b3)

C=(c1,c2,c3) D=(d1,d2,d3)

可得矩阵

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix} \tag{8.13}$$

矩阵的秩≤3¡4=向量的个数

由线性代数知识可得，这四个向量线性相关

所以三维空间中的任意四个向量必定线性相关

4: (1)系数行列式$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \end{vmatrix} = 3.$ 非退化.

(2) $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & x & 2 \end{vmatrix} = 0 \implies x = 1.$

5: (1)

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}|\, sin\theta = 1 \times 2 \times \left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

所以,

$$|\boldsymbol{a} \times \boldsymbol{b}|^2 = 3$$

(2)

$$|(\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{a} - \boldsymbol{b})| = |(\boldsymbol{a} + \boldsymbol{b})|\,|(\boldsymbol{a} - \boldsymbol{b})|\, sin\phi$$

分别计算

$$|(\boldsymbol{a} + \boldsymbol{b})| = \sqrt{(\boldsymbol{a} + \boldsymbol{b})^2} = \boldsymbol{a}^2 + \boldsymbol{b}^2 + 2\boldsymbol{ab} = 7$$

$$|(\boldsymbol{a} - \boldsymbol{b})| = \sqrt{(\boldsymbol{a} - \boldsymbol{b})^2} = \boldsymbol{a}^2 + \boldsymbol{b}^2 - 2\boldsymbol{ab} = 3$$

$$cos\phi = \frac{(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b})}{|\boldsymbol{a} + \boldsymbol{b}|\,|\boldsymbol{a} - \boldsymbol{b}|} = \frac{\boldsymbol{a}^2 - \boldsymbol{b}^2}{|\boldsymbol{a} + \boldsymbol{b}|\,|\boldsymbol{a} - \boldsymbol{b}|} = -\frac{\sqrt{21}}{7}$$

故

$$|(\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{a} - \boldsymbol{b})| = |(\boldsymbol{a} + \boldsymbol{b})|\,|(\boldsymbol{a} - \boldsymbol{b})|\, sin\phi = \sqrt{7} \times \sqrt{3} \times \frac{2\sqrt{7}}{7} = 2\sqrt{3}$$

6: 若$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$共面，则等式显然成立.若三者不共面，则构成空间中的一组基.利用混合积的轮换性，有:

$$\begin{aligned}&\boldsymbol{a}\cdot((\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}+(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}+(\boldsymbol{c}\times\boldsymbol{a})\times\boldsymbol{b})\\=&(\boldsymbol{a}\times(\boldsymbol{a}\times\boldsymbol{b}))\cdot\boldsymbol{c}+\boldsymbol{0}+(\boldsymbol{a}\times(\boldsymbol{c}\times\boldsymbol{a}))\cdot\boldsymbol{b}\\=&\boldsymbol{0}\end{aligned}$$

同理有:

$$\boldsymbol{b}\cdot((\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}+(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}+(\boldsymbol{c}\times\boldsymbol{a})\times\boldsymbol{b})=\boldsymbol{0}$$

$$\boldsymbol{c}\cdot((\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}+(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}+(\boldsymbol{c}\times\boldsymbol{a})\times\boldsymbol{b})=\boldsymbol{0}$$

由$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$构成空间中的一组基可知，原式为$\boldsymbol{0}$.

7: 在准线上任取一点A(x_0,y_0,z_0)

在柱面上任取一点$B(x,y,z)$且B为A沿母线移动后得到

则有$\begin{cases}x=x_0+2t\\y=y_0+t\\z=z_0+t\end{cases}$

由于$\begin{cases}y_0^2+z_0^2=1\\x_0=1\end{cases}$

则$\begin{cases}(y-t)^2+(z-t)^2=1\\x-2t=1\end{cases}$

消去t得$(y-\frac{x-1}{2})^2+(z-\frac{x-1}{2})^2=1$

亦即$x^2+2y^2+2z^2-2xy-2xz-2x+2y+2z-1=0$

8: 设顶点为$P_0(2,1,1)$，任取锥面上一点$P(x,y,z)$，直线PP_0与准线相交于$P_1(x_1,y_1,z_1)$，满足

$$\boldsymbol{P_0P_1} = t\boldsymbol{P_0P}$$

即

$$\begin{cases} x_1 = 2+(x-2)t \\ y_1 = 1+(y-1)t \\ z_1 = 1+(z-1)t \end{cases}$$

将准线方程表达式代入此方程组，得到

$$\begin{cases} x\left[1+(y-1)t\right]^2 + \left[1+(z-1)t\right]^2 = 1 \\ 2+(x-2)t = 1 \end{cases}$$

消去t，得

$$2y^2 + z^2 - 2xy - 2yz + 2x + 4y - 2z - 2 = 0$$

此为待求锥面的一般方程。

9: Select a point (x_1,y_1,z_1) on the line $\frac{x-1}{1} = \frac{y}{1} = \frac{z}{1}$ It's easy to select a point $(1,1,0)$ on $\frac{x-1}{0} = \frac{y-1}{0} = \frac{z}{1}$, then latitude circle can be represented by a sphere whose centre is $(1,1,0)$ intersect a plane.
The equation of sphere is $(x-1)^2 + (y-1)^2 + z^2 = (x_1-1)^2 + (y_1-1)^2 + z_1^2$ and the plane is $z - z_1 = 0$, then the equation of latitude circle is

$$\begin{cases} (x-1)^2 + (y-1)^2 + z^2 = (x_1-1)^2 + (y_1-1)^2 + z_1^2 \\ z - z_1 = 0 \end{cases}$$

since (x_1,y_1,z_1) on the line $\frac{x-1}{1} = \frac{y}{1} = \frac{z}{1}$, then $\begin{cases} x_1 - 1 = y_1 \\ y_1 = z_1 \end{cases}$

The above four equations eliminate x_1, y_1, z_1, we get the equation of revolution surface

$$(x-1)^2+(y-1)^2-2(z-\frac{1}{2})^2=\frac{1}{2}$$

The parametric equation is $\begin{cases} x=\frac{\sqrt{2}}{2}\sec\theta\cos\varphi+1 \\ y=\frac{\sqrt{2}}{2}\sec\theta\sin\varphi+1 \\ z=\frac{1}{2}\tan\theta+\frac{1}{2} \end{cases}$

10: 参数方程 $\begin{cases} x=(cos\theta+2)\cos\varphi \\ y=sin\theta \\ z=(cos\theta+2)\sin\varphi \end{cases}$ 一般方程 $(\sqrt{x^2+z^2}-2)^2+y^2=1$

11: 根据题意可得如下表格(坐标系绕e1旋转)

	e1	e2	e3
e11	0	π/2	π/2
e22	π/2	α	π/2-α
e33	π/2	π/2+α	α

坐标(e1,e2,e3)与坐标(e11,e22,e33)的变换公式如下

$$\begin{pmatrix} e11 \\ e22 \\ e33 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\alpha & sin\alpha \\ 0 & -sin\alpha & cos\alpha \end{pmatrix}\begin{pmatrix} e1 \\ e2 \\ e3 \end{pmatrix} \tag{8.14}$$

坐标系绕e22旋转时可得类似表格

	e1	e2	e3
e11	β	$\pi/2$	$\pi/2+\beta$
e22	$\pi/2$	0	$\pi/2$
e33	$\pi/2$-β	$\pi/2$	β

坐标(e11,e22,e33)与坐标(e111,e222,e333)的变换公式如下

$$\begin{pmatrix} e111 \\ e222 \\ e333 \end{pmatrix} = \begin{pmatrix} cos\beta & 0 & -sin\beta \\ 0 & 1 & 0 \\ sin\beta & 0 & cos\beta \end{pmatrix} \begin{pmatrix} e11 \\ e22 \\ e33 \end{pmatrix} \tag{8.15}$$

综上坐标(e1,e2,e3)与坐标(e111,e222,e333)的变换公式如下

$$\begin{pmatrix} e111 \\ e222 \\ e333 \end{pmatrix} = \begin{pmatrix} cos\beta & sin\alpha sin\beta & -cos\alpha sin\beta \\ 0 & cos\alpha & sin\alpha \\ sin\beta & -sin\alpha cos\beta & cos\alpha cos\beta \end{pmatrix} \begin{pmatrix} e1 \\ e2 \\ e3 \end{pmatrix} \tag{8.16}$$

注：若第二步是绕e2旋转，则坐标变换公式为

$$\begin{pmatrix} e111 \\ e222 \\ e333 \end{pmatrix} = \begin{pmatrix} cos\beta & 0 & -sin\beta \\ sin\alpha sin\beta & cos\alpha & sin\alpha cos\beta \\ cos\alpha sin\beta & -sin\alpha & cos\alpha cos\beta \end{pmatrix} \begin{pmatrix} e1 \\ e2 \\ e3 \end{pmatrix} \tag{8.17}$$

坐标变换矩阵的逆为旋转变换，可通过对应坐标轴的坐标改变求出坐标系的变换矩阵

12: 对称中心为$P(2,2,2)$, 三条直线的单位方向向量分别为$\vec{v_1}=(-\frac{1}{3},\frac{2}{3},\frac{2}{3})$,$\vec{v_2}=(\frac{2}{3},-\frac{1}{3},\frac{2}{3})$,$\vec{v_3}=(\frac{2}{3},\frac{2}{3},-\frac{1}{3})$. 那么椭球面上的点$A(x,y,z)$满足

$$\frac{(\overrightarrow{PA}\cdot\vec{v_1})^2}{a^2}+\frac{(\overrightarrow{PA}\cdot\vec{v_2})^2}{b^2}+\frac{(\overrightarrow{PA}\cdot\vec{v_3})^2}{c^2}=1.$$

即

$$\frac{(-x+2y+2z-6)^2}{9a^2}+\frac{(-y+2z+2x-6)^2}{9b^2}+\frac{(-z+2x+2y-6)^2}{9c^2}=1.$$

13:

$$\begin{aligned}
&\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\\
=&\sqrt{(\rho_2 sin\theta_2 cos\phi_2-\rho_1 sin\theta_1 cos\phi_1)^2+(\rho_2 sin\theta_2 sin\phi_2-\rho_1 sin\theta_1 sin\phi_1)^2+(\rho_2 cos\theta_2-\rho_1 cos\theta_1)^2}\\
=&\sqrt{\rho_1^2+\rho_2^2-2\rho_1\rho_2\left(sin\theta_1 sin\theta_2 cos\phi_1 cos\phi_2+sin\theta_1 sin\theta_2 sin\phi_1 sin\phi_2+cos\theta_1 cos\theta_2\right)}\\
=&\sqrt{\left(\rho_1-\rho_2\right)^2+2\rho_1\rho_2\left[1-cos\left(\phi_1-\phi_2\right)sin\theta_1 sin\theta_2-cos\theta_1 cos\theta_2\right]}
\end{aligned}$$

注：此题题中θ与ϕ记号应是弄反

14: 两点之间的球面距离即为连接两点的大圆的劣弧长，由大圆的半径等于球面半径a以及弧长公式知，只需验证γ为两点的位置向量的夹角即可.

$$\begin{aligned}
\cos\gamma=&\frac{(a\sin\theta_1\cos\phi_1,a\sin\theta_1\sin\phi_1,a\cos\theta_1)\cdot(a\sin\theta_2\cos\phi_2,a\sin\theta_2\sin\phi_2,a\cos\theta_2)}{a\cdot a}\\
=&(\cos\phi_1\cos\phi_2+\sin\phi_1\sin\phi_2)\sin\theta_1\sin\theta_2+\cos\theta_1\cos\theta_2\\
=&\cos(\phi_1-\phi_2)\sin\theta_1\sin\theta_2+\cos\theta_1\cos\theta_2
\end{aligned}$$

因此书中题目可能有误.

Chapter 9

多变量函数的微分学

9.1 多变量函数及其连续性

1:

$$\forall x \in (A \cap B)^c$$
$$\Leftrightarrow x \notin A \cap B$$
$$\Leftrightarrow x \notin A\text{或}x \notin B$$
$$\Leftrightarrow x \in A^c\text{或}x \in B^c$$
$$x \in (A \cup B)^c$$
$$\Leftrightarrow x \notin A \cup B$$
$$\Leftrightarrow x \notin A\text{且}x \notin B$$
$$\Leftrightarrow x \in A^c\text{且}x \in B^c$$
$$\Leftrightarrow x \in A^c \cap B^c$$

2: 略

3: 记满足y¿ax+b的所有点(x,y)组成的集合为E，任取E中一点M
显然,存在正数r, 使得B(M,r)⊆E
所以M是E的内点，由M的任意性可得，E为开集
绘图略，边界点满足的关系是y=ax+b

4: 由三角不等式

$$\rho(M_0,M_0')-\rho(M_n,M_0)-\rho(M_n',M_0') \le \rho(M_n,M_n') \le \rho(M_0,M_0')+\rho(M_n,M_0)+\rho(M_n',M_0').$$

从而根据夹逼准则得到$\lim \rho(M_n,M_n') = \rho(M_0,M_0')$.

5: 假设点列$\{M_n\}$是平面上的点列,收敛到点M_0,由点列收敛的定义,$\forall\epsilon > 0, \exists N \in \mathbb{N}, s.t. \forall n > N, \rho(M_n,M_0) < \epsilon$,即$M_n \subset B(0,\rho(M_0,0)+\epsilon), n > N$,取$R_1 = \rho(M_0,0)+\epsilon, R_2 = max\{\rho(M_i,0), i \le N\}$,令$R = max\{R_1,R_2\}$,则$M_n \subset B(0,R), \forall n$,故$\{M_n\}$为有界数列.

6: 若$\exists\gamma : [a,b] \to E$为连续函数s.t. $\gamma(a) = (0,0), \gamma(b) = (\frac{2}{\pi},1)$.取$c = \sup\{a \le t \le b : x(\gamma(t)) = 0\}$, 则$a \le c < b$. 取$t \to c^+$, 由$\gamma$的连续性知$\lim_{t\to c+} \sin\frac{1}{x(t)} = y(c)$.由$x(t) \to x(c) = 0$知前者极限是不存在的，得到矛盾.因而$E$不是道路连通的.
记$E_1 = \{a \in E : x(a) = 0\}, E_2 = \{a \in E : x(a) > 0\}$,$E_1,E_2$均为道路连通从而为连通集.若存在$U_1,U_2 \subset E, U_1 \ne \emptyset, U_2 \ne \emptyset, E = U_1 \cup U_2, U_1 \cap \overline{U_2} =$

$\overline{U_1} \cap U_2 = \emptyset$, 由$E_1, E_2$各自的连通性知，只可能为$U_1 = E_1, U_2 = E_2$或$U_1 = E_2, U_2 = E_1$.但$(0,0) \in E_1 \cap \overline{E_2}$,矛盾.因而$E$是连通集.

7: 以下画图均略

(1)$\{(x,y)\big|x+y \geqslant 0\}$是区域，是闭区域

(2)$\{(x,y)\big|x-2y^2 \geqslant 0\}$是区域，是闭区域

(3)$\{(x,y)\big|x^2+y^2+2x \geqslant 0$且$2x-x^2-y^2>0\} = (x,y)\big|2x-x^2-y^2>0$ 是区域，不是闭区域

(4)$\{(x,y)\big|2n\pi \leqslant x^2+y^2 \leqslant (2n+1)\pi, n \in Z\}$ 不是区域

(5)$\{(x,y)\big|\frac{x^2}{a^2}+\frac{y^2}{b^2}<1\}$是区域，不是闭区域

(6)$\{(x,y)\big|-\frac{\pi}{2}+2k_1\pi \leqslant x \leqslant \frac{\pi}{2}+2k_1\pi,$且$2k_2\pi \leqslant y \leqslant (2k_2+1)\pi, k_1 \in Z, k_2 \in Z\} \cup$

$\{(x,y)\big|\frac{\pi}{2}+2k_3\pi \leqslant x \leqslant \frac{3\pi}{2}+2k_3\pi,$且$(2k_4-1)\pi \leqslant y \leqslant 2k_4\pi, k_3 \in Z, k_4 \in Z\}$

不是区域

(7)$(x,y,z)\big|x^2+y^2 \leqslant z^2$且z$\neq 0$不是区域

(8)$(x,y,z)\big|x^2+y^2+(z-a)^2 \leqslant a^2$是区域，是闭区域

8: 图略.

等高线: $z=0: 2x+y=(k+\frac{1}{2})\pi, k \in \boldsymbol{Z}$

$z=1: 2x+y=2k\pi, k \in \boldsymbol{Z}$

$z=-1: 2x+y=(2k+1)\pi, k \in \boldsymbol{Z}$

$z=\frac{1}{2}: 2x+y=(2k\pi \pm \frac{\pi}{3}), k \in \boldsymbol{Z}$

$z = -\frac{1}{2} : 2x + y = (2k\pi \pm \frac{2\pi}{3}), k \in \boldsymbol{Z}$

9: (1) Consider the bivariate n-degree polynomials $x^r y^t$, where $r + t \leq n$

if $r = 0$ then $t \leq n$ there are $n + 1$ polynomials.

if $r = 1$ then $t \leq n - 1$ there are n polynomials.

...

if $r = n$ then $t \leq 0$ there is 1 polynomial.

then there are totally $\sum\limits_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$

(2) For the polynomial of degree n of k variable, let $p_k^{(n)}$ be the number of polynomials

then similar to the above method we have $p_k^{(n)} = \sum\limits_{i=0}^{n} p_{k-1}^{(i)}$ and $p_2^{(n)} = \frac{(n+1)(n+2)}{2} = C_{n+2}^2$

Lemma

$$\sum_{i=0}^{n} C_{m+i}^{m} = C_{m+n+1}^{m+1}$$

Proof

$\sum\limits_{i=0}^{n} C_{m+i}^{m} = C_m^m + C_{m+1}^m + ... + C_{m+n}^m = C_{m+1}^{m+1} + C_{m+1}^m + ... + C_{m+n}^m$

by the formula $C_n^m = C_{n-1}^{m-1} + C_{n-1}^m$

we have $(C_{m+1}^{m+1} + C_{m+1}^m) + C_{m+2}^m ... + C_{m+n}^m = C_{m+2}^{m+1} + C_{m+2}^m ... + C_{m+n}^m = ... = C_{m+n}^{m+1} + C_{m+n}^m = C_{m+n+1}^{m+1}$

$\square$

then $p_3^{(n)} = \sum\limits_{i=0}^{n} p_2^{(i)} = \sum\limits_{i=0}^{n} C_{i+2}^2 = C_{n+3}^3$

By the lemma, we can get $p_k^{(n)} = C_{n+k}^k$

10: $f(1,1)=1$，$f(y,x)=\frac{2xy}{x^2+y^2}$，$f(1,\frac{y}{x})=\frac{2xy}{x^2+y^2}$，$f(u,v)=\frac{2uv}{u^2+v^2}$，$f(\cos t,\sin t)=\sin 2t$

11: 根据题意可得

$$F(t)=\begin{cases}1 & 2k\pi+\pi/4\le t\le 2k\pi+5\pi/4, k\in Z,\\ 0 & else.\end{cases}$$

12: $f(2,3)=-2, f(x,y)=\frac{x^2(1-y)}{y+1}$.

13:

$$(x+y)^{x-y},\quad x^y+(x-y),\quad x+y-x^y$$

14: 1、

$$\frac{x^2+y^2}{|x|+|y|}\le\frac{(|x|+|y|)^2}{|x|+|y|}=|x|+|y|\le 2r\to 0$$

$$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2+y^2}{|x|+|y|}=0$$

2、

$$\lim_{\substack{x\to 0\\ y\to a}}\frac{\sin(xy)}{x}=\lim_{\substack{x\to 0\\ y\to a}}\frac{\sin(xy)}{xy}\cdot\lim_{\substack{x\to 0\\ y\to a}}y=1\cdot a=a$$

3、

$$(\frac{xy}{x^2+y^2})^{x^2}\le(\frac{1}{2})^{x^2}\to 0$$

$$\Rightarrow \lim_{\substack{x\to+\infty\\ y\to+\infty}}(\frac{xy}{x^2+y^2})^{x^2}=0$$

4、

$$\ln((1+\frac{1}{x})^{\frac{x^2}{x+y}}) = \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}\frac{x}{x+y} \to 1$$

$$\Rightarrow \lim_{\substack{x\to\infty\\ y\to a}}(1+\frac{1}{x})^{\frac{x^2}{x+y}} = e$$

5、

$$|\frac{x^3+y^3}{x^2+y^2}| = |\frac{r^3(\cos^3\theta+\sin^3\theta)}{r^2}| \le 2r \to 0$$

$$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3+y^3}{x^2+y^2} = 0$$

6、

$$|\frac{x^2+y^2}{x^4+y^4}| = |\frac{r^2}{r^4(\cos^4\theta+\sin^4\theta)}| \le \frac{1}{2r^2} \to 0$$

$$\Rightarrow \lim_{\substack{x\to\infty\\ y\to\infty}}\frac{x^2+y^2}{x^4+y^4} = 0$$

7、

$$(x^2+y^2)e^{-(x+y)} = r^2e^{-r(\cos\theta+\sin\theta)} \le r^2e^{-r} \to 0$$

$$\Rightarrow \lim_{\substack{x\to+\infty\\ y\to+\infty}}(x^2+y^2)e^{-(x+y)} = 0$$

8、

$$\lim_{\substack{x\to 1\\ y\to 0}}\frac{\ln(x+e^y)}{\sqrt{x^2+y^2}} = \frac{\ln(1+e^0)}{\sqrt{1+0}} = \ln 2$$

9、

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{\sqrt{xy+1}-1} = \lim_{t\to 0}\frac{t}{\sqrt{t+1}-1} = 2$$

10、

$$\frac{\sqrt{xy+1}-1}{x+y} = \frac{\sqrt{xy+1}-1}{xy}\frac{xy}{x+y}$$

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{\sqrt{xy+1}-1}{xy} = \frac{1}{2}$$

取$(x,y)=(t,-t+kt^2)$, 则当$t\to 0$时，$(x,y)\to(0,0)$,此时

$$\frac{xy}{x+y}=t-\frac{1}{k}\to-\frac{1}{k}$$

极限值与参数选取有关，因此$\lim_{\substack{x\to 0\\ y\to 0}}\frac{\sqrt{xy+1}-1}{x+y}$极限不存在

11、

$$\frac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2y^2}=\frac{1-\cos(r^2)}{r^4}\frac{r^2}{r^4\sin^2\theta\cos^2\theta}\geq\frac{1-\cos(r^2)}{r^4}\frac{1}{r^2}\to\infty$$

因此$\lim_{\substack{x\to 0\\ y\to 0}}\frac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2y^2}$极限不存在

12、

$$(1+xy)^{\frac{1}{x+y}}=(1+xy)^{\frac{1}{xy}\frac{xy}{x+y}}$$

由$\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x+y}$极限不存在知$\lim_{\substack{x\to 0\\ y\to 0}}(1+xy)^{\frac{1}{x+y}}=0$极限不存在

15: (1)

$$\lim_{\rho\to 0^+}e^{\frac{1}{x^2-y^2}}=\lim_{\rho\to 0^+}e^{\frac{1}{\rho^2 cos2\varphi}}$$

若极限存在,则$cos2\varphi<0$

又因为$0\leqslant\varphi\leqslant 2\pi$

则$\varphi\in(\frac{\pi}{4},\frac{3\pi}{4})\cup(\frac{5\pi}{4},\frac{7\pi}{4})$

(2)

$$\lim_{\rho\to+\infty}e^{x^2-y^2}sin(2xy)=\lim_{\rho\to+\infty}e^{\rho^2 cos2\varphi}sin(r^2 sin2\varphi)$$

若极限存在,则$sin2\varphi=0$或$cos2\varphi<0$

从而$\varphi\in(\frac{\pi}{4},\frac{3\pi}{4})\cup(\frac{5\pi}{4},\frac{7\pi}{4})\cup\{0,\pi,2\pi\}$