数分（B2）第10章综合习题解

1、计算

$$I=\iint_{x^2+y^2\leqslant 4}\operatorname{sgn}(y^2-x^2+2)\,\mathrm{d}x\,\mathrm{d}y$$

注: 这里与书上题目相比, 将 x 和 y 互换, 目的是为了计算方便.

解 由对称性得

$$I=4\iint_D\operatorname{sgn}(y^2-x^2+2)\,\mathrm{d}x\,\mathrm{d}y,$$

其中

$$D=\{(x,y)\mid x^2+y^2\leqslant 4,\ x\geqslant 0,\ y\geqslant 0\}.$$

两条曲线

$$\begin{cases} x^2+y^2=4,\\ y^2-x^2+2=0.\end{cases}$$

在第一象限的交点可解为 $y=1,\ x=\sqrt{3}$. 曲线 $y^2-x^2+2=0$ 将 D 分成两部分 $D=D_1\cup D_2$:

$$D_1=\{(x,y)\mid x^2+y^2\leqslant 4,\ y^2-x^2+2\geqslant 0\ x\geqslant 0,\ y\geqslant 0\}$$
$$D_2=\{(x,y)\mid x^2+y^2\leqslant 4,\ y^2-x^2+2\leqslant 0\ x\geqslant 0,\ y\geqslant 0\}.$$

所以

$$\operatorname{sgn}(y^2-x^2+2)=\begin{cases}1 & (x,y)\in D_1\\ -1 & (x,y)\in D_2\end{cases}$$

$$\begin{aligned}\Longrightarrow \iint_D\operatorname{sgn}(x^2-y^2+2)\,\mathrm{d}x\,\mathrm{d}y&=\iint_{D_1}\mathrm{d}x\,\mathrm{d}y-\iint_{D_2}\mathrm{d}x\,\mathrm{d}y\\ &=\sigma(D_1)-\sigma(D_2)=\sigma(D)-2\sigma(D_2).\end{aligned}$$

这里 $\sigma(D)$ 是以 2 为半径的四分之一个圆的面积 $\sigma(D)=\pi$, $\sigma(D_2)$ 是曲线 $x^2+y^2=4$ 和 $y^2-x^2+2=0$ 在第一象限围成的面积. 为了计算 D_2 的面积, 令

$$x=r\cos\theta,\ y=r\sin\theta.$$

在 D_2 中, $-r^2\cos 2\theta+2\leqslant 0$, 所以

$$0\leqslant\theta\leqslant\frac{\pi}{6},\quad \sqrt{\frac{2}{\cos 2\theta}}\leqslant r\leqslant 2.$$

$$\begin{aligned}
\Longrightarrow \sigma(D_2) &= \iint_{D_2} \mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{6}} \mathrm{d}\theta \int_{\sqrt{\frac{2}{\cos 2\theta}}}^{2} r\,\mathrm{d}r \\
&= \frac{1}{2}\int_0^{\frac{\pi}{6}} \mathrm{d}\theta \left(4 - \frac{2}{\cos 2\theta}\right) = \frac{\pi}{3} - \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{\mathrm{d}\varphi}{\cos\varphi} \\
&= \frac{\pi}{3} - \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{\mathrm{d}\sin\varphi}{1-\sin^2\varphi} = \frac{\pi}{3} - \frac{1}{4}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)\Big|_0^{\frac{\pi}{3}} \\
&= \frac{\pi}{3} - \frac{1}{4}\ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right).
\end{aligned}$$

最后有

$$I = 4(\sigma(D) - 2\sigma(D_2)) = \frac{4\pi}{3} + 2\ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right) = \frac{4\pi}{3} + 4\ln\left(2+\sqrt{3}\right).$$

2、计算

$$I = \iiint\limits_{[0,1]^3} \frac{\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w}{(1+u^2+v^2+w^2)^2}.$$

解 首先考虑被积函数关于区间的对称性，设

$$V = \{(u,v,w) \mid 0 \leqslant v \leqslant u \leqslant 1,\ 0 \leqslant w \leqslant 1\}$$

则原积分满足

$$I = 2\iiint\limits_{V} \frac{\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w}{(1+u^2+v^2+w^2)^2}.$$

作变量代换

$$u = r\cos\theta,\ v = r\sin\theta,\ w = \tan\varphi$$

其中变量 r,θ,φ 所在区域 V' 为

$$\sin\theta \leqslant \cos\theta \leqslant 1,\ 0 \leqslant \tan\varphi \leqslant 1,\ 0 \leqslant r \leqslant \frac{1}{\cos\theta} = \sec\theta$$

$$V' = \left\{(r,\theta,\varphi) \mid 0 \leqslant \theta \leqslant \frac{\pi}{4},\ 0 \leqslant r \leqslant \sec\theta,\ 0 \leqslant \varphi \leqslant \frac{\pi}{4}\right\}$$

这里，相当于先做一个柱坐标变换，再做变换 $w = \tan\varphi$

Jacobi 行列式为：

$$\frac{\partial(u,v,w)}{\partial(r,\theta,\varphi)} = r\sec^2\varphi$$

所以

$$
\begin{aligned}
I &= 2\int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_0^{\frac{\pi}{4}} \mathrm{d}\varphi \int_0^{\sec\theta} \frac{r\sec^2\varphi}{(1+r^2+\tan^2\varphi)^2}\,\mathrm{d}r \\
&= 2\int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_0^{\frac{\pi}{4}} \mathrm{d}\varphi \int_0^{\sec\theta} \frac{r\sec^2\varphi}{(r^2+\sec^2\varphi)^2}\,\mathrm{d}r \\
&= -\int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_0^{\frac{\pi}{4}} \mathrm{d}\varphi \left(\frac{\sec^2\varphi}{r^2+\sec^2\varphi}\right)\Big|_{r=0}^{r=\sec\theta} \\
&= \left(\frac{\pi}{4}\right)^2 - \int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_0^{\frac{\pi}{4}} \mathrm{d}\varphi \frac{\sec^2\varphi}{\sec^2\varphi+\sec^2\theta}
\end{aligned}
$$

下面的问题就是计算最后一个式子中的积分了。记

$$
A = \int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_0^{\frac{\pi}{4}} \mathrm{d}\varphi \frac{\sec^2\varphi}{\sec^2\varphi+\sec^2\theta} = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \frac{\sec^2\varphi}{\sec^2\varphi+\sec^2\theta}\,\mathrm{d}\theta\,\mathrm{d}\varphi
$$

注意到这个积分中，将积分变量θ,φ 进行互换，积分不变

$$
A = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \frac{\sec^2\varphi}{\sec^2\varphi+\sec^2\theta}\,\mathrm{d}\theta\,\mathrm{d}\varphi = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^2\theta+\sec^2\varphi}\,\mathrm{d}\varphi\,\mathrm{d}\theta
$$

因此

$$
2A = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} \frac{\sec^2\varphi+\sec^2\theta}{\sec^2\varphi+\sec^2\theta}\,\mathrm{d}\theta\,\mathrm{d}\varphi = \left(\frac{\pi}{4}\right)^2
$$

最终

$$
I = \left(\frac{\pi}{4}\right)^2 - A = \left(\frac{\pi}{4}\right)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \frac{\pi^2}{32}.
$$

3、设 $a>0,\ b>0$.计算

$$
\begin{aligned}
&(1)\ I = \int_0^1 \sin\left(\ln\frac{1}{x}\right)\frac{x^b-x^a}{\ln x}\,\mathrm{d}x; \\
&(2)\ J = \int_0^1 \cos\left(\ln\frac{1}{x}\right)\frac{x^b-x^a}{\ln x}\,\mathrm{d}x.
\end{aligned}
$$

解 这里只做(2). 因为

$$
\frac{x^b-x^a}{\ln x} = \int_a^b x^y\,\mathrm{d}y \quad \left(\frac{\partial}{\partial y}x^y = x^y\ln x\right)
$$

$$
\Longrightarrow J = \int_0^1 \mathrm{d}x\cos(\ln x)\int_a^b x^y\,\mathrm{d}y = \int_a^b \mathrm{d}y\int_0^1 \cos(\ln x)x^y\,\mathrm{d}x
$$

计算积分

$$\begin{aligned}\int_0^1 \cos(\ln x) x^y \,\mathrm{d}x &= \frac{1}{1+y}\int_0^1 \cos(\ln x)\,\mathrm{d}(x^{1+y}) \\ &= \frac{1}{1+y}\cos(\ln x)x^{1+y}\Big|_0^1 + \frac{1}{1+y}\int_0^1 \sin(\ln x)x^y\,\mathrm{d}x \\ &= \frac{1}{1+y} + \frac{1}{1+y}\int_0^1 \sin(\ln x)x^y\,\mathrm{d}x.\end{aligned}$$

这里, 注意极限 $\lim\limits_{x\to 0}\cos(\ln x)x^{1+y} = 0$, 因此上式中下限代入为零（下面情况类似）. 继续分部积分:

$$\begin{aligned}\int_0^1 \cos(\ln x)x^y\,\mathrm{d}x &= \frac{1}{1+y} + \frac{1}{(1+y)^2}\int_0^1 \sin(\ln x)\,\mathrm{d}(x^{1+y}) \\ &= \frac{1}{1+y} + \frac{1}{(1+y)^2}\sin(\ln x)x^{1+y}\Big|_0^1 - \frac{1}{(1+y)^2}\int_0^1 \cos(\ln x)x^y\,\mathrm{d}x \\ &= \frac{1}{1+y} - \frac{1}{(1+y)^2}\int_0^1 \cos(\ln x)x^y\,\mathrm{d}x.\end{aligned}$$

$$\Longrightarrow \int_0^1 \cos(\ln x)x^y\,\mathrm{d}x = \frac{1+y}{1+(1+y)^2}.$$

积分即得

$$J = \int_a^b \mathrm{d}y \int_0^1 \cos(\ln x)x^y\,\mathrm{d}x = \int_a^b \frac{1+y}{1+(1+y)^2}\,\mathrm{d}y = \frac{1}{2}\ln\left(\frac{1+(1+b)^2}{1+(1+a)^2}\right).$$

4、计算

$$I = \iint\limits_{x^2+y^2\leqslant 1} \left|\frac{x+y}{\sqrt{2}} - x^2 - y^2\right| \mathrm{d}x\,\mathrm{d}y$$

解 作变换

$$u = \frac{x+y}{\sqrt{2}},\quad v = \frac{x-y}{\sqrt{2}},\quad \frac{\partial(x,y)}{\partial(u,v)} = 1.$$

$$\Longrightarrow I = \iint\limits_{u^2+v^2\leqslant 1} \left|u - u^2 - v^2\right|\,\mathrm{d}u\,\mathrm{d}v$$

由于

$$u - u^2 - v^2 = \frac{1}{4} - \left(u - \frac{1}{2}\right)^2 - v^2$$

因此将 D 分解为两个区域的并: $D = D_1 \cup D_2$, 其中

$$D_1 = \{(u,v) \mid \left(u - \frac{1}{2}\right)^2 + v^2 \leqslant \frac{1}{4}\}$$
$$D_2 = \{(u,v) \mid \left(u - \frac{1}{2}\right)^2 + v^2 \geqslant \frac{1}{4},\ u^2 + v^2 \leqslant 1\}$$

所以, 在 D_1 和 D_2 上

$$|u - u^2 - v^2| = \begin{cases} u - u^2 - v^2 & (u,v) \in D_1 \\ u^2 + v^2 - u & (u,v) \in D_2 \end{cases}$$

因此

$$\begin{aligned} I &= \iint_{D_1} (u - u^2 - v^2)\,\mathrm{d}u\,\mathrm{d}v + \iint_{D_2} (u^2 + v^2 - u)\,\mathrm{d}u\,\mathrm{d}v \\ &= \iint_{D} (u^2 + v^2 - u)\,\mathrm{d}u\,\mathrm{d}v + 2\iint_{D_1} (u - u^2 - v^2)\,\mathrm{d}u\,\mathrm{d}v = a + 2b \end{aligned}$$

其中 a 和 b 分别是两个积分. 用极坐标变换

$$u = r\cos\theta,\ v = r\sin\theta,\ 0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant 2\pi.$$

得 a 的积分为

$$a = \iint_D (u^2 + v^2 - u)\,\mathrm{d}u\,\mathrm{d}v = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 (r^2 - r\cos\theta) r\,\mathrm{d}r = \frac{\pi}{2}.$$

在对 b 积分时, 用坐标变换

$$u = \frac{1}{2} + r\cos\theta,\ v = r\sin\theta\ \ 0 \leqslant r \leqslant \frac{1}{2},\ 0 \leqslant \theta \leqslant 2\pi,$$

$$\begin{aligned} \Longrightarrow b &= \iint_{D_1} (u - u^2 - v^2)\,\mathrm{d}u\,\mathrm{d}v = \iint_{D_1} \left(\frac{1}{4} - \left(u - \frac{1}{2}\right)^2 - v^2\right)\mathrm{d}u\,\mathrm{d}v \\ &= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\frac{1}{2}} \left(\frac{1}{4} - r^2\right) r\,\mathrm{d}r = \frac{\pi}{32}. \end{aligned}$$

所以

$$I = a + 2b = \frac{9\pi}{16}.$$

5、试求圆盘 $(x-a)^2 + (y-a)^2 \leqslant a^2$ 与曲线 $(x^2+y^2)^2 = 8a^2xy$ 所围部分相交的区域 D 的面积 S. 其中 $a > 0$.

解 设圆 $(x-a)^2+(y-a)^2=a^2$ 与曲线 $(x^2+y^2)^2=8a^2xy$ 的交点为 A, B. 解方程可得这两点的坐标 $A\left(\frac{3-\sqrt{7}}{8}a, \frac{3+\sqrt{7}}{8}a\right)$, $B\left(\frac{3+\sqrt{7}}{8}a, \frac{3-\sqrt{7}}{8}a\right)$.

设线段 OB 与 x 轴正向的夹角为 θ. 因为 OB 的长为 $\frac{\sqrt{2}}{2}a$, 所以

$$\sin\theta = \frac{3-\sqrt{7}}{8}a\Big/\frac{\sqrt{2}}{2}a = \frac{3\sqrt{2}-\sqrt{14}}{8}.$$

计算可得 $\sin\left(\frac{1}{2}\arcsin\frac{1}{8}\right) = \frac{3\sqrt{2}-\sqrt{14}}{8}$. 故, $\theta = \frac{1}{2}\arcsin\frac{1}{8}$.

在极坐标 $x = r\cos\varphi,\ y = r\sin\varphi$ 之下, D 为

$$D = \left\{(r,\varphi):\ \begin{array}{l} a[(\sin\varphi+\cos\varphi)-\sqrt{\sin 2\varphi}] \leqslant r \leqslant 2a\sqrt{\sin 2\varphi}; \\ \dfrac{1}{2}\arcsin\dfrac{1}{8} \leqslant \varphi \leqslant \dfrac{\pi}{2}-\dfrac{1}{2}\arcsin\dfrac{1}{8} \end{array}\right\}.$$

注意到 D 关于 $\varphi = \frac{\pi}{4}$ 对称, 有

$$\begin{aligned} S &= \iint_D \mathrm{d}x\,\mathrm{d}y = 2\int_{\frac{1}{2}\arcsin\frac{1}{8}}^{\pi/4} \mathrm{d}\varphi \int_{a[(\sin\varphi+\cos\varphi)-\sqrt{\sin 2\varphi}]}^{2a\sqrt{\sin 2\varphi}} r\,\mathrm{d}r \\ &= a^2\int_{\frac{1}{2}\arcsin\frac{1}{8}}^{\pi/4} \left[2\sin 2\varphi + 2(\sin\varphi+\cos\varphi)\sqrt{\sin 2\varphi} - 1\right]\mathrm{d}\varphi \\ &= a^2\left[\cos\left(\arcsin\frac{1}{8}\right) - \frac{\pi}{4} + \frac{1}{2}\arcsin\frac{1}{8}\right] + 2a^2\int_{\frac{1}{2}\arcsin\frac{1}{8}}^{\pi/4} (\sin\varphi+\cos\varphi)\sqrt{\sin 2\varphi}\,\mathrm{d}\varphi. \end{aligned}$$

因为 $\cos\left(\arcsin\frac{1}{8}\right) = \sqrt{1-\frac{1}{64}} = \frac{3\sqrt{7}}{8}$, 以及

$$-\frac{\pi}{4} + \frac{1}{2}\arcsin\frac{1}{8} = -\frac{1}{2}\left(\frac{\pi}{2} - \arcsin\frac{1}{8}\right) = -\frac{1}{2}\arccos\frac{1}{8},$$

作变换 $\varphi + \frac{\pi}{4} = t$, 我们有

$$S = a^2\left(\frac{3\sqrt{7}}{8} - \frac{1}{2}\arccos\frac{1}{8} + 2\sqrt{2}\int_{\frac{\pi}{4}+\frac{1}{2}\arcsin\frac{1}{8}}^{\pi/2} \sqrt{-\cos 2t}\sin t\,\mathrm{d}t\right).$$

记上式括号中的积分为 I, 我们有

$$I = 2\int_{\pi/2}^{\frac{\pi}{4}+\frac{1}{2}\arcsin\frac{1}{8}} \sqrt{1-(\sqrt{2}\cos t)^2}\,\mathrm{d}(\sqrt{2}\cos t).$$

作变换 $u = \sqrt{2\cos t}$, 得

$$I = 2\int_0^{\arcsin\frac{\sqrt{7}}{2\sqrt{2}}} \cos^2 u\,\mathrm{d}u = \arcsin\frac{\sqrt{7}}{2\sqrt{2}} + \frac{\sqrt{7}}{8}.$$

于是

$$S=a^2\left(\frac{\sqrt{7}}{2}+\arcsin\frac{\sqrt{7}}{2\sqrt{2}}-\frac{1}{2}\arccos\frac{1}{8}\right)=a^2\left(\frac{\sqrt{7}}{2}+\arcsin\frac{\sqrt{14}}{8}\right).$$

6. 计算曲面 $(x^2+y^2)^2+z^4=y$ 所围的区域 V 的体积 $\sigma(V)$.

解 设 V 在第一象限中的部分为 V_1, 则根据对称性, V 的体积是 V_1 的体积的4倍. V_1 在 xy 平面的投影趋于是 $D:\ (x^2+y^2)^2+z^4\leqslant y,\ x\geqslant 0, y\geqslant 0$. 因此,

$$\sigma(V)=4\iint_D\left(y-(x^2+y^2)^2\right)^{\frac{1}{4}}\,\mathrm{d}x\,\mathrm{d}y.$$

用极坐标变换 $x=r\cos\varphi,\ y=r\sin\varphi$. 有

$$\begin{aligned}\sigma(V)&=4\iint\limits_{\substack{0\leqslant\varphi\leqslant\frac{\pi}{2}\\0\leqslant r\leqslant\sin^{1/3}\varphi}}\left(r\sin\varphi-r^4\right)^{\frac{1}{4}}\cdot r\,\mathrm{d}r\\&=4\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{\sin^{1/3}\varphi}\left(\sin\varphi-r^3\right)^{1/4}\cdot r^{5/4}\,\mathrm{d}r\end{aligned}$$

对上式最右边的积分作变换 $r=(x\sin\varphi)^{1/3}$, 得

$$\begin{aligned}\sigma(V)&=\frac{4}{3}\int_0^{\frac{\pi}{2}}\sin\varphi\,\mathrm{d}\varphi\int_0^1 x^{-1/4}(1-x)^{1/4}\,\mathrm{d}x\\&=\frac{2}{3}\pi\int_0^1 x^{-1/4}(1-x)^{1/4}\,\mathrm{d}x.\end{aligned}$$

对上式作变换

$$t=x^{-1/4}(1-x)^{1/4}\Longrightarrow t^4=\frac{1-x}{x},$$

得

$$\int_0^1 x^{-1/4}(1-x)^{1/4}\,\mathrm{d}x=\int_0^\infty\frac{4t^4}{(1+t^4)^2}\,\mathrm{d}t=\int_0^\infty\frac{1}{1+t64}\,\mathrm{d}t.$$

因为

$$\int_0^\infty\frac{1}{1+t^4}\,\mathrm{d}t=\frac{\sqrt{2}}{4}\pi$$

故,

$$\sigma(V)=\frac{\sqrt{2}}{3}\pi.$$

7. 证明: $\displaystyle\iint\limits_{[0,1]^2}(xy)^{xy}\ \mathrm{d}x\,\mathrm{d}y=\int_0^1 t^t\ \mathrm{d}t.$

解 首先化为累次积分

$$\begin{aligned}\iint\limits_{[0,1]^2}(xy)^{xy}\ \mathrm{d}x\,\mathrm{d}y&=\int_0^1\ \mathrm{d}x\int_0^1(xy)^{xy}\ \mathrm{d}y=\int_0^1\ \mathrm{d}x\int_0^x\frac{t^t}{x}\ \mathrm{d}t\\&=\int_0^1\frac{f(x)}{x}\,\mathrm{d}x,\end{aligned}$$

其中 $f(x)=\int_0^x t^t\,\mathrm{d}t$. 由分部积分,

$$\int_0^1\frac{f(x)}{x}\,\mathrm{d}x=f(x)\ln x\Big|_0^1-\int_0^1 x^x\ln x\,\mathrm{d}x=-\int_0^1 x^x\ln x\,\mathrm{d}x$$

因为 $(x^x)'=x^x\ln x+x^x$, 所以

$$\int_0^1 x^x\ln x\,\mathrm{d}x=\int_0^1\left((x^x)'-x^x\right)\ \mathrm{d}x=-\int_0^1 x^x\,\mathrm{d}x.$$

于是

$$\iint\limits_{[0,1]^2}(xy)^{xy}\ \mathrm{d}x\,\mathrm{d}y=\int_0^1 t^t\ \mathrm{d}t.$$

8. 设 a,b 是不全为 0 的常数. 求证:

$$\iint\limits_{x^2+y^2\leqslant 1}f(ax+by+c)\ \mathrm{d}x\,\mathrm{d}y=2\int_{-1}^1\sqrt{1-t^2}\,f\left(t\sqrt{a^2+b^2}+c\right)\ \mathrm{d}t.$$

证明 作变换

$$x=\frac{a}{\sqrt{a^2+b^2}}t-\frac{b}{\sqrt{a^2+b^2}}s,y=\frac{b}{\sqrt{a^2+b^2}}t+\frac{a}{\sqrt{a^2+b^2}}s.$$

则有 $x^2+y^2=s^2+t^2$, 且 $\frac{\partial(x,y)}{\partial(t,s)}=1$. 因此

$$\begin{aligned}&\iint\limits_{x^2+y^2\leqslant 1}f(ax+by+c)\ \mathrm{d}x\,\mathrm{d}y\\&=\iint\limits_{t^2+s^2\leqslant 1}f\left(t\sqrt{a^2+b^2}+c\right)\ \mathrm{d}t\,\mathrm{d}s\\&=\int_{-1}^1 f\left(t\sqrt{a^2+b^2}+c\right)\ \mathrm{d}t\int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}}\ \mathrm{d}s\\&=2\int_{-1}^1\sqrt{1-t^2}f\left(t\sqrt{a^2+b^2}+c\right)\ \mathrm{d}t.\end{aligned}$$

9. 设 f 是连续可导的单变量函数. 令 $F(t)=\iint\limits_{[0,t]^2} f(xy)\,\mathrm{d}x\,\mathrm{d}y$. 求证:

$$(1)\ F'(t)=\frac{2}{t}\left(F(t)+\iint\limits_{[0,t]^2} xyf'(xy)\,\mathrm{d}x\,\mathrm{d}y\right);$$

$$(2)\ F'(t)=\frac{2}{t}\int_0^{t^2} f(s)\,\mathrm{d}s.$$

证明 先证明 (2) : 用累次积分

$$\begin{aligned}F(t)&=\int_0^t \mathrm{d}x\int_0^t f(xy)\,\mathrm{d}y=\int_0^t\frac{1}{x}\,\mathrm{d}x\int_0^{tx} f(s)\,\mathrm{d}s\\&=\int_0^t\frac{g(tx)}{x}\,\mathrm{d}x=\int_0^{t^2}\frac{g(u)}{u}\,\mathrm{d}u,\end{aligned}$$

其中 $g(u)=\int_0^u f(s)\,\mathrm{d}s$. 于是

$$F'(t)=2t\cdot\frac{g(t^2)}{t^2}=\frac{2}{t}\int_0^{t^2} f(s)\,\mathrm{d}s.$$

再证明 (1):

$$\begin{aligned}&\frac{2}{t}\iint\limits_{[0,t]^2} xyf'(xy)\,\mathrm{d}x\,\mathrm{d}y=\frac{2}{t}\int_0^t \mathrm{d}x\int_0^t xyf'(xy)\,\mathrm{d}y\\&=\frac{2}{t}\int_0^t\frac{1}{x}\,\mathrm{d}x\int_0^{tx} uf'(u)\,\mathrm{d}u=\frac{2}{t}\int_0^t\frac{1}{x}\left(txf(tx)-g(tx)\right)\,\mathrm{d}x\\&=\frac{2}{t}\int_0^{t^2}\frac{1}{s}\left(sf(s)-g(s)\right)\,\mathrm{d}s=F'(t)-\frac{2}{t}F(t).\end{aligned}$$

所以(1) 成立.

10. (Poincaré 不等式) 设 $\varphi(x),\psi(x)$ 是 $[a,b]$ 上的连续函数, $f(x,y)$ 在区域 $D=\{(x,y):a\leqslant x\leqslant b,\varphi(x)\leqslant y\leqslant\psi(x)\}$ 上连续可微, 且有 $f(x,\varphi(x))=0$, 则存在 $M>0$, 使得

$$\iint_D f^2(x,y)\,\mathrm{d}x\,\mathrm{d}y\leqslant M\iint_D (f'_y(x,y))^2\,\mathrm{d}x\,\mathrm{d}y.$$

证明 由 Newton-Leibeniz 公式和 Cauchy 不等式,

$$f^2(x,y)=\big[f(x,y)-f(x,\varphi(x)\big]^2=\left(\int_{\varphi(x)}^{y}\frac{\partial f}{\partial t}(x,t)\ \mathrm{d}t\right)^2$$
$$\leqslant (y-\varphi(x))\int_{\varphi(x)}^{y}\left(\frac{\partial f}{\partial t}(x,t)\right)^2\ \mathrm{d}t$$

因此

$$\begin{aligned}&\iint_D f^2(x,y)\ \mathrm{d}x\,\mathrm{d}y=\int_a^b \mathrm{d}x\int_{\varphi(x)}^{\psi(x)} f^2(x,y)\,\mathrm{d}y\\
\leqslant&\int_a^b \mathrm{d}x\int_{\varphi(x)}^{\psi(x)}(y-\varphi(x))\,\mathrm{d}y\int_{\varphi(x)}^{y}\left(\frac{\partial f}{\partial t}(x,t)\right)^2\ \mathrm{d}t\\
=&\int_a^b \mathrm{d}x\int_{\varphi(x)}^{\psi(x)}\left(\frac{\partial f}{\partial t}(x,t)\right)^2\ \mathrm{d}t\int_t^{\psi(x)}(y-\varphi(x))\,\mathrm{d}y\\
\leqslant&\int_a^b \mathrm{d}x\int_{\varphi(x)}^{\psi(x)}\left(\frac{\partial f}{\partial t}(x,t)\right)^2\ \mathrm{d}t\int_{\varphi(x)}^{\psi(x)}(y-\varphi(x))\,\mathrm{d}y\\
=&\int_a^b \mathrm{d}x\int_{\varphi(x)}^{\psi(x)}\frac{1}{2}(\psi(x)-\varphi(x))^2\left(\frac{\partial f}{\partial t}(x,t)\right)^2\ \mathrm{d}t\\
\leqslant& M\int_a^b \mathrm{d}x\int_{\varphi(x)}^{\psi(x)}\left(\frac{\partial f}{\partial t}(x,t)\right)^2\ \mathrm{d}t=M\iint_D\left(\frac{\partial f}{\partial y}(x,y)\right)^2\ \mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

这里 M 是满足 $M>\max\limits_{a\leqslant x\leqslant b}\frac{1}{2}(\psi(x)-\varphi(x))^2$ 的常数.

11. 设 $a>0$, $\Omega_n(a):\ x_1+\cdots+x_n\leqslant a,\ x_i\geqslant 0\ (i=1,2,\cdots,n)$. 求积分

$$I_n(a)=\int\cdots\int_{\Omega_n(a)} x_1x_2\cdots x_n\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\,\mathrm{d}x_n.$$

解 作变换 $x_i=at_i,\ i=1,2,\cdots,n$, 则

$$I_n(a)=a^{2n}\int\cdots\int_{\Omega_n(1)} t_1t_2\cdots t_n\,\mathrm{d}t_1\,\mathrm{d}t_2\cdots\,\mathrm{d}t_n=a^{2n}I_n(1). \tag{1}$$

用累次积分, 可得

$$\begin{aligned}I_n(1)&=\int\cdots\int_{\Omega_n(1)} t_1t_2\cdots t_n\,\mathrm{d}t_1\,\mathrm{d}t_2\cdots\,\mathrm{d}t_n\\
&=\int_0^1 t_n\,\mathrm{d}t_n\int\cdots\int_{t_1+\cdots+t_{n-1}\leqslant 1-t_n} t_1\cdots t_{n-1}\,\mathrm{d}t_1\cdots\,\mathrm{d}t_{n-1}\\
&=\int_0^1 t_nI_{n-1}(1-t_n)\,\mathrm{d}t_n=\int_0^1 t_n(1-t_n)^{2(n-1)}I_{n-1}(1)\,\mathrm{d}t_n.\end{aligned}$$

因此

$$I_n(1)=\frac{1}{2n(2n-1)}I_{n-1}(1).$$

注意到 $I_1(1)=\int_0^1 t\,\mathrm{d}t=\frac{1}{2}$. 由上面的递推公式, 可得

$$I_n(1)=\frac{1}{(2n)!}.$$

故,

$$I_n(a)=\frac{a^{2n}}{(2n)!}.$$

12. 设 $f(x_1,x_2,\cdots,x_n)$ 为 n 元连续函数. 证明:

$$\begin{aligned}&\int_a^b\mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_{n-1}}f(x_1,x_2,\cdots,x_n)\,\mathrm{d}x_n\\=&\int_a^b\mathrm{d}x_n\int_{x_n}^b\mathrm{d}x_{n-1}\cdots\int_{x_2}^b f(x_1,x_2,\cdots,x_n)\,\mathrm{d}x_1.\end{aligned}$$

证明 $n=1$ 时, 无需证明. $n=2$ 时, 就是要证

$$\int_a^b\mathrm{d}x_1\int_a^{x_1}f(x_1,x_2)\,\mathrm{d}x_2=\int_a^b\mathrm{d}x_2\int_{x_2}^b f(x_1,x_2)\,\mathrm{d}x_1.$$

上式左右两边都是 $f(x_1,x_2)$ 在区域 $D: a\leqslant x_1\leqslant b,\ 0\leqslant x_2\leqslant x_1$ 上的累次积分, 因而它们相等. 假设 $n-1$ 时结论成立.

记 $g(x_1,\cdots,x_{n-1})=\int_a^{x_{n-1}}f(x_1,x_2,\cdots,x_n)\,\mathrm{d}x_n$. 则

$$\begin{aligned}&\int_a^b\mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_{n-1}}f(x_1,x_2,\cdots,x_n)\,\mathrm{d}x_n\\=&\int_a^b\mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_{n-2}}g(x_1,x_2,\cdots,x_{n-1})\,\mathrm{d}x_{n-1}\\=&\int_a^b\mathrm{d}x_{n-1}\int_{x_{n-1}}^b\mathrm{d}x_{n-2}\cdots\int_{x_2}^b g(x_1,\cdots,x_{n-1})\,\mathrm{d}x_1\\=&\int_a^b\mathrm{d}x_{n-1}\int_{x_{n-1}}^b\mathrm{d}x_{n-2}\cdots\int_{x_2}^b\left(\int_a^{x_{n-1}}f(x_1,x_2,\cdots,x_n)\,\mathrm{d}x_n\right)\mathrm{d}x_1\\=&\int_a^b\mathrm{d}x_{n-1}\int_a^{x_{n-1}}\mathrm{d}x_n\int_{x_{n-1}}^b\mathrm{d}x_{n-2}\cdots\int_{x_2}^b f(x_1,x_2,\cdots,x_n)\,\mathrm{d}x_1\\=&\int_a^b\mathrm{d}x_{n-1}\int_a^{x_{n-1}}h(x_{n-1},x_n)\,\mathrm{d}x_n,\end{aligned}$$

这里 $h(x_{n-1},x_n)=\int_{x_{n-1}}^b \mathrm{d}x_{n-2}\cdots\int_{x_2}^b f(x_1,x_2,\cdots,x_n)\,\mathrm{d}x_1$. 再利用 $n=2$ 的结论, 得

$$\int_a^b \mathrm{d}x_{n-1}\int_a^{x_{n-1}} h(x_{n-1},x_n)\,\mathrm{d}x_n=\int_a^b \mathrm{d}x_n\int_{x_{n-1}}^b h(x_{n-1},x_n)\,\mathrm{d}x_{n-1}.$$

故,

$$\begin{aligned}&\int_a^b \mathrm{d}x_1\int_a^{x_1}\mathrm{d}x_2\cdots\int_a^{x_{n-1}} f(x_1,x_2,\cdots,x_n)\,\mathrm{d}x_n\\ =&\int_a^b \mathrm{d}x_n\int_{x_n}^b \mathrm{d}x_{n-1}\cdots\int_{x_2}^b f(x_1,x_2,\cdots,x_n)\,\mathrm{d}x_1.\end{aligned}$$

习题10.4 设 $f(x)$ 连续, 证明:

$$\int_0^a \mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\cdots\int_0^{x_{n-1}} f(x_1)f(x_2)\cdots f(x_n)\,\mathrm{d}x_n=\frac{1}{n!}\left(\int_0^a f(t)\,\mathrm{d}t\right)^n$$

证明 记

$$g_n(t)=\int_0^t \mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\cdots\int_0^{x_{n-1}} f(x_1)f(x_2)\cdots f(x_n)\,\mathrm{d}x_n.$$

则

$$g_1(t)=\int_0^t f(u)\,\mathrm{d}u.$$

假设

$$g_{n-1}(t)=\frac{1}{(n-1)!}\left(\int_0^t f(u)\,\mathrm{d}u\right)^{n-1}.$$

对 $g_n(t)$ 求导, 得

$$g_n'(t)=\int_0^t \mathrm{d}x_2\cdots\int_0^{x_{n-1}} f(t)f(x_2)\cdots f(x_n)\,\mathrm{d}x_n,$$

即,

$$\begin{aligned}g_n'(t)=f(t)g_{n-1}(t)&=\frac{1}{(n-1)!}f(t)\left(\int_0^t f(u)\,\mathrm{d}u\right)^{n-1}\\&=\frac{1}{n!}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t f(u)\,\mathrm{d}u\right)^n.\end{aligned}$$

于是

$$g_n(t)=\frac{1}{n!}\left(\int_0^t f(u)\,\mathrm{d}u\right)^n.$$

根据归纳原理, 结论得证.