A TREATISE

ON

OCTONIONS.

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OCTONIONS

A DEVELOPMENT

OF

CLIFFORD'S BI-QUATERNIONS

BY

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PREFACE.

I OWE a great debt of gratitude to an old pupil for the results of a casual conversation I had with him some six or seven years ago. On that occasion Mr P. à M. Parker discoursed of rotors and motors in such wise that it seemed to his tutor high time to rub the dust from the volume of Clifford's *Mathematical Papers* lying on the shelves; for otherwise the tutor and pupil bade fair to change places. Many days of most interesting work and thought have been the sequel of that talk within the walls of Ormond College.

The treatment below of what Clifford called Bi-quaternions runs on two sharply-defined lines. Quaternions and the Ausdehnungslehre have both been pressed into the service, and the help from them has led to very different kinds of development. Neither development could, in my opinion, be well spared. The first seems to be allied to metrical geometry and the second to descriptive. At any rate I do not see how, in few words, better to describe the essential characteristics of the two. For more precise ideas the reader must study the subject itself.

So far as the present treatise is concerned, these developments took place in two periods. I had done what I could on the quaternion model, but being dissatisfied because so many questions which presented themselves were thereby but imperfectly answered, I put the work aside. Meanwhile I had been led by Sir Robert Ball's *Theory of Screws* (Dublin, 1876)

to study the *Ausdehnungslehre* (1862), and was delighted to find that the gaps could apparently be filled from this source. On taking the subject up again it was found that this surmise was correct.

Perhaps the most striking fact that has come to light in the investigation is one that appeared almost at the outset, and one which mainly induced me to proceed. I mean the fact that every quaternion formula except such as involve ∇ admits of an octonion interpretation—a geometrical interpretation much more general than that which it was primarily meant to have. There is matter for reflection in that the founders of Quaternions, while they were busying themselves only with vector and quaternion conceptions, were, all the time, unknown to themselves, establishing motor and octonion truths.

There is a corresponding though less striking fact connected with the second method of development. The statements of Quaternions were always intended to have but one meaning—of course with many variations of form when put into words—but the Ausdehnungslehre was intended to be a general framework of symbols whose applications should be in many provinces of thought. It is therefore not surprising that there are geometrical interpretations which were not developed, even if seriously contemplated, by Grassmann. Such are the applications of the Ausdehnungslehre below. I cannot believe that Grassmann contemplated such applications of his calculus if only because apparently he never conceived of a magnitude other than zero whose "numerical value," in his own technical sense, was zero.

It may be asked why, in this treatise, I start *de novo*, instead of taking all that Clifford has done for granted. The reasons are (1) the desirability of making the treatise self-contained; (2) the fact that Clifford uses a method dependent on the properties of non-Euclidean space, whereas I regard the subject as referring

to Euclidean space; and (3) I do not altogether understand all of Clifford's arguments.

The treatise suffers in form, somewhat, from the fact that it was not, in the making, meant to be a book but a "paper," as will be directly explained. If I had from the beginning contemplated the book form, or if when the treatise became destined to take that form, I had thought myself justified, or indeed had had the courage, to recast the whole appropriately, it would have been at least doubled in length, without probably any material mathematical amplification. Not only is it too condensed, where the argument is fairly covered, but many steps of reasoning are left out which the reader will require patience to supply, as I myself have found in reading the proof-sheets. I can but apologise to the reader for these rather irritating defects.

There is a defect wholly unconnected with this, which qualified critics may help to remedy. I refer to the terminology. I have found myself compelled to invent quite a little vocabulary; and if ever there was an author in such an uncomfortable position whose ignorance of dead and other languages was more profound than the writer's, I pity him. About the term "Octonion" I shall speak directly. The three groups of terms *augmenter, tensor, additor, pitch*; and *twister, versor, translator*; and *velocity motor, force motor, momentum motor*; and their congeners I am (pending criticism) content with. The group of terms referring to linear motor functions of motors; *general function, commutative function, pencil function, energy function*; are passable. The terms *convert, convertor, axial quaternion* (or *axial*) and some less frequently used seem to me like unwilling conscripts begging at any price for substitutes. The term *variation* as used in the treatise is objectionable on the ground that it clashes with the technical meaning of the same term in Algebra. If I had thought that any serious inconvenience would result I should have used some such term as *replacement*, but I thought *variation* better. [It must be

remembered that in the technical use of this treatise the term *variation* is always qualified by some adjective such as *combinatorial* or *circular*.]

I have in the treatise itself tried to justify my deviation from Clifford's usage of *vector* and *quaternion* (replaced below by *lator* and *axial*), but I have given no reasons for the serious step of changing the name of the whole subject from Bi-quaternions to Octonions. The following reasons seemed to form sufficient justification. (1) I think it desirable to have a name for what Hamilton has called Bi-quaternions. For these there could scarcely be a better name. (2) I wish to imply that quaternions are not particular kinds of octonions but only very similar to such particular kinds. (3) Octonions like quaternions treat space impartially. By this I mean that they do not depend in any way on an arbitrarily chosen system of axes or arbitrary origin. But one of the two quaternions implied by Clifford's term does so depend on an arbitrary origin. This to me appears an absolute bar to the propriety of his term. If Clifford, in choosing his term, wished to emphasise his indebtedness to the inventor of Quaternions, this is scarcely a reason for one who merely follows Clifford to copy him in this respect, if there are intrinsic objections.

Most of the methods and some of the results which follow are to some extent, I believe, novel. But I fear that many references to the work of others which ought to occur are wanting. The treatise was written at a distance from all mathematical libraries. I believe I should have been able to improve it in many respects if I had been able to consult the many authorities cited by Sir Robert Ball in his *Theory of Screws*.

Explicit references to treatises on Quaternions are almost wholly omitted, as the reader must be assumed familiar with this subject. The references to the *Theory of Screws* and to the *Ausdehnungslehre* are copious. Except from these subjects and treatises I have, as far as I know, received no aid from the work of others.

The treatise was communicated to the Royal Society on 28 Nov. 1895 and read on 12 Dec. The Council of the Royal Society, considering its nature to be more that of a book than a "paper," offered to aid its publication in book form by making (from a fund voted by the Treasury for such purposes) a substantial pecuniary contribution to the expenses. The Cambridge Press Syndicate has borne all the other expenses. But for these aids, the treatise could not have been published. The deep gratitude I feel to both these bodies, and beg here to express, will be appreciated by all who have desired to present their reflections to the many sympathetically minded though difficult of access.

For obvious geographical reasons, and for others scarcely less obvious, I was wholly unable to act for myself in the arrangements which were terminated as above described. I am probably ignorant of some who disinterestedly did this work for me, but I know a large share was done by Dr Ferrers and Sir Robert Ball. My warmest thanks are due to them for this labour of kindness.

To Prof. Forsyth I am indebted for the valuable suggestion among others that the treatise should be preceded by a short sketch of the main argument.

There are probably many errors remaining in the book. That there are not more is due to the kindness of Mr P. à M. Parker of St John's College, Cambridge, and Mr G. H. A. Wilson of Clare College, Cambridge, in carefully revising the proof-sheets, and to the great care exercised at the Cambridge University Press.

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1 *August*, 1898.

DEFINITIONS AND EXPLANATIONS OF TERMS.

OCTONIONS.

SKETCH OF THE ARGUMENT.

THE following sketch is given—at the suggestion of Dr Forsyth—to enable the reader more easily to follow the main argument. It will probably also serve for the purposes of a table of contents. The numbers at the beginning of the following paragraphs refer to the sections of the text.

2. The subject of Quaternions may be described as a peculiar symbolic method of treating Geometry. But the formulae which occur may possibly have interpretations other than those which originally led up to them. A large part of the present treatise is concerned with the development of such other interpretations. It is desirable therefore to separate the purely symbolic laws of Quaternions from their geometrical interpretations. This has been done below under the heading "Fundamental formal laws of Quaternions." These so-called laws are below divided into six groups.

3. To convince the reader that these laws do really contain all the fundamental elements of quaternion formulae, some of the principal general formulae of Quaternions are deduced from them.

4. It is next shown that, connected with any system of symbols which obey the laws mentioned, there is a second system. These are called the primary and secondary systems of Formal Quaternions respectively. The two systems are connected by means of six equations in which an additional symbol, Ω, occurs. This last behaves exactly like a formal scalar of the primary system except that Ω^2 always $= 0$, i.e. it behaves in a sense like an infinitesimal

formal scalar of the primary system. Q denoting a formal quaternion of the secondary system, and q and r formal quaternions of the primary system, the equations are

$$Q = q + \Omega r \quad \text{(1)},$$

$$\mathbf{K}Q = \mathbf{K}q + \Omega\mathbf{K}r \quad \text{(2)},$$

$$\mathbf{S}Q = \mathbf{S}q + \Omega\mathbf{S}r \quad \text{(3)},$$

$$\mathbf{V}Q = \mathbf{V}q + \Omega\mathbf{V}r \quad \text{(4)},$$

$$\mathbf{T}Q = (1 + \Omega\mathbf{S}rq^{-1})\,\mathbf{T}q \quad \text{(5)},$$

$$\mathbf{U}Q = (1 + \Omega\mathbf{V}rq^{-1})\,\mathbf{U}q \quad \text{(6)}.$$

It is shown that if the primary system obeys the six fundamental groups of laws, so does the secondary system. It is therefore subsequently assumed that if every quaternion formula is true for the primary system it is also true for the secondary system. $\mathbf{V}$ is subsequently changed to $\mathbf{M}$ because the connotation of $\mathbf{V}$ does not hold in Octonions.

5, 6. These purely symbolic preliminary matters being disposed of, an octonion as a geometrical magnitude is described. Rotors and motors are defined in ways equivalent to Clifford's. An octonion is defined as a magnitude requiring for its specification a motor and two scalars; of which one is called its ordinary scalar and the other its convert. The term "lator" is used for Clifford's "vector" and "axial quaternion" or "axial" for his "quaternion" wherever these terms occur in his papers on Biquaternions. It is taken as a fundamental definition that axials through a given point O and their included system of rotors through O and ordinary scalars obey among themselves all the laws of the corresponding quaternions and their included system of vectors and scalars. Axials through O are then taken as the primary system spoken of above and octonions as the secondary system. Any octonion Q can be specified uniquely by means of two axials through O. Denoting these by q and r and assuming equations (1) to (6) above to hold ($\mathbf{M}$ replacing $\mathbf{V}$) the geometrical meanings of $\mathbf{K}Q$, $\mathbf{M}Q$, etc. are determinate, though not obvious. The specification of Q in terms of q and r and conversely of q and r in terms of Q may be expressed as follows.—Q is completely determined by its motor, its ordinary scalar and its convert. The motor of Q again is determined by its axis and the magnitudes (including the senses parallel to the axis) of its rotor and lator.

$$\left.\begin{array}{l}\text{Ordinary scalar of } Q = \mathbf{S}q,\ \text{convert} = \mathbf{S}r,\\ \text{rotor perpendicular from } O \text{ on axis of } Q = \mathbf{M}\,.\,\mathbf{M}r\mathbf{M}^{-1}q,\\ \text{rotor through } O \text{ parallel and equal to rotor of } Q = \mathbf{M}q,\\ \quad ,, \quad ,, \quad ,, \quad ,, \quad ,, \quad ,, \quad \text{lator } ,, \ = \mathbf{M}q\mathbf{S}\,.\,\mathbf{M}r\mathbf{M}^{-1}q.\end{array}\right\} \text{(7)},$$

$$\left.\begin{array}{l}q = (\text{ordinary scalar of } Q) + (\text{rotor through } O \text{ parallel and}\\ \qquad \text{equal to rotor of } Q),\\ r = (\text{convert of } Q) + \varpi\mathbf{M}q + \lambda, \text{ where } \varpi \text{ is the rotor per-}\\ \qquad \text{pendicular from } O \text{ on the axis of } Q \text{ and } \lambda \text{ is the}\\ \qquad \text{rotor through } O \text{ parallel and equal to the lator of } Q.\end{array}\right\} \ldots(8).$$

7. Every quaternion formula may now be read as an octonion formula and will have *some* geometrical interpretation in connection with octonions. It is not obvious that this interpretation will have any great utility. It is therefore shown that although O a fixed point is used in the above definitions the geometrical meaning of any such formula is quite independent of O. For instance (to take an example from a later part of the treatise) if A, B, C are three motors connected by the equation $C = \mathbf{M}AB$, the axis of C is the shortest distance between the axes of A and B, its rotor bears to the rotors of A and B the same relation as the vector $\mathbf{V}\alpha\beta$ does to the vectors α and β in Quaternions, and its pitch is the sum of $d \cot\theta$ and the pitches of A and B where d is the distance and θ the angle between A and B, d being reckoned positive or negative, according as the shortest twist which will bring the rotor of either A or B into coincidence, both as to axis and sense, with the rotor of the other is a right-handed or left-handed one.

8—15. The geometrical interpretations of octonion formulae, thus shown to be independent of an arbitrary origin, are examined in considerable detail.

16—19. A classification of linear motor functions of motors is made. The analogue of the corresponding quaternion function, which is not the most general type of the octonion function, is called a commutative function. Many of its properties are examined in detail. In particular the self-conjugate commutative function is examined.

20. The differentiation of all the octonion functions symbolised in the treatise is examined.

21—26. Octonions are adapted for use in physical questions. They are in this respect very similar to quaternions.

27. The quaternion analogy, fruitful as it is, proving—at any rate in my hands—nevertheless insufficient to furnish methods for answering various interesting questions that octonions present, recourse is had to Grassmann's *Ausdehnungslehre.* It seemed especially desirable to identify if possible Grassmann's meaning of "normal" with Sir Robert Ball's meaning of "reciprocal" as applied to screws. With Grassmann's own geometrical treatment of motors as quantities of the second order this is found to be impossible. But by treating motors as quantities of the first order the identification can be made. But it turns out that real motors with negative pitch must be regarded as imaginary quantities of the first order; and there are motors, viz. all rotors and all lators, which cannot in a calculus of octonions be regarded as zero, which nevertheless have in Grassmann's sense zero numerical value. Hence while his methods or extensions of them are applicable his theorems have to be carefully revised for our purposes.

28. Products and combinatorial products are defined in ways which though different in form from Grassmann's definitions yet harmonise with his. Two particular kinds of combinatorial products (of 5 and 6 motors respectively, the former product being a motor reciprocal to each of the five and the latter an ordinary scalar) directly suggested by some of Grassmann's products are introduced and used very frequently in subsequent parts of the treatise.

29. Grassmann's linear variation and circular variation are generalised to a new type called combinatorial variation. The uses, and they are many, that are subsequently made of this are all very similar to Grassmann's uses of his two species. The property of a combinatorial variation on which much of its usefulness depends is that a combinatorial product is unaltered by the variation.

30—34. By means of the methods thus introduced many properties of the general self-conjugate linear motor function, ϖ, of a motor are examined. They may all be expressed as properties of complexes of the second degree which are generally of arbitrary order. Putting $\varpi = 1$ we get also many properties of reciprocal

motors and reciprocal complexes of the first degree. A particular form of ϖ, called an energy-function, which is of primary importance in the consideration of the motion of a rigid body proves to have, comparatively, very simple properties.

35, 36. Passing to the general function which is not necessarily self-conjugate Grassmann's methods are further utilised.

37. The bearing of these results on the general self-conjugate and on commutative functions is examined. It is found that several properties of the general self-conjugate that might, by analogy with the corresponding quaternion case, be expected to exist are non-existent.

38. The commutative self-conjugate is returned to and several additional properties are proved by a method analogous to that used for the general self-conjugate.

39. Similarly the general function is returned to and treated by a method analogous to that used for the commutative function earlier in the treatise.

40. Combinatorial variation is used to establish some miscellaneous results especially in connection with second and third order complexes of the first degree.

41. The simplest forms of complexes of the first degree of all orders are established.

42—49. Several applications of Octonions are made. These are all suggested by Sir Robert Ball's 'Theory of Screws.'

CHAPTER I.

FORMAL QUATERNIONS.

1. **Quaternion formulae susceptible of new geometrical interpretations.** In the course of the development of our subject it will become apparent that every quaternion formula which involves any combination of symbols representing quaternions, scalars, vectors, linear quaternion functions of quaternions, linear vector functions of vectors, conjugate and self-conjugate ditto (i.e. the two separable ϕ's) and the symbols i, j, k, **K**, **S**, **T**, **U**, **V**, ζ is susceptible of a geometrical interpretation quite different from the ordinary one, an interpretation in the subject of Octonions. This interpretation like the ordinary one treats the different parts of and directions in space with perfect impartiality. Each symbol in the new interpretation involves just double the number of ordinary scalars that are involved in the old. Thus the symbol corresponding to a scalar involves two ordinary scalars, a vector six, a quaternion eight, a linear vector function of a vector eighteen, a self-conjugate ditto twelve, i four and so on. I have not been able to include in this interpretation formulae involving ∇ though two quite distinct octonion analogues of ∇ are used below.

It will be remembered by the readers of Clifford's Mathematical Papers that he continually speaks of the dual interpretation of certain quaternion equations. Below I shall give reasons for regarding this dual interpretation as in reality but a dual aspect of a single interpretation. But accepting Clifford's meaning of duality for the moment we must now add that there is a second dual interpretation also to be given to such equations. Whether or not in the future other symmetrical geometrical interpretations of quaternion formulae will come to light cannot of course be said, but there seems no reason for believing the contrary.

Since our interpretations of Quaternion formulae are to be different from the ordinary ones it is desirable to investigate the purely symbolic laws underlying those formulae.

2. Fundamental formal laws of Quaternions. Just as in ordinary Algebra there are certain fundamental formal laws on which the whole subject may be based quite apart from the applications to arithmetic quantity, so there are certain fundamental formal laws of quaternion symbols (i.e. the sixteen symbols just enumerated) from which all quaternion formulae can be derived, quite apart from their geometrical interpretation. We must collect these here because we wish to consider interpretations of the formulae different from the ordinary ones. The subject when thus considered apart from its geometrical interpretation will be called Formal Quaternions.

In the following few statements (§§ 2, 3) about formal quaternions Hamilton's conventions as to notation are adopted though this course has not been found advisable in the paper generally. The conventions referred to are that formal vectors shall be denoted by small Greek letters, the linear function by ϕ, formal quaternions by p, q, r, scalars by x, y, &c.

The following are the fundamental laws:—

(1) Every formal quaternion q can be expressed as the sum of two parts called its formal vector part and its formal scalar part, denoted by $\mathbf{V}q$ and $\mathbf{S}q$ respectively. A formal scalar means a formal quaternion whose formal vector part is zero and a formal vector one whose formal scalar part is zero. A particular case of a formal scalar is an ordinary scalar.

(2) All the fundamental laws of ordinary Algebra except the commutative law for multiplication are true of formal quaternions.

(3) Formal scalars and in particular ordinary scalars are commutative with formal quaternions (so that formal scalars obey among themselves *all* the fundamental laws of Algebra).

(4) Of the five symbols $\mathbf{K}q$, $\mathbf{S}q$, $\mathbf{V}q$, $\mathbf{T}q$, $\mathbf{U}q$, the two $\mathbf{S}q$ and $\mathbf{T}q$ are formal scalars and the five satisfy the following conditions:—

(*a*) $q = \mathbf{S}q + \mathbf{V}q = \mathbf{T}q \,.\, \mathbf{U}q$,

(*b*) $\mathbf{TU}q = 1$, $\mathbf{T}(qr) = \mathbf{T}q\mathbf{T}r$,

(*c*) $\mathbf{K}q = \mathbf{S}q - \mathbf{V}q = (\mathbf{T}q)^2 q^{-1}$,

(*d*) x, y being any formal scalars and α, β, any formal vectors, $x + y$ and xy are also formal scalars and $\alpha + \beta$ and $x\alpha$ formal vectors. [For instance $q\mathbf{K}q = (\mathbf{T}q)^2 =$ formal scalar, $\alpha^2 = -\alpha K\alpha = -(\mathbf{T}\alpha)^2 =$ formal scalar.] x^{-1} has a unique meaning. $(x^2)^{\frac{1}{2}} = \pm x$.

(5) α and β two formal vectors can be found such that $\mathbf{T}\alpha$, $\mathbf{TV}\alpha\beta$ and $\mathbf{T}(\alpha\mathbf{V}\alpha\beta)$ are all ordinary scalars not zero.

(6) $\phi q = \Sigma aqb$ (where a and b are given formal quaternions and q an arbitrary one) is defined as the linear formal quaternion function of a formal quaternion; and $\phi\rho = \Sigma\mathbf{V}a\rho b$ (where a and b are as before and ρ is an arbitrary formal vector) is defined as the linear formal vector function of a formal vector. [In Octonions which are formal quaternions we shall, however, have more general forms than these for linear functions.]

It may be remarked that conditions (1) to (4) are alone sufficient for all formulae that do not involve ϕ, i, j, k and that if (6) be added all formulae not involving i, j, k are true. (1) to (5) are sufficient for all formulae not involving ϕ.

3. Some general formulae deduced from the fundamental formal laws. It would involve too long a digression to attempt to give a satisfactory proof that these contain all the fundamental formal laws of quaternions. I am not sure that in the above the laws have been reduced to their simplest form. But though the above statements may to a certain extent be redundant, the following deductions from them of the chief general formulae of Quaternions will, I think, serve to convince the reader that they are at any rate sufficient.

We will then prove first that

$$\mathbf{V}\alpha\beta\gamma = \alpha\mathbf{S}\beta\gamma - \beta\mathbf{S}\gamma\alpha + \gamma\mathbf{S}\alpha\beta$$

(from which may be at once deduced in the ordinary way that $\rho\mathbf{S}\alpha\beta\gamma = \alpha\mathbf{S}\beta\gamma\rho + \ldots = \mathbf{V}\beta\gamma\mathbf{S}\alpha\rho + \ldots$); secondly that three formal vectors i, j, k can be found such that

$$jk = i,\ ki = j,\ ij = k,\ i^2 = j^2 = k^2 = ijk = -1\,;$$

and thirdly that if ζ_1 be defined by the equation

$$\psi(\zeta_1, \zeta_1) = \psi(i, i) + \psi(j, j) + \psi(k, k),$$

ζ_2 by

$$\chi(\zeta_1, \zeta_1, \zeta_2, \zeta_2) = \chi(\zeta_1, \zeta_1, i, i) + \chi(\zeta_1, \zeta_1, j, j) + \chi(\zeta_1, \zeta_1, k, k),$$

and so to any number of pairs of ζ's (ψ and χ being formal quaternion functions linear in each of their constituents) then

$$\omega \mathbf{S}\zeta_1\zeta_2\zeta_3 \mathbf{S}\phi\zeta_1\phi\zeta_2\phi\zeta_3 = -3\mathbf{V}\zeta_1\zeta_2\mathbf{S}\phi\omega\phi\zeta_1\phi\zeta_2,$$

where ω is any formal vector and ϕ any linear formal vector function of a formal vector. [Note that from the last can be deduced the cubic (with formal scalar coefficients) satisfied by ϕ just as in the case of Quaternions—*Utility of Quaternions in Physics*, p. 17, foot-note. And again, from the cubic in this form can be deduced the usual forms and in particular the various forms for ϕ^{-1}. This will be shown when we come to treat of the corresponding part of Octonions.] In proving these three general formulae we shall incidentally prove several other general formulae well known in Quaternions.

Notice first from (4) (c) that since it is assumed [(4) (d)] that when x is a formal scalar x^{-1} has a unique meaning, it follows that q^{-1} where q is a formal quaternion also has a unique meaning.

Next, notice that $\mathbf{K}x = x$, and therefore that

$$(\mathbf{T}x)^2 = x\mathbf{K}x = x^2.$$

Hence $\mathbf{T}x = \pm x$ by (4) (d). [In Quaternions it is usual to assume that if x be positive the upper sign is to be taken and if x be negative the lower sign. In fact $\mathbf{T}q$ in Quaternions is always assumed to be a *positive* scalar, but this assumption though convenient is not *necessary*. Similarly in Octonions an analogous convenient assumption is made though this again is not necessary.]

q^{-1} is of course defined as the formal quaternion which gives $qq^{-1} = 1$. From this we have $(qr)^{-1} = r^{-1}q^{-1}$. Also [(4) (b)] $\mathbf{T}(qr) = \mathbf{T}q\mathbf{T}r$ and $\mathbf{TU}q = 1$. From these and the equation

$$\mathbf{KK}q = \mathbf{S}q + \mathbf{V}q = q,$$

it follows that

$$\mathbf{T}(q^{-1}) = (\mathbf{T}q)^{-1}, \quad \mathbf{K}qr = \mathbf{K}r\mathbf{K}q.$$

From the second of these we have

$$\mathbf{K}(q_1q_2 \ldots q_n) = \mathbf{K}q_n\mathbf{K}q_{n-1} \ldots \mathbf{K}q_1.$$

Again if α be a formal vector

$$\mathbf{K}\alpha = \mathbf{S}\alpha - \mathbf{V}\alpha = -\alpha,$$

and therefore

$$\mathbf{K}(\alpha_1\alpha_2 \ldots \alpha_n) = (-1)^n \alpha_n\alpha_{n-1} \ldots \alpha_1.$$

From this since

$$q + \mathbf{K}q = 2\mathbf{S}q, \quad q - \mathbf{K}q = 2\mathbf{V}q,$$

$$2\mathbf{S}(\alpha_1 \ldots \alpha_n) = \alpha_1 \ldots \alpha_n + (-1)^n \alpha_n \ldots \alpha_1 = (-1)^n 2\mathbf{S}(\alpha_n \ldots \alpha_1).$$

$$2\mathbf{V}(\alpha_1 \ldots \alpha_n) = \alpha_1 \ldots \alpha_n - (-1)^n \alpha_n \ldots \alpha_1 = -(-1)^n 2\mathbf{V}(\alpha_n \ldots \alpha_1).$$

In particular

$$\alpha\beta + \beta\alpha = 2\mathbf{S}\alpha\beta = 2\mathbf{S}\beta\alpha, \quad \mathbf{V}\alpha\beta + \mathbf{V}\beta\alpha = 0.$$

Thus

$$\mathbf{V}\alpha(\beta\gamma + \gamma\beta) = 2\alpha\mathbf{S}\beta\gamma,$$

$$\mathbf{V}(\alpha\beta + \beta\alpha)\gamma = 2\gamma\mathbf{S}\alpha\beta,$$

$$\mathbf{V}(\alpha\gamma \,.\, \beta + \beta \,.\, \alpha\gamma) = 2\beta\mathbf{S}\gamma\alpha.$$

Adding the first two and subtracting the third of these we get

$$\mathbf{V}\alpha\beta\gamma = \alpha\mathbf{S}\beta\gamma - \beta\mathbf{S}\gamma\alpha + \gamma\mathbf{S}\alpha\beta.$$

Next choose α and β as in (5) § 2 and put

$$\mathbf{V}\alpha\beta = \alpha', \quad \alpha\mathbf{V}\alpha\beta = \alpha'' = \alpha\alpha'.$$

Thus by the equation just established,

$$\alpha'\alpha'' = \alpha'\alpha\alpha' = -\alpha\alpha'^2,$$

$$\alpha''\alpha = \alpha\alpha'\alpha = -\alpha'\alpha^2,$$

$$\alpha\alpha' = \alpha''.$$

Hence $\mathbf{U}\alpha$, $\mathbf{U}\alpha'$, $\mathbf{U}\alpha''$ are three formal vectors which being put respectively equal to i, j, k give

$$jk = i, \quad ki = j, \quad ij = k, \quad i^2 = j^2 = k^2 = ijk = -1.$$

Now notice that by the equations established if ρ be any formal vector

$$\rho = -i\mathbf{S}i\rho - j\mathbf{S}j\rho - k\mathbf{S}k\rho = -\zeta\mathbf{S}\zeta\rho,$$

$$\mathbf{V}\zeta\mathbf{V}\rho\zeta = -\rho\zeta^2 + \zeta\mathbf{S}\rho\zeta = 2\rho.$$

If $\phi\omega$ be a linear formal vector function of a formal vector ω the equation

$$m\mathbf{S}\lambda\mu\nu = \mathbf{S}\phi\lambda\,\phi\mu\,\phi\nu,$$

for the formal scalar m where λ, μ, ν are three formal vectors gives a value for m which is independent of the particular values of λ, μ, ν. For by the definition $\phi\rho = \Sigma\mathbf{V}a\rho b$, ϕ is commutative with formal scalars [Perhaps it should be explicitly stated that

formal scalars are commutative with $\mathbf{S}$ and $\mathbf{V}$. For x being a formal scalar

$$\mathbf{S}(xq) = \mathbf{S}(x\mathbf{S}q + x\mathbf{V}q) = x\mathbf{S}q.$$

Similarly for $\mathbf{V}$ and $\mathbf{K}$. Similarly also $\mathbf{S}$, $\mathbf{V}$ and $\mathbf{K}$ are distributive i.e. $\mathbf{S}(q+r) = \mathbf{S}q + \mathbf{S}r$, &c.]. Hence as can be easily seen, the meaning of m is unaltered by changing λ into $x\lambda + y\mu + z\nu$ where x, y, z are any three formal scalars. The proposition follows at once since, as we have seen, any formal vector can be expressed as the sum of (formal scalar) multiples of three formal vectors (say i, j, k).

With the meaning of m thus obtained we have in particular

$$m\mathbf{S}\zeta_1\zeta_2\zeta_3\mathbf{S}\zeta_1\zeta_2\zeta_3 = \mathbf{S}\zeta_1\zeta_2\zeta_3\mathbf{S}\phi\zeta_1\phi\zeta_2\phi\zeta_3,$$
$$m\mathbf{V}\zeta_1\zeta_2\mathbf{S}\omega\zeta_1\zeta_2 = \mathbf{V}\zeta_1\zeta_2\mathbf{S}\phi\omega\phi\zeta_1\phi\zeta_2.$$

But by repeated applications of the equations

$$\rho = -\zeta\mathbf{S}\zeta\rho, \quad \mathbf{V}\zeta\mathbf{V}\rho\zeta = 2\rho,$$

we have

$$\mathbf{S}\zeta_1\zeta_2\zeta_3\mathbf{S}\zeta_1\zeta_2\zeta_3 = 6,$$
$$\mathbf{V}\zeta_1\zeta_2\mathbf{S}\omega\zeta_1\zeta_2 = -2\omega.$$

Eliminating m we have

$$\omega\mathbf{S}\zeta_1\zeta_2\zeta_3\mathbf{S}\phi\zeta_1\phi\zeta_2\phi\zeta_3 = -3\mathbf{V}\zeta_1\zeta_2\mathbf{S}\phi\omega\phi\zeta_1\phi\zeta_2.$$

4. Primary and secondary systems of formal quaternions. We shall now assume that if laws (1) to (6) § 2 are satisfied by a set of formal quaternions all the formulae (always excepting such as involve ∇) of ordinary quaternions will be true of this formal set. We proceed to show that connected with any system (denoted by q, r, &c.) of formal quaternions is another system (denoted by Q, R, &c.). The former system we shall call the primary system and the latter the secondary system.

Let Ω be a symbol which behaves exactly like a formal scalar of the primary system except that Ω^2 always $= 0$. [Ω is what by Clifford is denoted by ω. We use Ω merely because it is a symbol whose want for other purposes is not so severely felt as that of ω. For many purposes the memory is greatly assisted by noticing that Ω behaves like a constant infinitesimal scalar of the primary system.] Further let Ω have the property that if $q + \Omega r = 0$, then $q = r = 0$. [From this it follows that if

$$q + \Omega r = q' + \Omega r',$$

then $q = q'$ and $r = r'$.]

Define as follows:

$$Q = q + \Omega r \quad \text{.................................(1)},$$

$$\mathbf{K}Q = \mathbf{K}q + \Omega \mathbf{K}r \quad \text{...........................(2)}.$$

$$\mathbf{S}Q = \mathbf{S}q + \Omega \mathbf{S}r \quad \text{...........................(3)},$$

$$\mathbf{V}Q = \mathbf{V}q + \Omega \mathbf{V}r \quad \text{...........................(4)},$$

$$\mathbf{T}Q = (1 + \Omega \mathbf{S} r q^{-1})\, \mathbf{T}q \quad \text{.....................(5)},$$

$$\mathbf{U}Q = (1 + \Omega \mathbf{V} r q^{-1})\, \mathbf{U}q \quad \text{.................(6)}.$$

It has been said that Ω behaves like a formal scalar infinitesimal of the primary system. It may here be noticed that in an extended sense this statement has a bearing on these definitions. First it must be carefully noted however that Ω is *not* such an infinitesimal. If it *were* such a scalar however Ωr would be an arbitrary infinitesimal formal quaternion of the primary system which might be put $= dq$. The definitions (1) to (6) would then give

$$Q = q + dq, \qquad \mathbf{K}Q = \mathbf{K}q + d\mathbf{K}q, \quad \mathbf{S}Q = \mathbf{S}q + d\mathbf{S}q,$$

$$\mathbf{V}Q = \mathbf{V}q + d\mathbf{V}q, \quad \mathbf{T}Q = \mathbf{T}q + d\mathbf{T}q, \quad \mathbf{U}Q = \mathbf{U}q + d\mathbf{U}q.$$

These relations might be made the basis of the proof of most that is required in the present section. But the most important use of them is to show as it were *why* the secondary system is a system of formal quaternions if the primary system is so.

Note that if equations (1) to (6) hold and lead to the secondary system obeying laws (1) to (6) of § 2, a formal scalar of the secondary system must be of the form $x + \Omega y$ and a formal vector of the form $\alpha + \Omega\beta$ where x, y are formal scalars and α, β formal vectors of the primary system. Further note that since (§ 3 above) q^{-1} has a unique meaning so has Q^{-1}. For putting

$$Q^{-1} = q_1 + \Omega r_1,$$

we have
$$(q + \Omega r)(q_1 + \Omega r_1) = 1,$$

or
$$q q_1 = 1, \quad q r_1 + r q_1 = 0,$$

i.e.
$$q_1 = q^{-1}, \quad r_1 = - q^{-1} r q^{-1}.$$

Hence
$$Q^{-1} = q^{-1} - \Omega q^{-1} r q^{-1} \quad \text{.....................(7)}.$$

[It may be noticed that from this if $Q = \Omega r$, $Q^{-1} = \Omega\infty$ or (let us say) Q^{-1} is unintelligible. A clear geometrical reason for this will appear in the case of Octonions. It is of course due to the fact that $\Omega^2 = 0$.]

With these definitions:—*If the primary system q, r, &c. is a system of formal quaternions, so also is the secondary system Q, R, &c.*

To prove this we must assume (1) to (6) § 2 above to hold for the primary system and deduce that they hold for the secondary system.

(1) is obviously true of the secondary system.

Since Ω behaves like a formal scalar of the primary system, and since formal scalars are commutative with formal quaternions in the primary system it follows that any ordinary algebraic law that holds for the primary system holds also for the secondary system. Hence (2) is true for the secondary system. [That is $P+Q=Q+P$, $P+(Q+R)=(P+Q)+R$, $P\,.\,QR=PQ\,.\,R$, $(P+Q)(R+S)=PR+PS+QR+QS$.]

(3) also follows for the secondary system from the similarity of the behaviour of Ω to that of a formal scalar of the primary system. [Ω is not a formal scalar of the primary system but it is of the secondary system viz. when $q=0$, $r=1$ in eq. (1) above.]

To prove (4) first notice that $\mathbf{T}Q$ and $\mathbf{S}Q$ are each of the form $x+\Omega y$ and are therefore formal scalars.

The equation (4)(a) $Q=\mathbf{S}Q+\mathbf{V}Q$ is obvious. To prove the second equation (4)(a)

$$Q=\mathbf{T}Q\mathbf{U}Q,$$

we have to prove that

$$q+\Omega r=(1+\Omega\mathbf{S}rq^{-1})\,\mathbf{T}q\,(1+\Omega\mathbf{V}rq^{-1})\,\mathbf{U}q.$$

Noticing that by definition Ω is commutative with every formal quaternion of the primary system and that $\Omega^2=0$, the expression on the right

$$=\mathbf{T}q\mathbf{U}q+\Omega\,(\mathbf{S}rq^{-1}\,.\,\mathbf{T}q\mathbf{U}q+\mathbf{V}rq^{-1}\,.\,\mathbf{T}q\mathbf{U}q)$$
$$=q+\Omega rq^{-1}\,.\,q=q+\Omega r.$$

To prove (4) (b) we have first to show that $\mathbf{TU}Q=1$. Now if $\mathbf{U}Q=q'+\Omega r'$ we have by equation (5)

$$\mathbf{TU}Q=(1+\Omega\mathbf{S}r'q'^{-1})\,.\,\mathbf{T}q'.$$

But by equation (6) $q'=\mathbf{U}q$, $r'=\mathbf{V}rq^{-1}\ \ \mathbf{U}q$. Hence

$$\mathbf{TU}Q=1+\Omega\mathbf{S}\,(\mathbf{V}rq^{-1}\,.\,\mathbf{U}q\,.\,\mathbf{U}q^{-1})=1.$$

Next we have to show that $\mathbf{T}(QR) = \mathbf{T}Q\mathbf{T}R$ or

$$\mathbf{T}(q + \Omega r) \, . \, \mathbf{T}(q' + \Omega r') = \mathbf{T} \, . \, (q + \Omega r)(q' + \Omega r'),$$

where q, r, q', r' are any formal quaternions of the primary system. Thus by equation (5) we have to show that

$$(1 + \Omega \mathbf{S} r q^{-1}) \mathbf{T} q \, . \, (1 + \Omega \mathbf{S} r' q'^{-1}) \mathbf{T} q' = \mathbf{T}(qq' + \Omega [qr' + rq'])$$
$$= \{1 + \Omega \mathbf{S} \, . \, (qr' + rq')(qq')^{-1}\} \mathbf{T}(qq').$$

The left of this equation is

$$\mathbf{T} q \mathbf{T} q' \{1 + \Omega \mathbf{S}(rq^{-1} + r'q'^{-1})\},$$

and the right is

$$\mathbf{T}(qq') \{1 + \Omega \mathbf{S}(rq^{-1} + r'q'^{-1})\},$$

so that the equation is true.

The equation (4) (c) $\mathbf{K}Q = \mathbf{S}Q - \mathbf{V}Q$ is obvious. To establish the equation $\mathbf{K}Q = (\mathbf{T}Q)^2 Q^{-1}$ it is necessary to show that

$$\mathbf{K} q + \Omega \mathbf{K} r = \{(1 + \Omega \mathbf{S} r q^{-1}) \mathbf{T} q\}^2 (q^{-1} - \Omega q^{-1} r q^{-1}).$$

Remembering that $\mathbf{K} q \, . \, q = (\mathbf{T} q)^2$ the right becomes

$$\begin{aligned} &(1 + 2\Omega \mathbf{S} r q^{-1}) \mathbf{K} q (1 - \Omega r q^{-1}) \\ &= \mathbf{K} q + \Omega \mathbf{K} q (-rq^{-1} + 2\mathbf{S} r q^{-1}) \\ &= \mathbf{K} q + \Omega \mathbf{K} q \mathbf{K}(rq^{-1}) \\ &= \mathbf{K} q + \Omega \mathbf{K}(rq^{-1} \, . \, q) \\ &= \mathbf{K} q + \Omega \mathbf{K} r. \end{aligned}$$

The statements of (4) (d) except those referring to x^{-1} and $(x^2)^{\frac{1}{2}}$ are obvious from the similarity of Ω to a formal scalar of the primary system. That Q^{-1} where Q is any octonion has a unique meaning has already been proved. It remains to prove that $(X^2)^{\frac{1}{2}} = \pm X$ when X is a formal scalar of the secondary system.

$Y^{\frac{1}{2}}$ where Y is such a formal scalar means of course a formal scalar whose square $= Y$. Let

$$Y = x + \Omega y, \quad Y^{\frac{1}{2}} = x' + \Omega y'.$$

Thus $$x + \Omega y = (x' + \Omega y')^2 = x'^2 + \Omega 2x'y'.$$

Hence $$x' = \pm x^{\frac{1}{2}}, \quad y' = \pm \tfrac{1}{2} x^{-\frac{1}{2}} y,$$

or $$\sqrt{x + \Omega y} = \pm \sqrt{x} \left(1 + \Omega \frac{y}{2x}\right) \text{.................} (8).$$

[This is obvious from the similarity of Ω to an infinitesimal scalar.]

Hence $\{(x+\Omega y)^2\}^{\frac{1}{2}} = (x^2 + 2\Omega xy)^{\frac{1}{2}} = \pm x \left(1 + \Omega \dfrac{y}{x}\right)$

$$= \pm (x + \Omega y),$$

i.e. $$(X^2)^{\frac{1}{2}} = \pm X.$$

(5) is obviously true for the secondary system if true for the primary for r may be put zero.

(6) consists of definitions only.

CHAPTER II.

OCTONIONS AS FORMAL QUATERNIONS.

5. **The meanings of certain words.** The following tabular comparison of the meanings of certain terms used in the present treatise with the corresponding terms used by Clifford and Sir Robert Ball will probably prove convenient to the reader.

Clifford.	*Ball.*	*Present Treatise.*
Bi-quaternion		Octonion
Motor		Motor
Rotor		Rotor
Vector		Lator
Quaternion		Axial quaternion or axial
Twist		Twist
	Screw	Screw, unit-motor
	Twist about a screw	Velocity motor
	Wrench on screw	Force motor
	Impulsive wrench	Momentum Motor
Pitch	Pitch	Pitch

With regard to the terms *lator* and *axial quaternion* it should be remarked that these are equivalent to Clifford's *vector* and *quaternion* only when the latter occur in his papers on Bi-Quaternions. Where he uses *vector* and *quaternion* strictly in Hamilton's sense the present writer would do the same.

On p. 182 of Clifford's *Mathematical Papers* he says :—“ The name *vector* may be conveniently associated with a velocity of *translation,* as the simplest type of the quantity denoted by it. In analogy with this, I propose to use the name *rotor* (short for *rotator*) to mean a quantity having magnitude, direction, and position, of which the simplest type is a velocity of *rotation* about

a certain axis. A rotor will be geometrically represented by a length proportional to its magnitude measured upon its axis in a certain sense. The rotor AB will be identical with CD if they are in the same straight line, of the same length, and in the same sense; i.e. a vector may move anywise parallel to itself, but a rotor *only* in its own line."

This meaning of *rotor* as also his meaning (explained later in the same treatise) of *motor* I propose to adopt, but not so his term *vector*. There is no objection generally to use *vector* to mean anything which has the same geometrical significance as Hamilton's *vector*. But when, as in the present treatise, we are using symbols to represent such geometrical quantities and when those symbols do not obey all the laws of Hamilton's symbols for vectors, and when further we frequently have to refer to Hamilton's symbols and their laws, it is necessary for clearness to use the term *vector* strictly in Hamilton's sense. Now Clifford does not use the term in quite that sense. Hamilton's vectors and Clifford's vectors have the same geometrical significance and obey the same laws of addition but they do not obey the same laws of multiplication. I propose then to use the term *lator* for Clifford's *vector*. Thus lators and vectors are only distinguishable by their laws of multiplication.

Exactly similar remarks apply to Clifford's use in his papers on Bi-Quaternions of the term *quaternion*. His quaternion is a particular kind of octonion and consistently with his use of *quaternion* he ought to call his rotor a vector; i.e. his quaternion has like his rotor a definite axis *fixed in space* and not like Hamilton's quaternion *fixed merely in direction*. For Clifford's *quaternion* I therefore substitute *axial quaternion*. This will generally be contracted to *axial*.

It will probably aid towards an understanding of the methods below if I here comment on another idea of Clifford's. He frequently speaks of the dual interpretation to be placed on certain quaternion equations. This dual interpretation exists and has been of great aid in the development of Quaternions. But I think that simplicity in our fundamental ideas is gained, and some doubtful metaphysics is avoided, when it is shown that the dual interpretation is but a dual aspect of a single fact.

To show this in the case of Quaternions is not difficult. Defining a vector as *anything* which (1) requires for its specification

and is completely specified by a direction and magnitude, (2) obeys certain assigned laws of addition, and (3) obeys certain laws of multiplication assigned later; we may define a quaternion as *anything* which (1) requires for its specification and is completely specified by a vector and a scalar, (2) obeys certain assigned laws of addition (depending on those of vectors), and (3) obeys certain assigned laws of multiplication. Among the other assigned laws is that the quaternion is the sum of the vector and scalar, and that the sum of any number of component quaternions is the quaternion whose vector part is the sum of the vector parts of the components and whose scalar part is the sum of the scalar parts of the components. Thus a vector is a particular case of a quaternion and a scalar is another particular case, and the laws of addition of quaternions harmonise with those of vectors and scalars. The laws of multiplication of quaternions likewise harmonise with those of scalars (by which of course it is *not* meant that the laws of quaternion multiplication are the same as those of scalar, but only that when quaternions degenerate into scalars, the quaternion laws degenerate into the scalar laws) and the laws of multiplication of vectors are now determinate by reason of vectors being particular quaternions.

From the assigned laws of multiplication of quaternions it can be shown that the product qr of two quaternions in the particular case when r reduces to a vector perpendicular to the vector part of q is another vector also perpendicular to q and definitely related to r. *In this sense* q is the quotient obtained by dividing the last vector by the vector r and may be looked upon as an operator which converts r into the other vector. We thus get the dual interpretation with one fundamental conception of a quaternion.

Of course historically this is not the order in which the quaternion conceptions emerged, but there is no reason why after we have ourselves reached by a tangled route a desired goal we should not point out to others a smoother way.

So with octonions. They *may* be defined as operators on motors just as quaternions may be defined as operators on vectors, and as in the case of quaternions we may later go back and enlarge our primitive conception of them as operators. But I think it is simpler to define them otherwise and show that from the definitions

QA where Q is an octonion and A a motor (a particular case of an octonion) which intersects Q at right angles is itself a motor which intersects Q at right angles. Moreover just as the conception of a quaternion as a geometrical magnitude requires first the conception of a vector as such a magnitude, so the conception of an octonion as a geometrical magnitude requires first the conception of a motor as such a magnitude.

6. Fundamental conceptions and laws of Octonions. The definitions about to be given are complete only when taken as a whole. In other words the definitions of Lator, Rotor, &c. are not complete till we have implied by the definition of the multiplication and addition of octonions what is meant by the multiplication and addition of lators, rotors, &c.

A lator is a quantity which requires for its specification and is completely specified by a direction and an ordinary magnitude (scalar).

A rotor is a quantity which requires for its specification and is completely specified by a direction, an indefinitely long straight line parallel to that direction and an ordinary magnitude. The line is called the axis of the rotor.

A motor is a quantity which requires for its specification and is completely specified by a rotor and a lator which are parallel to one another. The axis of the rotor is called the axis of the motor. [A particular case of a motor is a rotor and another particular case is a lator, and in the latter case the motor has no definite axis fixed in space, though of course any line (or the direction) parallel to the lator may be called the axis. Such an axis may be called an indefinite axis. The restriction that the rotor and lator should be parallel will be removed later. In this case the axis of the motor is not the axis of the rotor, though, as will appear, it is parallel to it.]

An octonion is a quantity which requires for its specification and is completely specified by a motor and two scalars of which one is called its ordinary scalar and the other its convert. The axis of the motor is called the axis of the octonion. [A particular case of an octonion is a motor and another particular case is a scalar, which is the ordinary scalar mentioned and not the convert. A third particular case is a 'scalar octonion' which involves both

the scalars of the definition. Of course if the motor have no definite axis, i.e. if it be a lator or if the motor be zero, the octonion has no *definite* axis or none at all respectively.]

An axial quaternion (or more shortly "an axial") is an octonion of which both the lator and convert are zero. [Thus it has a definite axis except when it reduces to an ordinary scalar. It will be observed that a quaternion has an indefinite axis except when it reduces to an ordinary scalar. If the scalar of the axial be zero the latter reduces to a rotor, so that rotors and ordinary scalars are particular forms of axials.]

It will thus be seen that if O be a given point a system of axials through O (i.e. with axes passing through O), rotors through O and ordinary scalars are specified—when once O is fixed—in precisely the same way as quaternions, vectors and ordinary scalars. Indeed just as the vectors and ordinary scalars are included in the system of quaternions, so the rotors through O and ordinary scalars are included in the system of axials through O.

If q stand for an axial through O and q' for the quaternion whose vector part is parallel and equal to the rotor specifying q and whose scalar part is equal to the scalar specifying q, q' may be conveniently called *the quaternion corresponding to* q. Thus to every axial in space there is one corresponding quaternion. To a quaternion corresponds an infinite number of axials however, with parallel axes. But if we limit ourselves to the axials through an assigned point O, there is but one axial corresponding to a given quaternion.

A system of axials through a given point O (including rotors through O and ordinary scalars) obey among themselves all the laws of the corresponding quaternions (and their included vectors and ordinary scalars).

This is our starting-point for the definitions of the multiplication and addition of octonions. A definite point O is used in the definitions. Thus the *definitions* of multiplication and addition do not apparently treat space impartially. [For instance starting with O we render definite the meaning of products and sums of any octonions in space. As particular cases we get the meanings of products and sums of axials through some second point P. So far as the definitions themselves go it is not evident that axials through P would obey among themselves all the laws of the

corresponding quaternions. As a matter of fact they do, but this must be *proved*.] The real meanings of the definitions however apart from their *form* are independent of the point O used; i.e. if Q and R be two octonions, although O is used in the definitions of QR and $Q+R$, these latter octonions are dependent solely on Q and R and not on O.

Since *when* we limit ourselves to axials through O we are to all intents and purposes dealing with ordinary quaternions we use $q, r\ldots$ for axials through O; $\rho, \sigma\ldots$ for rotors through O; $x, y\ldots$ for ordinary scalars. In other words we adopt Hamilton's conventions as to notation, using Hamilton's *vector* symbols for their corresponding *rotors* through O, his *quaternion* symbols for *axials* through O, and his scalar symbols for ordinary scalars. Whenever then we find it convenient as we sometimes shall to thus limit ourselves to a definite axial system we adopt the Theory of Quaternions *in toto*—in geometrical applications as well as notation.

The $\mathbf{V}$ that has hitherto been used will now be changed to $\mathbf{M}$ and read 'motor part of.' The symbolic properties of $\mathbf{M}$ are precisely those of the quaternion $\mathbf{V}$, but the connotation of the initial letter of *vector* does not hold in the present theory.

Consider now how an octonion Q can be supposed specified. It is determined when its axis, its rotor and lator parts and its two scalars are known. It is therefore completely specified by

$$\varpi,\ \omega,\ \lambda,\ x,\ y,$$

where ϖ, ω and λ are rotors through O and x and y are ordinary scalars; ϖ being the rotor perpendicular on the axis, ω the rotor parallel and equal to the rotor of (the motor of) Q, λ the rotor parallel and equal to the lator of (the motor of) Q, x the ordinary scalar of Q and y the convert. It will thus be seen that ω and λ are coaxial being both parallel to the axis of Q, and ϖ intersects each of these three parallels perpendicularly. Instead of ϖ, ω and λ we may use ω and σ where σ is the rotor through O defined by

$$\sigma = \varpi\omega + \lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which gives

$$\varpi = \mathbf{M}\sigma\omega^{-1}, \quad \lambda = \omega\mathbf{S}\sigma\omega^{-1} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

so that now Q is completely specified by

$$\omega,\ \sigma,\ x,\ y.$$

When the rotor of Q is zero, Q is still completely specified by ω, σ, x and y (i.e. by σ, x and y of which σ by equation (1) now equals λ) since there is now no definite axis so that we do not require to attach a meaning to the equation $\varpi = \mathbf{M}\sigma\omega^{-1}$, which in this case becomes what may be called unintelligible. [I do not mean by this that it is impossible to attach a geometrical interpretation to the result $\varpi = \infty$ that we here obtain. It is perfectly easy to do so.]

Put now

$$\omega + x = q, \quad \sigma + y = r,$$

so that q, r are axials through O. Q is now completely specified by q and r.

To define the laws of addition and multiplication of octonions we use the equation

$$Q = q + \Omega r \quad \text{.........................(3)},$$

with the proviso that Ω behaves exactly like an ordinary scalar except that $\Omega^2 = 0$.

Lastly, writing $\mathbf{M}$ instead of $\mathbf{V}$ in equations (1) to (6) § 4 above, those equations are taken as the definitions of $\mathbf{K}Q$, $\mathbf{S}Q$, $\mathbf{M}Q$, $\mathbf{T}Q$, $\mathbf{U}Q$.

Thus octonions are formal quaternions in that they obey all the formal laws of Quaternions.

7. The parts of an octonion, products of octonions and sums of octonions all independent of an arbitrary origin. Our first task must be to show that although a particular point O has been used in these definitions, the real meanings of the definitions are independent of O. To do this we have to show that if Q, R are any two octonions, the octonions denoted by $\mathbf{K}Q$, $\mathbf{S}Q$, $\mathbf{M}Q$, $\mathbf{T}Q$, $\mathbf{U}Q$, $Q+R$ and QR will be just the same whether O be used in the definitions or some other point P.

The definition of $\mathbf{S}Q$ is that it = the ordinary scalar of $Q+\Omega$ into the convert of Q. This definition does not involve O.

Again by definition $\mathbf{M}Q$ is the same as Q except that its scalars are both zero, i.e. $\mathbf{M}Q$ is the motor of Q. This definition is independent of O.

Similar reasoning applies to $\mathbf{K}Q$ but need not be given, as we are about to prove that in general $Q+R$ and therefore in particular $\mathbf{S}Q-\mathbf{M}Q$ is independent in meaning of O.

Next notice that though q is not independent in meaning of O, passing as it necessarily does through O, yet the quaternion corresponding" to it is by its definition independent of O. Hence $\mathbf{T}q$ is independent of O. Also

$$\mathbf{T}q\mathbf{S}rq^{-1} = (\mathbf{T}q)^{-1}\,\mathbf{S}\,.\,r\mathbf{K}q = (\mathbf{T}q)^{-1}\,\mathbf{S}\,(\sigma + y)\,(-\,\omega + x)$$
$$= (\mathbf{T}q)^{-1}\,(-\,\mathbf{S}\lambda\omega + xy),$$

in which again x and y are independent of O, and though λ and ω are not, yet $-\,\mathbf{S}\lambda\omega = -\,\lambda\omega$ is. Hence by eq. (5) § 4 $\mathbf{T}Q$ is independent in meaning of O.

We might prove directly that $\mathbf{U}Q$ is similarly independent but it is more easily deduced from the fact, to be proved immediately, that QR in general is and therefore that $Q\,(\mathbf{T}Q)^{-1}$ in particular is.

Suppose Q' is what Q becomes when its axis has been shifted through a distance equal and parallel to the rotor ϵ passing through O, but is otherwise unaltered. We are about to show that

$$Q' = Q + \Omega\mathbf{M}\epsilon\mathbf{M}Q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

By the definition of the octonion $\Omega\sigma$ (§ 6 above) we see that $\Omega\epsilon$ is a *lator* equal and parallel to ϵ. Thus equation (1) asserts that *if* ϵ_0 *be a lator the operator* $1 + \mathbf{M}\epsilon_0\mathbf{M}(\quad)$ *shifts any octonion operated upon through a distance equal and parallel to* ϵ_0 *but does not otherwise alter it.*

What may be called the rotor equation (origin O) to the axis of Q is

$$\rho = \varpi + t\omega,$$

where ρ is the rotor from O to any point on the axis and t is a scalar parameter. The rotor equation to the axis of Q' is

$$\rho = \varpi + \epsilon + t\omega.$$

To find the rotor perpendicular on this we put $\mathbf{S}\rho\omega^{-1}$ zero, in order to find t. The corresponding value, say ϖ', of ρ is the required rotor perpendicular. Thus $t = -\,\mathbf{S}\epsilon\omega^{-1}$, and therefore

$$\varpi' = \varpi + \epsilon - \omega\mathbf{S}\epsilon\omega^{-1} = \varpi + \omega\mathbf{M}\omega^{-1}\,\epsilon.$$

Hence the σ of Q', say σ', is

$$\sigma' = \varpi'\omega + \lambda = \sigma + \mathbf{M}\epsilon\omega = \mathbf{M}r + \mathbf{M}\epsilon\mathbf{M}q.$$

Hence $$Q' = q + \Omega r + \Omega\mathbf{M}\epsilon\mathbf{M}q = q + \Omega r + \Omega\mathbf{M}\epsilon\mathbf{M}\,(q + \Omega r)$$
$$= Q + \Omega\mathbf{M}\epsilon\mathbf{M}Q,$$

which is eq. (1).

Since $2\mathbf{M}\epsilon\mathbf{M}Q = \epsilon Q - Q\epsilon$ and since $(1+\frac{1}{2}\Omega\epsilon)^{-1} = 1 - \frac{1}{2}\Omega\epsilon$, eq. (1) may be put

$$Q' = (1+\tfrac{1}{2}\Omega\epsilon)\,Q\,(1+\tfrac{1}{2}\Omega\epsilon)^{-1} \quad\ldots\ldots\ldots\ldots\ldots(2).$$

Now any octonion Q has the same space relations with a point P as Q with its axis shifted a distance equal and parallel to PO has with O. Otherwise worded—any octonion viewed from P has the same aspect as the same octonion with its axis shifted a distance equal and parallel to PO viewed from O. Let

$$\overline{OP} = \rho \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where $\overline{OP}$ denotes, as it will throughout this treatise, the *rotor* OP.

It will now be seen that the fact that QR and $Q+R$ are octonions independent of O follows from the obvious truths

$$(1-\tfrac{1}{2}\Omega\rho)(QR)(1-\tfrac{1}{2}\Omega\rho)^{-1} = \{(1-\tfrac{1}{2}\Omega\rho)\,Q\,(1-\tfrac{1}{2}\Omega\rho)^{-1}\}\{(1-\tfrac{1}{2}\Omega\rho)\,R\,(1-\tfrac{1}{2}\Omega\rho)^{-1}\},$$

$$(1-\tfrac{1}{2}\Omega\rho)(Q+R)(1-\tfrac{1}{2}\Omega\rho)^{-1} = (1-\tfrac{1}{2}\Omega\rho)\,Q\,(1-\tfrac{1}{2}\Omega\rho)^{-1} + (1-\tfrac{1}{2}\Omega\rho)\,R\,(1-\tfrac{1}{2}\Omega\rho)^{-1},$$

for put into words the first of these equations asserts that if Q and R be two octonions, then defining multiplication by means of O, *QR shifted through a distance equal and parallel to PO is the same as Q similarly shifted multiplied into R similarly shifted*; i.e. defining multiplication by means of P instead of O the octonion now denoted by QR is the same as before. Similar remarks apply to the second of the equations.

The statement in § 1 above is now established, that there is a geometrical interpretation of every quaternion formula (not involving ∇) which treats the directions and regions of space impartially but which is quite different from the ordinary one. We have not yet found the principles on which that interpretation must be made. This is to be done by examining the geometrical meaning of the simpler combinations of octonions and especially motors.

8. **Motors in Mechanics. Velocity-motor, force-motor, momentum-motor, impulse-motor.** In § 4 it was assumed that if

$$q + \Omega r = q' + \Omega r',$$

where q, r, q', r' belong to the primary system, then $q = q'$ and

$r=r'$ without exception. It must now be remembered that the primary system (axials) that we have used to conduct to the secondary system (octonions) consists of a system of axials through one definite point O. In this case the assumption is justified and this is all that is required for the results based on the assumption.

But we may have $q+\Omega r=q'+\Omega r'$ when q and r are axials through O and q' and r' are axials through some other point P. In this case q' and r' do not belong to the primary system and we have not $q=q'$, $r=r'$ in general. Still it is geometrically evident that q' and r' are unique determinate functions of q, r and ρ where ρ stands for $\overline{OP}$. Now

$$q+\Omega r=(q+\Omega\mathbf{M}\rho\mathbf{M}q)+\Omega\{(r-\mathbf{M}\rho\mathbf{M}q)+\Omega\mathbf{M}\rho\mathbf{M}(r-\rho\mathbf{M}q)\}.$$

But q and $r-\mathbf{M}\rho\mathbf{M}q$ pass through O. Hence by equation (1) § 7 $q+\Omega\mathbf{M}\rho\mathbf{M}q$ and $r-\mathbf{M}\rho\mathbf{M}q+\Omega\mathbf{M}\rho\mathbf{M}(r-\rho\mathbf{M}q)$ pass through P. Hence

$$q'=q+\Omega\mathbf{M}\rho\mathbf{M}q,\ \ r'=r-\mathbf{M}\rho\mathbf{M}q+\Omega\mathbf{M}\rho\mathbf{M}(r-\rho\mathbf{M}q)\ldots(1).$$

We may notice in passing an important result. We see that q' is q with its axis shifted to pass through P. From this we easily deduce that *if q, q' are axials and Q, Q' octonions such that $q+\Omega Q=q'+\Omega Q'$ the quaternions corresponding to q and q' are identical*, i.e. q' is q with its axis shifted. This is quite easily proved directly from the definitions of q and r in § 6.

From (1) it follows that $\mathbf{S}q=\mathbf{S}q'$, $\mathbf{S}r=\mathbf{S}r'$, i.e. it is only that part of $q+\Omega r$ which is a motor that has its form changed. This is otherwise obvious. Supposing

$$q+\Omega r=\text{a motor}=\omega+\Omega\sigma=\omega'+\Omega\sigma',$$

where ω, σ are rotors through O and ω', σ' rotors through P, eq. (1) becomes

$$\omega'=\omega+\Omega\mathbf{M}\rho\omega,\ \ \sigma'=\sigma-\mathbf{M}\rho\omega+\Omega\mathbf{M}\rho(\sigma-\mathbf{M}\rho\omega)\ldots(2).$$

Thus ω' is parallel and equal to ω while σ' is parallel and equal to $\sigma+\mathbf{M}\omega\rho$.

From this we see that if the instantaneous motion of a rigid body is given by saying that its angular velocity is equal and parallel to ω and its velocity at O is equal and parallel to σ, it is equally given by saying that its angular velocity is equal and parallel to ω' and its velocity at P is equal and parallel to σ'.

Thus the motor $\omega + \Omega\sigma$ exactly specifies and specifies no more than the instantaneous motion. I therefore call it the *velocity-motor* of the body. By taking P on the axis of the motor we deduce that the axis of the motor is that one line about which the body is instantaneously twisting, that the magnitude of the rotor is the angular velocity, and that the magnitude and sense of the lator are those of the velocity of translation which partly makes up the twist-velocity. A velocity-motor is what Sir Robert Ball calls a twist about a screw.

Likewise if a system of forces be such that their resultant is a force through O equal and parallel to ω and a couple equal and parallel to σ, they are also such that their resultant is a force through P equal and parallel to ω' and a couple equal and parallel to σ'. A motor possessing the property thus described with reference to a system of forces we shall call the *force-motor* of the system. If the resultant is reduced to the canonical form of a force and a parallel couple the line of action of the force is the axis of the motor, the sense and magnitude of the force are those of the rotor, and those of the couple are those of the lator. When the system acts on a rigid body the force-motor is the wrench on a screw which Sir Robert Ball would say acted on the body.

Again if any system of moving matter is such that its momentum is equal and parallel to ω and its moment of momentum about O is equal and parallel to σ, then its momentum is equal and parallel to ω' and its moment of momentum about P is equal and parallel to σ'. A motor that possesses this property with reference to the system of matter we shall call the *momentum-motor* of the system. The line which is the locus of points about which the moment of momentum is parallel to the momentum is the axis of the motor. The momentum is equal and parallel to the rotor, and for these points the moment of momentum is equal and parallel to the lator. The motion is always such that it could be instantaneously produced from rest by a system of impulses. This system is related to a motor in the same way as a system of forces is related to the force-motor. The motor to which it is so related is the momentum-motor. When we are actually considering such a system of impulses we shall sometimes call the momentum-motor the *impulse-motor* of the system. An impulse-motor is what Sir Robert Ball calls an impulsive wrench on a screw.

It is now evident that the result of mechanically superposing two velocity, force, momentum, or impulse motors A and B is to obtain what in octonions is denoted by $A + B$.

Thus a distinct mental picture is formed of the sum of two motors, and indeed of the sum of two octonions since both scalars are added in the ordinary algebraic way.

We shall later obtain a similar mental picture of the product of two octonions based on the geometrical interpretation of the operator $Q(\ )Q^{-1}$ where Q is an octonion. Before attempting to do this it is best to introduce some new terms and symbols.

9. Various decompositions of an octonion into simpler elements. We have seen that the position of our original point of reference O does not alter the geometrical meaning of the various functions of an octonion that have been introduced. Take O then on the axis of Q and put

$$Q = q_Q + \Omega r_Q \dots\dots\dots\dots\dots\dots\dots\dots (1),$$

where q_Q and r_Q are axials through O.

From the original definition (§ 6) of $q + \Omega r$ we see that q_Q and r_Q are both coaxial with Q and are independent of the position of O on the axis of Q. By the definitions indeed

$$q_Q = \text{rotor of } Q + \text{ordinary scalar of } Q,$$
$$\Omega r_Q = \text{lator of } Q + \Omega \times \text{convert of } Q.$$

Thus q_Q and r_Q are axials coaxial with Q which are definite functions of Q. q_Q is called *the* axial of Q; Ωr_Q is called the convertor of Q. [When it is necessary to refer to r_Q by name it may be called the convertor-axial, but this is not a good name. It is sometimes convenient to refer to the octonion $\Omega r_Q q_Q^{-1}$ by a name. It may be called the pitch-translation convertor, but this is very clumsy. Similarly $r_Q q_Q^{-1}$ would be called the pitch-translation axial.]

Since in (1) q_Q and r_Q are coaxial we see that coaxial octonions like coaxial quaternions obey the commutative law of multiplication as well as the rest of the fundamental laws of ordinary algebra.

By equation (5) § 4 above we have

$$\mathbf{T}Q = \mathbf{T}q_Q(1 + \Omega \mathbf{S} r_Q q_Q^{-1}).$$

Put

$$\left.\begin{array}{l} \mathbf{T}_1 Q = \mathbf{T}q_Q,\ \mathbf{T}_2 Q = 1 + \Omega \mathbf{S} r_Q q_Q^{-1},\ \mathbf{t}Q = \mathbf{S} r_Q q_Q^{-1} \\ \mathbf{T}Q = \mathbf{T}_1 Q \mathbf{T}_2 Q = \mathbf{T}_1 Q(1 + \Omega \mathbf{t} Q) \end{array}\right\} \dots (2).$$

Here it is to be noticed that (just as $\mathbf{M}Q$ is not in octonions called the *vector* part of Q because though $\mathbf{M}Q$ behaves symbolically like Hamilton's vectors, yet the connotation of 'vector part' does not apply here) we do not in Octonions call $\mathbf{T}Q$ the tensor of Q *though it is* $\mathbf{T}Q$ *which is the exact symbolic analogue of the quaternion tensor.* $\mathbf{T}Q$, $\mathbf{T}_1Q$, $\mathbf{T}_2Q$ and $\mathbf{t}Q$ will be called the *augmenter*, the *tensor*, the *additor* and the *pitch* of Q respectively. [*Augmenter* may be regarded as an English contraction of the Latin *tensor-additor*.] It will be observed that the tensor and the pitch are ordinary scalars, whereas the augmenter and the additor are not in general but are of the form $x + \Omega y$, where x and y are ordinary scalars. An expression of the form $x + \Omega y$ will in the future be called a *scalar octonion.*

Again, by equation (6) § 4 above

$$\mathbf{U}Q = (1 + \Omega \mathbf{M} r_Q q_Q^{-1})\, \mathbf{U} q_Q.$$

Put

$$\left.\begin{aligned}&\mathbf{U}_1Q = \mathbf{U}q_Q,\ \ \mathbf{U}_2Q = 1 + \Omega\mathbf{M}r_Q q_Q^{-1},\ \ \mathbf{u}Q = \mathbf{M}r_Q q_Q^{-1}\\ &\mathbf{U}Q\ = \mathbf{U}_1Q\,.\,\mathbf{U}_2Q = \mathbf{U}_1Q\,(1 + \Omega\mathbf{u}Q)\end{aligned}\right\}\ \ldots (3).$$

Here again although $\mathbf{U}Q$ is the exact symbolic analogue of the quaternion versor it is not in Octonions called the versor. $\mathbf{U}Q$, $\mathbf{U}_1Q$ and $\mathbf{U}_2Q$ will be called the *twister*, the *versor* and the *translator* of Q. [*Twister* may be regarded as an English contraction of the Latin *versor-translator*.] It will be observed that the versor is an axial, the corresponding quaternion being a versor. The twister and the translator are not in general axials. [When it is necessary to refer to $\mathbf{u}Q$ by name it may be called the translation-rotor or simply the translation, but both names are bad.]

Again,

$$\mathbf{S}Q = \mathbf{S}q_Q + \Omega\mathbf{S}r_Q.$$

Put

$$\left.\begin{aligned}&\mathbf{S}_1Q = \mathbf{S}q_Q,\ \ \mathbf{S}_2Q = \Omega\mathbf{S}r_Q,\ \ \mathbf{s}Q = \mathbf{S}r_Q\\ &\mathbf{S}Q\ = \mathbf{S}_1Q + \mathbf{S}_2Q = \mathbf{S}_1Q + \Omega\mathbf{s}Q\end{aligned}\right\}\ \ldots\ldots\ldots\ldots (4).$$

$\mathbf{S}Q$, $\mathbf{S}_1Q$, $\mathbf{S}_2Q$, $\mathbf{s}Q$ will be called the *scalar-octonion part*, the *ordinary scalar*, the *scalar-convertor* and the *convert* of Q respectively.

Again,

$$\mathbf{M}Q = \mathbf{M}q_Q + \Omega\mathbf{M}r_Q.$$

Put

$$\left.\begin{aligned}&\mathbf{M}_1Q = \mathbf{M}q_Q,\ \ \mathbf{M}_2Q = \Omega\mathbf{M}r_Q,\ \ \mathbf{m}Q = \mathbf{M}r_Q\\ &\mathbf{M}Q\ = \mathbf{M}_1Q + \mathbf{M}_2Q = \mathbf{M}_1Q + \Omega\mathbf{m}Q\end{aligned}\right\}\ \ldots\ldots\ldots (5).$$

$\mathbf{M}Q$, $\mathbf{M}_1Q$ and $\mathbf{M}_2Q$ will be called the *motor*, the *rotor* and the *lator* of Q respectively. It will be observed that the rotor *being* a rotor is an axial, but the motor and the lator are not axials. [When it is necessary to refer to $\mathbf{m}Q$ by name it may be called the lator-rotor, but the name is bad.]

From these equations we have

$$q_Q = \mathbf{T}_1Q\mathbf{U}_1Q = \mathbf{S}_1Q + \mathbf{M}_1Q \quad \text{.....................(6)},$$

$$r_Q = \mathbf{s}Q + \mathbf{m}Q,\ \Omega r_Q = \mathbf{S}_2Q + \mathbf{M}_2Q \quad \text{...........(7)},$$

$$r_Q q_Q^{-1} = \mathbf{t}Q + \mathbf{u}Q,\ 1 + \Omega r_Q q_Q^{-1} = \mathbf{T}_2Q\mathbf{U}_2Q \ \text{........(8)},$$

from which it follows that

$$\mathbf{s}Q + \mathbf{m}Q = (\mathbf{t}Q + \mathbf{u}Q)\,\mathbf{T}_1Q\mathbf{U}_1Q = (\mathbf{t}Q + \mathbf{u}Q)\,(\mathbf{S}_1Q + \mathbf{M}_1Q)\text{...(9)}.$$

Some analogous symbols might be added such as $\mathbf{K}_1Q$, $\mathbf{K}_2Q$, $\mathbf{k}Q$, but it is best to symbolise only the absolutely necessary elements of Q.

It will be observed that of the symbols which are contained in the following list, each in the second line is always an axial, whereas each in the first line is in general not an axial, and so on.

Ωr_Q, $\mathbf{K}Q$, $\mathbf{S}Q$, $\mathbf{M}Q$, $\mathbf{T}Q$, $\mathbf{U}Q$,	$\mathbf{S}_2Q$, $\mathbf{M}_2Q$, $\mathbf{T}_2Q$,			$\mathbf{U}_2Q$	(not axials),
q_Q, r_Q,	$\mathbf{S}_1Q$, $\mathbf{M}_1Q$, $\mathbf{T}_1Q$, $\mathbf{U}_1Q$,	$\mathbf{s}Q$, $\mathbf{m}Q$,	$\mathbf{t}Q$,	$\mathbf{u}Q$	(axials),
	$\mathbf{M}_1Q$,	$\mathbf{m}Q$,		$\mathbf{u}Q$	(rotors),
	$\mathbf{S}_1Q$, $\mathbf{T}_1Q$,	$\mathbf{s}Q$,	$\mathbf{t}Q$,		(ordinary scalars)
Ωr_Q,		$\mathbf{S}_2Q$, $\mathbf{M}_2Q$,	$\mathbf{T}_2Q - 1$,	$\mathbf{U}_2Q - 1$	(convertors),
		$\mathbf{M}_2Q$,		$\mathbf{U}_2Q - 1$	(lators).

The only term in this list that has not been defined is *convertor*. *Any octonion of the form* Ωr where r is an axial, or what is the same ΩQ *where* Q *is an octonion, is called a convertor.* Thus in particular all lators and scalar-convertors such as $\mathbf{M}_2Q$, $\mathbf{S}_2Q$ and $\mathbf{T}_2Q - 1$ are convertors. Observe that a *convert* is an *ordinary* scalar; thus the convert of $x + \Omega y$ is not the scalar-convertor Ωy but the ordinary scalar y.

If Q is a convertor q_Q is zero and r_Q not zero. We may suppose $\mathbf{U}q_Q = 1$, $\mathbf{T}q_Q = 0$. In this case

$$\mathbf{T}Q = \mathbf{S}Q = \mathbf{S}_2Q = \Omega\mathbf{s}Q,\ \mathbf{T}_1Q = 0,\ \mathbf{U}_1Q = 1,\ \mathbf{t}Q = \mathbf{u}Q = \infty,$$

and no intelligible meaning can be attached to $\mathbf{T}_2Q$, $\mathbf{U}Q$, $\mathbf{U}_2Q$.

It may be observed that the octonion Q in the convertor $R = \Omega Q$ may without altering the meaning of R have (1) its axis translated arbitrarily since

$$\Omega Q = \Omega\,(Q + \Omega \mathbf{M}\rho \mathbf{M}Q),$$

and (2) its convertor part altered arbitrarily since

$$\Omega\,(q + \Omega r) = \Omega q.$$

To assist in remembering the relations between many of the symbols I give two figures. In these q and r are written instead of q_Q and r_Q. If the relations between the tensor, vector part,

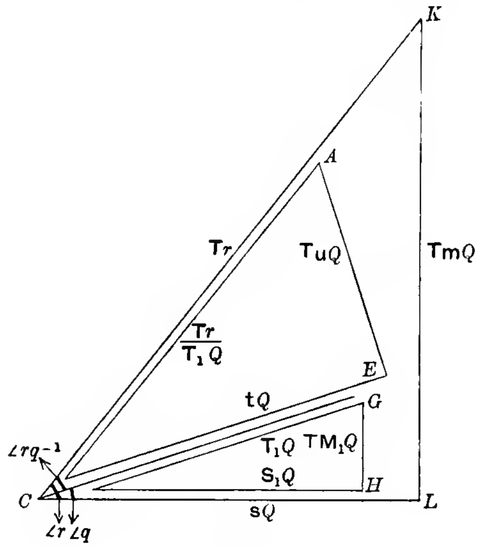
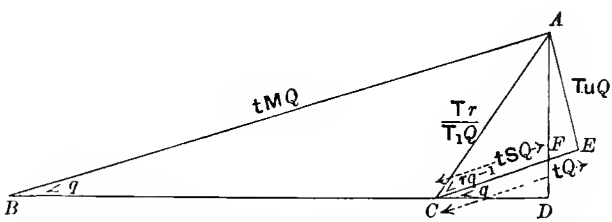


Fig. 1.



[$AC = \mathrm{T}r/\mathrm{T}_1 Q$, $BA = \mathrm{tM}Q$, $CF = \mathrm{tS}Q$, $CE = \mathrm{t}Q = \mathrm{tT}Q$.
$ABC = FCD = \angle q$, $ACD = \angle r$, $ACF = \angle rq^{-1}$].

Fig. 2.

scalar part and angle of a quaternion be remembered, and also the fact that for two coaxial quaternions q and r, $\angle (rq^{-1}) = \angle r - \angle q$,

it will be seen that fig. 1 is an immediate deduction from equations (6), (7), (8). To prove the relations implied by fig. 2, put $\angle q = \theta$, $\angle r = \phi$. Then

$$\mathbf{tS}Q = \frac{\mathbf{s}Q}{\mathbf{S}_1 Q} = \frac{\mathbf{T}r \cos\phi}{\mathbf{T}_1 Q \cos\theta} \text{ [fig. 1]},$$

and
$$\mathbf{tM}Q = \frac{\mathbf{Tm}Q}{\mathbf{TM}_1 Q} = \frac{\mathbf{T}r \sin\phi}{\mathbf{T}_1 Q \sin\theta} \text{ [fig. 1]},$$

from which fig. 2 at once follows.

It will be noticed that fig. 1 can be constructed when four of the data are given because $CK = CA \cdot CG$. For instance, if the magnitudes of the scalar and rotor parts of the axial ($\mathbf{S}_1 Q$ and $\mathbf{TM}_1 Q$), and the pitch and translation ($\mathbf{t}Q$ and $\mathbf{Tu}Q$), be given the other quantities are determined. For instance, the magnitude of the lator $\mathbf{Tm}Q$ is expressible in terms of these four by means of fig. 1.

Just as in Quaternions it is usual to adopt the convention that the tensor shall be a *positive* scalar, so here we may adopt the convention that $\mathbf{T}_1 Q$ the tensor of Q shall be positive. Thus *the ordinary scalar part of* $\mathbf{T}Q$ *is assumed to be positive.*

The following definitions are also convenient for some purposes:—

A scalar octonion $x + \Omega y$ *is said to be positive or negative according as its ordinary scalar part* x *is positive or negative.*

A scalar convertor Ωy *is said to be positive or negative according as its convert* y *is positive or negative.*

Thus $\mathbf{T}Q$ is always assumed to be a positive scalar octonion.

If $\dot{A}$ is a motor it may be noticed that in addition to the equation $\mathbf{S}A = 0$ (which involves $\mathbf{S}_1 A = \mathbf{S}_2 A = \mathbf{s}A = 0$) we have by equations (3) (since r_Q and q_Q are in this case coaxial rotors)

$$\mathbf{u}A = 0,\ \mathbf{U}_2 A = 1,\ \mathbf{U}A = \mathbf{U}_1 A = \mathbf{UM}_1 A \text{.........(10)}.$$

It is convenient here to notice the following deductions from § 6. If $Q = q + \Omega r$ where q and r are axials through any assigned point O, we have seen that the quaternion corresponding to q depends only on Q, i.e. it is the quaternion corresponding to the axial part of Q. Also by equation (5) § 4 $\mathbf{T}q\mathbf{S}rq^{-1}$ and therefore $\mathbf{S}rq^{-1}$ also depends only on Q. Hence we have the following:—

(1) *q is the axial part of Q with its axis translated to pass through O.*

(2) $\mathbf{MM}r\mathbf{M}^{-1}q$ *is the rotor perpendicular from O on the axis of Q.* [Eq. (2) § 6.]

(3) $\mathbf{S}rq^{-1}$ *is the pitch* $\mathbf{t}Q$ *of Q.*

(4) $\mathbf{S}_1Q = \mathbf{S}q$, $\mathbf{T}_1Q = \mathbf{T}q$, $\mathbf{M}q$ *is* $\mathbf{M}_1Q$ *with its axis translated to pass through O,* $\mathbf{U}q$ *is* $\mathbf{U}_1Q$ *with its axis translated to pass through O.*

We might by means of statement (2) and eq. (1) § 8 above express all the functions r_Q, $\mathbf{M}Q$, &c., of Q in terms of q and r. The formulae are too complicated to be of much use so I only write down those for q_Q and r_Q from which all the others *may* be derived, though some of them can be obtained by a shorter process

$$q_Q = q + \Omega\mathbf{M}^{-1}q\,.\,\mathbf{MM}q\mathbf{M}r \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

$$r_Q = \mathbf{S}r + (1 + \Omega\mathbf{MM}r\mathbf{M}^{-1}q)\,\mathbf{M}q\mathbf{SM}r\mathbf{M}^{-1}q\ldots\ldots(12).$$

10. Octonions as operators on Motors and as Motor quotients. We proceed now to those geometrical interpretations which are responsible for the above nomenclature. It is well to repeat here that the interpretations we are about to give although of prime importance in the theory of Octonions are yet not to be regarded as giving us the most general conception possible of an octonion. This general conception is given in the definitions of § 6 above.

Suppose A is a motor and Q an octonion at right angles to one another, and intersecting when both have definite axes. What are the geometrical relations subsisting between the three octonions Q, A and QA?

First assume that both Q and A have definite axes.

Let O be the point of intersection of Q and A and let

$$Q = q + \Omega r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$A = \omega + \Omega\sigma \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where the rotors ω and σ and the axials q and r all pass through O. Thus q and r are in this case what were in § 9 denoted by q_Q and r_Q and ω and σ are similarly related to A. Hence ω and σ are coaxial and so also are q and r, and each of the former intersects each of the latter perpendicularly.

If p is the pitch $\mathbf{t}Q$ and τ the translation rotor $\mathbf{u}Q$ of Q we have [eq. (8) § 9]

$$Q = \{1 + \Omega(p + \tau)\}\, q \quad\text{.....................(3)},$$

and if p' be the pitch $\mathbf{t}A$ of A we have similarly (since by eq. (10) § 9 the translator of A is unity)

$$A = (1 + \Omega p')\, \omega \quad\text{.......................(4)}.$$

It may be remarked that these transformations fail when Q and A cease to have definite axes [eq. (3) does not fail even when Q has no definite axis so long as the axial part which in this case must be a scalar $\mathbf{S}_1 Q$ is not zero], but are always legitimate (i.e. p, τ and p' are not infinite) when both have definite axes. Thus

$$\begin{aligned} QA &= \{1 + \Omega(p+\tau)\}\{1 + \Omega p'\}\, q\omega = \{1 + \Omega(p + p' + \tau)\}\, q\omega \\ &= \{1 + \Omega(p + p')\}\, q\omega + \Omega \mathbf{M} \,.\, \tau \mathbf{M} \,.\, \{1 + \Omega(p + p')\}\, q\omega, \end{aligned}$$

the last transformation being true since $q\omega$ and τ are perpendicular intersecting rotors.

Let the angle of q (the axial of Q) be called the angle of Q.

By quaternion interpretation the rotor $q\omega$ is obtained from the rotor ω by (1) rotation round the axis of Q through an angle equal to that of Q, (2) increase of the tensor in the ratio of $\mathbf{T}q = \mathbf{T}_1 Q$ to unity. By eq. (4) we see that multiplication by $1 + \Omega(p + p')$ changes this rotor into a motor whose rotor part is $q\omega$ and whose pitch is $p + p'$. Finally by eq. (1) § 7 we see that QA is this last motor translated through a distance equal and parallel to τ. More definitely we may suppose $A = (1 + \Omega p')\,\omega$, changed successively into

(1) $\mathbf{T}q\,(1 + \Omega p')\,\omega$.

(2) $q\,(1 + \Omega p')\,\omega = (1 + \Omega p')\, q\omega$.

(3) $\{1 + \Omega(p + p')\}\, q\omega$.

(4) $\{1 + \Omega(p + p')\}\, q\omega + \Omega \mathbf{M} \tau \mathbf{M} \,.\, \{1 + \Omega(p + p')\}\, q\omega$.

Hence QA is obtained from A by

(1) *multiplying* A's tensor ($\mathbf{T}_1 A = \mathbf{T}\omega$) by the tensor ($\mathbf{T}_1 Q = \mathbf{T}q$) of Q, keeping the pitch unaltered;

(2) *turning* the motor round Q through the angle of Q;

(3) *adding* to the motor's pitch ($\mathbf{t}A = p'$) the pitch ($\mathbf{t}Q = p$) of Q;

(4) *translating* the motor through a distance equal and parallel to the translation-rotor ($\mathbf{u}Q = \tau$) of Q.

These four commutative operations are performed by multiplying the motor A successively by (1) the *tensor* $\mathbf{T}_1Q$, (2) the *versor* $\mathbf{U}_1Q$, (3) the *additor* $\mathbf{T}_2Q$, (4) the *translator* $\mathbf{U}_2Q$. Operations (1) and (2) are performed simultaneously by multiplying the motor by the *axial* ($q = \mathbf{T}_1Q\mathbf{U}_1Q$) of Q, and (3) and (4) are performed by multiplying by $\mathbf{T}_2Q\mathbf{U}_2Q$. Operations (1) and (3) are performed simultaneously by multiplying by the *augmenter* ($\mathbf{T}Q$) of Q, and operations (2) and (4) by multiplying by the *twister* ($\mathbf{U}Q$) of Q.

The reasons for the names adopted are now evident.

If A have no definite axis, i.e. is a lator $\Omega\sigma$ where σ is a rotor and Q have a definite axis, the above phraseology requires further elucidation. We have seen (§ 9) that σ may have any position in space that is convenient. We may therefore suppose it to intersect Q. It is still supposed to be perpendicular to Q. Thus

$$QA = (q + \Omega r)\,\Omega\sigma = \Omega q\sigma.$$

If we call the tensor of the *rotor* σ the quasi-tensor of the *lator* $\Omega\sigma$ we see that in this case QA is obtained from A by operations (1) and (2) above, provided we read "quasi-tensor of A" for "tensor of A." Further we may suppose operations (3) and (4) superadded, for the pitch of A is now infinite and A has no definite position in space. With this elucidation then QA is in this case also obtained from A by the four operations, but the two of them which depend on the convertor of Q produce no effect. It will be seen indeed that in this case the convertor Ωr of Q does not appear in the product QA.

Now suppose A has a definite axis and Q has not. Q may or may not be a convertor. It is not a convertor if the axial (a very unsuitable term in this case) is an ordinary scalar not zero. In this case q is an ordinary scalar and Ωr [§ 9] is unaltered by translating r arbitrarily. r may therefore be made to intersect A. In this case rq^{-1} and therefore p and τ are both finite and the four operations are all intelligible. No modification of their wording is therefore required. The only thing to notice is that in this case there is no turning since the angle of Q is now zero.

If Q is a convertor $\Omega r, r$ may again be supposed to intersect A. The quaternion corresponding to the *axial* r may be called the quaternion corresponding to the *convertor* Ωr. The tensor and angle of r may be called the quasi-tensor and angle of Ωr. And the *direction* of the axis of r may be called the axis of Ωr. In the present case

$$QA = \Omega r (\omega + \Omega\sigma) = \Omega r\omega.$$

Only in a very forced sense can the above four operations be interpreted to apply to this case. Instead we have the following.— A convertor perpendicular to A operating on A (1) *destroys* the lator of A, (2) *converts* the rotor of A into a parallel equal lator, (3) *turns* this lator round the axis of the convertor through the angle of the convertor, (4) *increases* the quasi-tensor of the lator (i.e. the tensor of the original motor) in the ratio of the quasi-tensor of the convertor to unity. More generally any convertor Q_0 operating on any octonion Q produces another convertor R_0 and the quaternion corresponding to R_0 is the quaternion corresponding to Q_0 multiplied into the quaternion corresponding to the axial of Q. The characteristic property of these operations is the *conversion* mentioned.

Since $\Omega^2 = 0$ the product of any two convertors is zero. It follows that a convertor operating on a lator destroys it.

Thus *by the definition* (§ 6) *of the multiplication of octonions* it has been deduced that an octonion may always be looked upon as an operator on some motor. We have found that if A be *any* motor which (1) is perpendicular to Q, and (2) intersects Q when both have definite axes, then

$$QA = B,$$

where B is another motor which is also perpendicular to Q and also has a definite axis intersecting Q when both Q and A have definite axes. Further we have seen [eq. (7) § 4] that when A has a definite axis (and whatever be Q, A may always be so chosen as to have a definite axis and to satisfy the above conditions) A^{-1} has a definite intelligible meaning. Choosing A thus and multiplying the last equation into A^{-1} we get

$$Q = BA^{-1},$$

so that Q can *always* be expressed as the quotient of two motors. Conversely the quotient of any two motors of which either (1) the *divisor* has a definite axis, or (2) *both* are lators, is always an

octonion. In the last case (when both motors are lators) and in this case only the octonion is arbitrary to the extent that any convertor may be added to it. It is interesting to note that the quotient of a rotor by a lator is not an octonion.

In the typical case where Q has a definite axis it is to be noticed that either A or B above may be taken as *any* motor with definite axis intersecting Q perpendicularly, but the other motor is determined by the first chosen and Q.

This view of an octonion as an operator, though in no way lessening the generality of our conception of it, is very valuable in providing among other things a mental picture of the product of two octonions. Thus let Q and R be two octonions with definite axes. Let B be any motor which has the shortest distance between Q and R for axis. Choose A intersecting R perpendicularly and C intersecting Q perpendicularly such that

$$R = BA^{-1},\ Q = CB^{-1}.$$

Then
$$QR = CA^{-1},$$

so that viewed as an operator QR changes A into C. Its axis is the shortest distance between A and C, its tensor is the ratio of the tensor of C to that of A and its pitch is the pitch of C – the pitch of A. Also its translation rotor is the shortest distance from A to C, and its angle is the angle between A and C.

A general octonion involves eight scalars whereas a motor involves but six. The question arises, How does the absence of *two* scalars show itself when the motor is regarded as an operator? The answer is given by equation (10) § 9 above. Not only is the versor a quadrantal one, but the translation vanishes. That is, *both* elements of the twister, the angle and the distance translated are the same for all motors, the former being a right angle and the latter zero.

If the question be asked, what parts of the operation implied by Q are represented by $\mathbf{M}Q$ and $\mathbf{S}Q$ respectively, I do not know that a simpler answer can be given than that inferred from fig. 1 (p. 30). If A and B be given and $Q = BA^{-1}$ the figure can be at once constructed. Thus if the tensors and pitches and relative positions of B and A are given we have in the figure

$$CG = \frac{\text{tensor of } B}{\text{tensor of } A},\quad \angle GCH = \angle \text{ between } A \text{ and } B,$$

so that the triangle GCH can be constructed. And again

$$CE = \text{pitch of } B - \text{pitch of } A,\ AE = \text{distance between } A \text{ and } B.$$

Hence the triangle ACE can be constructed. Lastly

$$CK = CA \,.\, CG.$$

Hence the triangle KCL can be constructed. Hence the whole figure is constructed. Perhaps the shortest description of the operators $\mathbf{S}Q$ and $\mathbf{M}Q$ is by means of both figures 1 and 2. $\mathbf{S}Q$ regarded as an operator first magnifies the operand (keeping its pitch constant) in the ratio of CH (fig. 1) to unity and then increases the pitch by CF (fig. 2). $\mathbf{M}Q$ first magnifies the operand in the ratio of GH (fig. 1) to unity and turns it through a right angle and then increases the pitch by AB (fig. 2).

To fix the ideas more completely let the axis of Q be perpendicular to the plane of the paper, intersecting it in C, and consider the effect of Q on a unit rotor which has CL (fig. 1) for axis. The effect is to change the unit rotor into a motor intersecting the axis of Q. The rotor of this motor is equal and parallel to CG and is above the plane of the paper at a distance equal to EA. The pitch of the motor is equal to CE. $\mathbf{S}Q$ regarded as an operator changes the unit rotor into a coaxial motor whose rotor part is CH and whose lator part is CL, so that CH and CL represent in position and magnitude these two parts. Similarly $\mathbf{M}Q$ operating on the unit rotor changes it into a motor through C parallel to HG and LK. The rotor of this motor is equal to HG and the lator is equal to LK. Thus if the unit rotor be i, the rotor and lator parts of $\mathbf{S}Q \,.\, i$ are represented *in position and magnitude* by CH and CL; and $\mathbf{M}Q \,.\, i$ is a motor through C whose rotor and lator parts are represented *in direction and magnitude* by HG and LK.

11. On the operator $Q(\ \ )Q^{-1}$. Quaternion analogy suggests the examination of the meaning of the operator $Q(\ \ )Q^{-1}$ where Q is an octonion.

Suppose two octonions are precisely similar except that their axes are different, i.e. their scalars are equal each to each, their tensors are equal, their pitches are equal, and the distances to which they translate are equal; i.e. their *diagrams* (as fig. 1 may be called) are exact copies of each other. In such a case we shall say that one of them is the other *displaced*. It may be noted that the spatial relations between them are not quite

the same as those between two positions of a rigid body because of the infinite length of the axes. In other words if an octonion be displaced along its own axis it does not suffer change, whereas there is no direction in which a general rigid system can be displaced without suffering change. Nevertheless it is clear that given one octonion R and given two positions A and B of a rigid body, if R be displaced in exactly the same way as the rigid body must be to go from A to B, R' the octonion into which R is transformed will be a definite one. Further *in general* (and for our purposes it is not necessary to examine the exceptions) if R_1 and R_2 be two octonions which by such a displacement become R_1' and R_2' respectively, the four octonions will together *completely* specify the displacement. [If two infinite straight lines (1) and (2) become by such a displacement (1)′ and (2)′ the four in general completely specify the displacement.] Again *in general* if we are given R and R' (an octonion and the displaced octonion) and also the axis of the displacement, the displacement is determinate.

We are about to show that QRQ^{-1} *is obtained from R by such a displacement which is a function of Q only whatever be the value of R.* Q involves eight scalars and a displacement only six. It will be seen however that the augmenter $\mathsf{T}Q$ of Q does not occur in the product QRQ^{-1} and the augmenter involves two scalars, the tensor T_1Q and the pitch $\mathsf{t}Q$. Thus it is the twister $\mathsf{U}Q$ only which is really involved in this displacement-operator and a twister involves six scalars. The twister and the displacement are therefore each determinate functions of the other.

$$\text{Thus } QRQ^{-1} = \mathsf{U}Q \,.\, R \,.\, \mathsf{U}Q^{-1} = \mathsf{U}_2Q\mathsf{U}_1Q \,.\, R \,.\, \mathsf{U}_1Q^{-1}\mathsf{U}_2Q^{-1}.$$

Now $\mathsf{U}_2Q(\quad)\mathsf{U}_2Q^{-1}$ or $(1+\Omega\mathsf{u}Q)(\quad)(1+\Omega\mathsf{u}Q)^{-1}$ [where be it remembered $\mathsf{u}Q$ is a rotor] we have already seen [eq. (2) § 7 above] to be an operator which translates the operand through twice the translation $\mathsf{u}Q$. It only remains to find the meaning of $\mathsf{U}_1Q(\quad)\mathsf{U}_1Q^{-1}$. Putting $R = q + \Omega r$ where q and r are axials through some point O on the axis of Q we have

$$\mathsf{U}_1Q \,.\, R \,.\, \mathsf{U}_1Q^{-1} = \mathsf{U}_1Q \,.\, q \,.\, \mathsf{U}_1Q^{-1} + \Omega\mathsf{U}_1Q \,.\, r \,.\, \mathsf{U}_1Q^{-1}.$$

Here U_1Q and q are intersecting axials and again U_1Q and r are intersecting axials. Hence [by the definition of the laws of axials, § 6] $\mathsf{U}_1Q \,.\, q \,.\, \mathsf{U}_1Q^{-1}$ and $\mathsf{U}_1Q \,.\, r \,.\, \mathsf{U}_1Q^{-1}$ are subject to the corresponding quaternion interpretation, i.e. these two are axials

through O obtained by rotating q and r respectively round the axis of Q through twice the angle of Q. It follows that R is similarly rotated by the operator $\mathsf{U}_1 Q(\quad)\mathsf{U}_1 Q^{-1}$.

Hence $Q(\quad)Q^{-1}$ first rotates the operand as a rigid body round the axis of Q through twice the angle of Q and then translates it as a rigid body through twice the translation of Q. It therefore displaces the operand as a rigid body in the most general manner.

This finite displacement will for the future be referred to as the displacement Q. It is obvious as in Quaternions that the displacement $QR \ldots XY$ is the displacement that results from making in succession the displacements $Y, X, \ldots R, Q$.

I may remark that this meaning of $Q(\,)Q^{-1}$ was my starting-point in the consideration of octonions. Looking upon an octonion primarily as a quotient of two motors I sought the meaning of $Q(\,)Q^{-1}$ in order to give a simple geometrical definition of octonion multiplication. Splitting Q up into the four factors denoted above by $\mathsf{T}_1 Q$, $\mathsf{T}_2 Q$, $\mathsf{U}_1 Q$, $\mathsf{U}_2 Q$ it is easy to see that the factors must be commutative with each other and that the first two must be commutative with all octonions and therefore not appear in the product QRQ^{-1}. Putting then QRQ^{-1} in the form given in the last equation but one, I was led to study the two operators $\mathsf{U}_1 Q(\,)\mathsf{U}_1 Q^{-1}$ and $\mathsf{U}_2 Q(\,)\mathsf{U}_2 Q^{-1}$ separately, and without much difficulty proved by pure geometry what has just been established.

I was thus led to give a definition of the product of two octonions independent of their meaning as operators on motors and such that the equation

$$P\,.\,QR = PQ\,.\,R$$

followed obviously. I noticed however a difficulty which I had overlooked in the corresponding quaternion investigation ("Messenger of Mathematics," XVIII. [1889] p. 133), namely that corresponding to one finite displacement there are two operators on motors. But by a sort of stereographic chart of the varying orientations of three lines fixed in a rigid body which is subjected to successive finite displacements I got over this difficulty. [I now see that the difficulty can be got over much more easily.]

The sum of two octonions was defined by kinematical considerations in such a way that the equations

$$P + Q = Q + P,$$
$$P + (Q + R) = (P + Q) + R,$$

followed obviously. It remained to prove that

$$P(Q + R) = PQ + PR,$$
$$(Q + R)P = QP + RP.$$

I did succeed in proving these geometrically from the former definitions. But the process was so long and in places so analogous to corresponding quaternion processes that I reluctantly concluded that the only presentable way in which the results could be given was to utilise quaternions freely.

It was not till after this that I found what place the Ω of the present treatise (i.e. Clifford's ω) had in the subject as viewed from the present stand-point.

As already remarked Q contains more constants than are involved in a finite displacement but A, a motor, contains exactly the requisite number, namely six. A may therefore be made, of course in more ways than one, to specify such a displacement. One of these ways is very simple and exactly analogous to a corresponding quaternion specification. Since $\mathbf{M}_2 A$ is a convertor we have, when n is a positive integer,

$$A^n = (\mathbf{M}_1 A + \mathbf{M}_2 A)^n = \mathbf{M}_1{}^n A + n\mathbf{M}_1{}^{n-1} A \mathbf{M}_2 A \quad \ldots\ldots(1).$$

Hence defining e^Q by the equation

$$e^Q = 1 + Q + Q^2/2\,! + \ldots + Q^m/m\,! + \ldots\ldots\ldots\ldots(2),$$

we have

$$e^A = (1 + \mathbf{M}_2 A)\, e^{\mathbf{M}_1 A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Now the positive integral powers of a rotor such as $\mathbf{M}_1 A$ have meanings exactly analogous to the positive integral powers of a vector. Hence remembering that

$$\mathbf{TM}_1 A = \mathbf{T}_1 A, \quad \mathbf{UM}_1 A = \mathbf{U}_1 A,$$

we have

$$e^A = (1 + \mathbf{M}_2 A)(\cos \mathbf{T}_1 A + \mathbf{U}_1 A \sin \mathbf{T}_1 A) \quad \ldots\ldots\ldots(4),$$

or $\quad \mathbf{T}e^A = 1, \quad \mathbf{U}_1 e^A = \cos \mathbf{T}_1 A + \mathbf{U}_1 A \sin \mathbf{T}_1 A, \quad \mathbf{u}e^A = \mathbf{m}A \ \ldots(5).$

Hence the displacement e^A is a rotation round the axis of A through the angle $2\mathbf{T}_1 A$ combined with the translation $2\mathbf{M}_2 A$.

Suppose R_0 is a constant octonion and Q a function of the time and let

$$\left.\begin{aligned} R &= QR_0Q^{-1}, \\ \text{so that} \quad \dot{R} &= 2\mathbf{M}\,.\,\mathbf{M}\dot{Q}Q^{-1}\,.\,\mathbf{M}R \end{aligned}\right\} \quad \text{.................(6).}$$

Thus when R is subject to continuous displacement like a moving rigid body

$$\dot{R} = \mathbf{M}A\mathbf{M}R \text{ ....(7),}$$

where A is the motor given by

$$A = 2\mathbf{M}\dot{Q}Q^{-1} \text{ ....(8).}$$

One interpretation of equation (7) is that when R thus moves the only part of R which varies is $\mathbf{M}R$, which is otherwise obvious. Another is that the instantaneous motion of a rigid body can be specified by a motor A. This, of course, we already know, but the question arises, Is this A the motor which we have already (§ 8) called the velocity-motor? It is. Let B be the velocity-motor. By § 8 the motion in the time dt is a rotation $\mathbf{M}_1 B dt$ round the axis of B and a translation $\mathbf{M}_2 B dt$, i.e. it is what we have just denoted the displacement $e^{Bdt/2} = 1 + \frac{1}{2}Bdt$. Hence

$$\begin{aligned} R + dR &= (1 + \tfrac{1}{2}Bdt)\,R\,(1 - \tfrac{1}{2}Bdt) \\ &= R + \mathbf{M}B\mathbf{M}R\,.\,dt, \end{aligned}$$

or

$$\dot{R} = \mathbf{M}B\mathbf{M}R.$$

Comparing this with equation (7) it follows that $B = A$. $2\mathbf{M}\dot{Q}Q^{-1}$ is therefore the velocity-motor due to the finite displacement Q in the time t.

12. Analysis of the product and sum of two motors. We must now examine the geometrical relations between A, B and the various parts of AB such as $\mathbf{M}AB$, $\mathbf{s}AB$.

Before doing this however notice the physical meaning of $\mathbf{s}AB$. Putting $A = \omega + \Omega\sigma$, $B = \omega' + \Omega\sigma'$, where ω, σ, ω', σ' are rotors through some point O, we have

$$\mathbf{S}AB = \mathbf{S}\omega\omega' + \Omega\mathbf{S}(\omega\sigma' + \omega'\sigma).$$

Hence

$$\mathbf{s}AB = \mathbf{S}(\omega\sigma' + \omega'\sigma) \text{ ....(1).}$$

Hence $-\mathbf{s}AB$ is the rate of work done by a force-motor A or B acting on a rigid body moving with a velocity-motor B or A. The convert of AB with its sign changed ($-\mathbf{s}AB$) must therefore

be the product of the tensors of A and B and what Sir Robert Ball calls the virtual coefficient of the corresponding screws.

The geometrical relations now sought might be obtained by regarding AB as the operator which changes B^{-1} into A and utilising the results of § 10. They may also be obtained by regarding AB as an operator on a unit rotor which is perpendicular to B and the shortest distance of A and B and intersects them; but I think it is more instructive to proceed otherwise.

In the first place note that if A and B are both lators $AB = 0$. If one only of them is a lator AB is a convertor. It is convenient to discuss separately the two cases (1) when neither is a lator, i.e. each has a definite axis, and (2) when one is a lator and the other not, i.e. where one has not a definite axis and the other has.

Let

$$\mathbf{M}_1 A = \alpha_1,\ \mathbf{m} A = \alpha_2,\ \mathbf{M}_1 B = \beta_1,\ \mathbf{m} B = \beta_2 \ldots\ldots\ldots(2).$$

Let ϖ be the rotor shortest distance from A to B and let O be its point of intersection with A. Let β be the rotor through O equal and parallel to β_1. Finally let p, p' be the pitches of A and B respectively.

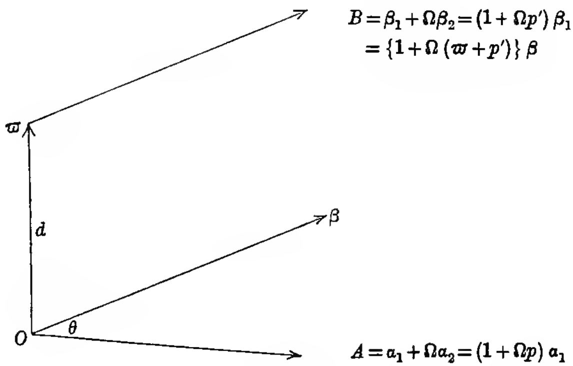


FIG. 3.

Thus

$$A = \alpha_1 + \Omega\alpha_2, \qquad B = \beta_1 + \Omega\beta_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$A = (1 + \Omega p)\,\alpha_1, \quad B = (1 + \Omega p')\,\beta_1 = \{1 + \Omega(\varpi + p')\}\,\beta \ldots(4).$$

The geometrical connections are roughly indicated in fig. 3.

It will be noticed that α_1, α_2, β and ϖ are all rotors through O and that ϖ is perpendicular to the other three. Expressions

involving these rotors and ordinary scalars are thus subject to interpretations similar to the corresponding quaternion ones.

For brevity of statement of some of the results it is convenient to call the angle between A and B, i.e. between α_1 and β, θ; and to call the shortest distance between A and B, d, d being reckoned positive or negative according as the shortest twist that will bring one of the two $\mathbf{U}A$ and $\mathbf{U}B$ (i.e. $\mathbf{U}\alpha_1$ and $\mathbf{U}\beta_1$) into coincidence with the other is a right-handed or left-handed twist. Thus $d = \pm\mathbf{T}\varpi$.

We have now a variety of expressions for AB. For instance, utilising equations (4),

$$AB = \alpha_1\beta + \Omega\,(p + p' - \varpi)\,\alpha_1\beta \quad \ldots\ldots\ldots\ldots\ldots(5),$$

where $-\varpi\alpha_1$ has been put for $\alpha_1\varpi$ since ϖ and α_1 are intersecting and perpendicular. Hence

$$\mathbf{M}AB = \mathbf{M}\alpha_1\beta + \Omega\,\{(p + p')\,\mathbf{M}\alpha_1\beta - \varpi\mathbf{S}\alpha_1\beta\} \quad \ldots\ldots\ldots(6),$$

$$\mathbf{M}_1AB = \mathbf{M}\alpha_1\beta,\;\; \mathbf{m}AB = (p + p')\,\mathbf{M}\alpha_1\beta - \varpi\mathbf{S}\alpha_1\beta \ldots\ldots(7),$$

$$\mathbf{tM}AB = p + p' - \varpi\mathbf{M}^{-1}\alpha_1\beta\mathbf{S}\alpha_1\beta = p + p' + d\cot\theta \ldots(8).$$

Hence $\mathbf{M}AB$ is a motor whose axis is the shortest distance between A and B, whose rotor part bears the same relation to the rotor parts of A and B as the vector product of two vectors bears to the vectors and whose pitch $= p + p' + d\cot\theta$. This, it will be observed, completely defines $\mathbf{M}AB$ except when A and B are parallel. In this case however $\mathbf{M}AB$ is the lator given by

$$\mathbf{M}AB = -\,\Omega\varpi\alpha_1\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

the interpretation of which is obvious. In this case observe that the pitches of A and B do not appear in $\mathbf{M}AB$.

Again,

$$\mathbf{S}AB = \mathbf{S}\alpha_1\beta + \Omega\,\{(p + p')\,\mathbf{S}\alpha_1\beta - \varpi\mathbf{M}\alpha_1\beta\} \ldots\ldots\ldots\ldots(10),$$

$$\mathbf{S}_1AB = \mathbf{S}\alpha_1\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

$$\mathbf{s}AB = (p + p')\,\mathbf{S}\alpha_1\beta - \varpi\mathbf{M}\alpha_1\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

$$\mathbf{tS}AB = p + p' - \varpi\mathbf{M}\alpha_1\beta\mathbf{S}^{-1}\alpha_1\beta = p + p' - d\tan\theta \ldots(13).$$

Thus $\mathbf{S}_1AB$ is the same function of the rotor parts of A and B as the scalar part of the product of two vectors is of the vectors. If A and B are *screws*, i.e. if their tensors

$$\mathbf{T}_1A = \mathbf{T}\alpha_1 \text{ and } \mathbf{T}_1B = \mathbf{T}\beta_1 = \mathbf{T}\beta$$

are each equal to unity the expression (12) for $-\mathbf{s}AB$ is the usual one for the "virtual coefficient."

From the above results it will be noticed that

$$\mathbf{M}_1 AB + \mathbf{S}_1 AB = \alpha_1\beta \qquad\qquad (14),$$

$$\mathbf{m}AB + \mathbf{s}AB = (p + p' - \varpi)\,\alpha_1\beta \qquad\qquad (15).$$

Hence by eq. (9) § 9

$$\mathbf{t}AB + \mathbf{u}AB = p + p' - \varpi,$$

or

$$\mathbf{t}AB = p + p', \quad \mathbf{u}AB = -\varpi \qquad\qquad (16).$$

These might be proved directly by regarding AB as an operator in the manner suggested above.

From eq. (14) and eq. (6) § 9 we also have

$$\mathbf{T}_1(AB) = \mathbf{T}(\alpha_1\beta), \quad \mathbf{U}_1(AB) = \mathbf{U}(\alpha_1\beta) \qquad\qquad (17).$$

Thus $\mathbf{S}_1 AB$, $\mathbf{M}_1 AB$, $\mathbf{T}_1 AB$ and $\mathbf{U}_1 AB$ are related to the rotor parts of A and B just as $\mathbf{S}\alpha\beta$, $\mathbf{V}\alpha\beta$, $\mathbf{T}\alpha\beta$ and $\mathbf{U}\alpha\beta$ are related to the vectors α, β. This is a particular case of a more general statement which will be given directly. Moreover this statement together with equations (8) and (13) and with the further statement that the axis of $\mathbf{M}_1 AB$ (and of $\mathbf{U}_1 AB$) is the shortest distance between A and B, virtually sums up all the geometrical interpretations just given. We give in fig. 4 therefore a geometrical construction for equations (8) and (13). It is for the present case merely a modification of fig. 2.

The case when either A or B is a lator is a very simple one. Since $\mathbf{S}AB = \mathbf{S}BA$ and $\mathbf{M}AB = -\mathbf{M}BA$ it is only necessary to

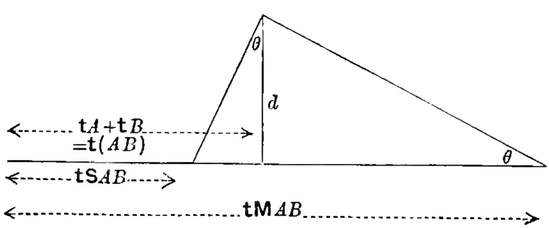


Fig. 4.

consider the case when B is a lator. Putting $B = \Omega\beta$ where β is a rotor we may suppose (§ 9) β to intersect A. Taking the form $\alpha_1 + \Omega\alpha_2$ for A we have $AB = \Omega\alpha_1\beta$, showing that AB is a convertor

related to the rotor part of A and the lator B exactly in the same way as the product of two vectors is related to the vectors. It is unnecessary therefore to examine separately $\mathbf{M}AB$, $\mathbf{S}AB$, &c.

From these results we gather that if $\mathbf{M}AB=0$ either (1) A and B are coaxial, or (2) one of them is a lator parallel to the axis of the other, or (3) they are both lators. If $\mathbf{M}_1AB=0$ either (1) A and B are parallel, or (2) one at least is a lator. If $\mathbf{m}AB=0$ either (1) $(p+p')\sin\theta+d\cos\theta=0$, or (2) one is a lator parallel to the axis of the other, or (3) they are both lators.

If $\mathbf{S}AB=0$ either (1) they intersect perpendicularly, or (2) one is a lator perpendicular to the axis of the other, or (3) they are both lators. If $\mathbf{S}_1AB=0$ either (1) they are perpendicular, or (2) one at least is a lator. If $\mathbf{s}AB=0$ either (1) $(p+p')\cos\theta=d\sin\theta$, or (2) one is a lator perpendicular to the axis of the other, or (3) they are both lators.

The geometrical relations between A, B and $A+B$ will be considered in detail when we come to consider complexes of different orders. We may here notice however that they may be treated in much the same way as $\mathbf{M}AB$, $\mathbf{S}AB$, &c. By eq. (4)

$$A+B=\alpha_1+\beta+\Omega\,(p\alpha_1+p'\beta+\varpi\beta),$$

or

$$A+B=\{1+\Omega\,(p''+\varpi')\}\,(\alpha_1+\beta)\ldots\ldots\ldots\ldots(18),$$

where $p''+\varpi'$ is the axial (whose axis is the shortest distance between A and B) given by

$$p''+\varpi'=(p\alpha_1+p'\beta+\varpi\beta)\,(\alpha_1+\beta)^{-1}\ \ldots\ldots\ldots(19).$$

p'' is of course assumed to be the scalar and ϖ' the rotor of this axial. Thus by eq. (18) p'' is the pitch of $A+B$. The rotor of $A+B$ is equal and parallel to $\alpha_1+\beta$ and intersects the shortest distance between A and B at a distance $\mathbf{T}\varpi'$ from O towards B. Eq. (19) gives

$$p''=\mathbf{S}\,(p\alpha_1+p'\beta)\,(\alpha_1+\beta)^{-1}-\varpi\mathbf{M}\alpha_1\beta\,.\,(\alpha_1+\beta)^{-2}\ldots(20),$$

$$\varpi'=\varpi\mathbf{S}\beta\,(\alpha_1+\beta)^{-1}+(p-p')\,\mathbf{M}\alpha_1\beta\,.\,(\alpha_1+\beta)^{-2}\ \ldots(21).$$

In particular we may notice that if $\mathbf{T}A=\mathbf{T}B$, i.e. if $\mathbf{T}\alpha_1=\mathbf{T}\beta$, and $p=p'$, $A+B$ is a motor through the middle point of the shortest distance which bisects the positive directions of A and B, and $A-B$ is a motor through the same point which bisects the

positive direction of A and the negative one of B. Both these statements can be deduced directly from the equations

$$Q + \mathbf{K}Q = 2\mathbf{S}Q, \quad Q - \mathbf{K}Q = 2\mathbf{M}Q.$$

Hence $\mathbf{U}A \pm \mathbf{U}B$ are perpendicular motors passing through the middle point of the shortest distance. Similarly $A\mathbf{T}_1^{-1}A \pm B\mathbf{T}_1^{-1}B$ are perpendicular motors but they do not in general intersect.

13. Some expressions involving more than two motors.

Before speaking of the products, &c. of three motors it is well to state the general theorem which is suggested by the similarity of the meanings of $\mathbf{S}_1AB$, $\mathbf{M}_1AB$, $\mathbf{T}_1AB$ and $\mathbf{U}_1AB$ to those of the corresponding quaternion expressions. Let us denote by the term "any formal quaternion function," any function of the octonions Q_1, Q_2... which involves beside the octonions themselves only the symbols which occur in ordinary quaternion formulae ($\mathbf{M}$ taking the place of $\mathbf{V}$). Thus $\mathbf{M}AB$, $\mathbf{S}AB$ are formal quaternion functions but $\mathbf{M}_1AB$, $\mathbf{s}AB$, &c. are not.

If Q be any formal quaternion function of the octonions Q_1, Q_2..., then q' the quaternion corresponding to the axial of Q is the same function of q_1', q_2'... the quaternions corresponding to the axials of Q_1, Q_2... as Q is of Q_1, Q_2....

Put

$$Q = q + \Omega r, \; Q_1 = q_1 + \Omega r_1, \;\; Q_2 = q_2 + \Omega r_2, \ldots$$

where $q, r, q_1, r_1, \ldots$ are axials through some one point O. By § 9 $q, q_1, q_2 \ldots$ are the axials of Q, Q_1, Q_2... translated to pass through O. Hence q', q_1', q_2' ... are the quaternions corresponding to q, q_1, q_2.... But since Ω behaves (in a formal quaternion function) like an infinitesimal scalar it follows that q is the same function of $q_1, q_2 \ldots$ as $q + \Omega r$ is of $q_1 + \Omega r_1$, $q_2 + \Omega r_2, \ldots$; i.e. q' is the same function of q_1', q_2'... as Q is of Q_1, Q_2....

A particular case of the above theorem is that $\mathbf{V}q'$, $\mathbf{S}q'$, $\mathbf{T}q'$ and $\mathbf{U}q'$ are the same functions of q_1', q_2',... as are $\mathbf{M}Q$, $\mathbf{S}Q$, $\mathbf{T}Q$ and $\mathbf{U}Q$ of Q_1, Q_2.... Considering now the axials of $\mathbf{M}Q$, Q_1, &c. it follows that:—

Making abstraction of the positions in space (but not of the directions) of the axes, $\mathbf{M}_1Q$, $\mathbf{S}_1Q$, $\mathbf{T}_1Q$ and $\mathbf{U}_1Q$ have precisely the

same geometrical relations with the axials of $Q_1, Q_2, \ldots$ *that* $\mathbf{V}q', \mathbf{S}q'$, $\mathbf{T}q'$ *and* $\mathbf{U}q'$ *have with the quaternions* $q_1', q_2' \ldots$.

The first of these theorems might have been proved in § 6 and the second in § 9.

Note that a result of the second is that *one* interpretation of any octonion formula which involves only formal quaternion functions is the ordinary quaternion interpretation.

Thus $\mathbf{S}_1ABC$ is the same function of the rotor parts of the motors A, B, C as $\mathbf{S}\alpha\beta\gamma$ is of the three vectors α, β, γ. Thus the geometrical meaning of $\mathbf{S}_1ABC$ is known. Similarly the geometrical meaning of $\mathbf{M}_1ABC$, except as to the position in space of the axis, may be said to be known. [It is known in the same sense as the geometrical meaning of $\mathbf{V}\alpha\beta\gamma$ is known. To get a clear notion of this last, one way is to regard it as a linear vector function of β.]

$\mathbf{S}ABC$ will be completely determined geometrically if in addition (to the knowledge of the meaning of $\mathbf{S}_1ABC$) we further know the meaning of $\mathbf{tS}ABC$. And $\mathbf{M}ABC$ will similarly (except for the position in space of its axis) be determined when $\mathbf{tM}ABC$ is determined.

Denote as in § 12 the angle and the distance between A and B by θ and d respectively. Also denote the angle and the distance between $\mathbf{M}AB$ and C (i.e. between C and the shortest distance of A and B) by ϕ and e. The conventions as to the signs of d and e are as in § 12.

By eq. (13) § 12 we have

$$\mathbf{tS}ABC = \mathbf{tS}(\mathbf{M}AB)C = \mathbf{tM}AB + \mathbf{t}C - e\tan\phi.$$

Hence by eq. (8) § 12

$$\mathbf{tS}ABC = \mathbf{t}A + \mathbf{t}B + \mathbf{t}C + d\cot\theta - e\tan\phi \quad \ldots\ldots(1).$$

Since $\mathbf{S}ABC = \mathbf{S}BCA = \mathbf{S}CAB$ the following deduction may be made from this:—*If* 1, 2, 3 *be three straight lines;* d_1, d_2, d_3 *the distances and* $\theta_1, \theta_2, \theta_3$ *the angles between the pairs* 23, 31, 12; e_1, e_2, e_3 *the distances and* ϕ_1, ϕ_2, ϕ_3 *the angles between* 1 *and* d_1, 2 *and* d_2, 3 *and* d_3; *then*

$$d_1\cot\theta_1 - e_1\tan\phi_1 = d_2\cot\theta_2 - e_2\tan\phi_2 = d_3\cot\theta_3 - e_3\tan\phi_3 \ldots(2).$$

Before finding $\mathsf{tM}ABC$, note that if Q, R, ... be any number of octonions

$$\mathsf{T}_1(QR...) = \mathsf{T}_1 Q \mathsf{T}_1 R..., \quad \mathsf{t}(QR...) = \mathsf{t}Q + \mathsf{t}R + ... \quad ...(3).$$

For two octonions Q and R these are at once deduced from the equation

$$\mathsf{T}(QR) = \mathsf{T}Q\mathsf{T}R$$

by putting $\mathsf{T}Q = \mathsf{T}_1 Q(1 + \Omega \mathsf{t}Q)$ and similarly for R and QR. The extension to any number of octonions is obvious.

Next note that from the equation

$$\mathsf{S}^2 Q - \mathsf{M}^2 Q = \mathsf{T}^2 Q,$$

we get by equating the ordinary scalar and the convertor parts [remembering that

$$\mathsf{S}Q = (1 + \Omega \mathsf{tS}Q)\,\mathsf{S}_1 Q, \ \mathsf{M}Q = (1 + \Omega \mathsf{tM}Q)\mathsf{M}_1 Q, \ \mathsf{T}Q = (1 + \Omega \mathsf{t}Q)\,\mathsf{T}_1 Q],$$

$$\mathsf{S}_1{}^2 Q - \mathsf{M}_1{}^2 Q = \mathsf{T}_1{}^2 Q, \ \ \mathsf{tS}Q \,.\, \mathsf{S}_1{}^2 Q - \mathsf{tM}Q \,.\, \mathsf{M}_1{}^2 Q = \mathsf{t}Q\mathsf{T}_1{}^2 Q ...(4).$$

For our purposes the last is put more conveniently in the form

$$\mathsf{tM}Q = \frac{\mathsf{t}Q\mathsf{T}_1{}^2 Q - \mathsf{tS}Q\mathsf{S}_1{}^2 Q}{\mathsf{T}_1{}^2 Q - \mathsf{S}_1{}^2 Q} \quad ....(5).$$

Now put $\qquad Q = ABC.$

By eq. (3) $\qquad \mathsf{t}Q = \mathsf{t}A + \mathsf{t}B + \mathsf{t}C,$

and by quaternion interpretation

$$\mathsf{T}_1 Q = \mathsf{T}_1 A \mathsf{T}_1 B \mathsf{T}_1 C, \quad \mathsf{S}_1 Q = -\,\mathsf{T}_1 A \mathsf{T}_1 B \mathsf{T}_1 C \sin\theta \cos\phi.$$

Hence by equations (1) and (5)

$$\mathsf{tM}ABC = \mathsf{t}A + \mathsf{t}B + \mathsf{t}C - \frac{d\cot\theta - e\tan\phi}{\cot^2\theta\tan^2\phi + \cot^2\theta + \tan^2\phi} ...(6).$$

Since $\mathsf{M}ABC = \mathsf{M}CBA$ a result similar to eq. (2) may be written down. By means of eq. (2) however it easily reduces to the well-known and easily proved equation

$$\sin\theta_1 \cos\phi_1 = \sin\theta_2 \cos\phi_2 = \sin\theta_3 \cos\phi_3.$$

$\mathsf{tM}(\mathsf{M}AB)C$ may be found by a process exactly similar to that used in establishing eq. (1). Thus by eq. (8) § 12

$$\mathsf{tM}(\mathsf{M}AB)C = \mathsf{tM}AB + \mathsf{t}C + e\cot\phi,$$

or $\qquad \mathsf{tM}(\mathsf{M}AB)C = \mathsf{t}A + \mathsf{t}B + \mathsf{t}C + d\cot\theta + e\cot\phi ......(7).$

From the geometrical interpretation of $\mathbf{S}_1 ABC$ just mentioned we see that in order that $\mathbf{S}_1 ABC$ may not be zero it is necessary and sufficient that not one of the motors A, B, C shall be a lator, and that they shall not all be parallel to one plane. Since [eq. (7) § 4]

$$\mathbf{S}^{-1}ABC = \mathbf{S}_1^{-1}ABC - \mathbf{S}_2 ABC\mathbf{S}_1^{-2}ABC = \mathbf{S}_1^{-1}ABC\,(1 - \Omega\mathbf{t}\mathbf{S}ABC),$$

we see that the conditions just mentioned are the necessary and sufficient conditions that $\mathbf{S}^{-1}ABC$ should have a definite intelligible meaning. Hence the equations

$$\left.\begin{aligned} E\mathbf{S}ABC &= A\mathbf{S}BCE + B\mathbf{S}CAE + C\mathbf{S}ABE \\ &= \mathbf{M}BC\mathbf{S}AE + \mathbf{M}CA\mathbf{S}BE + \mathbf{M}AB\mathbf{S}CE \end{aligned}\right\}\ldots(8),$$

serve to express any motor E in terms of motors coaxial with A, B, C or with $\mathbf{M}BC$, $\mathbf{M}CA$, $\mathbf{M}AB$ when A, B, C satisfy the conditions mentioned.

By putting $\mathbf{S}^{-1}ABC = \mathbf{S}_1^{-1}ABC\,(1 - \Omega\mathbf{t}\mathbf{S}ABC)$,

$$\mathbf{S}BCE = \mathbf{S}_1 BCE\,(1 + \Omega\mathbf{t}\mathbf{S}BCE),$$

$$\mathbf{S}AE = \mathbf{S}_1 AE\,(1 + \Omega\mathbf{t}\mathbf{S}AE), \text{ \&c.}$$

it will be seen that when E is expressed in terms of (ordinary scalar) multiples of A, B, C, ΩA, ΩB, ΩC, the A component is

$$A\mathbf{S}_1 BCE\mathbf{S}_1^{-1}ABC,$$

and the ΩA component is

$$\Omega A\mathbf{S}_1 BCE\mathbf{S}_1^{-1}ABC\,(\mathbf{t}\mathbf{S}BCE - \mathbf{t}\mathbf{S}ABC),$$

which may be transformed by eq. (1) to a fairly simple form. Similarly when expressed in terms of $\mathbf{M}BC$, $\Omega\mathbf{M}BC$, &c., the $\mathbf{M}BC$ component is

$$\mathbf{M}BC\mathbf{S}_1 AE\mathbf{S}_1^{-1}ABC,$$

and the $\Omega\mathbf{M}BC$ component is

$$\Omega\mathbf{M}BC\mathbf{S}_1 AE\mathbf{S}_1^{-1}ABC\,(\mathbf{t}\mathbf{S}AE - \mathbf{t}\mathbf{S}ABC),$$

which may be transformed by eq. (13) § 12 and eq. (1) of the present section.

14. Miscellaneous remarks. We collect here chiefly for future reference some miscellaneous statements most of which are almost obvious.

$\mathbf{S}$, $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{s}$ *are all distributive*, i.e.

$$\mathbf{S}(Q+R) = \mathbf{S}Q + \mathbf{S}R,\quad \mathbf{S}_1(Q+R) = \mathbf{S}_1 Q + \mathbf{S}_1 R,$$

$$\mathbf{S}_2(Q+R) = \mathbf{S}_2 Q + \mathbf{S}_2 R,\quad \mathbf{s}(Q+R) = \mathbf{s}Q + \mathbf{s}R \ldots\ldots(1).$$

$\mathbf{M}$ *is distributive* (*and also* $\mathbf{K}$) *but* $\mathbf{M}_1$, $\mathbf{M}_2$ *and* $\mathbf{m}$ *are not.*

In § 13 we saw that

$$\mathbf{T}(QR) = \mathbf{T}Q\mathbf{T}R,\ \ \mathbf{T}_1(QR) = \mathbf{T}_1Q\mathbf{T}_1R,\ \ \mathbf{T}_2(QR) = \mathbf{T}_2Q\mathbf{T}_2R,$$
$$\mathbf{t}(QR) = \mathbf{t}Q + \mathbf{t}R.$$

The similar equation in $\mathbf{U}$, *viz.* $\mathbf{U}(QR) = \mathbf{U}Q\mathbf{U}R$ *is true, but those in* $\mathbf{U}_1$, $\mathbf{U}_2$ *and* $\mathbf{u}$ *are not true.*

We have seen that Ω *and therefore every scalar octonion is commutative with each of the symbols* $\mathbf{K}$, $\mathbf{S}$, $\mathbf{M}$. *It is not commutative with* $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{s}$, $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{m}$, $\mathbf{T}$, $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{t}$, $\mathbf{U}$, $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{u}$. It may be remarked that a positive scalar octonion is commutative with $\mathbf{T}$, just as in Quaternions a positive scalar is.

The following relations are obvious

$$\mathbf{s}\Omega Q = \mathbf{S}_1 Q,\ \ \mathbf{S}_2\Omega Q = \Omega\mathbf{S}_1 Q,\ \ \mathbf{S}_1\Omega Q = 0,\ \ \Omega\mathbf{S}_2 Q = 0 \ \ldots(2).$$

From these we have

$$\mathbf{S}Q = \mathbf{s}\Omega Q + \Omega\mathbf{s}Q \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

so that *all the* $\mathbf{S}$'*s can be expressed by means of* $\mathbf{s}$ *and* Ω. *A similar remark is not true of the* $\mathbf{M}$'*s;* because $\mathbf{m}\Omega Q$ is indeterminate.

Suppose i, j, k are three mutually perpendicular intersecting unit rotors. Then A being any motor

$$A = -i\mathbf{S}iA - j\mathbf{S}jA - k\mathbf{S}kA \ \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

or by eq. (3)

$$A = -i\mathbf{s}\,.\,\Omega iA - j\mathbf{s}\,.\,\Omega jA - k\mathbf{s}\,.\,\Omega kA - \Omega i\mathbf{s}iA - \Omega j\mathbf{s}jA - \Omega k\mathbf{s}kA \ldots(5).$$

Let A be an independent variable motor given by

$$A = xi + yj + zk + l\Omega i + m\Omega j + n\Omega k \ldots\ldots\ldots\ldots(6),$$

where x, l, ... are ordinary scalars. Then ϑ is defined by the equation

$$\vartheta = i\partial/\partial l + j\partial/\partial m + k\partial/\partial n + \Omega i\partial/\partial x + \Omega j\partial/\partial y + \Omega k\partial/\partial z \ldots(7).$$

This gives

$$\mathbf{s}dA\vartheta = -d \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Hence if

$$A = x'i' + \ldots + l'\Omega i' + \ldots$$

where i', j', k' are any other set of mutually perpendicular intersecting unit rotors

$$i'\partial/\partial l' + \ldots + \Omega i'\partial/\partial x' + \ldots$$
$$= -i'\mathbf{s}\,.\,\Omega i'\vartheta - \ldots - \Omega i'\mathbf{s}i'\vartheta - \ldots = \vartheta,$$

so that ϑ is an invariant, i.e. is independent in meaning of the i, j, k used in defining it.

The following are easy deductions from equations (4) and (5):—

$A = B$ where A and B are motors

(1) if $\mathbf{S}\rho A = \mathbf{S}\rho B$ where ρ is an arbitrary rotor through a given point and *à fortiori* when ρ is an arbitrary rotor or motor;

(2) if $\mathsf{s}EA = \mathsf{s}EB$ when E is an arbitrary motor through a given point and *à fortiori* when E is an arbitrary motor;

(3) if $\mathsf{s}\rho A = \mathsf{s}\rho B$ when ρ is an arbitrary rotor.

The last of these statements may be deduced from the first of the following two:—

(4) If $\mathsf{s}\rho A = 0$ where ρ is an arbitrary rotor through a given point, A is a rotor through the same point.

(5) If $\mathsf{s}\rho_0 A = 0$ where ρ_0 is an arbitrary lator, A is a lator.

$\mathsf{s}\,.\,A^2$ is an expression that frequently occurs below. If A be a lator, $A^2 = 0$ and therefore $\mathsf{s}A^2 = 0$. If A be not a lator it can be put in the form $(1 + \Omega \mathsf{t}A)\mathbf{M}_1 A$. Hence

$$A^2 = (1 + 2\Omega \mathsf{t}A)\,\mathbf{M}_1{}^2 A \quad\quad (9).$$

In particular $$\mathsf{s}A^2 = 2\mathsf{t}A\mathbf{M}_1{}^2 A = -\,2\mathsf{t}A\mathbf{T}_1{}^2 A \quad\quad (10).$$

[This can be easily generalised to

$$\mathsf{s}\,.\,\mathbf{T}^2 Q = 2\mathsf{t}Q\mathbf{T}_1{}^2 Q \quad\quad (11),$$

but we shall not have any use for this more general form.] It follows that the necessary and sufficient condition to ensure that $\mathsf{s}A^2 = 0$ is that A shall be either a lator or a rotor.

n motors $A_1, A_2 \ldots A_n$ *are said to be independent* when no relation of the form $x_1 A_1 + \ldots + x_n A_n = \Sigma x A = 0$ (where $x_1 \ldots x_n$ are ordinary scalars not all zero) holds between them. When they are independent all motors of the form $\Sigma x A$ are said to form a *complex of order n*. The complex will generally be called the complex $A_1, A_2 \ldots A_n$, and when they are not independent the complex (though then of lower order than n), containing them, will be called the complex $A_1, A_2 \ldots A_r$.

Since [eq. (5)] any motor can be expressed in terms of six particular motors the complex of highest order is the sixth and all motors in space belong to it. If $A_1 \ldots A_6$ be any six independent motors, any motor in space can be expressed in the form

$$x_1 A_1 + \ldots + x_6 A_6.$$

This can be easily deduced from eq. (5).

Two motors A_1 and A_2 *are said to be reciprocal* when $\mathbf{s}A_1A_2=0$. Thus every lator and every rotor, but no other motor is self-reciprocal. The n motors $A_1 \ldots A_n$ are said to be co-reciprocal when every pair of them is a reciprocal pair. It is obvious that A_1 is then reciprocal to every motor of the complex $A_2 \ldots A_n$. More generally every motor of the complex $A_1 \ldots A_r$ is reciprocal to every motor of the complex $A_{r+1} \ldots A_n$. When two complexes are so related they are said to be *reciprocal complexes.*

Of six independent co-reciprocal motors not one can be self-reciprocal, for if it were it would be reciprocal to the complex of all six, i.e. to every motor in space. But this by statement (2) above requires that the motor should vanish. Thus *of six independent co-reciprocal motors not one is a lator or a rotor.*

If A, B, C are motors the necessary and sufficient condition to ensure that $\mathbf{S}ABC=0$ *is that either* (1) *two independent motors of the complex A, B, C are lators or* (2) $XA+YB+ZC=0$, *where X, Y, Z are scalar octonions whose ordinary scalar parts are not all zero.*

Before proving this, note the two following consequences of it.

A necessary condition is that $XA+YB+ZC=0$ where X, Y, Z are scalar octonions not all zero. For if $xA+yB+zC$ is a lator $\Omega xA+\Omega yB+\Omega zC=0$.

A sufficient condition is that $XA+YB+ZC=0$, where X, Y, Z are scalar octonions whose ordinary scalar parts are not all zero. This is merely a part of the general enunciation.

The condition is *sufficient.* If $XA+YB+ZC=0$ where $\mathbf{S}_1X$ is not zero, $A=-YX^{-1}B-ZX^{-1}C=Y'B+Z'C$ where Y', Z' are finite scalar octonions. Hence $\mathbf{S}ABC=0$. If A_0, B_0 be two independent lators of the complex A, B, C one of the three, (say C), motors A, B, C is independent of A_0, B_0 or they can all be expressed in terms of A_0 and B_0 in which case $\mathbf{S}ABC=0$. Expressing A and B in terms of A_0, B_0 and C, $\mathbf{S}ABC$ becomes an ordinary scalar multiple of $\mathbf{S}A_0B_0C$, i.e. it is zero since $A_0B_0=0$.

The condition is *necessary.* If $\mathbf{S}ABC=0$ we have [eq. (8) § 13]

$$A\mathbf{S}BCE+B\mathbf{S}CAE+C\mathbf{S}ABE=0$$

for all motor values of E. Hence a relation of the form

$$XA+YB+ZC=0$$

holds unless for all motor values of E

$$\mathbf{S}_1 BCE = \mathbf{S}_1 CAE = \mathbf{S}_1 ABE = 0.$$

By quaternion interpretation these last give

$$\mathbf{M}_1 BC = \mathbf{M}_1 CA = \mathbf{M}_1 AB = 0.$$

Hence those motors of A, B, C which are not lators are parallel. If they are all lators either two of them are independent or they are all multiples of any one of them; in each case the condition is satisfied. If they are not all lators let A have a definite axis. Then since B and C are either parallel to A or are lators we may put

$$B = bA + B', \quad C = cA + C',$$

where b and c are ordinary scalars and B' and C' are lators. [For first we can (eq. (1) § 7) by the addition of a suitable lator translate A so that its axis takes up any parallel position, say the axis of B. Then multiplying by a suitable scalar octonion, say $b + \Omega b'$, we can change it into any motor having that line for axis.] If B' and C' are independent, they are independent lators belonging to the complex A, B, C. If they are not independent or if either vanishes a relation of the form

$$XA + YB + ZC = 0$$

holds good. Hence the condition is necessary.

In connection with this it may be remarked that *if B and C have definite, not parallel, axes; $XB + YC$ is any motor which intersects a definite line (the axis of $\mathbf{M}BC$) perpendicularly.* For first $XB + YC$ does intersect $\mathbf{M}BC$ perpendicularly since

$$\mathbf{S}\,(XB + YC)\,\mathbf{M}BC = 0.$$

And secondly any motor which intersects $\mathbf{M}BC$ perpendicularly may be put in the form $XB + YC$, for if E be any motor we have, by putting $A = \mathbf{M}^{-1}BC$ in equation (8) § 13,

$$E = \mathbf{M}^{-1}BC\mathbf{S}BCE + B\mathbf{S}\,.\,EC\mathbf{M}^{-1}BC - C\mathbf{S}\,.\,EB\mathbf{M}^{-1}BC,$$

of which the first term on the right is zero when E intersects $\mathbf{M}BC$ perpendicularly. Under the same conditions as to B and C it follows that if $\mathbf{S}ABC = 0$ every motor of the form $XA + YB + ZC$

either intersects a definite line perpendicularly or is a lator perpendicular to the line.

15. Linear motor functions of motors. General, commutative and pencil functions. ϕ will be called a linear motor function of a motor, or more frequently a general function when ϕE is such that whatever be the motor value of E, ϕE is a motor, and that

$$\phi E + \phi F = \phi (E + F) \dots\dots\dots\dots\dots\dots (1),$$

whatever be the motor values of E and F.

ϕ will be called a commutative linear motor function of a motor, or more shortly a commutative function, when

$$\phi E = \mathbf{M} (Q_1 E R_1 + Q_2 E R_2 + \dots) = \Sigma \mathbf{M} Q E R \dots\dots\dots (2),$$

where E is an arbitrary motor and $Q_1, R_1, Q_2, R_2 \dots$ given octonions.

In Quaternions two such definitions would be precisely equivalent, but it is not so in Octonions. A commutative function is commutative with Ω and therefore with any scalar octonion, whence its name. The general function (except when it degenerates into a commutative function) is not thus commutative. It will be seen that when octonions are regarded as formal quaternions, the linear formal vector function of a formal vector [(6) § 2 above] is the commutative and not the general function.

We proceed to show that the commutative function involves eighteen scalars and the general function thirty-six.

Let A, B, C be any three constant motors such that $\mathbf{S}_1 ABC$ is not zero. Since the commutative ϕ is commutative with scalar octonions and therefore in particular with $\mathbf{S}ABC$, we have for it [eq. (8) § 13]

$$\phi E = (\phi A \mathbf{S} BCE + \phi B \mathbf{S} CAE + \phi C \mathbf{S} ABE) \mathbf{S}^{-1} ABC \dots (3).$$

Hence when the three motors ϕA, ϕB, ϕC are given, ϕ is completely specified. Moreover whatever values be given to these motors the expression on the right of (3) is of the same form as the expression on the right of (2); [put

$$2\mathbf{S}BCE = BCE + \mathbf{K}(BCE) = BCE - ECB, \text{ \&c.}];$$

and if any one of them be altered ϕE is altered for some value of E. Hence eighteen and only eighteen scalars are required to specify a commutative function.

Next let ϕ be a general function. A, B, C being as before, we have seen (§ 13) that any motor E can be expressed in the form

$$E = xA + yB + zC + l\Omega A + m\Omega B + n\Omega C \quad\text{.........(4)},$$

where x, y, z, l, m, n are ordinary scalars. Although ϕ is not commutative with scalar octonions in general, yet by its definition it is commutative with ordinary scalars. Hence

$$\phi E = x\phi A + y\phi B + z\phi C + l\phi(\Omega A) + m\phi(\Omega B) + n\phi(\Omega C) \text{...(5)}.$$

Hence ϕ is now completely given by six motors $\phi A \dots \phi(\Omega C)$, i.e. by thirty-six scalars. And it is clear that all these are required.

Let now $\psi(E, F)$ be an octonion function of E and F which is linear in the general sense in each of its constituents. Then Z_1 is defined by

$$\psi(Z_1, Z_1) = \psi(\vartheta_1, A_1) \quad\text{...................(6)},$$

where the suffixes of ϑ and A are to indicate the operand (A) of ϑ; and where ϑ and A have the meanings given to them in equations (6) and (7) of § 14 above. Similarly $\chi(E, F, G, H)$ being an octonion function linear in each of its constituents, Z_2 is defined by

$$\chi(Z_1, Z_1, Z_2, Z_2) = \chi(\vartheta_1, A_1, \vartheta_2, A_2) \quad\text{..... ......(7)},$$

and so to any number of pairs of Z's. If, as in equation (6), there is only one pair the suffix may be dropped.

ζ is defined as in the case (§ 3) of formal quaternions, i, j, k now standing for three mutually perpendicular intersecting unit rotors. In discussing formal quaternions we only considered what is now denoted by the commutative function. But the definition of ζ will be supposed to hold good with the more general meanings of ψ and χ now contemplated. Z is an invariant (§ 14). ζ is an invariant with regard to the directions of i, j, k but not with regard to their point of intersection, unless we restrict ourselves to commutative functions. When this restriction is not made, then, ζ must be regarded as a function of the point of intersection just mentioned.

It will be noticed that by equations (4) and (5) of § 14,

$$E = -\zeta \mathbf{S} E \zeta = -Z \mathrm{s} E Z \quad\text{.................(8)}.$$

The conjugate ϕ' of ϕ is defined by either of the equivalent equations

$$\mathbf{s}E\phi F = \mathbf{s}F\phi' E \quad\text{.......................(9)},$$

for all motor values of E and F or

$$\phi' E = - Z\mathbf{s}E\phi Z \quad\text{.......................(10)}.$$

In the case of the commutative function these may [eq. (3) § 14] be replaced by

$$\mathbf{S}E\phi F = \mathbf{S}F\phi' E \quad\text{.......................(11)},$$

$$\phi' E = - \zeta\mathbf{S}E\phi\zeta \quad\text{.....................(12)}.$$

If $\phi' = \phi$ the function is said to be self-conjugate. In this case $\mathbf{s}E\phi F = \mathbf{s}F\phi E$. There are fifteen such independent scalar equations that must be satisfied in order that the general ϕ may be self-conjugate, because the number of such independent equations is the number of pairs that can be chosen out of six independent motors. Hence a general self-conjugate function involves twenty-one scalars.

Equation (11) is equivalent to two scalar equations. Three independent equations of the form $\mathbf{S}E\phi F = \mathbf{S}F\phi E$ must be satisfied by a commutative self-conjugate function. Such a function therefore involves twelve scalars.

Rotors which pass through one point O may be said to form a pencil of rotors.

If a commutative function is such that it reduces every rotor of such a pencil to a rotor of the same pencil it will be called a pencil function, and the point through which the rotors pass will be called the centre of the function or of the pencil according to convenience. Since every motor in space can be put in the form $\omega + \Omega\sigma$ where ω and σ belong to the pencil and since Ω is commutative with a commutative function, the present function is known for every motor in space when it is known for every rotor of the pencil. So long as we restrict ourselves to rotors of the pencil, the pencil function has exactly the same properties as a linear vector function of a vector. Besides the three scalars required to specify the centre, a pencil function thus involves nine other scalars, and a self-conjugate pencil function six. A pencil function is therefore not the most general form of a commutative function.

We now proceed to express various functions (general, commutative, pencil, self-conjugate) in terms of each other.

Any general function Φ *can be expressed in the form*

$$\Phi E = \Upsilon_1 E + \Pi E \quad \ldots\ldots(13),$$

where Υ_1 *is a commutative function and* Π *is given in terms of another commutative function* Υ_2 *by the equations*

$$\Pi\omega = 0,\ \Pi\Omega\omega = \Upsilon_2\omega \quad \ldots\ldots(14),$$

where ω *is an arbitrary rotor of an assigned pencil.*

For taking i, j, k as unit perpendicular rotors belonging to the pencil, Φ can be expressed in the form

$$\begin{aligned}\Phi E &= -(A\mathbf{s}E\Omega i + A'\mathbf{s}Ei + B\mathbf{s}E\Omega j + B'\mathbf{s}Ej + C\mathbf{s}E\Omega k + C'\mathbf{s}Ek)\\ &= -\Sigma\{A\mathbf{S}Ei + (A' - \Omega A)\,\mathbf{s}Ei\}\\ &= \Upsilon_1 E + \Pi E,\end{aligned}$$

where Υ_1 is the commutative function $-\Sigma A\mathbf{S}(\ )i$ and Π is the function $-\Sigma(A' - \Omega A)\,\mathbf{s}(\ )i$. Here we clearly have $\Pi\omega = 0$ and $\Pi\Omega\omega = \Upsilon_2\omega$ where Υ_2 is the commutative function given by

$$\Upsilon_2 E = -\Sigma(A' - \Omega A)\,\mathbf{S}Ei.$$

It will be observed that Υ_1 involves eighteen scalars and Υ_2 eighteen so that the full number thirty-six of Φ is accounted for. Hence when the centre of the pencil is given Υ_1 and Υ_2 are unique.

It does not appear from the above that Υ_1 and Υ_2 are self-conjugate when Φ is. It will appear directly that in general they are not (though the two self-conjugates Υ_1 and Υ would involve twenty-four scalars and the self-conjugate Φ only twenty-one). The centre of the pencil however can be so chosen *in general* that they are self-conjugate.

The general function Φ *can always be put in the form*

$$\Phi\omega = \phi_1\omega + \Omega\phi_2\omega,\ \Phi\Omega\omega = \phi_3\omega + \Omega\phi_4\omega \quad \ldots\ldots(15),$$

where $\phi_1, \phi_2, \phi_3, \phi_4$ *are four pencil functions with a common assigned centre and* ω *is an arbitrary rotor through the centre.* For $\Phi\omega$ is a linear function of ω which can be expressed as the sum of two linear functions, the first being a rotor ($\phi_1\omega$) through the assigned centre and the second a lator ($\Omega\phi_2\omega$). Similarly for $\Phi\Omega\omega$. [If the legitimacy of this reasoning is questioned put

A above $= \alpha + \Omega\lambda$, $A' = \alpha' + \Omega\lambda'$, $B = \beta + \Omega\mu$, &c. where α, λ, &c. are rotors through the assigned centre. Then

$$\phi_1 E = -\Sigma\alpha \mathbf{S} Ei,\ \phi_2 E = -\Sigma\lambda\mathbf{S}Ei,\ \phi_3 E = -\Sigma\alpha'\mathbf{S}Ei,\ \phi_4 E = -\Sigma\lambda'\mathbf{S}Ei.]$$

This determination is also unique since each of the four pencil functions involves nine scalars and Φ thirty-six.

Comparing equations (13), (14), (15) we see that

$$\Upsilon_1 = \phi_1 + \Omega\phi_2,\ \Upsilon_2 = \phi_3 + \Omega(\phi_4 - \phi_1)\ldots\ldots\ldots\ldots(16).$$

The general self-conjugate Ψ can always be put in the form

$$\Psi\omega = \phi\omega + \Omega\psi_1\omega,\ \Psi\Omega\omega = \psi_2\omega + \Omega\phi'\omega\ldots\ldots\ldots(17),$$

where ψ_1, ψ_2 are self-conjugate pencil functions and ϕ is a pencil function whose conjugate is ϕ' with an assigned common centre and where ω is an arbitrary rotor through the centre.

To prove this first suppose Ψ to have the form of Φ in equation (15) and use the equation

$$\mathbf{S}(\omega + \Omega\sigma)\Psi(\omega' + \Omega\sigma') = \mathbf{S}(\omega' + \Omega\sigma')\Psi(\omega + \Omega\sigma),$$

where ω, σ, ω', σ' are rotors through the assigned centre. Putting σ and σ' zero and leaving ω and ω' arbitrary, we find that ϕ_2 is self-conjugate. Similarly putting ω and ω' zero we find that ϕ_3 is self-conjugate. Finally, putting ω and σ' zero we find that $\phi_4 = \phi_1'$.

ψ_1 and ψ_2 involve six scalars each and ϕ nine so that, since Ψ involves twenty-one, the determination of ψ_1, ψ_2 and ϕ is unique.

When Φ of equation (13) is self-conjugate and equal to the present Ψ we see from equation (16) that

$$\Upsilon_1 = \phi + \Omega\psi_1,\ \Upsilon_2 = \psi_2 + \Omega(\phi' - \phi) = \psi_2 - 2\Omega\mathbf{M}\epsilon(\ )\ldots(18),$$

where ϵ is put for the rotation rotor $\frac{1}{2}\mathbf{M}\zeta\phi\zeta$ of ϕ. Thus in general Υ_1 and Υ_2 are not self-conjugate.

ϕ of equation (17) *can in general be made self-conjugate by suitably choosing the centre.*

We see that this is probably the case by noticing that in the choice of the point there are three disposable scalars.

ϕ is self-conjugate if $\mathbf{S}_1\omega_1\Psi\omega_2 = \mathbf{S}_1\omega_2\Psi\omega_1$ for any two rotors through the centre. If the given centre O does not possess this

property suppose if possible that P does where $\overline{OP}=\rho$. Any two rotors through P may be expressed [eq. (1) § 7] as $\omega_1+\Omega\mathbf{M}\rho\omega_1$ and $\omega_2+\Omega\mathbf{M}\rho\omega_2$ where ω_1 and ω_2 are rotors through O. Now

$$\mathbf{S}_1(\omega_1+\Omega\mathbf{M}\rho\omega_1)\Psi(\omega_2+\Omega\mathbf{M}\rho\omega_2)=\mathbf{S}(\omega_1\phi\omega_2+\psi_2\omega_1\,.\,\rho\omega_2).$$

Hence the required condition is that

$$\mathbf{S}\omega_1(\phi-\phi')\,\omega_2+\mathbf{S}(\omega_1\rho\psi_2\omega_2-\omega_2\rho\psi_2\omega_1)=0,$$

or $$\mathbf{S}\omega_2\omega_1\mathbf{M}\zeta\phi\zeta=-\mathbf{S}\omega_2\omega_1\mathbf{M}\zeta\mathbf{M}\rho\psi_2\zeta=\mathbf{S}\omega_2\omega_1(\psi_2+\mathbf{S}\zeta\psi_2\zeta)\,\rho\,;$$

P satisfies the required conditions then if

$$(\psi_2+\mathbf{S}\zeta\psi_2\zeta)\,\rho=\mathbf{M}\zeta\phi\zeta\;\ldots\ldots\ldots\ldots\ldots\ldots(19),$$

which can generally but not invariably be satisfied.

When ϕ is self-conjugate Υ_1 and Υ_2 are also self-conjugate as appears by equation (18).

A commutative function Υ *can always be put in the form* $\phi_1+\Omega\phi_2$ *where* ϕ_1 *and* ϕ_2 *are pencil functions with a common assigned centre, and if* Υ *is self-conjugate* ϕ_1 *and* ϕ_2 *are self-conjugate.* This is obvious from what has gone before.

It may be noticed that if a commutative function Υ reduces every motor it acts on to a lator it is of the form $\Omega\phi_2$. Hence it reduces every lator it acts on to zero and may be said only to act on rotors. Moreover it reduces equal parallel rotors to the same lator. Its properties are precisely similar to those of the linear vector function of a vector corresponding to the pencil function ϕ_2.

I shall postpone the more detailed consideration of general functions to a later part of the treatise, as at present I wish to limit the discussion as far as convenient to those properties of octonions which are most immediately connected with the fact that octonions are formal quaternions.

16. Properties of the commutative function derived from Formal Quaternions. Confining our attention then to commutative functions we will in this section put down certain results, without proof, that flow from the fact that octonions are formal quaternions.

$$\phi E = -B\mathbf{S}AE - B'\mathbf{S}A'E - B''\mathbf{S}A''E = -\Sigma B\mathbf{S}AE \ldots (1)$$

is a perfectly general form for this case, even when the number of terms is only three. A, A', A'' may be assumed such that $\mathbf{S}_1 AA'A''$ is not zero. [If $\mathbf{S}_1 AA'A''$ is zero equation (1) defines a commutative function but not one of the most general form. *All* commutative functions are of the form given in (1) when A, A', A'' are arbitrarily chosen as long as $\mathbf{S}_1 AA'A''$ is not zero.]

Put $$\phi + \phi' = 2\bar{\phi} \ldots\ldots (2),$$

$\bar{\phi}$ is self-conjugate and is called the self-conjugate part of ϕ. Put

$$\mathbf{M}\zeta\phi\zeta = 2H \ldots\ldots (3).$$

Then $$\phi E = \bar{\phi}E + \mathbf{M}HE,\ \phi' E = \bar{\phi}E - \mathbf{M}HE \ldots\ldots (4).$$

$\psi(E, F)$ being a commutative linear motor function in both the motors E and F,

$$\psi(\zeta, \phi\zeta) = \psi(\phi'\zeta, \zeta) \ldots\ldots (5).$$

[Proved at once by putting $\phi'\zeta = -\zeta_1 \mathbf{S}\zeta\phi\zeta_1$. Note that putting $\phi = Q(\ )Q^{-1}$ it follows that ζ is an invariant when associated only with commutative functions.] When ϕ is given by equation (1)

$$\psi(\zeta, \phi\zeta) = \Sigma\psi(A, B) \ldots\ldots (6).$$

If χ be a perfectly arbitrary commutative function and if

$$\mathbf{S}\chi\zeta\phi_1\zeta = \mathbf{S}\chi\zeta\phi_2\zeta,$$

where ϕ_1 and ϕ_2 are commutative functions, then $\phi_1 = \phi_2$. And if the same equation hold when χ is a perfectly arbitrary self-conjugate commutative function, then $\bar{\phi}_1 = \bar{\phi}_2$.

The equation proved in § 3 above for formal quaternions

$$E\mathbf{S}\zeta_1\zeta_2\zeta_3\mathbf{S}\phi\zeta_1\phi\zeta_2\phi\zeta_3 = -3\mathbf{M}\zeta_1\zeta_2\mathbf{S}\phi E\phi\zeta_1\phi\zeta_2 \ldots\ldots (7),$$

may be put in the form

$$(\phi^3 - M''\phi^2 + M'\phi - M)E = 0 \ldots\ldots (8),$$

where $$6M = \mathbf{S}\zeta_1\zeta_2\zeta_3\mathbf{S}\phi\zeta_1\phi\zeta_2\phi\zeta_3 \ldots\ldots (9)$$

$$2M' = -\mathbf{S}\mathbf{M}\zeta_1\zeta_2\mathbf{M}\phi\zeta_1\phi\zeta_2 \ldots\ldots (10),$$

$$M'' = -\mathbf{S}\zeta\phi\zeta \ldots\ldots (11).$$

By means of equation (6) these may be put

$$M = \mathbf{S}AA'A''\mathbf{S}BB'B'' \ldots\ldots (12),$$

$$M' = -\mathbf{S}(\mathbf{M}A'A''\mathbf{M}B'B'' + \mathbf{M}A''A\mathbf{M}B''B + \mathbf{M}AA'\mathbf{M}BB') \ldots (13),$$

$$M'' = -\mathbf{S}(AB + A'B' + A''B'') \ldots\ldots (14).$$

By the same equation we see that if the form $-\Sigma B\mathbf{S}AE$ of ϕE be binomial or monomial respectively, we have

$$M=0,\ M'=-\mathbf{SM}AA'\mathbf{M}BB',\ M''=-\mathbf{S}(AB+A'B')\ldots(15),$$

$$M=0,\ M'=0,\ M''=-\mathbf{S}AB\ \ldots\ldots\ldots\ldots\ldots\ldots(16).$$

If we define $\bar{A}$, $\bar{A}'$, $\bar{A}''$ by the equations

$$\left.\begin{array}{l}\bar{A}=-\mathbf{M}A'A''\mathbf{S}^{-1}AA'A'',\ \bar{A}'=-\mathbf{M}A''A\mathbf{S}^{-1}AA'A'',\\ \bar{A}''=-\mathbf{M}AA'\mathbf{S}^{-1}AA'A''\end{array}\right\}\ldots(17),$$

we have

$$\mathbf{S}A\bar{A}=\mathbf{S}A'\bar{A}'=\mathbf{S}A''\bar{A}''=-1\ \ldots\ldots\ldots\ldots\ldots\ldots(18),$$

$$\mathbf{S}A\bar{A}'=\mathbf{S}A\bar{A}''=\mathbf{S}A'\bar{A}''=\mathbf{S}A'\bar{A}=\mathbf{S}A''\bar{A}=\mathbf{S}A''\bar{A}'=0\ldots(19).$$

[Note that equations (18), (19) are equivalent to eighteen equations among ordinary scalars and are therefore sufficient to determine the three motors $\bar{A}$, $\bar{A}'$, $\bar{A}''$. Also note that they are symmetrical in the two sets A, A', A'' and $\bar{A}$, $\bar{A}'$, $\bar{A}''$. Also note that when $\mathbf{S}EF=0$, E and F are both perpendicular and reciprocal.]

By equation (8) § 13,

$$\left.\begin{array}{l}E=-A\mathbf{S}\bar{A}E-A'\mathbf{S}\bar{A}'E-A''\mathbf{S}\bar{A}''E\\ \quad=-\bar{A}\mathbf{S}AE-\bar{A}'\mathbf{S}A'E-\bar{A}''\mathbf{S}A''E\end{array}\right\}\ldots\ldots\ldots\ldots(20).$$

From these we have again

$$\left.\begin{array}{l}A=-\mathbf{M}\bar{A}'\bar{A}''\mathbf{S}^{-1}\bar{A}\bar{A}'\bar{A}'',\ A'=-\mathbf{M}\bar{A}''\bar{A}\mathbf{S}^{-1}\bar{A}\bar{A}'\bar{A}'',\\ A''=-\mathbf{M}\bar{A}\bar{A}'\mathbf{S}^{-1}\bar{A}\bar{A}'\bar{A}''\end{array}\right\}\ldots(21),$$

which should be compared with equations (17). From equation (20) we see that $\bar{A}$, $\bar{A}'$, $\bar{A}''$ are any three motors for which $\mathbf{S}_1\bar{A}\bar{A}'\bar{A}''$ is not zero. Also

$$\mathbf{S}AA'A''\mathbf{S}\bar{A}\bar{A}'\bar{A}''=1\ \ldots\ldots\ldots\ldots(22).$$

From equation (1) $\phi\mathbf{M}A'A''=-B\mathbf{S}AA'A''$, &c. Hence

$$B=\phi\bar{A},\ B'=\phi\bar{A}',\ B''=\phi\bar{A}''\ldots\ldots\ldots\ldots(23),$$

[which are also given by the equation derived from equation (20),

$$\phi E=-\phi\bar{A}\mathbf{S}AE-\phi\bar{A}'\mathbf{S}A'E-\phi\bar{A}''\mathbf{S}A''E].$$

Hence by equations (12), (13), (14)

$$M\mathbf{S}\bar{A}\bar{A}'\bar{A}''=\mathbf{S}\phi\bar{A}\phi\bar{A}'\phi\bar{A}''\ \ldots\ldots\ldots\ldots\ldots\ldots(24),$$

$$M'\mathbf{S}\bar{A}\bar{A}'\bar{A}''=\mathbf{S}(\bar{A}\phi\bar{A}'\phi\bar{A}''+\bar{A}'\phi\bar{A}''\phi\bar{A}+\bar{A}''\phi\bar{A}\phi\bar{A}')\ldots(25),$$

$$M''\mathbf{S}\bar{A}\bar{A}'\bar{A}''=\mathbf{S}(\bar{A}'\bar{A}''\phi\bar{A}+\bar{A}''\bar{A}\phi\bar{A}'+\bar{A}\bar{A}'\phi\bar{A}'')\ldots\ldots\ldots(26).$$

Here $\bar{A}$, $\bar{A}'$, $\bar{A}''$ are any three motors whatever. Equation (24), it will be noticed, is the original definition (§ 2 above) of M. From the last three equations

$$X^3 - M''X^2 + M'X - M$$
$$\equiv \mathbf{S}(X-\phi)\bar{A}(X-\phi)\bar{A}'(X-\phi)\bar{A}''.\mathbf{S}^{-1}\bar{A}\bar{A}'\bar{A}'' \ldots (27),$$

where X is any scalar octonion.

By equation (5), equations (9), (10), (11) are unaltered when ϕ is changed to ϕ'. Hence the ϕ' cubic is the same as the ϕ cubic [eq. (8)]. It follows that ϕ may be changed to ϕ' in equations (24) to (27).

Since in equation (24) $\bar{A}$ is an arbitrary motor we have (§ 14)

$$M\mathbf{M}\bar{A}'\bar{A}'' = \phi'\mathbf{M}\phi\bar{A}'\phi\bar{A}'' \ldots\ldots\ldots\ldots\ldots\ldots (28).$$

Similarly
$$M\mathbf{M}\bar{A}'\bar{A}'' = \phi\mathbf{M}\phi'\bar{A}'\phi'\bar{A}'' \ldots\ldots\ldots\ldots\ldots\ldots (29).$$

Here again it must be remembered that $\bar{A}'$ and $\bar{A}''$ are arbitrary motors.

These results are exactly of the same form as corresponding quaternion results. We have therefore treated them very briefly. As the geometrical interpretations of octonion formulae have to be made independently of quaternion forms we must now enter into more detail.

17. The ϕ cubic, the ϕ_1 cubic, and their roots. At the end of § 15 we saw that any commutative function ϕ can be expressed in the form $\phi_1 + \Omega\phi_2$ where ϕ_1 and ϕ_2 are pencil functions with any assigned common centre. It is now important to observe that the *form* of ϕ_1 is independent of the position of this centre. By this is meant that if α, β, ... be given independent rotors and ρ an arbitrary rotor through the centre O and α', β', ..., ρ' be the parallel equal rotors through the centre O'; and if $\phi = \phi_1 + \Omega\phi_2 = \phi_1' + \Omega\phi_2'$ where ϕ_1 and ϕ_2 are pencil functions with centre O and ϕ_1' and ϕ_2' are pencil functions with centre O'; then if $\phi_1\rho = -\Sigma\beta\mathbf{S}\alpha\rho$, we shall have $\phi_1'\rho' = -\Sigma\beta'\mathbf{S}\alpha'\rho'$.

$$A = \omega + \Omega\sigma = \omega' + \Omega\sigma',\ \ \phi A = \omega_1 + \Omega\sigma_1 = \omega_1' + \Omega\sigma_1',$$

where A is any motor, where ω, σ, ω_1, σ_1 are rotors through O and where ω', σ', ω_1', σ_1' are rotors through O'. Then $\omega_1 = \phi_1\omega$, $\omega_1' = \phi_1'\omega'$;

and ω_1 and ω_1' are equal and parallel, being the rotor of ϕA translated to pass through O and O'; and also ω and ω' are equal and parallel, being the rotor of A translated to pass through O and O'.

The following may also be noticed in passing. As we do not propose to use these results we give them without proof. The form of ϕ_2 is of course altered. Let $\bar{\phi}_1$, $\bar{\phi}_2$ be the pure parts and $\mathbf{M}\epsilon_1(\ \ )$, $\mathbf{M}\epsilon_2(\ \ )$ the rotatory parts of ϕ_1, ϕ_2; and let $\overline{OO'}=\rho$. Then by changing the centre from O to O' the form of ϕ_2 is altered to the form of

$$\phi_2+\phi_1\mathbf{M}\rho(\quad)-\mathbf{M}\rho\phi_1(\quad),$$

the form of $\bar{\phi}_2$ to that of

$$\bar{\phi}_2+\bar{\phi}_1\mathbf{M}\rho(\quad)-\mathbf{M}\rho\bar{\phi}_1(\quad),$$

and ϵ_2 becomes changed to the rotor through O' equal and parallel to

$$\epsilon_2-\mathbf{M}\rho\epsilon_1.$$

On account of the similarity of ϕ_1 to a linear vector function of a vector we see that its cubic must have ordinary scalar coefficients. In accordance with equations (9), (10), (11) § 16 these are $-\mathbf{S}\zeta\phi_1\zeta$, &c. Substituting $\phi_1+\Omega\phi_2$ for ϕ in those equations we see by equating ordinary scalar and convertor parts that the cubic satisfied by ϕ_1 is

$$\phi_1^3-\mathbf{S}_1M''.\phi_1^2+\mathbf{S}_1M'.\phi_1-\mathbf{S}_1M=0 \ldots\ldots\ldots\ldots(1).$$

This cubic will be referred to as the ϕ_1 cubic.

One of the roots of the ϕ_1 cubic is always real when as we assume ϕ is real. The other two are both imaginary or both real. If any two are equal they are all real.

The question now arises,—has the ϕ cubic (8) § 16 always scalar octonion roots and if not always what are the conditions? In other words, when can we put

$$\phi^3-M''\phi^2+M'\phi-M\equiv(\phi-X)(\phi-X')(\phi-X'')\ldots(2),$$

where X, X', X'' are scalar octonions? Putting

$$X=x+\Omega y,\ X'=x'+\Omega y',\ X''=x''+\Omega y''\ldots\ldots\ldots(3),$$

where x, y, &c. are ordinary scalars and expanding the right of (2)

we find by equating the ordinary scalar part and the convertor part of each coefficient that

$$\left.\begin{aligned} xx'x'' &= \mathsf{S}_1 M \\ x'x'' + x''x + xx' &= \mathsf{S}_1 M' \\ x + x' + x'' &= \mathsf{S}_1 M'' \\ yx'x'' + y'x''x + y''xx' &= \mathsf{s}M \\ y(x' + x'') + y'(x'' + x) + y''(x + x') &= \mathsf{s}M' \\ y + y' + y'' &= \mathsf{s}M'' \end{aligned}\right\} \ldots\ldots (4).$$

Hence x, x', x'' are the roots of the ϕ_1 cubic. x will be always assumed real. Supposing these roots determined we have

$$y = \frac{n - n'x + n''x^2}{(x - x')(x - x'')},\ y' = \frac{n - n'x' + n''x'^2}{(x' - x'')(x' - x)},\ y'' = \frac{n - n'x'' + n''x''^2}{(x'' - x)(x'' - x')} \ldots\ldots\ldots\ldots (5),$$

where n, n', n'' have been put for $\mathsf{s}M$, $\mathsf{s}M'$, $\mathsf{s}M''$.

This shows that when x, x', x'' are all unequal, X, X' and X'' can all be determined. Further when x is real and x' and x'' are imaginary (in which case they are all unequal) both X and $(\phi - X')(\phi - X'')$ are real.

But when two of the ϕ_1 cubic roots are equal, say x' and x'', there are in general no corresponding roots X', X'' of the ϕ cubic; and when all three roots of the ϕ_1 cubic are equal there is in general no root of the ϕ cubic.

If $x' = x''$ and x is not equal to either it will be found by a similar process to the above that there is one root $X = x + \Omega y$ of the ϕ cubic, y being given by eq. (5). In this case and sometimes in others it is convenient to define the scalar octonions N and N' by the identity

$$\phi^3 - M''\phi^2 + M'\phi - M \equiv (\phi - X)(\phi^2 - N'\phi + N) \ \ldots\ (6).$$

If we put $\phi - Y = \psi$ where Y is a scalar octonion ψ is like ϕ a commutative function and therefore satisfies a cubic. It is of importance to remark that this cubic is obtained from the ϕ cubic by substituting $\psi + Y$ for ϕ. This, which is not a truism, appears from the identity (27) § 16.

If two roots or three roots of the ϕ_1 cubic are equal it is not always true that there are no corresponding roots of the ϕ cubic.

If two roots of the ϕ_1 cubic are equal to x and the third is not equal to x; in order that the ϕ cubic may have a root equal to $x + \Omega y$ the M of the $\phi - x$ cubic must be zero. There are then three

roots of the ϕ cubic and the two corresponding to x are arbitrary to the extent that their convertor parts may have any values consistent with the sum having a given value. For we have seen that the third root x'' of the ϕ_1 cubic has a root $x''+\Omega y''$ of the ϕ cubic corresponding to it. If $x+\Omega y$ is another root

$$(\phi - x - \Omega y)(\phi - x'' - \Omega y'')$$

is a factor of $(\phi - x)^3 - M''(\phi - x)^2 + M'(\phi - x) - M$. There must therefore be a third factor and this can only be $\phi - x - \Omega y'$. But the product $(\phi - x)^2 - \Omega(y+y')(\phi - x)$ of $(\phi - x) - \Omega y$ and $(\phi - x) - \Omega y'$ is unaltered if y and y' are changed arbitrarily so long as their sum remains unaltered. And in the product of these and $\phi - x'' - \Omega y''$ there is no term independent of $\phi - x$; i.e. the M of the $\phi - x$ cubic is zero. Conversely if the M of the $\phi - x$ cubic is zero it is easy to prove that there are three roots of the ϕ cubic.

If three roots of the ϕ_1 cubic are equal to x; in order that the ϕ cubic may have a root the M of the $\phi - x$ cubic must be zero. If $\phi - x - \Omega y$ is the one corresponding factor and

$$(\phi - x)^2 - \Omega z(\phi - x) + \Omega z'$$

is the other (quadratic) factor of $(\phi - x)^3 - M''(\phi - x)^2 + M'(\phi - x)$, y and z are indeterminate except as to their sum which is $\mathbf{s}M''$. In order that the ϕ cubic may have two roots both the M and M' of the $\phi - x$ cubic must be zero. There are then three roots of the ϕ cubic and these are arbitrary to the extent that their convertor parts may have any values consistent with their sum having a given value. The proof of this is exactly similar to the proof of the last proposition.

18. Geometrical properties of a commutative function. By discussion of the nature of the roots of the ϕ_1 cubic and of the ϕ cubic we are able to deduce many important geometrical properties of a commutative function.

For the sake of brevity the following terms will be used:—

An *axial motor* will mean a motor which is not a lator, i.e. a motor which has a definite axis.

Two *completely independent* axial motors are two axial motors which are not parallel. [According to the definition of § 14 two coaxial motors and two parallel motors are in general independent. Hence the necessity for the term *completely* independent.]

A *single root* of a cubic will mean a root which is not equal to any other root of the cubic.

A *repeated root* will mean a root which is equal to some other root.

The following conventions as to notation will be strictly adhered to in this section.

i, j, k will mean a set of mutually perpendicular intersecting unit rotors. A, B, A', E, F, E' &c. will stand for motors. $\alpha, \beta, \gamma, \alpha'$ &c. will stand for rotors.

X, Y, Z, Y', Z' will stand for scalar octonions. x, y, a, b, p &c. will stand for ordinary scalars.

For the sake of clearness it is convenient to arrange the most important assertions of this section in formal propositions.

PROP. I. *If X is a root of the ϕ cubic corresponding to a single root of the ϕ_1 cubic, ϕ can be put in the form*

$$(\phi - X) E = -B\mathbf{S}AE - B'\mathbf{S}A'E \ldots\ldots\ldots\ldots\ldots(1),$$

where $\mathbf{S}_1\mathbf{M}AA'\mathbf{M}BB'$ is not zero. In this case the axial motor $E = \mathbf{M}AA'$ is such that $\phi E = XE$. If X is a root of the ϕ cubic corresponding to a repeated root of the ϕ_1 cubic there is not always an axial motor E for which $\phi E = (X + \Omega y) E$ but when there is not, and when the root of the ϕ_1 cubic is not zero $\mathbf{M}_1\phi\rho$ and $\mathbf{M}_1 X\rho$ are equal and parallel where ρ is any rotor parallel to a certain plane. When the repeated root of the ϕ_1 cubic is zero $\phi\rho$ is a lator. [Note that the condition that $\mathbf{S}_1\mathbf{M}AA'\mathbf{M}BB'$ is not zero is equivalent to the conditions that not one of the rotors $\mathbf{M}_1A$, $\mathbf{M}_1A'$, $\mathbf{M}_1B$, $\mathbf{M}_1B'$, $\mathbf{M}_1AA'$, $\mathbf{M}_1BB'$ is zero and the two last are not perpendicular; i.e. A, A', B, B', $\mathbf{M}AA'$ and $\mathbf{M}BB'$ are all axial motors and the last two are not perpendicular.]

Since X is a root of the ϕ cubic we have by the identity (27) § 16 that

$$\mathbf{S}(X-\phi)\bar{A}(X-\phi)\bar{A}'(X-\phi)\bar{A}'' = 0 \ldots\ldots\ldots\ldots(2),$$

where $\bar{A}, \bar{A}', \bar{A}''$ are *any three* completely independent axial motors. Hence by § 14 either

$$X_1(X-\phi)\bar{A} + X_2(X-\phi)\bar{A}' + X_3(X-\phi)\bar{A}'' = 0,$$

where the ordinary scalars of X_1, X_2 and X_3 are not all zero, or else two independent motors of the complex

$$(X-\phi)\bar{A},\ (X-\phi)\bar{A}',\ (X-\phi)\bar{A}''$$

are lators. In the first case we may take $\bar{A}''$ for the axial motor $X_1\bar{A} + X_2\bar{A}' + X_3\bar{A}''$, when we have

$$\phi\bar{A}'' = X\bar{A}'' \text{.........................(3).}$$

Now choose A, A', A'', B, B', B'' to depend on $\bar{A}$, $\bar{A}'$, $\bar{A}''$, $(\phi - X)$ in the same way as in § 16 A, A', A'', B, B', B'' depend on $\bar{A}$, $\bar{A}'$, $\bar{A}''$, ϕ; that is to say define A, A', A'' by eq. (21) § 16 and in place of eq. (23) § 16 take

$$B = (\phi - X)\bar{A},\ B' = (\phi - X)\bar{A}',\ B'' = (\phi - X)\bar{A}'' \text{...(4).}$$

Thus $$(\phi - X)E = -B\mathbf{S}AE - B'\mathbf{S}A'E,$$

which is the same in form as eq. (1). If now the root $\mathbf{S}_1X$ of the ϕ_1 cubic corresponding to the root X of the ϕ cubic is not repeated, one root but not two of the $(\phi_1 - \mathbf{S}_1X)$ cubic is zero, i.e. the $\mathbf{S}_1M'$ of the $(\phi - X)$ cubic is not zero, i.e. $\mathbf{S}_1\mathbf{M}AA'\mathbf{M}BB'$ is not zero by eq. (15) § 16.

In the case when two independent motors of the complex $(X - \phi)\bar{A}$, $(X - \phi)\bar{A}'$, $(X - \phi)\bar{A}''$ are lators we may take these motors as $(X - \phi)\bar{A}'$ and $(X - \phi)\bar{A}''$, i.e. we may put

$$\phi\bar{A}' = X\bar{A}' + \Omega\beta',\ \phi\bar{A}'' = X\bar{A}'' + \Omega\beta'' \text{...........(5),}$$

and now defining A, A', A'', B, B', B'' as before we have $B' = \Omega\beta'$, $B'' = \Omega\beta''$ and therefore

$$(\phi - X)E = -B\mathbf{S}EA - \Omega\beta'\mathbf{S}E\alpha' - \Omega\beta''\mathbf{S}E\alpha'' \text{......(6),}$$

where α' and α'' are any rotors equal and parallel to $\mathbf{M}_1A'$ and $\mathbf{M}_1A''$. In this case we see by eq. (13) § 16 that the $\mathbf{S}_1M'$ of the $\phi - X$ cubic is zero and therefore $\mathbf{S}_1X$ is a repeated root of the ϕ_1 cubic. In this case we have by equation (5) that $\phi\bar{A}' = X\bar{A}' +$ a lator. Hence if ρ is any rotor parallel to $\bar{A}'$, $\phi\rho = X\rho +$ a lator or $\mathbf{M}_1\phi\rho$ is equal and parallel to $\mathbf{M}_1X\rho$ (unless $\mathbf{S}_1X = 0$ when $\phi\rho$ is a lator). Similarly for any rotor σ parallel to $\bar{A}''$. This can be seen perhaps still more easily from eq. (6) from which we see that if ρ is any rotor perpendicular to A $\phi\rho = X\rho +$ a lator.

That there is not always in the case of eq. (6) an axial motor E for which $(\phi - X - \Omega y)E = 0$ is seen by taking a particular case. If we put $\phi E = -j\mathbf{S}Ei - \Omega k\mathbf{S}Ej - \Omega i\mathbf{S}Ek$ the cubic is $\phi^3 = 0$ and it is easily proved in this case that there is no axial motor E for which $(\phi - \Omega y)E = 0$.

We may notice that equation (1) may be reduced to either of the forms

$$(\phi - X) E = - B\mathbf{S}jE - B'\mathbf{S}kE \ldots\ldots\ldots\ldots\ldots(7),$$

or

$$(\phi - X) E = - j\mathbf{S}AE - k\mathbf{S}A'E \ldots\ldots\ldots\ldots\ldots(8),$$

for taking i along the shortest distance of A and A' of eq. (1) we have

$$\mathbf{S}Ai = \mathbf{S}A'i = 0,$$

and therefore

$$A = - j\mathbf{S}jA - k\mathbf{S}kA, \; A' = - j\mathbf{S}jA' - k\mathbf{S}kA'.$$

Substituting these values in eq. (1) we get

$$(\phi - X) E = (B\mathbf{S}jA + B'\mathbf{S}jA')\,\mathbf{S}jE + (B\mathbf{S}kA + B'\mathbf{S}kA')\,\mathbf{S}kE,$$

which is of the form (7). Similarly taking i along the shortest distance of B and B' of eq. (1) we get the form (8).

In these forms the conditions that $\mathbf{S}_1X$ may not be a repeated root of the ϕ_1 cubic become respectively $\mathbf{S}_1iBB'$ not zero and $\mathbf{S}_1iAA'$ not zero.

Since $(\phi^3 - M''\phi^2 + M'\phi - M)\,E = 0$ where E is any motor, we see in particular that when a root of the ϕ_1 cubic is repeated three times there are *three* completely independent axial motors B, B', B'' such that for any motor $E = X_1B + X_2B' + X_3B''$, $(\phi^3 - \ldots)\,E = 0$. Similarly when $\mathbf{S}_1X$ is a single root of the ϕ_1 cubic we have now seen that there is *one* axial motor B such that if $E = X_1B$, $(\phi - X)\,E = 0$. A particular result of the next proposition is that when a root of the ϕ_1 cubic is repeated twice there are *two* completely independent motors B and B' such that if $E = X_1B + X_2B'$, $(\phi^2 - N'\phi + N)\,E = 0$ the repeated roots being the roots of $x^2 - x\mathbf{S}_1N' + \mathbf{S}_1N = 0$. It may be remarked that X_1B is any motor coaxial with B or any lator parallel to it; $X_1B + X_2B'$ is any motor which intersects a definite line (the shortest distance of B and B') perpendicularly or any lator perpendicular to that line; and $X_1B + X_2B' + X_3B''$ is any motor whatever.

PROP. II. *If X is the root of the ϕ cubic corresponding to a single root of the ϕ_1 cubic and if N', N are defined by the identity* (6) § 17 *there are two completely independent axial motors* (E) *for which*

$$(\phi^2 - N'\phi + N)\,E = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

This is not always true if X corresponds to a repeated root of the ϕ_1 cubic.

When X corresponds to a single root of the ϕ_1 cubic eq. (1) is true with the condition that $\mathbf{S}_1\mathbf{M}AA'\mathbf{M}BB'$ is not zero by Prop. I. By this condition it follows that $\mathbf{M}_1BB'$ is not zero and therefore that B and B' are two completely independent axial motors. But since $(\phi^2 - N'\phi + N)(\phi - X)E = 0$ for all values of E it follows that

$$(\phi^2 - N'\phi + N)(\phi - X)\bar{A} = 0,$$

$$(\phi^2 - N'\phi + N)(\phi - X)\bar{A}' = 0,$$

i.e.

$$(\phi^2 - N'\phi + N)B = 0,$$

$$(\phi^2 - N'\phi + N)B' = 0,$$

which proves the first part of the proposition.

That this is not always true when X corresponds to a repeated root (even when we change $\phi^2 - N'\phi + N$ to

$$\phi^2 - (N' - \Omega y)\phi + \{N + \Omega y(X - N')\},$$

where y is arbitrary in order to take account of the arbitrariness of the root $X + \Omega y$ [§ 17 above]) is seen by the particular case considered just now, viz. $\phi E = -j\mathbf{S}Ei - \Omega k\mathbf{S}Ej - \Omega i\mathbf{S}Ek$. The cubic is $\phi^3 = 0$ so that N', N and X are all zero and it will be found that

$$(\phi^2 + \Omega y\phi)E = -\Omega\{(k + yj)\mathbf{S}Ei + j\mathbf{S}Ek\},$$

so that in this case

$$[\phi^2 - (N' - \Omega y)\phi + \{N + \Omega y(X - N')\}]E = 0,$$

when E is any axial motor parallel to j but is not true for any other axial motor value of E, whatever be the value of y.

PROP. III. *If x is a repeated root of the ϕ_1 cubic, either*

$$(\phi - x)E = -B\mathbf{S}Ei - \Omega\beta'\mathbf{S}Ej - \Omega\beta''\mathbf{S}Ek \quad \text{.........(10)},$$

where

$$B = Y(i\cos\theta + j\sin\theta) \quad \text{.................(11)},$$

or

$$(\phi - x)E = -B\mathbf{S}Ei - B'\mathbf{S}Ej - \Omega\beta''\mathbf{S}Ek \quad \text{.........(12)},$$

where

$$B = Yj + Zk, \quad B' = Y'j + Z'k \quad \text{..............(13)},$$

and where $\mathbf{M}_1BB'$ is not zero. If x is repeated three times we have in the first case $\mathbf{S}_1Y\cos\theta = 0$ and in the second $\mathbf{S}_1Y' = 0$. [Note that eq. (13) is equivalent to the equations

$$\mathbf{S}iB = \mathbf{S}iB' = 0 \quad \text{......................(14)},$$

and that the condition $\mathbf{M}_1 BB'$ not zero is equivalent to the condition $\mathbf{S}_1(YZ' - Y'Z)$ not zero, and also equivalent to the condition that B and B' are completely independent axial motors.]

Fig. 5 shows graphically the geometrical meaning of equations (12), (13). The meanings of A, A', A_1', j' and k' will appear in the discussion. The rest of the figure is fully explained by equations (12), (13).

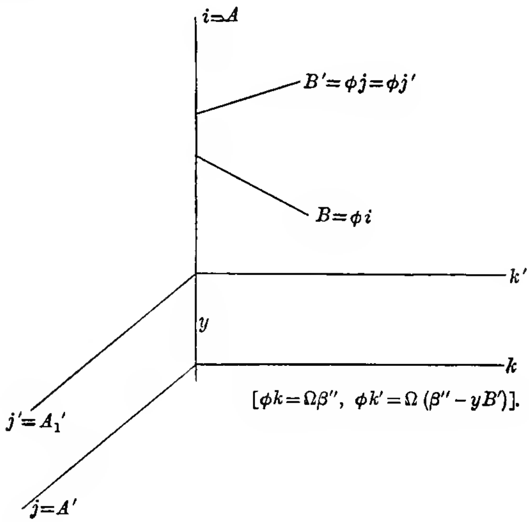


FIG. 5.

Since x is a repeated root of the ϕ_1 cubic we have by § 17 that the $\mathbf{S}_1 M$ and $\mathbf{S}_1 M'$ of the $\phi - x$ cubic are both zero. Putting then

$$(\phi - x)E = -B\mathbf{S}AE - B'\mathbf{S}A'E - B''\mathbf{S}A''E,$$

where A, A', A'' are any three completely independent axial motors, we have by eq. (12) § 16, since $\mathbf{S}_1 M = 0$,

$$\mathbf{S}_1 BB'B'' = 0.$$

Hence (by quaternion interpretation) B, B', B'' are axial motors parallel to one plane or one of them is a lator. Thus one of them (say B'') can be expressed in the form $yB + zB' + \Omega\beta''$. Substituting this in the last equation

$$(\phi - x)E = -B\mathbf{S}(A + yA'')E - B'\mathbf{S}(A' + zA'')E - \Omega\beta''\mathbf{S}\alpha''E,$$

where α'' is any rotor parallel and equal to $\mathbf{M}_1 A''$. Since A, A' and A'' are completely independent axial motors so are $A + yA''$, $A' + zA''$ and α''. Denoting the first two by A and A' respectively we have

$$(\phi - x) E = - B\mathbf{S}AE - B'\mathbf{S}A'E - \Omega\beta''\mathbf{S}\alpha''E \ldots\ldots (15),$$

where $\mathbf{S}_1 AA'\alpha''$ is not zero. Expressing now the condition that $\mathbf{S}_1 M'' = 0$ we have by equation (13) § 16

$$\mathbf{S}_1\mathbf{M}AA'\mathbf{M}BB' = 0.$$

Hence either $\mathbf{M}_1 BB'$ is zero or $\mathbf{M}_1 BB'$ is not zero and $\mathbf{M}AA'$ and $\mathbf{M}BB'$ are perpendicular to one another.

In the first case we may put $B' = z'B + \Omega\beta'$ whereupon changing $A + z'A'$ to A we get

$$(\phi - x) E = - B\mathbf{S}AE - \Omega\beta'\mathbf{S}\alpha'E - \Omega\beta''\mathbf{S}\alpha''E.$$

Here we may take i coaxial with A and express α' and α'' in terms of i, j, k. Doing this and collecting the terms in $\mathbf{S}iE$, $\mathbf{S}jE$, $\mathbf{S}kE$ we get an equation of the form

$$(\phi - x) E = - B\mathbf{S}iE - \Omega\beta'\mathbf{S}jE - \Omega\beta''\mathbf{S}kE,$$

(where of course the meanings of B, β' and β'' are in general different from their meanings in the last equation). In this form we notice that

$$(\phi - x)(i + \Omega\mathbf{M}\rho i) = B.$$

Hence if i' is any unit rotor parallel to i, $(\phi - x)\, i'$ has the constant value B. i may therefore when B has a definite axis be taken to intersect that axis. Lastly j and k may be so chosen that the plane of i, j is parallel to B. We thus get equations (10) and (11). [Eq. (11) is put in such a form that B may be a lator.]

When $\mathbf{M}_1 BB'$ and $\mathbf{M}_1 AA'$ are not zero but

$$\mathbf{S}_1\mathbf{M}AA'\mathbf{M}BB' = 0,$$

first in eq. (15) express $\Omega\alpha''$ in terms of ΩA, $\Omega A'$ and $\Omega\mathbf{M}AA'$ (which can always be done since $\mathbf{M}_1 AA'$ is not zero), and incorporate in the first two terms of eq. (15) the terms resulting from the first two components of $\Omega\alpha''$. The form of eq. (15) is not thereby altered but we may now assume α'' to be perpendicular to A and A'. Now put eq. (15) in the form

$$(\phi - x) E = - B\mathbf{S}A_1E - B'\mathbf{S}A_1'E - \Omega\beta_1''\mathbf{S}\alpha''E,$$

where

$$A_1 = A + y\Omega\alpha'',\ A_1' = A' + y'\Omega\alpha'',\ \Omega\beta_1'' = \Omega\,(\beta'' - yB - y'B').$$

The form of eq. (15) is not thereby altered and α'' is still perpendicular to both A_1 and A_1'. y and y' may be so chosen that not only is $\mathbf{S}_1\mathbf{M}A_1A_1'\mathbf{M}BB'$ zero but

$$\mathbf{SM}A_1A_1'\mathbf{M}BB' = 0,$$

for this last equation is true if

$$\mathbf{sM}AA'\mathbf{M}BB' - \mathbf{S}_1\{\alpha''\mathbf{M}BB' \, . \, (yA' - y'A)\} = 0.$$

Here α'' is parallel and $\mathbf{M}BB'$ is perpendicular to $\mathbf{M}AA'$ so that $\mathbf{M}_1(\alpha''\mathbf{M}BB')$ is parallel to the plane of A and A'. Since A and A' are not parallel it follows that y and y' can always be determined, and that in an infinite number of ways, as desired. Hence we may assume eq. (15) to hold with the condition

$$\mathbf{SM}AA'\mathbf{M}BB' = 0.$$

We may therefore take i along the shortest distance of B and B' and k along the shortest distance of A and A' so that α'' is parallel to k. Now express A, A', B, B' and α'' in terms of i, j and k (i.e. $A = -i\mathbf{S}iA - j\mathbf{S}jA$, &c.) and collect the terms in $\mathbf{S}Ei$, $\mathbf{S}Ej$ and $\mathbf{S}Ek$. We thus get equations of the form of (12) and (13). The condition $\mathbf{M}_1BB'$ not zero still holds with the new meanings of B and B'. [This can be proved directly or we may notice that if it do not hold we can by the above reduce equations (12), (13) to the form of equations (10), (11).]

Note that both equations (10) and (12) are of the form of eq. (15). Indeed as we see by the above proof eq. (15) with the condition $\mathbf{S}_1\mathbf{M}AA'\mathbf{M}BB' = 0$ may be taken as giving the general form for ϕ when x is a repeated root of the ϕ_1 cubic.

If x is a thrice repeated root of the ϕ_1 cubic we have further that the $\mathbf{S}_1M''$ of the $\phi - x$ cubic is zero, so that by equation (14) § 16 the further condition in the case of eq. (15) is

$$\mathbf{S}_1(AB + A'B') = 0.$$

In the case of eq. (10) this gives $\mathbf{S}_1Bi = 0$ or $\mathbf{S}_1Y\cos\theta = 0$. In the case of eq. (12) it gives $\mathbf{S}_1B'j = 0$ or $\mathbf{S}_1Y' = 0$. In this case B' in fig. 5 must be drawn parallel to k.

PROP. IV. *In the case of eq.* (10) *if* i' *is any unit rotor parallel to* i, $(\phi - x)\,i'$ *has the constant value* B. *If* E *is any axial motor perpendicular to* i, $\mathbf{M}_1\phi E$ *is equal and parallel to* $x\mathbf{M}_1 E$. The first statement has been already incidentally proved. The second follows at once from the fact, obvious from eq. (10), that when E is perpendicular to i, $(\phi - x)\,E$ is a lator.

PROP. V. *In the case of equation* (12) *if* j', k' *are unit rotors parallel to* j *and* k, *intersecting* i *at a distance* y *from the point of intersection of* i, j, k; $(\phi - x)\,j'$ *has the constant value* B' *and* $(\phi - x)\,k' = \Omega\,(\beta'' - yB')$. *If* E *is any axial motor parallel to* k, $\mathbf{M}_1\phi E$ *is equal and parallel to* $x\mathbf{M}_1 E$. We have

$$j' = (1 + \Omega yi)\,j = j + \Omega yk,\ \ k' = (1 + \Omega yi)\,k = k - \Omega yj,$$

from which the statements about j' and k' follow. If E is a motor parallel to k, $(\phi - x)\,E$ is a lator, from which the statement about E follows. [We saw in making the transformation $A_1 = A + y\Omega\alpha''$, &c., above that y and y' were to a certain extent arbitrary. It will be found that the present statements about j' and k' depend on this. The present y is the former $-y'$. The statements about j' and k' are represented graphically in fig. 5.]

From this proposition we see that j and k may be supposed to intersect i at any point which is convenient, for instance either at the point of intersection of i and B or that of i and B'. In the case of x thrice repeated the last is most convenient for we then have $B' = Z'k$. Other conditions might be satisfied, for instance β'' can always be made perpendicular to B'.

By a process exactly similar to the proof of Prop. V. we may prove :—

PROP. VI. *If* i', j', k' *be unit rotors parallel to* i, j, k *and intersecting* B *of eq.* (11) *at a distance* r *from the point of intersection of* i, j, k; *in the case of eq.* (10), $(\phi - x)\,i'$ *has the constant value* B, $(\phi - x)\,j'$ *has the constant value* $\Omega\beta'$, *and* $(\phi - x)\,k'$ *has the value* $\Omega\,(\beta'' + Br\sin\theta)$. Here one condition can be satisfied by properly choosing the point of intersection of i, j, k on B; for instance β'' can always be made perpendicular to B.

PROP. VII. *If there is no axial motor* E *for which* ϕE *is coaxial with* E, *either*

$$(\phi - x)\,E = -\,Yj\mathbf{S}Ei - \Omega\beta'\mathbf{S}Ej - \Omega\beta''\mathbf{S}Ek,\ \ \mathbf{S}_1 i\beta' = 0,$$
$$\mathbf{S}_1 j\beta'' = 0 \text{.........} (16),$$

where $\mathbf{S}_1 Y$, $\mathbf{S}_1 i\beta''$ *and* $\mathbf{S}_1 k\beta'$ *are not zero ; or*

$$(\phi - x) E = -(Yj + Zk)\mathbf{S}Ei - Z'k\mathbf{S}Ej - \Omega\beta''\mathbf{S}Ek \ldots\ldots (17),$$

where $\mathbf{S}_1 Y$, $\mathbf{S}_1 Z'$ *and* $\mathbf{S}_1 i\beta''$ *are not zero. Conversely if these conditions are satisfied there is no such axial motor.*

Note that in this proposition ϕE would be said to be coaxial with E if it were a lator parallel to E or zero.

We have seen (Prop. I.) that there is always such an axial motor E except when the roots of the ϕ_1 cubic are all equal. Calling each of them x, ϕ will be of one of the forms (10) or (12).

In the case of eq. (10) we have seen that when x is a thrice repeated root of the ϕ_1 cubic $\mathbf{S}_1 Y \cos\theta = 0$. If $\mathbf{S}_1 Y = 0$, B becomes a lator $\Omega\beta$ and in this case $(\phi - x) E$ is a lator function of E and bears to the rotor part of E exactly the same geometrical relations that a linear vector function of a vector bears to the vector. Hence in this case there is always an axial motor E of the kind mentioned (and any motor parallel to E is also of the same kind). If $\mathbf{S}_1 Y$ is not zero we have the first of equations (16). We have seen (Prop. VI.) that in this case we may suppose $\mathbf{S}_1 j\beta'' = 0$.

If now there is an axial motor E of the kind mentioned it cannot have a component parallel to i and therefore we may assume that it has the rotor form

$$E = yj + zk + \Omega\rho.$$

[Since ϕ is commutative, if E satisfies the required conditions so does the rotor $\mathbf{M}_1 E$ or $(1 - \Omega\mathbf{t}E) E$.] Thus

$$(\phi - x) E = \Omega\,(y\beta' + z\beta'' - Yj\mathbf{S}i\rho).$$

The lator on the right has to be parallel to $yj + zk$. It will be found by elementary algebra that when $\mathbf{S}_1 i\beta' = \mathbf{S}_1 i\beta'' = 0$ this condition can always be satisfied by real finite values of y, z and $\mathbf{S}i\rho$ of which the first two are not zero. [In this case indeed y and z may be given arbitrary values.] Since $yj + zk$ is perpendicular to i we see that when $\mathbf{S}_1 i\beta'$ and $\mathbf{S}_1 i\beta''$ are not both zero we must have

$$\frac{y}{-\mathbf{S}_1 i\beta''} = \frac{z}{\mathbf{S}_1 i\beta'} = c,$$

where c is not zero or infinite. Without loss of generality we may put $c = 1$. Thus

$$\mathbf{M}i\mathbf{M}\beta'\beta'' - Yj\mathbf{S}i\rho$$

is to be parallel to

$$-j\mathbf{S}i\beta'' + k\mathbf{S}i\beta';$$

i.e.
$$\frac{\mathbf{S}_1 k\beta'\beta'' - \mathbf{S}_1 Y\mathbf{S}_1 i\rho}{\mathbf{S}_1 i\beta''} = \frac{\mathbf{S}_1 j\beta'\beta''}{\mathbf{S}_1 i\beta'},$$

and this can always be satisfied by a real finite value of $\mathbf{S}i\rho$ unless $\mathbf{S}_1 i\beta' = 0$ and neither $\mathbf{S}_1 i\beta''$ nor $\mathbf{S}_1 j\beta'\beta''$ is zero. Conversely if these conditions ($\mathbf{S}_1 i\beta' = 0$, &c.) are satisfied the equation cannot be satisfied by a finite value of ρ. It is easy to prove that the condition $\mathbf{S}_1 j\beta'\beta''$ not zero may be replaced by the condition $\mathbf{S}_1 k\beta'$ not zero.

In the case of eq. (12) we have seen that when x is a thrice repeated root of the ϕ_1 cubic $\mathbf{S}_1 Y' = 0$ and by taking, as we have seen we may, the point of intersection of i, j, k on B this becomes $Y' = 0$. We thus have eq. (17) with the conditions $\mathbf{S}_1 Y$ and $\mathbf{S}_1 Z'$ not zero (since $\mathbf{M}_1 BB'$ is not zero). In this case we see that if E is a motor of the kind desired, it must be parallel to k so that we may put

$$E = k + \Omega\rho = k + \Omega\,(\xi i + \eta j + \zeta k).$$

Thus
$$(\phi - x)\,E = \Omega\,\{\beta'' + \xi\,(Yj + Zk) + \eta Z'k\},$$

so that the lator on the right is parallel to k. With the given conditions this can clearly be satisfied by real values of ξ and η (which may be zero) if and only if $\mathbf{S}_1 i\beta'' = 0$.

In the case of eq. (10) $x + \Omega y$ (where y is arbitrary) is a root of the ϕ cubic. In the case of eq. (11) $x + \Omega y$ is not always a root. For by equation (12) § 16 the M of $\phi - x$ is in the case of eq. (10) zero but is not generally zero in the case of eq. (12) [see end of § 17 above].

The following statements are not of sufficient importance to embody in formal propositions.

Both equations (10) and (12) are of the form

$$(\phi - x)\,E = -B\mathbf{S}iE - B'\mathbf{S}jE - \Omega\beta''\mathbf{S}kE.$$

Hence by equations (12), (13), (14) of § 16 we have in both cases for the $\phi - x$ cubic

$$M = -\Omega \mathbf{S} BB'\beta'',$$
$$M' = -\{\mathbf{S} kBB' + \Omega \mathbf{S} \beta'' (iB' - jB)\},$$
$$M'' = -\{\mathbf{S} iB + \mathbf{S} jB' + \Omega \mathbf{S} k\beta''\}.$$

In the case of eq. (10) these give

$$M = 0,\ M' = -\Omega Y \{\mathbf{S}\beta' (-i \sin\theta + j\cos\theta) + \mathbf{S}\beta'' k \cos\theta\},$$
$$M'' = -\{-Y\cos\theta + \Omega \mathbf{S} (j\beta' + k\beta'')\}.$$

Hence the cubic is

$$(\phi - x)\{(\phi - x)^2 + [-Y\cos\theta + \Omega \mathbf{S} (j\beta' + k\beta'')](\phi - x) + \Omega Y [\mathbf{S}\beta' (i\sin\theta - j\cos\theta) - \mathbf{S}\beta'' k\cos\theta]\} = 0,$$

which can generally be put in the form

$$(\phi - x)\{\phi - x - \Omega [\mathbf{S}\beta' (i\tan\theta - j) - \mathbf{S}\beta'' k]\} \{\phi - x - Y\cos\theta + \Omega \tan\theta\, \mathbf{S} i\beta'\} = 0.$$

The axial motor (when $\mathbf{S}_1 Y\cos\theta$ is not zero) E for which

$$\{\phi - x - Y\cos\theta + \Omega\tan\theta\, \mathbf{S} i\beta'\} E = 0,$$

is $(\phi - x)\{\phi - x - \Omega [\mathbf{S}\beta' (i\tan\theta - j) - \mathbf{S}\beta'' k]\} i$. E is however written down more simply by noticing that if

$$(\phi - x) E = -B\mathbf{S} EA - \Omega\beta' \mathbf{S} E\alpha' - \Omega\beta'' \mathbf{S} E\alpha'',$$

then $$(\phi - x) B = -B\mathbf{S} AB - \Omega\beta' \mathbf{S}\alpha' B - \Omega\beta'' \mathbf{S}\alpha'' B,$$

from which it is obvious that if we again operate by $(\phi - x)$ we shall get a scalar octonion multiple of $(\phi - x) B$. In fact putting $\Omega\beta' \mathbf{S}\alpha' B + \Omega\beta'' \mathbf{S}\alpha'' B = \Omega\epsilon$ we find that

$$(\phi - x)^2 B = -(\mathbf{S} AB + \Omega \mathbf{S} A\epsilon \mathbf{S}^{-1} AB)(\phi - x) B.$$

Hence in our present case the desired axial motor E is $(\phi - x) B$, i.e.

$$E = -B\mathbf{S} iB - \Omega\beta' \mathbf{S} jB - \Omega\beta'' \mathbf{S} kB.$$

The splitting of the cubic into three linear factors fails generally when x is a thrice repeated root, i.e. when $\theta = \frac{\pi}{2}$. [x is a thrice repeated root if $\mathbf{S}_1 Y = 0$ but in this case the above linear factor form of the cubic does not fail.] If however (§ 17) M' is zero we know that there are three linear factors. $M' = 0$ gives in the present case $\mathbf{S}_1 \beta' i = 0$ and then the above form of the cubic still holds and takes the simplified form

$$(\phi - x)^2 \{(\phi - x) + \Omega \mathbf{S} (j\beta' + k\beta'')\} = 0.$$

For eq. (12) we have

$$M = -\Omega(YZ' - Y'Z)\,\mathbf{S}i\beta'',\ M' = \Omega\mathbf{S}\beta''(iZ + jZ' - kY'),$$
$$M'' = Y' - \Omega\mathbf{S}k\beta'',$$

so that the cubic is

$$(\phi - x)^3 + (\phi - x)^2(-Y' + \Omega\mathbf{S}k\beta'')$$
$$+ (\phi - x)\,\Omega\mathbf{S}\beta''(iZ + jZ' - kY') + \Omega(YZ' - Y'Z)\,\mathbf{S}i\beta'' = 0,$$

which may when x is not thrice repeated be put in the form

$$\{\phi - x - Y' + \Omega Z'Y'^{-2}\,\mathbf{S}\beta''(jY' + iY)\}\,\{(\phi - x)^2$$
$$-(\phi - x)\,\Omega Y'^{-2}\,\mathbf{S}\beta''(iYZ' + jY'Z' - kY'^2)$$
$$-\Omega Y'^{-1}(YZ' - Y'Z)\,\mathbf{S}\beta''i\} = 0.$$

The axial motor which is reduced to zero by the first factor is

$$E = (\phi - x)B' - B\Omega Z'Y'^{-1}\mathbf{S}i\beta'' = Y'B' + \Omega Z'(\beta'' - BY'^{-1}\mathbf{S}i\beta'').$$

If x is a thrice repeated root of the ϕ_1 cubic $\mathbf{S}_1M'' = 0$ or $\mathbf{S}_1Y' = 0$ and the above factorising of the cubic fails. In this case in order that it may be possible to find one linear factor, M must be zero (§ 17), i.e. $\mathbf{S}_1i\beta'' = 0$. In this case we may put

$$\beta'' = b''j + c''k$$

and the cubic is

$$(\phi - x)\{(\phi - x)^2 - (Y' + \Omega c'')(\phi - x) + \Omega(c''Y' - b''Z')\} = 0.$$

In order that there may be three factors M' must be zero (§ 17), i.e. $\mathbf{S}_1(b''Z' - c''Y') = 0$. In this case the cubic is

$$(\phi - x)^2\{\phi - x - (Y' + \Omega c'')\} = 0.$$

These two factorisations are true when $\mathbf{S}_1i\beta'' = 0$ and when $\mathbf{S}_1i\beta'' = 0$ and $\mathbf{S}_1(b''Z' - c''Y') = 0$ respectively, whether or not $\mathbf{S}_1Y' = 0$. If the last be true, however, we may write $-\Omega b''Z'$ instead of $\Omega(c''Y' - b''Z')$ in the first of the two cases.

19. The self-conjugate commutative function. We proceed to show that when ϕ is self-conjugate it can always be put in the form

$$\phi E = -Xi\mathbf{S}iE - X'j\mathbf{S}jE - X''k\mathbf{S}kE \text{ ....(1).}$$

[In § 38 below a proof of this is given which is independent of § 18. The reader who chooses can at once pass to that proof.]

In the first place when ϕ is self-conjugate the cases of equations (16) and (17) of § 18 do not occur. For by equations (3), (4) § 16, when ϕ is self-conjugate

$$0 = \mathbf{M}\zeta\phi\zeta = \mathbf{M}\,(i\phi i + j\phi j + k\phi k).$$

In the case of eq. (16) this gives

$$Yk + \Omega\mathbf{M}\,(j\beta' + k\beta'') = 0,$$

which is impossible when $\mathbf{S}_1 Y$ is not zero. In the case of eq. (17)

$$Yk - Zj + Z'i + \Omega\mathbf{M}k\beta'' = 0,$$

which is impossible when $\mathbf{S}_1 Y$ and $\mathbf{S}_1 Z'$ are not zero.

Hence by Prop. VII. there is always in the case of a self-conjugate ϕ an axial motor E for which ϕE is coaxial with E. Let i be taken along the axis of this motor. Thus ϕ must be given by

$$\phi E = -Xi\mathbf{S}Ei - Yj\mathbf{S}Ej - Zk\mathbf{S}Ek - W\,(j\mathbf{S}kE + k\mathbf{S}jE).$$

[For put

$$\phi i = X_1 i + Y_3 j + Y_2 k,$$
$$\phi j = Y_3' i + X_2 j + Y_1 k,$$
$$\phi k = Y_2' i + Y_1' j + X_3 k.$$

Making ϕ self-conjugate we get $Y_1' = Y_1$, $Y_2' = Y_2$, $Y_3' = Y_3$, and making ϕi coaxial with i we get $Y_2 = Y_3 = 0$.]

Now Z' can always be determined so that $\phi\,(j + Z'k)$ is coaxial with $j + Z'k$. For

$$\phi\,(j + Z'k) = (Y + Z'W)\,j + (Z'Z + W)\,k,$$

and this is coaxial with $j + Z'k$ if

$$Z'^2 W + Z'\,(Y - Z) - W = 0 \text{..................(2)},$$

which gives

$$Z' = \tfrac{1}{2} W^{-1}\,\{Z - Y + [(Z - Y)^2 + 4W^2]^{\frac{1}{2}}\}.$$

By eq. (8) § 4 above this gives a real finite value for Z' except when W is a convertor. If W is a convertor eq. (2) is satisfied by $Z' = W/(Y - Z)$ unless $Y - Z$ is also a convertor. If both W and $Y - Z$ are convertors, say Ωw and Ωy where w and y are ordinary scalars, eq. (2) is satisfied by

$$Z' = \tfrac{1}{2} w^{-1}\,\{-y + [y^2 + 4w^2]^{\frac{1}{2}}\},$$

unless $w = 0$. But when $w = 0$, $W = 0$ and both ϕj and ϕk are coaxial with j and k respectively.

$j+Z'k$ is an axial motor which intersects i perpendicularly. We may therefore take its axis for that of j. Doing so we get eq. (1).

By equations (8) and (27) § 16 we see that the cubic of the ϕ of eq. (1) is

$$(\phi - X)(\phi - X')(\phi - X'') = 0 \text{ ....(3)},$$

so that when ϕ is self-conjugate there are always three real roots of its cubic.

In § 17 we saw that if any two, say $\mathbf{S}_1 X$ and $\mathbf{S}_1 X'$ of the ordinary scalar parts of X, X', X'' were equal, the cubic had an infinite number of roots of the form $X + \Omega y$. We will now show that if for any axial motor E

$$\phi E = YE,$$

Y must have one of the values X, X' or X'' of eq. (1). If $\phi E = YE$ we have by eq. (1)

$$i\mathbf{S}iE(Y-X) + j\mathbf{S}jE(Y-X') + k\mathbf{S}kE(Y-X'') = 0,$$

or $$(Y-X)\,\mathbf{S}iE = (Y-X')\,\mathbf{S}jE = (Y-X'')\,\mathbf{S}kE = 0.$$

Now since E is an axial motor one of the three ordinary scalars $\mathbf{S}_1 iE$, $\mathbf{S}_1 jE$, $\mathbf{S}_1 kE$ is not zero. If this is $\mathbf{S}_1 iE$ we have

$$Y - X = 0.$$

Hence Y must have one of the three values X, X', X''. Thus the X, X', X'' of eq. (1) have determinate values when ϕ is given even when the roots of the ϕ cubic are indeterminate. When the roots are indeterminate X, X' and X'' may be called the *principal roots.*

It may be noticed in passing that by the beginning of § 17 above the fact that it is always possible to put a self-conjugate commutative function in the real form (1) involves the following quaternion theorem:—If ϕ_1 and ϕ_2 are two self-conjugate linear vector functions of vectors it is always possible to determine a real vector ρ, in general uniquely so that the principal directions of ϕ_1 and $\phi_2 + \phi_1 \mathbf{M}\rho(\ ) - \mathbf{M}\rho\phi_1(\ )$ are the same.

Quaternion analogy suggests the examination of the self-conjugate ϕ given by

$$\phi E = \mathbf{M}AEB \text{ ....(4)}.$$

We have at once

$$\phi \mathbf{M}AB = -\mathbf{M}AB\mathbf{S}AB \text{ ....(5)},$$

$$\phi(\mathbf{U}A \pm \mathbf{U}B) = \mp \mathbf{T}A\mathbf{T}B(\mathbf{U}A \pm \mathbf{U}B) \text{ ....(6)}.$$

Hence the principal axes are the shortest distance of A and B and the two lines bisecting the shortest distance perpendicularly and bisecting the directions of A and B. The principal roots of the cubic are $-\mathbf{S}AB$ and $\pm\mathbf{T}A\mathbf{T}B$, i.e. the cubic is

$$(\phi^2 - A^2B^2)(\phi + \mathbf{S}AB) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

It is natural now to enquire whether every self-conjugate ϕ can be put in the form

$$\phi E = \mathbf{M}AEB + YE \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

The principal roots of the cubic are

$$Y + \mathbf{T}A\mathbf{T}B,\ Y - \mathbf{S}AB,\ Y - \mathbf{T}A\mathbf{T}B,$$

and the ordinary scalars are here in descending order. If A is parallel to B and they are both axial motors, the ordinary scalar of the intermediate root is equal to one of the others, but those of the first and last are not equal. If either A or B is a lator, say A, $\mathbf{T}A$ has no meaning. Equation (8) then becomes

$$\phi E = YE + \text{a lator},$$

so that the ordinary scalars of the roots are each equal to $\mathbf{S}_1 Y$.

Suppose now the ordinary scalars of X, X', X'' are in descending order. If these ordinary scalars are not all equal neither A nor B can be a lator, so that if equations (1) and (8) are the same we must have

$$X = Y + \mathbf{T}A\mathbf{T}B,\ X' = Y - \mathbf{S}AB,\ X'' = Y - \mathbf{T}A\mathbf{T}B \ldots(9).$$

From this it can be shown that in one case it is impossible to express ϕ in the form (8), viz. when two but not three of the roots of the ϕ_1 cubic are equal and when the corresponding principal roots of the ϕ cubic are unequal. If $\mathbf{S}_1 X = \mathbf{S}_1 X'$ and these are not equal to $\mathbf{S}_1 X''$,

$$X - X' = \mathbf{T}A\mathbf{T}B + \mathbf{S}AB = \mathbf{T}A\mathbf{T}B(1 + \mathbf{SU}A\mathbf{U}B).$$

Hence $\mathbf{T}_1 A\mathbf{T}_1 B + \mathbf{S}_1 AB$ is zero or A and B are parallel. Hence by eq. (10) § 12 $\mathbf{SU}A\mathbf{U}B = \pm 1$. The lower sign must be taken since $\mathbf{S}_1(X - X') = 0$. Thus $X - X' = 0$ or the transformation is impossible. Similarly the transformation is impossible when $\mathbf{S}_1 X' = \mathbf{S}_1 X''$ and these are not equal to $\mathbf{S}_1 X$ and X' and X'' are unequal.

In all other cases ϕ can be put in the form (8). It will be found that

$$-Xi\mathbf{S}Ei - X'j\mathbf{S}Ej - X''k\mathbf{S}Ek$$
$$\equiv -\mathbf{M}(Zi + Z'k)E(Zi - Z'k) + YE \ldots (10),$$

if $$2Z^2 = X - X', \; 2Z'^2 = X' - X'', \; 2Y = X + X'' \ldots\ldots (11),$$

and these equations give real finite values for Z, Z', Y when $\mathbf{S}_1X$, $\mathbf{S}_1X'$ and $\mathbf{S}_1X''$ are all different and in descending order. If $\mathbf{S}_1X = \mathbf{S}_1X' = \mathbf{S}_1X''$

$$-Xi\mathbf{S}Ei - X'j\mathbf{S}Ej - X''k\mathbf{S}Ek$$
$$\equiv -\Omega\mathbf{M}(zi + z'k)E(zi - z'k) + YE \ldots (12),$$

if $$2\Omega z^2 = X - X', \; 2\Omega z'^2 = X' - X'', \; 2Y = X + X'' \ldots (13),$$

and these equations give real finite values for z, z', Y when $\mathbf{s}X$, $\mathbf{s}X'$ and $\mathbf{s}X''$ are in descending order. Lastly

$$-Xi\mathbf{S}Ei - X'j\mathbf{S}Ej - X'k\mathbf{S}Ek$$
$$= \tfrac{1}{2}(X' - X)iEi + \tfrac{1}{2}(X' + X)E \ldots (14).$$

After the general linear function has been considered below several other theorems relating to commutative functions and especially self-conjugate commutative functions will be enunciated. For the present we leave this part of the subject.

20. Differentiation of octonion functions. We have not in the above considered all the different kinds of octonion functions that are even immediately suggested by quaternion analogy. For instance, we have not entered into interpretations of Q^R, where both Q and R are octonions, or of $\log Q$, or of the simpler conception A^n, where A is any motor and n any scalar not necessarily positive and integral. In this section we propose to provide materials for writing down the differential of any combination of octonions which can be made of such functions as have been considered.

In this section *explicit* references will not be made to § 9 above.

The following are obvious

$$d\mathbf{S}Q = \mathbf{S}dQ, \; d\mathbf{S}_1Q = \mathbf{S}_1dQ, \; d\mathbf{S}_2Q = \mathbf{S}_2dQ, \; d\mathbf{s}Q = \mathbf{s}dQ \ldots (1).$$

Also from the fact that octonions are formal quaternions,

$$d\mathbf{M}Q = \mathbf{M}dQ \text{ ....(2)},$$

$$d\mathbf{K}Q = \mathbf{K}dQ \text{ ....(3)},$$

$$d\mathbf{U}Q/\mathbf{U}Q = \mathbf{M}\,.\,dQ/Q \text{ ....(4)},$$

$$d\mathbf{T}Q/\mathbf{T}Q = \mathbf{S}\,.\,dQ/Q \text{ ....(5)}.$$

Putting in the last $\mathbf{T}Q = \mathbf{T}_1Q(1 + \Omega\mathbf{t}Q)$ we obtain

$$d\mathbf{T}_1Q/\mathbf{T}_1Q = \mathbf{S}_1\,.\,dQ/Q \text{ ....(6)},$$

$$d\mathbf{t}Q = \mathbf{s}\,.\,dQ/Q,\ d\mathbf{T}_2Q = \mathbf{S}_2\,.\,dQ/Q \text{ ....(7)}.$$

Since $$d\mathbf{M}_2Q = \Omega d\mathbf{m}Q,\ d\mathbf{U}_2Q = \Omega d\mathbf{u}Q \text{ ....(8)},$$

it only remains to find the differentials of $\mathbf{M}_1Q$, $\mathbf{m}Q$, $\mathbf{U}_1Q$ and $\mathbf{u}Q$.

From equation (2),

$$d\mathbf{M}Q = \mathbf{S}\frac{\mathbf{M}dQ}{\mathbf{M}Q}\,.\,\mathbf{M}Q + \mathbf{M}\frac{\mathbf{M}dQ}{\mathbf{M}Q}\,.\,\mathbf{M}Q.$$

Put now

$$\mathbf{M}dQ/\mathbf{M}Q = dR \text{ ....(9)},$$

so that dR is the octonion which, when viewed as an operator, changes the motor $\mathbf{M}Q$ into the motor $\mathbf{M}dQ$ and therefore intersects both Q and dQ perpendicularly. Thus

$$d\mathbf{M}Q = \mathbf{S}dR\,.\,\mathbf{M}Q + \mathbf{M}dR\,.\,\mathbf{M}Q,$$

and therefore

$$\left.\begin{aligned} Q + dQ &= Q + (\mathbf{S}dQ + \mathbf{S}dR\,.\,\mathbf{M}Q) + \mathbf{M}dR\,.\,\mathbf{M}Q \\ &= (1 + \tfrac{1}{2}dR)(Q + d'Q)(1 + \tfrac{1}{2}dR)^{-1} \end{aligned}\right\} \text{......(10)},$$

where $$d'Q = \mathbf{S}dQ + \mathbf{S}dR\,.\,\mathbf{M}Q \text{ ....(11)}.$$

Thus $Q + dQ$ is obtained from Q by first adding the *coaxial* increment $d'Q$ and then displacing $Q + d'Q$ as a rigid body. Denote the increments of the various functions of Q, such as $\mathbf{M}_1Q$, q_Q due to the increment $d'Q$ of Q, by the symbol d'. Since all these increments are coaxial with Q we have

$$d'\mathbf{M}_1Q = \mathbf{M}_1d'Q = \mathbf{S}_1dR\,.\,\mathbf{M}_1Q \quad\text{.........................(12)},$$

$$d'q_Q \quad = \mathbf{S}_1d'Q + \mathbf{M}_1d'Q = \mathbf{S}_1dQ + \mathbf{S}_1dR\,.\,\mathbf{M}_1Q \quad\text{...(13)},$$

$$\left.\begin{aligned} d'\mathbf{U}_1Q &= \mathbf{M}d'q_Qq_Q^{-1}\,.\,\mathbf{U}_1Q \\ &= \mathbf{T}_1^{-2}Q\mathbf{M}_1Q\,(-\,\mathbf{S}_1dQ + \mathbf{S}_1Q\mathbf{S}_1dR)\,\mathbf{U}_1Q \end{aligned}\right\} \text{...(14)},$$

[by putting $q_Q^{-1} = \mathbf{T}_1^{-2}Q\mathbf{K}q_Q = \mathbf{T}_1^{-2}Q\,(\mathbf{S}_1Q - \mathbf{M}_1Q)$]

$$d'\mathbf{m}Q = \mathbf{m}d'Q = \mathbf{S}_1dR\,.\,\mathbf{m}Q + \mathbf{s}dR\,.\,\mathbf{M}_1Q \quad\text{..................(15)},$$

$$d'r_Q \quad = \mathbf{s}d'Q + \mathbf{m}d'Q = \mathbf{s}dQ + \mathbf{S}_1dR\,.\,\mathbf{m}Q + \mathbf{s}dR\,.\,\mathbf{M}_1Q \text{...(16)},$$

$$d'\,(\mathbf{t}Q + \mathbf{u}Q) = d'\,(r_Qq_Q^{-1}) = q_Q^{-1}\,(d'r_Q - r_Qq_Q^{-1}d'q_Q)$$

[since q_Q, r_Q, $d'q_Q$ and $d'r_Q$ are all coaxial]

$$= \mathbf{T}_1^{-2}Q\,(\mathbf{S}_1Q - \mathbf{M}_1Q)\,\{d'r_Q - (\mathbf{t}Q + \mathbf{u}Q)\,d'q_Q\},$$

or $d'\,(\mathbf{t}Q + \mathbf{u}Q) = \mathbf{T}_1^{-2}Q\,(\mathbf{S}_1Q - \mathbf{M}_1Q)\,\{\mathbf{s}dQ + \mathbf{S}_1dR\,.\,\mathbf{m}Q$

$$+\,\mathbf{s}dR\,.\,\mathbf{M}_1Q - (\mathbf{t}Q + \mathbf{u}Q)\,(\mathbf{S}_1dQ + \mathbf{S}_1dR\,.\,\mathbf{M}_1Q)\}\text{......(17)}.$$

Hence utilising equation (10),

$$d\mathbf{M}_1Q = \tfrac{1}{2}\,(dR\mathbf{M}_1Q - \mathbf{M}_1QdR) + \mathbf{M}_1Q\mathbf{S}_1dR \quad\text{.......................(18)},$$

$$d\mathbf{m}Q = \tfrac{1}{2}\,(dR\mathbf{m}Q - \mathbf{m}QdR) + \mathbf{m}Q\mathbf{S}_1dR + \mathbf{M}_1Q\mathbf{s}dR \quad\text{............(19)},$$

$$d\mathbf{U}_1Q = \tfrac{1}{2}\,(dR\mathbf{U}_1Q - \mathbf{U}_1QdR) + \mathbf{T}_1^{-2}Q\mathbf{M}_1Q\mathbf{S}_1\,(dR\mathbf{S}Q - dQ)\,.\,\mathbf{U}_1Q\,(20),$$

$$\left.\begin{aligned} d\mathbf{u}Q &= \tfrac{1}{2}\,(dR\mathbf{u}Q - \mathbf{u}QdR) \\ &\qquad + \mathbf{T}_1^{-2}Q\mathbf{M}\,.\,(\mathbf{S}_1Q - \mathbf{M}_1Q)\,\{\mathbf{s}dQ + \mathbf{m}Q\mathbf{S}_1dR \\ &\qquad + \mathbf{M}_1Q\mathbf{s}dR - (\mathbf{t}Q + \mathbf{u}Q)\,(\mathbf{S}_1dQ + \mathbf{M}_1Q\mathbf{S}_1dR)\} \\ &= \tfrac{1}{2}\,(dR\mathbf{u}Q - \mathbf{u}QdR) \\ &\qquad + \mathbf{M}_1\,.\,Q^{-1}\,\{\mathbf{s}dQ + \mathbf{m}Q\mathbf{S}dR + \mathbf{M}Q\mathbf{s}dR \\ &\qquad - (\mathbf{t}Q + \mathbf{u}Q)\,(\mathbf{S}dQ + \mathbf{M}Q\mathbf{S}dR)\} \end{aligned}\right\}\text{......(21)},$$

$$dq_Q \quad = \tfrac{1}{2}\,(dRq_Q - q_QdR) + \mathbf{S}_1dQ + \mathbf{M}_1Q\mathbf{S}_1dR \quad\text{..................(22)},$$

$$dr_Q \quad = \tfrac{1}{2}\,(dRr_Q - r_QdR) + \mathbf{s}dQ + \mathbf{m}Q\mathbf{S}_1dR + \mathbf{M}_1Q\mathbf{s}dR \quad\text{......(23)}.$$

Equations (18), (19), (22) and (23) may also be written

$$d\mathbf{M}_1Q = (\mathbf{M} + \mathbf{S}_1)\,dR\,.\,\mathbf{M}_1Q \quad\text{.........................(24)},$$

$$d\mathbf{m}Q = (\mathbf{M} + \mathbf{S}_1)\,dR\,.\,\mathbf{m}Q + \mathbf{M}_1Q\mathbf{s}dR \quad\text{.........(25)},$$

$$dq_Q = (\mathbf{M} + \mathbf{S}_1)\, dR\,.\,\mathbf{M}_1 Q + \mathbf{S}_1 dQ \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26),$$

$$dr_Q = (\mathbf{M} + \mathbf{S}_1)\, dR\,.\,\mathbf{m}Q + \mathbf{M}_1 Q\mathbf{s}dR + \mathbf{s}dQ \quad\ldots\ldots\ldots(27).$$

The differential of Q^n, where n is any scalar constant, can be written down from the corresponding quaternion case (*Proc. R. S. E.* 1888–89, p. 201),

$$\left.\begin{aligned} d\,.\,Q^n &= nQ^{n-1}dQ + \tfrac{1}{2}(\mathbf{M}^{-1}Q - nQ^{n-1}\mathbf{M}^{-1}Q^n)\,(Q^n dQ - dQ\,.\,Q^n) \\ &= nQ^{n-1}dQ + (\mathbf{M}Q^n - nQ^{n-1}\mathbf{M}Q)\,\mathbf{M}\mathbf{M}^{-1}Q\mathbf{M}dQ \end{aligned}\right\}\ldots(28).$$

Also if A be a motor

$$d\,.\,A^n = nA^n\mathbf{S}A^{-1}dA + \mathbf{M}A^n\,.\,\mathbf{M}A^{-1}dA \quad\ldots\ldots\ldots(29),$$

and therefore

$$de^A = Ae^A\,\mathbf{S}A^{-1}dA + \mathbf{M}e^A\,.\,\mathbf{M}A^{-1}dA \ldots\ldots\ldots\ldots(30).$$

CHAPTER III.

ADAPTATION TO PHYSICAL APPLICATIONS.

21. Meaning and properties of ∇ in Octonions. We have for the sake of more readily utilising quaternion analogy altered the *geometrical* significance of many symbols which occur in Quaternions. For the same reason we now propose to take the same liberty with ∇.

It is first necessary to be precise in the meanings we shall in our physical applications attach to the symbols i, j, k and ζ. Let O be some fixed point and P a variable point. i, j, k will be supposed to be three mutually perpendicular unit rotors intersecting in P, and their *directions* will be supposed fixed. The values of i, j, k at O, which will be called the origin, will be denoted by i_0, j_0, k_0. Thus i, j, k are not constant rotors but i_0, j_0, k_0 are. On the other hand i, j, k are independent in meaning of an arbitrary origin, whereas i_0, j_0, k_0 are not. The rotor $\overline{OP}$ will be denoted by

$$\rho = i_0 x + j_0 y + k_0 z = ix + jy + kz \quad\text{..............(1).}$$

Thus x, y, z have their ordinary Cartesian meanings.

The symbolic rotor ∇ is defined by the equation

$$\nabla = i\partial/\partial x + j\partial/\partial y + k\partial/\partial z \quad\text{....................(2),}$$

and ∇_0 will denote what may be called the value of ∇ at the origin, i.e. $i_0\partial/\partial x + \ldots$. Thus ∇ is a symbolic rotor which in so far as it is a rotor is a function of the position of a point and is independent in meaning of an arbitrary origin; whereas ∇_0 is a symbolic rotor which in so far as it is a rotor is a constant depending in meaning on the selection of an arbitrary origin. ζ will still be defined by the equation

$$\psi(\zeta, \zeta) = \psi(i, i) + \psi(j, j) + \psi(k, k) \quad\text{...........(3),}$$

where $\psi(A, B)$ is any octonion function of two motors A and B which is linear in each. ζ_0 will be used for the value of ζ at the origin.

The *rotor* element of a curve will be denoted by $d\lambda$ and the *rotor* element of a surface by $d\Sigma$; that is to say, if P and Q are two near points on a curve $d\lambda = \overline{PQ}$ and if P, Q, R are three, not collinear, near points on a surface $d\Sigma$ will denote the rotor $\frac{1}{2}\mathbf{M}\overline{PQ}\,.\,\overline{PR}$ which passes through P, is normal to the surface at P and equal in tensor to the element PQR of surface. [More strictly, i.e. taking account of the third order of small quantities, $d\Sigma$ should be defined as the rotor equal and parallel to $\frac{1}{2}\mathbf{M}\overline{PQ}\,.\,\overline{PR}$ which passes through the centroid of the triangle PQR, i.e.

$$6d\Sigma = \mathbf{M}\,(\overline{PQ}\,\overline{QR} + \overline{QR}\,\overline{RP} + \overline{RP}\,\overline{PQ}).]$$

The usual conventions as to positive directions will be adopted when the boundary of a surface is compared with the surface and when the boundary of a volume is compared with the volume; i.e. in the first case $d\lambda$ will be in the direction of positive rotation round a proximate $d\Sigma$, and in the second case $d\Sigma$ will point away from the volume bounded.

With these definitions we have [eq. (1) § 7 above]

$$\left.\begin{array}{ll} i = i_0 + \Omega\mathbf{M}\rho i_0, & \nabla = \nabla_0 + \Omega\mathbf{M}\rho\nabla_0, \\ \zeta = \zeta_0 + \Omega\mathbf{M}\rho\zeta_0, & d\lambda = d\rho + \Omega\mathbf{M}\rho d\rho \end{array}\right\} \quad \ldots\ldots\ldots\ldots(4).$$

Thus $d\rho$ might for some purposes be appropriately denoted by $d\lambda_0$.

It will be observed that the properties of our present

$$i_0,\ j_0,\ k_0,\ \rho,\ \nabla_0,\ \zeta_0,$$

are practically identical with those of the corresponding quaternion symbols

$$i,\ j,\ k,\ \rho,\ \nabla,\ \zeta,$$

but it is obviously inappropriate in Octonions, on account of the arbitrary origin, to denote them by the latter symbols.

With the present notation

$$\mathbf{S}d\lambda\nabla = \mathbf{S}d\rho\nabla_0 = -d \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

There is a peculiarity in the use of the present ∇ which must be carefully attended to. It is well illustrated by the statement

that $\mathbf{S}\nabla(\mathbf{M}\nabla E)$, where E is any motor function of the position of a point, is not in general zero. At first sight this statement appears absurd, for ∇ is a motor operator and hence apparently

$$\mathbf{S}\nabla(\mathbf{M}\nabla E) = \mathbf{S}\nabla(\nabla E) = 0.$$

The last statement ($\mathbf{S}\nabla(\nabla E) = 0$) is not in general true because the present ∇ unlike the quaternion ∇ is a function of the position of a point and therefore is itself subject to space differentiations. Thus in the expression $\mathbf{S}\nabla(\nabla E)$ the left ∇ has two operands, viz. E and the right ∇.

We cannot then treat ∇ as a mere rotor in the same way as we may treat the quaternion ∇ as a mere vector, but we can do something very similar.

To enable us to symbolise the process here referred to we must indicate the operator and its operand in a rather peculiar way. Since a ∇ may be either an operator or an operand, or as with the second ∇ of $\mathbf{S}\nabla(\nabla E)$ simultaneously the one and the other, it is necessary to indicate what is its character by a system of suffixes. We shall indicate, when necessary, an operator by a capital letter suffix and its operand by the corresponding small letter suffix. Thus

$$\mathbf{S}\nabla(\nabla E) = \mathbf{S}(\nabla_A + \nabla_B)\nabla_{aB}E_b,$$

in which ∇_B and ∇_{aB} are operators both having E_b for operand and where ∇_A is an operator having ∇_{aB} for operand. Thus

$$\begin{aligned}\mathbf{S}\nabla(\nabla E) &= \mathbf{S}\nabla_A\nabla_{Ba}E_b + \mathbf{S}\nabla^2{}_B E_b \\ &= \mathbf{S}\nabla_A\nabla_{Ba}E_b.\end{aligned}$$

Here the suffixes might be removed if the usual convention be adopted that when not otherwise indicated the operator of a ∇ is the single symbol immediately succeeding it. Thus

$$\mathbf{S}\nabla(\mathbf{M}\nabla E) = \mathbf{S}\nabla(\nabla E) = \mathbf{S}\nabla\nabla E \quad\ldots\ldots\ldots\ldots(6),$$

which is *not* a truism as may be illustrated by the more general equation

$$\mathbf{S}\nabla(\nabla Q) = \mathbf{S}\nabla\nabla Q + \nabla^2{}_A\mathbf{S}Q_a \quad\ldots\ldots\ldots\ldots\ldots(7),$$

where Q is any octonion function of the position of a point.

It will be noticed that ∇^2 has not been used in the above. Adopting the ordinary convention by which for instance

$$\left(x\frac{\partial}{\partial x}\right)^2 y = x\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right),$$

∇^2 should be defined by the equation

$$\nabla^2 Q = \nabla(\nabla Q) \quad\text{.........................(8).}$$

With this meaning ∇^2 is not in general a scalar operator, though the square of a rotor is an ordinary scalar.

$\nabla^2{}_A Q_a$ or more generally $\nabla^n{}_A Q_a$ where n is a positive integer, is rather a cumbrous expression for the thing signified. It is easy to see that

$$\left.\begin{array}{l}\nabla^n{}_A\, Q_a = \nabla_0{}^n Q\,[n \text{ even}]\\ \nabla^n{}_A\, Q_a = \nabla\nabla_0{}^{n-1}\, Q\,[n \text{ odd}]\end{array}\right\} \quad\text{.................(9),}$$

but this alternative method of indication is if anything still more objectionable. I shall therefore use the notation defined by

$$\nabla^n{}_A Q_a = \rhd^n Q \quad\text{......................(10).}$$

We have not yet found the meaning of the expression $\mathbf{S}\nabla\nabla E$ or $\mathbf{S}\nabla_A\nabla_{aB}E_b$. Let ψ have the meaning just now given to it. Required the meaning of

$$\psi(\nabla_A, \nabla_a),$$

where ∇_a has some operand which however it is not necessary here to indicate.

Although ∇ cannot be treated as a constant rotor, ∇_0 can. Thus by eq. (4) the only variable part of ∇ is $\Omega\mathbf{M}\rho\nabla_0$. Thus

$$\begin{aligned}\psi(\nabla_A, \nabla_a) &= \psi(\nabla_{0A} + \Omega\mathbf{M}\rho\nabla_{0A}, \Omega\mathbf{M}\rho_a\nabla_0)\\ &= \psi(\zeta_0 + \Omega\mathbf{M}\rho\zeta_0, \Omega\mathbf{M}\zeta_0\nabla_0),\end{aligned}$$

or

$$\psi(\nabla_A, \nabla_a) = \psi(\zeta, \Omega\mathbf{M}\zeta\nabla) \quad\text{......................(11),}$$

where ∇ on the right has the same operand as ∇_a on the left.

As a particular result we have

$$\nabla^2 = (\nabla_A + \nabla_B)\nabla_{aB} = \Omega\zeta\mathbf{M}\zeta\nabla + \rhd^2,$$

or

$$\nabla^2 = -2\Omega\nabla + \rhd^2 \quad\text{...................... (12).}$$

Hence

$$\mathbf{S}\nabla^2 E = -2\Omega\mathbf{S}\nabla E \quad\text{....................(13).}$$

22. Line-surface integral and surface-volume integral. To lead up to the octonion theorems corresponding to the well-known quaternion integration theorems, first notice that for a closed curve and a closed surface respectively we have

$$\int d\lambda = 2\Omega\iint d\Sigma \quad\text{..........................(1),}$$

$$\iint d\Sigma = 0 \quad\text{.............................. (2).}$$

The truth of these are easily seen by their physical meanings. The first asserts that the system of forces which is represented by the sides of any closed polygon taken in order is equivalent to twice the system of couples represented by the area of the polygon—i.e. what would in Quaternions be called the vector area and what here must be called the lator area. The second asserts that the system of forces represented by a uniform hydrostatic pressure on a closed surface is in equilibrium.

It is easy to furnish octonion proofs of both statements. It is only necessary to prove equation (1) for a triangle and equation (2) for a tetrahedron, since a finite surface or volume can be split up into a series of elementary triangles or tetrahedra.

Let the triangle be OPQ. Then $\overline{PO} + \overline{OQ}$ is the rotor through O equal and parallel to $\overline{PQ}$, i.e. $\overline{PO} + \overline{OQ}$ is obtained by translating $\overline{PQ}$ along the rotor $\overline{PO}$ or [eq. (1) § 7]

$$\overline{PO} + \overline{OQ} = \overline{PQ} + \Omega \mathbf{M} \overline{PO}\,\overline{PQ},$$

or

$$\overline{PQ} + \overline{QO} + \overline{OP} = \Omega \mathbf{M} \overline{OP}\,\overline{PQ},$$

which proves the property for the triangle.

Next let $\rho_1, \rho_2 \ldots$ be the rotor perpendiculars from a fixed point O on a series of rotors equal and parallel to $m_1 i_0$, $m_2 i_0 \ldots$ where i_0 is a unit rotor through O. The sum of the rotors is by eq. (1) § 7 above

$$\Sigma m\,(1 + \Omega\rho)\, i_0 = (1 + \Omega\bar{\rho})\, i_0 \Sigma m,$$

where $\bar{\rho}$ is defined by the equation $\bar{\rho}\Sigma m = \Sigma m\rho$. Interpreting this sum by aid of equation (1) § 7 it follows that a rotor plane area is a rotor whose tensor is the area and which passes normally to the area through its centroid. If then α, β, γ be three of the rotor edges of a tetrahedron all starting from one angular point, the rotor areas of the faces are

$$-\tfrac{1}{2}\mathbf{M}\beta\gamma - \tfrac{1}{6}\Omega\mathbf{M}(\beta+\gamma)\,\mathbf{M}\beta\gamma, \quad -\tfrac{1}{2}\mathbf{M}\gamma\alpha - \tfrac{1}{6}\Omega\mathbf{M}(\gamma+\alpha)\,\mathbf{M}\gamma\alpha,$$
$$-\tfrac{1}{2}\mathbf{M}\alpha\beta - \tfrac{1}{6}\Omega\mathbf{M}(\alpha+\beta)\,\mathbf{M}\alpha\beta,$$

and $\tfrac{1}{2}\mathbf{M}(\beta\gamma + \gamma\alpha + \alpha\beta) + \tfrac{1}{6}\Omega\mathbf{M}(\alpha+\beta+\gamma)\,\mathbf{M}(\beta\gamma+\gamma\alpha+\alpha\beta)$,

the sum of which is easily seen to be zero (since

$$\mathbf{M}\alpha\mathbf{M}\beta\gamma + \mathbf{M}\beta\mathbf{M}\gamma\alpha + \mathbf{M}\gamma\mathbf{M}\alpha\beta = 0).$$

Let now ϕ be any linear octonion function of an octonion. Then

$$\int \phi d\lambda = \iint \{2\phi(\Omega d\Sigma) + \phi_a \mathbf{M} d\Sigma \nabla_A\} \ldots\ldots\ldots\ldots\ldots (3),$$

$$\iint \phi d\Sigma = \iiint \phi_a \nabla_A d\varsigma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

the first referring to a surface and its boundary and the second to a volume and its boundary, $d\varsigma$ in the latter case being the element of volume. These are quite easily proved from equation (5) § 21 and equations (1) and (2) of the present section by splitting the surface up into elementary triangles and the volume into elementary tetrahedra. [For the corresponding quaternion proof see *Utility of Quat. in Phys.* § 6.]

Equation (3) is more complicated in form than the corresponding quaternion theorem. It is of course more general than the latter. It may however be remarked that whenever the quaternion theorem is sufficiently general to furnish a convenient attack on a problem equation (3) can be put into a form which is practically identical with the quaternion form. Let the octonion function ϕ be commutative with Ω. Then equation (3) gives

$$\Omega \int \phi d\lambda = \Omega \iint \phi_a \mathbf{M} d\Sigma \nabla_A,$$

which is to all intents aud purposes the same as the corresponding quaternion theorem (since $\Omega\nabla = \Omega\nabla_0$).

Suppose σ is a rotor function of a point which passes through the point. From equations (3) and (4) we have

$$\int \mathbf{S} d\lambda\sigma = \iint \mathbf{S} d\Sigma (\mathbf{M}\nabla\sigma + 2\Omega\sigma) \ldots\ldots\ldots\ldots\ldots (5),$$

$$\iint \mathbf{S} d\Sigma\sigma = \iiint \mathbf{S}\nabla\sigma d\varsigma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

From these it follows that $\mathbf{S}\nabla\sigma$ has exactly the same physical significance as the corresponding quaternion expression, but not so $\mathbf{M}\nabla\sigma$. Since $d\lambda$ and σ are intersecting rotors $\mathbf{S}d\lambda\sigma$ is an ordinary scalar. Hence $\mathbf{S}d\Sigma(\mathbf{M}\nabla\sigma + 2\Omega\sigma)$ is an ordinary scalar. $d\Sigma$ may be regarded as an arbitrary rotor through the point under consideration, since equation (5) may be made to apply to *any* surface. Hence by statement (4) § 14 $\mathbf{M}\nabla\sigma + 2\Omega\sigma$ is a rotor through the point. It follows from equation (5) that it is a rotor equal and parallel to twice the spin when σ is taken for the rotor velocity of a fluid at the point. Calling this rotor spin ω, we have

$$\mathbf{M}\nabla\sigma = 2(\omega - \Omega\sigma) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Hence if A be the velocity motor of the element, i.e. $A = \omega + \Omega\sigma$ (§ 8 above), we see that $\frac{1}{2}\mathbf{M}\nabla\sigma$ is obtained from A by first translating A to the position symmetrically situated on the opposite side of the point and then changing the sign of its pitch.

From equation (7),

$$2\mathbf{S}\nabla\omega = \mathbf{S}\nabla^2\sigma + 2\Omega\mathbf{S}\nabla\sigma = 0,$$

by equation (13) § 21. Since $\mathbf{S}\nabla(\ )$ has the same physical significance as in quaternions, this equation merely asserts that the convergence of the spin is zero, a well-known fact. But it furnishes as it were a physical reason for equation (13) § 21; or rather it shows that there is a physical connection between the two results, first, equation (13) § 21, and second, the fact that twice the rotor spin is $\mathbf{M}\nabla\sigma + 2\Omega\sigma$.

As an illustration of the fact that equation (3) can always when convenient be reduced to a form practically identical with the quaternion form we may notice that

$$2\Omega\omega = \mathbf{M}\nabla\,(\Omega\sigma),$$

or twice the *lator* spin = $\mathbf{M}\nabla$ (the *lator* velocity).

The fact that when $\frac{1}{2}\mathbf{M}\nabla\sigma$ is expressed in the form $\omega + \Omega\tau$ where ω and τ are rotors through the point under consideration, ω is the spin and τ is the velocity reversed, may be generalised.

Let ϕ be a pencil function whose centre is at P and which is a function of the position of P. I proceed to show that if we put

$$\phi_a\nabla_A = \omega + \Omega\tau,$$

where ω and τ are rotors through P, $\tau = \mathbf{M}\zeta\phi\zeta$ and that the rotor ω has the same geometrical relations with the coordinates of ϕ that the vector $\phi_1'\nabla_1$ has with the coordinates of the linear vector function ϕ'. The last statement is seen at once from the equation $\Omega\phi_a\nabla_A = \Omega\omega$ coupled with the similarity of lators to vectors, and of a linear lator function of a rotor to a linear vector function of a vector. To prove that $\tau = \mathbf{M}\zeta\phi\zeta$ we have merely to prove that $\phi_a\nabla_A - \Omega\mathbf{M}\zeta\phi\zeta$ is a rotor through P. Now by equation (4),

$$\iint\mathbf{S}i\phi d\Sigma = \iiint\mathbf{S}i_a\phi_a\nabla_A d\varsigma = \iiint\mathbf{S}i\,(\phi_a\nabla_A - \Omega\zeta\phi\zeta)\,d\varsigma,$$

[since, as it is easy to prove, $\psi(i_a, \nabla_A) = \psi(\Omega\mathbf{M}\zeta i, \zeta)$]. Now $\mathbf{S}i\phi d\Sigma$ is an ordinary scalar. It follows that $\mathrm{s}i\,(\phi_a\nabla_A - \Omega\zeta\phi\zeta) = 0$, and thence from statement (4) § 14 that $\phi_a\nabla_A - \Omega\mathbf{M}\zeta\phi\zeta$ is a rotor

through P. Putting $\phi = \frac{1}{2}\mathbf{M}(\ )\sigma$ the above statements about the spin follow.

23. Strain. In treating of strain it is convenient to utilise the arbitrary origin O, though as might be expected many of the results are independent of it.

Let Q be a point near to P (fig. 6) and let

$$\overline{OP} = \rho,\ \overline{OQ} = \rho + d\rho,\ \overline{PQ} = d\lambda \ldots\ldots\ldots\ldots\ldots (1).$$

Suppose in a general displacement that P moves to P' and Q to Q'. Put

$$\overline{PP'} = \eta,\ \overline{QQ'} = \eta + d\eta \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

so that η and $\eta + d\eta$ are the rotor displacements of P and Q. Also put

$$\overline{OP'} = \rho',\ \overline{OQ'} = \rho' + d\rho',\ \overline{P'Q'} = d\lambda' \ldots\ldots\ldots\ldots (3).$$

Let Oq and PQ be equal and parallel, and again let Oq', PQ'' and $P'Q'$ be equal and parallel.

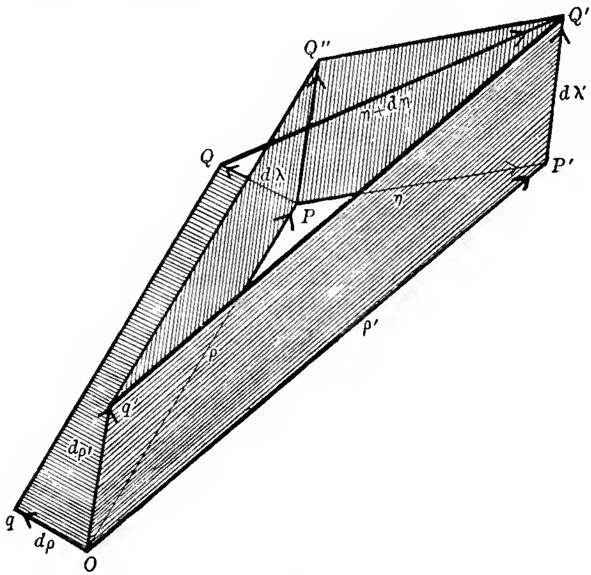


Fig. 6.

Thus $\quad \overline{Oq} = d\rho,\ \overline{PQ} = d\lambda = d\rho + \Omega\mathbf{M}\rho d\rho \ldots\ldots\ldots\ldots\ldots\ldots (4),$

$$\left.\begin{aligned} \overline{Oq'} &= d\rho',\ \overline{PQ''} = d\rho' + \Omega\mathbf{M}\rho d\rho' \\ \overline{P'Q'} &= d\lambda' = d\rho' + \Omega\mathbf{M}\rho' d\rho' = \overline{PQ''} + \Omega\mathbf{M}\eta\overline{PQ''} \end{aligned}\right\} \ldots (5),$$

$$\eta = \rho' - \rho + \Omega \mathbf{M} \rho\rho' \quad\text{.................................(6)},$$

$$d\rho' = -\mathbf{S} d\lambda \nabla \,.\, \rho' \quad\text{.................................(7)},$$

$$d\lambda' = -\{\mathbf{S} d\lambda\nabla \,.\, \rho' + \Omega \mathbf{M} \rho' (\mathbf{S} d\lambda \nabla \,.\, \rho')\} \quad\text{...........(8)}.$$

A being any motor define χ_0, χ_P, χ by the first of each of the following three sets of equations—

$$\chi_0 A = -\mathbf{S} A \nabla \,.\, \rho', \; d\rho' = \chi_0 d\lambda \quad\text{...............................(9)},$$

$$\chi_P A = \chi_0 A + \Omega \mathbf{M} \rho \chi_0 A, \; \overline{PQ''} = \chi_P d\lambda \quad\text{.......................(10)},$$

$$\chi A = \chi_0 A + \Omega \mathbf{M} \rho' \chi_0 A = \chi_P A + \Omega \mathbf{M} \eta \chi_P A, \; d\lambda' = \chi d\lambda \quad\text{...(11)}.$$

The rest of the equations are easily seen to follow from the defining equations.

From equations (10) and (11) it is obvious that χ and χ_P are independent in meaning of the origin. In fact

$$\chi_P A = A + \Omega \mathbf{M} \eta A - \mathbf{S} A \nabla \,.\, \eta \quad\text{..............(12)},$$

which may be proved from the above results or in a manner not involving the origin as follows.—Expressing the fact $\overline{PQ} + \overline{QQ'} + \overline{Q'P'} + \overline{P'P}$ = twice the lator area of the quadrilateral $PQQ'P'$, we get

$$d\eta + d\lambda - d\lambda' = \Omega \mathbf{M} (d\lambda + d\lambda') \, \eta,$$

whence putting $d\lambda' = \chi d\lambda = \chi_P d\lambda + \Omega \mathbf{M} \eta \chi_P d\lambda$ and $d\eta = -\mathbf{S} d\lambda \nabla \,.\, \eta$ we get

$$\chi_P d\lambda = d\lambda + \Omega \mathbf{M} \eta d\lambda - \mathbf{S} d\lambda \nabla \,.\, \eta,$$

which proves eq. (12) for A = any rotor $d\lambda$ through P. Operating by Ω and assuming that χ_P is a commutative function eq. (12) is true for A = any lator. Adding these results eq. (12) is true for A = any motor.

From the second of equations (10) it follows that χ_P is a pencil function (§ 15 above) with centre P. Hence by Tait's *Quaternions*, 3rd ed., § 381, we may put

$$\chi_P A = q \psi A q^{-1} \quad\text{......................(13)},$$

where q is an axial (§ 6 above) through P and ψ is a self-conjugate pencil function with centre P.

By eq. (11)

$$\chi A = (1 + \tfrac{1}{2}\Omega\eta) \, \chi_P A \, (1 + \tfrac{1}{2}\Omega\eta)^{-1}.$$

Hence

$$\chi A = Q \psi A Q^{-1} \quad\text{....................(14)},$$

where

$$Q = (1 + \tfrac{1}{2}\Omega\eta) \, q \quad\text{....................(15)}.$$

Similarly $$\chi_0 A = R\psi A R^{-1} \qquad \text{(16)},$$
where $$R = (1 - \tfrac{1}{2}\Omega\rho)\, q \qquad \text{(17)}.$$

If χ', χ_P' and χ_0' are the conjugates of χ, χ_P, χ_0 we have by equations (13), (14), (16)

$$\psi^2 = \chi'\chi = \chi_P'\chi_P = \chi_0'\chi_0 = \nabla_A \mathbf{S}(\quad)\nabla_B \mathbf{S}\rho_a'\rho_b' = \Psi \qquad \text{(18)},$$

say. From eq. (12) and eq. (18) Ψ may be expressed in terms of η and its derivatives. Eq. (12) may be written

$$\chi_P A = -(1 + \tfrac{1}{2}\Omega\eta)\, \mathbf{S}A(\nabla_A + \zeta) . (\eta_a - \Omega \mathbf{M}\eta\eta_a + \zeta)(1 + \tfrac{1}{2}\Omega\eta)^{-1} \qquad \text{(19)}.$$

Utilising eq. (12) directly

$$\Psi A = A - \left.\begin{array}{l}\{\nabla_A \mathbf{S}A(\eta_a - \Omega\mathbf{M}\eta\eta_a) + (\eta_a - \Omega\mathbf{M}\eta\eta_a)\mathbf{S}A\nabla_A\} \\ \qquad + \nabla_A \mathbf{S}A\nabla_B \mathbf{S}\eta_a\eta_b\end{array}\right\} \qquad \text{(20)}.$$

From either eq. (19) or eq. (20) we have

$$\Psi A = (\nabla_A + \zeta_1)\, \mathbf{S}A(\nabla_B + \zeta_2)\, \mathbf{S}(\eta_a - \Omega\mathbf{M}\eta\eta_a + \zeta_1)(\eta_b - \Omega\mathbf{M}\eta\eta_b + \zeta_2) \qquad \text{(21)}.$$

In expanding the expressions on the right of equations (19) and (21) it is understood that any term in which a ζ occurs without the corresponding ζ, is put zero.

It will be noticed from equations (18) and (21) that Ψ is the same function of

$$\nabla_A + \zeta_1, \quad \eta_a - \Omega\mathbf{M}\eta\eta_a + \zeta_1, \quad \nabla_B + \zeta_2, \quad \eta_b - \Omega\mathbf{M}\eta\eta_b + \zeta_2,$$

as it is of

$$\nabla_A, \qquad \rho_a', \qquad \nabla_B, \qquad \rho_b'.$$

In the present case the expression depending on the arbitrary origin is considerably simpler than the other.

In the case of small strain we may put

$$\chi_P = 1 + (\chi_P), \quad \chi_P' = 1 + (\chi_P)',$$

where (χ_P) and $(\chi_P)'$ are small and conjugate to one another. Thus

$$\Psi = \chi_P'\chi_P = 1 + (\chi_P) + (\chi_P)',$$

or from eq. (12)

$$(\Psi - 1)A = 2(\psi - 1)A = -\nabla_A \mathbf{S}A\eta_a - \eta_a \mathbf{S}A\nabla_A \qquad \text{(22)},$$

so that in the case of small strain the expression independent of the arbitrary origin is quite simple.

24. Strain (continued). We propose now to consider the straining of a volume element, the straining of a surface element, and differentiation with regard to the coordinates of strain. These are all now so similar to the corresponding quaternion considerations that they have been marked off by being put in a separate section.

The m [§ 3] or the M [eq. (9) § 16] of χ has the usual physical significance, viz. the ordinary scalar ratio of the strained element of volume at P' to the corresponding unstrained element at P. This is seen at once by the original definition [§ 3] of m for the λ, μ, ν of that definition are ordinary motors. Taking them to be three elementary rotors through the point P, $-\mathbf{S}\lambda\mu\nu$ will be the element of volume at P (contained by a parallelepiped of which λ, μ, ν are adjacent edges) and $-\mathbf{S}\chi\lambda\chi\mu\chi\nu$ will be the corresponding element at P'.

Change λ into $d\lambda$ and $\mathbf{M}\mu\nu$ into $d\Sigma$, $\chi\lambda$ into $d\lambda'$ and $\mathbf{M}\chi\mu\chi\nu$ into $d\Sigma'$ so that $d\lambda$, $d\lambda'$ have the meanings hitherto given and $d\Sigma$, $d\Sigma'$ are the unstrained and strained rotor values of an element of surface. Thus

$$m\mathbf{S}d\lambda d\Sigma = \mathbf{S}d\lambda' d\Sigma' = \mathbf{S}d\lambda\chi' d\Sigma'.$$

Since $d\lambda$ is an arbitrary rotor through P it follows [statement (1) of § 14 above] that $md\Sigma = \chi' d\Sigma'$ or

$$d\Sigma' = m\chi'^{-1}\, d\Sigma \qquad\qquad (1),$$

exactly as in Quaternions, though this result has a more general meaning than the quaternion one.

In connection with this last equation it may be noticed that we have as in Quaternions [§ 3 above and *Utility of Quat. in Phys.* eq. (6*e*) § 3 *a*]

$$\left.\begin{aligned} & m\chi'^{-1}\, d\Sigma \\ & = -\tfrac{1}{2}\mathbf{M}\chi\zeta_1\chi\zeta_2\mathbf{S}d\Sigma\zeta_1\zeta_2 \\ & = -\tfrac{1}{2}(1+\Omega\eta)\,\mathbf{M}\,(\eta_a - \Omega\mathbf{M}\eta\eta_a + \zeta_1)\,(\eta_b - \Omega\mathbf{M}\eta\eta_b + \zeta_2)\,.\,(1-\Omega\eta) \\ & \qquad\qquad \times\, \mathbf{S}d\Sigma\,(\nabla_A + \zeta_1)\,(\nabla_B + \zeta_2) \end{aligned}\right\} \qquad (2),$$

which may also be put into a number of other forms.

In many important physical applications a pencil function with a definite centre is an independent variable. Differentiation with regard to the nine independent scalars involved may be

treated in a manner practically identical with the same treatment in Quaternions [*Utility*, §§ 1, 5]. Thus for instance the potential energy of strain may be considered a function of Ψ and it might be required to find the stress in terms of such a function. This will involve differentiation of the type mentioned.

Let i, j, k have the usual meanings with regard to the centre of Ψ. Then the six ordinary scalars a, b, c, e, f, g, where

$$\left.\begin{aligned}\Psi A = &- ei\mathbf{S}iA - fj\mathbf{S}jA - gk\mathbf{S}kA \\ &- \tfrac{1}{2}\{a(j\mathbf{S}kA + k\mathbf{S}jA) + b(k\mathbf{S}iA + i\mathbf{S}kA) + c(i\mathbf{S}jA + j\mathbf{S}iA)\}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(3),$$

are independent variables and completely specify Ψ. Define $\mathcal{A}$ (or $_{\Psi}\mathcal{A}$ if there are more independent self-conjugate pencil functions than one to be dealt with) by the equation

$$\mathcal{A}A = -i\mathbf{S}iA\partial/\partial e - \ldots - (j\mathbf{S}kA + k\mathbf{S}jA)\,\partial/\partial a - \ldots \ldots(4).$$

The symbol to be operated upon by the differentiations of $\mathcal{A}$ can be indicated in the usual way by suffixes.

If Q be an octonion function of the independent Ψ we have as in Quaternions

$$dQ = -Q_1\mathbf{S}d\Psi\zeta\mathcal{A}_1\zeta \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Should it be required we may as in Quaternions define the $\mathcal{A}$ corresponding to a general (i.e. not self-conjugate) pencil function with definite centre.

25. Intensities and Fluxes. The similarity of many of the above formulae even when their significance is more extended to corresponding quaternion formulae naturally leads us to ask whether an equal similarity extends to the octonion treatment of intensities and fluxes analogous to the quaternion treatment in *Phil. Trans.* A, 1892, pp. 689 *et seq.* It will be found that the similarity of the formulae, almost amounting to identity, is quite remarkable.

It is probable that no great physical use could be made of the conception of a general motor intensity or a general motor flux. Nevertheless I propose to define these and examine their symbolic properties, as it is probable that it would prove convenient to regard an intensity (or a flux) sometimes as a rotor and sometimes as a lator.

It is sufficient here to give little more than the formulae, as the proofs are very similar to the quaternion proofs.

When we say that the motor E is an intensity we mean that E is a function of the position of the point P, and E' is a function of the position of P' connected by the equivalent equations

$$\mathsf{S}d\lambda E = \mathsf{S}d\lambda' E', \; E' = \chi'^{-1} E \;\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Similarly F is a motor flux if

$$\mathsf{S}d\Sigma F = \mathsf{S}d\Sigma' F', \; F' = m^{-1}\chi F \;\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

It should be noticed that if E or F is a rotor through P then E' or F' is a rotor through P' (from which it follows that if E or F is a motor through P then E' or F' is a motor *with the same pitch* through P'). For if E is a rotor through P, $\mathsf{s}d\lambda E = 0$ and therefore $\mathsf{s}d\lambda' E' = 0$. Since $d\lambda'$ is an arbitrary rotor through P' it follows that E' is a rotor through P' [statement (4) § 14]. Similarly when F is a rotor through P. If E is a motor through P it may be put equal to $(1 + \Omega p)\,\omega$ where p is its pitch and ω is its rotor part, i.e. a rotor through P. It follows that

$$E' = (1 + \Omega p)\,\chi'^{-1}\,\omega = (1 + \Omega p)\,\omega',$$

where ω' is a rotor through P'. Hence E' is a motor through P' whose pitch is p. Similarly when F is a motor through P.

From these statements and equations (1) and (2) it follows that χ'^{-1} and $m^{-1}\chi$ reduce any rotor through P to a rotor through P'. This is otherwise obvious from the equations

$$\chi A = \chi_P A + \Omega \mathsf{M}\eta\chi_P A, \; \chi'^{-1} A = \chi_P'^{-1} A + \Omega \mathsf{M}\eta\chi_P'^{-1} A \ldots (3),$$

and from remembering that χ_P is a pencil function with centre P and that $\eta = \overline{PP'}$.

If ∇' have the same meaning with regard to the strained space as ∇ has with regard to the unstrained space

$$\mathsf{S}d\lambda\nabla = \mathsf{S}d\lambda'\nabla' \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

[eq. (5) § 21], so that ∇ is a symbolic rotor intensity.

If E_1, E_2 are intensities $\mathsf{M}E_1E_2$ is a flux.

If E is an intensity $\mathsf{M}\nabla E + 2\Omega E$ is a flux.

This statement it will be noticed is not quite identical with the corresponding quaternion statement. It might be thought

that it was inconsistent with the two preceding statements. The explanation is that E in $\mathbf{M}\nabla E$ is not in all respects analogous to E_2 in $\mathbf{M}E_1E_2$. To be more precise, $\mathbf{M}\nabla_A'\chi'^{-1}E_a$ bears the same relation to $\mathbf{M}\nabla E$ as F' to F (and this is all that can be deduced from the facts that ∇ and E are intensities and $\mathbf{M}E_1E_2$ is a flux) whereas $\mathbf{M}\nabla'E'$ or $\mathbf{M}\nabla'(\chi'^{-1}E)$ does not because of the differentiations of χ'^{-1}. The reason that the same sort of thing does not occur in Quaternions is that the quaternion expression

$$\mathbf{V}\nabla_1'\chi_1'^{-1}\sigma$$

is identically zero, whereas the octonion expression

$$\mathbf{M}\nabla_A'\chi_a'^{-1}E$$

is not. The proof of these statements is obtained by noticing that in the quaternion case

$$\chi^{-1}d\rho' = d\rho = -\mathbf{S}d\rho'\nabla'\,.\,\rho,$$

and therefore
$$\chi'^{-1} = -\nabla_1'\mathbf{S}(\ )\rho_1,$$

whereas in the octonion case

$$\chi^{-1}d\lambda' = d\lambda = d\rho + \Omega\mathbf{M}\rho d\rho,$$
$$= -\mathbf{S}d\lambda'\nabla'\,.\,\rho + \Omega\mathbf{M}\rho\chi^{-1}d\lambda',$$

and therefore $\chi'^{-1} = -\nabla_A'\mathbf{S}(\ )\rho_a + \Omega\chi'^{-1}\mathbf{M}(\ )\rho.$

From this it does not follow that $\mathbf{M}\nabla_A'\chi_a'^{-1}E = 0$ but by eq. (12) § 21

$$\mathbf{M}\nabla_A'\chi_a'^{-1}E = \Omega(2\nabla_A'\mathbf{S}E\rho_a + \mathbf{M}\,.\,\nabla_A'\chi_a'^{-1}\mathbf{M}E\rho_a)\ldots\ldots(5).$$

We can however deduce a result of a different form fron. (5), a result independent of the arbitrary origin, from the facts that ∇ and E are intensities and $\mathbf{M}E_1E_2$ and $\mathbf{M}\nabla E + 2\Omega E$ are fluxes. For from these we have

$$\mathbf{M}\nabla'(\chi'^{-1}E) + 2\Omega\chi'^{-1}E = m^{-1}\chi(\mathbf{M}\nabla E + 2\Omega E),$$

and
$$\mathbf{M}\nabla_A'\chi'^{-1}E_a = m^{-1}\chi\mathbf{M}\nabla E,$$

from which by subtraction

$$\mathbf{M}\nabla_A'\chi_a'^{-1}E = 2\Omega(m^{-1}\chi - \chi'^{-1})E\ \ldots\ldots\ldots\ldots(6).$$

By a similar process or by observing the reciprocal relations between E' and E we obtain

$$\mathbf{M}\nabla_A\chi_a'E' = 2\Omega(m\chi^{-1} - \chi')E'\ \ldots\ldots\ldots\ldots\ldots(7).$$

These two results are not independent. The second can easily be obtained from the first by substituting for E and ∇' in terms of E' and ∇.

It is to be observed that since in equations (5), (6), (7), E and E' are perfectly arbitrary motors, these equations are not equations involving intensities, but differential identities satisfied by the strain function χ. Before going further we will deduce some more relations of the same type.

From the symmetrical relations between ∇, ρ, χ and ∇', ρ', χ^{-1} we can write down a similar equation to (5), viz.

$$\mathbf{M}\nabla_A\chi_a'E' = \Omega\,(2\nabla_A\mathbf{S}E'\rho_a' + \mathbf{M}\,.\,\nabla_A\chi_a'\mathbf{M}E'\rho_a') \quad\text{......(8).}$$

Two other relations may be written down by expressing the equation $\int d\lambda = 2\Omega\iint d\Sigma$ in terms of the dashed letters and the equation $\int d\lambda' = 2\Omega\iint d\Sigma'$ in terms of the undashed letters. They are

$$\chi_a^{-1}\mathbf{M}E'\nabla_A' = 2\Omega\,(m^{-1}\chi'E' - \chi^{-1}E') \quad\text{............(9),}$$

$$\chi_a\mathbf{M}E\nabla_A = 2\Omega\,(m\chi'^{-1}E - \chi E) \quad\text{..............(10).}$$

Each of these may be deduced from the other. Eliminating the expressions from the right of equations (6) and (10) and expressing ∇_A' in terms of ∇_A and E in terms of E', we have

$$\chi_a\mathbf{M}\nabla_A\chi_a'E' = 0 \quad\text{......................(11).}$$

Similarly $$\chi_a^{-1}\mathbf{M}\nabla_A'\chi_a'^{-1}E = 0 \quad\text{......................(12).}$$

Eq. (11) may be verified directly by expressing χ in terms of χ_0 by eq. (11) § 23 and putting $\chi_0 = -\mathbf{S}(\ )\nabla\,.\,\rho'$ by eq. (9) § 23.

Two other results may be obtained by expressing the equation $\iint d\Sigma = 0$ in terms of the dashed letters and $\iint d\Sigma' = 0$ in terms of the undashed letters. They are

$$m_a^{-1}\chi_a'\nabla_A' = 0 \quad\text{......................(13),}$$

$$m_a\chi_a'^{-1}\nabla_A = 0 \quad\text{......................(14),}$$

each of which may be derived from the other. Of the nine equations $\chi'\nabla' = \nabla$, (6), (7), (9), (10), (11), (12), (13), (14) only four are independent. They may be taken as $\chi'\nabla' = \nabla$, (7), (11), (14). If we generalise the meaning of "spin" so as to call $\mathbf{M}\nabla G + 2\Omega G$ the "spin" of the motor G, eq. (7) asserts that if E' is any constant motor the "spin" of $\chi'E'$ is the lator $2\Omega m\chi^{-1}E'$. Also the statement that $\mathbf{M}\nabla E + 2\Omega E$ is a flux may be put:—the spin of an intensity is a flux.

A curious property of some of these equations with regard to "dimensions" may be noticed [see Hamilton's *Elements of Quaternions*, § 347 (6) et seq.]. All the corresponding quaternion formulae are consistent with the assumption that χ is of arbitrary

homogeneous dimensions but some of the present formulae are not. They are only consistent in the matter of dimensions if χ be assumed of zero order in dimensions. This need not surprise us since the original physical definition necessarily implied this.

Equation (7) may be taken as a typical example. If χ be of n dimensions, m is of $3n$, and therefore $m\chi^{-1}$ is of $2n$ while χ' is of n.

There is a way other than that of regarding χ as of zero order in dimensions of avoiding the anomaly. We may assume χ to be of arbitrary homogeneous dimensions, but then we must not assume that $\int d\lambda' = 2\Omega \iint d\Sigma'$ where Ω has the same meaning as in the equation $\int d\lambda = 2\Omega \iint d\Sigma$. Instead we may put

$$\int d\lambda' = 2\Omega' \iint d\Sigma',$$

where Ω' is similar to Ω when it is in combination with itself and ordinary scalars but is of different dimensions from Ω. The relation between the *dimensions* of Ω and Ω' is the same as the relation between the dimensions of the Ω and Ω' of *Phil. Trans.* 1892 A, p. 691. Indeed [*ibid.* eq. (9)] eq. (7) above takes the consistent form

$$\mathbf{M}\nabla_A \chi_a' E' = 2\,(m\chi^{-1}\Omega' - \Omega\chi')\,E',$$

and the right would be identically zero if the relations between Ω and Ω' were precisely those of that paper.

This apparent anomaly in "dimensions" is ultimately due to the fact that Ω, though a constant octonion, is not of order zero in dimensions, but of the same dimensions as the reciprocal of a length.

If ρ' be assumed of arbitrary homogeneous dimensions different from ρ, η cannot be of homogeneous dimensions and therefore χ cannot. This may be illustrated by taking $\rho' = c\rho$, where c is a constant ordinary scalar, when the particular forms assumed by all the above equations can easily be written down. In the expressions for η and χ, c^{-1} will be found to occur so that if c be not of zero dimensions η and χ cannot be assumed homogeneous in dimensions.

We now return to the consideration of fluxes and intensities.

If E and F be an intensity and flux respectively

$$\mathbf{S}EF d\varsigma = \mathbf{S}E'F' d\varsigma',$$

where $d\varsigma$ and $d\varsigma'$ are corresponding unstrained and strained elements of volume.

A somewhat interesting result may be deduced from this. Suppose σ, σ_1 are two arbitrary rotor intensities passing through P and τ, τ_1 are two arbitrary rotor fluxes passing through P. Define the motors G, G', H, H' by the equations

$$G = \sigma + \Omega\tau, \qquad G' = \sigma' + \Omega\tau' \;\ldots\ldots\ldots\ldots\ldots(15),$$

$$H = \tau_1 + \Omega\sigma_1, \qquad H' = \tau_1' + \Omega\sigma_1' \;\ldots\ldots\ldots\ldots(16).$$

If we wish to give G and H names, G may be called an intensity-flux and H a flux-intensity.

By the result just given we have

$$\mathbf{S}\sigma\tau d\varsigma = \mathbf{S}\sigma'\tau' d\varsigma', \qquad \mathbf{S}\sigma_1\tau_1 d\varsigma = \mathbf{S}\sigma_1'\tau_1' d\varsigma',$$

$$\mathbf{S}\sigma\tau_1 d\varsigma = \mathbf{S}\sigma'\tau_1' d\varsigma', \qquad \mathbf{S}\sigma_1\tau d\varsigma = \mathbf{S}\sigma_1'\tau' d\varsigma'.$$

All these except the last can be expressed in terms of G and H and give

$$\mathbf{s}G^2 d\varsigma = \mathbf{s}G'^2 d\varsigma', \quad \mathbf{s}H^2 d\varsigma = \mathbf{s}H'^2 d\varsigma' \;\ldots\ldots\ldots\ldots(17),$$

$$\mathbf{S}_1 GH d\varsigma = \mathbf{S}_1 G'H' d\varsigma' \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18).$$

In particular if $\mathbf{s}G^2 = 0$, $\mathbf{s}G'^2 = 0$ and if $\mathbf{s}H^2 = 0$, $\mathbf{s}H'^2 = 0$; i.e. if G is a rotor G' is a rotor, if G is a lator G' is a lator [eq. (10) § 14] and similarly for H and H'.

If F be a flux $\mathbf{S}\nabla F d\varsigma = \mathbf{S}\nabla' F' d\varsigma'$. From this statement and the last italicised statement we may again deduce equations (13), (14).

Without going further in these theorems which are so analogous to quaternion theorems we may remark that §§ 9, 10, 53, 54 of the *Phil. Trans.* paper just alluded to can be paralleled with our present meanings of the symbols. In §§ 9, 10 we can deal with general motors, but in §§ 53, 54 we must limit ourselves to rotors.

We may as well however add the theorem corresponding to the proposition on p. 176 of the *Phil. Mag.* Aug. 1893. If the ratio of n' to n be defined as m so that n' and n may be taken as the 'specific volume' in the strained and unstrained states:—

If F_1 and F_2 are fluxes $n\mathbf{M}F_1F_2$ is an intensity.

26. Stress. Very little will be said about stress, as its treatment is very similar to the quaternion treatment.

Let ϕ be the stress pencil function at any point, i.e. let ϕ be a

pencil function whose centre is at the point considered and such that $\phi d\Sigma$ is the force due to stress exerted on any region at the rotor element $d\Sigma$ of its boundary.

The whole force motor exerted on the region by the rest of matter is

$$\iint \phi\, d\Sigma = \iiint \phi_a \nabla_A d\varsigma,$$

by eq. (4) § 22, and therefore the force motor exerted on the element $d\varsigma$ is $\phi_a \nabla_A d\varsigma$.

We saw in § 22 that the motor $\phi_a \nabla_A$ consisted of a rotor $\phi_a \nabla_A - \Omega \mathbf{M} \zeta \phi \zeta$ through the point considered and a lator $\Omega \mathbf{M} \zeta \phi \zeta$, and that the rotor bore to the stress function ϕ the same relation as in Quaternions the vector $\phi_1 \nabla_1$ bears to the stress function ϕ. We see then that as in quaternions the effect of a stress on the element is the force $\phi_a \nabla_A - \Omega \mathbf{M} \zeta \phi \zeta$ through the element and the couple $\Omega \mathbf{M} \zeta \phi \zeta$.

In particular the necessary and sufficient condition that there should be no molecular couple due to stress is that ϕ should be self-conjugate. And further if ϕ is any self-conjugate pencil function which is a function of the position of a point and has the point for centre, $\phi_a \nabla_A$ is a rotor through the point.

It is now evident that the discussion of the connection between stress and strain can be carried on by means of Octonions on lines practically identical with the quaternion ones.

CHAPTER IV.

MOTORS AS MAGNITUDES OF THE FIRST ORDER IN THE AUSDEHNUNGSLEHRE.

27. Can reciprocal motors be regarded as normal magnitudes of the first order? Just as, hitherto, we have freely drawn on quaternion theorems, we shall now utilise parts of Grassmann's *Ausdehnungslehre.* The references will always be to the 1862 (Berlin) edition. [These references are equally to "Hermann Grassmanns gesammelte Mathematische und Physikalische Werke. Ersten Bandes Zweiter Theil: Die Ausdehnungslehre von 1862." Leipzig, 1896. Both §§ and pages of the 1862 edition are indicated in the reprint.] References will also be frequently made to Sir Robert Ball's *Theory of Screws*, 1876 (Dublin). I shall for the future refer to these two treatises as '*Ausd.*' and '*Screws*' respectively.

In a note on p. 182 of *Screws* the author seems to imply that in the *Ausd.*, if two motors A and B are "normal" in Grassmann's sense they would be "reciprocal" in Ball's sense, i.e. $\mathbf{s}AB$ would be zero. Now as far as Grassmann's own geometrical applications are concerned this is not the case.

Grassmann says (*Ausd.* § 330) "Für die innere Multiplikation nehme ich als ursprüngliche Einheiten im Raume stets drei zu einander senkrechte und gleich lange Strecken, (e_1, e_2, e_3)." Thus he deliberately refuses to apply his theorems concerning inner multiplication and therefore in particular concerning normals to the more general system of primitive units E, e_1, e_2, e_3, where E is a simple point which he introduces in § 216. And the reason for his alteration is not far to seek. Taking the gross complex (Hauptgebiet) as that of E, e_1, e_2, e_3, we see by § 89 that the complements of $[Ee_1]$, $[Ee_2]$, $[Ee_3]$, $[e_2e_3]$, $[e_3e_1]$, $[e_1e_2]$ would be $[e_2e_3]$,

$[e_3 e_1]$, $[e_1 e_2]$, $[Ee_1]$, $[Ee_2]$, $[Ee_3]$ respectively. Now from the geometrical interpretations which Grassmann gives in § 239 *et seq.* we see that with our notation we must identify $[Ee_1]$, $[Ee_2]$, $[Ee_3]$ with i, j, k respectively and $[e_2 e_3]$, $[e_3 e_1]$, $[e_1 e_2]$ with Ωi, Ωj, Ωk respectively, where i, j, k are the unit rotors through E parallel to e_1, e_2, e_3 respectively. Hence i and Ωi are complementary. But a definite point is required for the specification of the system i, j, k and no such point for Ωi, Ωj, Ωk. Hence complementariness in this meaning does not treat space impartially. This indeed is pointed out by Grassmann himself [note to § 337 of *Ausd.*].

It does not follow from this however that all processes such as that of inner multiplication which are based on the idea of complements are similarly unsymmetrical. This is well illustrated by Grassmann's theorems concerning planimetric and stereometric multiplication (§ 288 *et seq.*). As a matter of fact $-\mathbf{s}AB$ is what Grassmann would call the stereometric product of the two motors A and B; but this has nothing to do with "normals."

Let us examine then the meaning of inner multiplication of motors. Denoting (*Ausd.* § 89) the complement by a vertical line (so that $|i = \Omega i$, $|\Omega i = i$, &c.) the inner product of two motors A and B is defined as $[A \mid B]$, and by § 142 we have

$$[i \mid i] = 1, \quad [i \mid j] = 0, \quad [i \mid \Omega i] = 0, \quad [\Omega i \mid \Omega i] = 1, \quad \&c.$$

Now the value of $\mathbf{s}i^2$ is zero and the value of $\mathbf{s}i\Omega i$ is not zero. Hence the inner product of two motors cannot be identified with $\pm\, \mathbf{s}AB$ and in particular two motors are not in general "normal" if they are "reciprocal," and conversely. [This of course does not prove that the inner multiplication of motors treats space unsymmetrically, but this is not hard to show since the inner product of

$$x_1 i + x_2 j + x_3 k + l_1 \Omega i + m_1 \Omega j + n_1 \Omega k,$$

and

$$x_2 i + y_2 j + z_2 k + l_2 \Omega i + m_2 \Omega j + n_2 \Omega k,$$

would (*Ausd.* § 143) be $x_1 x_2 + y_1 y_2 + z_1 z_2 + l_1 l_2 + m_1 m_2 + n_1 n_2$, from which it can be deduced that the meaning of $[A \mid B]$ is not independent of the point of intersection of i, j, k. As an example notice that the inner square (*Ausd.* § 145) $[(i + x\Omega j) \mid (i + x\Omega j)]$ of the unit rotor $i + \Omega j$ is not unity but $1 + x^2$.]

This want of symmetry follows from the fact that in Grassmann's own geometrical interpretations motors are magnitudes of the second order. But there is no obvious reason why they should not be treated as magnitudes of the first order.

Let us see if by so treating them we can make Grassmann's "normal" mean the same as Ball's "reciprocal." If E and F are two of the primitive units this would necessitate that $\mathbf{s}EF = 0$. If we are to identify $-\mathbf{s}AB$ with Grassmann's $[A \mid B]$ we must further make $\mathbf{s}E^2 = \mathbf{s}F^2 = -1$. Hence by eq. (10) § 14 above

$$2\mathbf{t}E\mathbf{T}_1^{\,2}E = 1 \quad \text{.......................(1).}$$

This shows that a real motor with negative pitch cannot be a primitive unit. But if the tensor $\mathbf{T}_1E''$ of a real motor E'' with negative pitch $\mathbf{t}E''$ be so chosen that $2\mathbf{t}E''\mathbf{T}_1^{\,2}E'' = -1$, then $E' = vE''$, where $v = \sqrt{(-1)}$, is such that $\mathbf{s}E'^2 = -1$.

We are thus led to make trial of the following coreciprocal motors as primitive units—

$$\left.\begin{aligned} E &= (i+\Omega i)/\sqrt{2}, \quad F = (j+\Omega j)/\sqrt{2}, \quad G = (k+\Omega k)/\sqrt{2} \\ E' &= v\,(i-\Omega i)/\sqrt{2}, \quad F' = v\,(j-\Omega j)/\sqrt{2}, \quad G' = v\,(k-\Omega k)/\sqrt{2} \end{aligned}\right\} \ldots(2).$$

$$\left.\begin{aligned} \text{Here} \quad \mathbf{s}E^2 = \mathbf{s}E'^2 = \mathbf{s}F^2 = \mathbf{s}F'^2 = \mathbf{s}G^2 = \mathbf{s}G'^2 = -1 \\ \mathbf{s}EE' = \mathbf{s}EF = \mathbf{s}EF' = \mathbf{s}E'F = \mathbf{s}E'F' = \ldots = 0 \end{aligned}\right\} \ldots(3).$$

It is now easy to see that if A_1, A_2 be any two motors

$$[A_1 \mid A_2] = [A_2 \mid A_1] = -\mathbf{s}A_1A_2 \ldots\ldots\ldots\ldots\ldots(4),$$

for we may put

$$A_1 = e_1E + e_1'E' + f_1F + f_1'F' + g_1G + g_1'G' \ldots\ldots\ldots(5),$$

$$A_2 = e_2E + e_2'E' + f_2F + f_2'F' + g_2G + g_2'G' \ldots\ldots\ldots(6),$$

where e_1 &c. are all ordinary scalars. Thus [*Ausd.* § 143]

$$\begin{aligned} -\mathbf{s}A_1A_2 &= e_1e_2 + e_1'e_2' + f_1f_2 + f_1'f_2' + g_1g_2 + g_1'g_2' \\ &= [A_1 \mid A_2] = [A_2 \mid A_1] \ldots\ldots(7). \end{aligned}$$

We are now in a position to utilise Grassmann's general theorems, but in doing so certain errors easily fallen into must be carefully avoided. The necessity for this caution is due to two connected facts:—(1) a real motor with negative pitch is not a real extensive magnitude of the first order but a simple imaginary one; (2) there are real motors which in our calculus we cannot regard as zero but whose numerical values [*Ausd.* § 151] in Grassmann's sense are zero. These last are rotors and lators.

With regard to (2) it is to be noted that Grassmann always assumes that a magnitude is zero if its numerical value is zero. In applying Grassmann's theorems then we have carefully to observe what parts of his proofs depend on (1) the assumed reality

of his magnitudes, and (2) the assumption that magnitudes with zero numerical values are themselves zero.

28. Some combinatorial products. If $A_1, A_2 \dots A_n$ are any n magnitudes (octonions, scalars, &c.) and if $\phi(A_1, A_2 \dots A_n)$ is such a function of them that if any one, say A_2, is expressed in the form $B_2 + C_2$ we always have

$$\phi(A_1, B_2+C_2, A_3 \dots A_n) = \phi(A_1, B_2, A_3 \dots A_n) + \phi(A_1, C_2, A_3 \dots A_n),$$

$\phi(A_1, A_2 \dots A_n)$ would by Grassmann be called a product of the magnitudes $A_1 \dots A_n$ [*Ausd.* § 44]. If further ϕ is of such a form that when any two of the magnitudes change places ϕ becomes changed to its own negative, the product is called a combinatorial product [*Ausd.* § 55. These are not Grassmann's *definitions*, but are most convenient for our definitions].

Thus if Q_1, Q_2, Q_3 are octonions and A, B, C, D, E motors,

$$Q_1Q_2Q_3 \text{ and } \mathsf{M}AB\mathsf{S}CDE$$

are products, but not in general combinatorial products. Again

$$\mathsf{M}AB,\ \mathsf{M}_1AB,\ \mathsf{S}CDE,\ \mathsf{s}CDE$$

are combinatorial products. Again if ϕ is a general linear motor function of a motor,

$$\mathsf{s}C\phi A\,\mathsf{s}D\phi B - \mathsf{s}D\phi A\,\mathsf{s}C\phi B$$

is a product, but not a combinatorial product in general, of A, B, C and D; it is however a combinatorial product of A and B and again of C and D.

There are thus many kinds of combinatorial products of magnitudes of any assigned kind. When the magnitudes are motors and their number is either five or six there is a particular meaning which can be attached to the combinatorial products which will make them symbolically harmonise with Grassmann's "Produkt in Bezug auf ein Hauptgebiet" [*Ausd.* § 94 *et seq.*] and which he denotes by $[A_1A_2A_3A_4A_5]$ and $[A_1A_2A_3A_4A_5A_6]$. The ideas we attach to the latter are precisely the same as Grassmann's, and those that we attach to the former are essentially the same.

In the first place Grassmann puts the product of the primitive units in an assigned order equal to the scalar unity, i.e. in our case [§ 27 above]

$$[EE'FF'GG'] = 1 \quad \dots\dots\dots\dots (1).$$

For all values of r let

$$\left.\begin{aligned} A_r &= e_r E + e_r' E' + f_r F + f_r' F' + g_r G + g_r' G' \\ &= x_r i + y_r j + z_r k + l_r \Omega i + m_r \Omega j + n_r \Omega k \end{aligned}\right\} \quad \text{.........(2).}$$

Thus [eq. (2) § 27]

$$x_r = (e_r + v e_r')/\sqrt{2},\ l_r = (e_r - v e_r')/\sqrt{2},\ y_r = (f_r + v f_r')/\sqrt{2}, \text{ \&c.}\ldots(3),$$

$$e_r = (x_r + l_r)/\sqrt{2},\ e_r' = v\,(l_r - x_r)/\sqrt{2}, \text{ \&c.}\ldots\ldots\ldots\ldots(4).$$

Grassmann then proves in § 63 that

$$[A_1 A_2 A_3 A_4 A_5 A_6] = \begin{vmatrix} e_1 & e_1' & f_1 & f_1' & g_1 & g_1' \\ e_2 & e_2' & f_2 & f_2' & g_2 & g_2' \\ e_3 & e_3' & f_3 & f_3' & g_3 & g_3' \\ e_4 & e_4' & f_4 & f_4' & g_4 & g_4' \\ e_5 & e_5' & f_5 & f_5' & g_5 & g_5' \\ e_6 & e_6' & f_6 & f_6' & g_6 & g_6' \end{vmatrix} \ldots\ldots\ldots\ldots(5).$$

[It is sufficient for our purposes to take this as the definition of $[A_1 \ldots A_6]$ and to notice that by the definition $[A_1 \ldots A_6]$ is a combinatorial product and that eq. (1) follows from this definition.]

The rules for combinatorial multiplication are given in the *Ausd.* § 52 *et seq.* The most important for our purpose is that to any factor of a combinatorial product we may add a multiple of any other factor without altering the value of the product. Thus

$$\begin{aligned} 8\,[EE'FF'GG'] &= -v\,[(i + \Omega i)(i - \Omega i)(j + \Omega j) \\ &\qquad\qquad (j - \Omega j)(k + \Omega k)(k - \Omega k)] \\ &= 8v\,[i \,.\, \Omega i \,.\, j \,.\, \Omega j \,.\, k \,.\, \Omega k], \end{aligned}$$

so that $$[i \,.\, j \,.\, k \,.\, \Omega i \,.\, \Omega j \,.\, \Omega k] = v \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Hence [*Ausd.* § 63]

$$[A_1 A_2 A_3 A_4 A_5 A_6] = v \begin{vmatrix} x_1 & y_1 & z_1 & l_1 & m_1 & n_1 \\ x_2 & y_2 & z_2 & l_2 & m_2 & n_2 \\ x_3 & y_3 & z_3 & l_3 & m_3 & n_3 \\ x_4 & y_4 & z_4 & l_4 & m_4 & n_4 \\ x_5 & y_5 & z_5 & l_5 & m_5 & n_5 \\ x_6 & y_6 & z_6 & l_6 & m_6 & n_6 \end{vmatrix} \ldots\ldots\ldots\ldots(7).$$

Equations (5) and (7) apparently depend for meaning on the arbitrary point of intersection of i, j, k. That in reality they do not so depend could easily be directly proved from eq. (7). It is unnecessary to give the proof, as the fact follows incidentally from a result we shall prove at the end of this section.

In Grassmann's mode of expression $[A_1A_2A_3A_4A_5]$ would be the complex (Gebiet) of A_1, A_2, A_3, A_4, A_5 [§ 14 above] associated with a numerical coefficient. Similarly $|A$ would be a complex of the fifth order all of whose motors were reciprocal to the motor A. This complex and its numerical coefficient are completely specified by A and therefore in our calculus it is more convenient to regard $|A$ as meaning, not the complex in question, but, the motor A itself. This of course is exactly parallel to the quaternion process of identifying Grassmann's line-vectors ("Strecken") with his surface-vectors (products of two "Strecken") and calling them both vectors.

This leads at once to the only meaning consistent with the *Ausd.* that we can give to $[A_1 \ldots A_5]$. For put

$$[A_1 \ldots A_5] = B,$$

where B is a motor. Then if A_0 be any motor whatever

$$-\mathsf{s}A_0B = -\mathsf{s}A_0 \,|\, B = [A_0 \,\|\, B] = -[A_0B] \ (Ausd. \ \S\ 93)$$

$$= -[A_0A_1A_2A_3A_4A_5]$$

$$= -\begin{vmatrix} e_0 & e_0' & \ldots & g_0' \\ e_1 & e_1' & \ldots & g_1' \\ \ldots & \ldots & \ldots & \ldots \\ e_5 & e_5' & \ldots & g_5' \end{vmatrix} = -v \begin{vmatrix} x_0 & y_0 & \ldots & n_0 \\ x_1 & y_1 & \ldots & n_1 \\ \ldots & \ldots & \ldots & \ldots \\ x_5 & y_5 & \ldots & n_5 \end{vmatrix}$$

$$= \mathsf{s}A_0 \begin{vmatrix} E & E' & \ldots & G' \\ e_1 & e_1' & \ldots & g_1' \\ \ldots & \ldots & \ldots & \ldots \\ e_5 & e_5' & \ldots & g_5' \end{vmatrix} = v\mathsf{s}A_0 \begin{vmatrix} \Omega i & \Omega j & \Omega k & i & j & k \\ x_1 & y_1 & z_1 & l_1 & m_1 & n_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x_5 & y_5 & z_5 & l_5 & m_5 & n_5 \end{vmatrix}$$

Since A_0 is arbitrary it follows from § 14 above that

$$[A_1 \ldots A_5] = -\begin{vmatrix} E & E' & \ldots & G' \\ e_1 & e_1' & \ldots & g_1' \\ \ldots & \ldots & \ldots & \ldots \\ e_5 & e_5' & \ldots & g_5' \end{vmatrix} = -v \begin{vmatrix} \Omega i & \Omega j & \Omega k & i & j & k \\ x_1 & y_1 & z_1 & l_1 & m_1 & n_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x_5 & y_5 & z_5 & l_5 & m_5 & n_5 \end{vmatrix} \ldots (8).$$

[For our purposes these equations may be taken as definitions. I thought it desirable to show what connection $[A_1 \ldots A_5]$ and $[A_1 \ldots A_6]$ had with the *Ausd.*]

Very often the imaginary v recurring in equations (7) and (8) is an inconvenience. We therefore define as follows

$$\{A_1 \ldots A_n\} = -v\,[A_1 \ldots A_n] \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

for the values 5 and 6 of n; and in the definitions about to be given of $\{A_1 \dots A_n\}$ for values from 1 to 4 of n the same equation may be supposed to hold.

Thus

$$\{A_1 \dots A_6\} = \begin{vmatrix} x_1 & y_1 & \dots & n_1 \\ x_2 & y_2 & \dots & n_2 \\ \dots & \dots & \dots & \dots \\ x_6 & y_6 & \dots & n_6 \end{vmatrix} \quad \dots\dots\dots\dots\dots\dots(10),$$

$$\{A_1 \dots A_5\} = -\begin{vmatrix} \Omega i & \Omega j & \dots & k \\ x_1 & y_1 & \dots & n_1 \\ \dots & \dots & \dots & \dots \\ x_5 & y_5 & \dots & n_5 \end{vmatrix} \quad \dots\dots\dots\dots\dots(11).$$

From these

$$\{A_1 \dots A_6\} = \mathbf{s}A_1\,\{A_2 \dots A_6\} \dots\dots\dots\dots\dots(12),$$

$$\{i\,.\,j\,.\,k\,.\,\Omega i\,.\,\Omega j\,.\,\Omega k\} = 1 \dots\dots\dots\dots\dots\dots(13),$$

$$\left.\begin{aligned} \{j\,.\,k\,.\,\Omega i\,.\,\Omega j\,.\,\Omega k\} &= -\,\Omega i \\ \{k\,.\,i\,.\,\Omega i\,.\,\Omega j\,.\,\Omega k\} &= -\,\Omega j \\ \{i\,.\,j\,.\,\Omega i\,.\,\Omega j\,.\,\Omega k\} &= -\,\Omega k \\ \{i\,.\,j\,.\,k\,.\,\Omega j\,.\,\Omega k\} &= i \\ \{i\,.\,j\,.\,k\,.\,\Omega k\,.\,\Omega i\} &= j \\ \{i\,.\,j\,.\,k\,.\,\Omega i\,.\,\Omega j\} &= k \end{aligned}\right\} \quad \dots\dots\dots\dots(14).$$

$\{A_1 \dots A_5\}$ may be regarded as a linear motor function of A_5; also as a motor function of A_4, A_5 linear in each; and so on. Thus

$$\left.\begin{aligned} \{A_1 \dots A_5\} &= \phi_4 A_5 = \phi_3(A_4,\, A_5) = \phi_2(A_3,\, A_4,\, A_5) \\ &= \phi_1(A_2,\, A_3,\, A_4,\, A_5) = \phi_0(A_1,\, A_2,\, A_3,\, A_4,\, A_5) \end{aligned}\right\} \dots\dots(15).$$

With these meanings of $\phi_1 \dots \phi_4$ we may put from $n = 1$ to $n = 4$,

$$\phi_n = \{A_1 \dots A_n\} \dots\dots\dots\dots\dots\dots\dots\dots(16),$$

and

$$\phi_0 = \{1\} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(17).$$

Thus $\{A_1 \dots A_n\}$ is for all values of n from 0 to 6 a combinatorial product of A_1, $A_2 \dots A_n$.

It is to be noticed that with the meanings of ψ and Z given in equation (6) § 15 above,

$$\left.\begin{aligned} \psi(Z,Z) = [-\,\psi(A_1,\, \{A_2 A_3 \dots A_6\}) + \psi(A_2,\, \{A_1 A_3 \dots A_5\}) - \dots \\ + \psi(A_6,\, \{A_1 A_2 \dots A_5\})]/\{A_1 A_2 \dots A_6\} \end{aligned}\right\} (18),$$

where A_1, $A_2 \dots A_6$ are any six independent motors. For if we change A_1 into *any* other motor $x_1 A_1 + x_2 A_2 + \dots + x_6 A_6$ (x_1 not

zero) independent of A_2, $A_3 \dots A_6$ the expression on the right is unaltered. Similarly for the other motors involved. We may therefore change $A_1 \dots A_6$ to i, j, k, Ωi, Ωj, Ωk, whereupon the right becomes $\psi(i, \Omega i) + \psi(\Omega i, i) + \dots$ or $\psi(Z, Z)$.

Since [eq. (8) § 15 above] $E = -Z\mathbf{s}EZ$ it follows from eq. (18) that

$$\left.\begin{aligned} E\{A_1 \dots A_6\} &= A_1\{EA_2 \dots A_6\} + A_2\{A_1EA_3 \dots A_6\} + \dots \\ &\qquad + A_6\{A_1 \dots A_5E\} \\ &= \{A_2A_3 \dots A_6\}\,\mathbf{s}EA_1 - \{A_1A_3 \dots A_6\}\,\mathbf{s}EA_2 + \dots \\ &\qquad - \{A_1A_2 \dots A_5\}\,\mathbf{s}EA_6 \end{aligned}\right\} \text{(19).}$$

Define $\bar{A}_1$, $\bar{A}_2 \dots \bar{A}_6$ by the equations

$$\bar{A}_1 = -\frac{\{A_2A_3 \dots A_6\}}{\{A_1A_2 \dots A_6\}},\ \bar{A}_2 = \frac{\{A_1A_3 \dots A_6\}}{\{A_1A_2 \dots A_6\}} \dots \bar{A}_6 = \frac{\{A_1A_2 \dots A_5\}}{\{A_1A_2 \dots A_6\}} \text{(20).}$$

Eq. (19) then gives

$$\left.\begin{aligned} E &= -A_1\mathbf{s}E\bar{A}_1 - A_2\mathbf{s}E\bar{A}_2 - \dots - A_6\mathbf{s}E\bar{A}_6 \\ &= -\bar{A}_1\mathbf{s}EA_1 - \bar{A}_2\mathbf{s}EA_2 - \dots - \bar{A}_6\mathbf{s}EA_6 \end{aligned}\right\} \dots\dots\dots\text{(21).}$$

Also by eq. (19)

$$E\{\bar{A}_1 \dots \bar{A}_6\} = \bar{A}_1\{E\bar{A}_2 \dots \bar{A}_6\} + \bar{A}_2\{\bar{A}_1E\bar{A}_3 \dots \bar{A}_6\} + \dots$$

Comparing this with the last equation we see by eq. (12) and § 14, that

$$A_1 = -\frac{\{\bar{A}_2\bar{A}_3 \dots \bar{A}_6\}}{\{\bar{A}_1\bar{A}_2 \dots \bar{A}_6\}},\ A_2 = \frac{\{\bar{A}_1\bar{A}_3 \dots \bar{A}_6\}}{\{\bar{A}_1\bar{A}_2 \dots \bar{A}_6\}}, \dots\dots\dots\text{(22),}$$

so that the relations between the two sets $A_1A_2 \dots A_6$ and $\bar{A}_1\bar{A}_2 \dots \bar{A}_6$ are symmetrical.

Again by eq. (12)

$$\mathbf{s}A_1\bar{A}_1 = \mathbf{s}A_2\bar{A}_2 = \mathbf{s}A_3\bar{A}_3 = \mathbf{s}A_4\bar{A}_4 = \mathbf{s}A_5\bar{A}_5 = \mathbf{s}A_6\bar{A}_6 = -1 \dots\text{(23).}$$

Again since $\{A_1A_1A_3 \dots A_6\} = 0$, $\mathbf{s}A_1\bar{A}_2 = 0$. Thus

$$\mathbf{s}A_1\bar{A}_2 = \mathbf{s}A_1\bar{A}_3 = \mathbf{s}A_1\bar{A}_4 = \mathbf{s}A_2\bar{A}_3 = \dots = 0 \quad \dots\dots\dots\text{(24).}$$

A_1, $A_2 \dots A_6$ have been assumed independent. [Otherwise $\{A_1 \dots A_6\} = 0$.] $\bar{A}_1$, $\bar{A}_2 \dots \bar{A}_6$ are also independent since any motor [eq. (21)] can be expressed in terms of them.

In § 14 we stated what was meant by reciprocal complexes. By equations (24) the complex $\bar{A}_{n+1} \dots \bar{A}_6$ is reciprocal to the complex $A_1 \dots A_n$. Moreover no motor which does not belong to the former complex is reciprocal to the latter. For any motor

can by eq. (21) be put in the form $E=\sum_1^6 x\bar{A}$, and expressing that this is reciprocal to each of the motors $A_1, A_2 \ldots A_n$ we get by equations (23) and (24) $x_1 = x_2 = \ldots = x_n = 0$, so that E belongs to the complex $\bar{A}_{n+1} \ldots \bar{A}_6$. Hence *to every complex of order n there is a reciprocal complex of order $6-n$ and no motor not belonging to the latter is reciprocal to the former.* In particular to every complex of order five there is one (and ordinary scalar multiples of it) and only one motor reciprocal.

We have just seen that $\bar{A}_1 \ldots \bar{A}_6$ are independent. Hence *the six motors which are reciprocal to each set of five out of six given independent motors are themselves independent.* More generally *if (n) be a complex of order n and $(6-n)$ an independent complex of order $(6-n)$, then if $(\overline{6-n})$ is the complex reciprocal to (n), and $(\bar{n})$ the complex reciprocal to $(6-n)$, $(\bar{n})$ and $(\overline{6-n})$ are independent complexes.* For (n) may be taken as $A_1 \ldots A_n$ and $(6-n)$ as $A_{n+1} \ldots A_6$. Then $(\bar{n})$ is $\bar{A}_1 \ldots \bar{A}_n$ and $(\overline{6-n})$ is $\bar{A}_{n+1} \ldots \bar{A}_6$.

If a motor is reciprocal to five out of six independent motors it is not reciprocal to the sixth. For $\mathbf{s}A_1\bar{A}_1$ is not zero.

These are well-known results or easily deducible from such results, but they serve the double purpose of exemplifying the present methods and of showing the physical meaning of the connections between the two sets of motors $A_1 \ldots A_6$ and $\bar{A}_1 \ldots \bar{A}_6$. [I am not sure that the second and third italicised statements are known.]

If $A_1, A_2 \ldots A_6$ are six independent coreciprocal motors and

$$E = x_1 A_1 + \ldots + x_6 A_6,$$

we have by operating by $\mathbf{s}A_1(\quad)$, $\mathbf{s}EA_1 = x_1 \mathbf{s}A_1^2$. Hence

$$E = A_1 \mathbf{s}EA_1/\mathbf{s}A_1^2 + A_2 \mathbf{s}EA_2/\mathbf{s}A_2^2 + \ldots + A_6 \mathbf{s}EA_6/\mathbf{s}A_6^2 \ldots (25).$$

Comparing this with eq. (21) we have

$$\mathbf{s}EA_1/\mathbf{s}A_1^2 = -\mathbf{s}E\bar{A}_1.$$

Hence in this case by § 14

$$A_1/\mathbf{s}A_1^2 = -\bar{A}_1 = \{A_2 A_3 \ldots A_6\}/\{A_1 A_2 \ldots A_6\} \ldots\ldots (26).$$

A relation that we shall require later is the following

$$\{ABC\Omega A'\Omega B'\Omega C'\} = \mathbf{S}_1 ABC \mathbf{S}_1 A'B'C' \ldots\ldots\ldots\ldots (27),$$

where A, B, C, A', B', C' are any six motors. The right vanishes if and only if either A, B, C or A', B', C' are not three completely independent axial motors. If A, B, C are completely independent axial motors and A', B', C' also are, the six motors on the left are independent and therefore the left expression does not vanish. If A', B', C' are not completely independent axial motors the lators on the left are not independent and therefore the expression on the left vanishes. If A, B, C are not completely independent axial motors one motor at least of the complex A, B, C is a lator, so that the six motors on the left are not independent (since four lators are never independent) and therefore the expression on the left vanishes. Hence if one of the two expressions of eq. (27) vanishes, the other does. If neither vanishes A, B, C are completely independent axial motors and $\Omega A'$, $\Omega B'$, $\Omega C'$ are independent lators. Hence $xA + yB + zC + l\Omega A' + m\Omega B' + n\Omega C'$ when x is not zero is any axial motor completely independent of B and C. If this be substituted for A both expressions are altered in the ratio of x to 1. Similarly if $\Omega A'$ be changed to any other lator $x\Omega A' + y\Omega B' + z\Omega C'$ (x not zero) independent of $\Omega B'$ and $\Omega C'$, the expressions are both altered in the ratio of x to 1. It follows that $\{ABC\Omega A'\Omega B'\Omega C'\}/\mathbf{S}_1 ABC\mathbf{S}_1 A'B'C'$ has the same value for any six motors A, B, C, A', B', C' of which the first three are completely independent axial motors and of which the last three are also. Changing A, B, C to i, j, k and A', B', C' also to i, j, k, we see by eq. (13) that this value is unity.

Since

$$\mathbf{S}_1 ABC\mathbf{S}_1 A'B'C' = \mathbf{s}A\,(\Omega\mathbf{M}BC)\,\mathbf{S}_1 A'B'C' = \mathbf{s}\,.\,(\Omega A')\,\mathbf{M}B'C'\mathbf{S}_1 ABC,$$

we see by equations (27) and (12) and § 14 that

$$\{BC\Omega A'\Omega B'\Omega C'\} = \Omega\mathbf{M}BC\mathbf{S}_1 A'B'C' \qquad\text{.............(28)},$$

and $$\{ABC\Omega B'\Omega C'\} = -\,\mathbf{M}B'C'\mathbf{S}_1 ABC + \text{a lator}.$$

If $\mathbf{S}_1 ABC$ is not zero this lator may be put in the form $\Omega\,(x\mathbf{M}BC + y\mathbf{M}CA + z\mathbf{M}AB)$. Operating on the last equation by $\mathbf{s}A\,(\quad)$ we get $x = \mathbf{s}AB'C'$ and similarly for y and z. Hence

$$\begin{aligned}\{ABC\Omega B'\Omega C'\} = {} & -\,\mathbf{M}B'C'\mathbf{S}_1 ABC \\ & + \Omega\,(\mathbf{M}BC\mathbf{s}AB'C' + \mathbf{M}CA\mathbf{s}BB'C' + \mathbf{M}AB\mathbf{s}CB'C') \ldots (29),\end{aligned}$$

when $\mathbf{S}_1 ABC$ is not zero. This equation is also true when $\mathbf{S}_1 ABC$ is zero, for then

$$xA + yB + zC = \text{a lator} = \Omega A',$$

say, where one at least (say x) of the ordinary scalars x, y, z, is not zero. Operating on the last equation by $\Omega\mathbf{M}C(\quad)$ we get

$$\Omega\mathbf{M}CA = \frac{y}{x}\,\Omega\mathbf{M}BC,$$

and similarly
$$\Omega\mathbf{M}AB = \frac{z}{x}\,\Omega\mathbf{M}BC.$$

Thus

$$\begin{aligned}\{ABC\Omega B'\Omega C'\} &= -x^{-1}\{BC\Omega A'\Omega B'\Omega C'\}\\ &= -x^{-1}\,\Omega\mathbf{M}BC\mathbf{S}_1A'B'C' \text{ [eq. (28)]}\\ &= -x^{-1}\,\Omega\mathbf{M}BC\mathbf{s}(xA+yB+zC)B'C'\\ &= -\Omega\Big\{\mathbf{M}BC\mathbf{s}AB'C' + \frac{y}{x}\mathbf{M}BC\mathbf{s}BB'C'\\ &\qquad\qquad + \frac{z}{x}\mathbf{M}BC\mathbf{s}CB'C'\Big\}\\ &= -\Omega\{\mathbf{M}BC\mathbf{s}AB'C' + \mathbf{M}CA\mathbf{s}BB'C' + \mathbf{M}AB\mathbf{s}CB'C'\},\end{aligned}$$

which proves eq. (29) when $\mathbf{S}_1ABC = 0$.

Eq. (29) may by eq. (8) § 13 be put in the form

$$\begin{aligned}\{ABC\Omega B'\Omega C'\} = \mathbf{M}B'C'\mathbf{S}_2ABC - \mathbf{M}BC\mathbf{S}_1AB'C'&\\ -\mathbf{M}CA\mathbf{S}_1BB'C' - \mathbf{M}AB\mathbf{S}_1CB'C'&\text{.........(30).}\end{aligned}$$

Putting in equations (29) and (30) $B' = B$, $C' = C$ we get

$$\{ABC\Omega B\Omega C\} = \mathbf{M}BC(-\mathbf{S}_1 + \mathbf{S}_2)ABC\text{.........(31).}$$

It will be observed that equations (28) to (31) are generalisations of equations (13) and (14). Also eq. (28) is a particular case of eq. (29) as we see by changing A of eq. (29) into $\Omega A'$.

These results are rather more general than the following which are also instructive. By eq. (8) § 13 if E is any motor and A, B, C are three given completely independent axial motors

$$\begin{aligned}E &= A\mathbf{S}EBC\mathbf{S}^{-1}ABC + \dots\\ &= A\{\mathbf{s}E(\Omega BC\mathbf{S}^{-1}ABC) + \Omega\mathbf{s}E(BC\mathbf{S}^{-1}ABC)\} + \dots.\end{aligned}$$

Hence if we take A, B, C, ΩA, ΩB, ΩC for the A_1, $A_2 \dots A_6$ of equations (20) to (24)

$$\bar{A} = -\Omega\mathbf{M}BC\mathbf{S}^{-1}ABC,\ \overline{\Omega A} = -\mathbf{M}BC\mathbf{S}^{-1}ABC\text{......(32),}$$

and similarly for $\bar{B}$, $\overline{\Omega B}$, $\bar{C}$, $\overline{\Omega C}$. [Notice that what was denoted in § 16 by $\bar{A}$ is here therefore denoted by $\overline{\Omega A}$.] The second of

equations (32) easily leads to eq. (31) and the first to the particular case of eq. (28) obtained by putting $B' = B$, $C' = C$.

Since $\quad E = -A\mathrm{s}E\bar{A} - \Omega A\mathrm{s}E\overline{\Omega A} - B\mathrm{s}E\bar{B} - \ldots,$

we see that

$$\{ABCB'C'\} = \{ABC\Omega\,(A\mathrm{s}B'\overline{\Omega A} + B\mathrm{s}B'\overline{\Omega B} + C\mathrm{s}B'\overline{\Omega C})$$
$$\Omega\,(A\mathrm{s}C'\overline{\Omega A} + B\mathrm{s}C'\overline{\Omega B} + C\mathrm{s}C'\overline{\Omega C})\}.$$

Equation (30) may now be utilised and we shall then have expressed $\{ABCB'C'\}$ in a manner independent of any arbitrary origin. Similarly for $\{ABCA'B'C'\}$. The expressions are however too cumbrous to be of much use.

29. Combinatorial, linear, circular and hyperbolic variation. In § 71 of *Ausd.* Grassmann explains what he means by a simple and a multiple linear variation. In § 154 he explains what he means by a simple and a multiple and also a positive and negative circular variation. In § 391 he somewhat extends the latter. The modification that we thus get for real motors will be called a (simple or multiple, positive or negative) hyperbolic variation.

These are all particular cases of a more general kind of variation which will be called a combinatorial variation (simple or multiple, positive or negative).

Any group of motors $A_1 \ldots A_p \ldots A_q \ldots A_n$ is said to be changed by simple combinatorial variation to the group $A_1 \ldots A_p' \ldots A_q' \ldots A_n$ (the only two motors changed being A_p and A_q) if

$$\left.\begin{aligned} A_p' &= cA_p + sA_q \\ A_q' &= s'A_p + cA_q \\ c^2 - ss' &= 1 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If further one of the motors A_p', A_q' have its sign changed this combined with the former is said to constitute a simple negative combinatorial variation. [By the latter variation A_p, A_q are changed to A_p', A_q' where $A_p' = cA_p + sA_q$, $A_q' = -s'A_p - cA_q$, $c^2 - ss' = 1$.] A series of combinatorial variations performed on the group are said to constitute a multiple combinatorial variation.

[So far as the fundamental property of combinatorial variation in connection with a combinatorial product is concerned (see

below) we might define it more generally. The above definition is the most convenient for our purposes. The more general definition is given by the equations

$$A_p' = x_1 A_p + y_1 A_q,\ A_q' = x_2 A_p + y_2 A_q,\ x_1 y_2 - x_2 y_1 = 1.]$$

When $c = 1$ (and therefore s or s' is zero) the variation is called a linear variation.

When $s' = -s$ the variation is called a circular variation. In this case we may put

$$c = \cos\theta,\ s = -s' = \sin\theta \qquad \text{...(2)}.$$

When $s' = s$ the variation is called a hyperbolic variation. In this case we may put

$$c = \cosh\theta,\ s = s' = \sinh\theta \qquad \text{...(3)}.$$

From eq. (1) we have

$$A_p = cA_p' - sA_q',\ A_q = -s'A_p' + cA_q' \qquad \text{...(4)}.$$

Thus
$$\frac{A_p',\ -A_q'}{A_p,\ A_q} = \frac{A_p,\ A_q}{A_p',\ -A_q'} \qquad \text{...(5)},$$

which in Grassmann's notation (*Ausd.* § 377) expresses the fact that A_p', $-A_q'$ are the same linear functions of A_p, A_q that the latter are of the former.

From this we see that if A, B be changed by *negative* combinatorial variation to A', B'

$$\frac{A',\ B'}{A,\ B} = \frac{A,\ B}{A',\ B'} \qquad \text{...(6)},$$

that is to say that if we put $A' = \phi A$, $B' = \phi B$; then $A = \phi A'$, $B = \phi B'$. This can be made the basis of the definition of combinatorial variation. On account of the relation (6) some few properties of negative variation are simpler than the corresponding properties of positive variation.

Since $c^2 - (-s)(-s') = 1$ we see from eq. (4) that A_p, A_q are obtained from A_p', A_q' by a simple positive combinatorial variation. And further, if the given variation (A_p, A_q into A_p', A_q') is linear, circular or hyperbolic, the derived variation (A_p', A_q' into A_p, A_q) is linear, circular or hyperbolic respectively. In the cases of circular and hyperbolic variations the derived variation is obtained from the given one by changing the θ of equations (2), (3) to $-\theta$.

The fundamental property of a combinatorial variation, on which all its usefulness may be said to depend, is that :—

Any combinatorial product of A_1, $A_2 \ldots A_n$ *is unaltered by combinatorial variation.*

In § 156 of *Ausd.* this is proved for the case of circular variation applied to the magnitudes of a "normal system." It is just as easy to prove it in general. Let the product be denoted by $(A_1 \ldots A_p \ldots A_q \ldots A_n)$. Then

$$(A_1 \ldots A_p' \ldots A_q' \ldots A_n) = (A_1 \ldots (cA_p + sA_q) \ldots (s'A_p + cA_q) \ldots A_n)$$

$$= (A_1 \ldots \{(cA_p + sA_q) - \frac{s}{c}(s'A_p + cA_q)\} \ldots (s'A_p + cA_q) \ldots A_n)$$

$$= (A_1 \ldots \frac{A_p}{c} \ldots (s'A_p + cA_q) \ldots A_n)$$

$$= (A_1 \ldots \frac{A_p}{c} \ldots cA_q \ldots A_n)$$

$$= (A_1 \ldots A_p \ldots A_q \ldots A_n).$$

[Here it has been assumed that c is not zero. The reader can easily supply the proof for the very simple case when c is zero.]

The proposition is therefore true for a simple variation. It follows at once for a multiple variation.

In particular if A_1, A_2 be combinatorially varied to A_1', A_2',

$$\mathbf{M}A_1A_2 = \mathbf{M}A_1'A_2'\,;$$

if $\quad A_1, A_2, A_3$ to A_1', A_2', A_3', $\mathbf{S}A_1A_2A_3 = \mathbf{S}A_1'A_2'A_3'$;

if $\quad E, F$ to E', F' and A, B to A', B',

$$\left.\begin{aligned}\mathbf{s}E\phi A\mathbf{s}F\phi B - \mathbf{s}F\phi A\mathbf{s}E\phi B &= \mathbf{s}E\phi A'\mathbf{s}F\phi B' - \mathbf{s}F\phi A'\mathbf{s}E\phi B' \\ &= \mathbf{s}E'\phi A'\mathbf{s}F'\phi B' - \mathbf{s}F'\phi A'\mathbf{s}E'\phi B'\end{aligned}\right\}\ldots(7),$$

where ϕ is a general linear motor function of a motor. A particular case of eq. (7) is extremely useful. Put $E = A$, $F = B$, $E' = A'$, $F' = B'$, and $\phi = \varpi$ where ϖ is self-conjugate. Then

$$\mathbf{s}A'\varpi A'\mathbf{s}B'\varpi B' - \mathbf{s}^2A'\varpi B' = \mathbf{s}A\varpi A\mathbf{s}B\varpi B - \mathbf{s}^2A\varpi B \ldots(8).$$

A still further restricted case is obtained by putting $\varpi = 1$ when we have

$$\mathbf{s}A'^2\mathbf{s}B'^2 - \mathbf{s}^2A'B' = \mathbf{s}A^2\mathbf{s}B^2 - \mathbf{s}^2AB \ldots\ldots\ldots\ldots\ldots(9).$$

The equivalent of this last (for any two magnitudes of the first order) is proved in § 391 of *Ausd.* for the cases of circular and hyperbolic variation. Grassmann's use of the equation is analogous

to our use below of eq. (8). Needless to say the use below was suggested by his.

In equations (7), (8), (9) we may put $\mathbf{S}$ or $\mathbf{S}_1$ for each $\mathbf{s}$. ϖ having the meaning just given to it we have from eq. (1)

$$\left.\begin{aligned}\mathbf{S}A'\varpi A' &= c^2\mathbf{S}A\varpi A + 2cs\mathbf{S}A\varpi B + s^2\mathbf{S}B\varpi B\\ \mathbf{S}B'\varpi B' &= s'^2\mathbf{S}A\varpi A + 2cs'\mathbf{S}A\varpi B + c^2\mathbf{S}B\varpi B\end{aligned}\right\}\text{.........(10)},$$

from which we deduce that

$$s'\mathbf{S}A'\varpi A' - s\mathbf{S}B'\varpi B' = s'\mathbf{S}A\varpi A - s\mathbf{S}B\varpi B \text{ ......(11)}.$$

Particular cases are

(circular variation) $\mathbf{S}A'\varpi A' + \mathbf{S}B'\varpi B' = \mathbf{S}A\varpi A + \mathbf{S}B\varpi B$...(12),
(hyperbolic variation) $\mathbf{S}A'\varpi A' - \mathbf{S}B'\varpi B' = \mathbf{S}A\varpi A - \mathbf{S}B\varpi B$...(13).

Putting $\varpi = 1$ we have in place of eq. (11)

$$s'A'^2 - sB'^2 = s'A^2 - sB^2 \text{ ....(14)}.$$

Similar deductions may be made from the other equations.

Again it may be noticed that if we put

$$\left.\begin{aligned}&\mathbf{S}A\varpi A + \frac{s}{s'}\mathbf{S}B\varpi B = 2b_1, \quad \mathbf{S}A\varpi B = b_2\\ &\mathbf{S}A'\varpi A' + \frac{s}{s'}\mathbf{S}B'\varpi B' = 2b_1', \ \mathbf{S}A'\varpi B' = b_2'\\ &C = c^2 + ss', \ S = 2cs, \ S' = 2cs'\end{aligned}\right\}\text{ ......(15)},$$

we have

$$b_1' = Cb_1 + Sb_2, \ b_2' = S'b_1 + Cb_2, \ C^2 - SS' = 1\text{......(16)},$$

so that b_1', b_2' are obtained from b_1, b_2 by a combinatorial variation. In particular, for circular variation,

$$\left.\begin{aligned}&\tfrac{1}{2}(\mathbf{S}A'\varpi A' - \mathbf{S}B'\varpi B')\\ &\qquad = \tfrac{1}{2}(\mathbf{S}A\varpi A - \mathbf{S}B\varpi B)\cos 2\theta + \mathbf{S}A\varpi B \sin 2\theta\\ &\mathbf{S}A'\varpi B' = -\tfrac{1}{2}(\mathbf{S}A\varpi A - \mathbf{S}B\varpi B)\sin 2\theta + \mathbf{S}A\varpi B\cos 2\theta\end{aligned}\right\}\text{ ...(17)},$$

and for hyperbolic variation

$$\left.\begin{aligned}&\tfrac{1}{2}(\mathbf{S}A'\varpi A' + \mathbf{S}B'\varpi B')\\ &\qquad = \tfrac{1}{2}(\mathbf{S}A\varpi A + \mathbf{S}B\varpi B)\cosh 2\theta + \mathbf{S}A\varpi B \sinh 2\theta\\ &\mathbf{S}A'\varpi B' = \tfrac{1}{2}(\mathbf{S}A\varpi A + \mathbf{S}B\varpi B)\sinh 2\theta + \mathbf{S}A\varpi B\cosh 2\theta\end{aligned}\right\}\text{ ...(18)}.$$

Since $\mathbf{S}Q = \mathbf{S}_1 Q + \Omega\mathbf{s}Q$, where Q is any octonion, we may in equations (10) to (18) write either $\mathbf{S}_1$ or $\mathbf{s}$ in place of $\mathbf{S}$.

Suppose $A_1, A_2 \dots A_6$ are six independent motors and $\overline{A}_1$, $\overline{A}_2 \dots \overline{A}_6$ have the meanings defined in § 28. $A_1', A_2' \dots A_6'$ will be defined in the enunciation about to be given. Let $\overline{A}_1', \overline{A}_2' \dots \overline{A}_6'$

have the same relations with $A_1, A_2 \dots A_6$ that $\bar{A}_1, \bar{A}_2 \dots \bar{A}_6$ have with $A_1, A_2 \dots A_6$. We proceed to show that:—

If by a series of negative circular variations $A_1, \dots A_6$ *become transformed to* $A_1', \dots A_6'$, *then* $\bar{A}_1', \dots \bar{A}_6'$ *will be exactly the same linear functions of* $\bar{A}_1, \dots \bar{A}_6$ *that* $A_1', \dots A_6'$ *are of* $A_1, \dots A_6$; *or in Grassmann's notation*

$$\frac{\bar{A}_1', \bar{A}_2', \bar{A}_3', \bar{A}_4', \bar{A}_5', \bar{A}_6'}{\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4, \bar{A}_5, \bar{A}_6} = \frac{A_1', A_2', A_3', A_4', A_5', A_6'}{A_1, A_2, A_3, A_4, A_5, A_6} \quad \dots(19).$$

It is sufficient to prove the theorem for a simple negative circular variation. Suppose then

$$\left.\begin{array}{r} A_3' = A_3,\ A_4' = A_4,\ A_5' = A_5,\ A_6' = A_6 \\ A_1' = cA_1 + sA_2,\ A_2' = -s'A_1 - cA_2,\ c^2 - ss' = 1 \end{array}\right\} \quad \dots(20).$$

[We at first take the variation as any negative combinatorial variation as, though the above theorem does not then hold good, certain very simple formulae hold for the general *simple* negative combinatorial variation.]

A combinatorial product which involves both A_1 and A_2 has its sign changed, but is otherwise unaltered by this negative variation. Hence

$$\bar{A}_3' = \frac{-\{A_1'A_2'A_4'A_5'A_6'\}}{\{A_1'A_2'A_3'A_4'A_5'A_6'\}} = \frac{\{A_1A_2A_4A_5A_6\}}{-\{A_1A_2A_3A_4A_5A_6\}} = \bar{A}_3,$$

and similarly for $\bar{A}_4', \bar{A}_5', \bar{A}_6'$. Again we have

$$\bar{A}_1' = \frac{-\{A_2'A_3' \dots A_6'\}}{\{A_1'A_2' \dots A_6'\}} = \frac{\{(s'A_1 + cA_2)A_3 \dots A_6\}}{-\{A_1A_2 \dots A_6\}} = c\bar{A}_1 - s'\bar{A}_2,$$

$$\bar{A}_2' = \frac{\{A_1'A_3' \dots A_6'\}}{\{A_1'A_2' \dots A_6'\}} = \frac{\{(cA_1 + sA_2)A_3 \dots A_6\}}{-\{A_1A_2 \dots A_6\}} = s\bar{A}_1 - c\bar{A}_2.$$

Thus

$$\left.\begin{array}{r} \bar{A}_3' = \bar{A}_3,\ \bar{A}_4' = \bar{A}_4,\ \bar{A}_5' = \bar{A}_5,\ \bar{A}_6' = \bar{A}_6 \\ \bar{A}_1' = c\bar{A}_1 - s'\bar{A}_2,\ \bar{A}_2' = s\bar{A}_1 - c\bar{A}_2,\ c^2 - (-s)(-s') = 1 \end{array}\right\} \quad \dots(21),$$

which expresses the connections between $\bar{A}_1', \dots \bar{A}_6'$ and $\bar{A}_1, \dots \bar{A}_6$ in the case of any simple negative combinatorial variation. We see that the former are obtained from the latter by a similar (linear, circular or hyperbolic) negative variation. Also by eq. (6) $\bar{A}_1 = c\bar{A}_1' - s'\bar{A}_2'$, $\bar{A}_2 = s\bar{A}_1' - c\bar{A}_2'$ express the converse relations.

In the case of circular variation $s' = -s$. Hence in this case $\bar{A}_1', \dots \bar{A}_6'$ are the same functions of $\bar{A}_1, \dots \bar{A}_6$ as $A_1', \dots A_6'$ are of

$A_1, \ldots A_6$. The theorem can be easily extended from simple to multiple variation.

30. Conjugacy with regard to a general self-conjugate function. Let ϖ be a real general self-conjugate linear motor function of a motor. We are about to establish certain properties of ϖ. Two particular cases will frequently be considered. (1) ϖ may be put equal to unity; conjugacy then reduces to reciprocity. (2) ϖ may be an energy function as we shall term it because of its intimate connection with the energy (kinetic and potential) of a rigid body. An energy function may be complete or partial. ϖ is a complete energy function when $\mathbf{s}E\varpi E$ is negative and not zero for every motor E in space. It is a partial energy function when $\mathbf{s}E\varpi E$ is zero for some values of E and is negative for all other values. We shall usually write ψ for ϖ when we restrict it to being an energy function.

When $\mathbf{s}E\varpi E = 0$ we shall find later that E belongs to what Sir Robert Ball (*Screws*, § 90) calls a complex of the fifth order and second degree. If E be restricted by this equation and further restricted by belonging to a complex (as defined in § 14 above) of the nth order it would be said to belong to a complex of the $(n-1)$th order and second degree (*Screws*, § 158). The theorems we are about to establish can all be expressed as theorems relating to such a complex of the second degree.

Two motors E and F are said to be conjugate with regard to ϖ when $\mathbf{s}E\varpi F = 0$. n motors are said to be conjugate when every pair of them is a conjugate pair.

Defining a complex of order n as in § 14 above, we proceed to show that:—

In any complex of order n, n real independent conjugate motors can be found.

Let $A_1 \ldots A_n$ be any n independent motors of the complex. If any pair is not already a conjugate pair it can be made conjugate by a simple circular variation. Let A_1 and A_2 be not conjugate. Circularly vary them to A_1', A_2' so that

$$A_1' = A_1 \cos\theta + A_2 \sin\theta,$$

$$A_2' = -A_1 \sin\theta + A_2 \cos\theta.$$

Thus

$$\mathbf{s}A_1'\varpi A_2' = \mathbf{s}A_1\varpi A_2 \cos 2\theta - (\mathbf{s}A_1\varpi A_1 - \mathbf{s}A_2\varpi A_2)\sin\theta\cos\theta,$$

so that A_1', A_2' will be conjugate if

$$\tan 2\theta = \frac{2\mathbf{s}A_1\varpi A_2}{\mathbf{s}A_1\varpi A_1 - \mathbf{s}A_2\varpi A_2},$$

and θ can always be determined so as to satisfy this equation.

When any such pair is thus made conjugate the product

$$\mathbf{s}A_1\varpi A_1\mathbf{s}A_2\varpi A_2 \ldots \mathbf{s}A_n\varpi A_n$$

is algebraically diminished. For the only factors of this product that are altered by the variation are the two involving the varied motors. And the product of these two factors ($\mathbf{s}A_1\varpi A_1$ and $\mathbf{s}A_2\varpi A_2$) is by eq. (8) § 29 diminished by the amount $\mathbf{s}^2A_1\varpi A_2$ which by hypothesis is not zero and since ϖ is real is not negative. There is an exception to this statement, viz. when one of the factors, not involving a varied motor, say $\mathbf{s}A_3\varpi A_3$ is zero; A_3 is then self-conjugate. The product then remains zero. But unless A_3 is conjugate not only to itself but to all the other motors it can be varied with one of them to which it is not conjugate, whereupon the product diminishes to less than zero by what has just been proved. If A_3 is self-conjugate and conjugate to all the other $n-1$ motors it is conjugate to the whole complex. In this case A_3 does not require to be varied. We now see that the following statement is true:—

If any pair of $A_1 \ldots A_n$ is not conjugate we can by a circular variation make it conjugate and the product $\mathbf{s}A_1\varpi A_1\mathbf{s}A_2\varpi A_2\ldots$ involving all the motors which are not conjugate to the whole complex thereby diminishes.

This last product then has a minimum value, and this value is only attained when the n motors are conjugate. Hence n such real motors exist. Moreover if any one of such a set of conjugate motors is self-conjugate it is conjugate to the whole complex.

There are in general an infinite number of such conjugate sets of n motors. It is easy to prove however that the complex consisting of the self-conjugate motors is a definite complex. In other words:—

If $A_1 \ldots A_l$, $B_{l+1} \ldots B_n$ are one set of conjugate motors and $A_1' \ldots A_m'$, $B'_{m+1} \ldots B_n'$ are another set, none of the A's being

self-conjugate but all of the B's being self-conjugate, then the complex $B_{l+1} \dots B_n$ *is the same as the complex* $B'_{m+1} \dots B_n'$. [Thus in particular $m = l$.] For while every motor of the complex $B_{l+1} \dots B_n$ is conjugate to the whole complex, no motor of the complex $A_1 \dots B_n$ which does not belong to the complex $B_{l+1} \dots B_n$ is conjugate to the whole complex. Suppose that

$$x_1 A_1 + \dots + y_{l+1} B_{l+1} + \dots = \Sigma xA + \Sigma yB$$

is such a motor conjugate to the whole complex. Operating by $\mathbf{s}A_1\varpi(\ )$, remembering that $\mathbf{s}A_1\varpi A_1$ is not zero since A_1 is not self-conjugate and that A_1 is conjugate to $A_2 \dots A_l B_{l+1} \dots B_n$, we see that $x_1 = 0$. Similarly $x_2 = \dots = x_l = 0$ or the motor belongs to the B complex. It follows that every motor of the complex $B'_{m+1} \dots B_n'$ belongs to the complex $B_{l+1} \dots B_n$ and every motor of the latter belongs to the former. Hence these complexes are identical.

If we put $\varpi = 1$ we obtain the following:—

In every complex of order n, n real coreciprocal motors can be found and the complex consisting of lators and rotors in such a set is a definite one.

[The reader should perhaps be cautioned against supposing that this means that there are no real self-reciprocal motors (lators and rotors) in the complex of the not self-reciprocal motors. The assertion only is that no such self-reciprocal motor can form one of a set of n coreciprocal motors.]

If $A_1 \dots A_l$ *be a conjugate set of motors not one of which is self-conjugate they must be independent, and if B be conjugate to the whole complex it cannot belong to the complex* $A_1 \dots A_l$. [Compare *Ausd.* § 157.] Suppose

$$yB + (x_1 A_1 + \dots + x_l A_l) = 0.$$

Operating by $\mathbf{s}A_1\varpi(\ )$ we obtain $x_1 = 0$ and similarly

$$x_2 = \dots = x_l = 0.$$

If $A_1 \dots A_l$, $B_{l+1} \dots B_n$ have the meanings they had just now:—

If E be a motor of the complex conjugate to each of the motors $A_1, A_2 \dots A_p$, *it must belong to the complex* $A_{p+1} \dots A_l, B_{l+1} \dots B_n$. [Compare *Ausd.* § 159.] Suppose

$$E = \Sigma xA + \Sigma yB.$$

Since E is conjugate to A_1 we have by operating by $\mathrm{S}A_1\varpi(\ )$, that $x_1=0$. Similarly $x_2=\ldots=x_p=0$, or E belongs to the complex $A_{p+1}\ldots A_l$, $B_{l+1}\ldots B_n$.

By putting $\varpi=1$, in both these statements we may read "reciprocal" instead of "conjugate."

31. Conjugate variation; positive, negative and zero norms; semi-conjugate complexes. Suppose A_1 and A_2 are two motors conjugate with regard to ϖ. Suppose A_1' and A_2' are derived from A_1, A_2 by the combinatorial variation

$$A_1'=cA_1+sA_2,\ A_2'=s'A_1+cA_2,\ c^2-ss'=1 \ldots\ldots (1),$$

with the condition

$$s'\mathrm{S}A_1\varpi A_1+s\mathrm{S}A_2\varpi A_2=0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Such a variation will be called a *ϖ-conjugate variation* or, when there is no risk of ambiguity, simply a *conjugate variation.* [The meaning will be very slightly extended directly.]

Since $\mathrm{S}A_1\varpi A_2=0$ we have at once from equations (1) and (2) that when A_1 and A_2 are conjugate and are by a conjugate variation transformed to A_1', A_2'; A_1' and A_2' are also conjugate, i.e.

$$\mathrm{S}A_1'\varpi A_2'=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and also

$$\mathrm{S}A_1'\varpi A_1'=\mathrm{S}A_1\varpi A_1,\ \mathrm{S}A_2'\varpi A_2'=\mathrm{S}A_2\varpi A_2 \ldots\ldots\ldots (4).$$

When $\mathrm{S}A_1\varpi A_1=\mathrm{S}A_2\varpi A_2$ *a conjugate variation is a circular variation.*

When $\mathrm{S}A_1\varpi A_1=-\mathrm{S}A_2\varpi A_2$ *a conjugate variation is a hyperbolic variation.*

When $\mathrm{S}A_2\varpi A_2=0$ *and* $\mathrm{S}A_1\varpi A_1$ *is not zero a conjugate variation is a linear variation.*

Note that the circular and hyperbolic variations here mentioned are perfectly arbitrary variations of those types. The linear variation can only be of the type $A_1'=A_1+sA_2$, $A_2'=A_2$ where s is however arbitrary; it must not be of the type $A_1'=A_1$, $A_2'=A_2+s'A_1$.

According to equations (1) and (2) if $\mathrm{S}A_1\varpi A_1=\mathrm{S}A_2\varpi A_2=0$ a conjugate variation is any combinatorial variation. In this case however, viz. that of A_1 and A_2 being both self-conjugate, it is convenient to regard conjugate variation as being of a more

arbitrary type than combinatorial variation according to the following definition:—

If A_1 and A_2 are conjugate, independent and both self-conjugate, and A_1', A_2' are any two independent motors of the complex A_1, A_2; the variation from A_1, A_2 to A_1', A_2' is said to be a conjugate variation.

This may be put into symbols thus:

$$A_1' = xA_1 + yA_2, \; A_2' = x'A_1 + y'A_2,$$

and $xy' - x'y$ is not zero.

If a series of conjugate variations be performed on a group of motors the whole variation may be called a multiple conjugate variation.

Whatever motor A be we can by multiplying it by a real ordinary finite (and not zero) scalar, make $\mathrm{s}A\varpi A$ take one of the three values $+1$, -1 or zero. For the sake of more olearly stating many of the results of this section it is convenient to define as follows:—

A is called a positive norm if $\mathrm{s}A\varpi A = -1$; a negative norm if $\mathrm{s}A\varpi A = +1$; and a zero norm if $\mathrm{s}A\varpi A = 0$.

Thus by § 30 any complex of order n can be expressed as a complex of n independent conjugate norms.

Equations (3) and (4) it will be observed remain true with the extended meaning just given to conjugate variation. Hence:—

By conjugate variation a set of n independent conjugate norms remains a set of n independent conjugate norms; and the number of positive norms remains unaltered, as also the number of negative norms and the number of zero norms. [Compare *Ausd.* §§ 155 and 391.] For the future we shall always suppose the conjugate norms to be independent. By § 30 we see that the positive norms and the negative norms *must* be independent. We here *define* the zero norms as independent. Also by § 30 the zero norms form a definite complex.

In this section as in § 30 we shall confine our attention to motors belonging to a given complex of order n. Let $A_1 \dots A_p$ be the positive norms, $B_1 \dots B_q$ the negative norms, and $C_1 \dots C_r$ the zero norms of a set of n conjugate norms belonging to this complex.

If any motor E of the complex is a positive or negative norm or is conjugate to the whole complex; $A_1 \ldots C_r$ can be so transformed by conjugate variation that E becomes one of the group. [Compare *Ausd.* § 160.]

If E be conjugate to the whole complex it belongs (§ 30) to the complex $C_1 \ldots C_r$. But by the definition of conjugate variation among the C's we can transform them by such variation so as to contain any motor of their own complex.

If E be a positive or negative norm $\mathrm{s}E\varpi E = \mp 1$. Let

$$E = x_1A_1 + \ldots + x_pA_p + y_1B_1 + \ldots + z_1C_1 + \ldots = \Sigma xA + \Sigma yB + \Sigma zC.$$

Since $\mathrm{s}A_1\varpi A_1 = -1$, $\mathrm{s}B_1\varpi B_1 = +1$, $\mathrm{s}C_1\varpi C_1 = 0$, &c, and the n motors on the right are conjugate,

$$\mathrm{s}E\varpi E = -\Sigma x^2 + \Sigma y^2 = \mp 1.$$

If the upper sign be taken we can successively bring into the group

$$E_2 = \frac{x_1A_1 + x_2A_2}{\sqrt{(x_1^2 + x_2^2)}},\ E_3 = \frac{x_1A_1 + x_2A_2 + x_3A_3}{\sqrt{(x_1^2 + x_2^2 + x_3^2)}},\ \ldots,$$

$$E_p = \frac{x_1A_1 + \ldots + x_pA_p}{\sqrt{(x_1^2 + \ldots + x_p^2)}},$$

E_2 being obtained from A_1 and A_2, E_3 from E_2 and A_3, &c., and E_p from E_{p-1} and A_p all by circular variations. And now we can successively bring into the group

$$E_1' = \frac{\Sigma xA + y_1B_1}{\sqrt{(\Sigma x^2 - y_1^2)}},\ E_2' = \frac{\Sigma xA + y_1B_1 + y_2B_2}{\sqrt{(\Sigma x^2 - y_1^2 - y_2^2)}},\ \ldots,$$

$$E_q' = \frac{\Sigma xA + \Sigma yB}{\sqrt{(\Sigma x^2 - \Sigma y^2)}} = \Sigma xA + \Sigma yB,$$

E_1' being obtained from E_p and B_1, E_2' from E_1' and B_2, &c., and E_q' from E'_{q-1} and B_q all by hyperbolic variations; for not one of these denominators is zero since $\Sigma x^2 - \Sigma y^2 = 1$. Finally we can successively bring into the group

$$E_1'' = \Sigma xA + \Sigma yB + z_1C_1,\ E_2'' = \Sigma xA + \Sigma yB + z_1C_1 + z_2C_2,\ \ldots,$$

$$E_r'' = \Sigma xA + \Sigma yB + \Sigma zC,$$

E_1'' being obtained from E_q' and C_1, E_2'' from E_1'' and C_2, &c., and E_r'' or E from E''_{r-1} and C_r all by linear variations. Similarly if $\mathrm{s}E\varpi E = +1$ we can bring E into the group by beginning with the B's.

It is to be observed that if E is a zero norm (e.g. $A_1 + B_1$) which is not conjugate to the whole complex it cannot be brought

into the group by conjugate variation. For it cannot be obtained from $C_1 \ldots C_r$ since it does not belong to their complex, and it cannot be obtained by any variation that involves a single positive or negative norm since every newly introduced norm is in that case a positive or negative norm.

Denote the given complex of order n by (n) and let (m) be a complex of order m which is included in (n). Then by § 30 (m) may be expressed as a complex of m conjugate norms. When expressed in such a form let $\alpha_1, \ldots \alpha_a$ be the positive norms and $\beta_1, \ldots \beta_b$ be the negative norms. The zero norms of (m) may have a complex in common with $C_1, \ldots C_r$, the zero norms of (n). Let this complex be that of $\gamma_1, \ldots \gamma_c$ and let the rest of the zero norms of (m) be $\delta_1, \ldots \delta_d$. Thus $\delta_1, \ldots \delta_d$ belong to the complex $A_1, \ldots A_p$, $B_1, \ldots B_q$, $C_1, \ldots C_r$, but the complexes $\delta_1, \ldots \delta_d$ and $C_1, \ldots C_r$ have no motor in common.

$A_1, \ldots A_p$, $B_1, \ldots B_q$, $C_1, \ldots C_r$ can by conjugate variation be transformed to $\alpha_1, \ldots \alpha_p$, $\beta_1, \ldots \beta_q$, $\gamma_1, \ldots \gamma_r$; where $\alpha_1, \ldots \alpha_p$ are positive norms, $\beta_1, \ldots \beta_q$ negative norms, and $\gamma_1, \ldots \gamma_r$ zero norms; where $\alpha_1, \ldots \alpha_a$, $\beta_1, \ldots \beta_b$, and $\gamma_1, \ldots \gamma_c$ have the meanings just given to them; and where

$$\delta_1 = x_1(\alpha_{a+1} + \beta_{b+1}),\ \delta_2 = x_2(\alpha_{a+2} + \beta_{b+2}),\ \&c. \ldots\ldots (5).$$

We first show [by a process essentially the same as that of *Ausd.* § 161] that $A_1 \ldots C_r$ can be so transformed as to bring $\alpha_1 \ldots \alpha_a \beta_1 \ldots \beta_b \gamma_1 \ldots \gamma_c$ into the group.

By the last proposition $A_1 \ldots A_p B_1 \ldots B_q C_1 \ldots C_r$ can be transformed to $\alpha_1 A_2' \ldots A_p' B_1' \ldots B_q' C_1 \ldots C_r$ where the A's, B's, and C's are positive, negative and zero norms respectively. α_2 belongs to the complex $A_1 \ldots C_r$ and therefore to $\alpha_1 A_2' \ldots C_r$ and it is conjugate to α_1. Hence (§ 30 above) it belongs to the complex $A_2' \ldots C_r$. Hence by the last proposition this last can be transformed to $\alpha_2 A_3'' \ldots A_p'' B_1'' \ldots B_q'' C_1 \ldots C_r$. α_3 belongs to the complex $A_1 \ldots C_r$ and therefore to $\alpha_1 \alpha_2 A_3'' \ldots C_r$. It is conjugate to α_1 and α_2 and therefore belongs to the complex $A_3'' \ldots C_r$. Hence this last can be transformed to $\alpha_3 A_4''' \ldots C_r$ and so on. Proceeding in this way we see that $\alpha_1 \ldots \alpha_a \beta_1 \ldots \beta_b$ can be brought into the group.

By hypothesis $\gamma_1 \ldots \gamma_c$ belong to the complex $C_1 \ldots C_r$. Hence $\gamma_1 \ldots \gamma_c$ can be brought into the group. It remains to prove that α_{a+1}, $\beta_{b+1} \ldots$ can be determined to form part of the group and satisfy eq. (5).

When $\alpha_1 \dots \alpha_a \beta_1 \dots \beta_b \gamma_1 \dots \gamma_c$ have been brought into the group let the rest of the positive norms be denoted by $A_1' \dots A'_{p-a}$, the rest of the negative norms by $B_1' \dots B'_{q-b}$ and the rest of the zero norms by $C_1' \dots C'_{r-c}$.

$\delta_1 \dots \delta_d$ are all conjugate to all the motors $\alpha_1 \dots \beta_b$. They therefore (§ 30 above) belong to the complex $A_1' .. A'_{p-a} B_1' \dots B'_{q-b} C_1 \dots C_r$. Thus

$$\delta_1 = \sum_1^{p-a} xA' + \sum_1^{q-b} yB' + C,$$

where C is a motor belonging to the complex $C_1 \dots C_r$. Since δ_1 is a zero norm we obtain

$$\Sigma x^2 - \Sigma y^2 = 0.$$

Σx^2 and Σy^2 are therefore neither of them zero; for if either were, all the x's and y's would be zero and δ_1 would, contrary to hypothesis, belong to the C complex. Thus

$$\delta_1 = x(A + B),$$

where A and B are a positive and negative norm respectively given by

$$A\sqrt{\Sigma x^2} = \Sigma xA', \; B\sqrt{\Sigma y^2} = \Sigma yB' + C.$$

This shows that α_{a+1} and β_{b+1} can be obtained as desired.

Suppose now for any value of e $(< d)$ $A_1' \dots B_1' \dots$ can be so chosen (consistently with the meanings just given to them) that

$$\delta_1 = x_1(A_1' + B_1'), \dots \delta_e = x_e(A_e' + B_e').$$

I proceed to show that the theorem is also true for $e + 1$.

Let
$$\delta_{e+1} = \sum_1^{p-a} \xi A' + \sum_1^{q-b} \eta B' + C,$$

where C belongs to the complex $C_1 \dots C_r$. Since δ_{e+1} is conjugate to δ_1 we get $\xi_1 = \eta_1$. Similarly

$$\xi_2 = \eta_2, \dots \xi_e = \eta_e.$$

Again since δ_{e+1} is self-conjugate we get

$$\Sigma \xi^2 = \Sigma \eta^2.$$

Here again were either $\sum_{e+1}^{p-a} \xi^2$ or $\sum_{e+1}^{q-b} \eta^2$ zero, δ_{e+1} would, contrary to hypothesis, belong to the complex $\delta_1 \dots \delta_e$, $C_1 \dots C_r$.

Thus

$$\delta_{e+1} = \sum_{e+1}^{p-a} \xi A' + \sum_{e+1}^{q-b} \eta B' + \xi_1 (A_1' + B_1') + \ldots + \xi_e (A_e' + B_e') + C$$

$$= x_{e+1} (A + B) + \xi_1 (A_1' + B_1') + \ldots + \xi_e (A_e' + B_e') + C,$$

where $\quad x_{e+1} = \sqrt{\sum_{e+1}^{p-a} \xi^2}, \; A = \sum_{e+1}^{p-a} \xi A'/x_{e+1}, \; B = \sum_{e+1}^{q-b} \eta B'/x_{e+1}.$

We now show how by successive conjugate variations, $\xi_1, \xi_2, \ldots \xi_e$ can be got rid of from the last equation.

ξ_1 can be got rid of by writing

$$\delta_1 = x_1' (A_1'' + B_1''),$$

$$\delta_{e+1} = x_{e+1} (A + B_0) + \xi_2 (A_2' + B_2') + \ldots + \xi_e (A_e' + B_e') + C,$$

where

$$x_1' = x_1 x_{e+1}/\sqrt{(x_{e+1}^2 + \xi_1^2)},$$

$$A_1'' = \{A_1' (x_{e+1}^2 + \xi_1^2) + B_1' \xi_1^2 + B \xi_1 x_{e+1}\}/x_{e+1} \sqrt{(x_{e+1}^2 + \xi_1^2)},$$

$$B_1'' = (B_1' x_{e+1} - B\xi_1)/\sqrt{(x_{e+1}^2 + \xi_1^2)},$$

$$B_0 = B + (A_1' + B_1') \xi_1/x_{e+1}.$$

Here A_1'', B_1'', A, B_0 are a conjugate set of motors belonging to the complex A_1', B_1', A, B such that

$$\mathsf{S} A_1'' \varpi A_1'' = \mathsf{S} A \varpi A = - \mathsf{S} B_1'' \varpi B_1'' = - \mathsf{S} B_0 \varpi B_0 = -1,$$

so that they can by a conjugate variation be obtained from A_1', B_1', A, B.

Thus ξ_1 can be got rid of and similarly $\xi_2 \ldots \xi_e$ can be successively got rid of. We may therefore assume them all to be zero. We then have

$$\delta_1 = x_1 (A_1' + B_1') \ldots \delta_e = x_e (A_e' + B_e'), \; \delta_{e+1} = x_{e+1} (A'_{e+1} + B'_{e+1}),$$

where $A'_{e+1} = A$, $B'_{e+1} = B + x_{e+1}^{-1} C$ (so that B'_{e+1} can be obtained from B, $C_1 \ldots C_r$ by a multiple conjugate variation).

It follows that α_{a+1}, $\beta_{b+1} \ldots$ can be determined as asserted.

Certain particular cases of this theorem are worth enunciating. First suppose $m = n$ so that the complexes (n) and (m) are the same. The theorem may then be thus put:—

If E be any one of n conjugate motors of a given complex of order n, the number of motors for which $\mathbf{s}E\varpi E$ *is positive, the number for which it is negative and the number for which it is zero, are all definite numbers characteristic of the complex.*

Putting $\varpi = 1$ we get:—

Of n independent motors of a given complex of order n, the number with positive pitch, the number with negative pitch and the number of lators and rotors, are all definite numbers characteristic of the complex. In this case we may also add that the number of completely independent (i.e. not parallel) rotors (of the rotor and lator complex which is a definite one) is also definite.

In order more easily to enunciate another result of the theorem multiply $\delta_1, \delta_2 \ldots$ by $(x_1\sqrt{2})^{-1}$, $(x_2\sqrt{2})^{-1}\ldots$ and denote the new values by $\delta_1, \delta_2 \ldots$. Also denote by $\delta_1', \delta_2' \ldots$ the motors defined by

$$\delta_1 = \frac{\alpha_{a+1} + \beta_{b+1}}{\sqrt{2}}, \quad \delta_2 = \frac{\alpha_{a+2} + \beta_{b+2}}{\sqrt{2}}, \text{ \&c.} \qquad \ldots\ldots\ldots\ldots(6),$$

$$\delta_1' = \frac{\alpha_{a+1} - \beta_{b+1}}{\sqrt{2}}, \quad \delta_2' = \frac{\alpha_{a+2} - \beta_{b+2}}{\sqrt{2}}, \text{ \&c.} \qquad \ldots\ldots\ldots\ldots(7).$$

Here it will be observed that δ_1, δ_1' are obtained from $\alpha_{a+1}, \beta_{b+1}$ by a negative circular variation. Hence (§ 29) $\alpha_{a+1}, \beta_{b+1}$ are obtained from δ_1, δ_1' by the same negative circular variation or

$$\alpha_{a+1} = \frac{\delta_1 + \delta_1'}{\sqrt{2}}, \quad \beta_{b+1} = \frac{\delta_1 - \delta_1'}{\sqrt{2}}, \quad \alpha_{a+2} = \frac{\delta_2 + \delta_2'}{\sqrt{2}}, \text{ \&c.} \qquad \ldots\ldots(8).$$

Now change the notation as follows:—change

$$\left.\begin{array}{l} \alpha_1 \ldots \alpha_a \text{ to } A_1A_2\ldots;\ \beta_1 \ldots \beta_b \text{ to } B_1B_2\ldots;\ \gamma_1 \ldots \gamma_c \text{ to } C_1C_2\ldots; \\ \delta_1\delta_1'\delta_2\delta_2' \ldots \text{ to } D_1D_1'D_2D_2'\ldots; \\ \left.\begin{array}{l} \alpha_p\alpha_{p-1} \ldots \text{ (not involved in } D_1D_2\ldots) \text{ to } A_1'A_2'\ldots; \\ \beta_q\beta_{q-1} \ldots \text{ (not involved in } D_1D_2\ldots) \text{ to } B_1'B_2'\ldots; \\ \gamma_r\gamma_{r-1} \ldots \gamma_{c+1} \text{ to } C_1'C_2'\ldots \end{array}\right\} \end{array}\right\} \ldots(9).$$

We then get the following:—

If (m) be any complex of order m included in the given complex (n) of order n, a complex $(n-m)$ of order $n-m$ independent of (m) can be found, such that (n) and $(n-m)$ make up (n).

(m) consists of the positive norms $A_1A_2\ldots$, the negative norms $B_1B_2\ldots$, and the zero norms $C_1C_2 \ldots D_1D_2\ldots$

$(n-m)$ consists of the positive norms $A_1'A_2'\ldots$, the negative norms $B_1'B_2'\ldots$, and the zero norms $C_1'C_2'\ldots D_1'D_2'\ldots$.

All pairs of these norms except the following pairs of zero norms (D_1D_1'), $(D_2D_2')\ldots$ are conjugate. These last are such that

$$\mathbf{s}D_1\varpi D_1' = \mathbf{s}D_2\varpi D_2' = \ldots = -1 \text{ ....(10).}$$

From eq. (10) it further follows for these exceptional pairs that:—

The pair of motors A_1'', B_1'' deduced from any such pair D_1, D_1' by the negative circular variation

$$A_1'' = \frac{D_1 + D_1'}{\sqrt{2}}, \quad B_1'' = \frac{D_1 - D_1'}{\sqrt{2}} \text{ .... (11)}$$

are conjugate and are respectively a positive and negative norm of (n). Therefore (n) consists of the following conjugate norms, (1) positive, $A_1A_2\ldots A_1'A_2'\ldots A_1''A_2''\ldots$, (2) negative, $B_1B_2\ldots B_1'B_2'\ldots B_1''B_2''\ldots$, and (3) zero, $C_1C_2\ldots C_1'C_2'\ldots$.

Two such complexes (m) and $(n-m)$ it will be observed are independent. They will be called *semi-conjugate complexes.*

For the sake of brevity denote the complex $A_1A_2\ldots$ by (A), the complex $A_1A_2\ldots B_1B_2\ldots$ by (AB), &c. Thus $(ABCD)$ and (m) have the same meanings. Similarly $(A'B'C'D')$ and $(n-m)$ have the same meanings.

Also order of (D) = order of (D').................(12).

The complex $(A'B'CC'D)$ contains all the motors of (n) that are conjugate to $(ABCD)$ or (m) and no others. For suppose E is a motor of $(A'B'CC'D)$ and let

$$E + A + B + D' = F$$

be any motor of (n) where A is a motor of (A), B a motor of (B) and D' a motor of (D'). Expressing that F is conjugate to $A_1, A_2\ldots$ we get $A = 0$. Expressing that F is conjugate to $B_1, B_2\ldots$ we get $B = 0$. Expressing that F is conjugate to $D_1, D_2\ldots$ we get $D' = 0$. Thus if F is conjugate to $(ABCD)$ it belongs to $(A'B'CC'D)$; and it is easy to see that every motor of the latter is conjugate to the former. [It should be remembered that (CC') is conjugate to the whole complex (n) and that $(A)(B)(A')(B')$ and (DD') are all conjugate to one another; also (D) is conjugate to $(ABCDA'B'C')$ and (D') to $(ABCA'B'C'D')$.]

Thus when (n) and (m) are given, the complex $(A'B'CC'D)$ is a determinate one. We shall call it *the conjugate of* (m) *with reference to* (n). Notice that *the sum of the orders of* (m) *and its conjugate exceeds* n *by the order of the complex,* (C), *which is common to* (m) *and the self-conjugate complex* (CC') *of* (n). Note that the conjugate of the conjugate of (m) is not (m) in general but $(ABCDC')$.

We are now in a position to establish the statements in the following table. The complexes (n) and (m) and nothing more are supposed given. The first column contains a list of complexes which are then determinate. The second column contains a list of complexes which are to a certain extent arbitrary and describes the extent of the arbitrariness.

	Determinate Complexes	Arbitrary Complexes
(1)	(C)	
(2)	(CC')	(C') is any complex which with (C) makes up the complex (CC')
(3)	$(CC'D)$	
(4)	(CD)	(D) ,, ,, ,, (C) ,, ,, (CD)
(5)	$(ABCD)$	(AB) ,, ,, ,, (CD) ,, ,, $(ABCD)$
(6)	$(ABCC'D)$	
(7)	$(A'B'CC'D)$	$(A'B')$,, ,, ,, $(CC'D)$,, ,, $(A'B'CC'D)$
(8)	$(ABA'B'CC'D)$	

Also (DD') *is any complex which contains* (D) *and is conjugate to* $(ABA'B')$ *and with* $(ABA'B'CC')$ *makes up the complex* (n). [The second and third complex in each statement of the second column are determinate complexes. This is not true of the statement about (DD'). Hence it is not included in the table.]

The statements in the first column are seen to be true by the following (now) obvious facts.

(1) $(ABCD)$ is the given complex (m);

(2) $(A'B'CC'D)$ is the conjugate of (m);

(3) (CC') and (CD) are the self-conjugate complexes of (n) and (m);

(4) (C) is the complex common to (CC') and (CD);

(5) $(CC'D)$ is the complex containing (CC') and (CD);

(6) $(ABCC'D)$ is the complex containing (CC') and (m), (and indeed is the conjugate of the conjugate of (m));

(7) $(ABA'B'CC'D)$ is the complex containing (m) and its conjugate.

That (C') is *any* complex which with (C) makes up (CC') follows from the fact that what have been denoted by $C_1C_2\ldots$ are any independent motors composing (C); and, with this restriction, that what have been denoted by $C_1C_2\ldots C_1'C_2'\ldots$ are any independent motors composing (CC').

(AB) has been defined as the complex of positive and negative norms of (m), when (m) is expressed as consisting of a set of conjugate norms. Now any complex which with (CD) makes up (m), can be [§ 31] expressed as consisting of conjugate norms; these will be conjugate to (CD) since (CD) is conjugate to the whole of (m); and there will be among them no zero norms, since in a set of conjugate norms composing (m), (CD) contains all the zero norms. The conjugate norms of this complex may therefore be taken as our $A_1A_2\ldots B_1B_2\ldots$. That is, (AB) is any complex which with (CD) makes up (m).

Exactly similar reasoning shows that $(A'B')$ is any complex which with $(CC'D)$ makes up the conjugate of (m).

It remains only to prove that $D_1D_2\ldots$ can be so chosen as to form *any* complex (δ) which with (C) makes up (CD), and that $D_1D_1'D_2\ldots$ can be so chosen as to form *any* complex $(\delta\delta')$ which contains (δ), is conjugate to $(ABA'B')$ and with $(ABA'B'CC')$ makes up (n). $D_1D_2\ldots$ belong to the complex containing (δ) and (C), and $D_1'D_2'\ldots$ belong to the complex containing $(\delta\delta')$ and (CC'). The first statement follows from the fact that $D_1D_2\ldots$ and (C) make up the complex (CD) as also do (δ) and (C); the second from the fact that $(\delta\delta')$ belongs to the complex $(CC'DD')$ since it is conjugate to $(ABA'B')$, and is independent of (CC') since with $(ABA'B'CC')$ it makes up (n). Put now

$$D_1 = \gamma_1 + \delta_1,\ D_1' = \gamma_1' + \delta_1',\ D_2 = \gamma_2 + \delta_2\ldots,$$

where $\gamma_1\gamma_1'\gamma_2\ldots$ belong to (CC') and $\delta_1\delta_1'\delta_2\ldots$ to $(\delta\delta')$. Thus $\delta_1\delta_2\ldots$ must belong to (δ) [and $\gamma_1\gamma_2\ldots$ to (C), though we do not require this]. Since $\gamma_1\gamma_1'\gamma_2\ldots$ belong to (CC') they are conjugate to every motor of (n). Hence for all values of p and q,

$$\mathbf{s}\delta_p\varpi\delta_q = \mathbf{s}D_p\varpi D_q,\ \mathbf{s}\delta_p'\varpi\delta_q' = \mathbf{s}D_p'\varpi D_q',\ \mathbf{s}\delta_p\varpi\delta_q' = \mathbf{s}D_p\varpi D_q'.$$

Hence $D_1D_1'D_2\ldots$ may be replaced by $\delta_1\delta_1'\delta_2\ldots$

This proves all that is required, but we may as well here prove the more general theorem:—*D_1D_2... may be taken as any independent motors which form a complex which with (C) makes up the complex (CD).* To prove this we have only to show in addition to what has just been proved that D_1 and D_2 may be replaced by any two independent motors of the complex D_1, D_2. If we put

$$\Delta_1 = x\,(cD_1 + sD_2),\ \Delta_2 = y\,(s'D_1 + cD_2),\ c^2 - ss' = 1,$$

Δ_1 and Δ_2 are (x and y not zero) any such independent motors. But if we further put

$$\Delta_1' = x^{-1}\,(cD_1' - s'D_2'),\ \Delta_2' = y^{-1}\,(-sD_1' + cD_2'),$$

then from the facts that D_1, D_2, D_1', D_2' are all self-conjugate and all conjugate to one another except the pairs (D_1D_1') (D_2D_2') for which $\mathbf{s}D_1\varpi D_1' = \mathbf{s}D_2\varpi D_2' = -1$, we deduce similar facts for Δ_1, Δ_2, Δ_1', Δ_2'. Hence D_1, D_1', D_2, D_2' may be replaced by Δ_1, Δ_1', Δ_2, Δ_2'.

As an example of the above theorems put $\varpi = 1$; let (n) be the complex i, Ωi, j, Ωj, k so that $n = 5$; and let (m) be the complex i, $(1 + \frac{1}{2}b^2\Omega)j$, k so that $m = 3$. No type of motor has more than one representative here and three of them, B, A' and C', are zero. The simplest values for the typical motors are

$$A = b^{-1}(1 + \tfrac{1}{2}b^2\Omega)j,\ B = 0,\ C = k,\ D = i,$$
$$A' = 0,\ B' = b^{-1}(1 - \tfrac{1}{2}b^2\Omega)j,\ C' = 0,\ D' = \Omega i.$$

First notice the definite complexes of the first column:—

(C) and (CC') are each the complex k;

$(CC'D)$ and (CD) are each the complex k, i;

$(ABCD)$ and $(ABCC'D)$ are each the complex $(1 + \frac{1}{2}b^2\Omega)j$, k, i;

$(A'B'CC'D)$ is the complex $(1 - \frac{1}{2}b^2\Omega)j$, k, i;

$(ABA'B'CC'D)$ is the complex j, Ωj, k, i.

Let us now give the more general values possible to the typical motors. The *number* of the motors of any type remains always the same. Hence in this case we must have $A' = B = C' = 0$. (C) is a determinate complex, so we may put generally

$$C = k.$$

[We might of course put $C = ck$, but this does not render things clearer.] (D) is any complex which with k makes up the complex k, i. Hence we may put

$$D = i + x_1k.$$

(AB) is any complex which with k, i makes up the complex $(1+\frac{1}{2}b^2\Omega)j$, k, i. Hence we may put

$$A = b^{-1}(1+\tfrac{1}{2}b^2\Omega)j + x_2k + x_3i.$$

[The coefficient b^{-1} is to ensure that $\mathbf{s}A^2 = -1$.] $(A'B')$ is any complex which with k, i makes up the complex $(1-\frac{1}{2}b^2\Omega)j$, k, i. Hence we may put

$$B' = b^{-1}(1-\tfrac{1}{2}b^2\Omega)j + x_4k + x_5i.$$

(DD') is any complex which is reciprocal to A and B' and which with A, B', k makes up the given complex i, Ωi, j, Ωj, k. To get the general value of this it is easiest to assume

$$D' = \xi i + \xi'\Omega i + \eta j + \eta'\Omega j + \zeta k.$$

Expressing the fact that $\mathbf{s}DD' = -1$ we get $\xi' = 1$. Expressing that D' is reciprocal to A and B' we get

$$x_3 + \tfrac{1}{2}\eta b + \eta' b^{-1} = 0,$$
$$x_5 - \tfrac{1}{2}\eta b + \eta' b^{-1} = 0,$$

so that

$$D' = \Omega i - x_3A + x_5B' + x_6k + \tfrac{1}{2}(x_3^2 - x_5^2)\,i,$$

the coefficient of i being determined by the condition $\mathbf{s}D'^2 = 0$.

It will be noticed that the sum of the orders of (m) and its reciprocal, i.e. of $(1+\frac{1}{2}b^2\Omega)j$, k, i and $(1-\frac{1}{2}b^2\Omega)j$, k, i is six, i.e. it exceeds five the order of (n) by one the order of (C).

All the above theorems lose their complexity when ϖ is an energy function as defined at the beginning of § 30. By that definition there are no negative norms in this case. This implies not only that the B's are zero but also the D's, for every D necessitates the existence of a negative norm $(D-D')/\sqrt{2}$. It will be noticed that *semi*-conjugacy only occurred by reason of the D's. Hence *in the case of an energy function all semi-conjugate complexes reduce to conjugate complexes.*

Connected with this is the fact that *when ϖ is an energy function, if $\mathbf{s}E\varpi E = 0$ for a motor E, then $\varpi E = 0$.* For whatever motor F be

$$\mathbf{s}(xE+yF)\varpi(xE+yF) = 2xy\mathbf{s}F\varpi E + y^2\mathbf{s}F\varpi F$$

is negative or zero for all values of x and y. Hence $\mathbf{s}F\varpi E = 0$ or, by § 14, $\varpi E = 0$. [If ϖ is any self-conjugate and if $\mathbf{s}E\varpi E = 0$ when *E is one of a set of six conjugate motors*, $\varpi E = 0$, for E is then by § 30 conjugate to every motor of a complex of the sixth order, i.e. to every motor in space, so that $\mathbf{s}F\varpi E = 0$ where F is

any motor. When ϖ is an energy function E need not be thus restricted.]

If we call $m\,(<n)$ independent conjugate norms of a complex of order n a partial set of conjugate norms, and n independent conjugate norms a complete set, we have by the above theorems for an energy function:—

Any complete set of conjugate norms can be transformed by conjugate variation so as to include any given partial set. And again

In a complex of order n, to any partial set of m conjugate norms can be added $n-m$ independent norms which are conjugate to one another and to the given partial set.

The table given above also takes a simpler form as follows:—

For an energy function the complexes (C) (CC') (AC) (ACC') $(A'CC')$ *are determinate;* (C') *is any complex which with* (C) *makes up* (CC'), (A) *any complex which with* (C) *makes up* (AC), *and* (A') *any complex which with* (CC') *makes up* $(A'CC')$. [It is unnecessary to say that $(AA'CC')$ is determinate, since $(ACA'C')$ is the whole complex (n). Similarly (AC) is the whole complex (m).]

When $\varpi=1$ the general theorems are not simplified in any such way. This was mainly our reason for not first considering the theorems relating to complexes which flow from putting $\varpi=1$. When $\varpi=1$ we have merely to remember that a positive norm is a motor E for which $\mathbf{s}E^2=-1$ (i.e. by eq. (10) § 14 $2\mathbf{t}E\mathbf{T}_1{}^2E=1$); a negative norm, one for which $\mathbf{s}E^2=1$ (so that positive and negative norms have positive and negative pitches respectively); and a zero norm, one for which $\mathbf{s}E^2=0$, i.e. a rotor or lator.

We may note for this case that D_1 and D_1' are two not parallel and not intersecting rotors or a rotor and lator not perpendicular to one another. They cannot be a pair of lators, a perpendicular rotor and lator, or two parallel or intersecting rotors, for in all these cases $\mathbf{s}D_1D_1'=0$. Also since D_1 and D_2 are reciprocal they must be either a pair of intersecting or parallel rotors, a pair of lators, or a perpendicular rotor and lator.

There is one important case ($\varpi=1$), viz. when $n=6$, in which the theorems *are* considerably simplified. For, since as we have just seen, $\varpi C=\varpi C'=0$ when $n=6$, there cannot be any C's in this case.

In this case the table above gives the following:—

The complexes (D) (ABD) $(A'B'D)$ *and* $(ABA'B'D)$ *are determinate ones;* (AB) *is any complex which with* (D) *makes up the complex* (ABD), $(A'B')$ *any complex which with* (D) *makes up* $(A'B'D)$, *and* (DD') *is the complex reciprocal to* $(ABA'B')$.

32. General expressions for self-conjugate functions and their reciprocals by means of sets of conjugate norms. Returning to the general meaning of ϖ, $A_1 \dots A_p$, $B_1 \dots B_q$, $C_1 \dots C_r$, let n or $p+q+r$ be 6.

Let $\bar{A}_1 \dots \bar{C}_r$ have the meanings with reference to $A_1 \dots C_r$ that $\bar{A}_1 \dots \bar{A}_6$ had in § 28 with reference to $A_1 \dots A_6$. We proceed to show that

$$\varpi A_1 = \bar{A}_1, \dots \varpi A_p = \bar{A}_p, \ \varpi B_1 = -\bar{B}_1, \dots \varpi B_q = -\bar{B}_q,$$
$$\varpi C_1 = \dots = \varpi C_r = 0 \dots\dots\dots\dots\dots (1).$$

The equations $\varpi C_1 = \dots = \varpi C_r = 0$ were established in last section. They may be established also by the methods of establishing the other equations of (1).

We saw in § 28 that $\bar{A}_1 \dots \bar{C}_r$ were six independent motors (since by their definitions $A_1 \dots C_r$ are six independent motors). Hence

$$\varpi A_1 = \Sigma x\bar{A} + \Sigma y\bar{B} + \Sigma z\bar{C}.$$

From this by § 28,

$$\mathbf{s}A_1\varpi A_1 = -x_1, \ \mathbf{s}A_2\varpi A_1 = -x_2, \dots \mathbf{s}C_r\varpi A_1 = -z_r,$$

i.e.
$$x_1 = 1, \ x_2 = \dots = z_r = 0.$$

Hence $\varpi A_1 = \bar{A}_1$. Similarly for the rest of eq. (1).

Now by eq. (21) § 28,

$$E = -\Sigma A\mathbf{s}E\bar{A} - \Sigma B\mathbf{s}E\bar{B} - \Sigma C\mathbf{s}E\bar{C}.$$

From eq. (1) it follows that

$$\varpi E = -\Sigma\bar{A}\mathbf{s}E\bar{A} + \Sigma\bar{B}\mathbf{s}E\bar{B} \dots\dots\dots\dots\dots\dots (2),$$

in which, be it observed, the total number of $\bar{A}$'s and $\bar{B}$'s is not greater than six.

More particularly if ϖ be an energy function, say ψ,

$$\psi E = -\Sigma\bar{A}\mathbf{s}E\bar{A} \dots\dots\dots\dots\dots\dots\dots (3).$$

If ψ is a partial energy function the number of $\bar{A}$'s is less than six. If it is a complete function they are six in number.

Thus if ψ is an energy function it is necessarily of the form (3). Conversely if ψ is of the form (3) it is an energy function as can easily be proved by expressing any motor E in the form ΣxA when we have $\mathbf{s}E\psi E = -\Sigma x^2$. Similarly if ϖ is of the form (2) it is a self-conjugate function.

If there are no C's we have from eq. (1),

$$\varpi^{-1}\bar{A}_1 = A_1, \ldots \varpi^{-1}\bar{B}_1 = -B_1 \ldots\ldots\ldots\ldots\ldots(4).$$

It is important to remark that these values for $\varpi^{-1}\bar{A}_1$ &c. are the only ones consistent with the equation $\varpi\varpi^{-1}E = E$, where E is any motor, for if we put $\varpi^{-1}\bar{A}_1 = A_1 + \Sigma xA + \Sigma yB$ we get from the equation $\varpi\varpi^{-1}E = E$ that $\bar{A}_1 = \bar{A}_1 + \Sigma x\bar{A} + \Sigma y\bar{B}$, so that all the x's and y's are zero. From eq. (4) and eq. (21) § 28 we deduce that

$$\varpi^{-1}E = -\Sigma A\mathbf{s}EA + \Sigma B\mathbf{s}EB \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Similarly when ψ is a complete energy function we deduce from eq. (3) that

$$\psi^{-1}E = -\Sigma A\mathbf{s}EA \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Thus ϖ^{-1} is a general self-conjugate such that for no motor E, $\varpi^{-1}E = 0$; and ψ^{-1} is a complete energy function. [That for *no* motor E is $\varpi^{-1}E = 0$ may be established by putting

$$E = \Sigma x\bar{A} + \Sigma y\bar{B}.]$$

If there are any C's ϖ^{-1} and ψ^{-1} when operating on a general motor are unintelligible. It is convenient then to *define* ϖ_{-1} and ψ_{-1} by the equations

$$\varpi_{-1}E = -\Sigma A\mathbf{s}EA + \Sigma B\mathbf{s}EB \ldots\ldots\ldots\ldots\ldots(7),$$

$$\psi_{-1}E = -\Sigma A\mathbf{s}EA \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

In this case if E is any motor in the complex $(\bar{A}\bar{B})$, $\varpi\varpi_{-1}E = E$, and if E is any motor in the complex (AB), $\varpi_{-1}\varpi E = E$; and generally

$$\varpi\varpi_{-1}E = -\Sigma\bar{A}\mathbf{s}EA - \Sigma\bar{B}\mathbf{s}EB \ldots\ldots\ldots\ldots\ldots(9),$$

$$\varpi_{-1}\varpi E = -\Sigma A\mathbf{s}E\bar{A} - \Sigma B\mathbf{s}E\bar{B} \ldots\ldots\ldots\ldots(10).$$

$$\left.\begin{array}{l}\text{Thus } \varpi \text{ reduces any motor it acts on to the complex } (\bar{A}\bar{B})\\ \varpi_{-1} \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad (AB)\\ \varpi\varpi_{-1}E \text{ is the component of } E \text{ in the complex } (\bar{A}\bar{B})\\ \varpi_{-1}\varpi E \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad (AB)\end{array}\right\} \ldots\ldots\ldots(11).$$

It is necessary to explain here what is meant by "the component in the complex." This has not a definite meaning when the complex in question only is given. The term implies that a second complex independent of the first is also given. If (n) is the given complex and $(6-n)$ the independent complex, E can uniquely be expressed as a motor of (n) + a motor of $(6-n)$. The former is called the component in the complex (n).

In the present case (AB) is not a determinate complex but (C) [§ 30] is. Thus $(\bar{A}\bar{B})$ which is the complex reciprocal to (C) is a determinate complex, but $(\bar{C})$ which is the complex reciprocal to (AB) is not. If however (AB) be determined in any way (say arbitrarily) the other complexes are all determinate. In this case (ϖ and the complex (AB) both given) there is no ambiguity in the meaning of (11).

From the above we see that when $A_1 \ldots C_r$ are a set of conjugate norms for ϖ, then $\bar{A}_1 \ldots \bar{C}_r$ are a set of conjugate norms for ϖ_{-1}. A less general statement is that *the six motors which are reciprocal to each set of five out of six independent motors conjugate with regard to ϖ are themselves six independent motors conjugate with regard to ϖ_{-1}.*

Still supposing $n=6$, let m, (m), $A_1A_2 \ldots B_1B_2 \ldots C_1C_2 \ldots D_1D_2 \ldots A_1'A_2' \ldots B_1'B_2' \ldots C_1'C_2' \ldots D_1'D_2' \ldots$ have the meanings they had in § 31 and let $A_1''B_1''A_2''B_2'' \ldots$ be defined in terms of $D_1D_1'D_2 \ldots$ by eq. (11) § 31. First let the bar have its ordinary meaning with regard to $A_1 \ldots B_1 \ldots C_1 \ldots A_1' \ldots B_1' \ldots C_1' \ldots A_1''B_1'' \ldots$. Then by eq. (2)

$$\varpi E = -\Sigma\bar{A}\mathbf{s}E\bar{A} - \Sigma\bar{A}'\mathbf{s}E\bar{A}' + \Sigma\bar{B}\mathbf{s}E\bar{B} + \Sigma\bar{B}'\mathbf{s}E\bar{B}'$$
$$- \Sigma(\bar{A}''\mathbf{s}E\bar{A}'' - \bar{B}''\mathbf{s}E\bar{B}'').$$

Now remember that D_1, D_1' are obtained from A_1'', B_1'' by a negative circular variation. Change the meaning of the bar so as to refer to the motors

$$A_1 \ldots B_1 \ldots C_1 \ldots A_1' \ldots B_1' \ldots C_1' \ldots D_1D_1' \ldots .$$

By § 29 the new meanings of $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{A}'$, $\bar{B}'$, $\bar{C}'$ will be the same as the old, and the new $\bar{D}_1$, $\bar{D}_1'$ will be the same functions of the old $\bar{A}_1''$, $\bar{B}_1''$ that D_1, D_1' are of A_1'', B_1''. In symbols

$$D_1 = (A_1'' + B_1'')/\sqrt{2},\ D_1' = (A_1'' - B_1'')/\sqrt{2},$$
$$\text{new } \bar{D}_1 = \text{old } (\bar{A}_1'' + \bar{B}_1'')/\sqrt{2},\ \text{new } \bar{D}_1' = \text{old } (\bar{A}_1'' - \bar{B}_1'')/\sqrt{2}.$$

The last expression for ϖE now gives

$$\left.\begin{aligned}\varpi E = -\Sigma\bar{A}\mathbf{s}E\bar{A} + \Sigma\bar{B}\mathbf{s}E\bar{B} - \Sigma\bar{D}'\mathbf{s}E\bar{D}\\ -\Sigma\bar{A}'\mathbf{s}E\bar{A}' + \Sigma\bar{B}'\mathbf{s}E\bar{B}' - \Sigma\bar{D}\mathbf{s}E\bar{D}'\end{aligned}\right\} \quad \ldots\ldots\ldots(12).$$

Similarly from eq. (7)

$$\left.\begin{aligned}\varpi_{-1}E = -\Sigma A\mathbf{s}EA + \Sigma B\mathbf{s}EB - \Sigma D\mathbf{s}ED'\\ -\Sigma A'\mathbf{s}EA' + \Sigma B'\mathbf{s}EB' - \Sigma D'\mathbf{s}ED\end{aligned}\right\} \quad \ldots\ldots(13).$$

The first line of eq. (12) gives the value of ϖE when E is confined to the given complex (m) or $(ABCD)$, and the second line the value of ϖE when E is confined to the semi-conjugate complex $(6-m)$ or $(A'B'C'D')$.

Thus if E is confined to (m), ϖE is confined to $(\bar{A}\bar{B}\bar{D}')$. The first line in eq. (13) gives the value of $\varpi_{-1}E$ when E is confined to $(\bar{A}\bar{B}\bar{D}')$. If E is confined to $(6-m)$, ϖE is confined to $(\bar{A}'\bar{B}'\bar{D})$. The second line in eq. (13) gives the value of $\varpi_{-1}E$ when E is confined to $(\bar{A}'\bar{B}'\bar{D})$.

Note that from equations (12), (13), we have as supplementary to equations (1), (4), (7) above

$$\left.\begin{aligned}\varpi D = \bar{D}', \qquad \varpi D' = \bar{D}\\ \varpi_{-1}\bar{D}' = D, \quad \varpi_{-1}\bar{D} = D'\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(14).$$

We may from the table of § 31 learn a good deal concerning the determinateness or arbitrariness of complexes involving $\bar{A}$, $\bar{B}$, &c.

First however note the following (in which it is to be understood that (E) and $[F]$ are independent complexes, and in which [ ] is used to denote an indeterminate complex and ( ) a determinate one):—

The statement that $[F]$ *is any complex which with a given complex* (E) *makes up another given complex* (EF) *is exactly equivalent to the statement that* $[\bar{F}]$ *is any complex which with the reciprocal of* (EF) *makes up the reciprocal of* (E). Let $[G]$ be any complex which is independent of (EF) such that the sum of the orders of $[G]$ and (EF) is 6. The reciprocals $(\bar{F}\bar{G})$ and $(\bar{G})$ of the determinate complexes (E) and (EF) are themselves determinate. Moreover from this form of the reciprocals we see that $[\bar{F}]$ is *a* complex which with the reciprocal of (EF) makes up the reciprocal of (E). That by suitably choosing $[F]$, $[\bar{F}]$ may be made *any* such complex is thus seen. Let

(1) $[\bar{\phi}]$ be any complex which with $(\bar{G})$ makes up $(\bar{F}\bar{G})$;

(2) $[\bar{\epsilon}]$ be any complex which is independent of $(\bar{F}\bar{G})$ and is such that the sum of the orders of the two $= 6$;

(3) (ϵ), $[\phi]$, $[\gamma]$ be the complexes deduced from $[\bar{\epsilon}]$, $[\bar{\phi}]$, $(\bar{G})$ by the rules of § 28 above.

Then (ϵ) is the reciprocal of $(\bar{F}\bar{G})$; or (ϵ) is (E). Also $(\epsilon\phi)$ is the reciprocal of $(\bar{G})$; or $(\epsilon\phi)$ is (EF). Thus $[F]$ may be taken as $[\phi]$; and if it is, $[\bar{F}]$ becomes $[\bar{\phi}]$, which by definition is *any* complex which with the reciprocal of (EF) makes up the reciprocal of (E). We have proved then that if $[F]$ is any complex which with (E) makes up (EF), $[\bar{F}]$ is any complex which with the reciprocal of (EF) makes up the reciprocal of (E). In the enunciation the converse of this is also stated, but the reader will see that this is the same proposition in other symbols; $[\bar{F}]$, $(\bar{G})$, $(\bar{G}\bar{F})$ replacing $[F]$, (E), (EF) respectively.

It is to be remarked that this proof depends on our complete liberty of choice of $[G]$ as a complex of the proper order which is independent of (EF). For if $[F]$ be taken as $[\phi]$, then $[G]$ must be taken as $[\gamma]$. This restriction militates against the utility of the theorem for our immediate purposes. But, if $[G]$ is not unrestricted, the above proof still shows that $[\bar{F}]$ is *a* complex which with the reciprocal of (EF) makes up the reciprocal of (E).

The reciprocals of all the complexes of the first column of the table of § 31 are themselves determinate complexes. That is, the following are determinate.

$$\text{(Determinate)}\ \left.\begin{array}{l} (\bar{A}\bar{B}\bar{D}\bar{A}'\bar{B}'\bar{C}'\bar{D}'),\ (\bar{A}\bar{B}\bar{D}\bar{A}'\bar{B}'\bar{D}'), \\ (\bar{A}\bar{B}\bar{A}'\bar{B}'\bar{D}'),\ (\bar{A}\bar{B}\bar{A}'\bar{B}'\bar{C}'\bar{D}'),\ (\bar{A}'\bar{B}'\bar{C}'\bar{D}'), \\ (\bar{A}'\bar{B}'\bar{D}'),\ (\bar{A}\bar{B}\bar{D}'),\ (\bar{D}') \end{array}\right\} \ldots(15).$$

From the second column we have that

$$\left.\begin{array}{llll} (\bar{C}') \text{ is a complex which with} & & & (\bar{A}\bar{B}\bar{D}\bar{A}'\bar{B}'\bar{D}') \\ & & & \text{makes up } (\bar{A}\bar{B}\bar{D}\bar{A}'\bar{B}'\bar{C}'\bar{D}') \\ (\bar{D}) \quad ,, & ,, & ,, & (\bar{A}\bar{B}\bar{A}'\bar{B}'\bar{C}'\bar{D}') \\ & & & \text{makes up } (\bar{A}\bar{B}\bar{D}\bar{A}'\bar{B}'\bar{C}'\bar{D}') \\ (\bar{A}\bar{B}) \quad ,, & ,, & ,, & (\bar{A}'\bar{B}'\bar{C}'\bar{D}') \\ & & & \text{makes up } (\bar{A}\bar{B}\bar{A}'\bar{B}'\bar{C}'\bar{D}') \\ (\bar{A}'\bar{B}') \quad ,, & ,, & ,, & (\bar{A}\bar{B}\bar{D}') \\ & & & \text{makes up } (\bar{A}\bar{B}\bar{A}'\bar{B}'\bar{D}') \end{array}\right\} \ldots(16).$$

It is to be remarked that in (16) we do not say "*any* complex" because of the restriction in the present cases upon the (G) of the theorem just proved. All the statements of (16) are indeed obvious, and the only use of them is as a record of the connections between the possibly indeterminate complexes and the certainly determinate ones.

When ϖ is an energy function there are (§ 31) no B's or D's and these lists become

$$\text{(Determinate) } (\bar{A}\bar{A}'\bar{C}'),\ (\bar{A}\bar{A}'),\ (\bar{A}'\bar{C}'),\ (\bar{A}'),\ (\bar{A}) \ldots (17),$$

$$(\bar{C}') \text{ is a complex which with } (\bar{A}\bar{A}') \text{ makes up } (\bar{A}\bar{A}'\bar{C}') \ldots (18).$$

The other statements of (16) do not in this case require to be made, since $(\bar{A})$ and $(\bar{A}')$ are definite complexes.

If $\varpi = 1$ we have by equations (1) and (14)

$$\bar{A} = A,\ \bar{A}' = A',\ \bar{B} = -B,\ \bar{B}' = -B',\ \bar{D} = D',\ \bar{D}' = D \ldots (19),$$

and as we saw in § 31 $C = C' = 0$. In this case the list (15) only gives over again the first column of the table and (16) does not contain any information not at once obvious from the table.

The following statements have a bearing on the matter of this section. Let (n) be a given complex of order n and $(\overline{6-n})$ the reciprocal of (n). Thus (n) and $(\overline{6-n})$ are determinate complexes. Also let $[n]$ be any given complex of order n which is independent of $(\overline{6-n})$. Suppose now $\mathbf{s}EA = 0$ where A belongs to (n) and E belongs to $[n]$ but is otherwise arbitrary. Then A must be zero. For any motor can be expressed as $E+F$ where F belongs to $(\overline{6-n})$. Hence

$$0 = \mathbf{s}EA = \mathbf{s}\,(E+F)\,A,$$

since F and A are reciprocal. Since $E+F$ is any motor whatever it follows from § 14 that $A=0$.

If (n) be the complex to which ϖ reduces any motor, i.e. the complex $(\bar{A}\bar{B})$ of eq. (2), (n) and $(\overline{6-n})$ are determinate complexes characteristic of ϖ. In this case F being any motor of $(\overline{6-n})$, $\varpi F = 0$. For if E is *any* motor, ϖE belongs to (n) and F to $(\overline{6-n})$. Hence

$$0 = \mathbf{s}F\varpi E = \mathbf{s}E\varpi F,$$

from which by § 14 it follows that $\varpi F = 0$. If any motor then be

expressed as a motor of $[n]$ + a motor of $\overline{(6-n)}$, ϖ only affects the $[n]$ component and reduces it to (n). Further if $\mathbf{s}E\varpi F = 0$ where E is an arbitrary motor belonging to $[n]$, $\varpi F = 0$.

It may be noticed that for an energy function equations (12) and (13) reduce to

$$\psi E = -\Sigma \bar{A} \mathbf{s} E \bar{A} - \Sigma \bar{A}' \mathbf{s} E \bar{A}' \quad\text{..............(20)},$$

$$\psi_{-1} E = -\Sigma A \mathbf{s} E A - \Sigma A' \mathbf{s} E A' \quad\text{..............(21)}.$$

It is of importance to remark that when (m) is given the *numbers* of the motors of the different types A, B, C, D, A', B', C', D' are determinate and may be said to be characteristic of ϖ. For by the table of § 31 the *orders* of the complexes (C), (C'), (D), (and therefore (D') which by § 31 is of the same order as (D)), (AB) and $(A'B')$ are all determinate; and also by § 31 the number of A's and the number of B's in $(ABCD)$ is determinate, and also the total number of A's and A''s and the total number of B's and B''s.

Self-conjugate functions may therefore be classified according to the numbers of motors of these types. They may be classified with regard to the given complex (m) in two ways; first according as they affect motors in (m) only, i.e. according to the numbers of motors of the types A, B, C, D; and secondly according as they affect motors in (m) and any semi-conjugate complex of order $6-m$.

It might be thought that in the first classification we ought to consider C's and D's as of the same type since they are both conjugate to the whole of (m). But this is not so; for by eq. (12) we see that while ϖC is zero, $\varpi D = \bar{D}'$.

For the first classification note that the order of (A), (B) or (C) is anything from 0 to m; and the order of (D) is any number not greater than the less of the two numbers m, $6-m$. For the second classification we have similarly that the order of (A), (B) or (C) is anything from 0 to m; the order of (A'), (B') or (C') is anything from 0 to $6-m$; and the order of (DD') is any even number not greater than the less of the two numbers $2m$, $12-2m$.

We find then according to the first classification that the number of types of ϖ is

$$\frac{(3+m)!}{3!\; m!} - \frac{(2m-4)!}{3!\; (2m-7)!} \text{ or } \frac{(3+m)!}{3!\; m!}$$

according as m is or is not greater than 3. From this we have the following table:

m	1	2	3	4	5	6
Number of types of ϖ when classified according as it affects motors in (m) only	4	10	20	31	36	28

The total number of types according to this classification is 129, but it must be remembered that one of the 28 cases when $m = 6$ is $\varpi = 0$.

According to the second classification the number of types when m has the given value m_0 is the same as when m has the value $6 - m_0$; for the number of types with reference to the given complex in the first case is the same as the number with reference to a semi-conjugate complex in the second case, and conversely. With this classification, rejecting the case $\varpi = 0$, we shall find that the number of types of ϖ when m has any value from 3 to 6 is the coefficient of x^6 in

$$(1 - x)^{-6}(1 - x^2)^{-1}(1 - x^{2(7-m)})(1 - x^{m+1})^3(1 - x^{7-m})^3,$$

diminished by 7, from which we have the following table:

m	6, 0	5, 1	4, 2	3
Number of types of ϖ when classified according as it affects motors in (m) and a semi-conjugate complex	27	208	407	477

The total number of types according to this classification is 1119.

33. The common conjugate systems of a general self-conjugate function and an energy function. If we examine the number of conditions that must be satisfied we shall find that if ϖ_1 and ϖ_2 are any two self-conjugate functions there are in general six motors forming a conjugate set both for ϖ_1 and ϖ_2. But by this process we are left in ignorance of the meaning of our results in certain limiting cases and also whether the six motors are real or not.

Let now ϖ have its usual general meaning and ψ be an energy function. Limiting ourselves to the motors of a given complex

(m) of order m let us enquire whether with reference to these two functions m such real motors exist within the complex.

Let then $A_1 \ldots B_1 \ldots C_1 \ldots$ be m conjugate norms with regard to ϖ, $A_1 \ldots$ being the positive norms, $B_1 \ldots$ the negative norms, and $C_1 \ldots$ the zero norms. We proceed to show that these can be so chosen that if the positive norms be divided into two groups $A_1 \ldots \alpha_1 \ldots$ and the negative norms into two groups $B_1 \ldots \beta_1 \ldots$ (the numbers in the α and β groups being the same), the following will be true.

$A_1 \ldots \alpha_1 \ldots$ are positive norms with regard to ϖ, $B_1 \ldots \beta_1 \ldots$ negative norms, and $C_1 \ldots$ zero norms, all conjugate with regard to ϖ; $A_1 \ldots B_1 \ldots C_1 \ldots (\alpha_1 + \beta_1)/\sqrt{2} \ldots (\alpha_1 - \beta_1)/\sqrt{2} \ldots$ are also a conjugate set of motors with regard to ψ, and of these $(\alpha_1 - \beta_1)/\sqrt{2} \ldots$ are self-conjugate with regard to ψ. Note that it is not here asserted that $A_1 \ldots$ are *norms* with regard to ψ. Also note that the motors of types A, B, C, $(\alpha + \beta)/\sqrt{2}$ on which alone ψ has any effect are a conjugate set with regard both to ϖ and ψ, being norms for ϖ, $(\alpha + \beta)/\sqrt{2}$ being a zero norm for ϖ. And again the motors of types A, B, C, $(\alpha - \beta)/\sqrt{2}$ are a conjugate set with regard both to ϖ and ψ, being norms for ϖ, $(\alpha - \beta)/\sqrt{2}$ being a zero norm both for ϖ and ψ.

We arrange the proof thus:—

(I) If for any motor E, $\mathbf{s}E\psi E = 0$, then $\psi E = 0$.

(II) For any two motors E and F

$$\text{either } 1 > \frac{2\mathbf{s}E\psi F}{\mathbf{s}E\psi E + \mathbf{s}F\psi F} > -1 \text{ or } \psi E = \pm \psi F \ldots\ldots(1).$$

(III) If any two of a set of m motors forming a conjugate set of norms with regard to ϖ are not already conjugate with regard to ψ they can be made so by a ϖ-conjugate variation unless they be a pair $\alpha\,\beta$ which are a positive and negative norm respectively with regard to ϖ, and are such that $\psi\alpha = \psi\beta$.

(IV) In every such transformation the product

$$\mathbf{s}A_1\psi A_1 \ldots \mathbf{s}C_r\psi C_r,$$

involving all the motors which are not self-conjugate with regard to ψ, diminishes so that this product has a minimum value which is only attained when the set of ϖ-conjugate norms is expressed as a series of motors $A_1 \ldots B_1 \ldots C_1 \ldots \alpha_1\,\beta_1, \ldots$ as described in the enunciation.

(I) has already been proved in § 31.

(II) Since ψ is an energy function we have that

$$\mathbf{s}(xE+yF)\,\psi\,(xE+yF) = x^2\mathbf{s}E\psi E + 2xy\mathbf{s}E\psi F + y^2\mathbf{s}F\psi F,$$

is negative or zero for all values of x and y. Also $\mathbf{s}E\psi E$ and $\mathbf{s}F\psi F$ are each negative or zero. Hence

$$\mathbf{s}E\psi E\mathbf{s}F\psi F \geqslant \mathbf{s}^2E\psi F.$$

Now unless $\mathbf{s}E\psi E = \mathbf{s}F\psi F$

$$(\mathbf{s}E\psi E + \mathbf{s}F\psi F)^2 > 4\mathbf{s}E\psi E\mathbf{s}F\psi F.$$

Hence either

$$(\mathbf{s}E\psi E + \mathbf{s}F\psi F)^2 > 4\mathbf{s}^2E\psi F \text{ or } \mathbf{s}E\psi E = \mathbf{s}F\psi F = \pm\,\mathbf{s}E\psi F.$$

If the latter be the case we have $\mathbf{s}(E \mp F)\,\psi\,(E \mp F) = 0$, so that by (I) $\psi(E \mp F) = 0$. Hence either

$$1 > \frac{2\mathbf{s}E\psi F}{\mathbf{s}E\psi E + \mathbf{s}F\psi F} > -1 \text{ or } \psi E = \pm\,\psi F.$$

This proves (II).

(III) If A_1 and A_2 are not already conjugate with regard to ψ they can be made so by the circular variation (which by § 31 is a ϖ-conjugate variation)

$$A_1' = A_1\cos\theta + A_2\sin\theta,\ A_2' = -A_1\sin\theta + A_2\cos\theta,$$

$$\tan 2\theta = \frac{2\mathbf{s}A_1\psi A_2}{\mathbf{s}A_1\psi A_1 - \mathbf{s}A_2\psi A_2},$$

for θ can always be determined to satisfy this equation.

Similarly if B_1 and B_2 are not already conjugate with regard to ψ they can be made so by a circular variation, and if C_1 and C_2 are not already conjugate with regard to ψ they can be made so by a circular variation.

If A and C are not already conjugate with regard to ψ they can be made so by the linear variation (which by § 31 is a ϖ-conjugate variation)

$$A' = A + sC,\ C' = C,\ s = -\mathbf{s}A\psi C/\mathbf{s}C\psi C,$$

for since A and C are not ψ-conjugate, ψC is not zero, i.e. by (I) $\mathbf{s}C\psi C$ is not zero; so that s can be always determined to satisfy this equation.

Similarly if B and C are not already conjugate they can be made so by a linear variation.

If A and B are not already conjugate with regard to ψ they can be made so except when $\psi A = \pm \psi B$ by the hyperbolic variation (which by § 31 is a ϖ-conjugate variation)

$$A' = A \cosh \theta + B \sinh \theta, \quad B' = A \sinh \theta + B \cosh \theta,$$

$$\tanh 2\theta = -\frac{2\mathrm{s}A\psi B}{\mathrm{s}A\psi A + \mathrm{s}B\psi B},$$

for by (II) except when $\psi A = \pm \psi B$, θ can be determined to satisfy this equation. [If $\psi A = \pm \psi B$ the equation is satisfied by $\theta = \mp \infty$ but the hyperbolic variation then becomes unintelligible.]

By changing the sign of B if necessary we can suppose this limiting case to occur only when $\psi A = \psi B$.

If $\psi A_1 = \psi B_1 = \psi B_2$ and each is not zero, the circular variation (which is a ϖ-conjugate variation)

$$B_1' = (B_1 + B_2)/\sqrt{2}, \quad B_2' = (-B_1 + B_2)/\sqrt{2},$$

makes B_2' self-conjugate with regard to ψ and also makes ψA_1 different from $\psi B_1'$.

Hence this limiting case can always be disposed of except for a series of corresponding pairs of positive and negative norms—$(\alpha_1\beta_1)(\alpha_2\beta_2)$ ...—as already described.

(IV) That the product $\mathrm{s}A_1\psi A_1 \ldots \mathrm{s}C_r\psi C_r$ always diminishes by such a transformation follows from eq. (8) § 29 above. The rest of (IV) now follows. Hence the theorem is true.

It might be thought that it would be simpler to start with a set of norms conjugate with regard to ψ and subject these to ψ-conjugate variations; for these last consist only of circular and linear variations. But it would be found that difficulties occurred with limiting cases of the linear variations not so easily surmountable as those of the above process.

The number of disposable constants in choosing m motors, apart from their tensors, is $5m$. The conditions that these must satisfy in order that the m motors may belong to a given complex is shown by Sir Robert Ball (*Screws*, § 49) to be $m(6-m)$. [This is shown by noticing that each of the m motors must be reciprocal to $6-m$ definite motors.] The number of conditions that must be further satisfied in order that these motors may be conjugate

with regard both to ϖ and ψ is $m(m-1)$. Thus the total number of conditions necessarily satisfied by the $5m$ scalars of such a common conjugate system is $m(6-m)+m(m-1)=5m$. Hence in general there is in a given complex of order m only one such common conjugate system. If now we take the (generally) independent and definite complex of order $6-m$ which is conjugate with regard to ψ we can in general find only $6-m$ definite motors in it which are conjugate with regard both to ϖ and ψ. The total of six motors thus found will form a conjugate set with regard to ψ but not in general with regard to ϖ. Hence when the complex of order m is an arbitrary one we cannot in general find another complex which is conjugate or semi-conjugate with regard both to ϖ and ψ.

We cannot then in the present case hope to express both ϖ and ψ in a manner similar to eq. (12) § 32, where the A's, A''s, B's and B''s have the same meanings for both ϖ and ψ.

Suppose however in addition to our present m motors

$$A_1 \ldots B_1 \ldots C_1 \ldots \alpha_1\, \beta_1 \ldots$$

(forming a conjugate set of norms with regard to ϖ) we take $6-m$ independent motors H_1, $H_2 \ldots H_{6-m}$ forming an independent complex of order $6-m$. For the sake of definiteness we may by § 31 suppose that this complex is conjugate to (m) with regard to ψ and that all six motors are conjugate with regard to ψ. Now apply the bar introduced in § 28 above to these six motors.

By the method of establishing eq. (2) § 32, we see that *when E belongs to (m)*

$$\left.\begin{aligned} \psi E &= -\Sigma a\bar{A}\mathbf{s}E\bar{A} - \Sigma b\bar{B}\mathbf{s}E\bar{B} - \Sigma c\bar{C}\mathbf{s}E\bar{C} \\ &\qquad -\tfrac{1}{2}\Sigma d(\bar{\alpha}+\bar{\beta})\,\mathbf{s}E(\bar{\alpha}+\bar{\beta}) \\ \varpi E &= -\Sigma(\bar{A}+A')\,\mathbf{s}E\bar{A} + \Sigma(\bar{B}+B')\,\mathbf{s}E\bar{B} \\ &\qquad -\Sigma C'\mathbf{s}E\bar{C} - \Sigma\{(\bar{\alpha}+\alpha')\,\mathbf{s}E\bar{\alpha} - (\bar{\beta}+\beta')\,\mathbf{s}E\bar{\beta}\} \end{aligned}\right\} \ldots(2),$$

where A', B', C', α', β' all belong to the complex $\bar{H}_1$, $\bar{H}_2 \ldots$, i.e. to the complex reciprocal to (m). We might generalise the expression for ψE by adding to the right of the equation $-\Sigma h\bar{H}\mathbf{s}E\bar{H}$ when it would be true for any value of E. Similarly the expression for ϖE might be generalised by adding $-\Sigma K\mathbf{s}E\bar{H}$ to the right of the equation, K standing for any motor whatever. Again, for most uses of eq. (2) it is unnecessary to distinguish between

the three types A, B, C of motors. Writing G for any (not zero) scalar multiple of any one of them we get

$$\left.\begin{aligned} \psi E &= -\Sigma g\bar{G}\mathbf{s}E\bar{G} \\ &\quad -\tfrac{1}{2}\Sigma d(\bar{\alpha}+\bar{\beta})\,\mathbf{s}E(\bar{\alpha}+\bar{\beta}) - \Sigma h\bar{H}\mathbf{s}E\bar{H} \\ \varpi E &= -\Sigma(f\bar{G}+G')\,\mathbf{s}EG - \Sigma\{(\bar{\alpha}+\alpha')\,\mathbf{s}E\bar{\alpha} \\ &\quad -(\bar{\beta}+\beta')\,\mathbf{s}E\bar{\beta}\} - \Sigma K\mathbf{s}E\bar{H} \end{aligned}\right\} \quad \ldots\ldots(3),$$

where

$$\left.\begin{array}{l} G_1,\ G_2 \ldots \alpha_1,\ \beta_1,\ \alpha_2 \ldots \text{ form the given complex } (m) \\ H_1,\ H_2 \ldots \text{ form an independent complex } (6-m) \\ G_1',\ \ldots \alpha_1',\ \beta_1' \ldots \text{ belong to the reciprocal of } (m) \\ \qquad\qquad (\text{i.e. } \bar{H}_1,\ \bar{H}_2 \ldots) \\ K_1,\ K_2 \ldots \text{ are any motors whatever} \\ g_1,\ g_2 \ldots d_1,\ d_2,\ \ldots \text{ are positive or zero scalars} \\ f_1, f_2 \ldots \text{ are any scalars whatever} \end{array}\right\} \quad \ldots\ldots\ldots(4).$$

For the important case when $m = 6$ there is no H, G', α', β' or K, and we get the simplified form

$$\left.\begin{array}{l} \psi E = -\Sigma g\bar{G}\mathbf{s}E\bar{G} - \tfrac{1}{2}\Sigma d(\bar{\alpha}+\bar{\beta})\,\mathbf{s}E(\bar{\alpha}+\bar{\beta}) \\ \varpi E = -\Sigma f\bar{G}\mathbf{s}E\bar{G} - \Sigma(\bar{\alpha}\mathbf{s}E\bar{\alpha} - \bar{\beta}\mathbf{s}E\bar{\beta}) \\ G_1,\ G_2,\ \ldots \alpha_1,\ \beta_1,\ \alpha_2,\ \ldots \text{ six independent motors;} \\ g_1,\ g_2 \ldots d_1,\ d_2 \ldots \text{ positive or zero scalars; } f_1, f_2, \ldots \\ \qquad\qquad \text{any scalars} \end{array}\right\} \quad \ldots\ldots\ldots(5).$$

For application to the dynamics of a rigid body there is one very simple case. Suppose ϖ and ψ are both energy functions either both complete or both partial in the following way.—(1) *If E be any motor in a certain complex neither* $\mathbf{s}E\psi E$ *nor* $\mathbf{s}E\varpi E$ *is zero;* (2) *both ϖ and ψ reduce every motor they act on to a second complex of the same order as the first.*

The case of ϖ and ψ both complete energy functions is clearly a particular one of the other. Take the first complex as the complex (m) of eq. (3). We see that there can be no α or β for $\psi(\alpha-\beta)=0$, so that putting $E=\alpha-\beta$, $\mathbf{s}E\psi E$ is zero. No g can be zero, for if $g_1=0$, $\mathbf{s}G_1\psi G_1=0$. $\bar{G}_1, \bar{G}_2 \ldots$ are (§ 28) independent, and therefore form a complex of order m. This must be the second complex mentioned above, since *any* motor $\Sigma x\bar{G}$ of this complex is a motor obtained by operating by ψ on some value of E, viz., $E=\Sigma xg^{-1}G$. Hence every h must be zero. Turning now to ϖ it follows that every f must be positive and not zero and every G' and K must be zero. We have then

$$\psi E = -\Sigma g\bar{G}\mathbf{s}E\bar{G},\ \varpi E = -\Sigma f\bar{G}\mathbf{s}E\bar{G} \ \ldots\ldots\ldots\ldots\ldots(6),$$

from which with the meanings of ψ_{-1} and ϖ_{-1} of equations (7) and (8) of § 32

$$\psi_{-1}E = -\Sigma g^{-1}G\mathbf{s}EG,\ \varpi_{-1}E = -\Sigma f^{-1}G\mathbf{s}EG \ldots\ldots\ldots (7).$$

From this again putting $\psi_{-1}\varpi = \phi$

$$\phi E = \psi_{-1}\varpi E = -\Sigma g^{-1}fG\mathbf{s}E\bar{G},\ \phi G_1 = g_1^{-1}f_1G_1, \ldots$$
$$\phi G_m = g_m^{-1}f_mG_m \ldots\ldots\ldots\ldots (8).$$

Also if $F(\phi)$ be any algebraic function of ϕ (such as $\phi^{\frac{1}{2}}$) $F(\phi)\,G_1 = F(g_1^{-1}f_1)\,.\,G_1$, &c., so that

$$F(\phi)\,E = -\Sigma F(g^{-1}f)\,.\,G\mathbf{s}E\bar{G} \ \ldots\ldots\ldots\ldots\ldots (9).$$

Another important case is when $m = 6$ and $\varpi = 1$. The motors conjugate with regard to ϖ are then reciprocal motors. It is in this case convenient to retain the A, B notation. We saw in § 31 that the number of motors of any type A, B... was definite. Now when $m = 6$ and $\varpi = 1$ there can be no C's, for $\varpi C = 0$. Also all the motors in space belong to the complex of six reciprocal motors $(1 \pm p\Omega)\,i$, $(1 \pm p\Omega)\,j$, $(1 \pm p\Omega)\,k$ of which three have positive and three negative pitch. It follows that the $A\alpha$ group must be three in number and $B\beta$ group also three in number. Also as in § 32 $\bar{A} = A$, $\bar{B} = -B$. Hence

$$\psi E = -\Sigma\,(aA\mathbf{s}EA + bB\mathbf{s}EB) - \tfrac{1}{2}\Sigma d\,(\alpha - \beta)\,\mathbf{s}E\,(\alpha - \beta) \ldots (10),$$

where A, B, α, β represent reciprocal motors such that

$$\mathbf{s}A^2 = \mathbf{s}\alpha^2 = -\,\mathbf{s}B^2 = -\,\mathbf{s}\beta^2 = -1.$$

From eq. (10)

$$\psi A = aA,\ \psi B = -bB,\ \psi(\alpha + \beta) = d\,(\alpha - \beta),\ \psi(\alpha - \beta) = 0 \ldots (11).$$

Unless the α, β *terms are absent there are not six real co-reciprocal motors which are conjugate with regard to* ψ, *nor are there six co-reciprocal motors for any one (say* E*) of which* ψE *is coaxial with* E. If these terms are absent there are six such motors. As examples the two cases may be considered

$$\psi E = -\,ai\mathbf{s}Ei - bj\mathbf{s}Ej - ck\mathbf{s}Ek \ \ldots\ldots\ldots\ldots\ldots\ldots (12),$$
$$\psi E = -\,a\Omega i\mathbf{s}E\Omega i - b\Omega j\mathbf{s}E\Omega j - c\Omega k\mathbf{s}E\Omega k \ldots\ldots (13).$$

34. Scalar functions of motors. Complexes of the second degree. A linear scalar function $f(E)$ of any motor E is defined as a scalar function such that for any two motors E, F

$$f(E + F) = f(E) + f(F).$$

From this it is quite easy to prove that f is commutative with ordinary scalars.

The most general form of such a function is

$$f(E) = -\mathbf{s}EA \quad\text{.........................(1)},$$

where A is some constant motor. For [eq. (8) § 15]

$$f(E) = -f(Z)\,\mathbf{s}EZ = -\mathbf{s}EA,$$

where

$$A = Zf(Z) \quad\text{.........................(2)}.$$

Let $f(E, F)$ be a scalar function of the two motors E, F, linear in each. The most general form of such a function is

$$f(E, F) = -\mathbf{s}E\phi F \quad\text{....................(3)},$$

where ϕ is a general linear motor function of a motor. For

$$f(E, F) = -f(Z, F)\,\mathbf{s}EZ = -\mathbf{s}E\phi F,$$

where

$$\phi F = Zf(Z, F) \quad\text{......................(4)}.$$

A homogeneous quadratic scalar function $f(E)$ of a motor E is defined as the function which is obtained by putting $F = E$ in a scalar function of E and F which is linear in each of them. From eq. (3) we see that the most general form of such a function is

$$f(E) = -\mathbf{s}E\varpi E \quad\text{......................(5)},$$

where ϖ is a general self-conjugate, ϖ being in fact the self-conjugate part $\frac{1}{2}(\phi + \phi')$ of ϕ.

The consideration then of such homogeneous quadratic functions may be made to depend on that of ϖ. More particularly such a scalar function which is always positive or zero has properties depending on what we have (§ 30) called an energy function, partial or complete according as the scalar function is zero for some motors or for none.

For instance (see § 30 above) in § 146 of *Screws*, Sir Robert Ball defines a motor complex of the $(n-1)$th order and second degree as consisting of all those motors of a complex of the nth order which satisfy the equation

$$\mathbf{s}E\varpi E = 0 \quad\text{.........................(6)}.$$

Thus they are those motors of the complex which are self-conjugate with regard to ϖ. What Sir Robert Ball calls the polar of any motor E of the complex is simply what we denote by ϖE.

In § 92 of *Screws* he uses the word "central" to denote any motor of the complex which is conjugate to the whole complex. What in the same section he terms conjugate screws of the complex are simply motors which are conjugate with regard to ϖ.

With the exception of the statement about the discriminant which will only appear in our work when we have considered the sextic satisfied by ϖ, all the theorems enunciated in § 92 of *Screws* are obvious from the above work. Most of them are not true in general for real motors, though they are for imaginary motors. A curious fact however is that just when the complex of the second degree itself becomes imaginary by reason of ϖ becoming a complete energy function, the theorems he gives concerning sets of screws are true of real screws.

An example of such a complex is examined in § 93 of *Screws.* We treat it here by the present methods as it serves as an illustration of some of the remarks above.

The necessary and sufficient condition that a motor E should have its pitch equal to h is by § 14 above that

$$\mathbf{s}E^2 = -2h\mathbf{T}_1{}^2E = 2h\mathbf{s}E\Omega E.$$

Hence it belongs to the motor complex of the fifth order and second degree for which $\varpi = 1 - 2h\Omega$. The polar ϖE of E is therefore $E - 2h\Omega E$, i.e. it is the coaxial motor with the same tensor as E and with pitch equal to that of E diminished by $2h$. If E belongs to the complex so that $\mathbf{s}E\varpi E = 0$, the polar is thus the reciprocal coaxial motor of equal tensor.

When E is any motor, $\mathbf{s}E\varpi E$ is negative or positive according as the pitch of E is greater or less than h, so that ϖ is not an energy function. On the other hand since it is commutative with Ω, it is a commutative function. Two coaxial motors are conjugate with regard to ϖ if the arithmetic mean of their pitches is h. If then we take three such pairs with axes on three mutually perpendicular intersecting lines they will form a conjugate system with regard to ϖ. In this case then we may put

$$\begin{aligned}
A_1 &= a^{-1}\{1 + \Omega(h + \tfrac{1}{2}a^2)\}\, i, & B_1 &= a^{-1}\{1 + \Omega(h - \tfrac{1}{2}a^2)\}\, \iota,\\
A_2 &= b^{-1}\{1 + \Omega(h + \tfrac{1}{2}b^2)\}\, j, & B_2 &= b^{-1}\{1 + \Omega(h - \tfrac{1}{2}b^2)\}\, j,\\
A_3 &= c^{-1}\{1 + \Omega(h + \tfrac{1}{2}c^2)\}\, k, & B_3 &= c^{-1}\{1 + \Omega(h - \tfrac{1}{2}c^2)\}\, k.
\end{aligned}$$

$\bar{A}_1$ is reciprocal to A_2, A_3, B_1, B_2, B_3, and is such that $\mathbf{s}A_1\bar{A}_1 = -1$. Hence

$$\begin{aligned}
\bar{A}_1 &= a^{-1}\{1 - \Omega(h - \tfrac{1}{2}a^2)\}\, i, & \bar{B}_1 &= -a^{-1}\{1 - \Omega(h + \tfrac{1}{2}a^2)\}\, i,\\
\bar{A}_2 &= b^{-1}\{1 - \Omega(h - \tfrac{1}{2}b^2)\}\, j, & \bar{B}_2 &= -b^{-1}\{1 - \Omega(h + \tfrac{1}{2}b^2)\}\, j,\\
\bar{A}_3 &= c^{-1}\{1 - \Omega(h - \tfrac{1}{2}c^2)\}\, k, & \bar{B}_3 &= -c^{-1}\{1 - \Omega(h + \tfrac{1}{2}c^2)\}\, k.
\end{aligned}$$

Equations (2) and (5) § 32 give

$$\begin{aligned}
\varpi E = -\bar{A}_1\mathbf{s}E\bar{A}_1 - \bar{A}_2\mathbf{s}E\bar{A}_2 - \bar{A}_3\mathbf{s}E\bar{A}_3&\\
+ \bar{B}_1\mathbf{s}E\bar{B}_1 + \bar{B}_2\mathbf{s}E\bar{B}_2 + \bar{B}_3\mathbf{s}E\bar{B}_3&,\\
\varpi^{-1} E = -A_1\mathbf{s}EA_1 - A_2\mathbf{s}EA_2 - A_3\mathbf{s}EA_3&\\
+ B_1\mathbf{s}EB_1 + B_2\mathbf{s}EB_2 + B_3\mathbf{s}EB_3&.
\end{aligned}$$

Put $a = b = c$. Suppose $E = \Sigma xA + \Sigma yB$. Then

$$\Sigma x^2 = \Sigma y^2$$

is the necessary and sufficient condition that E should belong to the complex $\mathbf{s}E\varpi E = 0$. If we examine the geometrical meaning of this we shall find that :—*Any motor in space whose pitch is h is the resultant of two motors through a given point O whose tensors are equal, and the arithmetic mean of whose pitches is h; and conversely any motor which is the resultant of two such motors has h for pitch.*

The following considerations are of importance in connection with complexes of the second degree.

Suppose that for every motor E of the complex (m) or $(ABCD)$ of eq. (12) § 32, $\mathbf{s}E\varpi E = 0$. Every A and B must be zero (since $-\mathbf{s}A\varpi A = \mathbf{s}B\varpi B = 1$). But the C's and D's are conjugate to the whole complex. Hence if E and F are any two motors of the complex $\mathbf{s}E\varpi F = 0$. [This may be proved directly from eq. 12 § 32. For we may put

$$E = \Sigma xC + \Sigma yD, \quad F = \Sigma x'C + \Sigma y'D,$$

so that $\varpi F = \Sigma y'\bar{D}'$. Now every $\bar{D}'$ is by § 28 reciprocal to every D. Hence $\mathbf{s}E\varpi F = 0$.]

Putting now for ϖ in this statement $\varpi - \varpi'$ where ϖ and ϖ' are two self-conjugates we get the following theorem :—*If $\mathbf{s}E\varpi E = \mathbf{s}E\varpi' E$ for every motor E belonging to a given complex, then $\mathbf{s}E\varpi F = \mathbf{s}E\varpi' F$ when E and F are any two motors of the complex.* In particular:—

If $\mathrm{S}E\varpi E = \mathrm{S}E\varpi' E$ *for every motor* E *belonging to a given complex, then any set of motors in the complex which are conjugate with regard to* ϖ *are also conjugate with regard to* ϖ'.

Let (m) and $(6-m)$ be two given independent complexes of orders m and $6-m$ respectively, and let $(\overline{6-m})$ and $(\overline{m})$ be the reciprocals of (m) and $(6-m)$ respectively.

If ϖ *be a given self-conjugate, a self-conjugate* ϖ' *can be uniquely determined so that* $\mathrm{S}E_1\varpi' E_1 = \mathrm{S}E_1\varpi E_1$, $\varpi' E_2 = 0$; *where* E_1 *is any motor belonging to* (m) *and* E_2 *any motor belonging to* $(6-m)$. *Also* $\varpi' E$ *belongs to* $(\overline{m})$ *whatever motor value* E *have.*

Let $G_1 \dots G_m$ be m independent ϖ-conjugate motors of (m) and $H_1 \dots H_{6-m}$, $6-m$ independent ϖ-conjugate motors of $(6-m)$. Using the bar introduced in § 28 with reference to these six motors, (m) will be the complex of $G_1 \dots G_m$, $(6-m)$ that of $H_1 \dots H_{6-m}$, $(\overline{m})$ that of $\overline{G}_1 \dots \overline{G}_m$ and $(\overline{6-m})$ that of $\overline{H}_1 \dots \overline{H}_{6-m}$.

By the method of establishing equations (1) and (2) § 32, we have

$$\varpi G_1 = g_1\overline{G}_1 + G_1', \ \dots \ \varpi G_m = g_m\overline{G}_m + G_m', \ \ \varpi H_1 = h_1\overline{H}_1 + H_1', \dots$$

where $g_1, \dots g_m, h_1, \dots h_{6-m}$ are ordinary scalars, and where $G_1' \dots G_m'$ belong to $(\overline{6-m})$ and $H_1' \dots H'_{6-m}$ to $(\overline{m})$. Thus

$$\varpi E = -\Sigma\,(g\overline{G} + G')\,\mathrm{S}E\overline{G} - \Sigma\,(h\overline{H} + H')\,\mathrm{S}E\overline{H}.$$

Define ϖ' by the equation

$$\varpi' E = -\Sigma g\overline{G}\mathrm{S}E\overline{G},$$

so that

$$\varpi' G_1 = g_1\overline{G}_1, \ \ \varpi' G_2 = g_2\overline{G}_2, \dots, \ \ \varpi' H_1 = \varpi' H_2 = \dots = 0,$$

ϖ' will then satisfy the conditions of the enunciation.

For with this definition it is evident that ϖ' is self-conjugate, that $\varpi' E_2 = 0$ and that $\varpi' E$ is confined to $(\overline{m})$. Also since G_1' belongs to $(\overline{6-m})$, $\mathrm{S}E_1\varpi G_1 = g_1\mathrm{S}E_1\overline{G}_1 = \mathrm{S}E_1\varpi' G_1$, and similarly for $G_2 \dots G_m$, from which it follows that $\mathrm{S}E_1\varpi E_1 = \mathrm{S}E_1\varpi' E_1$. Lastly no other self-conjugate $\varpi' + \varpi_0$ satisfies the conditions of ϖ' in the enunciation. For if it did we should have $\mathrm{S}E_1\varpi_0 E_1 = 0$, $\varpi_0 E_2 = 0$, from which

$$\mathrm{S}\,(E_1 + E_2)\,\varpi_0\,(E_1 + E_2) = 0,$$

or $\mathrm{S}E\varpi_0 E = 0$ for all motor values of E. From the general form of a self-conjugate given in equations (1) and (2) § 32, it follows that $\varpi_0 = 0$.

Putting $\varpi' = \varpi - \varpi''$ we have that ϖ'' can be determined uniquely to satisfy the conditions $\mathbf{s}E_1\varpi''E_1 = 0$, $\varpi''E_2 = \varpi E_2$. Also from the forms given above for ϖ and ϖ' we see that $\varpi''E_1$ is confined to $(\overline{6-m})$. Changing here m into $6-m$, and therefore interchanging E_1 and E_2, and writing ϖ' for the present ϖ'', we get the theorem:—

If ϖ be a given self-conjugate, a self-conjugate ϖ' can be uniquely determined, so that $\varpi'E_1 = \varpi E_1$, $\mathbf{s}E_2\varpi'E_2 = 0$, where E_1 is any motor belonging to (m) and E_2 any motor belonging to $(6-m)$. Also $\varpi'E_2$ is confined to $(\bar{m})$. As a matter of fact when ϖ is given as above we shall have

$$\varpi'E = -\Sigma\,(g\bar{G} + G')\,\mathbf{s}E\bar{G} - \Sigma H'\mathbf{s}E\bar{H}.$$

[When ϖ as given above is self-conjugate, ϖ' as given by this equation is also self-conjugate, since $(\varpi - \varpi')\,E = -\Sigma h\bar{H}\mathbf{s}E\bar{H}$.]

From these two theorems it is easy to see that ϖ' can be determined in an infinite number of ways to satisfy the two conditions $\mathbf{s}E_1\varpi'E_1 = \mathbf{s}E_1\varpi E_1$, $\mathbf{s}E_2\varpi'E_2 = 0$.

35. The general function when not necessarily self-conjugate. Let ϕ be a general linear motor function of a motor.

In the discussion of the properties of ϕ we start by practically reproducing for our own case § 390 of *Ausd.* There are two or three reasons for here reproducing that section. It has been seen that many of Grassmann's theorems are not applicable to our case, and it is perhaps just as easy to reproduce the proof as to show by general reasoning why this particular proof is valid. Moreover, Grassmann's theorem if immediately applied would not give quite so general a result as we seek, as it would only apply to the case where n below $=6$. Moreover, the theorem is of such fundamental importance that this alone is sufficient reason for giving a proof here.

Let (n) be a complex of the nth order and $(6-n)$ some independent complex of order $6-n$. Let at first $A_1 \dots A_n$ be any n independent motors of (n) and let $H_{n+1} \dots H_6$ be $6-n$ independent motors of $(6-n)$. Suppose ϕ operating on any motor of (n) reduces it to a motor of (n). The ϕ of equation (8) § 33 above is an example of such a function.

Consider the equation of the nth order in x,

$$\{(\phi - x) A_1 \dots (\phi - x) A_n H_{n+1} \dots H_6\} = 0 \quad \dots\dots\dots(1).$$

In the first place, if A_1 be changed to any other motor

$$x_1 A_1 + \dots + x_n A_n$$

of (n) which is independent of $A_2 \dots A_n$ (so that x_1 is not zero), the equation is unaltered. In fact the two expressions

$$\{(\phi - x) A_1 \dots (\phi - x) A_n H_{n+1} \dots H_6\},$$

and

$$\{A_1 \dots A_n H_{n+1} \dots H_6\}$$

are both thereby merely altered in the ratio of x_1 to 1. We may thus change $A_1 \dots A_n$ to any n independent motors $B_1 \dots B_n$ of (n) and we shall have

$$\frac{\{(\phi - x) A_1 \dots (\phi - x) A_n H_{n+1} \dots H_6\}}{\{A_1 \dots A_n H_{n+1} \dots H_6\}} \equiv \frac{\{(\phi - x) B_1 \dots (\phi - x) B_n H_{n+1} \dots H_6\}}{\{B_1 \dots B_n H_{n+1} \dots H_6\}} \quad \dots\dots(2).$$

[We may clearly on the right further change $H_{n+1} \dots H_6$ to $G_{n+1} \dots G_6$ any other $6 - n$ independent motors of $(6 - n)$, but this is unnecessary for our purpose.]

Let the n roots of (1) be a repeated r times, b repeated s times, c repeated t times, &c. Thus

$$\left.\begin{aligned} (x-a)^r (x-b)^s \dots &\equiv \frac{\{(x-\phi) A_1 \dots (x-\phi) A_n H_{n+1} \dots H_6\}}{\{A_1 \dots A_n H_{n+1} \dots H_6\}} \\ &\equiv x^n - m^{(n-1)} x^{n-1} + \dots + (-)^{n-1} m' x + (-)^n m \end{aligned}\right\} \dots(3),$$

where

$$\left.\begin{aligned} m &= \frac{\{\phi A_1 \dots \phi A_n H_{n+1} \dots H_6\}}{\{A_1 \dots A_n H_{n+1} \dots H_6\}} \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m^{(p)} &= \frac{\Sigma \{A_1 \dots A_p \phi A_{p+1} \dots \phi A_n H_{n+1} \dots H_6\}}{\{A_1 \dots A_n H_{n+1} \dots H_6\}} \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \dots(4),$$

where the summation sign implies that while $A_1 \dots A_n$ are always written in the same order the ϕ's are to be applied to every possible set of $n - p$ out of the n motors $A_1 \dots A_n$.

From the identity (2) it follows that the expressions on the right of equations (4) are the same whatever independent motors $A_1 \dots A_n$ be of the complex (n). This can also be shown independently.

The theorem in § 390 of *Ausd.* may be thus enunciated.—*In the complex* (n), $r+s+t\ldots(=n)$ *independent motors*

$$A_1 \ldots A_r B_1 \ldots B_s C_1 \ldots C_t \ldots$$

can be found such that

$$\left.\begin{array}{l} \phi A_1 = aA_1,\ \phi A_2 = aA_2 + A_2',\ \ldots\ \phi A_r = aA_r + A_r' \\ \phi B_1 = bB_1,\ \phi B_2 = bB_2 + B_2',\ \ldots\ \phi B_s = bB_s + B_s' \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots(5),$$

where A_p' *stands for some motor of the complex* $A_1, A_2 \ldots A_{p-1}$, *and similarly for* B_p', *&c.*

To prove this, § 66 of the *Ausd.* will be required. In our case it asserts that if for six motors $D_1 \ldots D_6$

$$\{D_1 \ldots D_6\} = 0,$$

$D_1 \ldots D_6$ are not independent. [If they are independent the primitive units $E_1 \ldots E_6$ can be expressed in terms of them. Expressing these units in this form we get $\{E_1 \ldots E_6\} = \{D_1 \ldots D_6\} \times$ a finite scalar $= 0$. But by § 28 $\{E_1 \ldots E_6\}$ is not zero.]

Since a is a root of eq. (1) we have

$$\{(\phi - a) A_1 \ldots (\phi - a) A_n H_{n+1} \ldots H_6\} = 0.$$

Hence some relation of the form

$$\sum_1^n y(\phi - a) A + \sum_{n+1}^6 zH = 0$$

must hold, where all the y's and z's are not zero. By hypothesis $\Sigma y(\phi - a) A$ belongs to (n) and ΣzH to $(6-n)$. Hence these two motors must separately vanish. Since $H_{n+1} \ldots H_6$ are independent, all the z's are zero and therefore $\Sigma y(\phi - a) A = 0$ where all the y's are not zero. Suppose that y_1 is not zero. Then instead of A_1 we may take ΣyA, and we shall have

$$(\phi - a) A_1 = 0.$$

Hence a motor A_1 can be found to satisfy eq. (5).

Now suppose p independent motors $A_1 \ldots A_p$ can be found to satisfy eq. (5), p being less than r. We proceed to show that $p+1$ such independent motors $A_1 \ldots A_p A_{p+1}$ can be found. Take the $A_1 \ldots A_p$ of eq. (1) to be our present $A_1 \ldots A_p$. Thus

$$(\phi - x) A_1 = (a - x) A_1,\ (\phi - x) A_2 = (a - x) A_2 + A_2', \ldots$$
$$(\phi - x) A_p = (a - x) A_p + A_p'.$$

Substituting these values in the expression on the left of eq. (1), $A_2' \dots A_p'$ may be successively cancelled because A_p' belongs to the complex $A_1 \dots A_p$, i.e. the complex $(a-x)A_1, \dots (a-x)A_p$ [except when $x=a$, but then the expression on the left of eq. (1) is zero whether $A_2' \dots A_p'$ be retained or not]. Thus eq. (1) becomes

$$(a-x)^p \{A_1 \dots A_p (\phi - x) A_{p+1} \dots (\phi - x) A_n H_{n+1} \dots H_6\} = 0.$$

But the root a occurs *more than* p times. Hence

$$\{A_1 \dots A_p (\phi - a) A_{p+1} \dots (\phi - a) A_n H_{n+1} \dots H_6\} = 0.$$

Hence some relation of the form

$$\sum_1^p yA + \sum_{p+1}^n z (\phi - a) A + \sum_{n+1}^6 wH = 0$$

holds, where the y's, z's and w's are not all zero. As before the terms in H separately vanish, so that

$$\sum_1^p yA + \sum_{p+1}^n z (\phi - a) A = 0,$$

where the y's and z's are not all zero. Further the z's are not all zero, for otherwise $A_1 \dots A_p$ would not be independent.

Suppose z_{p+1} is not zero. Then instead of A_{p+1} we may take $\sum_{p+1}^n zA$. We then have

$$(\phi - a) A_{p+1} = -\sum_1^p yA = A'_{p+1},$$

where A'_{p+1} belongs to the complex $A_1 \dots A_p$.

From this it follows that r independent motors $A_1 \dots A_r$ can be found in (n) to satisfy eq. (5). Similarly s independent motors $B_1 \dots B_s$ and t independent motors $C_1 \dots C_t$, &c., can be found to satisfy eq. (5). It remains to prove that these n motors are all independent of one another; i.e. for instance that not only are $C_1 \dots C_t$, t independent motors, but these t motors are also independent of the $r+s$ motors $A_1 \dots A_r\, B_1 \dots B_s$.

We shall not prove this last in the most direct way, but shall adopt a process that gives us incidentally two other useful propositions.

$(\phi - a)^p$ *acting on any motor of the complex* $A_1 \dots A_p$ *reduces it to zero,* p being less or equal to r. For $(\phi - a) A_1 = 0$. A_2' belongs to the complex A_1. Hence $(\phi - a) A_2' = 0$. But $(\phi - a) A_2 = A_2'$. Hence $(\phi - a)^2 A_2 = (\phi - a) A_2' = 0$. From this

we see that $(\phi - a)^2$ operating on any motor of the complex A_1, A_2 reduces it to zero. Hence $(\phi - a)^2 A_3' = 0$. But $(\phi - a) A_3 = A_3'$. Hence $(\phi - a)^3 A_3 = 0$. The proposition is now obvious.

If p be any positive integer and e any scalar different from a, $(\phi - e)^p A$ is not zero and belongs to the complex $A_1 \ldots A_r$ whenever A is any motor belonging to the complex $A_1 \ldots A_r$. Note first that $(\phi - e) A$ must belong to the complex $A_1 \ldots A_r$ by eq. (5). Also $(\phi - e) A$ is not zero. For suppose

$$A = x_1 A_1 + \ldots + x_l A_l,$$

where x_l is the x with highest suffix which is not zero. Then from eq. (5)

$$(\phi - e) A = (a - e) x_l A_l + \text{a motor of the complex } A_1 \ldots A_{l-1}.$$

The two motors on the right are independent, and the first is not zero since neither $a - e$ nor x_l is zero. Hence $(\phi - e) A$ is not zero. Thus $\phi - e$ operating on any motor of the complex $A_1 \ldots A_r$ reduces it to a non-evanescent motor of the same complex. It follows that $(\phi - e)^p A$ is a motor of $A_1 \ldots A_r$ not zero.

In particular $(\phi - b)^s B = 0$ if B is any motor belonging to the complex $B_1 \ldots B_s$. But b is different from a. Hence B cannot belong to the complex $A_1 \ldots A_r$. Hence the complexes $A_1 \ldots A_r$ and $B_1 \ldots B_s$ are independent. Similarly all the complexes $(A_1 \ldots A_r)$ $(B_1 \ldots B_s)$ $(C_1 \ldots C_t) \ldots$ are independent of one another, i.e. the $r + s + t + \ldots$ motors $A_1 \ldots B_1 \ldots C_1 \ldots$ are independent.

Let now E be any motor of the complex $A_1 \ldots A_r B_1 \ldots B_s$. It can be put in the form $E = A + B$ where A and B have the meanings just given to them. If then e is equal neither to a nor b $(\phi - e)^p E$ is a non-evanescent motor belonging to the complex $A_1 \ldots A_r B_1 \ldots B_s$; for $(\phi - e)^p A$ is not zero and belongs to $A_1 \ldots A_r$ and $(\phi - e)^p B$ is not zero and belongs to $B_1 \ldots B_s$. Proceeding in this way and calling the complexes $(A_1 \ldots A_r)$, $(B_1 \ldots B_s) \ldots$ the complexes corresponding to the roots $a, b, \ldots$ respectively, we get the following generalisation of the last theorem :—

If e be a scalar different from each of any assigned group of roots, $(\phi - e)^p$ operating on any motor belonging to the complex consisting of the complexes corresponding to those roots is a motor not zero belonging to the same complex. From this we have the following important theorem :—

The complex corresponding to any root is a perfectly definite complex. [The A's themselves may be chosen in various ways. Thus instead of A_2 we may invariably write $A_2 + yA_1$ where y is any scalar. The theorem asserts however that the whole complex $A_1 \dots A_r$ is a definite complex characteristic of ϕ.] Suppose A is any motor belonging to the complex $A_1 \dots A_r$. Then any motor of (n) may be expressed as $A + E$ where E belongs to the complex $B_1 \dots B_s C_1 \dots C_t \dots$. If this belongs to a complex corresponding in the sense just used to the root a we have $(\phi - a)^r (A + E) = 0$. Hence E must be zero or the motor $A + E$ must belong to the definite complex hitherto denoted by $A_1 \dots A_r$.

If $(\phi - e)^p E = 0$ *for every motor* E *of a complex included in* (n) *of order* q, e *must be a root of* (1) *repeated at least* q *times and the complex must be included in the complex corresponding to this root.* For we have seen that unless e is a root $(\phi - e)^p E$ is not zero for any motor E of (n). We may then assume that $e = a$.

In this case $(\phi - e)^p E$ is zero only if E belongs to the complex $A_1 \dots A_r$. Hence the complex must be included in $A_1 \dots A_r$. q is therefore not greater than r, and therefore the root e since it is repeated r times is repeated at least q times.

Any motor E belonging to the complex (n) can be expressed as $A + B + C + \dots$ where A belongs to the complex corresponding to the root a, B to that corresponding to b, and so on. Now we have seen that $(\phi - a)^r A = 0$, $(\phi - b)^s B = 0$, &c. It follows that

$$(\phi - a)^r (\phi - b)^s (\phi - c)^t \dots E = 0.$$

We may here as usual leave the operand E out, understanding that it is to be restricted to the given complex (n). Thus by the identity (3) we find that ϕ satisfies the n-tic

$$\phi^n - m^{(n-1)} \phi^{n-1} + \dots + (-)^{n-1} m' \phi + (-)^n m = 0 \dots\dots (6).$$

When $n = 6$ the given complex includes every motor in space. Hence ϕ invariably satisfies a sextic whatever motor the operand may be.

When we have found the roots of the n-tic satisfied by ϕ we can always find the complexes corresponding to these roots by the following simple property. $(\phi - b)^s (\phi - c)^t \dots E$ where E is any motor belonging to (n), and where all the roots except a are involved, is a motor belonging to the complex corresponding to the root a, for operating on it by $(\phi - a)^r$ it is reduced to zero by

eq. (6). Moreover, by giving E different values any motor in this complex can be expressed in the above form. To show this, it is only necessary to show that $(\phi - b)^s \dots A_1$, $(\phi - b)^s \dots A_2$, &c., are r independent motors. Now if $f(\phi)$ be any algebraic function of ϕ we see by equation (5) that

$$\left.\begin{array}{l} f(\phi)\, A_1 = f(a)\, A_1 \\ f(\phi)\, A_2 = f(a)\, A_2 + \text{a motor of the complex } A_1, \\ f(\phi)\, A_3 = f(a)\, A_3 + \text{a motor of the complex } A_1,\ A_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \dots(7).$$

But putting $f(\phi) \equiv (\phi - b)^s (\phi - c)^t \dots$, $f(a)$ is not zero. Thus in this case $f(\phi)\, A_1, f(\phi)\, A_2, \dots$ are r independent motors of the complex $A_1 \dots A_r$.

36. General forms of ϕ, its conjugate and its reciprocal. Express A_p' in terms of $A_1 \dots A_{p-1}$ as follows:

$$\left.\begin{array}{l} \phi A_1 = a A_1 \\ \phi A_2 = a\,(a_{21} A_1 + A_2) \\ \dots\dots\dots\dots\dots\dots\dots \\ \phi A_r = a\,(a_{r1} A_1 + \dots + a_{r\,.\,r-1}\, A_{r-1} + A_r) \\ \phi B_1 = b B_1 \\ \phi B_2 = b\,(b_{21} B_1 + B_2) \\ \dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \dots\dots\dots(1).$$

[If $a = 0$ these expressions are illegitimate. Except however when ϕ^{-1} is considered below (and ϕ^{-1} is unintelligible when $a = 0$) it is quite easy to correct all the formulæ below for this case, as no a_{pq} occurs except in the form aa_{pq}.]

Let the bar of § 28 refer to the six independent motors $A_1 \dots A_r B_1 \dots H_{n+1} \dots H_6$. Thus by equation (21) § 28

$$\phi E = -\,\phi A_1 \mathbf{s} E\bar{A}_1 - \phi A_2 \mathbf{s} E\bar{A}_2 - \dots$$

Hence

$$\phi E = -\,a\,\{A_1 \mathbf{s} E\bar{A}_1 + (a_{21} A_1 + A_2)\, \mathbf{s} E\bar{A}_2 + \dots \\ + (a_{r1} A_1 + \dots + a_{r\,.\,r-1} A_{r-1} + A_r)\, \mathbf{s} E\bar{A}_r\} - b\,\{\ \} - \dots \dots(2).$$

Hence if ϕ' is the conjugate of ϕ

$$\phi' E = -\,a\,\{\bar{A}_1 \mathbf{s} E A_1 + \bar{A}_2 \mathbf{s} E\,(a_{21} A_1 + A_2) + \dots \\ + \bar{A}_r \mathbf{s} E\,(a_{r1} A_1 + \dots + a_{r\,.\,r-1}\, A_{r-1} + A_r)\} - b\,\{\ \} - \dots.$$

Hence

$$\left.\begin{aligned}\phi'\bar{A}_1 &= a\,(\bar{A}_1 + a_{21}\bar{A}_2 + \ldots + a_{r1}\bar{A}_r)\\ \phi'\bar{A}_2 &= a\,(\bar{A}_2 + a_{32}\bar{A}_3 + \ldots + a_{r2}\bar{A}_r)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \phi'\bar{A}_r &= \qquad\qquad\qquad\qquad a\bar{A}_r\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\}\ldots\ldots\ldots\ldots(3).$$

The symmetry of the coefficients in (1) and (3) may be noticed. The expression for $\phi'E$ may be written

$$\phi'E = -a\,\{(\bar{A}_1 + a_{21}\bar{A}_2 + \ldots + a_{r1}\bar{A}_r)\,\mathbf{s}EA_1 + \ldots$$
$$+ (\bar{A}_{r-1} + a_{r.r-1}\bar{A}_r)\,\mathbf{s}EA_{r-1} + \bar{A}_r\mathbf{s}EA_r\} - \ldots \quad \ldots\ldots(4).$$

These show

(*a*) That there is a complex $(\bar{n})$ of order n and a complex $(\overline{6-n})$ of order $6-n$ standing towards ϕ' as the complexes (n) and $(6-n)$ stand towards ϕ. $(\bar{n})$ is the reciprocal of $(6-n)$ and $(\overline{6-n})$ is the reciprocal of (n).

(*b*) That the ϕ' n-tic is the same as the ϕ n-tic, since they have the same roots repeated the same number of times.

(*c*) That if ϕ be self-conjugate so that $\phi' = \phi$ the complexes corresponding to the different roots are reciprocal to one another and to $(6-n)$. [For since $\phi' = \phi$ the complex $A_1 \ldots A_r$ corresponding to the root a of the ϕ n-tic is the same as the complex $\bar{A}_1 \ldots \bar{A}_r$ corresponding to the root a of the ϕ' n-tic.]

(*d*) That if ϕ be self-conjugate every motor in the complex corresponding to one root is conjugate with regard to ϕ to every motor in the complex corresponding to a different root. [For if A be a motor in the complex corresponding to the root a, ϕA belongs to the same complex and is therefore reciprocal to B any motor belonging to the complex corresponding to another root b; i.e. A and B are conjugate.]

Suppose now that none of the roots of the n-tic is zero. Then $\phi^{-1}E$ where E is any motor of (n) has a definite meaning, namely, that one motor belonging to (n) which when operated on by ϕ gives E. ϕ^{-1} may be obtained from equations (1) by treating each motor in succession thus,

$$\phi^{-1}A_1 = a^{-1}A_1,$$

$$\phi^{-1}(a_{21}A_1 + A_2) = a^{-1}A_2,$$

whence

$$\phi^{-1}A_2 = a^{-1}(A_2 - a_{21}A_1).$$

Proceeding in this way we obtain

$$\left.\begin{aligned}
&\phi^{-1}A_1 = a^{-1}A_1\\
&\phi^{-1}A_2 = a^{-1}(-a_{21}A_1 + A_2)\\
&\phi^{-1}A_3 = a^{-1}\{(a_{32}a_{21} - a_{31})A_1 - a_{32}A_2 + A_3\}\\
&\phi^{-1}A_4 = a^{-1}\{-(a_{43}a_{32}a_{21} - a_{43}a_{31} - a_{42}a_{21} + a_{41})A_1\\
&\qquad\qquad\qquad + (a_{43}a_{32} - a_{42})A_2 - a_{43}A_3 + A_4\}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\phi^{-1}B_1 = b^{-1}B_1\\
&\cdots\cdots\cdots
\end{aligned}\right\}\ldots\ldots(5).$$

It is unnecessary to go further as the coefficients can always be at once written down in the form of determinants. It is easy apart from determinants however to see the law of the coefficients. For instance the coefficient of A_1 in $\phi^{-1}A_5$ is

$$a^{-1}(a_{54}a_{43}a_{32}a_{21} - a_{54}a_{43}a_{31} - a_{54}a_{42}a_{21} + a_{54}a_{41} - a_{53}a_{32}a_{21} + a_{53}a_{31} + a_{52}a_{21} - a_{51}).$$

The equations are also equivalent to the following in which the same coefficients occur

$$\left.\begin{aligned}
&\phi^{-1}A_1 = a^{-1}A_1\\
&\phi^{-1}A_2 = a^{-2}(-A_2' + aA_2)\\
&\phi^{-1}A_3 = a^{-2}(a_{32}A_2' - A_3' + aA_3)\\
&\phi^{-1}A_4 = a^{-2}\{-(a_{43}a_{32} - a_{42})A_2' + a_{43}A_3' - A_4' + aA_4\}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\}\ldots\ldots(6).$$

From these we see

(*a*) That ϕ^{-1} is a linear motor function of a motor.

(*b*) That the roots of the ϕ^{-1} n-tic are the reciprocals of the roots of the ϕ n-tic.

(*c*) That the complex corresponding to any root of the ϕ^{-1} n-tic is the same as the complex corresponding to the corresponding root of the ϕ n-tic.

(*d*) That ϕ'^{-1} is the conjugate of ϕ^{-1}.

37. Some properties of self-conjugate functions and commutative functions. In this and the following sections we propose to notice certain miscellaneous properties of different kinds of linear motor functions of motors.

In § 36 we saw that when ϕ is self-conjugate the complexes corresponding to the different roots of the sextic which it always

satisfies are reciprocal to one another. It is not true however that for a real self-conjugate ϖ they are always real.

For instance, put

$$\varpi E = x\,(\Omega i\mathbf{s}E\Omega i - i\mathbf{s}Ei) + y\,(\Omega j\mathbf{s}E\Omega j - j\mathbf{s}Ej) \\ + z\,(\Omega k\mathbf{s}E\Omega k - k\mathbf{s}Ek)\ldots\ldots(1),$$

where x, y, z are real ordinary scalars. Here

$$\varpi i = -x\Omega i, \quad \varpi\Omega i = xi.$$

Hence if υ be put for the imaginary $\sqrt{(-1)}$, we have

$$\varpi\,(i \pm \upsilon\Omega i) = \pm\,\upsilon x\,(i \pm \upsilon\Omega i).$$

Hence υx and $-\upsilon x$ are roots of the ϖ sextic, and the complexes corresponding to them are those of $i + \upsilon\Omega i$ and $i - \upsilon\Omega i$.

The sextic is

$$(\varpi^2 + x^2)(\varpi^2 + y^2)(\varpi^2 + z^2) = 0\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The complexes corresponding to the imaginary roots $\pm\,\upsilon x$ are both imaginary, but the complex consisting of these two imaginary complexes is itself real, being in fact that of i, Ωi

A similar statement may be made of the general real ϕ. For the coefficients of its sextic are real. Hence, if an imaginary root $a' + \upsilon a''$ occurs r times where a' and a'' are real, another imaginary root $a' - \upsilon a''$ also occurs r times. It is quite easy to prove that the complex of the $2r$th order consisting of the two imaginary complexes each of the rth order corresponding to these roots is itself real; that included in it is a real complex of the second order for each motor E of which $\{(\phi - a')^2 + a''^2\}\,E = 0$, a real complex of the fourth order for each motor E of which

$$\{(\phi - a')^2 + a''^2\}^2\,E = 0,$$

and so on; and that for every motor E of the complex of the $2r$th order, $\{(\phi - a')^2 + a''^2\}^r\,E = 0$.

Returning now to eq. (2) we see that there is no motor E in this case for which $\varpi E = aE$ where a is real, for if there were a would by § 35 be a root of the sextic, whereas in this case all the roots are imaginary.

Thus in Octonions a self-conjugate ϖ differs from the quaternion self-conjugate function and also differs from Grassmann's self-conjugate function (*Ausd.*, § 391) in that the sextic may have imaginary roots. In the present properties it is the energy

function ψ which is the complete analogue of the other self-conjugate functions. For the roots of an energy function sextic are always real. This is shown by eq. (11) § 33 above, in which by § 35 the root corresponding to A is a, to B, $-b$, and to $\alpha+\beta$ and $\alpha-\beta$ zero twice repeated. This last is seen by comparing the equations

$$\psi(\alpha-\beta)=0(\alpha-\beta),\ \ \psi(\alpha+\beta)=d(\alpha-\beta)+0(\alpha+\beta)$$

with eq. (5) § 35. We can clearly identify the present $\alpha-\beta$, $\alpha+\beta$, *zero*, with the A_1, A_2, a of that equation.

There is an allied property in which the present self-conjugate differs from the quaternion and Grassmann's self-conjugates. If a repeated root a is real or imaginary it does not follow that there are two independent motors A_1 and A_2 for which $\varpi A_1=aA_1$, $\varpi A_2=aA_2$.

This is illustrated by the self-conjugate commutative function. The general form of this is given by eq. (1) § 19 above. Changing the ϕ of that equation to ϖ we have

$$\varpi E=-Xi\mathbf{S}iE-X'j\mathbf{S}jE-X''k\mathbf{S}kE\ldots\ldots\ldots\ldots(3),$$

where X, X', X'' are scalar octonions. Put

$$X=x+\Omega y,\ \ X'=x'+\Omega y',\ \ X''=x''+\Omega y'',$$

where x, y, &c. are ordinary scalars. Then

$$\varpi i=(x+\Omega y)\,i,\ \ \varpi\Omega i=x\Omega i.$$

We may therefore identify the present Ωi, i, x with the A_1, A_2, a of eq. (5) § 35. Hence by § 35 x is a root of the ϖ sextic twice repeated. The sextic is

$$(\varpi-x)^2(\varpi-x')^2(\varpi-x'')^2=0\ \ldots\ldots\ldots\ldots\ldots(4).$$

Also the complex corresponding to the root x is that of Ωi, i. There is one motor E, viz. Ωi (and scalar multiples of it) for which $\phi E=xE$, but there is no other (except when $y=0$) as we can see by trying $E=ai+b\Omega i$.

The analogy in this case breaks down even for an energy function with a twice repeated zero root such as we considered just now, though it does not break down for any other root of an energy function sextic.

The reason that the present self-conjugate differs from Grassmann's is again that Grassmann assumes his extensive magnitudes to be such that if their "numerical values" are zero they are themselves zero.

It is easy to express the ϖ of eq. (3) in the form

$$-\Sigma\bar{A}\mathrm{s}E\bar{A}+\Sigma\bar{B}\mathrm{s}E\bar{B}$$

[eq. (2) § 32]. When x is not zero we have

$$-Xi\mathbf{S}Ei=-\bar{A}_1\mathrm{s}E\bar{A}_1+\bar{B}_1\mathrm{s}E\bar{B}_1 \quad\text{.............(5)},$$

where

$$\left.\begin{aligned}A_1&=a^{-1}x^{-1}\left\{x+\tfrac{1}{2}\Omega(-y+a^2)\right\}i\\ B_1&=a^{-1}x^{-1}\left\{x+\tfrac{1}{2}\Omega(-y-a^2)\right\}i\end{aligned}\right\} \quad\text{.............(6)},$$

$$\left.\begin{aligned}\bar{A}_1&=\quad a^{-1}\left\{x+\tfrac{1}{2}\Omega(y+a^2)\right\}i\\ \bar{B}_1&=-a^{-1}\left\{x+\tfrac{1}{2}\Omega(y-a^2)\right\}i\end{aligned}\right\} \quad\text{.................(7)},$$

where a is any scalar. Moreover it is easy to show by hyperbolic variation (§ 31) that with a arbitrary the above values for A_1 and B_1 are the most general values for two conjugate norms in the complex i, Ωi corresponding to the repeated root x of the sextic.

If $x=0$, we have

$$-Xi\mathbf{S}Ei=-y\Omega i\mathrm{s}E\Omega i \quad\text{.................(8)},$$

so that if y is positive, A_1, C_1 are here a conjugate positive and zero norm respectively in the complex i, Ωi, and if y is negative B_1 and C_1 are respectively a negative and zero norm, where

$$\left.\begin{aligned}A_1&=i/\sqrt{y}, & C_1&=\Omega i\\ \bar{A}_1&=\sqrt{y}\,.\,\Omega i, & \bar{C}_1&=i\end{aligned}\right\} \quad\text{....................(9)},$$

$$\left.\begin{aligned}B_1&=i/\sqrt{(-y)}, & C_1&=\Omega i\\ \bar{B}_1&=\sqrt{(-y)}\,.\,\Omega i, & \bar{C}_1&=i\end{aligned}\right\} \quad\text{..............(10)}.$$

We get the general expressions for the norms in the complex i, Ωi in the case of eq. (9) by the linear variation

$$\left.\begin{aligned}A_1'&=A_1+pC_1/\sqrt{y}=(1+\Omega p)\,i/\sqrt{y}, & C_1'&=C_1=\Omega i\\ \bar{A}_1'&=\bar{A}_1=\sqrt{y}\Omega i, & \bar{C}_1'&=-p\bar{A}_1/\sqrt{y}+\bar{C}_1=(1-p\Omega)\,i\end{aligned}\right\} \quad\text{...(11)},$$

the second of these lines being written down by the transformation at the end of § 29 above. A similar treatment may be accorded to eq. (10).

We may now treat $-X'j\mathbf{S}Ej$ and $-X''k\mathbf{S}Ek$ in the same way as we have treated $-Xi\mathbf{S}Ei$. Thus in all cases the ϖ of eq. (3) has been expressed in the form $-\Sigma\bar{A}\mathrm{s}E\bar{A}+\Sigma\bar{B}\mathrm{s}E\bar{B}$.

The following deductions may be made from these results:— For a commutative self-conjugate function

(a) The roots of the sextic are all real and consist of three pairs of equal roots.

(*b*) The three complexes corresponding to the three roots are three sets of coaxial motors. The axes of these sets are generally determinate and always form a set of three perpendicular intersecting lines.

(*c*) The function can never be a complete energy function and can only be a partial energy function when it degenerates into a lator function. [For unless $x = x' = x'' = 0$ there are B's.] The roots of the sextic are in this case all zero.

(*d*) Except when $\mathbf{s}X = \mathbf{s}X' = \mathbf{s}X'' = 0$ there are not six co-reciprocal motors forming a conjugate set also.

For suppose in eq. (1) § 32 the A's and B's are co-reciprocal. By eq. (26) § 28 $\bar{A}_1$ is an ordinary scalar multiple of A_1. Hence $\varpi A_1 = xA_1$ where x is an ordinary scalar. But we have just seen that except when $y = y' = y'' = 0$ there are not six independent motors for which this equation is true with the present ϖ.

It may be noticed that the ϖ considered at the end of § 34 above is a function of the kind now under consideration for which

$$x = x' = x'' = 1, \quad y = y' = y'' = -2h.$$

The sextic is therefore

$$(\varpi - 1)^6 = 0.$$

In the present case $(\varpi - 1)^2$ reduces every motor it acts on to zero.

The statement made about the roots of the ϖ sextic occurring in pairs is true of any commutative function. In fact, the sextic of a commutative ϕ is what in § 17 was called the ϕ_1 cubic squared and equated to zero. For let A, B, C be any three completely independent axial motors (§ 18). If the ϕ_1 cubic is

$$\phi_1^3 - n''\phi_1^2 + n'\phi_1 - n = 0,$$

we have by eq. (1) § 17 and eq. (27) § 16,

$$x^3 - n''x^2 + n'x - n \equiv \mathbf{S}_1 (x - \phi) A (x - \phi) B (x - \phi) C \mathbf{S}_1^{-1} ABC.$$

By equations (1) and (6) § 35, the sextic is

$$\phi^6 - m^{\text{v}}\phi^5 + m^{\text{iv}}\phi^4 - m'''\phi^3 + m''\phi^2 - m'\phi + m = 0,$$

where

$$x^6 - m^{\text{v}}x^5 + \ldots + m$$
$$\equiv \frac{\{(x-\phi)A\,(x-\phi)B(x-\phi)\,C(x-\phi)\,\Omega A\,(x-\phi)\,\Omega B(x-\phi)\,\Omega C\}}{\{ABC\Omega A\,\Omega B\Omega C\}}.$$

Hence by eq. (27) § 28

$$x^6 - m^{\text{v}}x^5 + \ldots + m \equiv (x^3 - n''x^2 + n'x - n)^2,$$

or $\phi^6 - m^{\text{v}}\phi^5 + m^{\text{iv}}\phi^4 - m'''\phi^3 + m''\phi^2 - m'\phi + m$

$$\equiv (\phi^3 - n''\phi^2 + n'\phi - n)^2 \ldots\ldots (12),$$

which proves the proposition.

It may be noticed that in this case the $A_1, B_1 \ldots$ of eq. (5) § 35 may invariably be taken as lators, for if a is a root of the sextic we have by what has just been shown

$$\mathbf{S}_1 (\phi - a) A (\phi - a) B (\phi - a) C = 0,$$

i.e. $(\phi - a) A$, $(\phi - a) B$, $(\phi - a) C$ are parallel to one plane or one is a lator. In either case

$$\Omega (\phi - a)(yA + zB + wC) = 0,$$

for some ordinary scalar values of y, z, w not all zero. Putting then $E = \Omega (yA + zB + wC)$, $\phi E = aE$, or E may be taken as A_1.

38. Another method of dealing with self-conjugate commutative functions. We give a sketch here of a method of dealing with commutative self-conjugates suggested by the methods above used for the general self-conjugate.

A *commutative* combinatorial product of two motors A and B is one which is commutative with Ω. Thus $\psi (A, B)$ is a commutative combinatorial product of A and B if it is a commutative linear function of both A and B such that $\psi (A, B) = - \psi (B, A)$.

A simple commutative combinatorial variation of any group of motors $A_1 \ldots A_n$ is one in which two of them A_p, A_q are replaced by A_p', A_q' where

$$A_p' = cA_p + sA_q,\ A_q' = s'A_p + cA_q \ldots\ldots\ldots\ldots\ldots (1),$$

where c, s, s' are any *scalar octonions* which satisfy the equation

$$c^2 - ss' = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Similarly for a negative and a multiple commutative combinatorial variation.

If $c = 1$ and therefore either s or s' is zero or both are convertors the variation is called linear; if $s' = -s$ it is called circular; and if $s' = s$ it is called hyperbolic. For a circular variation we may put

$$c = \cos\theta - \Omega r \sin\theta,\ s = -s' = \sin\theta + \Omega r \cos\theta \ldots\ldots (3),$$

where r is any ordinary scalar. Thus

$$\left.\begin{aligned} c^2 + s^2 &= 1 \\ c^2 - s^2 &= \cos 2\theta - \Omega 2r \sin 2\theta \\ 2cs &= \sin 2\theta + \Omega 2r \cos 2\theta \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

so that $c^2 - s^2$ and $2cs$ are obtained from c and s by changing r, θ into $2r$, 2θ. For a hyperbolic variation we may put

$$\left.\begin{aligned} c &= \cosh \theta + \Omega r \sinh \theta \\ s' = s &= \sinh \theta + \Omega r \cosh \theta \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

from which

$$\left.\begin{aligned} c^2 - s^2 &= 1 \\ c^2 + s^2 &= \cosh 2\theta + \Omega 2r \sinh 2\theta \\ 2cs &= \sinh 2\theta + \Omega 2r \cosh 2\theta \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(6).$$

Thus $c^2 + s^2$ and $2cs$ are obtained from c and s by changing r, θ into $2r$, 2θ.

Circular variation may also be defined by the equations

$$A_p' = (XA_p + YA_q)/\sqrt{(X^2 + Y^2)},$$
$$A_q' = (-YA_p + XA_q)/\sqrt{(X^2 + Y^2)}\ldots\ldots\ldots(7),$$

where X and Y are any two scalar octonions for which not both the ordinary scalars are zero. Hyperbolic variation may be defined as

$$A_p' = (XA_p + YA_q)/\sqrt{(X^2 - Y^2)},$$
$$A_q' = (YA_p + XA_q)/\sqrt{(X^2 - Y^2)}\ldots\ldots\ldots(8),$$

where X and Y are any two scalar octonions for which the ordinary scalar of X is numerically greater than that of Y.

A commutative combinatorial product of any number of motors is unaltered by a multiple commutative combinatorial variation. From this it may be deduced that if ϖ be a commutative self-conjugate function and if A, B be varied to A', B'

$$\mathbf{S}A'\varpi A'\mathbf{S}B'\varpi B' - \mathbf{S}^2A'\varpi B' = \mathbf{S}A\varpi A\mathbf{S}B\varpi B - \mathbf{S}^2A\varpi B\ldots(9),$$

from which by taking the ordinary scalar part

$$\mathbf{S}_1A'\varpi A'\mathbf{S}_1B'\varpi B' - \mathbf{S}_1{}^2A'\varpi B' = \mathbf{S}_1A\varpi A\mathbf{S}_1B\varpi B - \mathbf{S}_1{}^2A\varpi B\ldots(10).$$

[This of course may be generalised to statements similar to those in § 29 above, but equations (9) and (10) are sufficient for our purposes.]

Two motors A and B are said to be *fully* conjugate with regard to ϖ when

$$\mathbf{S}A\varpi B = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Thus they are conjugate in the ordinary sense and are also such that $\mathbf{S}_1 A\varpi B = 0$.

If A and B are not already fully conjugate with regard to ϖ they can invariably be made so by a commutative circular variation, and in this case the product $\mathbf{S}_1 A\varpi A \mathbf{S}_1 B\varpi B$ diminishes except when $\mathbf{S}_1 A\varpi B = 0$ when the product remains unaltered.

Three completely independent axial motors can always be found which are fully conjugate with regard to ϖ. For by the commutative circular variation just mentioned we can go on diminishing the product $\mathbf{S}_1 A\varpi A \mathbf{S}_1 B\varpi B \mathbf{S}_1 C\varpi C$ unless

$$\mathbf{S}_1 B\varpi C = \mathbf{S}_1 C\varpi A = \mathbf{S}_1 A\varpi B = 0.$$

In this case if we vary A, B to A', B' by a circular variation so as to make $\mathbf{S}A'\varpi B' = 0$ we shall find that $A' = A + \Omega rB$, $B' = -\Omega rA + B$, and therefore that

$$\mathbf{S}A'\varpi C = \mathbf{S}A\varpi C, \ \mathbf{S}B'\varpi C = \mathbf{S}B\varpi C.$$

Hence in this case we can make each pair conjugate without affecting the conjugacy or otherwise of the others.

If $\mathbf{S}A\varpi A = -1$, $\mathbf{S}B\varpi B = 1$, $\mathbf{S}A_0\varpi A_0 = -\Omega$,
$$\mathbf{S}B_0\varpi B_0 = \Omega, \ \mathbf{S}C\varpi C = 0 \ldots\ldots (12),$$
where A, B, A_0, B_0, C are axial motors, the motors will be called norms of types A, B, A_0, B_0, C respectively.

A set of three real norms fully conjugate with regard to ϖ can always be found.

The norms of types ΩA_0, ΩB_0, C, ΩC form a definite complex. The norms of types ΩA, ΩB, A_0, ΩA_0, B_0, ΩB_0, C, ΩC, also form a definite complex.

ϖA_0 and ϖB_0 are lators and $\varpi C = 0$.

By the *axial* complex of any number of motors A, B... is meant the complex of A, ΩA, B, ΩB....

If a motor be fully conjugate to an A or a B in a set of conjugate norms it belongs to the axial complex of the other norms. If a motor be fully conjugate to an A_0 or B_0 in a set of conjugate norms it belongs to the complex of ΩA_0 or ΩB_0 and the axial complex of the other norms. [Since every motor is fully conjugate to a C we have no corresponding property for a motor which is fully conjugate to a C.]

A motor of the form $\Sigma XA + \Sigma YB + \Sigma X_0 A_0 + \Sigma Y_0 B_0 + \Sigma ZC$ *where* $X \ldots Z$ *are scalar octonions cannot form one of a set of conjugate norms unless* (1) $\Sigma X^2 - \Sigma Y^2 + \Omega (\Sigma X_0^2 - \Sigma Y_0^2) = \pm 1$; *or* (2) *all the* $\mathbf{S}_1 X$*'s and* $\mathbf{S}_1 Y$*'s are zero and* $\mathbf{S}_1 (\Sigma X_0^2 - \Sigma Y_0^2) = \pm 1$; *or* (3) *all the* X*'s,* Y*'s,* $\mathbf{S}_1 X_0$*'s and* $\mathbf{S}_1 Y_0$*'s are zero.*

What is here meant by a ϖ-conjugate variation may be thus explained. A pair of zero norms C_1 and C_2 may be varied to any two completely independent motors C_1', C_2' belonging to the axial complex of C_1 and C_2, i.e. they can be varied (§ 14) to any two not parallel axial motors intersecting the shortest distance of C_1 and C_2 perpendicularly. Any other pair of norms E and F are varied according to the equations

$$\left.\begin{array}{r} E' = cE + sF, \;\; F' = s'E + cF, \\ c^2 - ss' = 1, \;\; s'\mathbf{S}E\varpi E + s\mathbf{S}F\varpi F = 0 \end{array}\right\} \ldots\ldots\ldots (13).$$

A multiple conjugate variation consists of a series of simple variations.

Any motor which satisfies one of the three conditions mentioned in the last enunciation can be brought into a group of fully conjugate norms by a multiple conjugate variation.

Any group of conjugate norms can be obtained from any other by a multiple conjugate variation.

The number of norms of any type in a set of conjugate norms is a definite number characteristic of the function ϖ. This important proposition may be enunciated otherwise. Refer back to § 9 for the meaning of positive and negative scalar octonions and positive and negative scalar convertors. Also refer to § 19 for the meaning of the *principal* roots of the ϕ cubic. [By the method of the present section we have not yet established the existence of these principal roots. That existence is however immediately deducible from the proposition below about the common conjugate systems of ϖ and an ellipsoidal function.] Then we have:—*If* A, B, C *be any three completely independent axial motors which are fully conjugate with regard to* ϖ, *i.e. are such that*

$$\mathbf{S}B\varpi C = \mathbf{S}C\varpi A = \mathbf{S}A\varpi B = 0:$$

the numbers of the scalar octonions $\mathbf{S}A\varpi A$, $\mathbf{S}B\varpi B$ *and* $\mathbf{S}C\varpi C$ *which are* (1) *positive scalar octonions,* (2) *negative scalar octonions,* (3) *positive convertors,* (4) *negative convertors,* (5) *zero, are the same as the numbers of the principal roots of the* ϖ *cubic of the same types.* [This connection with the principal roots is obvious from

the fact that i, j, k of eq. (1) of § 19 are fully conjugate with regard to the ϕ of that equation.]

If the self-conjugate commutative functions themselves be classified according to the numbers of conjugate norms of different types appertaining to them, rejecting the case $\varpi = 0$, it will be found that there are thirty-four kinds.

To see by analogy to a certain extent what these different kinds of ϖ imply geometrically, take the corresponding quaternion case. ϖ being a self-conjugate linear vector function of a vector we have, rejecting the case $\varpi = 0$, nine types of ϖ according to the numbers of roots of the ϖ cubic that are positive, negative, or zero. Interpreting this geometrically we have nine kinds of central conicoids not passing through their own centres represented by the quaternion equation $\mathbf{S}\rho\varpi\rho = -1$. These are (1) the ellipsoid, (2) the hyperboloid of one sheet, (3) the hyperboloid of two sheets, (4) the imaginary quadric which is not cylindrical (or conical), (5) the elliptic cylinder, (6) the hyperbolic cylinder, (7) the imaginary cylinder which is not a pair of planes, (8) a pair of real parallel planes, (9) a pair of imaginary parallel planes.

It might be thought that by analogy in the case of the general self-conjugate, the number of conjugate norms of any type (positive, negative or zero) would prove to be the same as the number of roots of the sextic of the same type, but curiously enough, this is not the case. This can easily be seen in the case of the commutative ϖ whose sextic was considered in § 37.

If ψ is a commutative self-conjugate function such that $\mathbf{S}_1 E\psi E$ is negative and not zero for every axial motor E in space, we will call ψ an ellipsoidal function. [We only require this not very appropriate name temporarily.] Thus as a particular case ψ may be put equal to unity.

A set of three real completely independent axial motors can invariably be found which are fully conjugate with regard both to ϖ any commutative self-conjugate and ψ an ellipsoidal function. The axes of these motors are in general unique.

Putting $\psi = 1$, we get:—*A set of three mutually perpendicular intersecting rotors can always be found which are fully conjugate with regard to ϖ.* These are of course the i, j, k of eq. (1) § 19.

If D_1, D_2, D_3 are three conjugate norms with regard to ϖ, and if the bar have the meaning with regard to these three that it

had with reference to A, A', A'' in § 16 (not the meaning it had with reference to six motors in § 28), the term in ϖE corresponding to A is $-\bar{A}\mathbf{S}E\bar{A}$; to B, $\bar{B}\mathbf{S}E\bar{B}$; to A_0, $-\Omega\bar{A}_0\mathbf{S}E\bar{A}_0$; to B_0, $\Omega\bar{B}_0\mathbf{S}E\bar{B}_0$; to C, zero. Thus

$$\varpi E = \Sigma\,(\pm\,\bar{D}\mathbf{S}E\bar{D}) + \Omega\Sigma\,(\pm\,\bar{D}'\mathbf{S}E\bar{D}')\text{.........}(14),$$

where the total number of terms under the two summation signs does not exceed three.

For an ellipsoidal function we have always

$$\psi E = -\bar{A}_1\mathbf{S}E\bar{A}_1 - \bar{A}_2\mathbf{S}E\bar{A}_2 - \bar{A}_3\mathbf{S}E\bar{A}_3\text{.........}(15).$$

For the common fully-conjugate system of ϖ and ψ we shall have equation (15) and also

$$\varpi E = -a_1\bar{A}_1\mathbf{S}E\bar{A}_1 - a_2\bar{A}_2\mathbf{S}E\bar{A}_2 - a_3\bar{A}_3\mathbf{S}E\bar{A}_3\text{......}(16),$$

where a_1, a_2, a_3 are any scalar octonions.

From eq. (15) we have always

$$\psi^{-1}E = -A_1\mathbf{S}EA_1 - A_2\mathbf{S}EA_2 - A_3\mathbf{S}EA_3\text{.........}(17),$$

and from eq. (16) when the ordinary scalars of a_1, a_2, a_3 are not any of them zero

$$\varpi^{-1}E = -a_1^{-1}A_1\mathbf{S}EA_1 - a_2^{-1}A_2\mathbf{S}EA_2 - a_3^{-1}A_3\mathbf{S}EA_3\text{...}(18).$$

Expressing equations (15) and (16) in full by means of § 16 in terms of the three conjugate norms (with regard to ψ) A_1, A_2, A_3 we have

$$\psi E\mathbf{S}^2A_1A_2A_3$$
$$= -\mathbf{M}A_2A_3\mathbf{S}EA_2A_3 - \mathbf{M}A_3A_1\mathbf{S}EA_3A_1 - \mathbf{M}A_1A_2\mathbf{S}EA_1A_2\text{...}(19),$$

$$\varpi E\mathbf{S}^2A_1A_2A_3$$
$$= -a_1\mathbf{M}A_2A_3\mathbf{S}EA_2A_3 - a_2\mathbf{M}A_3A_1\mathbf{S}EA_3A_1 - a_3\mathbf{M}A_1A_2\mathbf{S}EA_1A_2\,(20).$$

39. Another method of dealing with the general linear motor function of a motor. Just as the first method of dealing with the general self-conjugate function suggests a method of dealing with the commutative self-conjugate, so the first method of dealing with the commutative function suggests a method of dealing with the general function.

If ϕE be a general linear motor function of a motor E and $\psi(E, F)$ be any octonion function of the motors E, F, linear in the general sense in each,

$$\psi(Z, \phi Z) = \psi(\phi' Z, Z)\text{...................}(1),$$

where Z has the meaning defined in eq. (6) § 15. For

$$\psi(\phi' Z, Z) = -\psi(Z_1 \mathsf{s} Z \phi Z_1, Z) \text{ [eq. (10) § 15]}$$

$$= -\psi(Z_1, Z\mathsf{s}Z\phi Z_1)$$

[since ψ is commutative with ordinary scalars]

$$= \psi(Z_1, \phi Z_1) \text{ [eq. (8) § 15]}.$$

Particular cases of eq. (1) are

$$\mathbf{M}Z\phi Z = -\mathbf{M}Z\phi' Z, \quad \mathbf{S}Z\phi Z = \mathbf{S}Z\phi' Z \text{ ....(2).}$$

When $\phi E = -\Sigma B\mathsf{s}EA$, we have

$$\phi E = -\Sigma B\mathsf{s}EA, \quad \psi(Z, \phi Z) = \Sigma\psi(A, B) \text{.........(3),}$$

for

$$\psi(Z, \phi Z) = -\psi(Z, \Sigma B\mathsf{s}AZ) = -\Sigma\psi(Z\mathsf{s}AZ, B) = \Sigma\psi(A, B).$$

By eq. (21) § 28

$$\phi E = -\phi\bar{A}_1\mathsf{s}EA_1 - \ldots - \phi\bar{A}_6\mathsf{s}EA_6 \text{...........(4),}$$

where $A_1 \ldots A_6$ are any six independent motors; or

$$\phi E = -B_1\mathsf{s}EA_1 - \ldots - B_6\mathsf{s}EA_6 \text{..............(5),}$$

where

$$B_1 = \phi\bar{A}_1, \; B_2 = \phi\bar{A}_2, \ldots B_6 = \phi\bar{A}_6 \text{..............(6),}$$

so that the expression for ϕ in equations (3) is a perfectly general form of ϕ even when the number of terms is limited to six.

We shall in what follows, as just now, always suppose $A_1 \ldots A_6$ to be independent so that $\{A_1 \ldots A_6\}$ is not zero.

The equation

$$\{\phi A_1 \ldots \phi A_6\} = m\{A_1 \ldots A_6\} \text{ ....(7),}$$

gives a value for m independent of the particular values of $A_1 \ldots A_6$; for if we change A_1 to any motor $x_1 A_1 + \ldots + x_6 A_6$ (x_1 not zero) independent of $A_2 \ldots A_6$, both $\{\phi A_1 \ldots \phi A_6\}$ and $\{A_1 \ldots A_6\}$ are altered in the ratio of x_1 to unity. [The equation remains true for the same value of m when $A_1 \ldots A_6$ are not independent, for then both the expressions of eq. (7) involving $A_1 \ldots A_6$ are zero.]

In particular,

$$\{Z_1 \ldots Z_6\}\{\phi Z_1 \ldots \phi Z_6\} = m\{Z_1 \ldots Z_6\}\{Z_1 \ldots Z_6\} \text{......(8),}$$

$$\{Z_1 \ldots Z_5\}\{\phi E \phi Z_1 \ldots \phi Z_5\} = m\{Z_1 \ldots Z_5\}\{EZ_1 \ldots Z_5\} \text{ ...(9).}$$

The Z's may be eliminated from the right of these equations. To do this, first note that by eq. (3),

$$\{Z_1 \ldots Z_6\}\{\phi Z_1 \ldots \phi Z_6\} = 6!\,\{A_1 \ldots A_6\}\{B_1 \ldots B_6\} \text{ ....(10),}$$

$$\{Z_1 \ldots Z_5\}\{\phi E \phi Z_1 \ldots \phi Z_5\} = 5!\,\Sigma\{A_2 \ldots A_6\}\{\phi EB_2 \ldots B_6\} \text{...(11),}$$

there being six terms under the summation sign, obtained by omitting $A_1, A_2 \dots A_6$ successively.

A particular case of the last two equations is obtained by putting $i, j, k, \Omega i, \Omega j, \Omega k$ for $A_1 \dots A_6$ and $\Omega i, \Omega j, \Omega k, i, j, k$ for $B_1 \dots B_6$. Thus

$$\phi E = -\Omega i \mathbf{s} i E - \dots - i \mathbf{s} \Omega i E - \dots = E,$$

by eq. (5) § 14. In this particular case equations (10) and (11) become

$$\{Z_1 \dots Z_6\}\{\ Z_1 \dots Z_6\} = -6! \dots\dots\dots\dots\dots\dots (12),$$

$$\{Z_1 \dots Z_5\}\{EZ_1 \dots Z_5\} = \quad 5!\, E \dots\dots\dots\dots\dots (13),$$

by equations (13) and (14) § 28.

Thus equations (8) and (9) become

$$\{Z_1 \dots Z_6\}\{\phi Z_1 \dots \phi Z_6\} = -6!\, m \dots\dots\dots\dots (14),$$

$$\{Z_1 \dots Z_5\}\{\phi E \phi Z_1 \dots \phi Z_5\} = \quad 5!\, E \dots\dots\dots\dots (15),$$

or, eliminating m,

$$E\{Z_1 \dots Z_6\}\{\phi Z_1 \dots \phi Z_6\} = -6\{Z_1 \dots Z_5\}\{\phi E \phi Z_1 \dots \phi Z_5\} \dots (16),$$

which gives E explicitly in terms of ϕE, i.e. gives, except when m is zero, the reciprocal of ϕ^{-1} of ϕ.

Again, from equations (10) and (14)

$$m = -\{A_1 \dots A_6\}\{B_1 \dots B_6\} \dots\dots\dots\dots\dots\dots\dots (17),$$

$$mE = \Sigma\{A_2 \dots A_6\}\{\phi E B_2 \dots B_6\} \dots\dots\dots\dots\dots\dots (18),$$

from which again

$$E\{A_1 \dots A_6\}\{B_1 \dots B_6\} = -\Sigma\{A_2 \dots A_6\}\{\phi E B_2 \dots B_6\} \dots (19).$$

When m is not zero $\{B_1 \dots B_6\}$ is not zero. In this case we have by equations (12) and (20) § 28,

$$E = -\bar{A}_1 \mathbf{s} \phi E \bar{B}_1 - \dots - \bar{A}_6 \mathbf{s} \phi E \bar{B}_6 \dots\dots\dots\dots\dots (20),$$

which can easily be verified from equations (6) of the present section and (21) of § 28. This method of course is the simpler one of establishing eq. (20), but it does not lead to eq. (19) in the exceptional case when m is zero.

From equations (1) and (14) we see that the m of ϕ' is the same as the m of ϕ, where ϕ' is the conjugate of ϕ. Hence eq. (7) may be written

$$\{\phi' A_1 \dots \phi' A_6\} = m\{A_1 \dots A_6\} \dots\dots\dots\dots\dots (21),$$

or by eq. (12) § 28,

$$\mathbf{s} A_6 \phi\{\phi' A_1 \dots \phi' A_5\} = m \mathbf{s} A_6\{A_1 \dots A_5\}.$$

Hence by § 14,

$$\phi\,\{\phi' A_1 \dots \phi' A_5\} = m\,\{A_1 \dots A_5\} \dots\dots\dots\dots\dots (22),$$

or

$$\phi^{-1}\{A_1 \dots A_5\} = m^{-1}\{\phi' A_1 \dots \phi' A_5\} \quad \dots\dots (23),$$

which shows that to obtain $\phi^{-1}E$ we have only to determine $A_1 \dots A_5$ so that $E = \{A_1 \dots A_5\}$, and this can always be done explicitly in an infinite number of ways.

By a similar process we have

$$\phi'^{-1}\{A_1 \dots A_5\} = m^{-1}\{\phi A_1 \dots \phi A_5\} \dots\dots\dots\dots (24).$$

The ϕ sextic can be obtained in the following way. Put

$$\left.\begin{aligned} E_n &= \phi^n E \,.\, \{Z_1 \dots Z_6\}\,\{Z_1 \dots Z_n \phi Z_{n+1} \dots \phi Z_6\} \\ E_n' &= \mathbf{s} Z_1 \phi^n E \,.\, \{Z_2 \dots Z_6\}\,\{Z_1 \dots Z_n \phi Z_{n+1} \dots \phi Z_6\} \end{aligned}\right\} \dots (25).$$

We proceed to show that

$$E_n = nE_n' + (6-n)\,E'_{n+1} \dots\dots \dots\dots\dots (26).$$

By eq. (19) § 28

$$\begin{aligned} E_n = \{Z_1 \dots Z_n \phi Z_{n+1} \,..\, \phi Z_6\}\,[(\{Z_2 \dots Z_6\}\,\mathbf{s} Z_1 \phi^n E - \dots \\ \pm \{Z_1 \dots Z_{n-1} Z_{n+1} \dots Z_6\}\,\mathbf{s} Z_n \phi^n E) \\ + (\mp \{Z_1 \dots Z_n Z_{n+2} \dots Z_6\}\,\mathbf{s} Z_{n+1} \phi^n E + \dots - \{Z_1 \dots Z_5\}\,\mathbf{s} Z_6 \phi^n E)]. \end{aligned}$$

By interchange of suffixes we see that each of the first n terms is equal to the first, and each of the last $6-n$ to the last. Hence

$$E_n = nE_n' + (6-n)\,\mathbf{s} Z_6 \phi^n E \,.\, \{Z_1 \dots Z_5\}\,\{\phi Z_6 Z_1 \dots Z_n \phi Z_{n+1} \dots \phi Z_5\},$$

where ϕZ_6 has been shifted five places and therefore the sign has been changed. Now by eq. (1) change Z_6, ϕZ_6 into $\phi' Z_6$, Z_6, write $\mathbf{s} Z_6 \phi^{n+1} E$ instead of $\mathbf{s}\phi' Z_6 \phi^n E$, and finally change the suffixes from 6, 1, 2, … 5 to 1, 2 … 6. Eq. (26) follows.

Putting in that eq. $n = 0$ and 6, $E_0 = 6E_1'$, $E_6 = 6E_6'$. Also

$$E_n' = E_n/n - E'_{n+1}\,(6-n)/n.$$

Putting $n = 1, 2 \dots 5$ successively

$$E_0 = 6E_1' = \frac{6}{1}E_1 - \frac{5\,.\,6}{1}E_2' = \frac{6}{1}E_1 - \frac{5\,.\,6}{1\,.\,2}E_2 + \frac{4\,.\,5\,.\,6}{1\,.\,2}E_3'$$

$$= \dots = 6E_1 - 15E_2 + 20E_3 - 15E_4 + 6E_5 - E_6,$$

or

$$E_6 - 6E_5 + 15E_4 - 20E_3 + 15E_2 - 6E_1 + E_0 = 0 \dots\dots (27).$$

Substituting for the E's from eq. (25) we get the sextic

$$\phi^6 - m^{\mathrm{v}}\phi^5 + m^{\mathrm{iv}}\phi^4 - m'''\phi^3 + m''\phi^2 - m'\phi + m = 0 \dots\dots (28),$$

where remembering that $\{Z_1 \ldots Z_6\}\{Z_1 \ldots Z_6\} = -6\,!$

$$\left.\begin{array}{r}
6\,!\,m = -\{Z_1 \ldots Z_6\}\{\phi Z_1 \ldots \phi Z_6\} \\
1\,!\,5\,!\,m' = -\{Z_1 \ldots Z_6\}\{Z_1\phi Z_2 \ldots \phi Z_6\} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
p\,!\,(6-p)\,!\,m^{(p)} = -\{Z_1 \ldots Z_6\}\{Z_1 \ldots Z_p\phi Z_{p+1} \ldots \phi Z_6\} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right\} \ldots\ldots(29).$$

We have from the above

$$5\,!\,m' = -\mathsf{s}Z_1\{Z_2 \ldots Z_6\}\,.\,\mathsf{s}Z_1\{\phi Z_2 \ldots \phi Z_6\} = \mathsf{s}\,.\,\{Z_2 \ldots Z_6\}\{\phi Z_2 \ldots \phi Z_6\}$$

or
$$5\,!\,m' = \mathsf{s}\,.\,\{Z_1 \ldots Z_5\}\{\phi Z_1 \ldots \phi Z_5\}\ldots\ldots\ldots\ldots\ldots(30).$$

Also by eq. (13)

$$5\,!\,m^{\text{v}} = -\mathsf{s}Z_6\{Z_1 \ldots Z_5\}\,.\,\mathsf{s}\phi Z_6\{Z_1 \ldots Z_5\} = -5\,!\,\mathsf{s}Z_6\phi Z_6,$$

or
$$m^{\text{v}} = -\mathsf{s}Z\phi Z\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31).$$

I do not see how in the case of $m^{(p)}$ generally to get rid as in these two cases of $Z_1 \ldots Z_p$.

We may deduce from the sextic now obtained the n-tic satisfied by ϕ when acting on any motor of a complex of order n, when ϕ is related to the complex in the manner explained in § 35. But we have given enough to show the essential features of this method of dealing with ϕ.

40. Some further deductions from combinatorial variation. A few miscellaneous properties of motors readily deducible from combinatorial variation will now be established.

In a set of conjugate norms of ϖ, a general self-conjugate function, in a given complex let there be no zero norms (§ 31), and let the positive norms be denoted by $A_1A_2\ldots$ and the negative norms by $B_1B_2\ldots$. Remembering (§ 31) when a conjugate variation is circular and when hyperbolic we see by equations (12) and (13) § 29 that

$$\Sigma\mathbf{S}A\varpi A - \Sigma\mathbf{S}B\varpi B$$

has the same value for any set of conjugate norms in the complex. But for a positive norm $\mathsf{s}A\varpi A = -1$, and for a negative norm $\mathsf{s}B\varpi B = 1$. Hence denoting any norm of the conjugate set by H we have that

$$\Sigma\frac{\mathbf{S}H\varpi H}{\mathsf{s}H\varpi H}$$

has the same value for any set of conjugate norms. Since the value of this expression is unaltered when H is multiplied by any scalar, the expression is constant for any set of conjugate motors.

This again is fully expressed by saying that $\Sigma\,(\mathbf{S}_1 H\varpi H/\mathbf{s}H\varpi H)$ is constant, or finally

$$\Sigma\mathbf{t}^{-1}\mathbf{S}H\varpi H = \text{const.} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

for any set of conjugate motors in the complex.

By eq. (9) § 14, or eq. (3) § 13, $\mathbf{t}H^2 = 2\mathbf{t}H$. Hence putting $\varpi = 1$ we get that

$$\Sigma\mathbf{t}^{-1}H = \text{constant} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

for any set of co-reciprocal motors of a given complex. Eq. (2) is proved in *Screws*, § 136.

Let A', B', C' be *any* motors of a complex of the third order given by A, B, C. Expressing A', B', C' in terms of A, B, C we get

$$\mathbf{S}A'B'C' = x\mathbf{S}ABC \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where x is an ordinary scalar. Hence

$$\mathbf{tS}A'B'C' = \mathbf{tS}ABC \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

or $\mathbf{tS}ABC$ has the same value for any three motors of a complex of the third order.

Similarly if A', B' are any two motors of the complex of the second order given by A, B

$$\mathbf{M}A'B' = x\mathbf{M}AB \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where x is an ordinary scalar; and in particular

$$\mathbf{tM}A'B' = \mathbf{tM}AB \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Equations (3) and (4) are exactly equivalent, but eq. (5) expresses in addition to what is given by eq. (6) that the shortest distance of A' and B' is the shortest distance of A and B; i.e. all the motors of a complex of the second order intersect a definite line perpendicularly. [These equations might be apparently generalised by writing ϖA, ϖB, ϖC, $\varpi A'$, $\varpi B'$, $\varpi C'$ in place of A, B, C, A', B', C'. But if A, B, C belong to a complex of the third order it is obvious that ϖA, ϖB, ϖC also in general belong to a complex of the third order, so that we gain no additional information by the change.]

The geometrical interpretation of these results is given by eq. (1) § 13 and eq. (8) § 12. Adopting the notations of those sections for d, θ, e, ϕ we have

$$\mathbf{t}A + \mathbf{t}B + \mathbf{t}C + d\cot\theta - e\tan\phi = \text{const.} \quad\ldots\ldots\ldots(7),$$

for any three motors of a complex of the third order, and

$$\mathbf{t}A + \mathbf{t}B + d\cot\theta = \text{const.} \qquad\qquad (8),$$

for any two motors of a complex of the second order. Eq. (8) can clearly be deduced from eq. (7) if it is assumed as just proved that the shortest distance of any two motors of a complex of the second order is a definite line.

Another result may be deduced by combining equations (7) and (8). Call the definite line just mentioned the axis of the complex. Thus e is the distance and ϕ the angle between C and the axis of the complex A, B. Suppose now we are given a complex of the second order A, B and a complex of the third order including the first, viz. A, B, C. Then not only is $\mathbf{t}A+\mathbf{t}B+d\cot\theta$ constant for any two motors of the complex of the second order, but $\mathbf{t}C - e\tan\phi$ is constant for any motor of the complex of the third order.

Passing now to combinatorial variation, if A', B', C' are derived from A, B, C by this process the x of eq. (3) is unity, and similarly for A', B', A, B, x in eq. (5).

Let A, B, C (or A, B as the case may be) be conjugate norms of ϖ and let the variation be a conjugate one. Then $\mathbf{s}A\varpi A$, $\mathbf{s}B\varpi B$ and $\mathbf{s}C\varpi C$ remain individually constant. It follows that

$$\frac{\mathbf{S}^2 ABC}{\mathbf{s}A\varpi A\,\mathbf{s}B\varpi B\,\mathbf{s}C\varpi C}$$

is constant for any three conjugate motors (not necessarily norms) of a complex of the third order. Assuming the truth of eq. (4) this only in addition gives us the information that

$$\mathbf{S}_1{}^2 ABC/\mathbf{s}A\varpi A\,\mathbf{s}B\varpi B\,\mathbf{s}C\varpi C$$

is constant.

Putting a, b, c for $\mathbf{t}A$, $\mathbf{t}B$, $\mathbf{t}C$ we have when $\varpi = 1$

$$\sin^2\theta\cos^2\phi \propto abc \qquad\qquad (9).$$

Eq. (7) similarly expressed gives

$$a + b + c + d\cot\theta - e\tan\phi = \text{const.} \qquad\qquad (10)$$

and eq. (2)

$$a^{-1} + b^{-1} + c^{-1} = \text{const.} \qquad\qquad (11).$$

We shall see directly that in general there are three mutually perpendicular intersecting motors belonging to the complex.

For these $d = e = \phi = 0$ and $\theta = \frac{1}{2}\pi$. Calling their pitches a_0, b_0, c_0 the above equations give

$$\left.\begin{aligned} \sin^2\theta\cos^2\phi = \frac{abc}{a_0b_0c_0} = \frac{bc+ca+ab}{b_0c_0+c_0a_0+a_0b_0} \\ a+b+c+d\cot\theta - e\tan\phi = a_0+b_0+c_0 \end{aligned}\right\} \text{......(12).}$$

Similarly for a complex of the second order

$$\left.\begin{aligned} \sin^2\theta = \frac{ab}{a_0b_0} = \frac{a+b}{a_0+b_0} \\ a+b+d\cot\theta = a_0+b_0 \end{aligned}\right\} \text{..................(13),}$$

but eq. (13) is most easily deduced from eq. (12).

Eq. (12) is fully expressed by saying that a, b, c are the roots of the equation

$$\begin{aligned}(x-a_0)(x-b_0)(x-c_0)+x^2\{(\operatorname{cosec}^2\theta\sec^2\phi-1)(x-a_0-b_0-c_0) \\ +\operatorname{cosec}^2\theta\sec^2\phi\,(d\cot\theta-e\tan\phi)\}=0 \text{......(14),}\end{aligned}$$

and eq. (13) by saying that a, b are the roots of

$$(x-a_0)(x-b_0)+x^2\cot^2\theta=0 \text{..............(15),}$$

and that $$d=\tfrac{1}{2}(a_0+b_0)\sin 2\theta \text{....................(16).}$$

It may be noticed that eq. (10) gives for three perpendicular motors of a complex of the third order

$$a+b+c=\text{const.} \text{........................(17),}$$

and similarly for two perpendicular motors of a complex of the second order

$$a+b=\text{const.} \text{........................(18).}$$

41. Analysis of complexes of all orders into their simplest elements. We proceed to express every complex as a complex of reciprocal motors; the motors being whenever it is possible both perpendicular and intersecting.

Every motor (including rotors and lators) has a coaxial reciprocal motor, the latter having a pitch equal in magnitude and opposite in sign to that of the former. From this, by writing down in the most obvious way the complexes which are reciprocal to the complexes of orders zero, one, two, we shall obtain general forms of the complexes of orders six, five, four.

Thus the complex of the first order is that of one motor. If this is a lator the complex is that of Ωi, where i is any unit rotor parallel to the lator. If it is an axial motor, the unit rotor i may be taken on the axis and the complex is that of $(1 + a'\Omega) i$. Thus the complex may always be expressed as that of $(a + a'\Omega) i$, where a and a' are ordinary scalars, not both zero. The complex of the fifth order reciprocal to this is that of $(a - a'\Omega) i$ and of $(b \pm b'\Omega) j$ and $(c \pm c'\Omega) k$; where the five motors are reciprocal to one another and with axes on three mutually perpendicular intersecting lines; provided that j and k are *any* perpendicular intersecting unit rotors, intersecting i perpendicularly, and that not one of the ordinary scalars b, b', c, c' is zero (though they are otherwise arbitrary).

Similarly the complex of the sixth order is that of the six co-reciprocal motors $(a \pm a'\Omega) i$, $(b \pm b'\Omega) j$, $(c \pm c'\Omega) k$, where i, j, k are *any* set of mutually perpendicular intersecting unit rotors and a, a', b, b', c, c' are any ordinary scalars of which not one is zero.

We proceed now to show that every complex can be expressed as consisting of one of the lists of motors contained in the following table. It is to be understood that i, j, k are three mutually perpendicular intersecting unit rotors, and when in the table it is said that two, or three, of these are arbitrary, it is meant that they are arbitrary within the limits imposed by this condition.

Complexes expressed in the simplest form as consisting of co-reciprocal motors.

Order.	Complexes of orders up to the third.	Order.	The reciprocal complexes.
0	0	6	$(a \pm a'\Omega)\,i$, $(b \pm b'\Omega)\,j$, $(c \pm c'\Omega)\,k$. *Arbitrary.* $i, j, k, a, a', b, b', c, c'$. None of the scalars zero.
1	$(a + a'\Omega)\,i$.	5	$(a - a'\Omega)\,i$, $(b \pm b'\Omega)\,j$, $(c \pm c'\Omega)\,k$. *Arbitrary.* j, k, b, b', c, c'. None of the scalars zero.
2	(α) $(a + a'\Omega)\,i$, $(b + b'\Omega)\,j$. (β) $i \pm a'\Omega\,(i \cos\theta + j \sin\theta)$ $= \{1 \pm a'\Omega\,(\cos\theta + k \sin\theta)\}\,i$. a' is arbitrary but not zero. The two motors are parallel, in a definite plane (i, k) at arbitrary equal distances $(a' \sin\theta)$ from a definite line (i). Their pitches $(\pm\, a' \cos\theta)$ are equal and opposite and are proportional to the arbitrary distance mentioned.	4	(α) $(a - a'\Omega)\,i$, $(b - b'\Omega)\,j$, $(c \pm c'\Omega)\,k$. *Arbitrary.* c, c'. Neither zero. (β) $\left.\begin{array}{l} -i \sin\theta + j \cos\theta \pm b'\Omega j \\ = \{1 \pm b'\Omega(\cos\theta - k \sin\theta)\}(-i \sin\theta + j \cos\theta) \end{array}\right\}(c \pm c'\Omega)\,k$. *Arbitrary.* b', c, c'. None zero. Pitches of first two motors $= \pm b' \cos\theta$. Distance of each of first two motors from $-i \sin\theta + j \cos\theta$, $= b' \sin\theta$.
3	(γ) $(a + a'\Omega)\,i$, $(b + b'\Omega)\,j$, $(c + c'\Omega)\,k$. (δ) $\left.\begin{array}{l} i \pm a'\Omega\,(i \cos\theta + j \sin\theta) \\ = \{1 \pm a'\Omega(\cos\theta + k \sin\theta)\}\,i \end{array}\right\}(c + c'\Omega)\,k$. See fig. 7 and case ($\beta$) above.	3	(γ) $(a - a'\Omega)\,i$, $(b - b'\Omega)\,j$, $(c - c'\Omega)\,k$. (δ) $\left.\begin{array}{l} -i \sin\theta + j \cos\theta \pm b'\Omega j \\ = \{1 \pm b'\Omega(\cos\theta - k \sin\theta)\}(-i \sin\theta + j \cos\theta) \end{array}\right\}(c - c'\Omega)\,k$. *Arbitrary.* b'. Not zero.

Fig. 7 represents fully cases (β) and (δ) except when $\theta = 0$ or $\frac{1}{2}\pi$. It has only to be remembered that a' and b' are arbitrary but not zero; and in case (β) c and c' are also arbitrary but not zero. The two planes represented in the figure are definite planes.

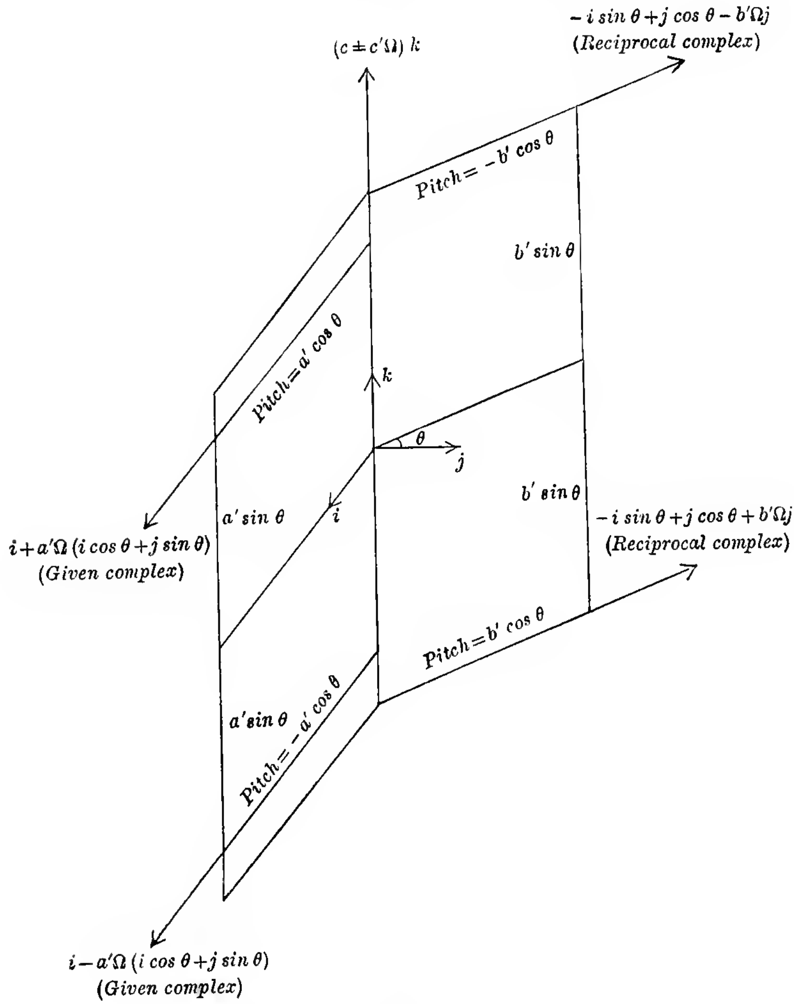


FIG. 7.

When $\theta = \frac{1}{2}\pi$ we get a particular case of (α) or of (γ) as the case may be. When $\theta = 0$, the two motors of (β) are the coaxial motors $(1 \pm a'\Omega)\,i$, and the reciprocal complex is that of $(1 \pm b'\Omega)\,j$ and $(c \pm c'\Omega)\,k$. Similarly for case (δ).

The statements of the table are all obvious from what has been said, except those which relate to the complexes of the second and third orders.

Put $$\varpi E = E, \quad \psi E = \Omega E.$$

Thus ϖ is a self-conjugate function and ψ is an energy function; for $-\mathbf{s}E\psi E$ is positive for every axial motor, being the square of the tensor of the motor, and zero for every lator.

Hence by § 33 any complex may be expressed as consisting of a conjugate set of norms of ϖ which are conjugate also with regard to ψ except possibly for certain pairs of positive and negative norms $(\alpha_1\beta_1)(\alpha_2\beta_2)(\alpha_3\beta_3)$. A motor of such a pair is conjugate with regard to ψ to all the norms except the other motor of that particular pair. Also for any such pair $\psi(\alpha-\beta)$ or $\Omega(\alpha-\beta)=0$, i.e. $\alpha-\beta$ is a lator. Now α and β are not lators, for since they are a positive and a negative norm of ϖ, $-\mathbf{s}\alpha^2=\mathbf{s}\beta^2=1$. Hence α and β are two parallel axial motors (§ 18) whose rotor parts are equal and similarly directed and whose lator parts are equal and oppositely directed.

Let now H_1 and H_2 be any two of the norms not forming such a pair. Since they are conjugate with regard to ϖ, $\mathbf{s}H_1H_2=0$, i.e. they are reciprocal. [*This* relation is of course also true for the two motors of a pair.] Since they are conjugate with regard to ψ, $\mathbf{s}H_1\Omega H_2$ or $\mathbf{S}_1H_1H_2=0$. Hence $\mathbf{S}H_1H_2=0$.

A lator may be said to intersect any line. Using for the moment this terminology we see from § 12 that H_1 and H_2 intersect perpendicularly or are both lators. A lator is a zero norm both of ϖ and of ψ. Hence (§ 30) any of the norms which are lators may be replaced by any independent lators (of the same number) of the same complex. They may therefore be replaced by mutually perpendicular lators. Hence all the norms may be so chosen as to intersect one another perpendicularly *except* that α_1 and β_1 do not, and α_2 and β_2 do not, and α_3 and β_3 do not. [Since not more than three lines can be intersecting and mutually perpendicular, it follows from this that in no complex can there be more than three norms which do not belong to the pairs; and if the order is even, not more than two. And in the complex of the sixth order all the norms belong to the pairs.]

In a complex of the second or third order there cannot be more than one pair. The statements in the table now all follow.

CHAPTER V.

EXAMPLES OF THE APPLICATION OF OCTONIONS.

THE examples about to be given are all either taken directly from *Screws* or immediately suggested by portions of that treatise.

42. Systems of forces. Finite displacements. By eq. (12) § 12, we see that if ρ, σ be two rotors, and if the tetrahedron which has ρ and σ for opposite edges be called the tetrahedron ρ, σ:—

The volume of the tetrahedron ρ, σ is $\pm \frac{1}{6}\mathbf{s}\rho\sigma$, the upper or lower sign being taken according as the shortest twist that will bring ρ into coincidence both as to position and sense with σ is a right-handed or a left-handed one.

In the Appendix of *Screws*, p. 178, certain propositions due to A. F. Möbius—*Lehrbuch der Statik* (Leipzig, 1837)—are cited. We take some of these as our first examples.

"If a number of forces acting on a free rigid body be in equilibrium, and if a straight line of arbitrary length and position be assumed, then the algebraic sum of the tetrahedra of which the line and each of the forces in succession are pairs of opposite edges is equal to zero." Let $\rho_1, \rho_2 \dots \rho_n$ be the rotors representing the forces and α that representing the arbitrary straight line. Since the forces are in equilibrium, $\Sigma\rho = 0$. Hence

$$0 = \mathbf{s}\alpha\Sigma\rho = \Sigma\mathbf{s}\alpha\rho.$$

This proves the proposition.

1, 2, 3, 4 being four straight lines and 12 representing the perpendicular distance between 1 and 2 multiplied by the sine of the angle between them, we have that if ρ_1, ρ_2 be rotors whose axes are 1 and 2,

$$12 = \mathbf{s}\mathbf{U}\rho_1\mathbf{U}\rho_2.$$

If ρ_1, ρ_2, ρ_3, ρ_4 be four forces in equilibrium along the lines 1, 2, 3, 4 we have

$$\rho_1 + \rho_2 + \rho_3 + \rho_4 = 0.$$

Hence $$\rho_1 + \rho_2 = -(\rho_3 + \rho_4).$$

Squaring, $$\rho_1^2 + \rho_2^2 + 2\mathbf{S}\rho_1\rho_2 = \rho_3^2 + \rho_4^2 + 2\mathbf{S}\rho_3\rho_4.$$

Equating the convertor parts,

$$\mathbf{s}\rho_1\rho_2 = \mathbf{s}\rho_3\rho_4.$$

This is quoted in the same place by Sir Robert Ball as a theorem due to Chasles. Similarly we have

$$\mathbf{s}\rho_1\rho_3 = \mathbf{s}\rho_2\rho_4, \quad \mathbf{s}\rho_1\rho_4 = \mathbf{s}\rho_2\rho_3.$$

Hence $$\mathbf{s}\rho_2\rho_3\mathbf{s}\rho_2\rho_4\mathbf{s}\rho_3\rho_4 = \mathbf{s}\rho_1\rho_3\mathbf{s}\rho_1\rho_4\mathbf{s}\rho_3\rho_4 = \ldots = \ldots,$$

each group being obtained by writing down all the products from which one suffix is absent. Dividing by $\rho_1^2\rho_2^2\rho_3^2\rho_4^2$ we get

$$\frac{23\,.\,24\,.\,34}{\rho_1^2} = \frac{13\,.\,14\,.\,34}{\rho_2^2} = \frac{12\,.\,14\,.\,24}{\rho_3^2} = \frac{12\,.\,13\,.\,23}{\rho_4^2},$$

another result quoted as due to Möbius.

A result somewhat similar to the above $\mathbf{s}\rho_1\rho_2 = \mathbf{s}\rho_3\rho_4$ is obtained by squaring both sides of the equation

$$-\rho_1 = \rho_2 + \rho_3 + \rho_4$$

and equating the convertor parts. It is

$$\mathbf{s}\rho_2\rho_3 + \mathbf{s}\rho_2\rho_4 + \mathbf{s}\rho_3\rho_4 = 0.$$

This is not independent of the two previous results ($\Sigma\mathbf{s}\alpha\rho = 0$ and $\mathbf{s}\rho_1\rho_2 = \mathbf{s}\rho_3\rho_4$), as can be seen by putting $\alpha = \rho_2$.

On p. 179 is quoted from Möbius that: "Any given displacement of a rigid body can be effected by two rotations." From the title of Möbius' paper this apparently refers only to small displacements and rotations. In this form (§ 11 above) it is only necessary to prove that any motor A can be expressed as the sum of two rotors ρ and σ. For then the small displacement $\mathbf{M}A\,(\;)$ will be compounded of the two small rotations $\mathbf{M}\rho\,(\;)$ and $\mathbf{M}\sigma\,(\;)$. As a matter of fact, A can be expressed in an infinite number of ways as $\rho + \sigma$. We will however prove the more general theorem for a displacement of any magnitude and rotations of any magnitude. We will show that such a displacement can always be effected by one such rotation followed by another, and this in an infinite number of ways.

To prove this we have by § 11 above to show that any octonion Q for which $\mathbf{T}Q=1$ can be expressed in the form qr, where q and r are (§ 6) axials for which $\mathbf{T}q=\mathbf{T}r=1$. [Note that an octonion Q in general cannot be expressed in the form qr where q and r are axials, since $\mathbf{T}Q$ is not in general an ordinary scalar, whereas $\mathbf{T}(qr)=\mathbf{T}q\mathbf{T}r$ is.]

Looking upon Q as an operator on motors we see that it can always (when $\mathbf{T}Q=1$) be expressed as $\rho\sigma^{-1}$, where ρ and σ are two rotors whose tensors are equal. Now let τ be *any* rotor whose tensor is the same as that of ρ and that of σ and which intersects both ρ and σ. Then

$$Q=\rho\sigma^{-1}=\rho\tau^{-1}\tau\sigma^{-1}=qr,$$

where $q=\rho\tau^{-1}$, $r=\tau\sigma^{-1}$. Here since ρ and τ are intersecting rotors with equal tensors q is an axial for which $\mathbf{T}q=1$, and similarly r is an axial for which $\mathbf{T}r=1$. This proves the theorem.

Note that τ is at right angles to the axes of both q and r, i.e. it is along the shortest distance of these axes. Starting with q and r it is easy to show that *any* two rotations which suffice to produce the given displacement can be obtained by the above construction.

The construction can be given in language not explicitly involving octonions. To do this, first define the double of a given twist as the twist obtained by doubling both the translation and rotation of the given twist (when that is reduced to the standard or canonical form of a translation along and a rotation around one and the same axis). Similarly for a half twist, a double rotation and a half rotation. Then we have the following:—

A twist can always be effected in an infinite number of ways by two rotations. The following construction suffices to find any two such rotations. Take any line 1 *intersecting the axis of the twist perpendicularly. Let* 1 *become* 2 *when it is subjected to half the given twist. Take any transversal* 3 *of* 1 *and* 2. *Then double the rotation which converts* 1 *into* 3 *followed by double the rotation which converts* 3 *into* 2 *will effect the given twist.*

A particular case of this is interesting. Let ρ and σ above be unit rotors and put $\sigma^{-1}=\rho'$. Thus

$$Q=\rho\rho',$$

so that a particular case of the above is obtained by putting $q=\rho$, $r=\rho'$. In this case the axials, viewed as operators, are quadrantal versors, and therefore the two rotations are rotations each through two right angles, or as we may call them semi-revolutions. A semi-revolution is completely specified by its axis. Thus we have:—

A twist can always be effected in an infinite number of ways by two semi-revolutions. The following suffices to obtain the axes in all cases. The axis of the first semi-revolution is any line intersecting the axis of the twist perpendicularly. The second axis is obtained from the first by giving to the latter half the given twist.

If the given twist degenerates into a translation this becomes:—

If the twist be a translation the first axis is any line at right angles to the direction of translation, and the second axis is obtained by giving to the first half the given translation.

It is interesting to note that of the four semi-revolutions thus possible to effect the twist obtained by superposing one given twist on another, the two intermediate ones can be made to cancel by taking the shortest distance between the axes of the twists as the axis of the second semi-revolution of the first twist and also as the axis of the first semi-revolution of the second twist.

In the same part of the Appendix we find:—"Two equal parallel and opposite rotations combine into a translation." This also is apparently meant only to apply to small displacements, but we will prove it for finite ones. It might be treated as a particular case of the above, but we will treat it separately.

Let q be the axial ($\mathbf{T}q=1$) which expresses the first rotation and let ρ be a rotor perpendicular from any point of the first axis on the second. Thus by eq. (1) § 7 the second rotation is $q^{-1}+\Omega\rho\mathbf{M}q^{-1}$ for $\mathbf{S}\rho q^{-1}=0$. Hence the whole displacement is $(q^{-1}+\Omega\rho\mathbf{M}q^{-1})\,q$ or $1+\Omega\rho\mathbf{M}q^{-1}\,.\,q$. Hence (§ 11) the displacement is a translation equal and parallel to $2\rho\mathbf{M}q^{-1}\,.\,q$. Since $q^{-1}=\mathbf{K}q$, we have

$$\begin{aligned}2\rho\mathbf{M}q^{-1}\,.\,q &= -\,2\mathbf{K}\,(\rho\mathbf{M}q^{-1}\,.\,q) = -\,2\mathbf{K}q\mathbf{M}q^{-1}\,.\,\rho\\ &= q^{-1}(q-q^{-1})\,\rho=(1-q^{-2})\,\rho.\end{aligned}$$

Hence *the translation is compounded of two translations, the one*

being equal and parallel to the perpendicular from the first axis on the second, and the other equal and parallel to the same perpendicular when it has been first rotated with the second rotation and then reversed.

In connection with this subject we may put into a form not explicitly involving octonions the statement that the displacement QR is the displacement obtained by superposing the displacement Q on the displacement R. It may be deduced from the remark above about the corresponding four semi-revolutions.

To combine two twists take two lines 1 *and* 2 *such that half the first twist brings* 1 *into coincidence with the shortest distance between the axes of the twists and half the second twist brings the shortest distance into coincidence with* 2. *Then the axis of the resultant twist is the shortest distance between* 1 *and* 2, *and the twist itself is double the twist about this axis which will bring* 1 *into coincidence with* 2.

43. Geometrical properties of the second order complex. Passing now to the text of *Screws* let us first establish the chief geometrical propositions there enunciated with regard to the complexes of different orders.

Beginning with the complex of the second order, case (β) of the table of § 41 and any case of (α) where one or more of the scalars a, a', b, b' is zero may be called singular cases. These are all simple. Let us dispose of them first.

First consider the singular cases of (α). If $a = b = 0$ the complex consists of all lators parallel to a certain plane. Every motor of the complex is self-reciprocal and any two motors are reciprocal.

If $a' = b = 0$ the complex consists of a plane of parallel rotors. Also the lator parallel to the plane and perpendicular to the rotors belongs to the complex. All the motors are self-reciprocal and any two are reciprocal. Similarly for the case $b' = a = 0$.

If b only $= 0$ the complex consists of a plane of parallel motors all of the same pitch. Also the lator parallel to the plane and perpendicular to the motors belongs to the complex. This lator is reciprocal to every motor of the complex and is the only motor of the complex which is reciprocal to any given motor of the plane of motors. Similarly for the case when a only $= 0$.

If $a' = b' = 0$ the complex consists of a plane pencil of rotors passing through a fixed point of the plane. All the rotors are self-reciprocal and any two are reciprocal.

If b' only $= 0$ the complex is that of $(1 + \Omega p)\, i,\ j$, where p is not zero. The rotor j is self-reciprocal and is *the* reciprocal of any motor of the complex. Any motor of the complex can be put in the form

$$r\{(1 + \Omega p)\, i \cos\theta + j \sin\theta\}$$
$$= r\{1 + \Omega\,(p\cos^2\theta - kp\sin\theta\cos\theta)\}\,(i\cos\theta + j\sin\theta),$$

where r is an ordinary scalar. Thus all the motors intersect k perpendicularly. If θ is the inclination of one to i, its pitch is $p\cos^2\theta$ and the perpendicular on it from the point of intersection O of i, j, k is $-kp\sin\theta\cos\theta$. They all lie on the ruled surface whose ordinary Cartesian equation is

$$z\,(x^2 + y^2) + pxy = 0,$$

which surface lies between the limits $z = \pm\tfrac{1}{2}p$ [the maximum and minimum values of $-p\sin\theta\cos\theta$]. If a rectangular hyperbola with i and j for asymptotes be drawn and also a straight line through any point of i except O parallel to j; and if any line OPQ be drawn through O in the plane of i, j cutting the hyperbola in P and the straight line in Q; the pitch of a motor of the complex parallel to OPQ is inversely proportional to OQ^2 and the distance of the motor from O is inversely proportional to OP^2. This case can easily be looked upon as a particular form of the non-singular case. Similarly for the case when a' only $= 0$.

This disposes of all the singular cases of (α) of the table. In case (β) if $\theta = \tfrac{1}{2}\pi$ we get the case just considered when $a' = b = 0$. When $\theta = 0$ we get a set of coaxial motors. The reciprocal of any motor is the motor with equal and opposite pitch. There are two self-reciprocal motors, the coaxial rotor and the parallel lator.

When θ is neither zero nor $\tfrac{1}{2}\pi$ the chief geometrical properties of the complex are contained in the table and represented in fig. 7. The two motors represented are reciprocal motors, and by varying a' we can get any motor of the complex except the lator

$$\Omega\,(i\cos\theta + j\sin\theta)$$

[which corresponds to $a' = \infty$].

The complex that remains for consideration is that of $(1 + a\Omega)i$, $(1 + b\Omega)j$, where neither a nor b is zero.

In discussing the properties of this and also those of the complex of the third order in the next section, octonion methods for the most part are to all intents and purposes quaternion methods. The examples now to be considered, if they serve no other purpose, will show how quaternion methods are practically a particular form of octonion methods.

The point of intersection of i, j, k will be denoted by O and called the origin. The rotor from O to any point P will be denoted by ρ and may be called the coordinate rotor of P. It will be used in a manner practically identical with that of using the quaternion ρ when that represents a coordinate vector.

Define the self-conjugate pencil function (§ 15) ψ by the equation

$$\psi E = -ai\mathbf{S}iE - bj\mathbf{S}jE \quad\text{.....................(1).}$$

Thus any motor belonging to the complex can be expressed as $\omega + \Omega\psi\omega$, where ω is any rotor through O perpendicular to k.

The equation

$$\mathbf{S}\rho\psi\rho = -1 \quad\text{..........................(2)}$$

represents a cylinder which is completely specified by its trace—a conic—on the plane of i, j. We shall refer to this conic as the conic (2). In fact (2) is the equation of the conic when ρ is confined to being perpendicular to k (i.e. $\mathbf{S}\rho k = 0$).

Similarly the equation

$$\mathbf{S}\rho\psi\rho = 1 \quad\text{.............................(3)}$$

also represents a conic in the same way. When both conics are real they are conjugate hyperbolas. In this case

$$\mathbf{S}\rho\psi\rho = 0 \quad\text{.............................(4)}$$

is the equation of their asymptotes.

Since (§ 9) $\mathbf{T}_1(\omega + \Omega\psi\omega) = \mathbf{T}\omega$, the pitch of $\omega + \Omega\psi\omega$ is by eq. (10) § 14, $\frac{1}{2}\mathbf{s}(\omega + \Omega\psi\omega)^2/\omega^2 = \mathbf{S}\omega\psi\omega^{-1}$. Hence if the ρ of eq. (2) is parallel to the motor $\omega + \Omega\psi\omega$ of the complex, the pitch of that motor is $-\rho^{-2}$; and if the ρ of eq. (3) is parallel to the motor the pitch is ρ^{-2}. (2) may be called *the* pitch conic, and (3) the conjugate pitch conic. [In *Screws* the pitch conic is defined as the conic $\mathbf{S}\rho\psi\rho = -H$, where H is any ordinary constant scalar.]

The pitch then of any motor of the complex is the inverse square of the parallel semi-diameter of the pitch conic, and is with sign reversed the inverse square of the parallel semi-diameter of the conjugate pitch conic. Also the two motors of the complex parallel to the asymptotes (4) have zero pitch, i.e. are rotors. (*Screws*, § 20.)

The equation of the cylindroid (the ruled surface to which motors of the complex are confined) and the distribution of pitch on it may be obtained in the usual form thus,

$$\omega + \Omega\psi\omega = (1 + \Omega\psi\omega \,.\, \omega/\omega^2)\,\omega = \{1 + \Omega\,(p + zk)\}\,\omega,$$

where p and z are scalars given by

$$p = \mathbf{S}\omega\psi\omega/\omega^2, \quad zk = -\mathbf{M}\omega\psi\omega/\omega^2.$$

Putting $\omega = xi + yj$, x, y, z will have their usual Cartesian meanings [eq. (1) § 7 above] and p will be the pitch [eq. (8) § 9]. Thus

$$p = \frac{ax^2 + by^2}{x^2 + y^2}, \quad z = \frac{(b-a)\,xy}{x^2 + y^2} \quad \ldots\ldots\ldots\ldots\ldots(5).$$

[Otherwise—by eq. (2) § 6 the pitch $= \mathbf{S}\psi\omega\omega^{-1}$ and the perpendicular rotor from O on the motor $= \mathbf{M}\psi\omega\omega^{-1}$.]

If two motors of the complex $\omega_1 + \Omega\psi\omega_1$ and $\omega_2 + \Omega\psi\omega_2$ are reciprocal,

$$0 = \mathsf{s}\,.\,(\omega_1 + \Omega\psi\omega_1)(\omega_2 + \Omega\psi\omega_2) = 2\mathbf{S}\omega_1\psi\omega_2.$$

Hence they are parallel to a pair of conjugate diameters of the pitch conic. [*Screws*, § 42. There is an error at the end of this section where it is stated that the sum *or difference* of the reciprocals of the pitches of two reciprocal motors of a complex of the second order is constant. It is the sum in every case that is constant, as we saw in § 40 above and as Sir Robert Ball himself proves in *Screws*, § 136. The cause of the error will be easily enough seen by the reader.]

Suppose $\omega + \Omega\sigma$, where ω is a unit rotor and σ any rotor through O, is a given "screw," i.e. a motor with unit tensor. Required the "screw" of our complex that has a given "virtual coefficient" with $\omega + \Omega\sigma$. [The virtual coefficient of two motors A and B is $-\mathsf{s}AB/\mathbf{T}_1A\mathbf{T}_1B$.] The virtual coefficient of two screws $\omega + \Omega\sigma$ and $\omega' + \Omega\sigma'$ is $-\mathbf{S}\,(\omega\sigma' + \sigma\omega')$. Hence the virtual coefficient of the screw $\omega + \Omega\sigma$ and the screw of our complex $(1 + a\Omega)\,i\cos\theta + (1 + b\Omega)\,j\sin\theta$ is

$$-\mathbf{S}\,\{\omega\,(ai\cos\theta + bj\sin\theta) + \sigma\,(i\cos\theta + j\sin\theta)\}$$
$$= -\cos\theta\mathbf{S}i\,(a\omega + \sigma) - \sin\theta\mathbf{S}j\,(b\omega + \sigma).$$

The two values of θ for which this is zero and a maximum or minimum differ by $\frac{1}{2}\pi$, i.e. the two corresponding screws of the complex are perpendicular (*Screws*, § 87).

In § 25 of *Screws* it is proved that there is a cone of the second degree with vertex at any point on which lie all the motors through the point which are reciprocal to a given complex of the second order. And in § 97 it is proved that these motors have pitches ranging from $+\infty$ to $-\infty$ and that the lator (motor of infinite pitch) is parallel to the nodal line of the cylindroid. To prove these statements let the point be taken for origin and let two of the motors of the complex to which the motors are to be reciprocal be $\omega_1 + \Omega\sigma_1$ and $\omega_2 + \Omega\sigma_2$, where ω_1, ω_2, σ_1, σ_2 are rotors through the point. The nodal line is the axis (§ 40 above) of $\mathbf{M}(\omega_1 + \Omega\sigma_1)(\omega_2 + \Omega\sigma_2)$, i.e. it is (§ 12 above) parallel to $\mathbf{M}\omega_1\omega_2$. Let $(1 + \Omega p)\omega$ be one of the motors whose pitch is p, where ω is a rotor through the point. Expressing that it is reciprocal to both $\omega_1 + \Omega\sigma_1$ and $\omega_2 + \Omega\sigma_2$, we have

$$\left.\begin{array}{l} p\mathbf{S}\omega\omega_1 + \mathbf{S}\omega\sigma_1 = 0 \\ p\mathbf{S}\omega\omega_2 + \mathbf{S}\omega\sigma_2 = 0 \end{array}\right\} \quad \text{.....................(6).}$$

These give as a necessary and sufficient condition

$$\omega = x\mathbf{M}(p\omega_1 + \sigma_1)(p\omega_2 + \sigma_2) \quad \text{.................(7),}$$

where x is an ordinary scalar. This can be satisfied for all values of p not infinite, but for this value we of course take the motor to be $\Omega\omega$ when we get $\omega = x\mathbf{M}\omega_1\omega_2$. This shows (1) that one motor (and ordinary scalar multiples of it) and only one can be found for each value of p from $+\infty$ to $-\infty$, and (2) that the motor of infinite pitch is parallel to the nodal line of the cylindroid. Also eliminating p from equations (6) we find that ω satisfies the homogeneous quadratic equation

$$\mathbf{S}\omega\omega_1\mathbf{S}\omega\sigma_2 - \mathbf{S}\omega\sigma_1\mathbf{S}\omega\omega_2 = 0 \quad \text{....................(8),}$$

i.e. it lies on a cone of the second degree.

The first of equations (6) proves other facts. For instance in § 80 of *Screws* it is stated that every line in space serves as the residence of a motor reciprocal to a given motor. Given ω and $\omega_1 + \Omega\sigma_1$, the first of equations (6) serves to determine p so that $(1 + \Omega p)\omega$ shall be thus reciprocal. Again in § 80 it is stated that all the motors of given pitch which pass through a given point and are reciprocal to a given motor lie in a plane.

This is obvious from the same equation above since now we must suppose p given when the equation becomes a homogeneous linear equation in ω, so that ω lies in a plane through the origin.

44. Geometrical properties of the third order complex. The complex of the third order can be treated in a way very like that of the second order. We shall consider the case of no singularities only, i.e. case (γ) of the table of § 41 where not one of the six scalars a, a', b, b', c, c' is zero. With the present as with the second order complex we shall change the notation so that the complex is that of $(1 + a\Omega) i$, $(1 + b\Omega) j$ and $(1 + c\Omega) k$.

Define the self-conjugate pencil function ψ by the equation

$$\psi E = -ai\mathbf{S}iE - bj\mathbf{S}jE - ck\mathbf{S}kE \quad \text{..............(1).}$$

Since the reciprocal complex is that of

$$(1 - a\Omega) i, \quad (1 - b\Omega) j, \quad (1 - c\Omega) k,$$

the corresponding pencil function for it is $-\psi$.

Any motor of the present complex is $\omega + \Omega\psi\omega$, where ω is any rotor through O.

The two quadrics

$$\mathbf{S}\rho\psi\rho = abc \quad \text{............................(2),}$$

$$\mathbf{S}\rho\psi\rho = -abc \quad \text{..........................(3)}$$

will be called the pitch quadric and the conjugate pitch quadric respectively. Their common asymptotic cone is

$$\mathbf{S}\rho\psi\rho = 0 \quad \text{............................(4).}$$

These quadrics have a fundamental property exactly similar to that of the pitch conics in the second order complex. For the pitch of the motor $\omega + \Omega\psi\omega$ of the present complex is $\mathbf{S}\omega^{-1}\psi\omega$, whence we have that if the ρ of (2) be parallel to the motor the pitch is $abc\rho^{-2}$, and if the ρ of (3) be parallel to the motor the pitch is $-abc\rho^{-2}$. Also the motors of zero pitch are those parallel to the generating lines of the asymptotic cone.

But in the present case not only is this last true but the motors of zero pitch, i.e. the rotors of the system, actually lie on the pitch quadric. To prove this we will find the locus of the motors of given pitch p. The pitch of $\omega + \Omega\psi\omega$ is $\mathbf{S}\omega^{-1}\psi\omega$. Hence if this is p, ω is confined to a cone of the second degree, as is evident from the equation

$$\mathbf{S}\omega(\psi - p)\omega = 0.$$

The rotor ρ from O to any point on this motor is by § 6 above

$$\rho = \mathbf{M}(\psi\omega + x)\,\omega^{-1},$$

where x is any scalar. Thus the extremity of ρ is confined to a ruled surface. We will show that this surface is a quadric. Putting $\psi - p = \varpi$, we have

$$\mathbf{S}\omega\varpi\omega = 0,\ \rho = \mathbf{M}(\varpi\omega + x)\,\omega^{-1} = (\varpi\omega + x)\,\omega^{-1}.$$

Hence $\mathbf{M}\rho\omega = \varpi\omega$. Hence

$$\omega = \varpi^{-1}\mathbf{M}\rho\omega = m_p{}^{-1}\mathbf{M}\varpi\rho\varpi\omega,$$

where m_p has the usual meaning with regard to the pencil function ϖ, i.e.

$$m_p = -\,\mathbf{S}\varpi i\varpi j\varpi k = (a-p)(b-p)(c-p)\,\ldots\ldots\ldots(5).$$

Substituting in the last equation $\mathbf{M}\rho\omega$ for $\varpi\omega$, we have

$$\omega = m_p{}^{-1}\mathbf{M}\varpi\rho\mathbf{M}\rho\omega = \omega m_p{}^{-1}\mathbf{S}\rho\varpi\rho,$$

since $\mathbf{S}\omega\varpi\rho = \mathbf{S}\rho\varpi\omega = \mathbf{S}\rho\mathbf{M}\rho\omega = 0$. Thus finally

$$\mathbf{S}\rho\,(\psi - p)\,\rho = \mathbf{S}\rho\varpi\rho = m_p = (a-p)\,(b-p)\,(c-p)\ldots(6).$$

Putting in this $p = 0$ we get eq. (2), which shows, as stated above, that the pitch quadric is the locus of the rotors of the complex.

Since for the reciprocal complex we have merely to change ψ into $-\psi$, eq. (6) is also the locus of those motors of the reciprocal complex whose pitch is $-p$. It is obvious from the fact that two motors, the sum of whose pitches is zero, are reciprocal only if they intersect that the motors of the given complex lie on one set of generators and the motors of the reciprocal complex on the other set. This also follows from the fact that though for both sets of motors $\mathbf{S}\omega\varpi\omega = 0$, we have for the motors of the given complex

$$\rho = (\varpi\omega + x)\,\omega^{-1},$$

and for those of the reciprocal complex

$$\rho = -\,(\varpi\omega + x)\,\omega^{-1}.$$

Thus a plane through the centre containing a motor of the given complex also contains at an equal distance on the opposite side of the centre a parallel motor of the reciprocal complex.

[It is deducible from the above that the generating lines of $\mathbf{S}\rho\phi\rho = -1$, where ϕ is any self-conjugate pencil function, are given by $\rho = \pm\,\{(-m)^{-\frac{1}{2}}\phi\sigma + x\}\,\sigma^{-1}$, where σ is any rotor gene-

rating line of the asymptotic cone $S\sigma\phi\sigma = 0$. That $(-m)^{-\frac{1}{2}}$ may be real all three roots or one and only one of the ϕ cubic must be negative. That the quadric and the asymptotic cone may themselves be real all three cannot be negative. Hence one and only one is negative or the quadric is an hyperboloid of one sheet.]

Eq. (6) may be looked at in a different way. Regarding p as given it represents, as we have seen, a quadric on which the motor must lie. But regarding ρ as given it is a cubic for p, the pitch of a motor of the complex passing through a given point. Thus there are three such pitches. And there are three such motors. These are given by the equation $\mathbf{M}\rho\omega = \varpi\omega$ which gives ω (as to direction but of course not as to tensor) when p and therefore ϖ is known. A general method of obtaining ω from this equation is to choose that one of the two values of ω given by the equations $S\omega\varpi\omega = 0$, $S\rho\varpi\omega = 0$ which makes $\mathbf{M}\rho\omega = \varpi\omega$. Another general method is to write down the cubic of ϕ, where $\phi\omega \equiv \varpi\omega - \mathbf{M}\rho\omega$,

$$\phi^3 - M''\phi^2 + M'\phi - M = 0,$$

and notice that M must be zero, since there is a rotor ω for which $\phi\omega = 0$, and therefore that $(\phi^2 - M''\phi + M')\alpha$, where α is any rotor through O, will serve for ω, since this gives $\phi\omega = 0$.

That the three motors of the complex $\omega_1 + \Omega\psi\omega_1$, $\omega_2 + \Omega\psi\omega_2$, and $\omega_3 + \Omega\psi\omega_3$ may be co-reciprocal the necessary and sufficient conditions are

$$S\omega_2\psi\omega_3 = S\omega_3\psi\omega_1 = S\omega_1\psi\omega_2 = 0,$$

i.e. the three motors are parallel to a triad of conjugate diameters of the pitch quadric.

That a motor $\omega + \Omega\psi\omega$ of the complex may be parallel to a fixed plane, we must have

$$\omega = x\alpha + y\beta,$$

where α and β are given rotors through O, and x and y are arbitrary scalars. The motor is therefore

$$x(\alpha + \Omega\psi\alpha) + y(\beta + \Omega\psi\beta),$$

i.e. it belongs to a complex of the second order. The axis and centre of the cylindroid of this complex can easily be found in terms of ϵ, the rotor through O perpendicular to the plane. For by § 40 above, the axis of the cylindroid is that of the motor

$$\mathbf{M}(\alpha + \Omega\psi\alpha)(\beta + \Omega\psi\beta) = \mathbf{M}\alpha\beta + \Omega\mathbf{M}(\alpha\psi\beta - \beta\psi\alpha).$$

Thus it is (of course) perpendicular to the plane of α, β, and the rotor perpendicular from O on it is (§ 6 above)

$$\mathbf{M} \,.\, \mathbf{M}(\alpha\psi\beta - \beta\psi\alpha)\,\mathbf{M}^{-1}\alpha\beta = \alpha\mathbf{S}\psi\beta\mathbf{M}^{-1}\alpha\beta - \beta\mathbf{S}\psi\alpha\mathbf{M}^{-1}\alpha\beta.$$

Take this rotor to be the same as α. Thus the coefficient of α in this expression must be unity, and the coefficient of β zero, or

$$\mathbf{S}\psi\beta\mathbf{M}^{-1}\alpha\beta = 1, \quad \mathbf{S}\epsilon\psi\alpha = 0, \quad \mathbf{S}\epsilon\alpha = 0.$$

From the last two we have $\alpha = x\mathbf{M}\epsilon\psi\epsilon$, and x is now obtained from the first of the three equations. Putting in that equation $\beta = \alpha^{-1}\epsilon$, thus making β a rotor in the plane perpendicular to α, we have $\mathbf{S}\alpha^{-1}\epsilon\psi\epsilon^{-1} = 1$, which gives

$$x = \epsilon^{-2}.$$

Thus the rotor perpendicular from the origin on the axis of the cylindroid of those motors of the given complex which are perpendicular to ϵ is $\mathbf{M}\epsilon^{-1}\psi\epsilon$. Similarly the perpendicular on the axis of the cylindroid similarly related to the reciprocal complex is $-\mathbf{M}\epsilon^{-1}\psi\epsilon$, so that these axes are situated symmetrically on opposite sides of the origin.

There is another meaning in connection with the complexes to be attached to these rotors $\pm\mathbf{M}\epsilon^{-1}\psi\epsilon$, where ϵ is any rotor through the origin. For the rotor perpendicular from the origin on any motor $\omega + \Omega\psi\omega$ of the given complex is $-\mathbf{M}\omega^{-1}\psi\omega$. Hence putting $\omega = \epsilon$, we see that the axis of the cylindroid of all motors of the complex perpendicular to a given direction is the residence of a motor of the reciprocal complex. This is otherwise obvious from the facts already proved: (1) that any motor whose axis is the nodal line is reciprocal to every motor of the cylindroid, and (2) that by giving this motor a suitable pitch it can be made reciprocal to a given third motor of the given complex.

Next supposing ω perpendicular to ϵ, we see that the motor of the complex parallel to ω is in the plane through O perpendicular to ϵ if $\mathbf{S}\epsilon\omega\psi\omega = 0$. But this is the necessary and sufficient condition that ω should be parallel to one of the principal axes of the section of the pitch quadric which is perpendicular to ϵ. Hence there are two such motors in this plane parallel to the two principal axes; these motors are therefore perpendicular; hence they are the principal motors of the cylindroid. Hence

the plane through O perpendicular to the axis of the cylindroid (i.e. perpendicular to ϵ) cuts the axis at the centre of the cylindroid.

These examples suffice to show how such purely geometrical results can be obtained by octonion methods which are essentially quaternion methods. In the examples now to be given the methods will be more characteristically octonion.

45. Miscellaneous simple results. Many of the results (besides those of the last three sections) of *Screws* have already in the present treatise been explicitly enunciated. Several others are almost obvious consequences of what has already been said.

Suppose $A_1 \dots A_6$ are six independent motors. Then we may express any motor E by the equation

$$E = x_1 A_1 + \dots + x_6 A_6 = \Sigma x A.$$

Thus
$$E^2 = \Sigma x^2 A^2 + 2\Sigma x_1 x_2 \mathbf{S} A_1 A_2.$$

Taking the ordinary scalar part and the convertor part, we get

$$\mathbf{T}_1{}^2 E = \Sigma x^2 \mathbf{T}_1{}^2 A - 2\Sigma x_1 x_2 \mathbf{S}_1 A_1 A_2,$$

which is § 37 of *Screws*, and

$$\mathbf{t} E \mathbf{T}_1{}^2 E = \Sigma x^2 \mathbf{t} A \mathbf{T}_1{}^2 A - \Sigma x_1 x_2 \mathbf{s} A_1 A_2,$$

which is § 32 of *Screws*.

If $A_1 \dots A_6$ are co-reciprocal,

$$\mathbf{t} E \mathbf{T}_1{}^2 E = \Sigma x^2 \mathbf{t} A \mathbf{T}_1{}^2 A,$$

which is § 33 of *Screws*.

If F be another motor which is equal to $\Sigma y A$ and the A's are co-reciprocal,

$$\mathbf{s} EF = \Sigma xy \mathbf{s} A^2 = -2\Sigma xy \mathbf{t} A \mathbf{T}_1{}^2 A,$$

which is § 35 of *Screws*.

§ 41 of *Screws* asserts that $\{A_1 \dots A_5\}$ (§ 28 above) is reciprocal to each of the motors $A_1, A_2 \dots A_5$ and also that the condition that $A_1 \dots A_6$ belong to a complex of lower order than the sixth is that $\{A_1 \dots A_6\} = 0$.

§ 45 asserts that if

$$\mathbf{s} E A_1 = \dots = \mathbf{s} E A_n = 0,$$

then
$$\mathbf{s} E (x_1 A_1 + \dots + x_n A_n) = 0.$$

If for some ordinary scalar values of x and y, $xA + yB$ is

reciprocal to both A' and B', it follows that for some ordinary scalar values of x' and y', $x'A' + y'B'$ is reciprocal to both A and B, for the necessary and sufficient condition for either of these is that

$$\mathbf{s}AA'\mathbf{s}BB' = \mathbf{s}AB'\mathbf{s}BA'.$$

This is § 88 of *Screws*.

§ 89 asserts that if A and B be two motors and x and y two ordinary scalars, the four motors xA, yB, $xA \pm yB$ are parallel to the four rays of an harmonic pencil. This is obvious (§ 13 above) from the fact that a similar statement is true of vectors.

§ 99 asserts that the anharmonic ratio of a pencil with rays parallel to the four motors

$$A, \quad xA + yB, \quad x'A + y'B, \quad B$$

is the same as the anharmonic ratio of the pencil with rays parallel to the four motors

$$\varpi A, \quad \varpi (xA + yB), \quad \varpi (x'A + y'B), \quad \varpi B,$$

where ϖ is a linear motor function of a motor (self-conjugate in the case considered in *Screws*). This is obvious from the fact that the anharmonic ratio of the pencil with rays parallel to the four vectors

$$\alpha, \quad x\alpha + y\beta, \quad x'\alpha + y'\beta, \quad \beta$$

is $x'y/xy'$.

§ 137 asserts that there is one line such that a motor of any pitch having that line for axis belongs to a given complex of the fourth order. This is of course k of cases (α) and (β) of the table of § 41 above. Similarly in a complex of the fifth order any line intersecting perpendicularly the motor reciprocal to the complex is the axis of a motor of arbitrary pitch belonging to the complex.

46. Equation of motion of a rigid body. Free body subject to no forces. We proceed now to the dynamics of a single rigid body. And first we shall not make the assumption that the motion is small or that the body is subject only to instantaneous impulses, though later one or other of these restrictions will be imposed and several propositions of *Screws* applying to these cases will be proved.

Take a certain position of the rigid body which may be considered the initial position as a standard of reference. Let $Q(\ )Q^{-1}$ be the operator (§ 11 above) which brings the body at one operation from its initial position to its actual position at any time. We assume, as by § 11 we may, that $\mathbf{T}Q = 1$, and that

$$Q = e^{E/2} \quad\text{.............................(1)},$$

where E is a motor. Thus the axis about which the body must be twisted is that of E (and also that of Q), the angle of the twist is $\mathbf{T}_1E$ (and also twice the angle of Q), and the translation is $\mathbf{M}_2E$ (and also is twice the translation of Q). E is therefore called the displacement motor.

Another motor E' will be used for some purposes defined by the equation

$$E' = 2\mathbf{M}Q = 2B \quad\text{.......................(2)}.$$

It will be observed that when the displacement is a small one

$$Q = 1 + \tfrac{1}{2}E = 1 + \tfrac{1}{2}E' \quad\text{....................(3)},$$

so that E and E' are in this case the same small (i.e. with small tensor) motor.

It may be noticed that since $\mathbf{T}Q = 1$

$$\mathbf{K}Q = Q^{-1} \quad\text{.........................(4)},$$

$$2B = E' = 2\mathbf{M}Q = Q - Q^{-1} = 2\sinh(\tfrac{1}{2}E),$$

$$2\mathbf{S}Q = Q + Q^{-1} = 2\cosh(\tfrac{1}{2}E) \text{......(5)}.$$

From these since $\cosh^2(\tfrac{1}{2}E) = 1 + \sinh^2(\tfrac{1}{2}E)$ we have

$$\mathbf{S}Q = \sqrt{(1+B^2)},\quad Q = B + \sqrt{(1+B^2)},\quad Q^{-1} = -B + \sqrt{(1+B^2)} \text{...(6)},$$

or

$$2\mathbf{S}Q = \sqrt{(4+E'^2)},\quad 2Q = E' + \sqrt{(4+E'^2)},$$

$$2Q^{-1} = -E' + \sqrt{(4+E'^2)} \text{........(7)}.$$

Notwithstanding that these equations are simpler when B is used instead of E', E' is the more convenient motor for our purposes mainly on account of the consequences of eq. (3).

E' instead of E may be taken to specify the displacement since Q which specifies it is given by means of equations (7) in terms of E'.

Let F, G, H be the velocity motor, the momentum motor and

the force motor respectively of the rigid body. These terms have all been explained in § 8 above.

Also let

$$F = QF_0Q^{-1}, \quad G = QG_0Q^{-1}, \quad H = QH_0Q^{-1} \ldots\ldots\ldots (8),$$

so that F_0, G_0, H_0 may be spoken of as the corresponding motors relative to the standard position or relative to the rigid body itself. In fact, in using in our equations F_0, G_0, H_0 instead of F, G, H, we are doing what is analogous in the Cartesian treatment of a rigid body with one point fixed to the well-known reference to axes fixed in the body.

We do not require two analogous symbols E_0 and E_0', for since Q, E and E' are coaxial

$$E = QEQ^{-1}, \quad E' = QE'Q^{-1} \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

We might assume from elementary Rigid Dynamics the existence of moments of inertia and principal axes of a rigid body. But we make a digression here to prove this, as a rather instructive series of octonion examples is involved.

Since impulses combine like forces, being of essentially the same nature (except as to time dimensions), we see that *the momentum motor of a system of moving particles is the sum of the individual momentum rotors of the particles.*

If G' is the momentum rotor of a single particle of a moving rigid body of which the velocity motor is F, twice the kinetic energy of the particle is $-\mathbf{s}FG'$, and if the position in space of the particle is given G' is a self-conjugate linear (energy-) function of F. If m is the mass of the particle, and $\omega + \Omega\sigma$ is the velocity motor F, where ω and σ are rotors through the particle, the momentum rotor G' is $m\sigma$. Hence $-\mathbf{s}FG' = -m\sigma^2 =$ twice the kinetic energy. Also if the position of the particle is given $m\sigma = G'$ is a linear function of $\omega + \Omega\sigma = F$ by the definition in § 15 above of a linear function. Putting $m\sigma = \psi(\omega + \Omega\sigma)$ we have

$$\mathbf{s}(\omega' + \Omega\sigma')\psi(\omega + \Omega\sigma) = m\mathbf{S}\sigma\sigma' = \mathbf{s}(\omega + \Omega\sigma)\psi(\omega' + \Omega\sigma'),$$

so that ψ is self-conjugate. From the last two propositions it follows that:—

If the velocity motor and momentum motor of a rigid body be F and G respectively, twice the kinetic energy is $-\mathbf{s}FG$, and if the position of the rigid body is given G is a self-conjugate linear

(*energy-*) *function of F.* [This might be proved if we assume some very elementary facts of Rigid Dynamics from the equation

$$\mathbf{s}\,(\omega + \Omega\sigma)\,(\omega' + \Omega\sigma') = \mathbf{S}\,(\omega\sigma' + \omega'\sigma).]$$

Hence with the meanings of F_0 and G_0 given just now

$$G_0 = \psi_0 F_0 \quad\text{..........................(10)},$$

where ψ_0 is a constant self-conjugate energy-function. Also if T is the kinetic energy

$$2T = -\,\mathbf{s}F_0G_0 = -\,\mathbf{s}F_0\psi_0F_0 \quad\text{..................(11)}.$$

When the velocity motor of a rigid body is a lator (velocity of translation) the momentum motor is a parallel and similarly directed rotor equal to $M \times$ the lator, where M is the mass of the body. This only asserts that the sum of a series of parallel similarly directed rotors is a similarly directed rotor whose tensor is the sum of the tensors of the components. This was incidentally proved by octonion methods in § 22 above. Without assuming any of the well-known geometrical properties of the position of the centre of mass it is easy to prove by octonion methods from this proposition alone, that this momentum rotor must pass through a point fixed in the rigid body. For if $M\rho$, $M\sigma$ be the two momentum rotors due to the two velocity lators $\Omega\rho$, $\Omega\sigma$; the momentum rotor due to the velocity lator $x\Omega\rho + y\Omega\sigma$, must be $M(x\rho + y\sigma)$ for all ordinary scalar values of x and y. Hence $x\rho + y\sigma$ is a rotor for all values of x and y. Hence $M\rho$ and $M\sigma$ and similarly all other momentum rotors due to velocity lators must intersect, i.e. they must pass through a point fixed in the rigid body. This point is of course what is known as the centre of mass, and will here be denoted by O.

Now let $\chi E = \Omega E$. Since both ψ_0 and χ are self-conjugate *energy*-functions there must, by § 33, be six independent motors forming a conjugate system with regard both to ψ_0 and χ. Two axial motors which are conjugate with regard to χ must be perpendicular. Since not more than three motors can be mutually perpendicular and since not more than three lators can be independent, it follows that the six motors must be three mutually perpendicular axial motors and three independent lators. Since every lator belongs to the complex of three independent lators, it follows that there are three mutually perpendicular axial motors A, B, C which together with *any* three

independent lators which form a conjugate system with regard to ψ_0, form a system of six conjugate motors with regard to ψ_0. Denote any lator by $\Omega\rho$ where ρ is any rotor through O. From the last proposition it follows that

$$\psi_0\Omega\rho = M\rho.$$

Since A and $\Omega\rho$ are conjugate with regard to ψ_0

$$0 = \mathsf{s}A\psi_0\Omega\rho = M\mathsf{s}A\rho.$$

Hence by statement (4) § 14, A and similarly B and C must be rotors through O. Take i, j, k as the unit coaxial rotors. It is easy to see that Ωi, Ωj, Ωk are conjugate to one another with regard to ψ_0. Hence i, j, k, Ωi, Ωj, Ωk are a conjugate system of motors with regard to ψ_0. Since $\mathsf{s}\Omega\rho\psi_0 i = 0$, $\psi_0 i$ is a lator; and since $\mathsf{s}j\psi_0 i = \mathsf{s}k\psi_0 i = 0$ this lator is parallel to i. Thus since ψ_0 is an energy-function

$$\psi_0 i = Ma^2\Omega i, \quad \psi_0\Omega i = Mi, \quad \psi_0 j = Mb^2\Omega j, \text{ \&c. } \ldots\ldots(12).$$

Putting the velocity-motor equal to Ωi, i, &c. successively we get all the ordinary theorems concerning moments of inertia, principal axes, and their connection with the kinetic energy of the rigid body. Interpreting the meaning of momentum motor we also get all their connections with moment of momentum.

i, j, k are of course the principal axes and Ma^2, Mb^2, Mc^2 the moments of inertia about them.

From equation (8)

$$G = Q\psi_0(Q^{-1}FQ)\,.\,Q^{-1} = \psi F \;\ldots\ldots\ldots\ldots\ldots(13),$$

which defines the self-conjugate energy-function ψ in terms of ψ_0 and Q. Thus whereas ψ_0 is of constant form ψ is not. We may also note that

$$2T = -\mathsf{s}FG = -\mathsf{s}F\psi F \;\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

Eq. (10) when expanded in full by aid of equations (12) gives

$$G_0 = \psi_0 F_0 = -M(i\mathsf{s}iF_0 + a^2\Omega i\mathsf{s}\Omega iF_0 + j\mathsf{s}jF_0 + \text{\&c.})\ldots(15).$$

It may be noticed that

$$\mathsf{S}_1F_0\psi_0F_0 = -M(\mathsf{s}\Omega iF_0\mathsf{s}iF_0 + \mathsf{s}\Omega jF_0\mathsf{s}jF_0 + \mathsf{s}\Omega kF_0\mathsf{s}kF_0) = \tfrac{1}{2}M\mathsf{s}F_0^2.$$

From this, since

$$\mathsf{S}FG = \mathsf{S}F_0G_0,\;\; F^2 = F_0^2,$$

we have
$$\left.\begin{aligned}\mathbf{S}F_0G_0 &= \tfrac{1}{2}M\mathrm{s}F_0^2 - 2\Omega T\\ = \mathbf{S}FG &= \tfrac{1}{2}M\mathrm{s}F^2 - 2\Omega T\end{aligned}\right\} \text{..............(16).}$$

i, j, k, Ωi, Ωj, Ωk are a *conjugate* set of motors of ψ_0, as we have seen, but they are not a co-reciprocal set. The set which are both conjugate and co-reciprocal we may call the absolute principal motors of ψ_0. These are what in *Screws* (§ 105) are called "the absolute principal screws of inertia." [A set of "conjugate screws of inertia" is a set of motors conjugate with regard to ψ_0. If a complex of order n be given, those n screws of it which are co-reciprocal and form a conjugate set with regard to ψ_0 are "the principal screws of inertia" of that complex. Thus the absolute principal screws of inertia are the principal screws of inertia for the case $n = 6$.]

The absolute principal motors of ψ_0 are

$$(1 \pm a\Omega)\,i,\ (1 \pm b\Omega)\,j,\ (1 \pm c\Omega)\,k \text{..............(17),}$$

for these are co-reciprocal and are such that

$$\psi_0(1 \pm a\Omega)\,i = \pm Ma(1 \pm a\Omega)\,i, \text{ \&c.............(18),}$$

from which it easily follows (from the fact that they are co-reciprocal) that they are also a conjugate set.

Equation (18) shows that the roots of the ψ_0 sextic are $\pm Ma$, $\pm Mb$, $\pm Mc$. The sextic is

$$(\psi_0^2 - M^2a^2)(\psi_0^2 - M^2b^2)(\psi_0^2 - M^2c^2) = 0 \text{.........(19).}$$

From these we see that the absolute principal motors of ψ are

$$(1 \pm a\Omega)\,QiQ^{-1},\ (1 \pm b\Omega)\,QjQ^{-1},\ (1 \pm c\Omega)\,QkQ^{-1} \text{......(20),}$$

and that the roots of the ψ sextic are the same as the roots of the ψ_0 sextic.

The equation of motion of a single particle is $\dot{G}' = H'$, where G' is its momentum rotor and H' the force rotor acting on it. Hence if G be the momentum motor of a system of particles $\dot{G} = \Sigma H'$. The "internal" forces of the system contribute zero to the sum $\Sigma H'$, so that *an* equation of motion of any system of moving matter is

$$\dot{G} = H \text{ ....(21),}$$

where G is its momentum motor and H the force motor of the system of "external" forces. For our rigid body eq. (21) is *the* equation of motion, since it suffices to determine consecutive (in

time) values of G when H is given at any instant, and therefore also suffices to determine consecutive values of Q (as will appear more obviously directly).

Since $\mathbf{T}Q = 1$ we have by eq. (5) § 20

$$\mathbf{S}\dot{Q}Q^{-1} = 0 \qquad \text{.........................(22).}$$

Hence by eq. (8) § 11

$$F = 2\dot{Q}Q^{-1} \qquad \text{.........................(23).}$$

Hence by eq. (8) of the present section

$$F_0 = 2Q^{-1}\dot{Q} \qquad \text{.........................(24).}$$

Hence $$H = d\,(QG_0Q^{-1})/dt = \mathbf{M}FG + Q\dot{G}_0Q^{-1}$$

[§ 11 above], or

$$H_0 = Q^{-1}HQ = \mathbf{M}F_0G_0 + \dot{G}_0.$$

Thus $$H_0 = \psi_0\dot{F}_0 + \mathbf{M}F_0\psi_0F_0 \qquad \text{....................(25).}$$

This can be expressed in terms of Q instead of F_0, for by eq. (24)

$$\dot{F}_0 = 2Q^{-1}\ddot{Q} - 2Q^{-1}\dot{Q}Q^{-1}\dot{Q} = 2Q^{-1}\ddot{Q} - \tfrac{1}{2}F_0^2.$$

Taking the scalar-octonion and motor parts we have

$$4\mathbf{S}Q^{-1}\ddot{Q} = F_0^2 = F^2 \qquad \text{....................(26),}$$

$$\dot{F}_0 = 2\mathbf{M}Q^{-1}\ddot{Q} \qquad \text{......................(27).}$$

Hence our equation of motion to determine Q when H_0 is given is

$$H_0 = 2\psi_0\mathbf{M}Q^{-1}\ddot{Q} + 4\mathbf{M}\,.\,Q^{-1}\dot{Q}\psi_0(Q^{-1}\dot{Q}) \qquad \text{........(28).}$$

It will be noticed that these equations are very analogous indeed to the quaternion equations which express the motion of a rigid body one point of which is fixed. It is worthy of remark however that the analogy here is *not* quite that of the general analogy so largely used above between Octonions and Quaternions. This is because ψ_0 is not a commutative function (§ 15 above).

The quaternion equations can be deduced from these. Without actually making this deduction it will be evident from what immediately follows how it may be made.

ψ_0 is by no means of the most general type of energy-functions, so we may expect it to have some special properties. These will now be examined.

In the first place though ψ_0 is not a commutative function, ψ_0^2 is. For by equation (18)

$$\psi_0^2(1 \pm a\Omega)\,i = M^2a^2(1 \pm a\Omega)\,i,$$

from which

$$\psi_0^2 i = M^2a^2 i,\ \ \psi_0^2\Omega i = M^2a^2\Omega i,$$

and similarly for j and k. Thus ψ_0^2 is a self-conjugate pencil function whose centre is O. Define ϖ_0 by the equation

$$\varpi_0 E = M^{-2}\,\psi_0^2 E = -(a^2 i\mathbf{S}iE + b^2 j\mathbf{S}jE + c^2 k\mathbf{S}kE)\ldots\ldots(29).$$

The relation between ψ^2 and ψ_0^2 is the same as the relation between ψ and ψ_0, so we naturally define ϖ by the equation

$$\varpi E = M^{-2}\psi^2 E = Q\varpi_0(Q^{-1}EQ)\,.\,Q^{-1}\ \ldots\ldots\ldots\ldots(30).$$

Now let $$F_0 = \omega_0 + \Omega\sigma_0,\ \ F = \omega + \Omega\sigma\ldots\ldots\ldots\ldots\ldots\ldots(31),$$

where ω_0 and σ_0 are rotors through O and $\omega = Q\omega_0 Q^{-1}$, $\sigma = Q\sigma_0 Q^{-1}$, so that ω and σ are rotors through the point to which O has been transported at any time. The relations between F_0, ω_0, σ_0, ψ_0 and ϖ_0 will clearly be precisely similar to those between F, ω, σ, ψ and ϖ. We restrict ourselves therefore to the former set.

Next notice the peculiar properties of the two (not self-conjugate and not commutative) linear functions $\psi_0\Omega$ and $\Omega\psi_0$. By equation (15)

$$M^{-1}\Omega\psi_0 F_0 = \Omega\sigma_0,\ \ M^{-1}\psi_0\Omega F_0 = \omega_0\ \ldots\ldots\ldots\ldots(32),$$

that is to say *if any motor be expressed as the sum of a rotor through O and a lator, $M^{-1}\Omega\psi_0$ operating on the motor picks out the lator and $M^{-1}\psi_0\Omega$ picks out the rotor.* We shall understand eq. (32) to have this general meaning and not to have special reference to F_0 the velocity motor.

As a result we may notice that

$$M^{-1}(\Omega\psi_0 + \psi_0\Omega) = 1\ldots\ldots\ldots\ldots\ldots\ldots\ldots(33),$$

$$M^{-1}\,\Omega\psi_0\Omega = \Omega\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(34),$$

so that both the linear functions on the left are both self-conjugate and commutative.

Since $M^{-1}\psi_0\Omega$ reduces any motor to its rotor part translated so as to pass through a given point, we see that if E be any formal quaternion motor function [§ 13 above] of the motors

A_1, A_2..., then $M^{-1}\psi_0\Omega E$ is the same function of $M^{-1}\psi_0\Omega A_1$, $M^{-1}\psi_0\Omega A_2$, ... For instance if A and B be any two motors

$$M^{-2}\mathbf{M}\psi_0\Omega A\psi_0\Omega B = M^{-1}\psi_0\Omega\mathbf{M}AB \qquad (35),$$

and indeed each of these is the rotor through O parallel and equal to the rotor part $\mathbf{M}_1AB$ of $\mathbf{M}AB$.

ψ_0F_0 may be expressed in terms of ω_0, σ_0 and ϖ_0, for by equation (15)

$$M^{-1}\psi_0\omega_0 = \Omega\varpi_0\omega_0,\ M^{-1}\psi_0\Omega\sigma_0 = \sigma_0 \qquad (36)$$

(the second of which is a particular case of eq. (32)). This again is true of any rotor and lator and will be quoted as meaning that. From this equation we have

$$G_0 = \psi_0F_0 = M(\sigma_0 + \Omega\varpi_0\omega_0) \qquad (37),$$

which expresses G_0 in the form of a rotor through O + a lator.

The physical meanings of the four rotors and lators ω_0, $\Omega\sigma_0$, $M\sigma_0$, $M\Omega\varpi_0\omega_0$ into which F_0 and G_0 have been decomposed are obvious from the meaning of a velocity motor and a momentum motor (§ 8 above). The instantaneous motion of the body has been decomposed into a rotation (rotor) ω_0 round an axis through O and a translation (lator) $\Omega\sigma_0$. The momentum in the ordinary sense is the rotor $M\sigma_0$ through O, and the moment of momentum about O is the lator $M\Omega\varpi_0\omega_0$. [We should naturally call the rotor momentum not $M\sigma_0$ but the equal and parallel, but not coincident, rotor $\mathbf{M}_1G_0 = M\mathbf{M}_1(\sigma_0 + \Omega\varpi_0\omega_0)$.]

As it is important that the geometrical relations between the rotors ω_0, σ_0 and $\varpi_0\omega_0$ and the motors F_0 and G_0 should be clearly grasped, I have endeavoured in fig. 8 to render them evident to the eye. First note that

$$\mathbf{M}F_0G_0 = M\mathbf{M}(\omega_0 + \Omega\sigma_0)(\sigma_0 + \Omega\varpi_0\omega_0) = \mathbf{M}\omega_0G_0 \qquad (38),$$

so that the shortest distances of F_0 and G_0 and of ω_0 and G_0 are coincident and are parallel to $\mathbf{M}\sigma_0\omega_0^{-1}$. The figure may be supposed constructed thus. First draw the two planes containing the pairs of rotors ω_0, σ_0 and σ_0, $\varpi_0\omega_0$. Then draw the two rotor normals to these planes from O, $\mathbf{M}\sigma_0\omega_0^{-1}$ and $\mathbf{M}\varpi_0\omega_0\sigma_0^{-1}$. F_0 and G_0 pass through the extremities of these normals (§ 6 above) and they are parallel to ω_0, σ_0 respectively, so that their axes can be drawn. Finally note that all three pairs of the three motors F_0, G_0, ω_0 have a common shortest distance, namely the axis of

$\mathsf{M}F_0G_0$ and that of $\mathsf{M}\omega_0G_0$. The sphere and the cylinder and the equator of σ_0 and other lines in the figure may for the present be regarded as given merely to help out the visual representation.

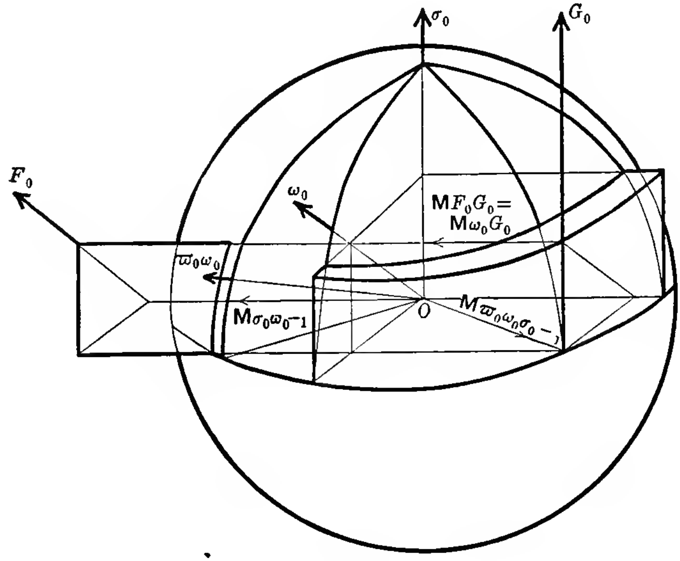


FIG. 8.

The instantaneous motion (referred to the rigid body) is a velocity-twist about F_0 of such a pitch that the motion of O is along σ_0. The system of impulses which would instantaneously produce the motion from rest is equivalent to an impulse along the axis of G_0 equal to M_1G_0 (i.e. equal and parallel to $\mathsf{M}\sigma_0$) and an impulsive couple in the plane perpendicular to G_0 equal to M_2G_0.

Now suppose there are no external forces. Poinsot's beautiful theorems concerning the rolling of an ellipsoid on a plane serve to obtain successive positions of the rigid body. By means of equation (25) we can similarly show how to construct successive positions, but with no such clear geometrical simple ideas involved as are involved in Poinsot's construction. Since $H_0=0$ we have

$$dG_0=\psi_0dF_0=-\mathsf{M}F_0G_0dt\dots\dots\dots\dots\dots\dots(39).$$

Suppose now that G_0 and F_0 and the position of the rigid body are known at any instant t. This equation enables us to find the values of G_0 and F_0 the next instant $t+dt$ and also to find the new

position of the rigid body; and thus enables us to construct an indefinite number of successive positions.

By eq. (7) § 11 the last equation asserts that the motion *in the rigid body* during the instant dt of the motor G_0 is the small twist $-F_0\,dt$. This gives us a *geometrical* construction for the change in G_0. It may be observed that if we *merely* notice the change in G_0 the possible small twist is to a certain extent arbitrary. Thus for instance by eq. (38) the change in G_0 can always be effected by the small *rotation* $-\omega_0 dt$. But there is only one twist about the axis of F_0 which will effect the change. Hence the axis of F_0 and the change in G_0 are alone sufficient to determine the small twist $-F_0 dt$. Meanwhile the rigid body receives a small twist in space and this can be referred to the rigid body. The twist in space is Fdt (by definition of F as the velocity motor) and this referred to the rigid body is $F_0 dt$. Thus the actual twist experienced by the rigid body is exactly the opposite of that experienced by G_0 about the axis of F_0. But in order to make successive constructions of the position on this plan we must also know the change in F_0. This of course is given by $dF_0 = \psi_0^{-1}\,dG_0$. I do not see how to determine a simple geometrical representation of this connection.

The sphere of fig. 8 is a sphere fixed in the rigid body. This can be seen from the figure; for since the new value of G_0 can be obtained by rotating it round ω_0, the point on the axis at the point of contact of G_0 and the sphere slides along the sphere. [If we give to G_0 the twist $-F_0 dt$ we must add to this rotation a small translation $\Omega\sigma_0 dt$. This of course slides the old point of contact off the sphere but leaves G_0 still touching the sphere since the translation mentioned is a mere sliding of G_0 along itself.]

In the present method as in the ordinary dynamical methods we gain much insight in the case of no external forces by considering the equations involving the actual motors F, G, &c., instead of the same referred to the rigid body. Thus in eq. (21) H is zero, so that G is an absolutely constant motor. But, more than this, σ is an absolutely constant rotor; and therefore since $G = M(\sigma + \Omega\varpi\omega)$, $\Omega\varpi\omega$ is also an absolutely constant lator. For, remembering the meaning of $\psi\Omega$, we see by eq. (37) that

$$M^2\sigma = \psi\Omega G = \psi\Omega\psi F = Q\psi_0\Omega\psi_0 F_0 \,.\, Q^{-1}.$$

Hence $$M^2\dot{\sigma} = Q\psi_0\Omega\psi_0\dot{F}_0Q^{-1} + \mathbf{M}\,.\,FQ\psi_0\Omega\psi_0F_0Q^{-1} \text{ [eq. (7) § 11]}$$
$$= Q(\psi_0\Omega\psi_0\dot{F}_0 + \mathbf{M}F_0\psi_0\Omega\psi_0F_0)\,Q^{-1}$$
$$= Q(-\psi_0\Omega\mathbf{M}F_0G_0 + \mathbf{M}F_0\psi_0\Omega G_0)\,Q^{-1} \text{ [eq. (39)]}$$
$$= Q\mathbf{M}(F_0 - M^{-1}\psi_0\Omega F_0)\,\psi_0\Omega G_0\,.\,Q^{-1} \text{ [eq. (35)]}$$
$$= 0 \text{ [equations (37) and (32)]},$$

so that σ is a constant rotor.

I have given this proof in order to avoid explicitly using the symbols σ, ω, σ_0, ω_0 as far as possible. The following proof is perhaps simpler and is certainly more analogous to a quaternion proof. Since $\sigma = Q\sigma_0Q^{-1}$, $\dot{\sigma} = Q\dot{\sigma}_0Q^{-1} + \mathbf{M}F\sigma$. But the equation $\dot{G}_0 = -\mathbf{M}F_0G_0$ gives the two equations

$$\dot{\sigma}_0 = -\mathbf{M}\omega_0\sigma_0,\quad \varpi_0\dot{\omega}_0 = -\mathbf{M}\omega_0\varpi_0\omega_0 \text{..............(40).}$$

Hence $$\dot{\sigma} = -Q\mathbf{M}\omega_0\sigma_0\,.\,Q^{-1} + \mathbf{M}F\sigma = -\mathbf{M}\omega\sigma + \mathbf{M}F\sigma = 0.$$

By dropping the zero suffixes in fig. 8 it serves to indicate the connections between F, G, ω, &c. The point O of the figure is now not a fixed point but the position of the centre of mass at any time. Regarding the figure in this way, since G and σ are both constant, the lines of the figure G and σ are absolutely fixed in space. Thus the centre of mass moves along a fixed straight line with uniform velocity. Also the lator $\Omega\varpi\omega$ is constant. Hence the rotor marked $\varpi\omega$ in the figure though not fixed in space is fixed in direction and tensor. The sphere twists about always inside the fixed cylinder, a portion of which is indicated in the figure. The mode of its twisting depends on the manner in which ω and F vary. Poinsot's construction, which (Tait's *Quaternions*, 3rd ed., § 407) is immediately deducible from the fact that $\varpi\omega$ is constant in magnitude and direction, indicates how ω varies and the relations shown in the figure are then sufficient to determine the variation in F. We shall investigate directly to a certain extent how the foot of the perpendicular from O on F moves, but for the present return to the equation (39).

Since G is G_0 displaced and $\varpi\omega$ is $\varpi_0\omega_0$ displaced and σ is σ_0 displaced, it is easy to see that four independent integrations of this equation can be effected. Three of these are obtained with ease. Operating by $\mathbf{S}G_0(\ \ )$ we at once obtain

$$G_0^2 = \text{const.}$$

This is equivalent to two ordinary scalar equations, so it involves two integrals. Operating by $\mathbf{s}F_0(\ )$ we get

$$0 = 2\mathbf{s}F_0 dG_0 = 2\mathbf{s}F_0\psi_0 dF_0 = d\mathbf{s}F_0\psi_0 F_0,$$

so that $\mathbf{s}F_0\psi_0F_0 = \text{const.}$ is another integral. The fourth integral may be obtained thus:—By eq. (32)

$$M^{-3}\psi_0\Omega\psi_0G_0 = \varpi_0\omega_0.$$

By equation (38)

$$\mathbf{M}F_0G_0 = \mathbf{M}\omega_0G_0 = M\mathbf{M}\omega_0(\sigma_0 + \Omega\varpi_0\omega_0).$$

Hence by eq. (39)

$$\begin{aligned}\mathbf{s}dG_0\psi_0\Omega\psi_0G_0 &= -\,\mathbf{s}F_0G_0\psi_0\Omega\psi_0G_0\,dt \\ &= -\,M^4\mathbf{s}\omega_0(\sigma_0 + \Omega\varpi_0\omega_0)\,\varpi_0\omega_0 = 0.\end{aligned}$$

Hence $\quad \text{Const.} = \mathbf{s}G_0\psi_0\Omega\psi_0G_0 = \mathbf{s}\Omega\,(\psi_0G_0)^2 = \mathbf{S}_1(\psi_0G_0)^2.$

By expressing G_0, &c., in terms of σ_0, &c., we can see what well-known result each of these integrals corresponds to. The integral $\mathbf{s}F_0\psi_0F_0 = \mathbf{s}F_0G_0 = \text{const.}$ gives

$$\sigma_0{}^2 + \mathbf{S}\omega_0\varpi_0\omega_0 = \text{const.},$$

i.e., it expresses that the kinetic energy remains constant;

$$\mathbf{S}_1(\psi_0G_0)^2 = \text{const.}$$

gives $\quad (\varpi_0\omega_0)^2 = \text{const.},$

i.e., the ordinarily called moment of momentum is constant. $G_0{}^2 = \text{const.}$ gives the two equations

$$\sigma_0{}^2 = \text{const.},\ \ \mathbf{S}\sigma_0\varpi_0\omega_0 = \text{const.}$$

From the first of these the translatory energy is constant; and since the whole is constant, the rotatory energy is also constant. From the last three integrals it follows also that the angle between σ_0 and $\varpi_0\omega_0$ is constant.

Since $\mathbf{S}G_0dG_0 = 0$ we have $\mathbf{s}G_0dG_0 = 0$ and $\mathbf{s}\Omega G_0dG_0 = 0$; i.e. dG_0 is reciprocal to G_0 and ΩG_0. Similarly the other two integrals express that dG_0 is reciprocal to F_0 and $\psi_0{}^3\Omega F_0$, i.e. to F_0 and $\varpi_0\omega_0$. But equation (39) gives both the axis and pitch of dG_0, i.e. expresses that dG_0 must be reciprocal to five motors, so that one motor independent of the four just mentioned can be found to which dG_0 is reciprocal. As a matter of fact since from that equation

$$\mathbf{S}G_0dG_0 = 0,\ \ \mathbf{S}F_0dG_0 = 0,\ \ \mathbf{s}dG_0\mathbf{M}^{-1}F_0G_0 = 0,$$

we see that dG_0 is reciprocal to the five motors G_0, ΩG_0, F_0, ΩF_0 and $\mathbf{M}^{-1}F_0G_0$. Either of the two last may be taken as the fifth motor independent of the four given by the four integrals.

Again, the statement that dG_0 is reciprocal to each of the motors F_0, G_0, ΩG_0 and $\psi_0\Omega\psi_0G_0$ may be expressed by saying that F_0 and dF_0 are conjugate with regard to each of the self-conjugate functions ψ_0, ψ_0^2, $\psi_0\Omega\psi_0$ and $\psi_0^2\Omega\psi_0^2$ respectively; or that G_0 and dG_0 are conjugate with regard to each of the functions ψ_0^{-1}, 1, Ω, $\psi_0\Omega\psi_0$ respectively.

Put $$\varpi\omega = \mu,\ \mathbf{S}\omega\varpi\omega = -c_1,\ \mu^2 = \mathbf{S}\omega\varpi^2\omega = -c_2 \ \ldots\ldots(41).$$

Thus μ is a rotor of constant magnitude and direction, and c_1 and c_2 are constant scalars; μ being the ordinarily called moment of momentum, c_1 twice the rotatory energy and c_2 the square of the moment of momentum. We may notice that if m, m', m'' are the coefficients of the ϖ cubic

$$\varpi^{n+3} - m''\varpi^{n+2} + m'\varpi^{n+1} - m\varpi^n = 0,$$

for all positive integral values of n from zero. This equation enables us to express $c_n = -\mathbf{S}\omega\varpi^n\omega$ in terms of c_2, c_1 and $c_0 = -\omega^2$, i.e. in terms of constants and the square of the angular velocity. For it gives

$$c_n = -\mathbf{S}\omega\varpi^n\omega,\ c_{n+3} - m''c_{n+2} + m'c_{n+1} - mc_n = 0 \ldots\ldots(42).$$

From this in particular we deduce a result required later, viz.

$$\left.\begin{aligned} &-c_1^2c_4 + 2c_1c_2c_3 - c_2^3 \\ &\quad = \{c_1^3(m'm'' - m) - c_1^2c_2(m''^2 + m') + c_1c_2^2 2m'' - c_2^3\} \\ &\quad + mc_1(m''c_1 - 2c_2)\,\omega^2 \end{aligned}\right\} \ldots(43),$$

and generally

$$c_n = -\frac{\omega^2b^2c^2 + c_1(b^2 + c^2) - c_2}{(a^2 - b^2)(a^2 - c^2)}a^{2n} - \frac{\omega^2c^2a^2 + c_1(c^2 + a^2) - c_2}{(b^2 - c^2)(b^2 - a^2)}b^{2n}$$
$$- \frac{\omega^2a^2b^2 + c_1(a^2 + b^2) - c_2}{(c^2 - a^2)(c^2 - b^2)}c^{2n} \ldots\ldots(44),$$

since a^2, b^2, c^2 are the roots of the ϖ cubic.

We may remind the reader here that the quaternion process (Tait's *Quaternions*, 3rd ed. § 407) of establishing Poinsot's construction is as follows. Let ρ_0 be a coordinate rotor through O

and $\rho = Q\rho_0 Q^{-1}$ the corresponding rotor through the point to which O has been transported at any time. Then the two equations

$$\mathbf{S}\rho\varpi\rho = -c_1,\ \mathbf{S}\rho_0\varpi_0\rho_0 = -c_1 \quad\ldots\ldots\ldots\ldots(45),$$

are exactly the same and therefore each is the equation of an ellipsoid fixed in the rigid body. The tangent plane at the point $\rho = \omega$ of this ellipsoid is

$$\mathbf{S}\rho\mu = -c_1 \quad\ldots\ldots\ldots\ldots\ldots\ldots(46).$$

Since μ is a rotor through the centre of mass of constant tensor and direction this plane is simply moving with the velocity σ of the centre of mass. Since relative to the centre of mass the rigid body, and therefore the ellipsoid which it carries, is moving with the angular velocity ω, the point of the ellipsoid in contact with the plane (46) is at rest relative to the plane. Hence the ellipsoid *rolls* on the plane. The ellipsoid (45) will be referred to as the *Poinsot ellipsoid*, or when there is no ambiguity *the* ellipsoid, and the plane (46) will be referred to as the *contact-plane*.

The curve traced out on the surface of this ellipsoid by the extremity of ω (the rotor drawn from the centre of mass representing the angular velocity) is called the polhode. It is by equations (41) the intersection of the Poinsot ellipsoid with the ellipsoid $\mathbf{S}\rho\varpi^2\rho = -c_2$. For our purposes it is more convenient to regard it as the intersection of the Poinsot ellipsoid with the quadric cone (vertex at centre of mass)

$$\mathbf{S}\rho\,(c_2 - c_1\varpi)\,\varpi\rho = 0.$$

This cone will be called the *polhode cone*. The usual definition of the polhode is that it is the locus of points on the Poinsot ellipsoid whose tangent planes are at the constant distance $\mathbf{T}\mu^{-1}c_1$ from the centre; i.e. which touch the sphere $\rho^2 = c_1^2\mu^{-2} = -c_1^2/c_2$. [The rotor perpendicular from the centre on the plane (46) is $-c_1\mu^{-1}$.]

It will be observed that $\dot\omega$ and $\dot\omega_0$ are not related in the same way as ω and ω_0; i.e. we have not $\dot\omega = Q\dot\omega_0 Q^{-1}$. But $Q\dot\omega_0 Q^{-1}$ is a rotor through the centre of mass at any time which we have occasion to consider, so we define ω' by the equation

$$\omega' = Q\dot\omega_0 Q^{-1} \quad\ldots\ldots\ldots\ldots\ldots\ldots(47).$$

Thus ω' is the value of $\dot\omega_0$ when the standard fixed position is taken as that of the position at the instant under consideration. It is parallel and equal to the velocity relative to the contact

plane of the point of contact of the Poinsot ellipsoid with the contact plane.

By equation (40)

$$\dot{\omega}_0 = -\varpi_0^{-1}\mathbf{M}\omega_0\varpi_0\omega_0,$$

which gives the following expressions for ω'

$$\omega' = -\varpi^{-1}\mathbf{M}\omega\varpi\omega = -m^{-1}\mathbf{M}\varpi\omega\varpi^2\omega \quad\text{.............(48)},$$

or

$$\omega' = -\varpi^{-1}\mathbf{M}\omega\mu = -m^{-1}\mathbf{M}\mu\varpi\mu \quad\text{..............(49)}.$$

The first of each of these expressions shows that ω' is along that diameter of the Poinsot ellipsoid whose conjugate plane contains the angular velocity and the moment of momentum. Call the centre of mass at any instant O', the point of contact of the ellipsoid and contact-plane P, and the foot of the perpendicular from O' on the contact plane T. Then the conjugate plane of the diameter along ω' is $O'PT$; in other words the normal at the point where ω' cuts the ellipsoid is perpendicular to the plane $O'PT$. The second expression shows that ω' is perpendicular both to μ and $\varpi\mu$. That it is perpendicular to μ of course follows from the fact that it is by its definition parallel to the contact plane.

For applications to be made immediately it is necessary to notice that $2\mathbf{M}\omega'\omega$ can be expressed as a self-conjugate pencil function $\psi\omega$ of ω where ψ is fixed in the rigid body; i.e. where $\psi = Q\psi_0(Q^{-1}[\ ]Q).Q^{-1}$, ψ_0 being an absolutely constant self-conjugate pencil function. For

$$\left.\begin{aligned} 2\mathbf{M}\omega'\omega &= 2m^{-1}\mathbf{M}\omega\mathbf{M}\varpi\omega\varpi^2\omega \\ &= 2m^{-1}(-\varpi\omega\mathbf{S}\omega\varpi^2\omega + \varpi^2\omega\mathbf{S}\omega\varpi\omega) = \psi\omega \end{aligned}\right\} \dots (50),$$

where

$$\psi = 2m^{-1}(c_2 - c_1\varpi)\varpi \quad\text{.................(51)}.$$

[In the rest of this section and in the following section (§ 47) we shall have no occasion to refer to the ψ and ψ_0 of equations (10) to (39) above. The former ψ was not like the present ψ, a *pencil* function, though it was self-conjugate.]

The polhode cone is

$$\mathbf{S}\rho\psi\rho = 0 \quad\text{.........................(52)}.$$

As P traces out the polhode the extremity of $\rho = \psi\omega$, a rotor drawn from O' traces out another curve. The point on this curve corresponding to P we will call Q. Thus

$$\overline{O'P} = \omega,\ \overline{O'Q} = \psi\omega \quad\text{...................(53)}.$$

Since [eq. (52)] $\mathbf{S}\omega\psi\omega = 0$, $O'P$ and $O'Q$ are perpendicular. And further, the normal to the polhode cone at P is parallel to $\psi\omega$, i.e. to $O'Q$. Hence Q lies on the cone (vertex O') of lines normal to the polhode cone. This cone will be referred to as the *normal cone* and the curve traced out by Q will be called the *normal cone curve*. Since

$$\mathbf{S}\omega\psi\omega = 0,\ \mathbf{S}\omega\varpi\omega = -c_1,$$

or

$$\mathbf{S}\psi\omega\psi^{-1}\psi\omega = 0,\ \mathbf{S}\psi\omega\psi^{-2}\varpi\psi\omega = -c_1,$$

the equation of the normal cone is

$$\mathbf{S}\rho\psi^{-1}\rho = 0 \qquad\qquad (54),$$

and the equations of the normal cone curve are (54) and

$$\mathbf{S}\rho\psi^{-2}\varpi\rho = -c_1 \qquad\qquad (55).$$

From the equation $\mathbf{S}\omega\varpi^2\omega = -c_2$ we also have for any point on the normal cone curve

$$\mathbf{S}\rho\psi^{-2}\varpi^2\rho = -c_2 \qquad\qquad (56),$$

and its equations may be taken as any two independent combinations of the three last equations. We will examine the geometrical properties of the normal cone curve later. Meanwhile we return to the dynamics involved.

Suppose now R is any octonion and suppose its time-changes are observed by three observers—the first, whom we will call the outsider, being fixed in space; the second, whom we will call the plane-resident, moving with the velocity of the body's centre of mass; and the third, whom we will call the resident, residing on the rigid body. We may suppose the plane-resident to be unconscious of his own motion and also the resident to be unconscious of the motion of the rigid body on which he resides. The evolutions of R will appear to the resident to be those of the actual octonion $Q^{-1}RQ$. To the plane-resident they will appear to be those of $(1+\tfrac{1}{2}\Omega\sigma t)^{-1}R(1+\tfrac{1}{2}\Omega\sigma t) = R - t\Omega\mathbf{M}\sigma\mathbf{M}R$. Let us then define R_0, R_1, R' and R'' by the equations

$$\left.\begin{aligned} R_0 &= Q^{-1}RQ, & R_1 &= R - t\Omega\mathbf{M}\sigma\mathbf{M}R \\ R' &= Q\dot{R}_0Q^{-1}, & R'' &= \dot{R}_1 + t\Omega\mathbf{M}\sigma\mathbf{M}\dot{R}_1 \end{aligned}\right\} \qquad (57).$$

Thus R_0 and R_1 are the aspects of R to the resident and plane-resident respectively, *reduced to their respective initial positions*; and R' and R'' are the apparent rates of change of R to these observers *reduced to the actual position of the rigid body*. [If the

body is subject to external forces and $\Omega\tau$ is its integral translation at any time we ought to write $\Omega\tau$ in place of $\Omega\sigma t$ in the last equation. In this case $\Omega\dot{\tau}$ takes the place of $\Omega\sigma$ in eq. (58).]

By eq. (6) § 11 we have

$$\dot{R} = R' + \mathbf{M}F\mathbf{M}R = R'' + \Omega\mathbf{M}\sigma\mathbf{M}R \quad\text{............(58)},$$

i.e., the actual rate of change $\dot{R}$ which is noted by the outsider consists of the rate R' observed by the resident combined with the rate $\mathbf{M}F\mathbf{M}R$ due to the resident's motion; and also consists of the rate R'' observed by the plane-resident combined with the rate $\Omega\mathbf{M}\sigma\mathbf{M}R$ due to the plane-resident's motion.

It will be noticed that since $\mathbf{M}F\omega = \Omega\mathbf{M}\sigma\omega$, the meaning [eq. (47)] above given to ω' is consistent with the present meaning of R' and is also such that

$$\omega' = \omega'' \quad\text{..........................(59)}.$$

If we define the axials q and r by the equations

$$\sigma\omega^{-1} = q = f + \beta,\ r = \varpi\omega\sigma^{-1} = \mu\sigma^{-1} = g + \gamma \quad\text{......(60)},$$

β and γ being rotors and f and g scalars; then $\mathbf{S}q$ or f is the pitch of F, and $\mathbf{S}r$ or g is the pitch of G; and $\mathbf{M}q$ or β is the rotor perpendicular from O' on F, and $\mathbf{M}r$ or γ the rotor perpendicular from O' on G. It will be noticed that

$$rq = \mu\omega^{-1} = \varpi\omega\omega^{-1} \quad\text{...................(61)}.$$

These equations enable us to express various rotors, &c., in terms of the relative positions of O', F and G. Note that r is a constant except as to the *position* of its axis which always passes through O'; but q is a variable.

For instance by eq. (49)

$$\omega' = \omega^2\varpi^{-1}\mathbf{M}\mu\omega^{-1} = \omega^2\varpi^{-1}\mathbf{M}rq,$$

which gives

$$\omega' = \omega^2\varpi^{-1}(\mathbf{M}\gamma\beta + g\beta + f\gamma) \quad\text{.................(62)}.$$

The expression in the brackets admits of quite simple interpretation by means of figure (8); thus we have an expression for the velocity of the point of contact of the ellipsoid with the contact-plane in terms of the relative positions of O', F, G and the ellipsoid.

Again by eq. (50)

$$\psi\omega^{-1} = -2\mathbf{M}\omega\varpi^{-1}\mathbf{M}\mu\omega^{-1} = -2m^{-1}\varpi\mathbf{M}\mu\mathbf{M}\mu\omega^{-1}.$$

Putting in this equation

$$\mathbf{M}\mu\omega^{-1} = \mathbf{M}rq = \mathbf{M}\gamma\beta + g\beta + f\gamma,$$

we get
$$\psi\omega^{-1} = -2m^{-1}\varpi\mathbf{M}\mu\,(\mathbf{M}\gamma\beta + g\beta + f\gamma) \quad \text{.........(63)},$$

which gives an expression for $\psi\omega^{-1}$ in terms of the relative positions of O', F, G, the constant rotor μ and the ellipsoid. Again putting in the last

$$\mu = r\sigma = (\gamma + g)\,\sigma,$$

we get

$$\psi\omega^{-1} = -2m^{-1}\varpi\,\{-\sigma S\gamma\,(g\beta + f\gamma)$$
$$+\,\sigma\,[(g^2 - \gamma^2)\,\beta + (fg + \mathbf{S}\beta\gamma)\,\gamma]\} \quad \text{......(64)},$$

which gives an expression in terms of the same things except that the constant rotor μ has been replaced by the constant rotor σ. Notice that the expression in the brackets is in terms of components along the three rotors σ, $\sigma\beta$ and $\sigma\gamma$; and that the coefficient $g^2 - \gamma^2$ of the variable rotor $\sigma\beta$ is constant; whereas the coefficients of the constant rotors σ and $\sigma\gamma$ are variable.

To the plane-resident the rotors μ and σ and the motor G would appear invariable and the ellipsoid would appear to roll on the plane according to Poinsot's construction. The variation of the rotor ω is given since this is the rotor from O' to the point of contact. Similarly to the resident the ellipsoid appears at rest, the system consisting of μ, σ, G, and the plane appears as a moving rigid system whose motion is governed by the fact that the plane appears to roll on the ellipsoid while it is constrained to touch a fixed sphere at a point fixed in itself. Here the variation of ω is given as before. The question now naturally arises—how does F in position and pitch appear to vary to each of these observers? Since F is completely given in terms of σ and ω we have one construction for its motion from the above, but it is interesting to obtain others. The direction and tensor of F being the same as those of ω are sufficiently simply given by the above constructions. When in addition to these the pitch f and the rotor perpendicular β from O' on F are known, F is known. We therefore now proceed to find various expressions for the rate of change of these two.

σ is absolutely constant. Hence by eq. (60)

$$\dot{q} = -\sigma\omega^{-1}\dot{\omega}\omega^{-1} = -\sigma\omega^{-1}\omega'\omega^{-1} - \Omega\sigma\omega^{-1}\mathbf{M}\sigma\omega\,.\,\omega^{-1},$$

by substituting from eq. (58) $\omega' + \Omega \mathbf{M}\sigma\omega$ for $\dot{\omega}$, since $\omega' = \omega''$. Now $\sigma\omega^{-1}\mathbf{M}\sigma\omega \,.\, \omega^{-1} = -\sigma\mathbf{M}\sigma\omega^{-1}$. Hence by equation (58)

$$q'' = -\sigma\omega^{-1}\omega'\omega^{-1} = -q\omega'\omega^{-1} \quad\text{..............(65).}$$

Taking the scalar part we get $\dot{f}$ the rate of change of the pitch of F. Thus

$$\dot{f} = -\mathbf{S}\sigma\omega^{-1}\omega'\omega^{-1} = -\mathbf{S}q\omega'\omega^{-1} \quad\text{..............(66).}$$

Each of these expressions for $\dot{f}$ admits of simple interpretation. $\omega\omega'\omega^{-1}$ is ω' turned conically round ω through two right angles. Call such a conical rotation a semi-revolution (§ 41 above). By the first expression of eq. (66) then, $\dot{f}$ is the scalar product of the velocity of the rigid body σ, and of the velocity of the point of contact revolved half round ω, multiplied by $-\omega^{-2}$.

Putting in the second expression $q = f + \beta$ we have

$$\dot{f} = -f\mathbf{S}\omega'\omega^{-1} - \mathbf{S}\beta\mathbf{M}\omega'\omega^{-1} \quad\text{..............(67).}$$

Here $\mathbf{S}\omega'\omega^{-1}$ is the rate of increase of the logarithm of $\mathbf{T}\omega$, and $\mathbf{M}\omega'\omega^{-1}$ is the rotor angular velocity of the point of contact about O'. This gives a second interpretation.

Remembering that $\mathbf{M}\omega'\omega^{-1} = \frac{1}{2}\psi\omega^{-1}$ we get other simple expressions from equations (63) and (64) which can be written down by the reader. And again we may substitute from equations (48) and (49) for ω'. Thus taking the first expression of eq. (49)

$$\begin{aligned}\dot{f} &= -\mathbf{S}\sigma\omega\varpi^{-1}\mathbf{M}\mu\omega^{-1} \,.\, \omega^{-1} \\ &= \mathbf{S}\sigma\varpi_\omega{}^{-1}\mathbf{M}\mu\omega^{-1} = \mathbf{S}\sigma\varpi_\omega{}^{-1}\mathbf{M}rq,\end{aligned}$$

[eq. (61)] where ϖ_ω is what ϖ would become if the rigid body were revolved half round ω, i.e.

$$\varpi_\omega E = \omega\varpi(\omega^{-1}E\omega) \,.\, \omega^{-1} \quad\text{.................(68).}$$

Thus substituting for $\mathbf{M}rq$ in terms of β, &c.

$$\dot{f} = \mathbf{S}\sigma\varpi_\omega{}^{-1}(\mathbf{M}\gamma\beta + g\beta + f\gamma) \quad\text{..............(69).}$$

Again in the expression $-\mathbf{S}\sigma\omega^{-1}\varpi^{-1}\mathbf{M}rq \,.\, \omega$ if we substitute q for $\sigma\omega^{-1}$ and $q^{-1}\sigma$ for ω we get

$$\dot{f} = -\mathbf{S}\sigma q\varpi^{-1}\mathbf{M}rq \,.\, q^{-1},$$

or $$\dot{f} = -\mathbf{S}\sigma\varpi_q{}^{-1}\mathbf{M}qr = -\mathbf{S}\sigma\varpi_q{}^{-1}(\mathbf{M}\beta\gamma + g\beta + f\gamma) \quad\text{......(70),}$$

where ϖ_q is what ϖ would become if the rigid body were turned

in the plane of σ, ω through twice the angle between these two; i.e.

$$\varpi_q E = q\varpi (q^{-1}Eq) . q^{-1} \quad\text{.................(71).}$$

The velocity of the foot of the perpendicular from O' on F as observed by the plane-resident is equal and parallel to β''. Taking the motor part of equation (65) we get

$$\beta'' = -\omega^{-2}\mathbf{M}\sigma\omega\omega'\omega^{-1} \quad\text{....................(72).}$$

Thus this velocity is the rotor product of the velocity σ of the rigid body, and the velocity ω' of the point of contact revolved half round ω, multiplied by $-\omega^{-2}$.

Another interesting expression for β'' may be obtained thus. By eq. (48)

$$\omega' = -m^{-1}\mathbf{M}\varpi\omega\varpi^2\omega = m^{-1}\mathbf{M}\varpi\omega\,(c_2c_1{}^{-1}\varpi\omega - \varpi^2\omega)$$
$$= \tfrac{1}{2}c_1{}^{-1}\mathbf{M}\varpi\omega\psi\omega,$$

or by eq. (51)
$$\omega' = \tfrac{1}{2}c_1{}^{-1}\mathbf{M}\mu\psi\omega \quad\text{......................(73).}$$

Now $\psi\omega$ is [eq. (50)] perpendicular to ω. Hence

$$\omega\psi\omega \,.\, \omega^{-1} = -\psi\omega.$$

Hence
$$\omega^{-1}\omega'\omega^{-1} = -\tfrac{1}{2}c_1{}^{-1}\mathbf{M} \,.\, \omega\mu\omega^{-1}\psi\omega^{-1},$$

so that
$$\beta'' = \tfrac{1}{2}c_1{}^{-1}\mathbf{M}\sigma\mathbf{M}\omega\mu\omega^{-1}\psi\omega^{-1} \quad\text{..............(74).}$$

Here $\omega\mu\omega^{-1}$ is μ revolved half round ω, and $\psi\omega^{-1}$ may be substituted for from equation (63) or (64).

Again taking the second expression of eq. (65) and putting it in the form (since $\omega = q^{-1}\sigma$)

$$q'' = -\omega^{-2}q\omega'q^{-1}\sigma,$$

we have
$$\beta'' = \omega^{-2}\mathbf{M}\sigma q\omega'q^{-1} \quad\text{....................(75).}$$

Here $q\omega'q^{-1}$ is ω' rotated conically round β so that the equation admits of simple interpretation. Putting [eq. (49)]

$$\omega^{-2}\omega' = \varpi^{-1}\mathbf{M}\mu\omega^{-1} = \varpi^{-1}\mathbf{M}rq$$

[eq. (61)], we get [eq. (71)]

$$\beta'' = \mathbf{M}\sigma\varpi_q{}^{-1}\mathbf{M}qr = \mathbf{M}\sigma\varpi_q{}^{-1}(\mathbf{M}\beta\gamma + g\beta + f\gamma) \quad\text{......(76).}$$

By eq. (58)

$$\beta' = \beta'' - \mathbf{M}\omega\beta \quad\text{......................(77),}$$

so that each of these expressions for β'' serves to give an expression for β' the velocity of the foot of the perpendicular from O' on F as observed by the resident.

The introduction above of ϖ_ω and ϖ_q suggests a brief examination of the kinematics of a body first subjected to the displacement $Q(\ )Q^{-1}$ and then to one or other of the displacements $\omega(\ )\omega^{-1}$ or $q(\ )q^{-1}$.

If a body is first subjected to the displacement $Q(\ )Q^{-1}$ and then to the displacement $R(\ )R^{-1}$, its whole displacement is $RQ(\ )Q^{-1}R^{-1}$. If this combined displacement is made in the time t and if F_R is the velocity motor at time t we have by eq. (8) § 11

$$F_R = 2\mathbf{M}(\dot{R}Q + R\dot{Q})\,Q^{-1}R^{-1} = 2\mathbf{M}\dot{R}R^{-1} + RFR^{-1} \ldots\ldots(78),$$

which admits of a simple enough kinematical interpretation.

Putting $R = \omega$ we have

$$F_\omega = 2\mathbf{M}\dot{\omega}\omega^{-1} + \omega F\omega^{-1}$$
$$= 2\mathbf{M}(\omega' + \Omega\mathbf{M}\sigma\omega)\,\omega^{-1} + \omega F\omega^{-1},$$

by equations (58) and (59). Now $\Omega\mathbf{M}\sigma\omega = \mathbf{M}F\omega$ since $F = \omega + \Omega\sigma$.

Hence $\quad 2\Omega\mathbf{M}\sigma\omega\,.\,\omega^{-1} + \omega F\omega^{-1} = 2\mathbf{M}F\omega\,.\,\omega^{-1} + \omega F\omega^{-1} = F.$

Hence $\quad F_\omega = F + 2\mathbf{M}\omega'\omega^{-1} = F + \psi\omega^{-1} \ldots\ldots\ldots\ldots\ldots(79).$

F_q may be obtained independently or from this result. For the displacement $q(\ )q^{-1} = \sigma\omega^{-1}(\ )\omega\sigma^{-1}$. Hence the displacement $qQ(\ )Q^{-1}q^{-1}$ may be obtained by first making the displacement $\omega^{-1}Q(\ )Q^{-1}\omega$ [or $\omega Q(\ )Q^{-1}\omega^{-1}$] and then the displacement $\sigma(\ )\sigma^{-1}$. Using eq. (78) for this case

$$F_q = 2\mathbf{M}\dot{\sigma}\sigma^{-1} + \sigma F_\omega\sigma^{-1},$$

or since $\dot{\sigma} = 0$,

$$F_q = \sigma F_\omega \sigma^{-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(80).$$

The decomposition here of the rotation $q(\ )q^{-1}$ into the two semi-revolutions $\omega(\ )\omega^{-1}$ and $\sigma(\ )\sigma^{-1}$ is a particular case of the more general decompositions of § 42 above.

If the velocity motor F_ω be decomposed into a rotation about O' and a translation, the rotation is $\omega + \psi\omega^{-1}$ and the translation $\Omega\sigma$ that of the rigid body. Similarly decomposing F_q the rotation is $\sigma(\omega + \psi\omega^{-1})\,\sigma^{-1}$ and the translation $\Omega\sigma$.

47. **The polhode and the normal cone curve.** The frequent appearance above of the function ψ [eq. (51) § 46] leads us to enquire into the geometrical properties of what was called the normal cone curve and its relations to the polhode.

In this section we shall drop all consideration of ω as representing the rotation of the rigid body and also all consideration of the motion of the body. We shall regard ω merely as the coordinate rotor of any point on the polhode and ω' (though as a matter of fact it is equal and parallel to the velocity of the point of contact of the polhode with the contact plane) as a rotor parallel to the tangent of the polhode at the point ω. For the sake of clearness we here briefly recapitulate the geometrical properties and equations established in § 46.

ϖ is a given self-conjugate pencil function whose centre is the origin O. Its cubic is

$$\varpi^3 - m''\varpi^2 + m'\varpi - m = 0 \qquad\text{.....................(1).}$$

The roots of this cubic which are all positive will be taken as α, β, γ. [They were a^2, b^2, c^2, in § 46.] Thus

$$\alpha + \beta + \gamma = m'',\ \beta\gamma + \gamma\alpha + \alpha\beta = m',\ \alpha\beta\gamma = m \qquad\text{......(2).}$$

We shall have occasion to require the half-sum of the roots, so define s by the equation

$$2s = \alpha + \beta + \gamma = m'' \qquad\text{.....................(3).}$$

ψ is a pencil function defined by the equation

$$\psi = 2m^{-1}(c_2 - c_1\varpi)\,\varpi \qquad\text{.....................(4),}$$

where c_1 and c_2 are given positive constants.

The polhode is defined as the curve given by the two equations

$$\mathbf{S}\omega\varpi\omega = -c_1,\ \mathbf{S}\omega\varpi^2\omega = (\varpi\omega)^2 = -c_2 \qquad\text{...........(5).}$$

[Thus the perpendicular from O on the tangent plane of the ellipsoid $\mathbf{S}\omega\varpi\omega = -c_1$ at the point ω has when this point is on the polhode the constant value $c_1/\sqrt{c_2}$.] Thus the polhode lies on the quadric cone

$$\mathbf{S}\omega\psi\omega = 0 \qquad\text{..........................(6).}$$

The normal cone curve is defined as the curve any point of which has $\rho = \psi\omega$ for coordinate rotor. Thus P being a point on the polhode and Q the corresponding point on the normal curve

$$\overline{OP} = \omega,\ \overline{OQ} = \rho = \psi\omega \qquad\text{.....................(7).}$$

By eq. (6)

$$\mathbf{S}\rho\omega = 0 \qquad\text{..........................(8),}$$

so that ρ is perpendicular to ω. It is also perpendicular to the

tangent plane at P of the polhode cone (6). It therefore lies on the cone normal to the polhode cone. Substituting for ω in terms of ρ equations (5) and (6) give

$$\mathbf{S}\rho\psi^{-2}\varpi\rho = -c_1,\ \mathbf{S}\rho\psi^{-2}\varpi^2\rho = -c_2,\ \mathbf{S}\rho\psi^{-1}\rho = 0 \ldots\ldots(9),$$

and any two independent combinations of these three equations may be taken as the equations of the normal cone curve. Thus the equation of the normal cone is $\mathbf{S}\rho\psi^{-1}\rho = 0$.

When ω is given ρ^2 can be written down from the following:—

$$\rho^2 = \mathbf{S}\omega\psi^2\omega = 4m^{-2}\mathbf{S}\omega\,(c_2 - c_1\varpi)^2\,\varpi^2\omega$$
$$= 4m^{-2}(-c_2^3 - 2c_1c_2\mathbf{S}\omega\varpi^3\omega + c_1^2\mathbf{S}\omega\varpi^4\omega).$$

Now by eq. (1) $\mathbf{S}\omega\varpi^3\omega$ and $\mathbf{S}\omega\varpi^4\omega$ can (§ 46) be expressed in terms of ω^2, c_1 and c_2. The relation that we thus get is eq. (43) § 46 which for present purposes may be written

$$\rho^2 = 4m^{-1}c_1\,(m''c_1 - 2c_2)\,\omega^2 + \text{const.} \ldots\ldots\ldots\ldots(10).$$

If ω' is a rotor through O parallel to the tangent of the polhode at P we have by equations (5)

$$\mathbf{S}\omega'\varpi\omega = 0,\ \mathbf{S}\omega'\varpi^2\omega = 0.$$

Thus $\omega' = x\mathbf{M}\varpi\omega\varpi^2\omega$ where x is any scalar. If x is put equal to $-m^{-1}$ [eq. (48) § 46], ω' is the velocity of P along the polhode if P is always the point of contact of the ellipsoid and contact-plane. We therefore put

$$\omega' = -m^{-1}\mathbf{M}\varpi\omega\varpi^2\omega = -\varpi^{-1}\mathbf{M}\omega\varpi\omega \ldots\ldots\ldots\ldots(11).$$

By substituting here for ϖ^2 in terms of ψ and ϖ by means of eq. (4) we get

$$\omega' = \tfrac{1}{2}c_1^{-1}\mathbf{M}\varpi\omega\psi\omega = -\tfrac{1}{2}c_1^{-1}\mathbf{M}\rho\varpi\omega \ldots\ldots\ldots\ldots(12).$$

From the last expression we have by equations (8) and (5)

$$2\mathbf{M}\omega'\omega = \rho \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

These results except (10) have already been proved in § 46.

As P moves along the polhode with velocity ω', Q moves along the normal cone curve with some definite velocity ρ'. Thus $\rho' = \psi\omega'$. [This velocity is of course the velocity as it appears to the resident (of § 46). The velocity as it appears to the plane-resident $= \rho'' = \rho' + \mathbf{M}\omega\rho$.] Thus by the first of equations (12)

$$\rho' = -\tfrac{1}{2}c_1^{-1}m_\psi\mathbf{M}\omega\psi^{-1}\varpi\omega = -\tfrac{1}{2}c_1^{-1}m_\psi\mathbf{M}\omega\psi^{-2}\varpi\rho \ldots\ldots(14),$$

where m_ψ is the product of the roots of the ψ cubic, so that by equations (1) and (4)

$$m_\psi = 8m^{-2}(c_2^3 - c_2^2c_1m'' + c_2c_1^2m' - c_1^3m) \qquad (15).$$

From the first of equations (14) we have

$$\mathbf{M}\rho'\rho = \tfrac{1}{2}c_1^{-1}m_\psi\mathbf{M}\rho\mathbf{M}\omega\psi^{-1}\varpi\omega = -\tfrac{1}{2}c_1^{-1}m_\psi\omega\mathbf{S}\rho\psi^{-1}\varpi\omega,$$

by eq. (8). Thus by eq. (7) and eq. (5)

$$2\mathbf{M}\rho'\rho = m_\psi\omega \qquad (16).$$

Thus from eq. (13)

$$\frac{\rho\mathbf{M}\rho'\rho}{\omega\mathbf{M}\omega'\omega} = -m_\psi \qquad (17),$$

i.e. $\mathbf{T}\rho^3 \times$ the angular velocity of Q about the origin bears a constant ratio to $\mathbf{T}\omega^3 \times$ the angular velocity of P about the origin; or $\mathbf{T}\rho \times$ the areal velocity of Q about the origin bears a constant ratio to $\mathbf{T}\omega \times$ the areal velocity of P about the origin.

From eq. (10)

$$\frac{\mathbf{S}\rho'\rho}{\mathbf{S}\omega'\omega} = 4m^{-1}c_1(m''c_1 - 2c_2) \qquad (18).$$

Combining the last two equations, or better combining (13), (16) and (18)

$$\frac{\mathbf{M}\omega'\omega}{\mathbf{S}\omega'\omega}\bigg/\frac{\mathbf{M}\rho'\rho}{\mathbf{S}\rho'\rho} = 4m^{-1}m_\psi^{-1}c_1(m''c_1 - 2c_2)\,\rho/\omega \qquad (19),$$

or the tangent of the angle between the radius vector (from O) and the polhode at P bears a ratio to the tangent of the angle between the radius vector and the normal cone curve at Q which is a constant multiple of the ratio of $\mathbf{T}\omega^{-1}$ to $\mathbf{T}\rho^{-1}$. This, along with the fact that Q lies on the cone normal to the polhode cone, suffices to geometrically construct the normal cone curve when one point on it is given and the polhode is given.

These relations it will be observed are symmetrical with respect to the two curves. This symmetry is further borne out by a comparison between the two equations of each pair of the following which have been established above or which may be at once deduced from the established results.

$$\mathbf{S}\omega\psi\omega = 0, \quad \mathbf{S}\rho\psi^{-2}\varpi\rho = 0,$$

$$\omega' = -m^{-1}\mathbf{M}\varpi\omega\varpi^2\omega, \quad \rho' = -\tfrac{1}{2}c_1^{-1}m_\psi\mathbf{M}\psi^{-1}\rho\psi^{-2}\varpi\rho,$$

$$\omega' = -\varpi^{-1}\mathbf{M}\omega\varpi\omega, \quad \rho' = -\tfrac{1}{2}c_1^{-1}\psi\mathbf{M}\rho\psi^{-1}\varpi\rho,$$

$$\omega' = m\varpi^{-2}\mathbf{M}\omega\varpi^{-1}\omega, \quad \rho' = \tfrac{1}{2}c_1^{-1}mm_\psi^{-1}\psi^2\varpi^{-1}\mathbf{M}\rho\psi\varpi^{-1}\rho,$$

$$2\mathbf{M}\omega'\omega = \rho, \quad 2\mathbf{M}\rho'\rho = m_\psi\omega.$$

This symmetry suggests that perhaps the normal cone curve is a polhode on some quadric and that the original polhode is its normal cone curve. We will now prove that something very like this is the case.

If four scalars y, z, y', z' can be determined such that

$$(y + z\varpi)^2 = \psi^2\varpi^{-1}(y' + z'\varpi) \quad \ldots\ldots\ldots\ldots\ldots(20),$$

we shall have

$$(y\psi^{-2}\varpi + z\psi^{-2}\varpi^2)^2 = y'\psi^{-2}\varpi + z'\psi^{-2}\varpi^2 \ldots\ldots\ldots(21).$$

Thus if we put

$$\varpi_1 = y\psi^{-2}\varpi + z\psi^{-2}\varpi^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(22),$$

the equations of the normal cone curve can by equations (9) be put in the form

$$\left.\begin{aligned} \mathbf{S}\rho\varpi_1\rho &= -yc_1 - zc_2 = -c_1' \\ (\varpi_1\rho)^2 = \mathbf{S}\rho\varpi_1^2\rho &= -y'c_1 - z'c_2 = -c_2' \end{aligned}\right\} \ldots\ldots\ldots(23),$$

so that the normal cone curve is a polhode on the quadric $\mathbf{S}\rho\varpi_1\rho = -c_1'$ whose contact plane is at the distance $c_1'/\sqrt{c_2'}$ from the centre.

Assuming that y, z, y' and z' can be so determined, m_1 the product of the roots of the ϖ_1 cubic is by equation (22) given by

$$m_1 = mm_\psi^{-2}(y^3 + m''y^2z + m'yz^2 + mz^3) \ldots\ldots\ldots(24),$$

and we have

$$\begin{aligned} 2m_1^{-1}(c_2' - c_1'\varpi_1)\varpi_1 &= 2m_1^{-1}\{c_2'(y + z\varpi) - c_1'(y' + z'\varpi)\}\psi^{-2}\varpi \\ &= 2m_1^{-1}(yz' - y'z)(c_2 - c_1\varpi)\psi^{-2}\varpi \\ &= m_1^{-1}m(yz' - y'z)\psi^{-1} = x\psi^{-1} = \psi_1, \end{aligned}$$

say, by eq. (4). Thus the normal cone curve of this new polhode is given by the equation

$$\rho_1 = \psi_1\rho = x\omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25),$$

where ρ_1 is the coordinate vector of Q_1 the point on the new normal cone curve corresponding to Q on the new polhode. Thus

OQ_1P is a straight line; and OQ_1/OP has the constant value x. Hence the new normal cone curve is similar and similarly situated to the original polhode, the centre of similarity being the origin.

It may be noticed that though only the three ratios $y^2:z^2:y':z'$ are determined by eq. (20) the quadrics of eq. (23) are not altered by altering the absolute magnitudes of these four quantities. Thus we cannot by such an alteration reduce x of eq. (25) to unity. Also since c_2 is of two dimensions and c_1 of one and m of three in ϖ, ψ is of no dimensions in ϖ. Similarly ψ_1 is of no dimensions in ϖ_1. ψ is however of two dimensions in ω and ψ_1 of two dimensions in ρ. This is illustrated in § 46 where $\psi\omega^{-1}$ appears as an angular velocity which is therefore of the same dimensions as ω.

It is to be observed that if y and z and therefore ϖ_1 are real each of the quadrics $\mathbf{S}\rho\varpi_1\rho = -c_1'$ and $\mathbf{S}\rho\varpi_1^2\rho = -c_2'$ must be real, for each equation is satisfied by an infinite number of real values of ρ, namely the coordinate rotors of the (real) normal cone curve. Under these circumstances the quadric $\mathbf{S}\rho\varpi_1^2\rho = -c_2'$ must be an ellipsoid and c_2' must be positive since the roots of the ϖ_1^2 cubic are all positive. [It is necessary to note this because ϕ may be real and yet the quadric $\mathbf{S}\rho\phi\rho = -c$ imaginary. This last is the case if all the roots of the ϕ cubic have the same sign as $-c$.]

We shall find that the determination of the ratios $y^2:z^2:y':z'$ depends on the solution of a quadratic equation. This equation has in general two roots which give rise to two sets of values of the ratios. There are thus two and only two values which satisfy the required conditions except when the roots of this quadratic equation are equal, when there is only one. We will now show that in the limiting case when the roots are equal (1) $\mathbf{S}\rho\varpi_1\rho = -c_1'$ is a sphere and therefore (2) $\mathbf{S}\rho\varpi_1^2\rho = -c_2'$ is also a sphere as that the two equations (23) [since they are both satisfied by at least one and the same value of ρ] become identical; and (3) we will find the second quadric which always exists when $\mathbf{S}\rho\varpi_1\rho = -c_1'$ is not a sphere, of which the normal cone curve is a polhode.

Putting in eq. (20) $\psi = 2m^{-1}(c_2 - c_1\varpi)\varpi$ it becomes

$$(y + z\varpi)^2 = 4m^{-2}(c_2 - c_1\varpi)^2\varpi(y' + z'\varpi).$$

This can be satisfied by constant scalar values of y, z, y', z' if and only if

$$(y + z\varpi)^2 - 4m^{-2}(c_2 - c_1\varpi)^2\,\varpi\,(y' + z'\varpi)$$
$$\equiv (y'' + z''\varpi)(\varpi^3 - m''\varpi^2 + m'\varpi - m),$$

where y'' and z'' are two other constant scalars. Equating coefficients of the powers of ϖ it is easy to see as stated above that we are eventually led to a quadratic equation for the ratio of $y : z$. This shows that there are *not more than* two values of ϖ_1 of the form $(y + z\varpi)\,\psi^{-2}\varpi$ satisfying the required conditions. We now show that there *are* two functions of this form except when $\mathbf{S}\rho\varpi_1\rho = -c_1'$ is a sphere. We shall postpone till later the proof that when the two values of the ratio $y : z$ are equal the quadric in question is a sphere.

Suppose one set of values of y, z, y', z', c_1', c_2', ϖ_1 has been determined to satisfy the required conditions. Then the normal cone curve is given by the two equations

$$\mathbf{S}\rho\varpi_1\rho = -c_1',\quad \mathbf{S}\rho\varpi_1^2\rho = -c_2',$$

for these are always independent except when the roots of the ϖ_1 cubic are equal, i.e., except when $\mathbf{S}\rho\varpi_1\rho = -c_1'$ is a sphere. [It will appear later that this is always a *real* sphere.]

Let α_1, β_1, γ_1 be the roots of the ϖ_1 cubic and s_1 their half sum so that

$$(\varpi_1 - \alpha_1)(\varpi_1 - \beta_1)(\varpi_1 - \gamma_1) = 0,\ s_1 = \tfrac{1}{2}(\alpha_1 + \beta_1 + \gamma_1)\ldots\ldots(26).$$

If the constant scalars x, x' and y' can be determined to satisfy

$$(x\varpi_1 - \varpi_1^2)^2 = (x'\varpi_1 - y'\varpi_1^2),$$

the normal cone curve is also given by the two equations

$$\left.\begin{array}{l}\mathbf{S}\rho\varpi_2\rho = -c_1'' = -(xc_1' - c_2'),\\ \mathbf{S}\rho\varpi_2^2\rho = -c_2'' = -(x'c_1' - y'c_2')\end{array}\right\}\ \ldots\ldots\ldots\ldots(27),$$

where
$$\varpi_2 = x\varpi_1 - \varpi_1^2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28),$$

and expressing ϖ_1 and ϖ_1^2 in terms of ϖ and ψ, it is clear that ϖ_2 and ϖ_2^2 will be expressed in terms of ϖ and ψ by equations of the same form as those for ϖ_1 and ϖ_1^2.

Now $\varpi_1(x - \varpi_1)^2 - (x' - y'\varpi_1) \equiv (\varpi_1 - \alpha_1)(\varpi_1 - \beta_1)(\varpi_1 - \gamma_1)$,

if $\qquad x = s_1,\ x^2 + y' = \beta_1\gamma_1 + \gamma_1\alpha_1 + \alpha_1\beta_1,\ x' = \alpha_1\beta_1\gamma_1,$

i.e. if $x = s_1$, $x' = \alpha_1\beta_1\gamma_1$,

$$\left.\begin{array}{l} \\ y' = -\tfrac{1}{2}\{(s_1-\alpha_1)^2 + (s_1-\beta_1)^2 + (s_1-\gamma_1)^2 - s_1^2\}\end{array}\right\} \quad \text{......(29)},$$

so that these values of x, x' and y' satisfy the required conditions and we have

$$\varpi_2 = \varpi_1(s_1 - \varpi_1) \quad \text{......................(30)}.$$

From eq. (30) the roots α_2, β_2, γ_2 of the ϖ_2 cubic are given by

$$\alpha_2 = \alpha_1(s_1-\alpha_1),\ \beta_2 = \beta_1(s_1-\beta_1),\ \gamma_2 = \gamma_1(s_1-\gamma_1) \quad \text{......(31)},$$

and these are proportional to the roots α_1, β_1, γ_1 of the ϖ_1 cubic if and only if

$$\alpha_1 = \beta_1 = \gamma_1,$$

i.e. if and only if the quadric $\mathbf{S}\rho\varpi_1\rho = -c_1'$ is a sphere.

This proves that the normal cone curve is a polhode of both the quadrics $\mathbf{S}\rho\varpi_1\rho = -c_1'$ and $\mathbf{S}\rho\varpi_2\rho = -c_1''$, and that these quadrics though coaxial are not similar except when the first is a sphere. Also when one quadric is real the other is also.

It is clear from the above that (if we put $m_2 = \alpha_2\beta_2\gamma_2$)

$$\psi_2 = 2m_2^{-1}(c_2'' - c_1''\varpi_2)\varpi_2$$

is a constant scalar multiplied by ψ_1 [eq. (25)]. As a matter of fact it is quite easy to prove from the above results that

$$\psi_2 = -\psi_1 \quad \text{..........................(32)}.$$

The direct determination of the ratios $y^2 : z^2 : y' : z'$ is tedious. The following process is therefore preferable.

Changing the above ϖ to $c_1\varpi$ the original polhode has for equations

$$\mathbf{S}\omega\varpi\omega = -1,\quad \mathbf{S}\omega\varpi^2\omega = -b \quad \text{..............(33)},$$

where b is put for c_2/c_1^2. This is of course equivalent to taking in the above $c_1 = 1$, $c_2 = b$. According to eq. (4) ψ would now be $2m^{-1}\varpi(b - \varpi)$, but we will take the slightly more general form

$$\psi = em^{-1}\varpi(b - \varpi) \quad \text{....................(34)},$$

and we shall assume that the normal cone curve is given by $\rho = \psi\omega$ with this new value of ψ.

Thus the roots of the ϖ cubic [equations (1) and (2)] are now the inverse squares of the axes of the original quadric, if the ordinary conventions be adopted as to the imaginary axes of an

hyperboloid; and b is the inverse square of the perpendicular on the contact plane.

Define ϖ_0 by the equation

$$\varpi_0 = \frac{\varpi(s-\varpi)}{s-b} \quad\quad\text{.......................(35).}$$

By equation (33) $\mathbf{S}\omega\varpi_0\omega = -1$. Also

$$(s-b)^2\varpi_0^{\,2} = \varpi(\varpi^3 - 2s\varpi^2 + s^2\varpi),$$

or by eq. (1)

$$\varpi_0^{\,2} = \varpi\{m - (m' - s^2)\varpi\}/(s-b)^2 \quad\text{...........(36).}$$

Hence $$\mathbf{S}\omega\varpi_0\omega = -1,\ \mathbf{S}\omega\varpi_0^{\,2}\omega = -b_0 \text{..............(37),}$$

where $$b_0 = \{-b(m'-s^2) + m\}/(s-b)^2 \quad\text{...........(38).}$$

Hence the given polhode is also a polhode on the quadric $\mathbf{S}\omega\varpi_0\omega = -1$ and the inverse square of the perpendicular on the new contact plane is b_0.

If the ϖ_0 cubic is

$$\varpi_0^{\,3} - 2s_0\varpi_0^{\,2} + m_0'\varpi_0 - m_0 = 0 \text{.................(39),}$$

we have

$$2s_0 = \{\alpha(s-\alpha) + \beta(s-\beta) + \gamma(s-\gamma)\}/(s-b) = 2(m'-s^2)/(s-b),$$

$$m_0' = \{\beta\gamma(s-\beta)(s-\gamma) + \ldots\}/(s-b)^2$$
$$= \{s^2m' - s(2m's - 3m) + (m'^2 - 4ms)\}/(s-b)^2,$$

$$m_0 = \alpha\beta\gamma(s-\alpha)(s-\beta)(s-\gamma)/(s-b)^3$$
$$= m(s^3 - 2s^3 + m's - m)/(s-b)^3,$$

or
$$\left.\begin{aligned} s_0 = \frac{m'-s^2}{s-b},\quad m_0' &= \frac{-s^2m' - ms + m'^2}{(s-b)^2},\\ m_0 &= \frac{m(-s^3 + m's - m)}{(s-b)^3}\end{aligned}\right\}\text{......(40).}$$

From the first of these and equation (38)

$$s_0 - b_0 = \frac{s(m'-s^2) - m}{(s-b)^2} \quad\text{.................(41).}$$

From this we may notice in passing that

$$\frac{m_0}{s_0 - b_0} = \frac{m}{s-b} \text{.......................(42).}$$

Also $$s_0 - \varpi_0 = (m' - s^2 - \varpi s + \varpi^2)/(s-b),$$
whence

$$\varpi_0 (s_0 - \varpi_0)/(s_0 - b_0)$$
$$= \varpi \{(s-\varpi)(m'-s^2) - \varpi (s-\varpi)^2\}/\{s(m'-s^2) - m\},$$

or by eq. (1)

$$\varpi = \frac{\varpi_0 (s_0 - \varpi_0)}{s_0 - b_0} \quad \text{....................(43).}$$

Comparing this with equation (35) we see that the relations between ϖ_0, b_0 and the coefficients of the ϖ_0 cubic on the one hand and ϖ, b and the coefficients of the ϖ cubic on the other are symmetrical.

When the given quadric is a sphere b must be equal to ϖ in order that the polhode may be real. In this case $\varpi_0 = \varpi$ and the two quadrics are the same.

When $s = b$ eq. (35) gives $\varpi_0 = \infty$. In this case there are not two different quadrics on which the given polhode is a polhode. Putting however

$$\varpi' = \varpi (s - \varpi) = e^{-1} m\psi \text{.................(44),}$$

[eq. (34)] we have

$$\mathbf{S}\omega\varpi'\omega = 0, \; \mathbf{S}\omega\varpi'^2\omega = s(m' - s^2) - m$$
$$= (s-\alpha)(s-\beta)(s-\gamma) \text{ ......(45).}$$

From this it follows that $\mathbf{T}\psi\omega$ is constant or the normal cone curve lies on a sphere. As we shall see directly this is the limiting case mentioned just now, when $\mathbf{S}\rho\varpi_1\rho = \text{const.}$ is a sphere.

From equations (35) and (43) we have

$$\varpi_0\varpi^{-1} = (s-\varpi)/(s-b), \; \varpi\varpi_0^{-1} = (s_0 - \varpi_0)/(s_0 - b_0) \text{......(46).}$$

Hence

$$(s-\varpi)(s_0 - \varpi_0) = (s-b)(s_0 - b_0) = \{s(m' - s^2) - m\}/(s-b) \text{...(47),}$$

by eq. (41). This very simple relation between $s - \varpi$ and $s_0 - \varpi_0$ makes it more convenient for many purposes to regard $\phi = s - \varpi$ and $\phi_0 = s_0 - \varpi_0$ as the two fundamental pencil functions. ϖ is a determinate function of ϕ, for s is the sum of the roots of the ϕ cubic. We will not here however work with ϕ, ϕ_0 and their cubics.

By eq. (36)

$$\begin{aligned}\varpi_0 (b_0 - \varpi_0)(s-b)^3 &\\ &= \varpi \{b_0 (s-b)^2 (s-\varpi) - (s-b)[m - \varpi (m' - s^2)]\} \\ &= \varpi \{[m - b(m' - s^2)](s-\varpi) - (s-b)[m - \varpi (m' - s^2)]\} \\ &= -\varpi (b-\varpi)\{s(m'-s^2) - m\},\end{aligned}$$

or by eq. (40)

$$m_0^{-1}\varpi_0 (b_0 - \varpi_0) = - m^{-1}\varpi (b - \varpi) \qquad\text{.............(48).}$$

Hence if we define ψ_0 by the equation

$$\psi_0 = e_0 m_0^{-1}\varpi_0 (b_0 - \varpi_0) \qquad\text{....................(49),}$$

ψ_0 and ψ will have the same value if

$$e_0 = -e \qquad\text{..............................(50),}$$

and therefore $\rho = \psi_0\omega$. Thus the normal cone curve will have the same meaning for both the quadrics of which the given polhode is a polhode. [If we put $e_0 = e$ the same statement is true, but the point Q on the normal cone curve corresponding to the point P on the polhode when that curve is defined with reference to the quadric $\mathbf{S}\omega\varpi\omega = -1$, is diametrically opposite to Q_0 the point on the curve corresponding to P when the curve is defined with reference to the quadric $\mathbf{S}\omega\varpi_0\omega = -1$.]

By equation (35) we have

$$\varpi + \varpi_0 = \varpi (2s - b - \varpi)/(s-b) \qquad\text{...................(51),}$$

$$\varpi - \varpi_0 = \varpi (\varpi - b)/(s-b) = -\psi m/e(s-b) \qquad\text{......(52).}$$

Now define the pencil functions ϖ_1 and ϖ_2 by the equations

$$\varpi_1 + \varpi_2 = 2x(\varpi + \varpi_0)\psi^{-1} = -\frac{2xm}{e(s-b)}\frac{\varpi + \varpi_0}{\varpi - \varpi_0} \qquad\text{......(53),}$$

$$\varpi_1 - \varpi_2 = 2y\psi^{-1} = -\frac{2ym}{e(s-b)}\frac{1}{\varpi - \varpi_0} \qquad\text{..............(54),}$$

where x and y are ordinary scalars to be determined directly by certain conventions.

Since $\rho = \psi\omega$,

$$\varpi_1\rho = \{x(\varpi + \varpi_0) + y\}\,\omega, \quad \varpi_2\rho = \{x(\varpi + \varpi_0) - y\}\,\omega \qquad\text{...(55).}$$

Also
$$\mathbf{S}\rho(\varpi_1 - \varpi_2)\rho = 2y\mathbf{S}\omega\psi\omega = 0.$$

x is determined by making each of the equal quantities $\mathbf{S}\rho\varpi_1\rho$ and $\mathbf{S}\rho\varpi_2\rho$ equal to -1. Thus

$$\begin{aligned} -1 &= \tfrac{1}{2}\mathbf{S}\rho\,(\varpi_1+\varpi_2)\,\rho = x\mathbf{S}\omega\,(\varpi+\varpi_0)\,\psi\omega \\ &= -xe\,(s-b)\,m^{-1}\,\mathbf{S}\omega\,(\varpi^2-\varpi_0{}^2)\,\omega \text{ [eq. (52)]} \\ &= xe\,(s-b)\,m^{-1}\,(b-b_0) \text{ [equations (33) and (37)].} \end{aligned}$$

Hence
$$x = \frac{m}{e\,(s-b)\,(b_0-b)} \quad\text{..................(56).}$$

In connection with this note that

$$\begin{aligned} b_0 - b &= (m - bm' + 2sb^2 - b^3)/(s-b)^2 \\ &= (\alpha - b)\,(\beta - b)\,(\gamma - b)/(s-b)^2, \end{aligned}$$

or
$$(b_0-b)\,(s-b)^2 = (\alpha-b)\,(\beta-b)\,(\gamma-b) = -m^2 m_\psi e^{-3} \text{...(57).}$$

To determine y first notice that

$$\begin{aligned} \mathbf{S}\omega\varpi\varpi_0\omega\,.\,(s-b) &= \mathbf{S}\omega\varpi^2\,(s-\varpi)\,\omega \\ &= \mathbf{S}\omega\,(s\varpi^2 - 2s\varpi^2 + m'\varpi - m)\,\omega \text{ [eq. (1)]} \\ &= sb - m' - m\omega^2 \qquad \text{[eq. (33)].} \end{aligned}$$

Thus by equation (55)

$$\begin{aligned} (\varpi_1\rho)^2 &= \mathbf{S}\omega\,\{x\,(\varpi+\varpi_0)+y\}^2\,\omega \\ &= \mathbf{S}\omega\,\{x^2\,(\varpi^2+\varpi_0{}^2) + 2xy\,(\varpi+\varpi_0) + 2x^2\varpi\varpi_0 + y^2\}\,\omega \\ &= -x^2\,(b+b_0) - 4xy + 2x^2\,(sb-m')/(s-b) \\ &\qquad + \{y^2 - 2x^2 m/(s-b)\}\,\omega^2. \end{aligned}$$

Putting then
$$y = x\,\surd\{2m/(s-b)\} \text{.....................(58),}$$

we shall have

$$\mathbf{S}\rho\varpi_1{}^2\rho = -b_1,\ \mathbf{S}\rho\varpi_2{}^2\rho = -b_2 \text{..............(59),}$$

where
$$\left.\begin{aligned} b_1 &= x^2\,\{(b+b_0) + 2\,(s+s_0) + 4\,\surd[2m/(s-b)]\} \\ b_2 &= x^2\,\{(b+b_0) + 2\,(s+s_0) - 4\,\surd[2m/(s-b)]\} \end{aligned}\right\} \text{...(60).}$$

Since [eq. (50)] $e_0 = -e$, $e\,(b_0-b)$ may be regarded as expressed symmetrically in terms of ϖ and ϖ_0. It will be seen that x, b_1 and b_2 are also thus symmetrically expressed when $m/(s-b)$ is so expressed [see eq. (42) above]. By equations (40) and (41) we have

$$m/(s-b) = ss_0 - (s-b)\,(s_0-b_0) \text{..............(61).}$$

From these we have

$$\varpi_1 = x\,\{\varpi + \varpi_0 + \surd[2m/(s-b)]\}\,\psi^{-1} \text{...........(62),}$$

and ϖ_2 is obtained from ϖ_1 by altering the sign of the radical. Putting in this expression the value of x, substituting for ψ in terms of $\varpi - \varpi_0$, and utilising equations (57) and (61), we have

$$\varpi_1 = \frac{e}{m_\psi} \frac{\varpi + \varpi_0 + \sqrt{\{2ss_0 - 2(s-b)(s_0 - b_0)\}}}{\varpi - \varpi_0} \quad \text{......(63).}$$

This is a symmetrical expression in ϖ and ϖ_0 for $e = -e_0$ and $\psi = \psi_0$. Again ϖ_1 may be expressed explicitly in terms of ϖ, e, b and the coefficients of the ϖ cubic. Modifying in this way everything but the scalar m_ψ, we get by equations (51) and (52)

$$\varpi_1 = \frac{e}{m_\psi} \frac{\varpi(2s - b - \varpi) + \sqrt{\{2m(s-b)\}}}{\varpi(\varpi - b)} \quad \text{.........(64).}$$

Since $$\mathbf{S}\rho\varpi_1\rho = -1,\ \mathbf{S}\rho\varpi_1^2\rho = -b_1 \quad \text{..............(65),}$$

and similarly for ϖ_2, it follows that the normal cone curve is a polhode of the quadric $\mathbf{S}\rho\varpi_1\rho = -1$ and also of the quadric $\mathbf{S}\rho\varpi_2\rho = -1$.

When the normal cone curve is thus regarded as a polhode, its own normal cone curve is the original polhode magnified in a definite ratio, as we see from the above. It is well, notwithstanding, to show that this follows from the results just established.

From the equations

$$\mathbf{S}\rho\varpi_1\rho = -1,\ \mathbf{S}\rho\varpi_1^2\rho = -b_1,$$

the equation of the normal cone must be

$$\mathbf{S}\rho(b_1 - \varpi_1)\varpi_1\rho = 0;$$

and from the equations

$$\mathbf{S}\rho\varpi_1\rho = -1,\ \mathbf{S}\rho\varpi_2\rho = -1,$$

it must be $$\mathbf{S}\rho(\varpi_1 - \varpi_2)\rho = 0.$$

Hence the pencil function $\varpi_1 - \varpi_2$ must be a simple multiple of the function $(b_1 - \varpi_1)\varpi_1$. It follows that ϖ_2 is of the form $y\varpi_1 + z\varpi_1^2$. From this again it follows by the above that ϖ_2 can only have the unique value $\varpi_1(s_1 - \varpi_1)/(s_1 - b_1)$, where s_1 is the half-sum of the roots of the ϖ_1 cubic. If then we put

$$\psi_1 = e_1 m_1^{-1} \varpi_1 (b_1 - \varpi_1) \quad \text{....................(66),}$$

where m_1 is the product of the roots of the ϖ_1 cubic, the relations between ψ_1, ϖ_1, ϖ_2, e_1, m_1, b_1, s_1 must be exactly the same as the relations between ψ, ϖ, ϖ_0, e, m, b, s. Hence by eq. (52)

$$\psi_1 = -m_1^{-1} e_1 (s_1 - b_1)(\varpi_1 - \varpi_2),$$

or
$$\psi_1 = -2y\psi^{-1}m_1^{-1}e_1(s_1 - b_1).$$

Hence if we put
$$e_1 = -m_1/2y\,(s_1 - b_1) = -(2y)^{-1}\{s_1s_2 - (s_1 - b_1)(s_2 - b_2)\}\ldots(67),$$

[eq. (61)] we shall have
$$\psi_1 = \psi^{-1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(68),$$

and the normal cone curve of the original normal cone curve becomes identical with the original polhode.

Let us now examine the cases where the above results become unintelligible by reason of certain of the constants becoming infinite, zero or imaginary.

For this purpose it is convenient to write the values of x and y [equations (56) and (58)] in the form [eq. (57)]
$$x = m(s - b)/\{e(\alpha - b)(\beta - b)(\gamma - b)\}\ \ldots\ldots\ldots\ldots(69),$$
$$y = m\sqrt{\{2m(s - b)\}}/\{e(\alpha - b)(\beta - b)(\gamma - b)\}\ \ldots(70).$$

We shall assume that *the original polhode and quadric are real and that e is not zero, infinite or imaginary.* Thus the normal cone curve is always real. It is never infinite (if we exclude the possibility of the original quadric being a cylinder) but may be evanescent.

Of the roots α, β, γ of the ϖ cubic let
$$\alpha > \beta > \gamma.$$

We assume that none of these are zero. That the quadric $\mathbf{S}\omega\varpi\omega = -1$ may be real α must be positive. Also (by geometrical interpretation) that the original polhode may be real b must be positive; it must lie between α and γ when $\mathbf{S}\omega\varpi\omega = -1$ is an ellipsoid, be greater than β when the quadric is an hyperboloid of one sheet, and be greater than α when it is an hyperboloid of two sheets.

y, and therefore both the quadrics on which the normal cone curve is a polhode, are imaginary if $m(s-b)$ is negative; i.e. if $s - b$ is negative when $\mathbf{S}\omega\varpi\omega = -1$ is an ellipsoid or hyperboloid of two sheets, and if $s-b$ is positive when this equation represents an hyperboloid of one sheet. From this it is quite easy to see in any particular case between what limits b must lie in order that the two normal cone curve quadrics may be real. Thus if the

original quadric is an hyperboloid of two sheets $b > \alpha$ and therefore $s - b$ is negative. In this case, then, the two normal cone quadrics are always imaginary.

When $s = b$, x and y both vanish, but as we have already seen ϖ_0 becomes infinite. Thus equations (53) and (54) become unintelligible and this case requires separate consideration. We have for it [equations (44) and (45)]

$$\rho^2 = e^2 m^{-2} (s - \alpha)(s - \beta)(s - \gamma) = -p_1^{-1} \quad \text{........(71)},$$

say. Thus as already remarked the normal cone curve is a polhode on a sphere

$$\mathbf{S}\rho p_1 \rho = -1.$$

That ρ^2 is constant can also be seen from equation (10) since the coefficient of ω^2 in that equation vanishes.

The normal cone curve is in the present case generally not a polhode on any second quadric. For if it lie on the quadric

$$\mathbf{S}\rho\varpi_2\rho = -1$$

the normal cone must have for equation

$$\mathbf{S}\rho(\varpi_2 - p_1)\rho = 0,$$

so that this equation is the same as the equation $\mathbf{S}\rho\psi^{-1}\rho = 0$. Hence $\varpi_2 - p_1$ is a simple multiple of ψ^{-1}. We may therefore put

$$\varpi_2 = p_1 + z\psi^{-1}.$$

Hence $$\varpi_2\rho = (p_1\psi + z)\,\omega,$$

so that $$(\varpi_2\rho)^2 = p_1^2\rho^2 + z^2\omega^2.$$

Here ρ^2 is constant, but ω^2 is constant only in the extreme case when the original quadric is a surface of revolution. For if $\omega^2 = -p^{-1}$, a constant, the normal cone has for equation either

$$\mathbf{S}\omega(\varpi - p)\,\omega = 0 \text{ or } \mathbf{S}\omega(sp - \varpi^2)\,\omega = 0.$$

Hence $\varpi - p$ is a simple multiple of $sp - \varpi^2$. Hence a relation of the form

$$x\varpi^2 + y\varpi + z = 0,$$

where x, y, z are constant scalars, holds good. Now ϖ satisfies such a quadratic equation if and only if two of the roots of its cubic are equal, i.e. if and only if $\mathbf{S}\omega\varpi\omega = -1$ is a surface of revolution. When it is a surface of revolution $(\varpi_2\rho)^2$ above is constant for any constant value of z. Hence in this case there is

a whole family of quadrics on each of which the normal cone curve is a polhode.

It is not hard to see by similar reasoning that the last case is a particular one of a more general case. If $S\omega\varpi\omega = -1$ is a surface of revolution ω^2 is constant whatever constant value b have and therefore by equation (10) ρ^2 is also constant. Both the polhode cone and the normal cone are coaxial right circular cones; and the polhode and normal cone curve are generating circles of these cones. Hence each of them is a polhode on each of a whole family of quadrics of revolution having the given centre for common centre.

Returning to the singular case when $b = s$ and $S\omega\varpi\omega = -1$ is not a surface of revolution we may notice that s cannot be equal to α or γ but it may be equal to β. We then have

$$\gamma = \beta - \alpha,$$

so that γ must be negative and α positive. Moreover since b is positive and $= \beta$, β is positive. The surface is therefore an hyperboloid of one sheet and ω is along the greatest axis. In this case we see by geometrical interpretation that the polhode reduces to a point. We also have $\rho = \psi\omega = 0$, so that the normal cone curve reduces to a point at the centre. This again is a particular case of a more general one now to be considered.

The only other singular case when the above general solution breaks down is when x and y both become infinite by reason of b being equal to one of the roots of the ϖ cubic.

When the given quadric $S\omega\varpi\omega = -1$ is an ellipsoid it is easy to see geometrically that when $b = \alpha$ or γ the polhode reduces to a point. Similarly when the quadric is an hyperboloid of one sheet and $b = \beta$, or when it is an hyperboloid of two sheets and $b = \alpha$, the polhode reduces to a point. In each of these cases, $(b - \varpi)\,\omega = 0$ and therefore the normal cone curve vanishes at the centre of the quadric. The only other possible cases of b being equal to a root of the ϖ cubic are when $b = \beta$ for the ellipsoid and when $b = \alpha$ for the hyperboloid of one sheet.

In both these cases one root of the ψ cubic is zero and the other two are of opposite signs so that the polhode cone $S\omega\psi\omega = 0$ becomes a pair of planes, i.e. the polhode consists of two plane sections of the given quadric whose intersection is the axis of the

quadric corresponding to the root of the ϖ cubic which is equal to b. These planes are equally inclined to a second axis of the quadric. The normal cone curve therefore reduces to the two straight lines through the centre which are normal to these two planes. It is quite easy to prove that the distance of Q a point on the normal cone curve from the first of these axes varies as the distance of P the corresponding point on the polhode from the same axis.

We now return to the cases when the general solution is applicable.

There is an important reciprocal relation between the pair of quadrics ϖ_1, ϖ_2 and the pair ϖ, ϖ_0 which shows (as we might otherwise anticipate) that the original polhode and its two quadrics are related to the normal cone curve and its two quadrics in exactly the same way as the latter are to the former. If we substitute from equations (56) and (58) in eq. (67) for y we get

$$m_1^{-1} e_1 (s_1 - b_1) = - e (b_0 - b) \{(s - b)/2m\}^{\frac{3}{2}}.$$

But by eq. (60)

$$b_1 - b_2 = 2e^{-2} (b_0 - b)^{-2} \{2m/(s - b)\}^{\frac{5}{2}}.$$

Hence
$$\frac{e_1 (s_1 - b_1)(b_2 - b_1)}{2m_1} \cdot \frac{e (s - b)(b_0 - b)}{2m} = 1 \quad \text{......(72)}.$$

If then we put

$$\frac{e (s - b)(b_0 - b)}{2m} = g \quad \text{.....................(73)},$$

and similarly for g_0, g_1, g_2, we shall have

$$g = g_0, \ g_1 = g_2, \ g g_1 = 1 \quad \text{..................(74)},$$

$$x = \tfrac{1}{2} g^{-1} \quad \text{..........................(75)},$$

$$\left.\begin{aligned} \psi = \psi_0 &= 2g \frac{\varpi (b - \varpi)}{(s - b)(b_0 - b)}, \\ \psi_1 = \psi_2 = \psi^{-1} &= 2g_1 \frac{\varpi_1 (b_1 - \varpi_1)}{(s_1 - b_1)(b_1 - b_2)} \end{aligned}\right\} \text{.........(76)},$$

$$\psi = 2g \frac{\varpi - \varpi_0}{b - b_0} \quad \text{.....................(77)}.$$

If we put $g = 1$ and therefore $g_1 = 1$ we may call the resulting normal cone curve the principal normal cone curve. Thus the given polhode is the principal normal cone curve of its own principal normal cone curve when the last curve is regarded as a polhode. Or we may call in this case the given polhode and its normal cone curve reciprocal or conjugate polhodes.

To give a notion of the possible magnitudes of the quantities appearing above, consider the four quadrics corresponding to each of two polhodes on one ellipsoid for which

$$\alpha = 8,\ \beta = 4,\ \gamma = 2,\ b = 5 \text{ and } \tfrac{5}{2}.$$

On this ellipsoid we have real polhodes for all values of b from 8 to 2, but since $s = 7$ their normal cone curves and in particular their conjugate polhodes are not polhodes on real quadrics for values of b between 8 and 7. Some of the eight quadrics in the above two cases are ellipsoids and some are hyperboloids of one sheet. They cannot possibly be hyperboloids of two sheets, as we have already seen. The following tabulated results are easily calculated from the equations above.

SPECIFICATION IN TWO CASES OF THE TWO POLHODE QUADRICS AND THE TWO CONJUGATE POLHODE QUADRICS.

Quadric	(Semi-axes)$^{-2}$			(Perp.)$^{-2}$	Semi-axes			Perp.
Ellip. ϖ	8	4	2	5	·35355	·50000	·70711	·44721
Hyp. ϖ_0	-4	6	5	$\frac{29}{4}$	[·50000]	·40825	·44721	·37139
Hyp. ϖ_1	$-\frac{9}{16}$	$\frac{81}{16}$	$\frac{45}{16}$	$\frac{261}{16}$	[1·33333]	·44444	·59629	·24759
Hyp. ϖ_2	$\frac{3}{16}$	$\frac{9}{16}$	$-\frac{3}{16}$	$\frac{5}{16}$	2·30940	1·33333	[2·30940]	1·78886

2ND CASE.

Quadric	(Semi-axes)$^{-2}$			(Perp.)$^{-2}$	Semi-axes			Perp.
Ellip. ϖ	8	4	2	$\frac{5}{2}$	·35355	·50000	·70711	·63246
Hyp. ϖ_0	$-\frac{16}{9}$	$\frac{8}{3}$	$\frac{20}{9}$	$\frac{62}{27}$	[·75000]	·61237	·67082	·65991
Hyp. ϖ_1	$\frac{13}{216}$	$\frac{11}{24}$	$-\frac{473}{216}$	$\frac{2335}{216}$	4·07620	1·47710	[·67577]	·30415
Ellip. ϖ_2	$\frac{1}{216}$	$\frac{11}{216}$	$\frac{55}{216}$	$\frac{31}{216}$	14·69694	4·43130	1·98173	2·63965

Under the heading "semi-axes" the square brackets indicate the imaginary axes of the hyperboloids.

In figure 9 is given an indication of the relative positions of the four quadrics of the second case. It represents, roughly to scale, three of the semi-axes of each quadric, and the traces of each

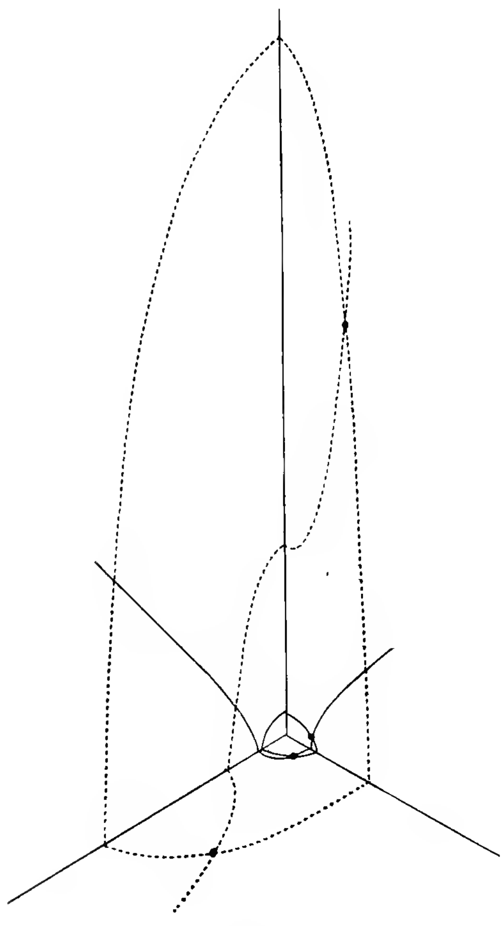


FIG. 9.

quadric in one of the corresponding "octants." To distinguish between the polhode quadrics on the one hand and the conjugate polhode quadrics on the other, the traces of the latter are put in dotted lines. Since in the present case, of each of these two pairs

of quadrics one quadric is an ellipsoid and one an hyperboloid of one sheet, the four quadrics are easily distinguishable. The four points where the two polhodes intersect the parts of the planes included in the octant are indicated by thick marks.

The conjugate polhode it will be remembered is obtained by taking $g=1$. The curve first denoted above, "the normal cone curve," was the one for which $e=2$ and therefore

$$g=m^{-1}(s-b)(b_0-b).$$

It will be found that to obtain this curve and its corresponding quadrics in the second case just considered we must diminish each axis of the two quadrics ϖ_1 and ϖ_2 in the ratio of $69{\cdot}\dot{8}\dot{1}$ to unity. The figure that we thus get shows about the same disproportion between the dimensions of the two pairs of quadrics as does Fig. 9, but the disproportion is in the opposite direction, the axes of ϖ_1 and ϖ_2 being all smaller than the smallest of the axes of ϖ and ϖ_0.

We conclude this section by establishing certain results which are symmetrical with regard to the four quadrics.

Remembering [eq. (75)] that $x=\frac{1}{2}g^{-1}$ we get from eq. (60)

$$2g^2(b_1+b_2)=b+b_0+2(s+s_0).$$

Similarly since $gg_1=1$

$$2(b+b_0)=g^2\{b_1+b_2+2(s_1+s_2)\}.$$

By cross addition and subtraction we obtain from these

$$g^2\{3(b_1+b_2)+2(s_1+s_2)\}=3(b+b_0)+2(s+s_0)\;\ldots(78),$$

$$g^2\{2(s_1+s_2)-(b_1+b_2)\}+\{2(s+s_0)-(b+b_0)\}=0\;\ldots(79).$$

Again from equation (60)

$$b_1-b_2=\frac{2}{g^2}\sqrt{\frac{2m}{s-b}}.$$

Similarly
$$b-b_0=2g^2\sqrt{\frac{2m_1}{s_1-b_1}}\;\ldots\ldots\ldots\ldots\ldots\ldots(80).$$

Multiplying and dividing

$$(b_1-b_2)(b-b_0)=8\sqrt{\left\{\frac{m}{s-b}\,\frac{m_1}{s_1-b_1}\right\}}\;\ldots\ldots\ldots\ldots(81),$$

$$g^4(b_1-b_2)\sqrt{\frac{m_1}{s_1-b_1}}=(b-b_0)\sqrt{\frac{m}{s-b}}\;\ldots\ldots\ldots(82).$$

The symmetry of the last two results is rendered more evident by substituting from equation (61)

$$m_1/(s_1 - b_1) = s_1 s_2 - (s_1 - b_1)(s_2 - b_2), \quad m/(s-b) = ss_0 - (s-b)(s_0 - b_0).$$

Again by eq. (77)

$$\rho = 2g\,(b - b_0)^{-1}(\varpi - \varpi_0)\,\omega.$$

Hence

$$\begin{aligned}\rho^2 (b - b_0)^2/4g^2 &= -b - b_0 - 2\mathbf{S}\omega\varpi\varpi_0\omega \\ &= -(b + b_0) + 2\{m\omega^2 + (m' - s^2) + s(s-b)\}/(s-b),\end{aligned}$$

by the value found above for $\mathbf{S}\omega\varpi\varpi_0\omega$. Thus by equations (40) and (80)

$$g^2\rho^2 \frac{2m_1}{s_1 - b_1} = \omega^2 \frac{2m}{s-b} + 2(s + s_0) - (b + b_0) \ldots\ldots\ldots(83),$$

which by eq. (79) may be written

$$\omega^2 \frac{2m}{s-b} = g^2 \left\{\rho^2 \frac{2m_1}{s_1 - b_1} + 2(s_1 + s_2) - (b_1 + b_2)\right\} \ldots\ldots(84).$$

[Both equations (79) and (80) may be deduced from eq. (83) and the one it is deduced from by reason of the symmetry of the relations between the two pairs of quadrics.] A symmetrical form of these equations is

$$\begin{aligned}g^2 &\left\{\rho^2 \frac{4m_1}{s_1 - b_1} + 2(s_1 + s_2) - (b_1 + b_2)\right\} \\ &= \left\{\omega^2 \frac{4m}{s-b} + 2(s + s_0) - (b + b_0)\right\} \ldots(85).\end{aligned}$$

The last three equations are of course only eq. (43) § 46 in a simplified form. They may in fact be deduced from that equation.

It is not hard to prove from equations (40) and (57) that

$$\begin{aligned}&2(s + s_0) - (b + b_0) \\ &\quad = (\beta + \gamma - b)(\gamma + \alpha - b)(\alpha + \beta - b)/(s-b)^2 \ldots(86).\end{aligned}$$

Since the left is symmetrical in ϖ and ϖ_0, and by eq. (79), we have

$$\left.\begin{aligned}\frac{(\beta+\gamma-b)(\gamma+\alpha-b)(\alpha+\beta-b)}{(s-b)^2}&\\ =\frac{(\beta_0+\gamma_0-b_0)(\gamma_0+\alpha_0-b_0)(\alpha_0+\beta_0-b_0)}{(s_0-b_0)^2}&\\ =-g^2\frac{(\beta_1+\gamma_1-b_1)(\gamma_1+\alpha_1-b_1)(\alpha_1+\beta_1-b_1)}{(s_1-b_1)^2}&\\ =-g^2\frac{(\beta_2+\gamma_2-b_2)(\gamma_2+\alpha_2-b_2)(\alpha_2+\beta_2-b_2)}{(s_2-b_2)^2}&\end{aligned}\right\}\ldots\ldots(87).$$

Thus if any one of these vanishes they all vanish; i.e. if b is equal to the sum of two roots of the ϖ cubic, b_0 is equal to the sum of two roots of the ϖ_0 cubic &c. In this rather curious case we see by eq. (83) that $\mathbf{T}\rho/\mathbf{T}\omega$ is constant.

From eq. (79) it follows that at least one of the four quadrics ϖ, ϖ_0, ϖ_1, ϖ_2 when all are real is an hyperboloid, i.e. as we have already seen, an hyperboloid of one sheet. For if ϖ is an ellipsoid, s is positive, and if ϖ_1 and ϖ_2 are real, $s-b$ is also positive. It follows that if all four were ellipsoids $2(s+s_0)-(b+b_0)$ and $2(s_1+s_2)-(b_1+b_2)$ would both be positive and not zero, and this is impossible by eq. (79).

They may however all be hyperboloids though the case is sufficiently rare to require careful artificial construction. The following properties of this case are stated for brevity without proof. As usual above, $\alpha>\beta>\gamma$ so that α and β are positive and γ negative. Assuming this only we have for the present case

α_0 and γ_0 positive, β_0 negative, $\alpha_0>\gamma_0>\beta_0$,

$b-s$, $\alpha-s$, $s-\beta$, and $s-\gamma$ all positive,

b_0-s_0, α_0-s_0, $s_0-\beta_0$, and $s_0-\gamma_0$ all positive.

Of the four quadrics ϖ, ϖ_0, ϖ_1, ϖ_2 we may take the first pair such that $2(s+s_0)-(b+b_0)$ is positive. Assuming this

b is between $\alpha+\gamma$ and $\alpha+\beta$.

[If we assumed that $2(s+s_0)-(b+b_0)$ was negative we should have b between s and $\alpha+\gamma$ or between $\alpha+\beta$ and $+\infty$.] b is not equal to b_0, for if it were we should have the singular case where b is equal to a root of the ϖ cubic. We may then

choose the quadric ϖ such that $b > b_0$. Assuming this we shall have

$$2\alpha + \gamma > \alpha + \beta > b > \alpha > \alpha_0 > b_0 > \alpha_0 + \beta_0 > \gamma_0,$$

$$\alpha_1 \text{ and } \beta_1 \text{ positive, } \gamma_1 \text{ negative, } \alpha_1 > \beta_1 > \gamma_1,$$

$$\alpha_2 \text{ and } \gamma_2 \text{ positive, } \beta_2 \text{ negative, } \alpha_2 > \gamma_2 > \beta_2.$$

Thus the real smaller semi-axes of all four hyperboloids are coincident (α, α_0, α_1, α_2). The larger real semi-axis of ϖ is coincident with the imaginary semi-axis of ϖ_0 and conversely. A similar statement also holds with regard to the semi-axes of ϖ_1 and ϖ_2.

These statements may be illustrated by taking

$$\alpha = 50, \ \beta = 6, \ \gamma = -16, \ b = \frac{158}{3},$$

when it will be found that

$$\alpha_0 = \frac{18}{49} . 125, \quad \beta_0 = -\frac{18}{49} . 7, \quad \gamma_0 = \frac{18}{49} . 48,$$

$$b_0 = \frac{18}{49} . \frac{23 . 37}{7} = \frac{18}{49} . \frac{851}{7},$$

and also

$$g^2\alpha_1 = \frac{206 . 277}{21 . 49}, \quad g^2\beta_1 = \frac{206 . 24}{21 . 49}, \quad g^2\gamma_1 = -\frac{50 . 23}{21 . 49},$$

$$g^2\alpha_2 = \frac{206 . 193}{21 . 49}, \quad g^2\beta_2 = -\frac{206 . 16}{21 . 49}, \quad g^2\gamma_2 = \frac{50 . 19}{21 . 49};$$

b_1 and b_2 can also be found from eq. (60). They are greater than α_1 and α_2 respectively.

From equation (78) we can get another symmetrical result similar to eq. (87). It is

$$\left.\begin{aligned} &(s-b)^{-2}\{(\beta+\gamma-b)(\gamma+\alpha-b)(\alpha+\beta-b)+4(\alpha-b)(\beta-b)(\gamma-b)\}+8b \\ &= \ (s_0-b_0)^{-2}\{(\beta_0+\gamma_0-b_0)(\ )(\ )+4(\alpha_0-b_0)(\ )(\ )\}+8b_0 \\ &= g^2(s_1-b_1)^{-2}\{(\beta_1+\gamma_1-b_1)(\ )(\ )+4(\alpha_1-b_1)(\ )(\ )\}+g^2 8b_1 \\ &= g^2(s_2-b_2)^{-2}\{(\beta_2+\gamma_2-b_2)(\ )(\ )+4(\alpha_2-b_2)(\ )(\ )\}+g^2 8b_2 \end{aligned}\right\} \quad \text{..............(88).}$$

48. Motion of rigid body resumed. Potential. Freedom and constraint. Reactions. Impulses. Return now

to the discussion of the mechanics of a rigid body as considered in equations (1) to (28) § 46.

F_0, G_0, F, G and their rates of variation may be expressed in terms of E the displacement motor and its time derivatives. The formulae are too complicated to be of much use in a general discussion, so I content myself with writing them down without proof. [See eq. (30) § 20.]

$$F = E\mathbf{S}E^{-1}\dot{E} + 2\mathbf{M}e^{\frac{1}{2}E}.\mathbf{M}E^{-1}\dot{E}.e^{-\frac{1}{2}E} \quad \text{......... (1)},$$

$$F_0 = E\mathbf{S}E^{-1}\dot{E} + 2e^{-\frac{1}{2}E}\mathbf{M}e^{\frac{1}{2}E}.\mathbf{M}E^{-1}\dot{E} \quad \text{............(2)}.$$

This last may also be written

$$F_0 = \dot{E} + (1 - E - e^{-E})\mathbf{M}E^{-1}\dot{E} = \dot{E} + \left(-\frac{1}{2!} + \frac{E}{3!} - \frac{E^2}{4!} + \dots\right)\mathbf{M}E\dot{E} \quad \text{............(3)},$$

from which

$$\dot{F}_0 = \ddot{E} + (1 - E - e^{-E})\mathbf{M}E^{-1}\ddot{E} - \{\dot{E} + (2 - 2E - 2e^{-E} - Ee^{-E}) \times \mathbf{S}E^{-1}\dot{E} + \mathbf{M}e^{-E}\mathbf{M}E^{-1}\dot{E}\}\mathbf{M}E^{-1}\dot{E} \quad \text{...... (4)}.$$

This might perhaps prove useful in writing down to any required degree of approximation equations of motion when E is small.

Suppose that the external forces have a potential v. v may be regarded as a function of E but it is simpler to regard it as a function of $2\mathbf{M}Q$ or E' [eq. (2) § 46].

If Q be varied by the infinitesimal δQ the small displacement that the rigid body receives is $2\delta QQ^{-1}$ [eq. (8) § 11]. The work done in this displacement is $-2\mathbf{s}H\delta QQ^{-1}$. Hence by eq. (8) § 14

$$2\mathbf{s}\delta QQ^{-1}H = \delta v = -\mathbf{s}\delta E'\vartheta v = -2\mathbf{s}\delta Q\vartheta v,$$

where now the independent variable motor implied by ϑ is E'. Since δQQ^{-1} is an arbitrary infinitesimal motor we have by § 14

$$H = -\mathbf{M}Q\vartheta v \quad \text{......................... (5)}.$$

Hence by eq. (8) § 46

$$H_0 = -\mathbf{M}\vartheta vQ \quad \text{........................(6)}.$$

These may be substituted for the force motor in the various equations of § 46.

It may be noticed that

$$\dot{v} = -2\mathbf{s}\dot{Q}\vartheta v = -\mathbf{s}FQ\vartheta v = \mathbf{s}FH.$$

Also by eq. (11), § 46

$$\dot{T} = -\mathbf{s}F_0\psi_0\dot{F}_0 = -\mathbf{s}F_0H_0 \text{ [eq. (25), § 46]}$$
$$= -\mathbf{s}FH \text{ [eq. (8), § 46]}.$$

Thus $$\dot{v} + \dot{T} = 0,$$

i.e. the sum of the potential and kinetic energies remains constant.

When the displacement is small we may in all these equations put $Q = 1$. Thus

$$H = H_0 = -\vartheta v \quad \text{.......................} \quad (7).$$

We have seen (§ 46) that in this case $E' = E$, so that we may suppose the independent variable motor implied by ϑ to be the displacement motor.

H and H_0 were originally defined as the external force motor. If the body is constrained it is necessary to state whether or not the reaction of the constraints is to be included under the term "external." Let us suppose that it is so included. Then equations (21) and (25), § 46, remain true for a constrained body but equations (5), (6) and (7) of the present section do not. Let P be the external force motor exclusive of the reactions and let

$$P = QP_0Q^{-1} \quad \text{.........................} \quad (8).$$

In this case the system of forces is due to two causes, the "field," as it may be called, with potential v, and the reactions. Of these P is due to the field. Hence in place of equations (5) and (6) we have

$$P = -\mathbf{M}Q\vartheta v, \quad P_0 = -\mathbf{M}\vartheta vQ \quad \text{..............} \quad (9).$$

Also if $R = QR_0Q^{-1}$ is the force motor due to the reaction of the constraints,

$$H = P + R, \quad H_0 = P_0 + R_0 \quad \text{..............} \quad (10).$$

On account of the constraint F (and therefore also F_0, G and G_0) is for a given value of Q confined to a definite complex whose order is the number of degrees of freedom. Let us call the complex to which F is confined the velocity complex, and that to which G is confined the momentum complex. The constraint is assumed to be smooth. Hence the reaction is such that it would do no work on the body whatever were its instantaneous motion consistent with the given constraints. Hence R is confined to the complex reciprocal to the velocity complex. Call this reciprocal complex the reaction complex.

The complexes to which F_0, G_0 and R_0 are confined may similarly be called the velocity, momentum and reaction complexes referred to the standard position respectively. More shortly we may call the complexes, the F complex, the R_0 complex, &c.

Eq. (25) § 46 and the last equation give

$$P_0 + R_0 = \psi_0 \dot{F}_0 + \mathsf{M} F_0 \psi_0 F_0 \ldots\ldots\ldots\ldots\ldots (11).$$

Here P_0 and ψ_0 are given, F_0 is confined to a definite complex and R_0 to the reciprocal complex. Thus F_0 and R_0 together involve six unknown scalars; and equation (11) is equivalent to six scalar equations; it is therefore sufficient to determine R_0 and F_0 and therefore the motion.

If $P' = QP_0'Q^{-1}$ is a given external *impulse* motor and $R' = QR_0'Q^{-1}$ the impulsive reaction motor, R' is still confined to the reaction complex. If F_0 and $F_0 + \Delta F_0 = Q^{-1}(F + \Delta F)Q$, G_0 and $G_0 + \Delta G_0 = Q^{-1}(G + \Delta G)Q$ are the values of F_0, G_0, just before and just after the impulse, ΔF and ΔG are confined to the velocity and momentum complexes respectively. Eq. (11) gives in this case

$$P_0' + R_0' = \psi_0 \Delta F_0 = \Delta G_0 \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

This admits of the following simple interpretation:—ΔG *the increment in the momentum motor = the component of P' the given impulse in the momentum complex when P' is expressed as such a component + a component in the reaction complex.* In other words $\Delta G - P'$ *is reciprocal to the velocity complex.* This last statement is sufficient to determine ΔG since it involves the same number of scalar equations as there are disposable scalars in ΔG (the number of degrees of freedom).

By the definition of the momentum complex every motor of that complex can be expressed as ψF where F is a motor of the velocity complex. By the definition of the reaction complex $\mathsf{S}RF = 0$ where R is any motor of that complex. Now $-\frac{1}{2}\mathsf{S}F\psi F$ being the kinetic energy for a possible motion is not zero. Hence no values of F and R can be found for which $R = \psi F$; i.e. there is no motor common to the momentum and reaction complexes, or:—

The momentum and reaction complexes are independent. This is otherwise evident since P' above is any motor and is resolved into the two finite components ΔG and $-R$ in the momentum

and reaction complexes; and by their definitions the sum of the orders of these complexes is six.

In order to apply the general methods of dynamics to the motion of a constrained rigid body it is convenient to re-define ϑ. Let the number of degrees of freedom be n and let us denote the velocity complex when $Q=1$ by (n). Let $A_1 \dots A_n$ be n independent motors of (n), and $B_1 \dots B_{6-n}$ $6-n$ independent motors of any given independent complex $(6-n)$ of order $6-n$. Let the bar introduced in § 28 be used with reference to these six motors. Let C, any independent variable motor of (n), be given by the equation

$$C = x_1 A_1 + \dots + x_n A_n = \Sigma x A \quad\dots\dots\dots\dots\dots (13).$$

Then define ϑ by the equation

$$\vartheta = \Sigma \bar{A} \partial/\partial x \quad\dots\dots\dots\dots\dots\dots\dots (14).$$

By equations (23), (24) § 28 we have

$$\mathrm{s} dC\vartheta = -d \quad\dots\dots\dots\dots\dots\dots\dots (15),$$

and therefore by eq. (8) and statement (2) of § 14, when $n=6$ the present and former meanings of ϑ are the same. It is important to notice that except when $n=6$ this meaning of ϑ does not depend only on the complex (n) but also upon the independent complex $(6-n)$. (n) is fixed by the constraints. $(6-n)$ may be any complex independent of (n) which is convenient for the matter in hand. $(\overline{6-n})$ will denote the complex reciprocal to (n), i.e. the reaction complex which is perfectly definite. $(\bar{n})$ will denote the complex reciprocal to $(6-n)$. Since $\bar{A}_1 \dots \bar{A}_n$, $\bar{B}_1 \dots \bar{B}_{6-n}$ are (§ 28) always independent we may either take $(\bar{n})$ as any complex convenient for our purpose which is independent of $(\overline{6-n})$ or as above we may take $(6-n)$ as any complex independent of (n). Thus, as the momentum complex is independent of $(\overline{6-n})$, we may take $(\bar{n})$ as the momentum complex when $Q=1$. This we shall frequently do below. If y is any scalar function of C, ϑy is confined to $(\bar{n})$.

When (n) contains no motors reciprocal to the whole of (n) it is often convenient to take $(6-n)$ as the reciprocal of (n) since this reciprocal is then independent of (n). $(\bar{n})$ and (n) are then the same complex, as also are $(\overline{6-n})$ and $(6-n)$. Even when this is not the case it is often convenient to take $(6-n)$ as a

complex "semi-reciprocal" (§ 31) to (n). We may remind the reader that (n) and $(6-n)$ are semi-reciprocal when (n) is the complex of the motors $A_1A_2 \ldots D_1D_2 \ldots$ and $(6-n)$ that of $B_1B_2 \ldots D_1'D_2' \ldots$; where all pairs of these motors are reciprocal except the pairs (D_1D_1') $(D_2D_2')\ldots$ which are such that

$$\mathbf{s}D_1D_1' = \mathbf{s}D_2D_2' = \ldots = -1\,;$$

and where D_1, D_1', $D_2 \ldots$ are all self-reciprocal. The complex $D_1D_2 \ldots$ is a definite complex; and when $(6-n)$ is given so also is the complex $D_1'D_2' \ldots$. With this notation and that of § 31 for (AD), &c., we here have:—the complex (n) is (AD), $(6-n)$ is (BD'), $(\bar{n})$ is (AD'), and $(\overline{6-n})$ is (BD). Also by eq. (26) § 28 and eq. (14) § 32, if

$$C = \Sigma xA + \Sigma yD,$$

$$\vartheta = -\Sigma\,(A\mathbf{s}^{-1}A^2 \,.\, \partial/\partial x) + \Sigma\,(D'\partial/\partial y) \ldots\ldots\ldots\ldots (16).$$

In § 32 $\mathbf{T}A$ is chosen so that $\mathbf{s}A^2 = -1$.

Q is a function of n independent variables. It is not in general such that E or E' is confined to the definite complex (n). But Q may be supposed a single-valued function of C and the functional form may be so chosen that

$$\text{when } Q-1 \text{ is small, } E = E' = C \quad \ldots\ldots\ldots\ldots (17).$$

Thus when, as in next section, the displacement is small throughout all time, the new independent variable C is not required. E the displacement motor and $\dot{E}$ the velocity motor in this case both belong to the given complex (n).

Let L be the Lagrangian function. L is thus a single-valued function of C and $\dot{C}$ only, and so far as it depends on $\dot{C}$ it only contains a positive quadratic term T. This term may by § 34 (p. 152) be put in the form

$$T = -\tfrac{1}{2}\mathbf{s}\dot{C}\psi_1\dot{C} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18),$$

where ψ_1 is a partial energy function such that (1) $\psi_1 C' = 0$, where C' is any motor belonging to $(6-n)$, (2) $\mathbf{s}C\psi_1 C$ is negative and not zero for any motor C belonging to (n), and (3) $\psi_1 C''$ belongs to $(\bar{n})$ for all values of C''.

The connection between ψ_1 and ψ_0 of § 46 may be given. Since [eq. (15)]

$$\dot{Q} = -\,\mathbf{s}\dot{C}\vartheta \,.\, Q,$$

we have by equations (11) and (24) § 46

$$T = -2\mathsf{s}\,(Q^{-1}\mathsf{s}\dot{C}\vartheta\,.\,Q)\,\psi_0\,(Q^{-1}\mathsf{s}\dot{C}\vartheta\,.\,Q),$$

so that $$\psi_1 = 4\vartheta_1\mathsf{s}\,(\;)\,\vartheta_2\mathsf{s}Q^{-1}Q_1\psi_0\,(Q^{-1}Q_2) \quad\text{...........(19)},$$

for ψ_1 as defined by this equation is (1) self-conjugate, (2) such that $T = -\frac{1}{2}\mathsf{s}\dot{C}\psi_1\dot{C}$, (3) such that $\psi_1 C''$ is confined to $(\bar{n})$ [eq. (14)].

Using now the general dynamical principle

$$\delta\int L dt = 0,$$

where the initial and final positions are not varied, we get

$$0 = \int \mathsf{s}\,(\delta\dot{C}\psi_1\dot{C} + \delta C\vartheta L)\,dt$$

$$= \int \mathsf{s}\delta C\,(\vartheta L - d\psi_1\dot{C}/dt)\,dt.$$

Here $\vartheta L - d\psi_1\dot{C}/dt$ is a motor belonging to $(\bar{n})$ and δC is an arbitrary motor belonging to (n). Hence by § 32 (p. 140)

$$\frac{d\psi_1\dot{C}}{dt} = \vartheta L \quad\text{.........................(20)}.$$

This equation is equivalent to n scalar equations and is therefore sufficient to determine C and therefore the motion as a function of the time.

$\dot{C}$ may appropriately be called the generalised velocity motor and $\psi_1\dot{C}$ the generalised momentum motor.

49. Small motions. Let now the rigid body never depart more than a small way from an absolutely stable position.

The potential energy v is then a positive quadratic function of the displacement. We may therefore put

$$v = -\tfrac{1}{2}\mathsf{s}E_0\varpi_0E_0 \quad\text{.......................(1)},$$

where E_0 is the displacement motor and ϖ_0 is a given complete energy function. The force motor due to this displacement is by eq. (7), § 48, $-\varpi_0E_0$.

It is convenient here to recapitulate and in some slight degree alter our notation. [The *alterations* refer only to E_0 and ψ. ψ as defined directly is for the case of small motions, our previous ψ_1 and not what has hitherto been denoted by ψ. For small

motions our previous ψ_0 and ψ have the same meaning and we continue to use ψ_0 with that meaning.]

When the body is *free* the displacement motor, the velocity motor, the momentum motor and the force motor are denoted by E_0, $\dot{E}_0$, $\psi_0\dot{E}_0$, $-\varpi_0 E_0$ respectively, where ψ_0 and ϖ_0 are complete energy functions; the kinetic and potential energies are respectively $-\frac{1}{2}\mathrm{S}\dot{E}_0\psi_0\dot{E}_0$ and $-\frac{1}{2}\mathrm{S}E_0\varpi_0 E_0$.

When the body is *constrained* the displacement motor (and therefore also the velocity motor) is confined to the given complex (n) called either the displacement complex or the velocity complex. $(6-n)$ is any complex independent of (n); $(\bar{n})$ and $(\overline{6-n})$ are the reciprocals of $(6-n)$ and (n) respectively and are independent of one another. E_n and E_{6-n} will be used to denote *arbitrary* motors of (n) and $(6-n)$, respectively. The complex to which $\psi_0 E_n$ is confined is called the momentum complex, that to which $\varpi_0 E_n$ is confined is called the force complex, that to which the reaction motor due to the constraints is confined is called the reaction complex. The reaction complex is $(\overline{6-n})$. $(\bar{n})$ *may* but will not always be taken as the momentum complex. The displacement motor, the velocity motor, the momentum motor and the force motor are E, $\dot{E}$, $\psi_0\dot{E}$ and $-\varpi_0\dot{E}$. The reaction motor is R.

The partial energy functions ψ and ϖ are uniquely (§ 34) defined by the equations

$$\mathrm{S}E_n\psi E_n = \mathrm{S}E_n\psi_0 E_n,\ \ \psi E_{6-n} = 0 \ldots\ldots\ldots\ldots\ldots (2),$$

$$\mathrm{S}E_n\varpi E_n = \mathrm{S}E_n\varpi_0 E_n,\ \ \varpi E_{6-n} = 0 \ldots\ldots\ldots\ldots\ldots (3).$$

By § 34 it follows that ψF and ϖF are both confined to $(\bar{n})$ whatever motor value F have. $\psi\dot{E}$ and $-\varpi E$ are called the generalised momentum motor and the generalised force motor respectively. They are for small motions the $\psi_1\dot{C}$ and ϑL of eq. (20) § 48. For constrained motion the kinetic energy $= -\frac{1}{2}\mathrm{S}\dot{E}\psi_0\dot{E} = -\frac{1}{2}\mathrm{S}\dot{E}\psi\dot{E}$, and the potential energy $= -\frac{1}{2}\mathrm{S}E\varpi_0 E = -\frac{1}{2}\mathrm{S}E\varpi E$.

From these definitions we see that ϖ_0 and ψ_0 may be regarded as particular forms of ϖ and ψ, viz. when $n = 6$.

The equation of motion is by eq. (20) § 48

$$\psi\ddot{E} = -\varpi E \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

or

$$\ddot{E} = -\psi_{-1}\varpi E = -\phi E \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

See equations (6) to (9) § 33. The general real solution of this equation is

$$E = \cos(t\sqrt{\phi}) \,.\, E_1 + \sin(t\sqrt{\phi})/\sqrt{\phi} \,.\, E_2 \;\;\ldots\ldots\ldots(6),$$

where E_1 and E_2 are the initial values of E and $\dot{E}$ respectively.

A physical definition of the generalised momentum and force motors may be given. The present n, ψ_0 or ϖ_0, ψ or ϖ, E_n and E_{6-n} may be identified with the m, ϖ, ϖ', E_1 and E_2 of the last proposition but one of § 34. It follows from the forms there given for ϖ and ϖ' that, since E and $\dot{E}$ are confined to (n), $\psi\dot{E}$ and ϖE are the components of $\psi_0\dot{E}$ and $\varpi_0 E$ respectively in $(\bar{n})$ when the last motors are each expressed as an $(\bar{n})$ component + a $(\overline{6-n})$ component. Thus the generalised momentum motor is the $(\bar{n})$ component of the actual momentum motor, and similarly for the force motor. The important result follows that when $(\bar{n})$ is taken as the momentum complex the generalised momentum motor is the actual momentum motor. The force complex is not necessarily independent of the reaction complex $(\overline{6-n})$, so that $(\bar{n})$ cannot always be identified with the force complex. If it can however, by so identifying it, the generalised force motor becomes the actual force motor. It will be observed that only in very special circumstances, viz. when the momentum complex and force complex are identical, can the generalised momentum motor and the generalised force motor be simultaneously regarded as the actual momentum motor and the actual force motor respectively.

By § 33 there are always n real coreciprocal motors of (n) forming a conjugate set with regard to ψ. By § 34 they also form a conjugate set with regard to ψ_0. They are what Sir Robert Ball calls the n principal motors of inertia. Similarly the n real coreciprocal motors of (n) which form a conjugate set with regard to ϖ (and therefore also with regard to ϖ_0) are what he calls the n principal motors of the potential. Again by § 33 there are n real motors ($G_1G_2\ldots$ of equations (6) to (9) § 33) which form a conjugate set both with regard to ϖ and ψ. These are what he calls the n harmonic motors. The important property which gives them their name is at once proved from eq. (9) § 33 and eq. (6) of the present section.

Most of the theorems of *Screws* not already explicitly considered above now follow almost obviously. I content myself with considering a few of the less obvious.

In § 55 of *Screws* the question is asked—When the velocity motor $\dot{E}$ is given, what motors will serve as impulses each of which will generate $\dot{E}$ from rest? If the body is free the answer is $\psi_0\dot{E}$ only. If it is constrained the answer is $\psi_0\dot{E}+R$ where R is any motor of the reaction complex $(\overline{6-n})$.

If A_1, $A_2 \dots A_n$ are n conjugate motors of inertia and if the velocity motor $\dot{E}$ is given by

$$\dot{E} = x_1A_1 + \dots = \Sigma xA,$$

the kinetic energy is

$$-\tfrac{1}{2}\,\mathsf{s}\dot{E}\psi\dot{E} = -\tfrac{1}{2}\,\Sigma x^2\mathsf{s}A\psi A.$$

This is part of § 58 and § 59. In the rest of § 58 is considered the effect on the kinetic energy of the superposition of two impulse motors P_1' and P_2'. Resolving these each into two components (§ 48 above), the one in the momentum complex and the other in the reaction complex, let G_1 and G_2 be the former components. The kinetic energy acquired is

$$\begin{aligned}-\tfrac{1}{2}\,\mathsf{s}\,(G_1+G_2)\,\psi_0^{-1}(G_1+G_2)&\\ = -\tfrac{1}{2}\,\mathsf{s}G_1\psi_0^{-1}G_1 &- \mathsf{s}G_1\psi_0^{-1}G_2 - \tfrac{1}{2}\,\mathsf{s}G_2\psi_0^{-1}G_2.\end{aligned}$$

It is therefore the sum of the kinetic energies due to P_1' and P_2' separately if and only if G_1 and G_2 are conjugate with regard to ψ_0^{-1}, i.e. if and only if the velocity motors $\psi_0^{-1}G_1$ and $\psi_0^{-1}G_2$ acquired are conjugate motors of inertia (i.e. conjugate with regard to ψ_0).

In § 60 is considered how the twist velocity acquired due to a given impulse P' when the body is constrained to twist about A depends on A. By the method of § 48 (that $\Delta G - P'$ is reciprocal to the velocity complex) we see that if xA is the velocity motor acquired

$$x\psi_0A = P' + R,$$

where R is reciprocal to A. Thus

$$x\mathsf{s}A\psi_0A = \mathsf{s}AP',$$

or x varies directly as $\mathsf{s}AP'$ and inversely as $\mathsf{s}A\psi_0A$, which is the theorem enunciated for this case.

In § 64 a theorem due to Euler is proved, viz. that if (m) be a complex included in (n) the kinetic energy acquired by a given impulse P' is greater when $\dot{E}$ is allowed the freedom of (n) than when it is further restricted to (m). Let F' be the velocity motor acquired when the freedom is (m) and F when the freedom is (n). Thus $P' - \psi_0 F$ is reciprocal to every motor of (n) and $P' - \psi_0 F'$ is reciprocal to every motor of (m). They are both therefore reciprocal to F' which belongs to (m) and therefore to (n). Thus

$$\mathbf{s}F'\psi_0 F = \mathbf{s}F'P' = \mathbf{s}F'\psi_0 F',$$

or

$$\mathbf{s}(F - F')\,\psi_0 F' = 0.$$

Hence $F - F'$ and F' are conjugate motors of inertia. The kinetic energy due to F is therefore (see above) the sum of those due to F' and $F - F'$, i.e. (except when $F = F'$ and therefore the restriction to (m) is really no restriction) the kinetic energy due to F is greater than that due to F'.

In §§ 65, 66 of *Screws* there are certain errors due to the assumption which is explicitly stated in § 66 but which is not true, viz. that "a given wrench can always be resolved into two wrenches—one on a screw of any given complex and the other on a screw of the reciprocal complex." If the given complex contains any screw which is reciprocal to the whole complex this is not true, though it is otherwise. For instance if the given complex is that of the third order consisting of all the rotors through a given point, the reciprocal complex is identical with the given complex and therefore the statement is obviously untrue.

Thus the attempted proof in § 65 that "one screw can always be found upon a screw complex of the nth order reciprocal to $n - 1$ screws of the same complex" is unsound. For the $6 - n$ screws reciprocal to the given complex that are taken are not necessarily independent of the $n - 1$ screws of the given complex. The statement itself is true, but sometimes more than one screw can be found since there may be [§ 30 above] a complex (of not higher order than the third) in the given complex which is reciprocal to the whole complex. The proposition may be proved thus:—Let (n) be the given complex of order n and let $(n - 1)$ be the complex included in (n) consisting of the given $n - 1$ motors. Let $(7 - n)$ of order $7 - n$ be the complex reciprocal to $(n - 1)$. Then the two complexes (n) and $(7 - n)$ must contain at least one motor in common, for otherwise we should have 7 independent

motors. Thus there is at least one motor of (n) which is reciprocal to $(n-1)$. This does not prove that this nth motor is independent of the given $n-1$ motors. And as a matter of fact it is not necessarily thus independent. For instance the only motor of the complex i, Ωi which is reciprocal to Ωi is Ωi itself (and ordinary scalar multiples of it).

The main proposition of § 66 is erroneous, viz. that "a wrench which acts upon a constrained rigid body may always be replaced by a wrench on a screw belonging to the screw complex which defines the freedom of the body." For instance if the complex is that of i the freedom enjoyed is that of rotating about i. But the only wrench on the screw i is a force along i, and this cannot replace any given system of forces, such as a couple whose plane is perpendicular to i.

Thus (*Screws*, § 66) Sir Robert Ball's "reduced wrench" is not always intelligible. When it is intelligible it is a useful conception. It is intelligible when the reciprocal $(\overline{6-n})$ above of (n) is independent of (n), i.e. when (n) contains no motor which is reciprocal to (n) (i.e. no rotor or lator intersecting every other motor of the complex perpendicularly). In this case $(6-n)$ may as we saw in § 48 be defined as the reciprocal of (n); and then $(\bar{n})$ becomes identical with (n) and $(\overline{6-n})$ with $(6-n)$. Further, in this case our generalised force motor becomes identical with Sir Robert Ball's reduced wrench.

The main object apparently of the introduction of the reduced wrench is to obtain a *definite* motor function of the given force motor which has the same mechanical effect as the force motor itself. This may be done in a way that is always intelligible, viz. by identifying $(\bar{n})$ above with the momentum complex. The generalised force motor that we then get may be called the virtual force motor. It may be defined as the component of P the force motor in the momentum complex when P is expressed as such a component + a component in the reaction complex. Similarly the virtual impulse motor G of a given impulse motor P' may be defined as the component of P' in the momentum complex when P' is expressed as such a component + a component in the reaction complex. The virtual impulse motor is then the given impulse combined with the impulsive reaction, but a similar statement does not hold for the virtual force motor.

When $n = 2$ Sir Robert Ball establishes in §§ 102 and 103 the existence of what he calls the ellipse of inertia and the ellipse of the potential. Similarly when $n = 3$ he establishes the existence of the ellipsoid of inertia and the ellipsoid of the potential. Since by present methods (see §§ 43, 44 above) the treatment of these two cases—$n = 2$ and $n = 3$—are very similar, we content ourselves with the consideration of the ellipsoids only.

Suppose then $n = 3$, and suppose as in § 44 above that the case is not what is there called a singular one. Thus the complex (n) contains no motors which are reciprocal to the whole complex, so that in this case $(\bar{n})$ and (n) *may* be taken as identical. Putting χ for the ψ of § 44, so as to enable us to retain our present meaning of ψ, we see that every motor of (n) can be expressed as $(1 + \Omega\chi)\,\omega$, where ω is an arbitrary rotor through a definite point and χ is a given self-conjugate pencil function with this point for centre. Thus we may put

$$E_n = (1 + \Omega\chi)\,\omega \quad \text{.......................(7).}$$

If E_n' be any other motor $(1 + \Omega\chi)\,\omega'$ of (n) we have

$$\mathsf{s}E_n\psi E_n' = \mathsf{s}\omega\,(1 + \Omega\chi)\,\psi\,(1 + \Omega\chi)\,\omega' = \mathsf{s}\omega'\,(1 + \Omega\chi)\,\psi\,(1 + \Omega\chi)\,\omega,$$

i.e. $\mathsf{s}E_n\psi E_n'$ is a symmetrical function of ω and ω' linear in each. Hence

$$\mathsf{s}E_n\psi E_n' = \mathbf{S}\omega\tau\omega' \quad \text{..................... (8),}$$

where τ is a definite self-conjugate pencil function with the point just mentioned for centre.

The ellipsoid

$$\mathbf{S}\rho\tau\rho = -1 \quad \text{.......................... (9)}$$

is what is called the ellipsoid of inertia. [That it is an *ellipsoid* follows from the fact that $\mathsf{s}E_n\psi E_n$ is always negative and not zero.]

Similarly

$$\mathsf{s}E_n\varpi E_n' = \mathbf{S}\omega\upsilon\omega' \quad \text{.................... (10),}$$

where υ is a function of the same nature as τ. The ellipsoid

$$\mathbf{S}\rho\upsilon\rho = -1 \quad \text{....................... (11)}$$

is what is called the ellipsoid of the potential.

When $\mathsf{s}E_n\psi E_n' = 0$, $\mathbf{S}\omega\tau\omega' = 0$; i.e. two conjugate motors of inertia are any two motors of (n) which are parallel to a pair of conjugate diameters of the ellipsoid of inertia. Similarly two conjugate motors of the potential are any two motors of (n) which

are parallel to a pair of conjugate diameters of the ellipsoid of the potential.

Twice the kinetic energy is $-\mathbf{s}\dot{E}\psi\dot{E}$, or if $\dot{E}=(1+\Omega\chi)\,\omega$, $-\mathbf{S}\omega\tau\omega$. It is therefore the square of the tensor of the velocity motor multiplied by the inverse square of the parallel semi-diameter of the ellipsoid of inertia. Similarly twice the potential energy is the square of the tensor of the displacement motor multiplied by the inverse square of the parallel semi-diameter of the ellipsoid of the potential.

E_n and E_n' are conjugate with regard to ψ when $\mathbf{S}\omega\tau\omega'=0$, and are reciprocal when $\mathbf{S}\omega\chi\omega'=0$ [§ 44 above]. Hence the principal motors of inertia are those three of (n) which are parallel to the system of common conjugate diameters of the ellipsoid of inertia (τ) and the pitch quadric (χ). Similarly the principal motors of the potential are those three of (n) which are parallel to the system of common conjugate diameters of the ellipsoid of the potential (υ) and the pitch quadric (χ).

E_n and E_n' are conjugate with regard to ψ when $\mathbf{S}\omega\tau\omega'=0$, and are conjugate with regard to ϖ when $\mathbf{S}\omega\upsilon\omega'=0$. Hence the harmonic motors are those three of (n) which are parallel to the system of common conjugate diameters of the ellipsoid of inertia and the ellipsoid of the potential.

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