

NT-5 — Digital-Root Power Tables and Cross-Base Tiling

Geometry, Cycles, Mirrors, and Base-Portable Invariants in the Finite Ring Substrate

Justin Grieshop
Marithmetics / Public-Arch
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Abstract

Digital-Root Power Tables (DRPTs) are the minimal, finite visual trace of exponentiation in the residue ring \mathbb{Z}_{b-1} . When rendered as a table $(u, k) \mapsto \text{dr}_b(u^k)$, they exhibit stable, repeatable geometric phenomena—stripes, echoes, mirror panels, and attractor collapse—that admit complete explanation by elementary residue arithmetic: (i) unit rows are purely periodic, with period $\text{ord}_d(u)$ dividing the Carmichael exponent $\lambda(d)$; (ii) inverse structure appears as rigid rotation within each cycle class; and (iii) every non-unit row collapses into the unique absorbing residue class (the “zero channel”) and cannot return to the units. These facts are not numerology; they are the deterministic geometry of \mathbb{Z}_{b-1} .

This paper (NT-5) formalizes DRPTs as a substrate object within Marithmetics: it fixes the canonical DRPT definition, the “reading protocol” that maps a plate to a mathematical statement, and the Rosetta-normalized invariants (“hats”) that make DRPT-derived quantities base-portable. In addition, it provides audit-grade reproducibility guidance anchored to the AoR bundle and the canonical ledgers used throughout the suite.

Keywords: digital root; residue ring; multiplicative order; Carmichael exponent; modular exponentiation; periodicity; cross-base invariance; Rosetta normalization; reproducibility; authority of record.

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1 Document Control and How to Cite This Work

1.1 Canonical AoR Anchors

All empirical outputs and demonstration artifacts cited by the Marithmetics papers are bound to a single Authority-of-Record tag and bundle. The canonical citation surface for the current rewrite cycle is given in `URL_MAP.md` and includes (among others) the report PDF, the report manifest, and the claim ledger.

Canonical artifacts (stable URLs):

- Master zip (AoR snapshot):

`https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z/gum/authority_archive/AOR_20260209T040755Z_0fc79a0/MARI_MASTER_RELEASE_20260209T040755Z_0fc79a0.zip`

- GUM Report v32 (organized results):

`https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z/gum/authority_archive/AOR_20260209T040755Z_0fc79a0/report/GUM_Report_v32_2026-02-09_04-27-46Z.pdf`

- claim_ledger.jsonl (hash-locked claim ledger):

`https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z/gum/authority_archive/AOR_20260209T040755Z_0fc79a0/claim_ledger.jsonl`

- constants_master.csv (index of derived constants/results):

`https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z/gum/authority_archive/AOR_20260209T040755Z_0fc79a0/GUM_BUNDLE_v30_20260209T040755Z/constants_master.csv`

1.2 Citation Format (Audit-Grade)

For any claim in this paper that depends on an AoR-produced artifact (a computed table, a figure, a certified constant, a demo output), cite all three of the following:

1. The AoR tag: `aor-20260209T040755Z`
2. The AoR bundle SHA-256: `c299b1a7a8ef77f25c3ebb326cb73f060b3c7176b6ea9eb402c97273dc3cf66c`
3. The specific artifact URL (report section, ledger line, table path, or demo path) from `URL_MAP.md`

For purely mathematical results in this paper (lemmas/theorems proved from residue arithmetic), standard citation practice applies; they do not require AoR anchoring because they are deductive theorems.

2 Why DRPTs Are Foundational

2.1 The Object That the Eye Can Verify

The Number Theory track begins by asserting that a large fraction of “structure” commonly attributed to magnitude is already present at the residue level. DRPTs are the simplest object that makes this claim visible: a DRPT plate is not a plot of “big numbers,” but a finite table whose size is determined only by $d = b - 1$ (the modulus induced by the numeral base) and an exponent horizon K .

A DRPT plate therefore has two crucial properties:

- It is *finite and exhaustible*: every entry is a deterministic value in a finite ring.
- It is *canonical*: once b and K are fixed, the table is fixed.

These are precisely the kinds of objects on which a deterministic operator calculus can act without ambiguity, and from which base-portable invariants can be defined. In Marithmetics, DRPTs form the substrate layer from which “hats” (dimensionless invariants) are extracted and transported across bases.

2.2 DRPTs as the “Visible Trace” of \mathbb{Z}_{b-1}

The primary claim of this paper is not interpretive; it is structural:

Claim 2.1 (Substrate identification). *A DRPT plate in base b is a rendering of exponentiation in the residue ring \mathbb{Z}_d with $d = b - 1$. Its repeating bands, mirror panels, and attractor rows are the direct consequence of (i) finiteness, (ii) the unit/non-unit dichotomy, and (iii) the exponent bound $\lambda(d)$ that controls all unit orders.*

This claim is fully proved in §4 below.

3 Canonical Definitions: DRPTs, Units, and the Two “Zeros”

3.1 Base, Modulus, Residue Ring

Fix an integer base $b \geq 2$ and define the associated modulus

$$d := b - 1.$$

We work in the finite residue ring

$$\mathbb{Z}_d := \mathbb{Z}/d\mathbb{Z}.$$

Write $U_d := (\mathbb{Z}_d)^\times$ for the unit group.

3.2 Digital-Root Projector and the Residue Projector

There are two closely related maps that must be distinguished in Marithmetics:

1. **The digital-root projector** into a base- b digit alphabet Σ_b :

$$T_b(n) := \begin{cases} 0, & n = 0, \\ 1 + ((n - 1) \bmod d), & n \geq 1, \end{cases} \quad \Sigma_b = \{0, 1, \dots, d\}.$$

This is the canonical “projector” definition used in the UFET/USNE authority stream.

2. **The residue-level projector** (used for ring-level DRPT algebra): Define $\rho_b : \mathbb{Z} \rightarrow \mathbb{Z}_d$ by

$$\rho_b(n) \equiv n \pmod{d}.$$

Relationship and the “two zeros.” In base-digit space Σ_b , both symbols 0 and d represent residue class 0 (mod d): $T_b(d) = d$, yet $d \equiv 0 \pmod{d}$. Much of the DRPT geometry becomes visually clearer if one displays the residue class 0 uniformly as “0,” collapsing d and 0 at the rendering layer (while preserving the distinction in digit semantics when needed). This is the standard “0/d identification” used in the Number Taxonomy’s digital-root compatibility lemma.

Accordingly, in this paper we define DRPT entries in \mathbb{Z}_d (ring-level), and we allow a display convention that maps the class 0 (mod d) to the printed symbol “0”.

3.3 The DRPT Definition (Canonical)

Definition 3.1 (Digital-Root Power Table, ring-level). Fix base $b \geq 2$, $d = b - 1$, and a finite exponent horizon $K \in \mathbb{N}$. The *DRPT plate* is the function

$$R_{b,K} : \mathbb{Z}_d \times \{0, 1, \dots, K\} \rightarrow \mathbb{Z}_d, \quad R_{b,K}(u, k) := \rho_b(u^k) \in \mathbb{Z}_d.$$

When u is interpreted as a digit in Σ_b , the corresponding digit-level entry is $T_b(u^k)$, and the ring-level entry is obtained by reducing $T_b(u^k)$ modulo d (equivalently, identifying d with 0).

Remark 3.2 (Choice of $k = 0$). Including $k = 0$ makes the unit/non-unit dichotomy explicit: $u^0 \equiv 1$ for all $u \neq 0$, while 0^0 is convention-dependent. In all computational DRPT artifacts used in Marithmetics, $k \geq 1$ is the default plate range unless otherwise stated; this matches the standard “power table” convention in the DRPT authority entry and avoids the 0^0 ambiguity.

3.4 Units, Non-Units, and the Irreversible Boundary

Lemma 3.3 (Unit closure and non-unit absorption). *If $u, v \in U_d$, then $uv \in U_d$. If $a \notin U_d$, then for every $x \in \mathbb{Z}_d$, the product $ax \notin U_d$.*

This lemma is the algebraic reason for the “attractor collapse” phenomenon: once a power enters a non-unit ideal component, subsequent powers cannot return to the unit group.

4 The Three Structural Laws of Every DRPT Plate

This section states and proves the three facts that generate essentially all of the visible DRPT geometry: periodicity, inverse rotation, and attractor collapse.

4.1 Periodicity of Unit Rows

Lemma 4.1 (Multiplicative order and Carmichael exponent). *For $u \in U_d$, let $\text{ord}_d(u)$ denote its multiplicative order in \mathbb{Z}_d . The Carmichael exponent $\lambda(d)$ is the least positive integer such that $u^{\lambda(d)} \equiv 1 \pmod{d}$ for all $u \in U_d$. Every order divides $\lambda(d)$.*

Corollary 4.2 (Unit-row periodicity). *If $u \in U_d$ and $p = \text{ord}_d(u)$, then for all $k \geq 0$,*

$$R_{b,K}(u, k + p) = R_{b,K}(u, k).$$

In particular, the visible horizontal repetition length for a unit row is exactly its order (or a divisor if the rendered horizon is too short).

Proof. By definition of p , $u^{k+p} \equiv u^k u^p \equiv u^k \pmod{d}$. Applying ρ_b yields the identity. \square

Interpretation (tiling along the exponent axis). A DRPT unit row is the infinite repetition of a finite word of length p , hence the plate tiles horizontally (in k) by blocks whose widths divide $\lambda(d)$. This “tiling across infinity” is not a metaphor; it is the literal statement of periodicity in a finite group.

4.2 Inverse Rotation on Unit Cycles

The second structural law explains mirror panels and rotated copies that appear within DRPT plates.

Lemma 4.3 (Inversion as a shift). *Let $u \in U_d$ have order p . Let u^{-1} denote the inverse of u in U_d . Then for all integers k ,*

$$u^{-k} \equiv u^{p-k} \pmod{d}.$$

Proof. Since $u^p \equiv 1$, we have $u^{-k} \equiv u^{p-k}$ for all k modulo p . \square

Corollary 4.4 (Inverse-row rotation). *Let $u \in U_d$ have order p . The sequence $k \mapsto R_{b,K}(u^{-1}, k)$ is a cyclic rotation of the sequence $k \mapsto R_{b,K}(u, k)$ with period p . In particular, any functional on a row that is invariant under cyclic rotation (e.g., an average with uniform weights on the cycle) takes the same value on u and u^{-1} .*

Proof. For $k \geq 1$, $(u^{-1})^k = u^{-k} \equiv u^{p-k} \pmod{d}$. Thus the u^{-1} row is exactly the u row read backward modulo a cyclic shift; equivalently it is a rotation when indexed on the cycle. \square

4.3 Attractor Structure: Non-Units, Valuations, and the Algebraic “Zero” Regime

The third structural law concerns what happens once a residue is not invertible. Visually, DRPT plates exhibit an evident boundary between “unit behavior” (clean cycles) and “non-unit behavior” (collapse, truncation, or confinement). Algebraically, this boundary is governed by prime-power valuations and Chinese-remainder gluing.

4.3.1 Non-Unit Permanence (No Return to the Unit Group)

Lemma 4.5 (Non-unit permanence). *Let $d = b - 1$ and $u \in \mathbb{Z}_d$. If $u \notin U_d$, then $u^k \notin U_d$ for every integer $k \geq 1$.*

Proof. If $u \notin U_d$, then $\gcd(u, d) > 1$, so there exists a prime $p \mid d$ with $p \mid u$. Then $p \mid u^k$ for all $k \geq 1$, so $\gcd(u^k, d) \geq p > 1$; hence $u^k \notin U_d$. \square

Interpretation. Once a DRPT row begins outside the unit group, it cannot “re-enter” the unit-cycle regime. This is the algebraic basis for the visually sharp boundary between cyclic unit bands and the non-unit region.

4.3.2 Prime-Power Classification and the Minimal Hitting Time to 0

The phenomenon commonly described informally as “collapse to zero” is not uniform across all non-units. It is precise, local (prime-power), and then global (CRT). The correct classification is in terms of nilpotence.

Fix a prime power p^a and consider \mathbb{Z}_{p^a} . Write $v_p(n)$ for the p -adic valuation of an integer representative n (the largest $t \geq 0$ such that $p^t \mid n$).

Lemma 4.6 (Prime-power nilpotence criterion and hitting time). *Let $u \in \mathbb{Z}_{p^a}$ and choose an integer representative n .*

- If $v_p(n) = 0$, then $u^k \not\equiv 0 \pmod{p^a}$ for all $k \geq 1$.
- If $v_p(n) = t \geq 1$, then $u^k \equiv 0 \pmod{p^a}$ if and only if $kt \geq a$.

In particular, the minimal exponent at which $u^k \equiv 0 \pmod{p^a}$ is

$$K_{p^a}(u) := \left\lceil \frac{a}{t} \right\rceil.$$

Proof. If $v_p(n) = 0$, then $p \nmid n$, so n^k is not divisible by p , hence not divisible by p^a .

If $v_p(n) = t \geq 1$, then $v_p(n^k) = kt$. Therefore $p^a \mid n^k$ if and only if $kt \geq a$. The minimal such k is $\lceil a/t \rceil$. \square

4.3.3 CRT Gluing: Which Residues Reach 0 (mod d)

Write the prime-power factorization

$$d = \prod_{j=1}^r p_j^{a_j}.$$

By CRT,

$$\mathbb{Z}_d \cong \prod_{j=1}^r \mathbb{Z}_{p_j^{a_j}}.$$

Definition 4.7 (Algebraic zero-reachability and global hitting time). For $u \in \mathbb{Z}_d$, define the component hitting times

$$K_{p_j^{a_j}}(u) \in \mathbb{N} \cup \{\infty\}$$

via Lemma 4.6 applied to the j -th component of u under the CRT isomorphism. Define the *global hitting time*

$$K_d(u) := \max_{1 \leq j \leq r} K_{p_j^{a_j}}(u) \in \mathbb{N} \cup \{\infty\}.$$

Proposition 4.8 (Exact criterion for reaching 0 (mod d)). Let $u \in \mathbb{Z}_d$. Then $u^k \equiv 0 \pmod{d}$ for some $k \geq 1$ if and only if $K_d(u) < \infty$; equivalently, u has strictly positive valuation at every prime $p_j \mid d$ (i.e., $p_j \mid u$ for all j). In this case,

$$u^k \equiv 0 \pmod{d} \iff k \geq K_d(u),$$

so $K_d(u)$ is the exact minimal hitting time to 0 (mod d).

Proof. By CRT, $u^k \equiv 0 \pmod{d}$ holds if and only if $u^k \equiv 0 \pmod{p_j^{a_j}}$ holds for every j . By Lemma 4.6, this occurs for a given j if and only if $K_{p_j^{a_j}}(u) < \infty$ and $k \geq K_{p_j^{a_j}}(u)$. Therefore $u^k \equiv 0 \pmod{d}$ holds if and only if k exceeds the maximum of the component thresholds, i.e., $k \geq K_d(u)$, and this is possible if and only if all component thresholds are finite. \square

Remark 4.9 (Non-units bifurcate into nilpotent and non-nilpotent). • Every nilpotent element is a non-unit and reaches 0 (mod d) at an explicitly computable finite time $K_d(u)$.

- A non-unit that is not divisible by every prime factor of d is not nilpotent and does not reach 0 (mod d). It remains permanently outside U_d (Lemma 4.5) but can exhibit periodic behavior on the CRT components where it remains a unit.

This bifurcation is the mathematically correct refinement of the informal phrase “non-unit collapse.” The algebraic “zero” regime is precisely the nilpotent sector, and it is completely classified by valuations and CRT gluing (a point made explicitly in the Zero-Atlas authority stream).

4.3.4 Eventual Periodicity and Finite Codebooks (Tiling Beyond a Transient)

Even when u is not a unit, the exponent axis still becomes finite in the following sense: after a finite transient, $k \mapsto u^k$ is periodic in \mathbb{Z}_d . This is the precise algebraic statement behind the observed “tiling across infinity.”

Theorem 4.10 (Eventual $\lambda(d)$ -periodicity). Let $d = b - 1$, let $\lambda(d)$ be the Carmichael exponent of U_d , and define the transient bound

$$K_0(d) := \max_{1 \leq j \leq r} a_j \quad \text{for } d = \prod_{j=1}^r p_j^{a_j}.$$

Then for every $u \in \mathbb{Z}_d$ and every integer $k \geq K_0(d)$,

$$u^{k+\lambda(d)} \equiv u^k \pmod{d}.$$

Proof. Work modulo each prime power $p_j^{a_j}$ and then glue by CRT. Fix j .

If $p_j \nmid u$, then u is a unit modulo $p_j^{a_j}$ and $u^{\lambda(d)} \equiv 1 \pmod{p_j^{a_j}}$, so $u^{k+\lambda(d)} \equiv u^k \pmod{p_j^{a_j}}$ for all $k \geq 1$.

If $p_j \mid u$, then for every $k \geq a_j$ we have $u^k \equiv 0 \pmod{p_j^{a_j}}$. Since $k \geq K_0(d) \geq a_j$, both u^k and $u^{k+\lambda(d)}$ vanish modulo $p_j^{a_j}$, hence are equal.

Thus the congruence holds modulo every $p_j^{a_j}$ for $k \geq K_0(d)$, and CRT yields the congruence modulo d . \square

Corollary 4.11 (Finite codebook). *For any fixed base b , the infinite exponent axis of the DRPT table is determined by a finite set of columns:*

$$k \in \{1, 2, \dots, K_0(d) + \lambda(d)\}.$$

Beyond the transient region $k \geq K_0(d)$, the table tiles with period $\lambda(d)$ in the exponent coordinate.

Interpretation. This is the strongest finitary statement a DRPT can make about “infinity”: for each base b , exponentiation in \mathbb{Z}_{b-1} does not generate unbounded novelty. After a finite transient, it repeats with a base-dependent period controlled by $\lambda(d)$.

5 How to Read a DRPT Plate: Cycles, Mirrors, and Attractors

This section fixes the *reading protocol*: a deterministic procedure that maps a DRPT visualization to explicit algebraic statements. The protocol is deliberately finite and checkable; it is not interpretive.

5.1 Plate Conventions and Coordinate Choices

A DRPT plate can be presented with either axis orientation. In this paper we adopt:

- **row index:** $u \in \{0, 1, \dots, d-1\} \subset \mathbb{Z}_d$,
- **column index:** exponent $k \in \{1, 2, \dots, K\}$,
- **cell value:** $R_{b,K}(u, k) = u^k \pmod{d}$ displayed with the convention that residue 0 is printed as “0”.

All structural statements are invariant under transposition; only the visual reading changes.

5.2 Cycles (Unit Rows) and the Canonical Period Parameter

Protocol step 1 (Cycle detection). Choose a row u . Compute (or visually infer) the smallest period p such that

$$R_{b,K}(u, k+p) = R_{b,K}(u, k) \quad \text{for all } k \text{ in a range long enough to verify.}$$

If such a p exists within the observed horizon and $u \in U_d$, then $p = \text{ord}_d(u)$ (Lemma 3.1 and Corollary 4.2).

Operationally:

- If u is a unit, the row is purely periodic from $k = 1$ onward.
- The period is a divisor of $\lambda(d)$.
- The set of distinct values appearing in the row is exactly the cyclic subgroup generated by u .

This is the mathematical content of the “striping” and “repeat blocks” seen across DRPT unit bands.

5.3 Mirrors (Inverse Structure) as Rotations, Not Coincidences

Protocol step 2 (Mirror identification). Within a fixed order- p cycle class, compute the inverse $u^{-1} \in U_d$. Then the row u^{-1} is a cyclic rotation of the row u (Corollary 4.4). Hence any statistic invariant under cyclic rotation (e.g., the uniform row-average over a full period) takes identical values on u and u^{-1} .

This explains why DRPT plates exhibit mirrored panels or rotated echoes: what appears as a “new” structure is often the same cycle read from a different phase origin.

5.4 Attractors (Non-Unit Regime) and the Two Algebraic Outcomes

Protocol step 3 (Attractor classification). Given a non-unit $u \notin U_d$, classify it by prime-power valuations:

- If $K_d(u) < \infty$, then u is nilpotent and the row reaches the value 0 exactly at exponent $K_d(u)$ (Proposition 4.8).
- If $K_d(u) = \infty$, then the row never reaches 0 (mod d) but remains permanently outside U_d (Lemma 4.5) and becomes eventually periodic (Theorem 4.10).

This is the precise algebraic content of the authority statement that “zero in DRPT is simultaneously an algebraic attractor (valuation/CRT) and a geometric collar,” with the CRT-glued valuation law providing the exact certification of which residues reach 0 and which do not.

5.5 Practical Note: Why “Mirror Columns” Appear in Some Bases

In many plotted plates one observes that exponent indices k and $p - k$ (or $\lambda(d) - k$ on a chosen cycle length) produce visually related columns. The algebraic source is the identity

$$u^{p-k} \equiv u^{-k} \quad \text{in a cycle of order } p,$$

which is simply Lemma 4.3 applied within each cyclic subgroup. Whether this appears as an obvious “mirror” depends on (i) the chosen plotting horizon, (ii) the display convention for residue 0, and (iii) how the plate is filtered or highlighted.

In subsequent sections, the “mirror” phenomenon will be treated only as a derived corollary of inversion/rotation, not as a primitive symmetry assumption.

6 Rosetta Normalization: Base-Portable Invariants (“Hats”)

6.1 Motivation: Comparisons Across Bases Require a Normalized Alphabet

A DRPT plate is base-dependent in two distinct ways:

1. **Indexing dependence.** The row axis is the residue ring \mathbb{Z}_{b-1} , whose size changes with b . The exponent axis tiles with a base-dependent period dividing $\lambda(b-1)$.
2. **Symbol dependence.** The digit-level projector T_b uses the base- b alphabet $\Sigma_b = \{0, 1, \dots, b-1\}$, which is not comparable across bases without a reduction convention.

The purpose of Rosetta normalization is to remove these contingencies at the level of statements and dimensionless invariants. The guiding principle is:

Rosetta principle. *Any quantity intended to be transported across bases must be expressed as a rational assembly of normalized invariants (“hats”), rather than in raw digits or base-specific indices. Any violation is detectable as base drift and is treated as a falsifier in the AoR workflow.*

In this paper we distinguish *Rosetta coordinates* (for cross-base visualization) from the *Rosetta invariant alphabet* (for cross-base claims).

6.2 Rosetta Coordinates: Putting Any DRPT Plate into a Unit Square

Fix $b \geq 2$ and $d = b - 1$. Let L be a chosen exponent window length (typically a multiple of $\lambda(d)$). Define the coordinate normalization map

$$(u, k) \mapsto (\hat{u}, \hat{k}) \quad \text{where} \quad \hat{u} := \frac{u}{d} \in [0, 1), \quad \hat{k} := \frac{k}{L} \in (0, 1]. \quad (5.1)$$

If one chooses $L = \lambda(d)$, then \hat{k} parameterizes a single fundamental exponent tile for the unit-cycle regime; if one chooses $L = k_{\text{mult}}\lambda(d)$, then \hat{k} parameterizes a controlled number of repeated tiles used to stabilize visual and statistical functionals (as in the Zero Atlas and related DRPT geometry extractions).

For display values, define the residue normalization

$$\hat{R}(u, k) := \frac{R_{b,L}(u, k)}{d} \in [0, 1], \quad (5.2)$$

with the convention that the residue class $0 \pmod{d}$ is displayed uniformly (digit symbol 0). This renders any base's DRPT plate as a function $(\hat{u}, \hat{k}) \mapsto \hat{R}$ on a common domain, enabling meaningful overlays between bases.

Remark 6.1 (Coordinate hats are not the invariant hats). The symbols $\hat{u}, \hat{k}, \hat{R}$ are normalized coordinates/values. The Rosetta invariant alphabet is $\hat{\chi}, \hat{\theta}, \hat{\Psi}, \hat{\kappa}$ below; these are the dimensionless quantities used in claims and in the Φ -template.

6.3 The Rosetta Invariant Alphabet

The suite standardizes four dimensionless invariants (the “hats”) as the alphabet from which base-portable claims are assembled:

$$\hat{\chi} = \frac{\text{ord}}{\lambda}, \quad \hat{\theta} = \frac{\theta}{\theta_*}, \quad \hat{\Psi} = \frac{\Psi}{\text{circ}_*}, \quad \hat{\kappa} = \frac{\kappa}{\kappa_*}. \quad (5.3)$$

Here ord is a unit-row multiplicative order (a DRPT cycle length), λ is the Carmichael exponent of U_d , θ denotes a survivor density for a declared wheel/filter, Ψ denotes a canonical circulation functional on a fixed cycle, and κ denotes a universal envelope constant for the legal operator class; the starred quantities θ_* , circ_* , κ_* are fixed references declared in the Authority and held constant across bases.

6.3.1 The DRPT-Native Invariant $\hat{\chi}$

Among the hats, $\hat{\chi}$ is the one directly native to DRPT plates without further apparatus.

Definition 6.2 (Order fraction). Let $u \in U_d$. Define

$$\chi(u) := \text{ord}_d(u), \quad \hat{\chi}(u) := \frac{\chi(u)}{\lambda(d)} \in (0, 1]. \quad (5.4)$$

For non-units $u \notin U_d$, one may leave $\hat{\chi}(u)$ undefined or set $\hat{\chi}(u) = 0$ by convention when constructing distributions that mix unit and non-unit populations.

Interpretation. $\hat{\chi}$ records the cycle length in units of the maximal universal exponent tile $\lambda(d)$. Two different bases may have different $\lambda(d)$, yet $\hat{\chi} \in (0, 1]$ lives on a common normalized scale. This is the first step in turning a base-dependent cycle into a base-portable datum.

6.3.2 $\hat{\theta}, \hat{\Psi}, \hat{\kappa}$ as DRPT-Derived but Operator-Mediated Invariants

The remaining hats depend on lawful operators and/or lawful measurements applied to DRPT-derived fields (for example, a “zero collar” extracted from a smoothed DRPT field, or a survivor set extracted under a declared filter). In order to keep NT-5 self-contained and audit-correct, we adopt the following discipline:

- NT-5 defines $\hat{\theta}, \hat{\Psi}, \hat{\kappa}$ as named invariants with a declared production protocol and uses them only in statements explicitly framed as “Rosetta-template statements.”
- The complete legality framework that governs their extraction (windows, PSD/unit-mass gates, commutator requirements, residual budgets) is treated in the DOC-governed papers and in the Authority entries that define the corresponding audit objects.

For the purpose of this paper, it suffices to note the intended meaning:

- θ (survivor density) is a normalized count of a declared survivor set W (often $W \subseteq U_d$) under a specified selection rule; θ_* is the Authority reference density for that selection.
- Ψ (circulation) is a signed, cycle-based functional computed on a canonical loop/cycle within a normalized DRPT-derived field; circ_* is the Authority reference circulation.
- κ (envelope) is a certified bound parameter controlling a universal legality envelope for an operator family; κ_* is its Authority reference.

These are the invariants used downstream in Φ -templates and SCFP selection; they are introduced here because DRPT plates are the substrate on which their base-portable extraction is demonstrated.

6.4 Algebraic Hat of a Base: A Portable Summary of DRPT Structure

Beyond individual row invariants, a base b may be summarized by base-normalized distributions derived from \mathbb{Z}_{b-1} . This object is not “base-invariant” (it changes with b); rather, it is base-portable in the sense that it lives in a common normalized schema and can be compared across bases without ambiguity.

Definition 6.3 (Algebraic hat of $d = b - 1$). Define the following dimensionless quantities:

- **Nilpotent fraction:**

$$\text{nilp_frac}(d) := \frac{|\{u \in \mathbb{Z}_d : u \text{ divisible by every prime } p \mid d\}|}{d}. \quad (5.5)$$

- **Hitting time histogram** K_{hist} : the distribution of global nilpotent hitting times $K_d(u)$ (Definition 3.7) among the nilpotent residues, normalized by d .
- **Order histogram** ord_hist : the distribution of $\text{ord}_d(u)$ over units $u \in U_d$, normalized by $|U_d|$.
- **Squarefree flag:** $\mathbf{1}_{\text{squarefree}}(d)$.

A canonical serialized schema for these quantities is fixed in the Zero/DRPT authority stream and is intended to be emitted as JSON/CSV with explicit gate checks and SHA-256 manifests for audit and replication.

Remark 6.4 (Why this belongs in a DRPT paper). The algebraic hat is the minimal “fingerprint” of the residue ring that generates the DRPT plate. It captures, in a single normalized record, (i) how much of the plate is inevitably non-invertible, (ii) how quickly nilpotent rows hit the absorbing class, and (iii) how unit cycles distribute across divisors of $\lambda(d)$. These three features control essentially all of the visible DRPT geometry (striping, mirror-rotation redundancy, and attractor collars).

6.5 The Φ -Template and the Formal Meaning of Base Portability

The Rosetta hats are not introduced merely for visualization. They are introduced to support a theorem-level claim:

Φ -template principle. *Any dimensionless invariant used for scientific claims in the Marithmetics suite is expressed as a rational assembly*

$$\Phi = F(\hat{\chi}, \hat{\theta}, \hat{\Psi}, \hat{\kappa}), \quad (5.6)$$

and consequently is identical across admissible bases when computed from the corresponding hat values; dimensional anchors (SI or observational mappings) are applied only after Φ is fixed.

This principle is stated here because DRPTs provide the cleanest laboratory for observing base portability: one can compute $\hat{\chi}$ directly from a finite table and verify that any claim framed purely in hats exhibits no base drift under cross-base translation.

7 Cross-Base Tiling and the Visual Meaning of $\lambda(d)$

7.1 $\lambda(d)$ as the Exponent Tile Width for the Unit-Cycle Regime

Section 4 established that:

- every unit row is purely periodic from $k = 1$, with period $p = \text{ord}_d(u)$ dividing $\lambda(d)$;
- every row (unit or non-unit) becomes $\lambda(d)$ -periodic after a finite transient $k \geq K_0(d)$.

This makes $\lambda(d)$ the canonical “tile width” along the exponent axis. In the DRPT figures, this tile is what the eye detects as repeated vertical band structure.

A mathematically precise restatement is:

For fixed b , there exists a finite “codebook window” of columns of length $K_0(d) + \lambda(d)$ that determines the entire infinite DRPT plate, and the repeating tile beyond the transient has width $\lambda(d)$ (Theorem 4.10 and Corollary 4.11).

The term “tiling across infinity” refers to this exact periodic extension.

7.2 Mirror Columns as Inversion, Not Coincidence

The “mirror columns” frequently observed in DRPT plates (and highlighted in the accompanying visualizations) are a direct consequence of inversion on unit cycles.

Let $u \in U_d$ have order p . Then

$$u^{p-k} \equiv u^{-k} \pmod{d} \quad (6.1)$$

(Lemma 4.3). If u has full order $p = \lambda(d)$, then the exponent-tile itself carries an intrinsic reflection: the map $k \mapsto \lambda(d) - k$ corresponds to inversion along the cycle.

Two concrete implications follow.

Proposition 7.1 (Column pairing in full-order rows). *Suppose $u \in U_d$ has $\text{ord}_d(u) = \lambda(d)$. Then within a single exponent tile $k \in \{1, \dots, \lambda(d)\}$,*

$$R_{b,\lambda(d)}(u, \lambda(d) - k) \equiv R_{b,\lambda(d)}(u^{-1}, k) \pmod{d}. \quad (6.2)$$

Thus, in a full-order row, reflecting columns corresponds to reading the inverse row (up to a cyclic shift).

Proof. By (6.1), $u^{\lambda(d)-k} \equiv u^{-k}$. But $u^{-k} = (u^{-1})^k$. Applying ρ_b yields (6.2). \square

Corollary 7.2 (Mirror-invariant statistics). *Any row functional invariant under cyclic rotation (e.g., a uniform average over a full period) takes equal values on u and u^{-1} , and hence cannot distinguish a row from its mirror partner (Corollary 4.4).*

This is not a philosophical statement; it is a concrete warning: any measurement that does not break cyclic symmetry cannot “see” the difference between a cycle and its inverse rotation. This is why lawful measurement design (in later papers) must declare what symmetries it respects.

7.3 A Worked Base-10 Example (Tile Width, Inverse Rotation, and Mirror Pairing)

To ground the above in a familiar base, take $b = 10$, so $d = b - 1 = 9$. The unit group is

$$U_9 = \{1, 2, 4, 5, 7, 8\}, \quad \lambda(9) = 6.$$

The full-order units are 2 and 5, with $2^{-1} \equiv 5 \pmod{9}$.

The corresponding DRPT rows over one full tile $k = 1, \dots, 6$ are:

u	1	2	3	4	5	6
$2^k \pmod{9}$	2	4	8	7	5	1
$5^k \pmod{9}$	5	7	8	4	2	1

(6.3)

One sees directly that the $u = 5$ row is the inverse-rotation of $u = 2$, consistent with Corollary 4.4, and that reflection across the tile corresponds to inversion behavior as in Proposition 7.1.

Base 8 (mod 7) $\lambda(7) = 6$													Base 10 (mod 9) $\lambda(9) = 6$																	
u	1	2	3	4	5	6	7	8	9	10	11	12	u	1	2	3	4	5	6	7	8	9	10	11	12					
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1						
2	2	4	1	2	4	1	2	4	1	2	4	1	2	4	1	2	4	8	7	5	1	2	4	8	7	5	1			
3	3	2	6	4	5	1	3	2	6	4	5	1	3	2	6	4	5	1	3	2	6	4	5	1	3	2	6	4	5	1
4	4	2	1	4	2	1	4	2	1	4	2	1	4	2	1	4	2	1	4	7	1	4	7	1	4	7	1	4	7	1
5	5	4	6	2	3	1	5	4	6	2	3	1	5	4	6	2	3	1	5	7	8	4	2	1	5	7	8	4	2	1
6	6	1	6	1	6	1	6	1	6	1	6	1	6	1	6	1	6	1	6	1	8	1	8	1	8	1	8	1	8	1

Base 12 (mod 11) $\lambda(11) = 10$													Base 14 (mod 13) $\lambda(13) = 12$																
u	1	2	3	4	5	6	7	8	9	10	11	12	u	1	2	3	4	5	6	7	8	9	10	11	12				
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
2	2	4	8	5	10	9	7	3	6	1	2	4	2	4	8	5	10	9	7	3	6	12	11	9	5	10	7	1	
3	3	9	5	4	1	3	9	5	4	1	3	9	5	4	1	3	9	5	12	8	1	5	12	8	1	5	12	8	1
4	4	5	9	3	1	4	5	9	3	1	4	5	9	3	1	4	5	12	8	1	5	12	8	1	5	12	8	1	
5	5	3	4	9	1	5	3	4	9	1	5	3	4	9	1	5	3	4	9	1	9	3	1	9	3	1	9	3	1
10	10	1	10	1	10	1	10	1	10	1	10	1	10	1	10	1	10	1	12	1	12	1	12	1	12	1	12	1	

Figure 1: Identity Element (Value 1) across four bases. Orange cells mark positions where $u^k \equiv 1 \pmod{d}$, corresponding to cycle completion points (Proposition 7.9). Row u shows identity at all multiples of $\text{ord}_d(u)$. Only unit rows ($u \in U_d$) are shown; non-unit rows collapse to the attractor and never return to 1.

Remark 7.3 (Digit-level display). If one displays the plate as digital roots in base 10, the residue class 0 (mod 9) appears as 9 for nonzero inputs in the conventional digital-root rule. In this paper, we work ring-level and display 0 (mod 9) uniformly as 0 for geometric clarity. The underlying algebra is unchanged (Section 2.2).

7.4 Cross-Base Overlays: Aligning Tiles by Rosetta Coordinates

The first cross-base fact one can verify without invoking any SCFP/physics layer is:

Different bases have different tile widths $\lambda(b - 1)$, but each base's tile is normalized to the same unit interval under $\hat{k} = k/\lambda(b - 1)$.

This is why cross-base DRPT visualizations should be plotted on \hat{k} , not on raw k . The Authority record explicitly uses such hat annotations to demonstrate cross-base equality of structure under Rosetta normalization.

Concretely, for $b \in \{7, 10, 16\}$, the moduli are 6, 9, 15, with tile widths $\lambda(6) = 2$, $\lambda(9) = 6$, $\lambda(15) = 4$. A raw exponent axis makes these incomparable; the normalized axis $\hat{k} \in (0, 1]$ makes them directly overlay-compatible.

7.5 Cross-Base “Tiling Across Infinity” as a Finite Statement

For each base b , Theorem 4.10 implies eventual periodicity with period $\lambda(b - 1)$ beyond a finite transient. Therefore, for any fixed finite set of bases $\{b_i\}$, the joint object

$$k \mapsto (u^k \bmod (b_1 - 1), \dots, u^k \bmod (b_r - 1)) \quad (6.4)$$

is also eventually periodic (with period dividing $\text{lcm}_i \lambda(b_i - 1)$) after a finite transient determined by the maximal prime-power exponent among the moduli. In this sense, “tiling across infinity” is stable under cross-base stacking: the infinite axis remains finite-codebook after a finite start-up cost.

This is the correct mathematical statement behind the visual intuition that “the codebook is the same across infinity”: novelty does not accumulate in the exponent axis; it repeats in a controlled way.

7.6 Diagonal Echoes and Column Resonances (The Source of the “Mirrored” and “Cadence” Columns)

The visual structures that a reader often describes informally as “mirrors in columns” or “special columns” are not independent phenomena. They are consequences of two elementary facts:

1. the unit group U_d has finite exponent $\lambda(d)$, and
2. the map $u \mapsto u^k$ is a group endomorphism on U_d whose image size depends on $\gcd(k, \lambda(d))$.

These facts produce (i) *echo families* (columns separated by $\lambda(d)$ coincide on unit rows), and (ii) *resonant columns* (exponent indices with large common factors with $\lambda(d)$ collapse values into small subgroups).

7.6.1 Echo Congruence (Columns Repeat on the Unit Band)

Lemma 7.4 (Echo congruence on unit rows). *Let $d = b - 1$ and let $\lambda(d)$ be the Carmichael exponent of U_d . If $k \equiv k' \pmod{\lambda(d)}$, then for every $u \in U_d$,*

$$R_{b,K}(u, k') = R_{b,K}(u, k).$$

Proof. By definition of $\lambda(d)$, $u^{\lambda(d)} \equiv 1 \pmod{d}$ for all $u \in U_d$. If $k' = k + m\lambda(d)$, then $u^{k'} = u^k (u^{\lambda(d)})^m \equiv u^k \pmod{d}$. Applying ρ_b yields the claimed equality. \square

Definition 7.5 (Echo class and echo family). Fix a residue class $c \in \{0, 1, \dots, \lambda(d) - 1\}$. The *echo class* E_c is the set of exponent indices

$$E_c := \{k \geq 1 : k \equiv c \pmod{\lambda(d)}\}.$$

The corresponding *echo family* in the DRPT plate is the set of cells

$$\mathcal{E}_c := \{(u, k) : u \in U_d, k \in E_c\}.$$

By Lemma 7.4, all columns whose indices lie in the same echo class agree entrywise on the unit band.

Echo Classes and Resonant Columns

Column structure in a single $\lambda(d)$ -tile

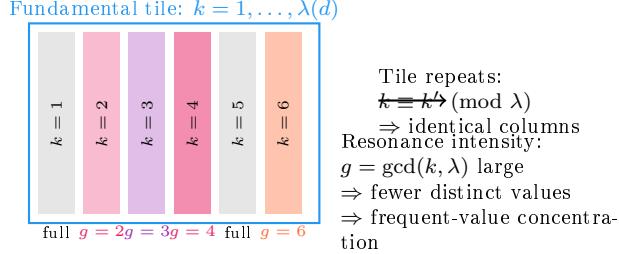


Figure 2: Schematic of echo classes and resonant columns within a single $\lambda(d)$ -tile. Columns k with larger $\gcd(k, \lambda(d))$ exhibit fewer distinct values (higher resonance intensity), concentrating the power map into smaller subgroups. This mechanism produces the “frequent value” patterns visible in Figure 3.

Interpretation. When a plate is drawn over multiple exponent tiles (several multiples of $\lambda(d)$), the unit band is a repeated copy of a single $\lambda(d)$ -tile. Echo classes are the rigorous statement of that repetition.

7.6.2 Resonant Columns (Collapse into Power Subgroups)

The next phenomenon is subtler: within a single $\lambda(d)$ -tile, some columns exhibit noticeably fewer colors/values across the unit band. This is a direct consequence of the power map $u \mapsto u^k$ collapsing U_d onto a proper subgroup when k has a nontrivial common factor with the group exponent.

Proposition 7.6 (Resonance at divisors of $\lambda(d)$). *Let $d = b - 1$. Fix $k \geq 1$ and consider the map*

$$\varphi_k : U_d \rightarrow U_d, \quad \varphi_k(u) = u^k.$$

Then $\varphi_k(U_d)$ is a subgroup of U_d . Moreover, when U_d is cyclic of order $\lambda(d)$ (in particular, when d is an odd prime), the image size satisfies

$$|\varphi_k(U_d)| = \frac{\lambda(d)}{\gcd(k, \lambda(d))}. \quad (6.6)$$

Consequently, exponent indices k for which $\gcd(k, \lambda(d))$ is large produce “resonant columns” containing comparatively few values, and these values are precisely the k -th powers in U_d .

Proof (cyclic case). Assume U_d is cyclic of order $\lambda = \lambda(d)$ and let g be a generator, so every $u \in U_d$ is $u = g^t$ for some t . Then $\varphi_k(g^t) = g^{kt}$. Hence $\varphi_k(U_d) = \langle g^{\gcd(k, \lambda)} \rangle$, which has order $\lambda / \gcd(k, \lambda)$. \square

Remark 7.7 (General composite moduli). For general d , U_d need not be cyclic, but it remains a finite abelian group. The map φ_k is still a group endomorphism; its image is still a subgroup, and resonance persists. The cyclic formula (6.6) is included because it is the cleanest mechanism behind the “frequent value” panels in prime-modulus bases (e.g., bases 8 and 14 in the attached figures).

7.7 Frequent-Value Fields and the Identity Field (Formal Definitions + Exact Explanation)

The visualizations supplied with the DRPT explorer highlight two derived fields:

- an “identity element” field (cells where the value is 1), and
- a “frequent values” field (cells whose values belong to a small set of globally frequent residues within a chosen window).

Both fields admit precise, finite definitions, and both reduce to the order and resonance structure in U_d .

7.7.1 The Identity Field

Definition 7.8 (Identity indicator field). Fix base b , $d = b - 1$, and exponent horizon K . Define the identity indicator

$$\mathbf{1}_{\text{id}}(u, k) := \begin{cases} 1, & R_{b,K}(u, k) = 1, \\ 0, & \text{otherwise,} \end{cases} \quad u \in \mathbb{Z}_d, \quad 1 \leq k \leq K. \quad (6.7)$$

Proposition 7.9 (Cycle completion points). *Let $u \in U_d$ and let $p = \text{ord}_d(u)$. Then*

$$R_{b,K}(u, k) = 1 \iff p \mid k. \quad (6.8)$$

Equivalently, the identity field on a unit row marks precisely the cycle completion points.

Proof. By definition of order, $u^k \equiv 1 \pmod{d}$ if and only if $p \mid k$. \square

Worked example (base 9, i.e., modulus 8). Here $d = 8$ and $U_8 = \{1, 3, 5, 7\}$. Every nontrivial unit has order 2, so (6.8) says the identity field appears exactly at even exponents for rows 3, 5, 7. This is exactly the pattern shown in the “Identity Element (Value 1)” visualization (positions where $\text{base}^{\text{power}} \equiv 1 \pmod{8}$).

7.7.2 The Frequent-Value Field

Definition 7.10 (Value frequency in a DRPT window). Fix base b , modulus $d = b - 1$, and a window $W \subseteq \mathbb{Z}_d \times \{1, \dots, K\}$. For a residue value $v \in \mathbb{Z}_d$, define its frequency in the window by

$$\text{freq}_W(v) := |\{(u, k) \in W : R_{b,K}(u, k) = v\}|. \quad (6.9)$$

A *frequent-value set* of size q (with any declared tie-break rule) is then

$$\text{Top}_q(W) := \{v \in \mathbb{Z}_d : v \text{ is among the } q \text{ largest } \text{freq}_W(v)\}. \quad (6.10)$$

The corresponding *frequent-value indicator field* is

$$\mathbf{1}_{\text{freq}}(u, k) := \begin{cases} 1, & R_{b,K}(u, k) \in \text{Top}_q(W), \\ 0, & \text{otherwise.} \end{cases} \quad (6.11)$$

Remark 7.11 (Protocol discipline). To make “frequent value” plots comparable, the window W , the horizon K , and the choice of excluding/including special values (e.g., whether to exclude the identity 1) must be declared. Otherwise the visualization is not an invariant; it is a data-dependent rendering choice.

7.7.3 Why Small-Order Residues Become Frequent (Prime Modulus Case)

When d is an odd prime p , the unit group U_p is cyclic of order $p - 1 = \lambda(p)$. In that setting, resonant columns force values into small power subgroups, and these subgroup values become frequent because many different bases map into the same small image.

Proposition 7.12 (Uniform preimage multiplicity for cyclic groups). *Let U be cyclic of order λ . Fix $k \geq 1$ and let $g = \gcd(k, \lambda)$. Then:*

1. $|\varphi_k(U)| = \lambda/g$.
2. Every element of $\varphi_k(U)$ has exactly g preimages under φ_k .
3. Elements outside $\varphi_k(U)$ have 0 preimages.

Proof. The kernel of φ_k has size g in a cyclic group of order λ , and $|\text{im}| = |U|/|\ker| = \lambda/g$. Every fiber has size $|\ker| = g$. \square

Interpretation. At resonant exponents k with large $g = \gcd(k, \lambda)$, the image is small and the fiber multiplicity is large. In a column-wise scan, values in the small image appear repeatedly across many rows.

Worked example (base 14, i.e., modulus 13). Here $d = 13$ is prime and $\lambda(13) = 12$. Consider the resonant exponent $k = 4$, for which $\gcd(4, 12) = 4$. By Proposition 7.12 the image has size $12/4 = 3$. Explicitly, the fourth-power subgroup in \mathbb{Z}_{13}^\times is

$$\{u^4 \pmod{13} : 1 \leq u \leq 12\} = \{1, 3, 9\}. \quad (6.12)$$

Thus the column $k = 4$ (and its echo-class companions $k \equiv 4 \pmod{12}$) can only contain the values $\{1, 3, 9\}$ on unit rows; these values therefore recur frequently in aggregate frequency plots. Likewise, $k = 6$ has $\gcd(6, 12) = 6$ and the sixth-power subgroup is

$$\{u^6 \pmod{13}\} = \{1, 12\}. \quad (6.13)$$

This explains, in purely algebraic terms, why the “Frequent Value Patterns” visualization in base 14 prominently highlights residues associated to the order-3 and order-2 substructures (3, 9 and 12).

Worked example (base 8, i.e., modulus 7). Here $d = 7$ is prime and $\lambda(7) = 6$. For $k = 2$, $\gcd(2, 6) = 2$ and the image size is $6/2 = 3$. Explicitly,

$$\{u^2 \pmod{7} : 1 \leq u \leq 6\} = \{1, 2, 4\}. \quad (6.14)$$

Thus the even-exponent columns (squares and their echoes) concentrate values into $\{1, 2, 4\}$, yielding the repeated highlighting visible in the base-8 “Frequent Value Patterns” panel.

7.8 Cross-Base Figure Protocol (Four-Panel Family Overlays)

The cross-base claim of this paper is not “the tables are identical across bases.” The tables are not identical because $d = b - 1$ changes. The claim is:

Cross-base structural portability. *The mechanisms that produce visible DRPT geometry—echo classes, resonant columns, inverse rotations, and attractor boundaries—are invariant under base change when expressed in Rosetta-normalized coordinates and invariants.*

To support this claim visually without ambiguity, the following four-panel figure protocol is recommended.

Figure protocol (family overlay). Choose a finite base set $\mathcal{B} = \{b_1, b_2, b_3, b_4\}$. For each $b \in \mathcal{B}$:

Base 8 (mod 7)													Base 10 (mod 9)																	
u	1	2	3	4	5	6	7	8	9	10	11	12	u	1	2	3	4	5	6	7	8	9	10	11	12					
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1						
2	2	4	1	2	4	1	2	4	1	2	4	1	2	4	1	2	4	8	7	5	1	2	4	8	7	5	1			
3	3	2	6	4	5	1	3	2	6	4	5	1	3	2	6	4	5	1	3	2	6	4	5	1	3	2	6	4	5	1
4	4	2	1	4	2	1	4	2	1	4	2	1	4	2	1	4	2	1	4	7	1	4	7	1	4	7	1	4	7	1
5	5	4	6	2	3	1	5	4	6	2	3	1	5	4	6	2	3	1	5	7	8	4	2	1	5	7	8	4	2	1
6	6	1	6	1	6	1	6	1	6	1	6	1	6	1	6	1	6	1	6	1	8	1	8	1	6	1	8	1	6	1
Base 12 (mod 11)													Base 14 (mod 13)																	
u	1	2	3	4	5	6	7	8	9	10	11	12	u	1	2	3	4	5	6	7	8	9	10	11	12					
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	3	9	5	4	1	3	9	5	4	1	3	9	5	4	1	3	9	5	4	1	3	9	5	4	1	3	9	5	4	1
4	4	5	9	3	1	4	5	9	3	1	4	5	9	3	1	4	5	9	3	1	4	5	9	3	1	4	5	9	3	1
5	5	3	4	9	1	5	3	4	9	1	5	3	4	9	1	5	3	4	9	1	5	3	4	9	1	5	3	4	9	1
9	9	4	3	5	1	9	4	3	5	1	9	4	3	5	1	9	4	3	5	1	9	4	3	5	1	9	4	3	5	1
10	10	1	10	1	10	1	10	1	10	1	10	1	10	1	10	1	10	1	10	1	10	1	10	1	10	1	10	1	10	1

Figure 3: Frequent Value Patterns across four bases. Highlighted cells show values that appear with above-average frequency due to resonance at columns where $\gcd(k, \lambda(d))$ is large. Only unit rows are shown; non-units collapse to the attractor. The specific frequent values vary by base (they are elements of proper subgroups of U_d), but the *mechanism*—concentration at resonant columns—is base-invariant.

1. compute $d = b - 1$, $\lambda(d)$, and a tile-aligned horizon $K = m\lambda(d)$ (e.g., $m = 2$ or $m = 3$);
2. compute the DRPT plate $R_{b,K}(u, k)$ for $u \in \{1, \dots, d\}$ and $1 \leq k \leq K$;
3. plot either:
 - the identity field $\mathbf{1}_{\text{id}}$ (Definition 6.8), or
 - the frequent-value field $\mathbf{1}_{\text{freq}}$ with declared (W, q) (Definitions 6.10–6.11);
4. render using Rosetta coordinates (\hat{u}, \hat{k}) (Section 5.2) so that each base occupies the same unit square.

This protocol ensures that a reader can see, directly, that:

- identity loci correspond to cycle completion points (Proposition 7.9) in every base, and
- resonant columns correspond to large $\gcd(k, \lambda(d))$ collapse (Propositions 7.6 and 7.12) in every base.

Remark 7.13 (Interactive verification tool). An interactive implementation of the DRPT visualization framework is available as a supplementary computational artifact at:

<https://claude.ai/public/artifacts/3aac6f21-8ad0-42ff-82df-52c14d6a42b2>

This “Digital Root Pattern Explorer” allows readers to verify the structural claims of this paper by generating identity-element fields, frequent-value patterns, and echo-class visualizations for arbitrary bases $b \in \{3, \dots, 20\}$. The tool computes $R_b(u, k) = u^k \bmod (b - 1)$ directly and highlights the patterns described in Sections 5–6, enabling independent confirmation that the mechanisms are indeed base-invariant. All computations are performed client-side using the definitions from this paper; no pre-computed tables are used.

8 Reproducibility and Audit Artifacts (AoR Linkage)

NT-5 contains two kinds of statements:

1. **deductive theorems** (Sections 2–6), provable from residue arithmetic, and
2. **reproducible visual/statistical artifacts** (identity/frequent-value fields, cross-base overlays), generated deterministically from the definitions.

Accordingly, the reproducibility contract is:

- The deductive statements are verified by proof.
- The visual/statistical statements are verified by deterministic re-generation from a declared AoR bundle identity and declared generation scripts.

8.1 AoR Anchors and Chain-of-Custody Format

All Marithmetics AoR-bound citations use:

- **AoR tag:** `aor-20260209T040755Z`
- **AoR folder:** `gum/authority_archive/AOR_20260209T040755Z_0fc79a0`
- **Bundle SHA-256:** `c299b1a7a8ef77f25c3ebb326cb73f060b3c7176b6ea9eb402c97273dc3cf66c`

The canonical citation surface for these artifacts is `URL_MAP.md` (in the release repository).

8.2 Deterministic Generation Protocol (Language-Independent)

Algorithm 8.1 (DRPT generation). **Input:** base $b \geq 2$, horizon $K \in \mathbb{N}$.

Output: plate $R_{b,K}(u, k) \in \mathbb{Z}_{b-1}$.

1. Set $d = b - 1$.
2. For each $u \in \{0, 1, \dots, d - 1\}$:
 for each $k \in \{1, 2, \dots, K\}$:
 compute $R_{b,K}(u, k) := \text{pow}(u, k) \bmod d$.
3. (Optional display convention) Map residue class 0 to a uniform symbol “0” in the visualization layer.

Algorithm 8.2 (Carmichael exponent). **Input:** modulus d .

Output: $\lambda(d)$.

1. Factor $d = \prod p_i^{a_i}$.
2. Compute $\lambda(p^a)$ for each prime power (standard number-theory formula).
3. Return $\lambda(d) = \text{lcm}_i \lambda(p_i^{a_i})$.

Algorithm 8.3 (Order map). **Input:** modulus d , unit $u \in U_d$.

Output: $\text{ord}_d(u)$.

1. Compute $\lambda = \lambda(d)$.
2. Factor $\lambda = \prod q_j^{e_j}$.
3. Set $p := \lambda$.

4. For each prime q_j : while p/q_j is an integer and $u^{p/q_j} \equiv 1 \pmod{d}$, set $p := p/q_j$.
5. Return p .

Algorithm 8.4 (Nilpotent hitting time $K_d(u)$). **Input:** modulus $d = \prod p_i^{a_i}$, residue $u \in \mathbb{Z}_d$. **Output:** $K_d(u) \in \mathbb{N} \cup \{\infty\}$.

1. For each $p_i^{a_i}$, compute $t_i = v_{p_i}(u)$ using an integer representative.
2. If any $t_i = 0$, return ∞ (not nilpotent).
3. Else return $K_d(u) = \max_i \lceil a_i/t_i \rceil$.

8.3 Auditor Checklist (Finite, Deterministic)

Given a generated plate $R_{b,K}$, an auditor can certify the main structural claims by the following checks:

- **Unit periodicity:** for each $u \in U_d$, compute $\text{ord}_d(u)$ and verify that $R_{b,K}(u, k + \text{ord}_d(u)) = R_{b,K}(u, k)$ for all k in range.
- **Echo classes:** compute $\lambda(d)$ and verify that for unit rows, columns k and $k + \lambda(d)$ are identical (Lemma 7.4).
- **Resonant collapse:** for prime moduli (or cyclic U_d), verify that the set of values in column k has size $\lambda(d)/\gcd(k, \lambda(d))$ on unit rows (Proposition 7.6).
- **Attractor classification:** compute $K_d(u)$ and verify that $u^k \equiv 0 \pmod{d}$ if and only if $k \geq K_d(u)$ for nilpotent rows (Proposition 4.8).
- **Cross-base overlays:** re-render identity/frequency fields using Rosetta coordinates (\hat{u}, \hat{k}) to ensure that the same mechanism is visible in each base (Section 6.8).

8.4 AoR Linkage to Cross-Base Integrity Demonstrations

While the DRPT theorems are deductive, Marithmetics additionally maintains AoR-bound evidence that base-portable invariants and base-gauge mechanisms behave deterministically under the Rosetta layer.

For readers auditing cross-base integrity in the AoR bundle, the following AoR entries are the canonical starting points:

- **Demo index (all demos, IDs, and metadata):**

https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755z/authority_archive/aor_20260209T040755z_0fc79a0/demo_index.csv	/gum/
	/tables/

- **Base-gauge invariance demonstration (DEMO-64), including stdio/stderr and vendored artifacts:**

https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755z/authority_archive/aor_20260209T040755z_0fc79a0/substrate__demo-64-base-gauge-invariance-integer-selector.out.txt	/gum/
	/logs/
https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755z/authority_archive/aor_20260209T040755z_0fc79a0/substrate__demo-64-base-gauge-invariance-integer-selector.err.txt	/gum/
	/logs/

- **Continuous-lift paradox demonstration (DEMO-65)** as the suite’s AoR anchor for “finite infinity” in the workflow:

```
https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z           /gum/
authority_archive/AOR_20260209T040755Z_0fc79a0          /GUM_BUNDLE_v30_20260209T040755Z   /logs/
infinity__demo-65-continuous-lift-paradox.out.txt

https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z           /gum/
authority_archive/AOR_20260209T040755Z_0fc79a0          /GUM_BUNDLE_v30_20260209T040755Z   /logs/
infinity__demo-65-continuous-lift-paradox.err.txt
```

These artifacts are cited here only to connect NT-5’s cross-base portability intent to the suite’s AoR chain-of-custody. NT-5’s internal claims remain deductive unless explicitly flagged as AoR-derived measurements.

8.5 Reference Implementation for Visual Exploration

A companion interactive tool, “Digital Root Pattern Explorer,” may be used to generate the identity/frequency panels shown in this paper and to explore the echo/resonance structures discussed in Sections 6.6–6.8. For audit purposes, the tool must be treated as a renderer only: the authoritative generator is Algorithm 7.1 (modular exponentiation), and any visualization must be reproducible from the definition independent of UI code.

9 Scope Notes and Relationship to the Suite

9.1 What NT-5 Establishes

NT-5 establishes the DRPT substrate as an explicit finite object and formalizes the three mechanisms that generate its geometry:

- periodicity (unit cycles),
- inversion/rotation redundancy (mirror structure), and
- attractor behavior (non-unit permanence and nilpotent hitting times),

together with the two derived visualization fields (identity and frequency) used to communicate these mechanisms to a reader.

9.2 What NT-5 Does Not Claim

NT-5 does not claim that digital roots are “new physics,” nor that DRPTs alone determine constants. The role of DRPTs is structural:

- DRPTs provide the smallest finite object on which cross-base portability can be made visible.
- DRPTs provide a substrate from which Rosetta-normalized invariants can be extracted without invoking floating-point numerics or continuum approximations.
- DRPTs provide an audit-friendly entry point for readers: the entire plate is computable by modular arithmetic.

9.3 Position Relative to NT-1 Through NT-4

NT-5 is conceptually substrate-level but is placed after NT-4 for a practical reason: the suite's interpretive discipline requires that "infinite extension" be handled as residual law or stabilized tiling (NT-4), not as informal extrapolation. DRPT tiling is the clearest example of this discipline in finite arithmetic, and NT-5 provides the visual vocabulary that supports the later physics-track demos.

9.4 Reading Order Guidance

- Readers seeking the legality discipline that governs all later extraction operators should begin with NT-1 (DOC).
- Readers seeking the finitary measurement and window grammar should read NT-2.
- Readers seeking the invariant and selection discipline should read NT-3.
- Readers seeking the formal contract for "infinity" should read NT-4.
- Readers seeking the substrate geometry and cross-base visual portability should read NT-5.

References

- [1] Hardy, G. H., *Divergent Series*, Oxford University Press, 1949.
- [2] Ireland, K., and Rosen, M., *A Classical Introduction to Modern Number Theory*, 2nd ed., Springer, 1990.
- [3] Niven, I., Zuckerman, H., and Montgomery, H., *An Introduction to the Theory of Numbers*, 5th ed., Wiley, 1991.
- [4] Rosen, K. H., *Elementary Number Theory and Its Applications*, 6th ed., Pearson, 2010.

Marithmetics internal references (Authority / AoR):

- Justin Grieshop, *Deterministic Operator Calculus (DOC)*, master baseline document (ZFC-conservative operator calculus).
- Justin Grieshop, *Authority on DRPTs*, Marithmetics Authority Series (definitions, visual grammar, auditor protocol).
- Justin Grieshop, *Authority on Zero*, Marithmetics Authority Series (zero regimes, idempotents, and attractor logic).
- Justin Grieshop, *Cross-Base Translator (Rosetta Layer)*, Marithmetics Authority Series (base portability and invariant translation).
- URL_MAP.md (AoR citation surface), tag `aor-20260209T040755Z c299b1a7a8ef77f25c3ebb326cb73f060b3c7176b6ea9eb402c97273dc3cf66c`, bundle SHA-256