18: 略

9.5 多变量函数的Taylor公式与极值

1: (1)$F(t)=\sin((x+th)^2+y+tk), F'(t)=\cos((x+th)^2+y+tk)(2(x+th)h+k), F'(1)=\cos((x+h)^2+y+k)(2(x+h)h+k)$

(2)$F(t)=(x+th)^2+2(x+th)(y+tk)^2-(y+tk)^4, F'(t)=2h(x+th)+2h(y+tk)^2+4k(x+th)(y+tk)-3k(y+tk)^3$

2: (1)$f(x_0+h,y_0+k)-f(x_0,y_0)=106-39(5+h)+(5+h)^3+18(6+k)-6(5+h)(6+k)+(6+k)^2=15h^2+h^3-6hk+k^2$

(2)$f(x_0+h,y_0+k)-f(x_0,y_0)=-2-2(1+h)(-1+k)+(1+h)^2(-1+k)+(1+h)(-1+k)^2=h^2(-1+k)+h(-1+k)^2+(-3+k)k$

4: (1)成立区域：$\{(x,y)|y>-1\}$.

$$\begin{aligned}f(x,y)&=(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+o(x^3))(y-\frac{1}{2}y^2+\frac{1}{3}y^3+o(y^3))\\&=y+xy-\frac{1}{2}y^2+\frac{1}{2}x^2y-\frac{1}{2}xy^2+\frac{1}{3}y^3+o(\rho^3).\end{aligned}$$

(2)成立区域：$\{(x,y)|x^2+y^2<1\}$.

$$\begin{aligned}f(x,y)=\sqrt{1-\rho^2}&=1-\frac{1}{2}\rho^2-\frac{1}{8}\rho^4+o(\rho^4)\\&=1-\frac{1}{2}(x^2+y^2)-\frac{1}{8}(x^2+y^2)^2+o(\rho^4).\end{aligned}$$

(3)成立区域：$\{(x,y)|x<-1,y<-1\}$.

$$f(x,y)=\frac{1}{(1-x)(1-y)}=(\sum_{i=0}^{n}x^i+o(x^n))(\sum_{i=0}^{n}y^i+o(y^n))$$
$$=\sum_{k=0}^{n}\sum_{i=0}^{k}x^iy^{k-i}+o(\rho).$$

(4)成立区域：$\{(x,y)|1-x+y>0\}$.

$$f(0,0)=\frac{\pi}{4},\quad \frac{\partial f}{\partial x}(0,0)=1,\quad \frac{\partial f}{\partial y}=0,$$
$$\frac{\partial^2 f}{\partial x^2}(0,0)=0,\quad \frac{\partial^2 f}{\partial x\partial y}(0,0)=-1,\quad \frac{\partial^2 f}{\partial y^2}(0,0)=0.$$

从而

$$f(x,y)=\frac{\pi}{4}+x-xy+o(\rho^2).$$

(5)成立区域：$\mathbb{R}^2$. 下面$\rho=\sqrt{x^2+y^2}$, $m\in\mathbb{Z}$.

$$f(x,y)=\sin\rho^2=\begin{cases}\sum\limits_{k=0}^{m}\dfrac{\rho^{4k+2}}{(2k+1)!}+o(\rho^{4m+2}), & n=4m+2,4m+3,4m+4.\\ \sum\limits_{k=0}^{m}\dfrac{\rho^{4k-2}}{(2k-1)!}+o(\rho^{4m-2}), & n=4m+1.\end{cases}$$

(6)成立区域：$\mathbb{R}\times(-\frac{\pi}{2},\frac{\pi}{2})$.

$$f(0,0)=1,\quad \frac{\partial f}{\partial x}(0,0)=0,\quad \frac{\partial f}{\partial y}=0,$$
$$\frac{\partial^2 f}{\partial x^2}(0,0)=-1,\quad \frac{\partial^2 f}{\partial x\partial y}(0,0)=0,\quad \frac{\partial^2 f}{\partial y^2}(0,0)=1.$$

从而

$$f(x,y)=1-\frac{1}{2}x^2+\frac{1}{2}y^2+o(\rho^2).$$

(7)成立区域：$\mathbb{R}^2$. 配方得：

$$f(x,y)=2(x-1)^2-(y+2)^2-(x-1)(y+2)+5.$$

5: 用多元函数Taylor公式将$z=z(x,y)$展开至二阶:

$$z=z_0+\frac{\partial z}{\partial x}(x-x_0)+\frac{\partial z}{\partial y}(y-y_0)+\frac{1}{2}\frac{\partial^2 z}{\partial x^2}(x-x_0)^2+\frac{1}{2}\frac{\partial^2 z}{\partial y^2}(y-y_0)^2+\frac{\partial^2 z}{\partial x\partial y}(x-x_0)(y-y_0)+o\left(\rho^2\right)$$

方程$z^3-2xz+y=0$两边同时对x求导,得

$$3z^2\frac{\partial z}{\partial x}-2x\frac{\partial z}{\partial x}-2z=0$$

再将此式对x求导,得

$$\left(6z\frac{\partial z}{\partial x}-2\right)\frac{\partial z}{\partial x}+(3z^2-2x)\frac{\partial^2 z}{\partial x^2}-2\frac{\partial z}{\partial x}=0$$

从上两式中解得

$$\frac{\partial z}{\partial x}=\frac{2z}{3z^2-2x},\quad \frac{\partial^2 z}{\partial x^2}=\frac{2\frac{\partial z}{\partial x}-\left(6z\frac{\partial z}{\partial x}-2\right)\frac{\partial z}{\partial x}}{3z^2-2x}$$

再将方程$z^3-2xz+y=0$两边同时对y求导,得

$$3z^2\frac{\partial z}{\partial y}-2x\frac{\partial z}{\partial y}+1=0$$

再将此式分别对x,y求导,得

$$\left(6z\frac{\partial z}{\partial x}-2\right)\frac{\partial z}{\partial y}+(3z^2-2x)\frac{\partial^2 z}{\partial x\partial y}=0,\quad 6z\frac{\partial z}{\partial y}\frac{\partial z}{\partial y}+(3z^2-2x)\frac{\partial^2 z}{\partial y^2}=0$$

从上三式中解得

$$\frac{\partial z}{\partial y}=-\frac{1}{3z^2-2x},\ \frac{\partial^2 z}{\partial x\partial y}=-\frac{\left(6z\frac{\partial z}{\partial x}-2\right)\frac{\partial z}{\partial y}}{3z^2-2x},\ \frac{\partial^2 z}{\partial y^2}=-\frac{6z\frac{\partial z}{\partial y}\frac{\partial z}{\partial y}}{3z^2-2x}$$

代入点$(1,1,1)$,得

$$\frac{\partial z}{\partial x}=2,\quad \frac{\partial z}{\partial y}=-1,\quad \frac{\partial^2 z}{\partial x^2}=-16,\quad \frac{\partial^2 z}{\partial x\partial y}=10,\quad \frac{\partial^2 z}{\partial y^2}=-6$$

因此,

$$z(x,y)=1+2(x-1)-(y-1)-8(x-1)^2-3(y-1)^2+10(x-1)(y-1)+o\left(\rho^2\right)$$

6: 证明:$\cos x, \cos y, \cos z$在(0,0)点的二阶泰勒展开式为:

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$
$$\cos y = 1 - \frac{y^2}{2} + o(y^2)$$
$$\cos z = 1 - \frac{z^2}{2} + o(z^2)$$

$\sin x, \sin y$的二阶泰勒展开式为:

$$\sin x = x + o(x^2)$$
$$\sin y = x + o(y^2)$$

$\therefore \cos x \cos y + \sin x \sin y \cos\theta$在(0,0)处的二阶泰勒展开式为:

$$\cos x \cos y + \sin x \sin y \cos\theta = 1 - \frac{x^2}{2} - \frac{y^2}{2} + xy\cos\theta + o(x^2 + y^2)$$

又$\because \cos z = \cos x \cos y + \sin x \sin y \cos\theta$,$\therefore$在原点的邻域内有:

$$1 - \frac{z^2}{2} = 1 - \frac{x^2}{2} - \frac{y^2}{2} + xy\cos\theta$$

即$z^2 = x^2 + y^2 - 2xycos\theta$.

7:

(1) $\frac{\partial f}{\partial x} = y - \frac{50}{x^2}, \frac{\partial f}{\partial y} = x - \frac{20}{y^2}, \frac{\partial^2 f}{\partial x^2} = \frac{100}{x^3}, \frac{\partial^2 f}{\partial x \partial y} = 1, \frac{\partial^2 f}{\partial y^2} = \frac{40}{y^3}$.

令$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$,

可解得$x = 5, y = 2$

当$x > 0, y > 0$时，$Q(h,k) = 0.8h^2 + 2hk + 5k^2$ 是正定的,

因此$(x,y) = (5,2)$ 是小极值点，极小值为30.

(2) $\frac{\partial f}{\partial x}=4-2x, \frac{\partial f}{\partial y}=-4-2y, \frac{\partial^2 f}{\partial x^2}=-2, \frac{\partial^2 f}{\partial x\partial y}=0, \frac{\partial^2 f}{\partial y^2}=-2.$

令$\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0,$

可解得$x=2, y=-2.$

由于$Q(h,k)=-2h^2-2k^2$是负定的. 因此$(x,y)=(2,-2)$ 是极大值点，极大值为8

(3) $\frac{\partial f}{\partial x}=2e^{2x}(x+2y+y^2)+e^{2x}, \frac{\partial f}{\partial y}=e^{2x}(2+2y), \frac{\partial^2 f}{\partial x^2}=e^{2x}(4x+8y+4y^2+4), \frac{\partial^2 f}{\partial x\partial y}=e^{2x}(4+4y), \frac{\partial^2 f}{\partial y^2}=2e^{2x}.$

令$\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0$, 可解得$x=0.5, y=-1.$

由于$Q(h,k)=e(2h^2+2k^2)$ 是正定的, 因此$(x,y)=(0.5,-1)$ 是极小值点，极小值为$-frace2$

(4) 记$f(x,y)=(x^2+y^2)^2-a^2(x^2-y^2),$

则$\frac{\partial f}{\partial x}=4x(x^2+y^2)-2a^2x, \frac{\partial f}{\partial y}=4y(x^2+y^2)+2a^2y.$

因此$\frac{dy}{dx}=-\frac{2x(x^2+y^2)-a^2x}{2y(x^2+y^2)+a^2y}=0 \Leftrightarrow x=0$, 或$2(x^2+y^2)=a^2.$

若$x=0$, 那么$f(x,y)=0\rightarrow y=0$, 从而$\frac{\partial f}{\partial y}=0$, 这说明$y(x)$ 不存在.

若$2(x^2+y^2)=a^2$, 那么$f(x,y)=0\rightarrow x^2=\frac{3}{8}a^2, y^2=\frac{1}{8}a^2, a\neq 0$. 再通过计算$\frac{d^2y}{dx^2}$可知, $(\pm\sqrt{\frac{3}{8}}|a|, \pm\sqrt{\frac{1}{8}}|a|)$是极值点，$y$极大值为$\sqrt{\frac{1}{8}}|a|)$,极小值为$-\sqrt{\frac{1}{8}}|a|)$

(5) 原方程可化为$(x-1)^2+(y+1)^2+(z-2)^2=16.$

这是以$(1,-1,2)$ 为圆心半径为4的圆.

故$z(x,y)$ 再$(1,-1)$.处取得极值，极大值为6，极小值为-2

8: 设该三角形的两个内角分别为 $x, y, z, 0 < x, y, z < \pi, x + y + z = \pi$. 记 $f(x, y, z) = \sin x \sin y \sin z, F(x, y, z) = f(x, y, z) - \lambda(x + y + z - \pi)$, 对$F$分别关于$x, y, z, \lambda$ 求导并令其为0有

$$\frac{\partial F}{\partial x} = \cos x \sin y \sin z - \lambda = 0 \tag{9.10}$$
$$\frac{\partial F}{\partial y} = \sin x \cos y \sin z - \lambda = 0 \tag{9.11}$$
$$\frac{\partial F}{\partial z} = \sin x \sin y \cos z - \lambda = 0 \tag{9.12}$$
$$\frac{\partial F}{\partial \lambda} = x + y + z - \pi = 0 \tag{9.13}$$

解上述方程得$x = y = z = \frac{\pi}{3}$，显而易见，此即为$f$ 得条件最值点. 故正三角形的三个内角的正弦乘积最大，为$(\frac{\sqrt{3}}{2})^3$.

9: 按照$xy = 1, 2, ...$作图，易见和圆的切点处是最大最小值

10: (1)极小值$f(\frac{ab^2}{a^2+b^2}, \frac{a^2b}{a^2+b^2}) = \frac{a^2b^2}{a^2+b^2}$
(2)极小值$f(3, 3, 3) = 9$
(3)极小值$f(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = \frac{1}{8}$
(4)极大值$\frac{\sqrt{6}}{18}$,极小值$-\frac{\sqrt{6}}{18}$

12: 只需求出函数

$$f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2 + (x-2)^2 + (y-3)^2 + (z-4)^2,$$

在约束条件$3x-2z=0$下的最小值即可.

取Lagrange函数为$F(x,y,z,\lambda)=f(x,y,z)+\lambda(3x-2z)$. 那么

$$\begin{cases}\dfrac{\partial F}{\partial x}=4x-6+3\lambda=0,\\ \dfrac{\partial F}{\partial y}=4y-8=0,\\ \dfrac{\partial F}{\partial z}=4z-10-2\lambda=0,\\ \dfrac{\partial F}{\partial \lambda}=3x-2z=0.\end{cases} \Longrightarrow \begin{cases}x=2,\\ y=2,\\ z=3.\end{cases}$$

依题意知最小值一定存在，从而最小距离为$f(2,2,3)=8$.

13: 联立二式得$x^2+y^2-2=0$,此为约束条件,为求$z=x^2+2y^2$的最值,令

$$F(x,y,\lambda)=x^2+2y^2+\lambda(x^2+y^2-2)$$

求得驻点方程组

$$\begin{cases}F'_x=2x(1+\lambda)=0\\ F'_y=2y(2+\lambda)=0\\ F'_\lambda=x^2+y^2-2=0\end{cases}$$

解得驻点有4个:$(0,\pm\sqrt{2}),(\pm\sqrt{2},0)$.一一代入,可得到使$z$为最大值的点为$(0,\pm\sqrt{2})$,最大值为4;使$z$为最小值的点为$(\pm\sqrt{2},0)$,最小值为2.

14: 证明:

对点$O(0,0)$的任意$B(0,\epsilon)$邻域，取$x=0, 0<y<\epsilon$，则有:

$$f(x,y)=-2y^2<f(0,0)=0$$

而取$0<x<\epsilon, 0<y<\epsilon$，由基本不等式可知：

$$f(x,y)=3x^2y-x^4-2y^2\leqslant 3x^2y-2\sqrt{2}x^2y=(3-2\sqrt{2})x^2y>f(0,0)=0$$

因为原点的任意邻域内既存在$f(x,y)>0$的点，亦存在$f(x,y)<0$的点，所以原点不是极值点。
过原点的直线的参数式方程为：

$$x=t\cos\alpha, y=t\sin\alpha$$

易知：

$$f(x,y)=f(t\cos\alpha,t\sin\alpha)=3t^3\cos^2\alpha\sin\alpha-t^4\cos^4\alpha-2t^2\sin^2\alpha$$

求其二阶导数为：

$$f^{''}(t)=18t\cos^2\alpha\sin\alpha-12t^2\cos^4\alpha-4\sin^2\alpha$$

由：

$$f^{''}(0)=-4\sin^2\alpha<0(\alpha\neq 0)$$

和：

$$f(t)=-t^4(\alpha=0), Max(f)=f(0)=0$$

（最大值必为极大值）可知：沿过原点的每条直线，原点均是其极大值点。证毕。

15: 帐篷的表面积$S(H,R,h)=\pi R(2H+\sqrt{R^2+h^2})$, 体积是$V(H,R,h)=\pi R^2(H+\frac{1}{3}h)=V_0$.

记$f(H,R,h)=S-\lambda(V-V_0)$,

令$\begin{cases} f_H = 2\pi R - \lambda\pi R^2 = 0, \\ f_R = 2\pi H + \pi\sqrt{R^2+h^2} + \pi R^2 \frac{1}{\sqrt{R^2+h^2}} - \lambda(2\pi R(H+\frac{1}{3}h)) = 0, \\ f_h = \pi Rh\frac{1}{\sqrt{R^2+h^2}} - \lambda(\frac{1}{3}\pi R^2) = 0, \\ \pi R^2(H+\frac{1}{3}h) - V_0 = 0, \end{cases}$

因此$\lambda R = 2\pi, \frac{\pi h}{\sqrt{h^2+R^2}} = \frac{\lambda R}{3} = \frac{2\pi}{3} \Rightarrow h = \frac{2R}{\sqrt{5}}$,

故$R = \sqrt{5}H, h = 2H.$

由于极值点是唯一的，且该问题是中最小值存在，则$R = \sqrt{5}H, h = 2H$即为最小值

16: 设该平行六面体的底面平行四边形的边长分别为x, y, 侧棱长为z,则有

$$4(x+y) + 4z = 12a, \quad i.e. \;\; x+y+z-3a = 0.$$

易见，同底面时，四棱柱比平行六面体的体积大, 因此只需考虑四棱柱的体积. 更进一步, 边长对应相等的长方形比平行四边形面积更大，因此只需考虑长方体的情形. 设体积$v(x,y,z) = xyz$, 问题等价于求V 在条件$x+y+z-3a=0$ 下的最大值. 记

$$F(x,y,z) = V(x,y,z) + \lambda(x+y+z-3a)$$

F 分别对x, y, z, λ 求导并令其等于0有

$$yz + \lambda = 0 \tag{9.14}$$

$$xz + \lambda = 0 \tag{9.15}$$

$$xy + \lambda = 0 \tag{9.16}$$

$$x + y + z - 3a = 0 \tag{9.17}$$

于是得$x=y=z=a$, 由题知，V 必存在最大值，于是$V_{max}=V(a,a,a)=a^3$, 此时该平行六面体为棱长为a 的正方体, 体积为a^3.

17: 不妨在第一象限讨论

$\frac{x_0dx}{a^2}+\frac{y_0dy}{b^2}=0, dy/dx=-\frac{x_0b^2}{y_0a^2}$

$x=0, y_0+\frac{x_0^2b^2}{y_0a^2}; y=0, x_0+\frac{y_0^2a^2}{x_0b^2}$

$S(x,y)=(y+\frac{x^2b^2}{ya^2})(x+\frac{y^2a^2}{xb^2})$

18: $(\frac{\sum_{i=1}^n x_i}{n}, \frac{\sum_{i=1}^n y_i}{n})$

20. 点(x,y,z)到平面$x+2y+z=9$的距离为$\dfrac{|x+y+2z-9|}{\sqrt{6}}$.

因此只需求出函数$f(x,y,z)=d^2=\dfrac{(x+y+2z-9)^2}{6}$

在约束条件$\dfrac{x^2}{4}+y^2+z^2-1=0$下取最大值、最小值的点.

取Lagrange函数为$F(x,y,z,\lambda)=f(x,y,z)+\lambda(\frac{x^2}{4}+y^2+z^2-1)$. 那么

$$\begin{cases} \dfrac{\partial F}{\partial x}=\dfrac{x+y+2z-9}{3}+\dfrac{\lambda}{2}x=0, \\ \dfrac{\partial F}{\partial y}=\dfrac{x+y+2z-9}{3}+2\lambda y=0, \\ \dfrac{\partial F}{\partial z}=\dfrac{2(x+y+2z-9)}{3}+2\lambda z=0, \\ \dfrac{\partial F}{\partial \lambda}=\dfrac{x^2}{4}+y^2+z^2-1=0. \end{cases} \Longrightarrow \begin{cases} x=\dfrac{4}{3}, \\ y=\dfrac{1}{3}, \\ z=\dfrac{2}{3}. \end{cases} \quad 或 \quad \begin{cases} x=-\dfrac{4}{3}, \\ y=-\dfrac{1}{3}, \\ z=-\dfrac{2}{3}. \end{cases}$$

代入得$f\big(\dfrac{4}{3},\dfrac{1}{3},\dfrac{2}{3}\big)=6, f\big(-\dfrac{4}{3},-\dfrac{1}{3},-\dfrac{2}{3}\big)=24$.

依题意知f一定有最大值、最小值. 因此距离平面最近、最远的点分别是$\big(\dfrac{4}{3},\dfrac{1}{3},\dfrac{2}{3}\big)$和$\big(-\dfrac{4}{3},-\dfrac{1}{3},-\dfrac{2}{3}\big)$.

21: (1)曲面S在(x_0,y_0,z_0)处的切平面方程为

$$\frac{1}{2\sqrt{x_0}}(x-x_0)+\frac{1}{2\sqrt{y_0}}(y-y_0)+\frac{1}{2\sqrt{z_0}}(z-z_0)=0$$

即

$$\frac{x}{\sqrt{x_0}}+\frac{y}{\sqrt{y_0}}+\frac{z}{\sqrt{z_0}}=\sqrt{a}$$

与x,y,z轴截距分别为$\sqrt{ax_0},\sqrt{ay_0},\sqrt{az_0}$,截距之和为$\sqrt{a}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})=\sqrt{a}\cdot\sqrt{a}=a.$

(2)记切平面与各坐标轴截距分别为r,s,t,不妨考虑$r,s,t>0$的情形.围成四面体体积为$V(r,s,t)=\frac{1}{6}rst$.由(1)知$r+s+t-a=0$.令

$$F(r,s,t,\lambda)=\frac{1}{6}rst+\lambda(r+s+t-a)$$

求得驻点方程组

$$\begin{cases}F'_r=\dfrac{1}{6}st+\lambda=0\\ F'_s=\dfrac{1}{6}rt+\lambda=0\\ F'_t=\dfrac{1}{6}rs+\lambda=0\\ F'_\lambda=r+s+t-a=0\end{cases}$$

解得

$$r=s=t=\frac{a}{3}$$

最大体积为

$$V=\frac{1}{6}\left(\frac{a}{3}\right)^3=\frac{a^3}{162}$$

由(1)结果可知切点为$\left(\frac{a}{9},\frac{a}{9},\frac{a}{9}\right)$,从而切平面方程为$x+y+z=\frac{a}{3}$.

注:或用基本不等式求解,简单快捷

9.6 向量场的微商

4: (1)

$$\mathrm{div}[(\boldsymbol{r}\cdot\boldsymbol{w})\boldsymbol{w}] = (\boldsymbol{r}\cdot\boldsymbol{w})(\nabla\cdot\boldsymbol{w}) + \boldsymbol{w}\cdot\nabla(\boldsymbol{r}\cdot\boldsymbol{w}) = \boldsymbol{w}\cdot\boldsymbol{w}.$$

(2)

$$\mathrm{div}\frac{\boldsymbol{r}}{r} = \frac{y^2+z^2}{r^3} + \frac{x^2+z^2}{r^3} + \frac{y^2+x^2}{r^3} = \frac{2}{r}.$$

(3)

$$\mathrm{div}(\boldsymbol{w}\times\boldsymbol{r}) = \boldsymbol{r}\cdot\nabla\times\boldsymbol{w} - \boldsymbol{w}\cdot\nabla\times\boldsymbol{r} = 0.$$

(4)

$$\mathrm{div}(r^2\boldsymbol{w}) = r^2\nabla\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\nabla r^2 = 2\boldsymbol{w}\cdot\boldsymbol{r}.$$

5: (1)

$$\mathrm{rot}\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2z\boldsymbol{i} - 2x\boldsymbol{j} - 2y\boldsymbol{k}$$

(2)

$$\mathrm{rot}\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^y + y & z + e^y & y + 2ze^y \end{vmatrix} = (2ze^y)\boldsymbol{i} - (xe^y + 1)\boldsymbol{k}$$

6: 1、$\mathbf{rot}(\mathbf{w}\times\mathbf{r}) = 2\mathbf{w}$

2、$\mathbf{rot}[f(r)\mathbf{r}] = \mathbf{0}$

3、$\mathbf{rot}[f(r)\mathbf{w}] = \frac{f'(r)}{r}\mathbf{r}\times\mathbf{w}$

4、$div[\mathbf{r}\times f(r)\mathbf{w}] = 0$

7: (1)

$$\begin{aligned}\nabla(\overrightarrow{\omega}\cdot f(r)\overrightarrow{r}) &= \overrightarrow{\omega}\cdot\overrightarrow{r}\nabla f(r)+f(r)\overrightarrow{\omega}\\ &= (\overrightarrow{\omega}\cdot\overrightarrow{r})f'(r)\frac{\overrightarrow{r}}{|\overrightarrow{r}|}+f(r)\overrightarrow{\omega}\end{aligned}$$

(2)

$$\begin{aligned}\nabla\cdot(\overrightarrow{\omega}\times f(r)\overrightarrow{r}) &= f(r)\overrightarrow{r}\cdot\nabla\times\overrightarrow{\omega}-\overrightarrow{\omega}\cdot\nabla\times[f(r)\overrightarrow{r}]\\ &= f(r)\overrightarrow{r}\cdot\nabla\times\overrightarrow{\omega}-\overrightarrow{\omega}\cdot(\nabla f(r)\times\overrightarrow{r}+f(r)\nabla\times\overrightarrow{r})\\ &= f(r)\overrightarrow{r}\cdot\nabla\times\overrightarrow{\omega}\\ &= 0\end{aligned}$$

(3).

$$\begin{aligned}\overrightarrow{\nabla}\times(\overrightarrow{\omega}\times f(r)\overrightarrow{r}) &= \nabla f(r)\times(\overrightarrow{\omega}\times\overrightarrow{r})+f(r)\nabla\times(\overrightarrow{\omega}\times\overrightarrow{r})\\ &= f'(r)\frac{\overrightarrow{r}}{|\overrightarrow{r}|}\times(\overrightarrow{\omega}\times\overrightarrow{r})+f(r)\nabla\times(\omega_2 z-\omega_3 y,\omega_3 x-\omega_1 z,\omega_1 y-\omega_2 x)\\ &= f'(r)r\overrightarrow{\omega}-f'(r)(\frac{\overrightarrow{r}}{r}\cdot\overrightarrow{\omega})\overrightarrow{r}+2f(r)\overrightarrow{\omega}\end{aligned}$$

8: 设$\phi=\phi(x,y,z),\psi=\psi=\psi(x,y,z),\boldsymbol{a}=P_1\boldsymbol{i}+Q_1\boldsymbol{j}+Z_1\boldsymbol{k},\boldsymbol{b}=P_2\boldsymbol{i}+Q_2\boldsymbol{j}+Z_2\boldsymbol{k}$.

(1) 按定义计算可得.

(2) 按定义计算可得.

(3) 由行列式的知识知

$$\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{x} & \frac{\partial}{y} & \frac{\partial}{z}\\ P_1+P_2 & Q_1+Q_2 & R_1+R_2\end{vmatrix}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P_1 & Q_1 & R_1\end{vmatrix}+\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P_2 & Q_2 & R_2\end{vmatrix} \tag{9.18}$$

于是$\nabla \times (\boldsymbol{a}+\boldsymbol{b}) = \nabla \times \boldsymbol{a} + \nabla \times \boldsymbol{b}$

(4) 由乘积的求导法则易得.

(5) 由乘积的求导法则易得.

(6) 直接计算得.

(7) 直接计算得.

9.7 微分形式*

9.8 综合习题

4: 由齐次函数的Euler定理, f是n次其次函数等价于

$$xf'_x + yf'_y + zf'_z = nf.$$

当$f(x,y,z)=0$时, 有

$$z = z - nf = -\frac{xf;_x + yf'_y}{f'_z} = x\phi'_x + y\phi'_y.$$

这意味着$\phi(x,y)$是一次齐次函数.

6: 证明:

令:

$$f\left(x,y\right) = \frac{1}{4}\left(x^2+y^2\right) - e^{x+y-2}$$

易知:

$$f_x^{'} = \frac{1}{2}x - e^{x+y-2}, f_y^{'} = \frac{1}{2}y - e^{x+y-2}$$

联立其偏导均为0，可以得到其在定义域上无驻点，且在边界有：

$$f(x,0)=\frac{1}{4}x^2-e^{x-2}\leqslant 0, f(0,y)=\frac{1}{4}y^2-e^{y-2}\leqslant 0$$

$$\lim_{x\to\infty,y\to\infty} f(x,y)=-\infty$$

所以$f(x,y)$在定义域上不大于0，由此：

$$\frac{x^2+y^2}{4}\leq e^{x+y-2}$$

证毕

7: 任取$(x_0,y_0,z_0)\in R^2$

记$D=\{(x,y,z)\big||x-x_0|\leqslant 1,|y-y_0|\leqslant 1,|z-z_0|\leqslant 1\}$

则$\exists M,s.t.((x,y,z)\in D\implies |\frac{\partial f(x,y,z)}{\partial x}|\leqslant M,|\frac{\partial f(x,y,z)}{\partial y}|\leqslant M)$

$\forall\epsilon>0,$

由f关于z的连续性,$\exists\delta_0>0,s.t.(|z-z_0|<\delta_0\implies|f(x_0,y_0,z_0+\Delta z)-f(x_0,y_0,z_0)|<\frac{\epsilon}{3})$

取$\delta=min\{\delta_0,\frac{\epsilon}{3M},1\}$

则当$|(x,y,z)-(x_0,y_0,z_0)|<\delta$时,$(x,y,z)\in D$

记$\Delta x=x-x_0,\Delta y=y-y_0,\Delta z=z-z_0$

则$|\Delta x|<\frac{\epsilon}{3M},|\Delta y|<\frac{\epsilon}{3M},|\Delta z|<\delta_0$

故

$$
\begin{aligned}
&|f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z)-f(x_0,y_0,z_0)|\\
&\leqslant |f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z)-f(x_0, y_0+\Delta x, z_0+\Delta z)|+|f(x_0, y_0+\Delta y, z_0+\Delta z)\\
&\quad -f(x_0,y_0,z_0+\Delta z)|+|f(x_0,y_0,z_0+\Delta z)-f(x_0,y_0,z_0)|\\
&=\Big|\frac{\partial f(x+p\Delta x, y+\Delta y, z+\Delta z)}{\partial x}\Delta x\Big|+\Big|\frac{\partial f(x, y+q\Delta y, z+\Delta z)}{\partial x}\Delta y\Big|\\
&\quad +|f(x_0,y_0,z_0+\Delta z)-f(x_0,y_0,z_0)(\text{其中}(0\leqslant p,q\leqslant 1))\\
&<\epsilon
\end{aligned}
$$

由此即说明了f的连续性

8: 对任意$(x,y)\in\mathcal{D}$, 由二元函数的中值定理有

$$f(x,y)-f(0,0)=xf'_x(\theta x,\theta y)+yf'_y(\theta x,\theta y),\quad \theta\in[0,1],$$

由题设知$f(x,y)=f(0,0)$. 由(x,y) 的任意性知f为常值函数.

12: 令$\phi(t)=(x(b)-x(a))x(t)+(y(b)-y(a))y(t)$, 则由微分中值定理, 存在$\theta\in(a,b)$, 使得

$$\phi(b)-\phi(a)=\phi'(\theta)(b-a).$$

又因为

$$
\begin{aligned}
|\boldsymbol{r}(b)-\boldsymbol{r}(a)|^2 &= (x(b)-x(a))^2+(y(b)-y(a))^2 \\
&= \phi(b)-\phi(a)=\phi'(\theta)(b-a) \\
&= [(x(b)-x(a))x'(\theta)+(y(b)-y(a))y'(\theta)](b-a) \\
&\leq \sqrt{(x(b)-x(a))^2+(y(b)-y(a))^2}\cdot\sqrt{x'(\theta)+y'(\theta)}(b-a) \\
&= |\boldsymbol{r}(b)-\boldsymbol{r}(a)||\boldsymbol{r}'(\theta)|(b-a).
\end{aligned}
$$

移项就有

$$|\boldsymbol{r}(b)-\boldsymbol{r}(a)|\leq|\boldsymbol{r}'(\theta)|(b-a).$$

14: 证明：对于：

$$f\left(x,y\right)=x^2+xy^2-x$$

分别对x,y求偏导为：

$$f_x^{'}=2x+y^2-1,f_y^{'}=2xy$$

由此，求得驻点为：

$$\left(0,\pm1\right),\left(\frac{1}{2},0\right)$$

函数在各个驻点取值分别为：

$$f\left(0,\pm1\right)=0,f\left(\frac{1}{2},0\right)=-\frac{1}{4}$$

而在边界$x^2+y^2=2$上，令：

$$g\left(x,y\right)=f\left(x,y\right)-\lambda\left(x^2+y^2-2\right)$$

求其偏导为：

$$g_x^{'}=f_x^{'}-2\lambda x,g_y^{'}=f_y^{'}-2\lambda y,g_\lambda^{'}=x^2+y^2-2$$

得到驻点为：

$$\left(-\frac{1}{3},\pm\frac{\sqrt{17}}{3}\right),(1,\pm1),\left(\sqrt{2},0\right),\left(-\sqrt{2},0\right)$$

求得在边界的最大值和最小值分别为：

$$Min\left(f,x^2+y^2=2\right)=\frac{11}{27},Max\left(f,x^2+y^2=2\right)=2+\sqrt{2}$$

综上，可以得到函数在定义域上的最大值和最小值分别为：

$$Min\left(f\right)=-\frac{1}{4},Max\left(f\right)=2+\sqrt{2}$$

15: $f(x_1,x_2,\dots x_n)=\prod_{i=1}^n x_i\sum_{j=1}^n\frac{1}{x_j}$

$\varphi(x_1,x_2,\dots x_n)=\sum_{j=1}^n x_j-n$

$F(x_1,x_2,\dots x_n)=f(x_1,x_2,\dots x_n)+\lambda\varphi(x_1,x_2,\dots x_n)$

令

$$\begin{aligned}\frac{\partial F}{\partial x_i}&=\prod_{k=1,k\neq i}^n x_k\sum_{j=1}^n\frac{1}{x_j}+\prod_{j=1}^n x_j(-\frac{1}{x_i^2})+\lambda\\&=\prod_{k=1,k\neq i}^n x_k\sum_{j=1,j\neq i}^n\frac{1}{x_j}=0\end{aligned}$$

$$\varphi(x_1,x_2,\dots x_n)=0$$

从而可解得$x_1=x_2=\cdots=x_n=1$为唯一极值点，从而为极大值

则$\prod_{i=1}^n x_i\sum_{j=1}^n\frac{1}{x_j}\leqslant n$

等号成立当且仅当$x_1=x_2=\cdots=x_n=1$

16: 考虑函数

$$f(x_1,x_2,...,x_n)=\frac{x_1^p+x_2^p+\cdots+x_n^p}{n}$$

在条件$\frac{x_1+x_2+\cdots+x_n}{n}=A\geq 0$ 下的极值. 不妨设$A>0$, 令

$$F(x_1,x_2,...,x_n)=f(x_1,x_2,...,x_n)+\lambda\left(\frac{x_1+x_2+\cdots+x_n}{n}-A\right)$$

分别对$x_1,x_2,...,x_n,\lambda$ 求导有

$$\frac{1}{n}(px_i^{p-1}+\lambda)=0,\quad i=1,...,n \tag{9.19}$$

$$\frac{x_1+x_2+\cdots+x_n}{n}-A=0 \tag{9.20}$$

解的$x_1=x_2=\cdots=x_n=A$. 由题意知, f 的最小值一定存在, 故$(A,A,...,A)$为最小值点, 且

$$f(x_1,...,x_n)\geq f(A,\cdots,A)=A^p=\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^p$$

且等号当且仅当

$$x_i=\begin{cases}\dfrac{x_1+x_2+\cdots+x_n}{1(x_1>0)+1(x_2>0)+\cdots+1(x_n>0)}, & x_i\neq 0\\ 0 & \end{cases}$$

或者考虑$Hessian$ 矩阵的正定性: $H(A,A,...,A)=\frac{p(p-1)}{n}A^{p-2}\boldsymbol{I}>0\ \ (p>1,A>0)$.