Deterministic Operator Calculus
ZFC-Internal Framework for Finite Analytic and PDE Arguments

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Abstract

We develop a finite, ZFC-internal operator calculus—the Deterministic Operator Calculus (DOC)—and show how a broad class of analytic and PDE-motivated arguments can be expressed and checked within it using explicit residual bounds. The DOC framework consists of a ZFC-definable finite substrate (residue rings, Fourier–Fejér kernels, convolution operators), together with transfer theorems (UFET/CDUT) that connect finite DOC operators to standard Hilbert-space operators under classical consistency hypotheses. On a formally specified analytic fragment, DOC is a definitional extension of ordinary ZFC analysis and is therefore conservative over ZFC.

We then describe a higher-level modeling layer (SPDE), built on top of DOC, designed to reflect structural features of classical PDE discretizations such as locality, stability, and exact arithmetic. SPDE is not part of the conservativity theorem; rather, it is a DOC-based methodological framework for expressing discrete models and their classical interpretations using explicit finite operators and residuals.

1 Introduction

This paper develops a finite operator calculus, together with a collection of axioms and theorems, that allows analytic and PDE arguments to be expressed and checked entirely using finite objects inside ZFC. The aim is not to replace classical analysis, but to provide a law-governed substrate in which familiar operations—smoothing, truncation, discretization, and lifting—are carried out by explicit operators with explicit residual bounds. Many arguments in analysis and numerical PDEs share a common pattern:

- one smooths sharp quantities by convolving with kernels such as the Fejér kernel, gaining stability and uniform bounds at the cost of a controlled residual;
- one discretizes continuous operators, evolves a finite state (on a grid or spectral basis), and then reconstructs a continuous approximation.

In practice, these steps often rely on a mixture of local operators, global norms, and informal phrases like "the error is small" or "the approximation is good enough". Our goal is to capture the shared structure of these constructions inside a single finite calculus, equipped with clear axioms and transfer theorems, so that both the operators and the error mechanisms are explicit.

1.1 Motivation and guiding questions

The work is motivated by three guiding questions:

1. Unification of discrete operators. Can we describe a broad class of discrete and smoothed operators—convolutions with kernels, projection operators, discrete stencils for PDEs—inside a single finite framework, with clear axioms for legality, positivity, and bounds, rather than treating each method in isolation?

2. A universal finite-to-continuum transfer law. Can we state and prove a universal finite-to-continuum transfer law, with an explicit residual schedule, that covers many familiar arguments in harmonic analysis and numerical PDEs, including smoothing with Fejér-type kernels and discrete approximations of differential operators?

3. A substrate PDE framework with exact interior computation. Can we formalize a substrate PDE framework in which interior computation is exact (no rounding, no hidden tolerances), cost per degree of freedom is independent of analytic error tolerance, and yet the framework remains compatible with classical PDE models through a controlled lift?

From an applied perspective, discrete methods are everywhere: spectral schemes, finite differences, finite volume and finite element methods, multigrid, and adaptive mesh refinement (AMR) are standard tools in analysis and simulation. At the same time, analytic smoothing arguments—convolving with Fejér kernels, applying Cesàro means, or using localized "windows" and "facades"—are central in analytic number theory and harmonic analysis. What is missing is a single, finitistic, axiomatized calculus that captures:

- the algebraic substrate beneath these methods,
- the structure of admissible operators,
- and the interface between the finite world and the continuum.

The Deterministic Operator Calculus is designed to fill this gap.

1.2 Overview of the Deterministic Operator Calculus

DOC is built on three layers:

1. Finite substrate layer. We work in finite residue rings $\mathbb{Z}_d$ (with $d = b - 1$ for a chosen base b), complemented by finite index sets and discrete Fourier transforms. Digital-root power tables (DRPTs) describe the dynamics of residues under powering and distinguish units (which move on finite cycles) from non-units (which collapse into attractors). This layer is governed by a collection of substrate axioms that formalize closure, indexing, cycle bounds, and basic density laws.

2. Operator layer. On top of the substrate, we define a class of deterministic operators: Fejér-type PSD kernels, projection operators, and local stencils. We impose legality axioms describing which kernels and windows are admissible, envelope axioms that provide uniform operator bounds, and normalization axioms that define a base-invariant alphabet of dimensionless invariants (often called Rosetta "hats"). In this layer we prove basic operator theorems about positivity, bandlimit, rank, and "only-zero" cancellation regimes.

3. Transfer and PDE layer. Using these operators, we define a universal finite-to-continuum transfer law (UFET/CDUT) that provides an explicit residual schedule $\eta(h)$ for lifting finite energies and norms to the continuum under standard consistency and coercivity hypotheses. We then build a substrate PDE framework (SPDE) where classical PDE instances are represented as finite operator systems (X_d, L_d, B_d, f_d) acting on residue-valued states. In this layer we prove interior exactness, ε-independent cost, and method-class saturation theorems.

The final piece is a conservativity meta-theorem: we formulate a simple formal theory T representing DOC and its axioms, restrict attention to an analytic fragment $\mathcal{C}$ (statements about norms, energies, and residuals), and prove that T is conservative over ZFC with a concrete model. In other words, DOC is not a replacement for ZFC; it is a reusable subsystem that can be interpreted entirely inside ZFC.

1.3 Scope and non-claims

Because DOC is ambitious in scope, it is important to state explicitly what this paper does not claim:

- We do not claim to solve any major open problem in analysis or PDE theory.

- We do not introduce any new axioms beyond ZFC.
- We do not propose DOC as an alternative foundation for mathematics.
- We do not claim new physical laws or interpretations in this paper.

What we do claim is that:

- the finite substrate, together with the operator and substrate PDE axioms, can express many familiar analytic and numerical constructions;
- the UFET/CDUT transfer theorems provide explicit, reusable bounds connecting finite and continuous quantities;
- the SPDE framework gives a structured way to view PDE discretizations with exact interior arithmetic;
- on a natural analytic fragment, DOC is conservative over ZFC.

Applications of this framework to specific analytic questions, or to particular PDEs, can be pursued in separate work. Here, our focus is on defining the calculus and proving its core theorems.

1.4 Organization of the paper

The paper is organized into two main parts.

- Part I (Finite substrate and operator calculus). In Section 2 we introduce the finite substrate: residue rings $\mathbb{Z}_d$, digital-root power tables, and discrete Fourier transforms on finite index sets. We formulate a set of substrate axioms that capture closure, units vs non-units, cycle bounds, and basic density laws. In Section 3 we build the operator layer: Fejér-type kernels, convolution operators, legality and envelope axioms, and the base-invariant invariant alphabet. In Section 4 we state and prove the UFET/CDUT transfer theorem and derive an explicit residual schedule. In Section 5 we formulate the formal theory T, define the analytic fragment $\mathcal{C}$, and prove the conservativity meta-theorem relating DOC to ZFC.
- Part II (Substrate PDE framework and geometric laws). In Section 6 we introduce the substrate PDE framework (SPDE): we describe how to represent classical PDE instances as finite operator systems on the substrate, state the SPDE axioms, and prove interior exactness and cost theorems. In Section 7 we show how standard geometric and PDE laws—such as area-density laws in two dimensions, surface flux laws in three dimensions, and parabolic local energy inequalities on four-dimensional space–time cylinders—can be expressed and recovered within DOC with explicit finite residuals. Section 8 provides practical checklists for authors and referees using DOC, and the appendices collect the more technical algebraic and model-theoretic proofs.

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2 The finite substrate: rings, digital roots, and Fourier analysis

In this section we introduce the finite algebraic substrate on which DOC is built. The basic objects are residue rings $\mathbb{Z}_d$, digital-root maps, and finite index sets equipped with a discrete Fourier transform. We also state a collection of axioms describing the behavior we require from this substrate. These axioms are all satisfied in the concrete models we work with, and they will serve as the foundation for the operator and PDE constructions that follow.

2.1 Finite rings and index sets

Fix an integer base $b \geq 2$ and set $d = b - 1$. We work in the finite commutative ring $R = \mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$, with addition and multiplication induced from $\mathbb{Z}$. The canonical projection $\pi : \mathbb{Z} \to R, \quad \pi(n) \equiv n \pmod{d}$, maps each integer to its residue class modulo d. We will also use finite index sets of the form $I_N = \mathbb{Z}/N\mathbb{Z}$, for integers $N \geq 1$. A (real- or complex-valued) function on I_N is simply a map $f : I_N \to$

$\mathbb{R}$ or $f : I_N \to \mathbb{C}$. We equip the space of complex-valued functions on I_N with the standard ℓ^2 inner product $\langle f, g\rangle = \sum_{n\in I_N} f(n), \overline{g(n)}$, $\quad |f|_2 = \langle f, f\rangle^{1/2}$. When we combine R and I_N, we will often consider R-valued functions on I_N, $u : I_N \to R$, representing substrate states, and real- or complex-valued functions $f : I_N \to \mathbb{R}$, $f : I_N \to \mathbb{C}$ representing lifted fields.

2.2 Digital roots and digital-root power tables

The base-b digital-root map is

$$\mathrm{dr}_b : \mathbb{Z}_{\geq 0} \to \{0, 1, \ldots, d\}$$

defined by

$$\mathrm{dr}_b(n) = \begin{cases} 0, & n = 0, \\ d, & n \equiv 0 \pmod{d}, \\ 1 + \big((n-1) \bmod d\big), & \text{otherwise.} \end{cases}$$

Equivalently, for $n \geq 1$, $\mathrm{dr}_b(n)$ is the unique element of $\{1, \ldots, d\}$ congruent to n modulo d, and $\mathrm{dr}_b(0) = 0$.

Lemma 2.1 (Digital-root homomorphism)**.** *For all integers $m, n \geq 0$ we have*

$$\mathrm{dr}_b(m+n) \equiv \mathrm{dr}_b(m) + \mathrm{dr}_b(n) \pmod{d}, \qquad \mathrm{dr}_b(mn) \equiv \mathrm{dr}_b(m)\, \mathrm{dr}_b(n) \pmod{d}.$$

In particular, $\mathrm{dr}_b(n) \equiv n \pmod{d}$ *for all* $n \geq 0$*.*

Proof. This follows immediately from the fact that $\mathrm{dr}_b(n)$ is congruent to n modulo d, combined with the compatibility of congruences with addition and multiplication. □ □

The digital-root map induces a simple dynamical system on R under powering. For $x \in R$ and $k \geq 1$, consider the sequence $x, x^2, x^3, \ldots$ in R.

Definition 2.2 (Digital-root power table)**.** *For an integer $n \geq 1$, the digital-root power table (DRPT) of n in base b is the sequence* $\mathrm{DRPT}_b(n) = \big(\mathrm{dr}b(n^k)\big)k \geq 1$*. Equivalently, if we identify* $\mathrm{dr}_b(n^k)$ *with its residue class modulo d, we are tracking the powers of $\pi(n)$ in R:* $\mathrm{DRPT}b(n) = \big(\pi(n^k)\big)k \geq 1$*.*

The DRPT sequence stabilizes into a finite cycle, and its behavior depends on whether $\pi(n)$ is a unit or a non-unit in R.

Definition 2.3 (Units, non-units, cycles, attractors)**.** *Let $R^\times$ denote the multiplicative group of units in R, i.e. those residues $u \in R$ with $\gcd(u, d) = 1$. For $u \in R^\times$, the sequence $(u^k)_{k\geq 1}$ is periodic. We define the cycle length $p(u)$ to be the minimal $p \geq 1$ such that $u^{k+p} = u^k$ for all $k \geq 1$.*

For a general element $a \in R$ (unit or non-unit), Lemma 2.5 shows that there exist integers $m_0 \geq 1$ and $p \geq 1$ such that $a^{m+p} = a^m$ for all $m \geq m_0$. The finite set

$$\{a^m : m \geq m_0\}$$

is called the attractor cycle of a. If this cycle has length 1, its unique element is called a fixed-point attractor.

Thus every residue in R has a finite attractor cycle under powering; units move on cycles in $R^\times$, and non-units may have attractor cycles of length 1 or greater.

Lemma 2.4 (Cycle bound for units)**.** *If $u \in R^\times$, then the cycle length $p(u)$ divides $\varphi(d)$, where φ is Euler's totient function.*

Proof. The multiplicative group $R^\times$ has order $\varphi(d)$. The order of u in this finite group is $p(u)$, and by Lagrange's theorem it must divide the group order. □ □

Lemma 2.5 (Attractor cycles for non-units)**.** *Let R be a finite commutative ring and let $a \in R$. Then there exist integers $m_0 \geq 1$ and $p \geq 1$ such that*

$$a^{m+p} = a^m \quad \textit{for all } m \geq m_0.$$

The finite set $\{a^m : m \geq m_0\}$ is called the attractor cycle of a. If the cycle has length 1, its unique element is a fixed-point attractor.

Proof. Since R is finite, the sequence $(a^m)_{m\geq 1}$ takes only finitely many values, so there exist integers $m_0 < m_1$ with $a^{m_0} = a^{m_1}$. Putting $p := m_1 - m_0 \geq 1$, we have $a^{m+p} = a^m$ for all $m \geq m_0$ by induction, so the tail of the sequence is periodic with period p. The attractor cycle is exactly the finite set of values taken on this periodic tail. If $p = 1$, then a^m is constant for all $m \geq m_0$, and the unique value in the cycle is a fixed-point attractor. □

These lemmas show that, when we view powers of integers through dr_b and π, the behavior is completely finite: unit residues generate finite cycles of bounded length, and every residue has a finite attractor cycle under powering. In the next subsection we will collect the basic properties of this finite ring layer in the form of axioms. These substrate axioms will underlie the operator and PDE constructions in subsequent sections.

2.3 Substrate axioms for the finite ring layer

We now summarize the basic properties of the finite ring layer in the form of axioms. These axioms are all satisfied in the concrete model $R = \mathbb{Z}_d$ with the digital-root map dr_b and projection π, and they will be assumed throughout the rest of the paper. For clarity, we label the axioms A1–A10.

Axiom [A1 (Finite substrate and lift)] For each base $b \geq 2$, there is a finite commutative ring $R = \mathbb{Z}_d, \qquad d = b - 1$, and a projection map $\pi : \mathbb{Z} \to R$. All substrate computations occur in R. There exists a lift functor that regards elements of R as integers in $0, \ldots, d-1 \subset \mathbb{Z}$, and hence as real numbers when needed.

Axiom [A2 (Digital-root homomorphism)] There is a map $\mathrm{dr}_b : \mathbb{Z}_{\geq 0} \to \{0, \ldots, d\}$ such that for all $m, n \geq 0$,

$$\mathrm{dr}_b(m+n) \equiv \mathrm{dr}_b(m) + \mathrm{dr}_b(n) \pmod{d}, \qquad \mathrm{dr}_b(mn) \equiv \mathrm{dr}_b(m)\, \mathrm{dr}_b(n) \pmod{d},$$

and $\mathrm{dr}_b(n) \equiv n \pmod{d}$. In particular, dr_b induces the same residue class as π.

Axiom [A3 (Units vs non-units; cycles vs attractors)] The ring R decomposes as a disjoint union of a multiplicative unit group $R^\times$ and a set of non-units $R \setminus R^\times$.

- For each $u \in R^\times$, the powering sequence $u, u^2, u^3, \ldots$ is periodic with some cycle length $p(u) \geq 1$.
- For each $a \in R$, the powering sequence $(a^m)_{m\geq 1}$ is eventually periodic: there exist integers $m_0 \geq 1$ and $p \geq 1$ such that $a^{m+p} = a^m$ for all $m \geq m_0$. The resulting finite set $\{a^m : m \geq m_0\}$ is called the attractor cycle of a. If the cycle has length 1, its unique element is called a fixed-point attractor.

Axiom [A4 (Power-cycle bound)] There exists a positive integer $\Lambda(d)$ (the exponent of $R^\times$) such that for every unit $u \in R^\times$, the cycle length $p(u)$ divides $\Lambda(d)$. For the concrete model $R = \mathbb{Z}_d$, one may take $\Lambda(d)$ to be any common multiple of the orders of units, for example the Carmichael exponent or $\varphi(d)$.

Axiom [A5 (CRT indexing and factorization)] For any finite collection of pairwise coprime moduli $m_1, \ldots, m_k$, there is an isomorphism $\mathbb{Z}m_1 \cdots m_k \cong \mathbb{Z}m_1 \times \cdots \times \mathbb{Z}m_k$ and substrate index sets can be factored accordingly. Any finite predicate on residues modulo the m_i induces a translation-invariant subset of $\mathbb{Z}m_1 \cdots m_k$. In particular, we may factor substrate masks and classification problems across prime-power components.

Axiom [A6 (Window exactness at primorial levels)] Let $M_y = \prod_{p\leq y} p$ be the primorial up to a prime cutoff y. There exists, for each level y, a substrate mask (projector) that identifies residues coprime to M_y exactly up to the next prime square p_{y+1}^2. That is, up to this range there are no false positives or false negatives in the classification of residues as units modulo M_y.

Axiom [A7 (Density law for units and masks)] For primorial moduli M_y, the density of units in $\mathbb{Z}M_y$ is given by $\theta_y = \frac{\varphi(M_y)}{M_y} = \prod p \leq y\big(1 - \frac{1}{p}\big)$. More generally, the density of residues surviving a finite collection of forbidden residue classes modulo the prime factors of M_y is multiplicative in the prime factors.

Axiom [A8 (Algorithmic finiteness)] All substrate classifications—such as deciding whether an element is a unit, determining its cycle length, or testing membership in a mask—can be decided by finite tables whose size is bounded by a function polynomial in d and the relevant moduli. In particular, all substrate-level decisions can be implemented as finite computations inside ZFC.

Axiom [A9 (Cross-base injectivity)] Let $b_1, \ldots, b_k$ be a finite collection of pairwise coprime bases with corresponding moduli $d_i = b_i - 1$. For any integer n in a bounded range, the tuple of digital-root images $\big(\mathrm{dr}b_1(n), \ldots, \mathrm{dr}b_k(n)\big)$ determines n uniquely via the Chinese Remainder Theorem and base-change isomorphisms. Thus cross-base DRPT signatures are injective on bounded integer ranges.

Axiom [A10 (Inverse cycles)] For each unit $u \in R^\times$, there exists an inverse $u^{-1} \in R^\times$ with $uu^{-1} = 1$. The powering sequence $(u^{-k})_{k\geq 1}$ traces the same finite cycle as (u^k) but in reverse order. Consequently, each unit cycle admits a well-defined inverse cycle, and the set of cycles is closed under inversion.

These axioms are simple formalizations of standard facts about finite residue rings, the digital-root projection, and multiplicative dynamics. They ensure that the substrate layer is fully finite, computable, and structurally well behaved. In the concrete model $R = \mathbb{Z}_d$, all ten axioms hold. In the next subsection we introduce the discrete Fourier transform on finite index sets and the Fejér-type kernels that will serve as basic smoothing operators in DOC.

2.4 Discrete Fourier transform and Fejér kernels

We now recall the discrete Fourier transform on finite index sets and introduce a family of Fejér-type kernels that will act as canonical smoothing operators in DOC.

2.4.1 Discrete Fourier transform

Let $N \geq 1$ and let $I_N = \mathbb{Z}/N\mathbb{Z}$. A complex-valued function on I_N is a map $f : I_N \to \mathbb{C}$.

Definition 2.6 (Discrete Fourier transform)**.** *The discrete Fourier transform (DFT) of f is the function $\widehat{f} : I_N \to \mathbb{C}$ defined by $\widehat{f}(k) = \sum_{n\in I_N} f(n), e^{-2\pi ikn/N}, \qquad k \in I_N$. The inversion formula is $f(n) = \frac{1}{N}\sum_{k\in I_N} \widehat{f}(k), e^{2\pi ikn/N}, \qquad n \in I_N$, and Parseval's identity takes the form $\sum_{n\in I_N} |f(n)|^2 = \frac{1}{N}\sum_{k\in I_N} |\widehat{f}(k)|^2$.*

The DFT extends to vector-valued functions componentwise. It is unitary up to the normalization factor $1/\sqrt{N}$.

Definition 2.7 (Circular convolution)**.** *For functions $f, g : I_N \to \mathbb{C}$, the circular convolution $fg : I_N \to \mathbb{C}$ is defined by $(fg)(n) = \sum_{m\in I_N} f(n-m), g(m), \qquad n \in I_N$, where subtraction is taken modulo N. The Fourier transform of the convolution satisfies $\widehat{f * g}(k) = \widehat{f}(k), \widehat{g}(k), \qquad k \in I_N$.*

Convolution by a fixed kernel g can thus be understood as multiplying by $\widehat{g}(k)$ in the frequency domain.

2.4.2 Fejér-type kernels on I_N

We now describe a class of nonnegative, symmetric convolution kernels with good smoothing properties. Let $N \geq 1$ and let $1 \leq r \leq \lfloor N/2 \rfloor$ be an integer. We view each index $n \in I_N$ via its least absolute value representative in $-\lfloor N/2 \rfloor, \dots, \lfloor N/2 \rfloor$ and write $|n|$ for that representative's absolute value.

Definition 2.8 (Triangular Fejér-type kernel)**.** *The triangular Fejér-type kernel of span r is the function $F_r : I_N \to \mathbb{R}_{\geq 0}$ defined by $F_r(n) = \frac{1}{r}, \max\Big(0, 1 - \frac{|n|}{r}\Big), \qquad n \in I_N$. Equivalently, F_r is supported on the indices with $|n| \leq r - 1$ and has a symmetric triangular profile on that range.*

This kernel has the following elementary properties.

Lemma 2.9 (Basic properties of F_r)**.** *For $1 \leq r \leq \lfloor N/2 \rfloor$, the kernel F_r satisfies:*

1. *$F_r(n) \geq 0$ for all $n \in I_N$.*
2. *$F_r(-n) = F_r(n)$ for all $n \in I_N$.*
3. *$\displaystyle\sum_{n \in I_N} F_r(n) = 1$.*

Sketch of proof. Nonnegativity and symmetry follow directly from the definition. For the sum, note that only the indices with $|n| \leq r - 1$ contribute. Writing the sum as a central value at $n = 0$ plus pairs $(\pm n)$ for $1 \leq n \leq r - 1$, one finds $\sum_{n \in I_N} F_r(n) = \frac{1}{r}\Big(1 + 2\sum_{n=1}^{r-1}\big(1 - \frac{n}{r}\big)\Big) = \frac{1}{r}\Big(1 + 2 \cdot \frac{r-1}{2}\Big) = 1$. □ □

Thus F_r is a nonnegative, symmetric, unit-mass kernel. Its Fourier transform has a simple closed form.

Proposition 2.10 (Fourier symbol of F_r)**.** *Let $1 \leq r \leq \lfloor N/2 \rfloor$ and $k \in I_N := \{0, \dots, N-1\}$. Then*

$$\widehat{F_r}(k) = \sum_{n \in I_N} F_r(n)\, e^{-2\pi i k n/N} = \begin{cases} \left(\dfrac{\sin(\pi r k/N)}{r\,\sin(\pi k/N)}\right)^2, & k \not\equiv 0 \pmod{N}, \\ 1, & k \equiv 0 \pmod{N}. \end{cases}$$

In particular, $\widehat{F_r}(k) \geq 0$ and $\widehat{F_r}(k) \leq 1$ for all $k \in I_N$.

Sketch of proof. Let $g : I_N \to 0, 1$ be the boxcar function $g(n) = 1$ for $n \in 0, \dots, r-1$ and $g(n) = 0$ otherwise. One checks that $F_r(n) = \frac{1}{r^2}\sum_{m \in I_N} g(m), g(n+m)$, so $\widehat{F_r}(k) = r^{-2}|\widehat{g}(k)|^2$. A geometric-series computation gives $\widehat{g}(k) = \sum_{n=0}^{r-1} e^{-2\pi i k n/N} = e^{-\pi i k (r-1)/N}, \frac{\sin(\pi r k/N)}{\sin(\pi k/N)}$ for $k \not\equiv 0$, and $\widehat{g}(0) = r$. Dividing by r^2 yields the claimed expression. Nonnegativity follows from the squared-modulus form, and $|\widehat{g}(k)| \leq r$ implies $\widehat{F_r}(k) \leq 1$. □ □

Remark 2.11. *The kernel F_r is thus a positive semi-definite, self-adjoint convolution kernel whose Fourier multipliers lie in $[0, 1]$. Convolution with F_r defines a smoothing operator on $\ell^2(I_N)$ that preserves the mean and contracts the ℓ^2 norm. These Fejér-type kernels will serve as the basic smoothing operators in DOC, and many of our operator axioms and transfer theorems will be stated in terms of such kernels and their mixtures.*

In the next section we move from the substrate layer to the operator layer: we define the class of DOC operators, state axioms for their legality and envelopes, and introduce a normalized, base-invariant alphabet of substrate invariants.

3 DOC operators, legality, and envelopes

We now move from the substrate layer (finite rings, digital roots, and discrete Fourier analysis) to the operator layer. In the Deterministic Operator Calculus (DOC), the basic objects are deterministic linear (and later, nonlinear) operators acting on finite functions over the index sets I_N, built from kernels such as the Fejér-type kernels introduced above. The purpose of this section is threefold:

1. to define the class of linear DOC operators we will work with;
2. to state axioms describing when such operators and their associated "windows" are legal;
3. to state envelope axioms that provide uniform bounds on these operators.

The substrate axioms (A1–A10) ensure that the underlying ring and indexing structure behave well. The operator axioms introduced here will ensure that the DOC operators we consider are well behaved in terms of positivity, normalization, and boundedness.

3.1 Linear DOC operators

We begin with the linear case. Let $N \geq 1$ and $I_N = \mathbb{Z}/N\mathbb{Z}$. A linear operator $T : \ell^2(I_N) \to \ell^2(I_N)$ can be represented by a kernel $K_T : I_N \times I_N \to \mathbb{C}$ via $(Tf)(n) = \sum_{m \in I_N} K_T(n,m), f(m)$. In DOC, we are particularly interested in translation-invariant operators of convolution type.

Definition 3.1 (DOC convolution operators)**.** *A linear operator $T : \ell^2(I_N) \to \ell^2(I_N)$ is called a DOC convolution operator if there exists a kernel $k : I_N \to \mathbb{C}$ such that $(Tf)(n) = \sum_{m \in I_N} f(n-m), k(m) = (f * k)(n)$, for all $f : I_N \to \mathbb{C}$ and all $n \in I_N$, where $*$ denotes circular convolution.*

When k is real-valued and satisfies the symmetry and normalization properties of the Fejér-type kernels F_r, the associated operator T will be our canonical notion of a smoothing operator. We will sometimes write T_k for the convolution operator induced by kernel k. The Fejér-type kernels from Section 2.4 provide natural examples.

Example 3.2 (Fejér smoothing operators)**.** *For each choice of N and $1 \leq r \leq \lfloor N/2 \rfloor$, the Fejér-type kernel $F_r : I_N \to \mathbb{R}_{\geq 0}$ defines a convolution operator $(S_r f)(n) = (f * F_r)(n)$. By Lemma 2.9 and Proposition 2.10, S_r is self-adjoint, positive semi-definite, preserves the mean, and has operator norm at most 1 on $\ell^2(I_N)$.*

More generally, DOC allows mixtures of such kernels, projectors onto subspaces, and local stencils corresponding to discrete PDE operators. All such operators will be subject to legality and envelope axioms defined below.

3.2 Operator axioms: legality and windows

To keep the operator calculus under control, we impose a small set of axioms on DOC operators and the "windows" (weight functions) they are allowed to use. These axioms formalize the idea that smoothing and truncation should be performed by positive, normalized kernels, and that certain exact cancellation patterns are not allowed unless they arise from trivial symmetries. We denote these operator-level axioms by O1–O4.

Axiom [O1 (Positive semi-definiteness)] A linear DOC operator $T : \ell^2(I_N) \to \ell^2(I_N)$ is admissible only if it is self-adjoint and positive semi-definite (PSD). Equivalently, for all $f \in \ell^2(I_N)$, $\langle Tf, f \rangle \geq 0, \qquad \langle Tf, g \rangle = \langle f, Tg \rangle$. For convolution operators T_k, this means the Fourier multipliers $\widehat{k}(k)$ are real and nonnegative for all frequencies.

Axiom [O2 (Unit-mass smoothing kernels)] Admissible smoothing kernels $k : I_N \to \mathbb{R}{\geq}\, 0$ (including Fejér-type kernels and their mixtures) must satisfy:

1. nonnegativity: $k(n) \geq 0$ for all $n \in I_N$;
2. symmetry: $k(-n) = k(n)$ for all $n \in I_N$;
3. normalization: $\sum n \in I_N k(n) = 1$.

This ensures that smoothing operators preserve the global mean and do not introduce bias.

Axiom [O3 (Legal windows and weight functions)] A window or weight function $w : I_N \to \mathbb{R}{\geq}\, 0$ used in sums of the form $\sum n \in I_N f(n), w(n)$ is called legal if it can be expressed as either:

- a nonnegative, unit-mass Fejér-type kernel (or a finite convex combination of such kernels), or
- a sharp indicator of a contiguous interval (or finite union of such intervals) of length at most N, optionally normalized by its length.

Windows outside this class (for example, highly oscillatory or sign-changing weights designed to force exact cancellation) are not admitted in DOC. Intuitively, O3 restricts us to the kinds of windows that arise naturally in smoothing arguments (Fejér) and in localized counting (interval indicators), but excludes artificial constructions designed to produce fragile exact cancellations.

Axiom [O4 (Only-zero cancellation and sharp +1 floor)] Let F_r be a Fejér-type smoothing kernel as in O2–O3, with nonnegative weights and unit mass, and let 1_I denote the sharp indicator of an interval $I \subset I_N$ of length $|I| = r$.

(1) (Only-zero cancellation.) If W is an admissible DOC kernel whose Fourier symbol $\widehat{W}(k)$ satisfies $0 \le \widehat{W}(k) \le 1$ for all k and $\widehat{W}(k_0) = 0$ for some $k_0 \neq 0$, then any DOC-smooth field whose Fourier support is contained in $\{k_0\}$ must vanish identically. In other words, the only mean-free field that is annihilated by an admissible smoothing kernel at a single nonzero frequency is the zero field.

(2) (Sharp +1 floor.) When using a sharp interval window $w = \mathbf{1}_I$ (or a normalized version) of length $|I| = r$, there is an irreducible integer floor of size 1 in the discrepancy between the sharp sum and its smoothed analogue for bounded integer-valued data. More precisely:

   (a) For general bounded $f : I_N \to \mathbb{R}$,

$$\left| \sum_{n \in I_N} f(n)\big(\mathbf{1}_I(n) - w_{\text{smooth}}(n)\big) \right| \le C \, \|f\|_\infty$$

   for some absolute constant $C > 0$, whenever w_{smooth} is a Fejér-type smoothing window of span r as in O2–O3.

   (b) In the core counting applications of DOC (for example when f takes values in $\{0, 1\}$ or $\{-1, 0, 1\}$), this implies a sharp ± 1 bound

$$\left| \sum_{n \in I_N} f(n)\, \mathbf{1}_I(n) - \sum_{n \in I_N} f(n)\, w_{\text{smooth}}(n) \right| \le 1.$$

   This reflects the fact that passing from smooth to sharp intervals in a discrete setting inevitably incurs at least one unit of error at the boundary when counting with integer-valued test functions.

Axiom O4 encodes two structural facts: that only very symmetric windows can produce exact cancellation for broad classes of mean-zero functions, and that sharp counting always carries an irreducible discrete error term when approximated by smooth weights.

These operator axioms ensure that DOC operators are well behaved: smoothing operators are PSD, symmetric, and mean-preserving; windows are drawn from a controlled class; and sharp counts carry an explicit, minimal error when compared to their smoothed counterparts. In the next subsection we introduce envelope axioms that provide uniform bounds on families of DOC operators, and we define a normalized, base-invariant alphabet of invariants (the "hats") that will allow us to express operator quantities in a dimensionless, portable way.

3.3 Envelope axioms and operator bounds

To make the transfer theorems in later sections quantitative, we need uniform bounds on the size of DOC operators. These bounds will be expressed in terms of a small collection of scalar parameters, which we call envelopes. The idea is that once an operator (or a family of operators) is certified to lie under a given envelope, we can invoke generic suppression and residual estimates without re-deriving them from scratch. We denote the envelope-level axioms by E1–E3.

Axiom [E1 (Spectral envelope for smoothing kernels)] There exists a constant $\kappa_{\text{smooth}} \in (0, \infty)$, depending only on the choice of Fejér-type kernels and their mixtures, such that for every admissible smoothing kernel k on I_N, the associated convolution operator T_k satisfies $|\widehat{k}|\infty \le 1$, $|T_k|\ell^2 \to \ell^2 \le 1, \qquad |T_k|\ell^1 \to \ell^1 \le \kappa$smooth. In particular, the ℓ^2-operator norm of every admissible smoothing operator is at most 1, and the ℓ^1-operator norm is uniformly bounded by κ_{smooth}.

Axiom [E2 (Envelope for local stencil operators)] For each admissible discrete differential operator (or local stencil) $L : \ell^2(I_N) \to \ell^2(I_N)$, there exists an envelope constant $\kappa_L \in (0, \infty)$ such that $|L|\ell^2 \to \ell^2 \le \kappa_L, \qquad |L|\ell^1 \to \ell^1 \le \kappa_L$. Moreover, for any finite family of such stencils L_j, there exists a universal envelope κ_* (depending on the family but not on N) such that $\kappa_L \le \kappa_*$ for all L in the family. This axiom reflects the fact that local stencils acting on finite grids have bounded operator norms that do not grow with the grid size, as long as the stencil itself is fixed.

Axiom [E3 (Uniform envelope for DOC operator families)] Given any finite, admissible family $\mathcal{T}$ of DOC operators (smoothing operators, projectors, and local stencils), there exists a constant $\kappa_{\mathcal{T}} \in (0, \infty)$ such that for all $T \in \mathcal{T}$, $|T|\ell^2 \to \ell^2 \le \kappa\mathcal{T}, \qquad |T|\ell^1 \to \ell^1 \le \kappa\mathcal{T}$. In addition, the adjoints and finite compositions of operators from $\mathcal{T}$ that remain admissible are also bounded by a (possibly larger) envelope constant depending only on $\mathcal{T}$.

The precise values of the envelope constants are not important; what matters is that they are finite and depend only on the operator family, not on the grid size or the particular function being acted on. In concrete settings, one can bound these constants in terms of the underlying kernels and stencils; in the abstract calculus, E1–E3 package those bounds into a simple set of assumptions.

3.4 Normalized invariants and Rosetta "hats"

To compare operators and masks across different bases and scales, it is useful to introduce a small alphabet of normalized invariants. These invariants are dimensionless quantities that remain stable under changes of base, grid size, and similar choices. We will denote them with a "hat", and collectively they form what we call the Rosetta alphabet. At a high level:

- $\hat{\chi}$ will encode normalized cycle information from the DRPT layer (how a unit's cycle length compares to the maximal cycle length);
- $\hat{\theta}$ will encode normalized densities for masks (e.g. survivor densities modulo primorials, scaled by a reference);
- $\hat{\Psi}$ will encode normalized circulation or smoothing weights on unit cycles;
- $\hat{\kappa}$ will encode normalized envelope constants.

We introduce these at the level of definitions here; their base-invariance and formal properties will be developed in more detail in later sections.

Definition 3.3 (Cycle invariant $\hat{\chi}$)**.** *Let $\Lambda(d)$ be the exponent from Axiom A4. For a unit $u \in R^\times$ with cycle length $p(u)$, define $\hat{\chi}(u) = \frac{p(u)}{\Lambda(d)} \in (0, 1]$. We regard $\hat{\chi}(u)$ as a normalized cycle invariant attached to u. For non-units, we set $\hat{\chi}(a) = 0$.*

Definition 3.4 (Density invariant $\hat{\theta}$)**.** *Let θ_y be the density of units modulo the primorial M_y as in A7, and let θ^{ref} be a fixed reference density (for example, the density at some reference level). The normalized density invariant is $\hat{\theta}_y = \frac{\theta_y}{\theta^{\text{ref}}}$. More generally, for any admissible mask $S \subseteq \mathbb{Z}_{M_y}$ with density $\theta(S)$, we define $\hat{\theta}(S) = \theta(S)/\theta^{\text{ref}}$.*

Definition 3.5 (Circulation invariant $\hat{\Psi}$)**.** *Given a Fejér-type kernel F_r or a mixture thereof, we can consider its restriction to a cycle (for example, a unit cycle in the DRPT layer) and define a circulation quantity—such as the average of the kernel over the cycle or a normalized sum of its Fourier multipliers on that cycle. We denote any such normalized, dimensionless circulation quantity by $\hat{\Psi}$. Precise choices will be made later, but the key point is that $\hat{\Psi}$ is constructed to be invariant under scaling of the grid and changes of base.*

Definition 3.6 (Envelope invariant $\hat{\kappa}$)**.** *For a given operator family $\mathcal{T}$ with envelope constant $\kappa_{\mathcal{T}}$ (as in E3) and a fixed reference envelope $\kappa^{\mathrm{ref}} > 0$, we define the normalized envelope invariant $\hat{\kappa}(\mathcal{T}) = \frac{\kappa_{\mathcal{T}}}{\kappa^{\mathrm{ref}}}$.*

In later sections, we will show that rational expressions in these invariants (the "Rosetta alphabet" $\hat{\chi}, \hat{\theta}, \hat{\Psi}, \hat{\kappa}$) are insensitive to the choice of base and grid size, as long as the underlying operator families remain admissible. This base-invariance is one of the reasons DOC can be viewed as a portable calculus: the same dimensionless expressions describe operator behavior across different substrate realizations. For now, it is enough to note that DOC operators come equipped with:

- legality properties (O1–O3), ensuring positivity, normalization, and admissible window shapes;
- envelope bounds (E1–E3), providing uniform operator norms;
- and a small alphabet of normalized invariants that will be used to express transfer and suppression laws.

In the next section, we use these ingredients to state and prove a universal finite-to-continuum transfer theorem (UFET/CDUT) that quantifies how DOC operators approximate their continuum analogues.

4 Finite–to–continuum transfer (UFET/CDUT)

We now turn to the problem of relating finite DOC operators to their continuum analogues. In many applications, one starts from a continuous operator acting on functions on a domain (e.g. a differential operator), discretizes it to obtain a finite operator on a grid, and then uses the discrete solution to approximate the continuous one. The purpose of this section is to place this process in a general framework and to state a universal finite–to–continuum transfer theorem with an explicit residual schedule. We refer to this transfer theorem as UFET (Universal Finite–to–Exact Transfer), and to its specialization to DOC operators as CDUT (Concrete DOC–UFET Transfer).

4.1 Discrete models, reconstructions, and energies

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let X be a Banach space of functions on Ω (for example, $L^2(\Omega)$ or a Sobolev space $H^1(\Omega)$). Suppose we have:

- a family of finite index sets I_h (e.g. grid points) indexed by a mesh parameter $h > 0$;
- a sampling or projection operator $\Pi_h : X \to \mathbb{R}^{I_h}$ that approximates a function $u \in X$ by a finite collection of values (e.g. pointwise samples or cell averages);
- a reconstruction operator $R_h : \mathbb{R}^{I_h} \to X$ that lifts a finite state back to a function on Ω.

In DOC, the finite states will typically live in a ring-valued or real-valued space (for example, $\mathbb{R}^{I_h}$ or R^{I_h}), but for the transfer theorem we can treat them as $\mathbb{R}^{I_h}$ without loss of generality, since the ring elements can be embedded into $\mathbb{R}$ via the lift. We will also consider a family of finite energies E_h approximating a continuous energy E:

- $E : X \to [0, \infty]$, a continuous energy functional on X;
- $E_h : \mathbb{R}^{I_h} \to [0, \infty]$, a finite energy defined on discrete states.

Typical examples include:

- for the heat equation, $E(u) = \int_\Omega |\nabla u|^2, dx$ and E_h the discrete Dirichlet energy of the grid function;
- for a smoothing operator, $E(u) = |Tu - f|_2^2$ and E_h the corresponding discrete least-squares functional.

The transfer problem is: under what conditions can we say that minimizers or near-minimizers of E_h (when reconstructed via R_h) approximate minimizers of E, with an error that can be bounded in terms of a small parameter? Analogous questions are treated in the theory of Γ-convergence. Here, we adopt a slightly more concrete approach tailored to the DOC setting, but conceptually similar.

4.2 UFET: abstract transfer theorem

We first state UFET in a general form, without committing to a particular choice of DOC operators. Let X be a Banach space and $I_h h > 0$ a family of finite index sets with sampling maps Π_h and reconstructions R_h as above. Assume:

- (T1) Consistency. There exists a function $\eta_1(h) \to 0$ as $h \to 0$ such that for all $u \in \mathcal{D} \subset X$ (a dense domain), $|R_h \Pi_h u - u|X \leq \eta_1(h), \Phi(u)$, for some control functional $\Phi(u)$ depending on u but not on h.
- (T2) Stability. There exists $C\text{stab} > 0$ such that for all discrete states $v_h \in \mathbb{R}^{I_h}$, $|R_h v_h|X \leq C\text{stab}, |v_h|\ell^2(I_h)$.
- (T3) Energy convergence. The energies E_h approximate E in the sense that there exists a function $\eta_2(h) \to 0$ such that for all $u \in \mathcal{D}$, $|E_h(\Pi_h u) - E(u)| \leq \eta_2(h), \Psi(u)$, for some control functional $\Psi(u)$.

Under these assumptions, one can prove a standard type of transfer result: discrete minimizers converge (up to subsequences) to continuous minimizers, and the energy values converge with an explicit error depending on η_1 and η_2. Rather than reproducing the full Γ-convergence formalism here, we encapsulate the conclusion in a single theorem tailored for our purposes.

Theorem 4.1 (UFET: Universal Finite–to–Exact Transfer)**.** *Suppose (T1)–(T3) hold and that E is coercive and lower semi-continuous on X. Let $u \in X$ be a minimizer of E, and let $v_h \in \mathbb{R}^{I_h}$ be (approximate) minimizers of E_h in the sense that $E_h(v_h) \leq \inf_{w_h} E_h(w_h) + \delta_h$, with $\delta_h \to 0$. Then there exists a function $\eta(h) = \eta_1(h) + \eta_2(h)$ (up to multiplicative constants depending on Φ, Ψ) such that, along a subsequence if necessary, $|R_h v_h - u|_X \to 0 \quad$ as $h \to 0$, and $|E_h(v_h) - E(u)| \leq C, \eta(h)$ for some constant $C > 0$ independent of h.*

Informal proof idea. The consistency and energy convergence assumptions imply that the discrete energies $\tilde{E}_h(u) := E_h(\Pi_h u)$ Γ-converge to $E(u)$ as $h \to 0$, with rate controlled by η_1 and η_2. Coercivity and stability ensure precompactness of minimizing sequences, and the lower semi-continuity of E identifies the limit as a minimizer. The explicit bounds then follow from the control provided by η_1 and η_2. □ □

The main point of UFET is that the error between discrete and continuous objects is capped by an explicit residual $\eta(h)$, which depends on the sampling/reconstruction scheme and the way the discrete energy approximates the continuous one. In DOC, we will specialize this theorem to the case where:

- sampling and reconstruction are implemented via Fejér-type smoothing and DOC operators;
- the energies are built from DOC operators (e.g. discrete Dirichlet energies, residual norms).

This specialization is what we call CDUT.

4.3 CDUT: transfer for DOC operators

We now describe the DOC-specific version of UFET. Here, the discretization parameter h can be thought of as:

- a mesh size in physical space,
- and/or the inverse of a grid cardinality N,

- and/or the size of a smoothing window $r(h)$.

For concreteness, consider a one-dimensional periodic domain of length 1 and grids with N equally spaced points (so $h = 1/N$), and let $I_N = \mathbb{Z}/N\mathbb{Z}$. The DOC operators we use are:

- smoothing operators $S_{r(h)}$ defined by convolution with Fejér-type kernels $F_{r(h)}$;
- discrete differential operators L_h (e.g. discrete Laplacians) defined by local stencils;
- projectors built from these pieces.

We choose:

- $\Pi_h u$ as the sampled values of u at grid points, possibly pre-smoothed by a Fejér kernel;
- $R_h v_h$ as the reconstruction of a grid function v_h by piecewise-polynomial interpolation or an inverse DFT, possibly followed by a Fejér smoothing.

Under reasonable assumptions on the smoothness of u and the order of the stencil, standard numerical analysis gives estimates of the form $|R_h \Pi_h u - u|_{L^2} \le c_1 h^p, \qquad |E_h(\Pi_h u) - E(u)| \le c_2 \Big(h^p + \frac{r(h)}{N} \Big)$, where:

- p is the order of accuracy of the discretization;
- $r(h)$ is the span of the Fejér kernel in grid units;
- the term $r(h)/N$ reflects the residual coming from smoothing over a finite window on a finite grid.

DOC packages these standard estimates into a single residual schedule.

Definition 4.2 (DOC residual schedule)**.** *For a family of DOC operators discretizing a given continuous operator (or energy), we define the DOC residual schedule by* $\eta(h) = c_1 h^p + c_2 \frac{r(h)}{N(h)}$*, where:*

- *h is a mesh parameter;*
- *p is the discretization order;*
- *$r(h)$ is the Fejér span used in the smoothing step;*
- *$N(h)$ is the grid size;*
- *$c_1, c_2 > 0$ are constants depending on the operator family but not on h.*

In many cases, we choose $r(h)$ as a function of $N(h)$ to optimize the tradeoff between discretization error and smoothing error. For instance, taking $r(h) \sim \frac{\sqrt{N(h)}}{\log N(h)}$ keeps $r(h)/N(h)$ small while providing effective smoothing at scale $r(h)$. The following theorem is the DOC specialization of UFET.

Theorem 4.3 (CDUT: Concrete DOC–UFET Transfer)**.** *Let E be a coercive, lower semi-continuous energy on X, and let E_h be DOC energies constructed from admissible operators (in the sense of O1–O4 and E1–E3). Suppose that:*

- *the sampling/reconstruction operators Π_h, R_h are built from admissible DOC operators and satisfy consistency and stability estimates with errors bounded by $c_1 h^p$;*
- *the DOC energies E_h approximate E with errors bounded by $c_2(h^p + r(h)/N(h))$.*

Then the conclusions of UFET (Theorem 4.1) hold with residual schedule $\eta(h) = c_1 h^p + c_2 \frac{r(h)}{N(h)}$. In particular, discrete minimizers (or near-minimizers) of E_h, when lifted via DOC reconstructions, converge to minimizers of E in X, and the energy error is controlled by $\eta(h)$ up to a constant factor.

Informal proof sketch. The DOC assumptions ensure that the operators used in Π_h, R_h, and E_h satisfy the hypotheses of UFET: they are bounded with respect to the envelope constants, positive semi-definite where needed, and their discrete actions approximate the continuous operator in a controlled way. The consistency and energy approximation bounds feed directly into $\eta_1(h)$ and $\eta_2(h)$ in Theorem 4.1, yielding the DOC residual schedule. □ □

4.4 Interpretation and role in DOC

The UFET/CDUT transfer theorem plays a central role in DOC:

- it provides a single, reusable bound on how far a finite DOC computation can stray from its continuous target;
- it depends only on the mesh scale h, the discretization order p, and the Fejér span $r(h)$, all of which are under the user's control;
- it links the finite substrate (where all computations are exact) to the continuum space X, where analytic statements are made.

In subsequent sections, we will use CDUT to:

- justify the use of DOC operators as surrogates for continuous operators in PDE settings;
- define acceptance sets for substrate PDE solutions based on continuous norms and residuals;
- and establish that the substrate PDE framework (SPDE) faithfully captures the behavior of classical discretizations at the level of energies and norms, with explicit residuals $\eta(h)$.

In the next section, we turn to the formalization of the DOC theory itself and to the conservativity meta-theorem that shows DOC is a ZFC-internal calculus rather than a new foundation.

5 Formal theory and conservativity over ZFC

Up to this point we have worked at the level of concrete finite models: residue rings, grids, Fejér kernels, and discrete operators defined on $\ell^2(I_N)$. In this section we briefly formalize the Deterministic Operator Calculus as a theory T, identify a natural fragment of statements $\mathcal{C}$ that we care about (the "analytic fragment"), and state a conservativity meta-theorem: On $\mathcal{C}$, the theory T is conservative over ZFC with a concrete model built from the substrate and DOC operators. This means that DOC is not a new foundation: it is a structured, reusable way to talk about finite operators and their continuous limits, and any theorem it proves (in $\mathcal{C}$) is already a theorem of ordinary set-theoretic mathematics.

5.1 The theory T (DOC as a formal system)

We do not need a fully formal syntactic presentation of T for our purposes; it is enough to describe its basic ingredients. The theory T has symbols for:

- the finite substrate: a family of rings $R_b = \mathbb{Z}b - 1$, digital-root maps dr$b$, projections π_b, and index sets I_N;
- the DRPT dynamics: predicates expressing "$x \in R_b^\times$ is a unit", "cycle length $p(x)$", "x is a non-unit attractor";
- DOC operators: symbols for admissible kernels k, convolution operators T_k, local stencils L, and projectors;
- discrete Fourier transforms on I_N;
- envelope constants κsmooth$, \kappa_L, \kappa\mathcal{T}$, and normalized invariants $\hat{\chi}, \hat{\theta}, \hat{\Psi}, \hat{\kappa}$;
- sampling and reconstruction operators Π_h, R_h;
- substrate PDE objects (X_d, L_d, B_d, f_d) and acceptance predicates based on energies and norms.

The non-logical axioms of T are exactly the substrate axioms A1–A10, the operator axioms O1–O4, the envelope axioms E1–E3, and the SPDE axioms that will be introduced in the next section. In addition, T includes standard axioms for real numbers and norms sufficient to talk about ℓ^2-spaces and Banach spaces on the lifted side. Intuitively, T is just "ZFC plus a list of named structures and properties" that codify the finite substrate and operator layer. Because all of these structures are finite or built from finite pieces, they can be interpreted inside ZFC in a straightforward way.

5.2 The analytic fragment $\mathcal{C}$

The full language of T allows us to talk about finite rings, DRPT cycles, operator kernels, and so on. In practice, the conclusions we care about have a specific form:

- inequalities between norms of functions and their images under DOC operators;
- bounds on energies $E_h(v_h)$ vs $E(u)$;
- statements of the form "if a discrete state lies in a certain acceptance set, then its reconstruction has small residual".

We package these kinds of statements into a fragment $\mathcal{C}$ of the language of T.

Definition 5.1 (Analytic fragment $\mathcal{C}$)**.** *The analytic fragment $\mathcal{C}$ consists of all sentences of T that:*

- *quantify only over finite index sets I_N, substrate rings R, finite functions v_h, and elements of Banach spaces X;*
- *express relationships between:*
  - *norms $|R_h v_h|_X$, $|R_h \Pi_h u - u|X$;*
  - *energies $E_h(v_h)$, $E(u)$;*
  - *operator norms $|T|\ell^2 \to \ell^2$;*
  - *normalized invariants $\hat{\chi}, \hat{\theta}, \hat{\Psi}, \hat{\kappa}$;*
- *are built from equalities and inequalities involving real-valued expressions in these quantities.*

In other words, $\mathcal{C}$ captures exactly the sort of numerical and functional-analytic conclusions we state in DOC: "this operator norm is bounded by κ", "this energy converges with residual $\eta(h)$", "this discrete minimizer approximates a continuous minimizer up to error ε", and so on.

We do not include arbitrary set-theoretic statements, nor do we attempt to encode the entire substrate structure in $\mathcal{C}$. Instead, $\mathcal{C}$ focuses on the observable, analytic side of DOC.

5.3 Concrete models inside ZFC

To speak about conservativity, we need an interpretation of T inside ZFC. Given a base b, a ring $R = \mathbb{Z}_{b-1}$, and a family of grids I_N, we can build a concrete model $\mathcal{M}$ as follows:

- the ring symbols of T are interpreted as the actual finite rings $\mathbb{Z}_d$;
- the digital-root maps dr_b are interpreted as the concrete functions defined in Section 2;
- the DRPT predicates (cycle length, attractor status) are interpreted via actual powering in $\mathbb{Z}_d$;
- DOC kernels and operators are interpreted as actual functions $k : I_N \to \mathbb{R}$ and linear maps $T_k : \ell^2(I_N) \to \ell^2(I_N)$;
- envelope constants are interpreted as concrete real numbers bounding the operator norms;
- sampling and reconstruction operators Π_h and R_h are defined in terms of actual sampling (e.g. pointwise evaluation) and reconstruction (e.g. interpolation or inverse DFT);
- energies E_h and E are defined as actual functionals, e.g. discrete Dirichlet energies and continuous Dirichlet integrals.

All of these objects are definable in ordinary set theory. The axioms A1–A10, O1–O4, E1–E3, and the transfer statements from Section 4, when interpreted in $\mathcal{M}$, are simply theorems of standard mathematics (elementary number theory, finite group theory, Fourier analysis, numerical analysis, and functional analysis). Thus, $\mathcal{M}$ is a ZFC model of the theory T: every axiom of T holds in $\mathcal{M}$, and every symbol of T has a concrete interpretation inside ZFC.

5.4 Conservativity meta-theorem

We can now state the main model-theoretic claim about DOC: it is conservative over ZFC on the analytic fragment $\mathcal{C}$. Roughly, this means:

- If T proves a statement $\varphi \in \mathcal{C}$, then ZFC proves that φ holds in the concrete model $\mathcal{M}$.
- Conversely, if ZFC proves that a concrete analytic inequality or convergence statement holds for the structures described in Section 2–4, then there is a corresponding sentence $\varphi \in \mathcal{C}$ that T proves.

Formally:

Theorem 5.2 (Conservativity of DOC on $\mathcal{C}$)**.** *Let T be the theory consisting of:*

- *the substrate axioms A1–A10;*
- *the operator axioms O1–O4;*
- *the envelope axioms E1–E3;*
- *the UFET/CDUT transfer theorems from Section 4;*
- *and the standard axioms needed to talk about $\mathbb{R}$, norms on Banach spaces, and finite-dimensional vector spaces.*

Let $\mathcal{C}$ be the analytic fragment defined in Definition 5.1, and let $\mathcal{M}$ be the concrete ZFC model described in Section 5.3. Then:

1. *(Soundness on $\mathcal{M}$.) If $T \vdash \varphi$ for some $\varphi \in \mathcal{C}$, then ZFC proves that φ holds in $\mathcal{M}$. In particular, φ is true of the concrete DOC operators and substrates defined in Sections 2–4.*
2. *(Relative completeness on $\mathcal{C}$.) Conversely, if ZFC proves that a concrete analytic statement ψ about the structures in $\mathcal{M}$ holds, and ψ can be expressed as a sentence $\varphi \in \mathcal{C}$, then $T \vdash \varphi$.*

In other words, on the analytic fragment $\mathcal{C}$, T is conservative over ZFC with respect to the model $\mathcal{M}$.

Informal proof idea. Because all the non-logical symbols of T have concrete interpretations in $\mathcal{M}$ built from finite rings, finite index sets, and standard operators, the axioms of T are simply definitional extensions and gathered theorems in ordinary mathematics. Soundness (1) follows from the fact that any proof in T can be interpreted inside ZFC as a proof about $\mathcal{M}$. For (2), any analytic statement ψ about $\mathcal{M}$ that uses only the structures and operations present in T can be translated into the language of T as $\varphi \in \mathcal{C}$. If ZFC proves ψ, then standard completeness and definitional extension arguments show that T proves φ. $\square$ $\square$

5.5 Interpretation and use

The conservativity theorem has two key implications for how DOC should be understood:

- First, DOC is not a new foundation for mathematics. It is a structured way of talking about specific finite objects (rings, kernels, discrete operators, energies) and their continuous limits, and all of its analytic conclusions are theorems of ordinary set-theoretic mathematics.
- Second, DOC can be treated as a reusable lawbook: once a particular DOC operator, substrate, or SPDE construction is verified to satisfy the axioms of T, any analytic conclusion about it that lies in $\mathcal{C}$ is guaranteed to be sound in the usual sense.

In practice, this means that later papers can use DOC as a black box:

- define a finite substrate and a family of DOC operators;
- check the axioms A1–A10, O1–O4, E1–E3 (and the SPDE axioms when PDEs are involved);
- apply the UFET/CDUT transfer theorem;

- and then work entirely inside the analytic fragment $\mathcal{C}$, knowing that any conclusions about norms, energies, and residuals are ZFC theorems about concrete objects.

In the next part of the paper, we move from the abstract operator calculus to the substrate PDE framework (SPDE), where we apply DOC to finite models of PDEs and develop exactness and cost theorems in that setting.

6 Substrate PDE framework (SPDE)

We now apply the substrate and operator calculus to the setting of partial differential equations. The goal of this section is to formalize a substrate PDE framework (SPDE) in which:

- classical PDE instances are represented as finite operator systems acting on substrate states;
- interior computation is exact (no rounding, no iterative tolerance loops);
- cost per degree of freedom is independent of analytic error tolerance;
- and, for a broad class of classical discrete methods, the substrate method saturates their acceptance sets.

We begin by describing how to map a classical PDE instance to a substrate instance, and then state axioms for the resulting discrete system. In this section we focus on the structural properties; detailed examples and geometric laws will appear later.

6.1 From PDE instances to substrate systems

A classical PDE instance can be represented abstractly as a quadruple $\mathbf{P} = (\Omega, \mathcal{L}, \mathcal{B}, f)$, where:

- $\Omega \subset \mathbb{R}^n$ is a domain;
- $\mathcal{L}$ is a (possibly nonlinear) differential operator acting on functions $u : \Omega \to \mathbb{R}^m$;
- $\mathcal{B}$ encodes boundary or initial conditions (e.g. Dirichlet, Neumann, periodic, or initial data in time);
- f is a source term (right-hand side).

Examples include:

- Poisson's equation $-\Delta u = f$ with Dirichlet or periodic boundary conditions;
- the heat equation $\partial_t u - \Delta u = f$;
- the incompressible Navier–Stokes equations with divergence-free constraints;
- linear wave equations, Schrödinger equations, and others.

To represent $\mathbf{P}$ on the substrate, we proceed in three steps:

1. Choose a finite index set X_d. This might be a Cartesian grid, a spectral index set, or a more general connectivity structure. In simple cases, X_d can be identified with I_N or I_N^n for some N, but more general index sets are allowed as long as they are finite and admit local neighborhoods.
2. Define a discrete operator L_d. We choose an R-linear operator $L_d : R^{X_d} \to R^{X_d}$ that represents the action of $\mathcal{L}$ on substrate states. Typically, L_d is defined by a stencil: $(L_d u)(i) = \sum_{j \in \mathcal{N}(i)} a_{ij}, u(j)$, where $\mathcal{N}(i) \subset X_d$ is a finite neighborhood and $a_{ij} \in R$ are coefficients chosen to approximate $\mathcal{L}$ in a consistent way (e.g. centered finite differences, spectral multipliers, or discrete divergence/gradient pairs).

3. Discretize boundary data $\mathcal{B}$ and sources f. The boundary/initial conditions and source terms are sampled, scaled to integers, and projected to R via π. This yields discrete boundary/initial data B_d and discrete sources f_d, so the full substrate PDE instance is $F(\mathbf{P}) = (X_d, L_d, B_d, f_d)$.

The evolution or solution process then consists of computing substrate states $u_d \in R^{X_d}$ that satisfy a discrete equation (steady-state) or update rule (time-stepping):

- steady-state: $L_d u_d = f_d$, with u_d respecting B_d;
- time-stepping: $u_d^{t+1} = \Phi_d(u_d^t, L_d, B_d, f_d)$ for some DOC-defined update map Φ_d.

On top of this, sampling and reconstruction operators Π_h and R_h (as in Section 4) provide the link between continuous fields u and substrate states u_d.

6.2 SPDE axioms

We now state a set of axioms governing the behavior of substrate PDE systems. These axioms are intended to capture the essential structural properties of reasonable discretizations—locality, closure, exact arithmetic in R, bounded growth, and compatibility with boundary conditions and refinement. To avoid confusion with the substrate axioms A1–A10, we label the SPDE axioms by S1–S12.

Axiom [S1 (Index closure and locality)] The substrate index set X_d is finite. For each $i \in X_d$, the neighborhood $\mathcal{N}(i) \subset X_d$ used by L_d is finite and fixed (independent of the particular state). The update at i depends only on $u_d(j)$ for $j \in \mathcal{N}(i)$.

Axiom [S2 (Ring-linearity and exact arithmetic)] The operator L_d is R-linear: $L_d(\alpha u_d + \beta v_d) = \alpha L_d u_d + \beta L_d v_d, \qquad \alpha, \beta \in R$. All arithmetic in the interior (multiplications by a_{ij}, additions) is carried out exactly in $R = \mathbb{Z}_d$, with no rounding or floating-point operations.

Axiom [S3 (Boundary/initial data compatibility)] The discrete boundary/initial data B_d is represented by a subset $X_d^{\mathrm{bdry}} \subseteq X_d$ and prescribed values $u_d(i) \in R$ for $i \in X_d^{\mathrm{bdry}}$, or by constraints on the stencil coefficients at those indices. Updates respect these constraints, i.e. boundary/initial data is not overwritten by interior stencils except in ways explicitly encoded in B_d.

Axiom [S4 (Update map and determinism)] In the time-dependent case, there is a deterministic update map $\Phi_d : R^{X_d} \times L_d, B_d, f_d \to R^{X_d}$ such that the substrate evolution is given by $u_d^{t+1} = \Phi_d(u_d^t, L_d, B_d, f_d)$. In the steady-state case, we interpret Φ_d as a single-step map whose fixed points satisfy the discrete PDE $L_d u_d = f_d$ under the boundary/initial constraints.

Axiom [S5 (Spectral/stability bound)] There exists an envelope constant $\kappa_{\mathrm{SPDE}} \in (0, \infty)$ such that, when L_d and any associated time-stepping operators are viewed as DOC operators on $\ell^2(X_d)$, their operator norms are bounded by κ_{SPDE} uniformly in the state: $|L_d|\ell^2 \to \ell^2 \leq \kappa\mathrm{SPDE}, \qquad |\Phi_d(\cdot, L_d, B_d, f_d)|\ell^2 \to \ell^2 \leq \kappa\mathrm{SPDE}$.

Axiom [S6 (Conservation and symmetry compatibility)] When the continuous PDE has conserved quantities (e.g. mass, divergence-free constraints, or quadratic invariants), the corresponding discrete system (X_d, L_d, B_d, f_d) is constructed so that:

- the discrete invariants are preserved exactly in the interior (subject to boundary/forcing);
- symmetries such as translation, reflection, or rotation (in a discrete sense) are respected by the stencil and update map.

This mirrors Axioms O1–O3 at the PDE level.

Axiom [S7 (Finite state space and recurrence)] The state space R^{X_d} is finite. For any fixed PDE instance (X_d, L_d, B_d, f_d) and any initial state u_d^0, the trajectory $u_d^0, u_d^1, u_d^2, \ldots$ under repeated application of Φ_d eventually becomes periodic (possibly after entering a fixed point or an attractor cycle).

Axiom [S8 (Cost per update and ε-independence)] The computational cost of a single substrate update (computing $\Phi_d(u_d^t, \cdot)$) is proportional to $|X_d|$ times a fixed stencil cost. In particular, the cost per degree of freedom per update does not depend on any analytic error tolerance parameter ε.

Axiom [S9 (Sampling and reconstruction compatibility)] The sampling operators $\Pi_h : X \to \mathbb{R}^{X_d}$ and reconstruction operators $R_h : \mathbb{R}^{X_d} \to X$ used to link continuous fields and substrate states are constructed from DOC operators (smoothing kernels and stencils) and satisfy the consistency and stability conditions (T1)–(T2) from Section 4, with residuals captured by the DOC residual schedule $\eta(h)$.

Axiom [S10 (Residual-preserving lift)] When comparing the discrete residual $r_d = L_d u_d - f_d$ to its continuous analogue, the lift preserves residuals in the sense that $|R_h r_d - (\mathcal{L}u - f)|_X \leq C, \eta(h)$, for some $C > 0$ and suitable choices of Π_h, R_h and norms.

Axiom [S11 (Adaptive refinement closure)] If the index set X_d is refined locally (e.g. via AMR) into smaller cells or finer grids, the resulting refined index set X_d' and operator L_d' still satisfy S1–S10, and the refinement is compatible with the substrate axioms A1–A10 (e.g. CRT indexing, mask densities) in each partition.

Axiom [S12 (Acceptance sets)] For each tolerance $\varepsilon > 0$, there is an acceptance set $\mathsf{Acc}\varepsilon$ of substrate states defined in terms of DOC energies and norms, such that:

- if a classical solution u of the PDE satisfies a given error criterion (residual or observable) at level ε, then there exists a substrate state $u_d \in \mathsf{Acc}\varepsilon$ whose reconstruction $R_h u_d$ also satisfies that criterion;
- conversely, if $u_d \in \mathsf{Acc}_\varepsilon$, then $R_h u_d$ is an acceptable approximation to a classical solution up to error ε.

These SPDE axioms encode a simple but powerful idea: the substrate PDE framework represents classical PDE instances by finite, exact dynamical systems whose behavior can be related back to the continuum via DOC transfer theorems, with residuals controlled by $\eta(h)$. In the next section, we will state and prove several SPDE theorems that follow from S1–S12 and the DOC framework:

- interior exactness (no rounding inside R^{X_d});
- ε-independent cost (per-degree-of-freedom cost does not depend on analytic tolerance);
- and a method-class saturation result, showing that for a wide class of classical discrete schemes, the acceptance sets of those schemes are contained within (and often strictly contained in) the acceptance sets of SPDE.

6.3 SPDE theorems: exactness, cost, and method-class saturation

We now state several basic theorems about the SPDE framework, derived from axioms S1–S12 and the DOC structure. The first theorem formalizes the idea that interior computation is exact at the substrate level: all arithmetic is carried out in the finite ring R, so there is no rounding error inside the discrete evolution.

Theorem 6.1 (Interior exactness)**.** *Let (X_d, L_d, B_d, f_d) be a substrate PDE instance satisfying S1–S4 and S2 in particular. Let $u^t_{dt\geq 0}$ be the sequence of substrate states generated by $u_d^{t+1} = \Phi_d(u_d^t, L_d, B_d, f_d)$. Then for each $t \geq 0$ and each index $i \in X_d$, the value $u_d^t(i)$ is computed exactly in the ring $R = \mathbb{Z}_d$, using only the ring operations and coefficients specified by L_d, B_d, and f_d. In particular, no rounding or real-number approximations occur in the interior computation.*

Proof (sketch). By S1, the update at each index i depends only on the finitely many neighbors in $\mathcal{N}(i)$. By S2, the operator L_d and the update map Φ_d are R-linear combinations of the existing state entries and the fixed discrete data B_d, f_d, with all coefficients in R. Thus each new state $u_d^{t+1}(i)$ is obtained from the previous state values and data by a finite composition of additions and multiplications in R. Since R is a finite ring and we never leave R, the interior computation is exact. □ □

The next theorem formalizes the idea that the computational cost per degree of freedom per update does not depend on any analytic error tolerance ε: SPDE does not iterate to drive residuals below ε; instead, it updates deterministically according to Φ_d.

Theorem 6.2 (Cost per update and ε-independence)**.** *Suppose S1, S2, and S8 hold. Let $|X_d|$ denote the number of substrate degrees of freedom and let C_{stencil} be the maximum number of ring operations required to update a single site $i \in X_d$. Then:*

1. *the cost of a single SPDE update $u_d^{t+1} = \Phi_d(u_d^t, \cdot)$ is $O(C_{\text{stencil}}, |X_d|)$;*
2. *the cost per degree of freedom per update is $O(C_{\text{stencil}})$, independent of any analytic error tolerance parameter ε.*

Proof (sketch). By S1, each update at $i \in X_d$ depends on values in $\mathcal{N}(i)$ of bounded size and on fixed data. By S2, the update involves a fixed number of ring operations per neighbor; we may absorb this into C_{stencil}. Applying Φ_d to the entire state therefore requires at most $C_{\text{stencil}}|X_d|$ operations. No additional iterations are required to meet an analytic tolerance ε; those tolerances are handled at the lifting level via DOC residuals (Section 4), not by changing the number of substrate updates. Thus the per-update cost is linear in $|X_d|$, with per-degree-of-freedom cost independent of ε. □ □

The third theorem compares SPDE to a class of classical discrete methods. Informally, it says that for any method in such a class, and any fixed tolerance ε, every discrete solution accepted by that method at tolerance ε can be matched (or improved) by an SPDE solution whose reconstruction passes the same acceptance test. In this sense, SPDE saturates the method class. Let $\mathcal{M}$ be a family of classical discrete methods that approximate the same PDE problem class as SPDE. Each $M \in \mathcal{M}$ is assumed to produce a discrete state v_h at a given resolution and tolerance ε, and to have an associated acceptance set $\mathsf{Acc}_\varepsilon(M)$ defined in terms of continuous residuals or observables after reconstruction.

Theorem 6.3 (Method-class saturation)**.** *Let $\mathcal{M}$ be a family of classical discrete PDE methods targeting the same PDE instances as SPDE, and let $\mathsf{Acc}_\varepsilon(M)$ be their acceptance sets for a fixed tolerance $\varepsilon > 0$. Assume:*

- *each $M \in \mathcal{M}$ uses a discretization and reconstruction compatible with DOC and UFET/CDUT (Section 4);*
- *SPDE satisfies S1–S12;*
- *the DOC residual schedule $\eta(h)$ can be chosen so that, for sufficiently fine resolutions, $\eta(h) \leq \varepsilon$.*

Then for each $M \in \mathcal{M}$, $\mathsf{Acc}\varepsilon(M) \subseteq \mathsf{Acc}\varepsilon(\text{SPDE})$, where $\mathsf{Acc}_\varepsilon(\text{SPDE})$ is the SPDE acceptance set defined in S12. In words, any state accepted by a classical method at tolerance ε has an SPDE counterpart whose reconstruction also passes the same tolerance test.

Proof (sketch). Consider a PDE instance $\mathbf{P}$ and a classical method $M \in \mathcal{M}$ producing a discrete state v_h that lies in $\mathsf{Acc}_\varepsilon(M)$. By assumption, M's discretization and reconstruction are compatible with DOC and UFET/CDUT; in particular, the reconstruction $R_h^M v_h$ has residual or observable error at most ε when measured against the continuous PDE. On the SPDE side, construct the substrate instance $F(\mathbf{P}) = (X_d, L_d, B_d, f_d)$. By S9–S12 and CDUT, we can choose an SPDE state u_d whose reconstruction $R_h u_d$ matches the continuous solution norm and residual to within $\eta(h)$. By choosing the resolution fine enough so that $\eta(h) \leq \varepsilon$, we obtain $|R_h u_d - u|X \leq \varepsilon$, or $|\mathcal{L}(R_h u_d) - f| \leq \varepsilon$, depending on the acceptance criterion. Thus $R_h u_d$ lies in the same acceptance region as $R_h^M v_h$, and by the definition of $\mathsf{Acc}\varepsilon$(SPDE) we have $u_d \in \mathsf{Acc}_\varepsilon$(SPDE). This shows that for each accepted state of M, there is an accepted SPDE state at the same tolerance. Taking the union over all $M \in \mathcal{M}$ yields the inclusion of acceptance sets. □ □

Theorem 6.3 does not say that SPDE is always strictly better than every classical method in every metric. Rather, it says that:

- for any method in a compatible class, each of its accepted outputs at a given tolerance can be matched by an SPDE output at the same (or better) tolerance;
- SPDE has the structural advantage of exact interior computation and ε-independent per-update cost, while classical methods often rely on floating-point arithmetic and iterative solvers whose cost depends on the desired tolerance.

In later sections, we will combine these SPDE results with the DOC transfer theorems to derive concrete geometric and PDE laws (area-density, surface flux, and parabolic local energy inequalities) in the DOC/SPDE setting, with explicit finite residuals.

7 Geometric and PDE laws in the DOC/SPDE framework

We now illustrate how the DOC/SPDE framework captures standard geometric and PDE laws in a finite, ZFC-internal setting. The emphasis here is not on new regularity results, but on showing that familiar balance laws can be expressed and recovered inside DOC, with explicit finite residuals controlled by the DOC residual schedule $\eta(h)$. We focus on three representative cases:

- a 2D area–density law, expressing that windowed lattice sums approximate a continuous area integral;
- a 3D surface flux law, relating discrete flux across a surface to a continuous flux integral;
- a 4D parabolic local energy inequality, encoding the structure of a space–time energy balance on a parabolic cylinder.

In each case, the DOC/SPDE machinery provides a finite representation and an explicit error term.

7.1 A 2D area–density law

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $f : \Omega \to \mathbb{R}$ be a continuous (or integrable) function. Consider a sequence of 2D grids $I_{N_x} \times I_{N_y}$ with mesh sizes h_x, h_y, and let h denote a generic mesh scale (e.g. $h = \max h_x, h_y$). Let $\Pi_h f$ be the sampled values of f on the grid, and R_h a reconstruction operator as in Section 4. We also choose a family of 2D Fejér-type windows $\rho_{R,h}$ that play the role of smooth cutoffs:

- each $\rho_{R,h}$ is a DOC-legal window (built from tensor products or mixtures of 1D Fejér kernels);
- $\rho_{R,h}$ is supported in a square (or disk) of radius R;
- $\sum_{i,j} \rho_{R,h}(i,j)$ approximates 1 over a region of linear size R.

The continuous area–density we want to approximate is $A_R(f) = \int_\Omega f(x), \rho_R(x), dx$, where ρ_R is the continuum counterpart of the discrete window $\rho_{R,h}$. Inside DOC/SPDE we represent this by the finite sum $A^{\text{disc}}_{R,h}(f) = \sum_{(i,j)\in I_{N_x}\times I_{N_y}} f_{i,j}, \rho_{R,h}(i,j)$, where $f_{i,j} = (\Pi_h f)(i,j)$ are the substrate samples.

Theorem 7.1 (2D area–density law in DOC/SPDE). *Under the consistency and stability assumptions of UFET/CDUT (Section 4), and assuming f is regular enough for standard quadrature error estimates to apply, there exists a function $\eta_{\text{area}}(h) \to 0$ such that $\left|A^{\text{disc}}_{R,h}(f) - A_R(f)\right|; \leq; C, \eta_{\text{area}}(h)$, where $C > 0$ depends on f and the choice of windows, and $\eta_{\text{area}}(h)$ has the same structure as the DOC residual schedule: $\eta_{\text{area}}(h) \sim c_1 h^p + c_2 \frac{r(h)}{N(h)}$.*

Interpretation. The theorem states that the DOC-legal windowed sum over the grid approximates the 2D area integral with an explicitly controlled finite residual. The error arises from two sources:

- discretization error due to sampling/reconstruction (captured by $c_1 h^p$);
- smoothing/window error on the finite grid (captured by $c_2 r(h)/N(h)$).

The proof is a direct application of CDUT (Theorem 4.3) to the integral functional A_R and its discrete realization.

7.2 A 3D surface flux law

Next, consider a vector field $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ (for example, a velocity field) and a smooth surface $\Sigma \subset \Omega$ with unit normal n. The classical surface flux of a scalar quantity transported by u is $\mathcal{F}\Sigma(u) = \int \Sigma\phi(x, u(x)), u(x)\cdot n(x), dS_x$, for some scalar function ϕ (e.g. $\phi = |u|^2/2$ in kinetic energy flux). On the substrate, Σ is approximated by a collection of grid faces or cells, and the flux integral is approximated by a DOC-legal sum. The finite flux might be written schematically as $\mathcal{F}\Sigma, h(u) = \sum \sigma \in \Sigma_h \phi_\sigma, (u_\sigma \cdot n_\sigma), w_\sigma$, where:

- Σ_h is a discrete representation of the surface;
- u_σ and n_σ are discrete velocity and normal vectors;
- w_σ are legal weights (possibly Fejér-smoothed) associated with the surface elements.

Theorem 7.2 (3D surface flux law in DOC/SPDE)**.** *Under DOC/SPDE discretization compatible with UFET/CDUT and assuming u and Σ are sufficiently regular, there exists a function $\eta_{\text{flux}}(h) \to 0$ such that $\big|\mathcal{F}\Sigma, h(u) - \mathcal{F}\Sigma(u)\big|; \leq; C, \eta_{\text{flux}}(h)$, with $\eta_{\text{flux}}(h)$ of the same structural form as $\eta(h)$ in Theorem 4.3, and C depending on bounds on u, ϕ, and the geometry of Σ.*

Interpretation. Just as in the area case, the DOC-legal surface flux sum converges to the continuous flux integral with an explicit finite residual. The assumptions ensure that:

- the discrete normals and weights approximate their continuous counterparts;
- the DOC reconstruction of the discrete field approximates the continuous field in a controlled way.

From the DOC viewpoint, $\mathcal{F}_{\Sigma,h}$ is just an energy-like functional built from DOC operators, so CDUT applies.

7.3 A 4D parabolic local energy inequality

Finally, we consider a parabolic PDE such as the heat equation or the (formal) local energy balance for a fluid. The archetypal structure is an inequality on a space–time cylinder $Q_{R,\tau}(x_0, t_0) = B_R(x_0) \times (t_0 - \tau, t_0)$, with parabolic scaling $\tau \sim R^2$. A typical local energy inequality has the form $\int_{t_0-\tau}^{t_0} \int_{B_R(x_0)} |u|^3, \phi^2; \leq ; C\Big(\text{energy on } Q_{R,\tau} + \text{pressure terms}\Big)$, where ϕ is a smooth cutoff supported in $Q_{R,\tau}$. In DOC/SPDE, we represent space–time by an index set X_d with both spatial and temporal components, and we represent the cutoff ϕ by a DOC-legal window (a product or mixture of Fejér-type kernels in space and time). The discrete version of the local energy expression is a finite sum over space–time indices, and the continuous inequality becomes a statement about the DOC reconstructed fields and pressures.

Theorem 7.3 (Parabolic local energy inequality in DOC/SPDE)**.** *Let u be a sufficiently regular solution of a parabolic PDE on $\Omega \times (0, T)$, and let u_d be its SPDE substrate representation. Let $\phi_{R,\tau,h}$ be a DOC-legal space–time window supported on a discrete approximation of $Q_{R,\tau}$. Then, under suitable DOC/SPDE discretization assumptions and using CDUT, there exists a finite residual $\eta_{\text{par}}(h) \to 0$ such that $\sum_{(i,t)\in Q_{R,\tau,h}} |u_d(i, t)|^3, \phi_{R,\tau,h}(i, t)^2; \leq; C\Big(\text{discrete energy on } Q_{R,\tau,h} + \text{discrete pressure terms}\Big); +; \eta_{\text{par}}(h)$, and the DOC reconstruction of this inequality converges to the corresponding continuous local energy inequality as $h \to 0$, with error controlled by $\eta_{\text{par}}(h)$.*

Interpretation. This theorem says that:

- the form of the parabolic local energy inequality can be expressed in the DOC/SPDE setting as a finite inequality involving sums over the discrete cylinder and DOC-legal windows;
- the inequality is stable under the DOC lifting process, with a finite residual that vanishes as the resolution is refined.

The precise structure of the pressure terms and the function spaces for u depend on the PDE under consideration. The key point for DOC is that such an inequality is an element of the analytic fragment $\mathcal{C}$: it is a relation between norms and integrals (or sums) that can be written in terms of DOC operators and windows. CDUT then transfers it between the substrate and the continuum.

7.4 Summary

The examples in this section illustrate how:

- DOC/SPDE provides a finite representation of familiar geometric and PDE laws;
- UFET/CDUT supplies explicit control of the finite residuals;
- and the resulting statements live comfortably inside the analytic fragment $\mathcal{C}$ for which DOC is conservative over ZFC.

We emphasize again that we are not claiming new regularity or existence theorems for specific PDEs here. Rather, we are showing that standard balance laws—area-density, surface flux, parabolic local energy—can be formulated and recovered inside the Deterministic Operator Calculus with explicit, finite error terms. In the next section we turn to more practical considerations: how to use DOC as a lawbook in applications, and how authors and referees can systematically check that a given construction fits within the DOC/SPDE framework and inherits its guarantees.

8 Using DOC as a lawbook: checklists and practice

The Deterministic Operator Calculus is designed to be used as a lawbook rather than a one-off construction. In this final section we give practical guidance for using DOC in applications and for reviewing DOC-based work. The intent is to make it straightforward to:

- verify that a proposed finite construction fits within the DOC/SPDE framework;
- know which theorems (e.g. UFET/CDUT, SPDE exactness, method-class saturation) can be invoked;
- and ensure that all conclusions lie in the analytic fragment $\mathcal{C}$, where conservativity over ZFC is guaranteed.

8.1 Checklist for authors

Suppose you wish to use DOC in an analytic or PDE context—for example, to analyze a particular discrete scheme, to design a new operator, or to give a finite proof of a balance law. The following checklist summarizes the steps.

1. Specify the substrate.
   - Choose a base b and modulus $d = b - 1$; set $R = \mathbb{Z}_d$.
   - Describe the finite index sets I_N or X_d you will use.
   - Ensure Axioms A1–A10 are satisfied in your setting (this is automatic for $R = \mathbb{Z}d$ and standard index sets).
2. Define the DOC operators.
   - Specify the kernels k (e.g. Fejér-type) and stencils L you will use.

- Check that your smoothing kernels satisfy O2 (nonnegative, symmetric, unit mass).
- Check that your operators are self-adjoint and PSD when required (O1), or otherwise clearly bounded by an envelope (E1–E3).
- Verify that your windows (weights) are legal in the sense of O3 (Fejér-type or sharp intervals), and that any cancellation claims respect the Only-zero and sharp +1 floor principles (O4).

3. Identify envelope constants and invariants.
   - For each operator family $\mathcal{T}$ you intend to use, identify an envelope constant $\kappa\mathcal{T}$ as in E3.
   - When needed, compute or estimate normalized invariants $\hat{\chi}, \hat{\theta}, \hat{\Psi}, \hat{\kappa}$ to express your results in a dimensionless form.

4. Set up sampling and reconstruction.
   - Define sampling $\Pi_h : X \to \mathbb{R}^{I_h}$ and reconstruction $R_h : \mathbb{R}^{I_h} \to X$ using DOC operators (Fejér smoothing, inverse DFT, simple interpolation).
   - Check the consistency and stability conditions (T1)–(T2) in Section 4, and identify the discretization order p and Fejér span $r(h)$.

5. Apply UFET/CDUT.
   - Define your continuous energy $E(u)$ and discrete energies $E_h(v_h)$.
   - Verify the energy convergence condition (T3) and plug in your $\eta_1(h)$ and $\eta_2(h)$ to obtain the DOC residual schedule $\eta(h) = c_1 h^p + c_2 r(h)/N(h)$.
   - Use Theorem 4.3 to transfer statements about discrete minimizers, residuals, or norms back to the continuum.

6. For PDE applications, instantiate SPDE.
   - Map your PDE instance $\mathbf{P} = (\Omega, \mathcal{L}, \mathcal{B}, f)$ to a substrate instance (X_d, L_d, B_d, f_d) as in Section 6.1.
   - Check the SPDE axioms S1–S12: locality, ring-linearity, boundary compatibility, stability, conservation, finite state space, ε-independent cost, and acceptance sets.
   - Apply Theorems 6.1–6.3 to assert interior exactness, cost bounds, and method-class saturation.

7. Check that your conclusions lie in $\mathcal{C}$.
   - Ensure that the final statements you assert are of the analytic type: inequalities between norms, energies, residuals, and DOC invariants—i.e. they belong to the fragment $\mathcal{C}$.
   - When this is the case, Theorem 5.2 guarantees that your DOC-based conclusions are ZFC theorems about the concrete model.

Following this checklist helps ensure that your construction is fully within the DOC/SPDE framework and that your conclusions are backed by the conservativity meta-theorem.

8.2 Checklist for referees and readers

For referees evaluating a DOC-based manuscript, or readers wishing to verify a DOC argument, the following checklist mirrors the author's perspective.

1. Substrate layer.
   - Does the paper clearly specify the substrate (ring $R = \mathbb{Z}_d$, index sets, digital-root map)?
   - Are the substrate axioms A1–A10 explicitly invoked or clearly satisfied?

2. Operator layer.

- Are all operators acting on finite functions built from admissible kernels and stencils?
- Do smoothing kernels satisfy O2 (nonnegative, symmetric, unit mass)?
- Are self-adjointness and PSD properties used only where O1 guarantees them?
- Are windows explicitly shown to be legal as in O3?
- When sharp counting is approximated by smoothing, does the argument respect the +1 floor from O4?

3. Envelopes and bounds.
   - Are envelope constants identified and used consistently (E1–E3)?
   - Do operator norms or energy bounds rely on envelope assumptions that are stated and justified?
4. Transfer layer (UFET/CDUT).
   - Are sampling and reconstruction operators defined clearly?
   - Are the needed consistency and stability estimates for Π_h and R_h stated?
   - Is UFET/CDUT (Theorem 4.1 or 4.3) invoked with a clear residual schedule $\eta(h)$?
5. SPDE layer (if PDEs are involved).
   - Is the mapping from a classical PDE instance to a substrate instance clearly described?
   - Are the SPDE axioms S1–S12 either directly checked or inherited from standard discretizations?
   - Are Theorems 6.1–6.3 used appropriately (e.g. interior exactness, cost, method-class saturation)?
6. Analytic fragment and conservativity.
   - Do the main conclusions fit within the analytic fragment $\mathcal{C}$?
   - If so, the conservativity theorem (Theorem 5.2) guarantees that those conclusions are valid in a concrete ZFC model.
7. Clarity and separation of layers.
   - Is it clear when a statement pertains to the finite substrate, to DOC operators, to the transfer layer, or to SPDE?
   - A well-structured DOC paper should label which axioms and theorems are used in each major step.

A DOC-based manuscript that follows these patterns can be evaluated using standard mathematical tools and does not rely on any hidden assumptions beyond ZFC and the explicitly stated axioms.

8.3 Concluding remarks

The Deterministic Operator Calculus provides a way to talk about finite operators and their continuous limits under a single, tightly controlled set of axioms. The substrate layer ensures that all computations are carried out in finite rings and index sets; the operator layer enforces positivity, normalization, and boundedness via legality and envelope axioms; the transfer layer (UFET/CDUT) provides a universal residual schedule for lifting finite results to the continuum; and the SPDE layer shows how classical PDE discretizations fit inside this framework with exact interior arithmetic and ε-independent cost. The conservativity meta-theorem confirms that DOC is not a new foundation: it is a structured, reusable piece of ordinary mathematics. Any analytic statement proved in DOC's fragment $\mathcal{C}$ is a ZFC theorem about concrete finite matrices, kernels, and energies. Future work can build on this lawbook in several directions:

- applying DOC/SPDE to specific PDEs and numerical schemes;
- exploring optimized choices of Fejér spans and grids for particular problems;

- and using the DOC framework as a base for studying more complex emergent structures, while remaining grounded in finite, ZFC-internal constructions.

This paper has focused on defining the substrate, stating the axioms, and proving the core theorems needed to treat DOC as a stable platform. The intent is that subsequent work can refer to DOC by invoking its axioms and theorems, rather than re-deriving these structural facts each time.

Appendix A — ZFC Compliance of the DOC/MARI Law Book

This appendix gives a self-contained, systematic verification that the axioms used in the main text are theorems or definable schemes inside ordinary ZFC (with classical analysis). All discrete structures are realized as finite sets, functions, and matrices; all continuum objects live in standard function spaces on compact domains (e.g. tori). No set-theoretic assumptions beyond ZFC are required.

Throughout, all "axioms" in the body of the paper are reinterpreted here as theorems about explicit ZFC models. Wherever the main text refers to an axiom by name, the corresponding result below is the justification.

A.1 Discrete Fourier–Fejér Substrate

A.1.1 Underlying sets and operations

Fix integers $b \geq 2$ and $M \geq 2$. Define:

- The finite ring $D \coloneqq \mathbb{Z}_d, \quad d \coloneqq b-1$, with addition and multiplication modulo d.
- The finite cyclic group $I \coloneqq \mathbb{Z}_M = \{0, 1, \ldots, M-1\}$ with addition modulo M.
- The real sequence space $V \coloneqq \{f : I \to \mathbb{R}\}$, with pointwise addition and scalar multiplication. Equip V with the inner product $\langle f, g \rangle_V \coloneqq \sum_{x \in I} f(x)g(x), \qquad \|f\|_V \coloneqq \big(\langle f, f \rangle_V\big)^{1/2}$.

These objects are definable as finite sets and functions in ZFC.

A.1.2 Digital-root map and ring homomorphism properties

Define the base-b digital-root map $\mathrm{dr}_b : \mathbb{Z}_{\geq 0} \to \{0, 1, \ldots, d\}$ by $\mathrm{dr}_b(0) = 0, \qquad \mathrm{dr}_b(n) = 1 + ((n-1) \bmod d) \quad (n \geq 1)$.

Lemma A.1.1 (Digital-root congruence). For all integers $m, n \geq 0$,

$$\mathrm{dr}_b(m+n) \equiv m+n \pmod d,$$
$$\mathrm{dr}_b(mn) \equiv mn \pmod d.$$

In particular,

$$\mathrm{dr}_b(m+n) \equiv \mathrm{dr}_b(\mathrm{dr}_b(m) + \mathrm{dr}_b(n)) \pmod d,$$
$$\mathrm{dr}_b(mn) \equiv \mathrm{dr}_b(\mathrm{dr}_b(m)\,\mathrm{dr}_b(n)) \pmod d.$$

Proof. Reduction modulo d is a ring homomorphism $\mathbb{Z} \to \mathbb{Z}_d$. The map dr_b coincides with this reduction followed by the canonical identification of class 0 with digit 0 and classes $1, \ldots, d-1$ with themselves. Hence both congruences hold. □

Thus the digital-root axioms in the body of the paper are simply rephrasings of this lemma.

A.1.3 Survivor sets and densities

Fix a prime cutoff $y \geq 2$ and define the primorial $M_y := \prod_{p \leq y} p$. Define the survivor indicator $S_y : \mathbb{Z}/M_y\mathbb{Z} \to \{0,1\}$, $\quad S_y(n) = \begin{cases} 1, & \gcd(n, M_y) = 1, \\ 0, & \text{otherwise,} \end{cases}$ and its mean density $\theta_y := \frac{1}{M_y} \sum_{n=0}^{M_y-1} S_y(n) = \frac{\varphi(M_y)}{M_y} = \prod_{p \leq y} \bigl(1 - \frac{1}{p}\bigr)$.

Lemma A.1.2 (CRT survivor structure). In ZFC:

1. $(\mathbb{Z}/M_y\mathbb{Z})^\times = \{n \in \mathbb{Z}/M_y\mathbb{Z} : \gcd(n, M_y) = 1\}$ is a finite multiplicative group.
2. There is a ring isomorphism $\mathbb{Z}/M_y\mathbb{Z} \cong \prod_{p \leq y} \mathbb{Z}_p$, under which the unit group corresponds to $\prod_{p \leq y} \mathbb{Z}_p^\times$.
3. The density formula above is valid and follows from Euler's totient and CRT.

Proof. Standard number theory: properties of $\varphi(M_y)$ and the Chinese Remainder Theorem. □

The "window exactness" statement used in the main text (that survivors coincide with primes up to p_{y+1}^2) is the classical CRT-based prime sieve fact: in ZFC one proves by contradiction that any composite in $[2, p_{y+1}^2)$ has a prime factor $\leq p_y$ and hence is not a survivor.

A.1.4 Discrete Fourier transform and Plancherel

Fix $M \geq 2$. Define the discrete Fourier transform (DFT) on V by $\widehat{f}(k) := \sum_{x \in I} f(x)\, e^{-2\pi i k x/M}, \qquad k \in I,$ and its inverse by $f(x) = \frac{1}{M} \sum_{k \in I} \widehat{f}(k)\, e^{2\pi i k x/M}$.

Lemma A.1.3 (Plancherel). For all $f \in V$, $\sum_{x \in I} |f(x)|^2 = \frac{1}{M} \sum_{k \in I} |\widehat{f}(k)|^2$.

Proof. This is the standard finite Fourier transform identity; it follows from orthogonality of exponentials and is a routine calculation in ZFC. □

A.1.5 Fejér kernel

Define the discrete Fejér kernel with span $r \in \mathbb{N}$ by $F_r : \mathbb{Z}/M\mathbb{Z} \to \mathbb{R}_{\geq 0}$, $\qquad F_r(n) := \frac{1}{r} \sum_{j=-(r-1)}^{r-1} \bigl(1 - \frac{|j|}{r}\bigr)\, \mathbf{1}_{\{n \equiv j \pmod M\}}$.

Lemma A.1.4 (Fejér positivity, symmetry, unit mass, symbol). For all r and M:

1. $F_r(n) \geq 0$ for all $n \in \mathbb{Z}/M\mathbb{Z}$.
2. $F_r(n) = F_r(-n)$ (evenness).
3. $\displaystyle\sum_{n \in \mathbb{Z}/M\mathbb{Z}} F_r(n) = 1.$
4. Defining $\widehat{F}_r(k) = \sum_n F_r(n) e^{-2\pi i k n/M}$, one has the closed form $\widehat{F}_r(k) = \begin{cases} \dfrac{1}{r}\Bigl(\dfrac{\sin(\pi r k/M)}{\sin(\pi k/M)}\Bigr)^2, & k \not\equiv 0 \bmod M, \\ 1, & k \equiv 0 \bmod M, \end{cases}$

   and $0 \leq \widehat{F}_r(k) \leq 1$ for all k.

Proof. Each summand in F_r is ≥ 0; symmetry follows from the range of j and the symmetry of $|j|$; the mass computation is a finite summation over j, giving total r, divided by r. The symbol formula is the classical Fejér kernel DFT; positivity of $\widehat{F}_r$ follows from expressing F_r as the convolution of Dirichlet kernels with themselves, or directly from the squared sine representation. The bound $\widehat{F}_r \leq 1$ is a standard trigonometric estimate. All steps are finite arithmetic, justified in ZFC. □

Corollary A.1.5 (Energy contraction). For all $g \in V$, $\|F_r * g\|_V \leq \|g\|_V$, where $(F_r * g)(x) = \sum_n F_r(x - n) g(n)$ is circular convolution.

Proof. Plancherel and Lemma A.1.4(4) give $\|F_r * g\|_V^2 = \frac{1}{M} \sum_k |\widehat{F}_r(k)\, \widehat{g}(k)|^2 \leq \frac{1}{M} \sum_k |\widehat{g}(k)|^2 = \|g\|_V^2$. □

Lemma A.1.6 (Two-term envelope, discrete form). For any mean-free $g \in V$ (i.e. $\sum_x g(x) = 0$), there exist constants $A, B > 0$, independent of M, such that $\max_{x \in I} |(F_r * g)(x)| \leq A\sqrt{r}\, \|g\|_V + B\, \frac{r}{M}\, \|g\|_V$.

Proof sketch. Split the Fourier sum into low frequencies $|k| \le r$ and high frequencies $|k| > r$; bound each part via Cauchy–Schwarz and the decay properties of $\widehat{F}_r(k)$. All steps use only finite sums and standard inequalities; details are routine and purely combinatorial/analytic. □

This lemma is the substrate form of the "two-term error" used in the main text.

A.2 DOC Operators on the Fejér Substrate

A.2.1 Legal kernels and DOC operators

A legal kernel is any function $W : \mathbb{Z}/M\mathbb{Z} \to \mathbb{R}_{\ge 0}$ of the form $W = \sum_{j=1}^{J} \lambda_j F_{r_j}, \qquad \lambda_j \ge 0, \quad \sum_j \lambda_j = 1.$

Lemma A.2.1 (Legality of convex Fejér mixtures). Any such W satisfies:

1. $W(n) \ge 0$ and $\sum_n W(n) = 1$.
2. Its DFT $\widehat{W}(k) = \sum_j \lambda_j \widehat{F}_{r_j}(k)$ lies in $[0, 1]$ for all k.
3. W commutes with translations and preserves constants.

Proof. Nonnegativity and unit mass follow from convexity and Lemma A.1.4. The symbol bound follows from convexity and $0 \le \widehat{F}_{r_j}(k) \le 1$. Translation invariance and preservation of constants are immediate from the definition of convolution with a periodic kernel. □

Define the DOC kernel operator $T_W : V \to V$ by $T_W f := W * f$.

Lemma A.2.2 (DOC kernel operator properties). For any legal W:

1. T_W is linear and self-adjoint on $(V, \langle \cdot, \cdot \rangle_V)$.
2. T_W is positive semidefinite: $\langle T_W f, f \rangle_V \ge 0$ for all $f \in V$.
3. T_W is a contraction: $\|T_W f\|_V \le \|f\|_V$.

Proof. Self-adjointness follows from symmetry $W(n) = W(-n)$; positivity and contraction from Lemma A.1.4 and Plancherel, as in Corollary A.1.5. □

Let $\Omega : V \to V$ be the mean projector $\Omega f := \frac{1}{M}\Big(\sum_x f(x)\Big)\mathbf{1}, \qquad \Pi := I - \Omega.$

Lemma A.2.3 (Projector properties). In ZFC:

1. $\Omega^2 = \Omega, \Omega^\top = \Omega; \Pi^2 = \Pi, \Pi^\top = \Pi.$
2. $\mathrm{im}(\Omega)$ is the 1-dimensional subspace of constant sequences; $\mathrm{im}(\Pi)$ is the mean-free subspace.
3. $T_W \Omega = \Omega T_W = \Omega$ and $T_W \Pi = \Pi T_W$.

Proof. All claims are straightforward matrix identities. Commutation with Ω and Π follows from the fact that convolution with a unit-mass kernel preserves the mean and commutes with translations; Ω is the translation-invariant rank-1 projector. □

The DOC operator class is the smallest set of linear operators on V that contains $\{T_W\} \cup \{\Omega, \Pi\}$ and is closed under linear combinations and composition. Every such operator is a finite matrix with real entries, definable and manipulable in ZFC.

A.3 UFET: Finite→Continuum Transfer

A.3.1 Discrete and continuum spaces

Let Ω (now the domain, not the projector) be a compact subset of $\mathbb{R}^d$ with periodic boundary conditions, e.g. $\mathbb{T}^d$. Let $X := L^2(\Omega)$ with its usual inner product. For a mesh parameter $h > 0$, define a grid $\mathcal{G}_h \subset \Omega$ and discrete space $V_h := \{u : \mathcal{G}_h \to \mathbb{R}\}$ with inner product $\langle f, g\rangle_{V_h} := h^d \sum_{x \in \mathcal{G}_h} f(x)g(x)$.

Define sampling and reconstruction operators:

- $S_h : X \to V_h$, $(S_h u)(x) := u(x)$ for $x \in \mathcal{G}_h$.
- $R_h : V_h \to X$, $(R_h f)(x) := f(x_0)$ for x in the cell centered at $x_0 \in \mathcal{G}_h$.

Lemma A.3.1 (Sampling/reconstruction stability). For each h:

1. $\|R_h f\|_X^2 = \|f\|_{V_h}^2$ for all $f \in V_h$.
2. If $u \in H^1(\Omega)$, then $\|R_h S_h u - u\|_X \le Ch\|u\|_{H^1}$ for some constant C independent of h.

Proof. (1) is a computation with cell volumes; (2) follows from standard quadrature error estimates and the Sobolev embedding of cell-average approximations. □

A.3.2 Discrete operators and Γ-convergence

Let $A : D(A) \subset X \to X$ be a densely defined linear operator (e.g. $-\Delta$ with periodic boundary conditions). Let $A_h : V_h \to V_h$ be discrete operators.

Definition A.3.2 (UFET hypotheses). We say (A_h) approximates A if:

1. (Consistency) There exists a norm $\|\cdot\|_{X_1}$ and $C_1 > 0$ such that for all $u \in D(A) \cap X_1$, $\|R_h A_h S_h u - Au\|_X \le C_1 \eta(h)\, \|u\|_{X_1}$, where $\eta(h) \to 0$ as $h \to 0$.
2. (Stability) There exists $C_2 > 0$ such that $\|A_h f\|_{V_h} \le C_2 \|f\|_{V_h}$ for all $f \in V_h$.
3. (Energy Γ-convergence) The quadratic forms $E_h(f) := \langle A_h f, f\rangle_{V_h}$ Γ-converge to $E(u) := \langle Au, u\rangle_X$ in the sense of Γ-convergence on X.
4. (Equi-coercivity) There exist $c, C \ge 0$ such that $E_h(f) \ge c\|R_h f\|_X^2 - C$ for all h, f.

These are exactly the standard assumptions used in numerical analysis and Γ-convergence theory, and each is a definable statement in ZFC.

Theorem A.3.3 (UFET residual law). Under the above hypotheses, there exist constants $C_3, C_4 > 0$ such that for solutions $A_h u_h = S_h f$ and $Au = f$, $\|R_h u_h - u\|_X \le C_3 \eta(h) \Big(\|u\|_{X_1} + \|f\|_X \Big)$, with $\eta(h)$ as in Definition A.3.2. In particular, $R_h u_h \to u$ in X.

Proof sketch. This is a standard consequence of Γ-convergence and resolvent stability: consistency bounds yield an $O(\eta(h))$ defect in the discrete equation, coercivity and stability control the inverse, and Γ-convergence together with compactness (precompactness of bounded-energy sequences) then give the claimed limit for the attractor cycles.

Concrete choices (e.g. $A = -\Delta$, A_h as finite differences, $\eta(h) = c_1 h^2 + c_2 r(h)/M_y$) are supplied by standard finite-difference/spectral error estimates and Fejér alias bounds; these are again classical theorems.

A.4 UCL, KUEC, and GUM: Legality, Envelopes, Uniformity

A.4.1 UCL: Only-Zero and sharp +1

Let $g : \mathbb{Z}/M_y\mathbb{Z} \to \mathbb{R}$ be mean-free ($\sum g = 0$), and let $w \geq 0$ be a period-M_y weight with $\sum w = 1$.

Theorem A.4.1 (Sharp transfer with +1 for integer-valued data). Let $g : \mathbb{Z}/M_y\mathbb{Z} \to \mathbb{Z}$ be integer-valued with $|g(n)| \leq 1$ and mean-free $\sum g = 0$. For every interval $I \subset \mathbb{Z}/M_y\mathbb{Z}$ of length r, there exists a translate x such that

$$\left| \sum_{n\in I} g(n) - \sum_{n\in\mathbb{Z}/M_y\mathbb{Z}} g(n)\, F_r(x-n) \right| \leq 1.$$

No uniform constant $c < 1$ works in place of 1 for all r, M_y and all such g.

Proof. Because F_r is nonnegative and sums to 1, there is a translate whose support lies inside I except possibly on one boundary point. On the indices where both windows are fully contained in I the two sums agree; the only discrepancy comes from the (at most one) boundary index. Since $|g(n)| \leq 1$ for all n, the difference between the sharp sum and the smoothed sum is bounded in absolute value by 1.

Extremal "spike" examples (where g places a unit mass near the boundary) show that any $c < 1$ fails for sufficiently large r, so the constant 1 is sharp. □

Theorem A.4.2 (Only-Zero classification). Let $g = S_y - \theta_y$. Then $\sum_n g(n)w(n) = 0$ if and only if w is either:

1. class-uniform on residue classes modulo some divisor of M_y, or
2. the indicator (normalized) of a union of intervals that tile the period aligned with residue boundaries.

Proof sketch. Expand g and w in the Dirichlet character basis modulo M_y; observe that g has no principal character component, and that nonprincipal characters detect any deviation from uniformity/tiling. Algebraic details are handled entirely in finite Fourier analysis on $\mathbb{Z}/M_y\mathbb{Z}$, within ZFC. □

These theorems justify the "legality" of windows and the necessity of the sharp +1 floor.

A.4.2 KUEC: kernel envelopes

KUEC asserts the existence of kernel-independent bounds:

Lemma A.4.3 (Fejér envelope). There exist universal constants $A, B > 0$ such that for all spans r and moduli M, $\max_{x\in I_M} |(F_r * g)(x)| \leq A\sqrt{r}\, \|g\|_V + B\, \frac{r}{M}\, \|g\|_V$ for all mean-free $g \in V$.

This is a direct restatement of Lemma A.1.6 with constants tracked uniformly; all proofs are finite trigonometric/DFT calculations.

A.4.3 GUM: general uniformity mechanism

GUM encodes uniform operator bounds over families:

Lemma A.4.4 (Uniform operator norm bound). Let $\{T_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of DOC kernel operators built from Fejér mixtures with a fixed envelope. Then there exists $C > 0$ such that $\|T_\alpha\|_{V\to V} \leq C$ for all α.

Proof. Each T_α is a convex combination of contractions on V; the norm is bounded by 1. More elaborate families (e.g. compositions with a priori bounded projectors) have norms bounded by the product of individual bounds. All arguments are finite-dimensional Hilbert space estimates. □

A.5 USNE and UAIP: Equivalence and Identification

A.5.1 USNE: skew/normalization equivalence

Let H be a Hilbert space, and let $T, S : H \to H$ be bounded linear operators.

Definition A.5.1 (Skew–normalization equivalence). Operators T and S are USNE-equivalent if there exists a bounded invertible $B : H \to H$ such that $S = B^{-1}TB$ and $\|B\|, \|B^{-1}\|$ are uniformly bounded by a prescribed constant.

Lemma A.5.2 (Preservation of norm inequalities). If $\|Tf\| \leq C\|f\|$ for all $f \in H$ and $S = B^{-1}TB$ with $\|B\|, \|B^{-1}\| \leq C_B$, then $\|Sg\| \leq CC_B^2\|g\|$ for all $g \in H$.

Proof. Put $f = Bg$ and compute: $\|Sg\| = \|B^{-1}Tf\| \leq \|B^{-1}\| \, \|Tf\| \leq CC_B\|f\| \leq CC_B^2\|g\|$. $\square$

This is standard similarity invariance of operator inequalities in Hilbert spaces.

A.5.2 UAIP: analytic identification templates

The UAIP used in the body of the paper splits into:

1. A generic operator-theoretic part (proved in ZFC) stating that if a family of discrete operators satisfies symbol convergence, energy bounds, and UFET conditions, then there exists a unique continuum operator with those properties.
2. Problem-specific hypotheses (UAIP-X for a given domain X), which assert that a particular classical operator (e.g. the Laplacian, Maxwell operator, or a Dirichlet/ζ operator) matches the limit of a given discrete DOC family.

Part (1) is a standard application of functional calculus and spectral theory; part (2) is explicitly marked as a hypothesis in the main text and is not assumed as an axiom in this appendix.

Lemma A.5.3 (Generic UAIP schema). Let (K_h) be a family of bounded DOC operators on $L^2(\Omega)$ with symbols converging pointwise to a bounded measurable $m(\xi)$ and satisfying the UFET residual law. Then there exists a bounded operator K with symbol $m(\xi)$ such that, for all $f \in L^2(\Omega)$, $\|K_h f - Kf\|_2 \to 0$ as $h \to 0$.

This is proved by standard spectral convergence arguments in ZFC: for each Fourier mode, $\widehat{K}_h$ converges to $m(\xi)$; Plancherel then yields convergence in L^2.

Problem-specific identification statements (e.g. "K is the Laplacian resolvent" or "K is the zeta operator") are handled as explicit hypotheses outside this appendix.

A.6 Demos, Code Capsules, and δs

The demos and code capsules provided with the paper operate exclusively on the finite structures described above:

- Rings $\mathbb{Z}_d$, grids $\mathbb{Z}_M$, and finite arrays of real or complex numbers.
- DOC operators implemented as explicit matrices or convolution routines with Fejér kernels.
- Discrete energies and UFET residual estimates computed as sums and products of floating-point representations of rational and real numbers.

Lemma A.6.1 (ZFC-interpretability of demos). Each demo script defines only:

- finite sets and sequences;
- functions computable by finitely many arithmetic and transcendental evaluations;
- norms, energies, and residuals expressible as finite sums.

Thus every demo corresponds to a finite ZFC computation on the concrete models described in Sections A.1–A.3. A δ is simply a finite record of inequalities of the form observed_error$(h) \leq$ bound_rhs(h) for finitely many h; these inequalities are decidable statements in ZFC and can be turned into formal proofs by expanding the definitions of the operators and kernels.

A.7 Summary and Scope

- The discrete substrate (rings, groups, DRPT, survivors, DFT, Fejér kernels) is constructed entirely inside ZFC, using finite sets and functions. All "axioms" at this level are theorems (Lemmas A.1.1–A.1.6).
- DOC operators (Fejér mixtures, projectors, compositions) are bounded linear operators on finite-dimensional Hilbert spaces; their properties (positivity, contraction, commutation) are elementary matrix identities or consequences of Plancherel (Lemmas A.2.1–A.2.3).
- UFET and CLP+ are standard Γ and discretization theorems in Hilbert spaces; the residual laws used in the main text follow from classical functional analysis (Theorem A.3.3).
- UCL, KUEC, and GUM are finite Fourier/Fejér lemmas: sharp transfer with mandatory $+1$, Only-Zero classification, and uniform operator bounds (Theorems A.4.1–A.4.2, Lemmas A.4.3–A.4.4).
- USNE and the generic UAIP schema are standard operator-theoretic facts in ZFC (Lemmas A.5.2–A.5.3); problem-specific UAIP-X assumptions are explicitly marked as hypotheses, not axioms.
- All demos and δs are finite computations carried out inside these ZFC models (Lemma A.6.1).

Consequently, every invariant, operator, and residual used in the main text is ZFC-compliant; every demo is a computation on a fully specified ZFC model of the DOC/MARI law book, and any analytic statement that is proven in this framework can be rephrased as an ordinary ZFC theorem about these concrete structures.

Appendix B — Formal Conservativity of the DOC/MARI Theory over ZFC on the Analytic Fragment

This appendix gives a fully formal version of the "conservativity" claim used in Section 5. We:

1. Specify a formal language $\mathcal{L}_T$ for the DOC/MARI theory T.
2. Define precisely the analytic fragment $\mathcal{C}$ of sentences in $\mathcal{L}_T$.
3. Explain how T is obtained from a ZFC-axiomatized base theory by definitional extension.
4. Prove that T is conservative over ZFC for $\mathcal{C}$: if $T \vdash \varphi \in \mathcal{C}$, then ZFC proves the corresponding statement about the concrete model constructed in Appendix A.

We work throughout in standard first-order logic with equality, and we take ZFC as the underlying set theory in which all metatheoretic arguments are carried out.

B.1 Formal Language $\mathcal{L}_T$

We use a many-sorted first-order language $\mathcal{L}_T$ with the following sorts:

- **D**: digit-ring elements.
- **I**: index elements.
- **V**: discrete vectors (finite sequences).
- **K**: kernels (weights on **I**).
- **X**: continuum functions.
- **H**: mesh/scale parameters (a subset of the reals).

For each sort, we have variables $d, d', \dots$ of sort $\mathbf{D}$, $i, i', \dots$ of sort $\mathbf{I}$, etc.
Function symbols and constants.
We include:

- Ring operations on $\mathbf{D}$: $+_D, \cdot_D : \mathbf{D} \times \mathbf{D} \to \mathbf{D}$, constants $0_D, 1_D$.
- Group operation on $\mathbf{I}$: $+_I : \mathbf{I} \times \mathbf{I} \to \mathbf{I}$, constant 0_I.
- Vector-space operations on $\mathbf{V}$: $+_V : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$, scalar multiplication $\cdot : \mathbb{R} \times \mathbf{V} \to \mathbf{V}$.
- Kernel evaluation: $\mathrm{val}_K : \mathbf{K} \times \mathbf{I} \to \mathbb{R}$.
- Vector evaluation: $\mathrm{val}_V : \mathbf{V} \times \mathbf{I} \to \mathbb{R}$.
- Inner products and norms: $\langle \cdot, \cdot \rangle_V : \mathbf{V} \times \mathbf{V} \to \mathbb{R}, \|\cdot\|_V : \mathbf{V} \to \mathbb{R}_{\geq 0}$; $\langle \cdot, \cdot \rangle_X : \mathbf{X} \times \mathbf{X} \to \mathbb{R}, \|\cdot\|_X : \mathbf{X} \to \mathbb{R}_{\geq 0}$.
- Convolution: $* : \mathbf{V} \times \mathbf{K} \to \mathbf{V}$, written $(f, k) \mapsto f * k$.
- Projectors: $\Omega, \Pi : \mathbf{V} \to \mathbf{V}$.
- Sampling and reconstruction: $S : \mathbf{H} \times \mathbf{X} \to \mathbf{V}$, $R : \mathbf{H} \times \mathbf{V} \to \mathbf{X}$.
- Discrete operators: $A : \mathbf{H} \times \mathbf{V} \to \mathbf{V}$.
- Continuum operator: $A^* : \mathbf{X} \to \mathbf{X}$.
- Residual function: $\eta : \mathbf{H} \to \mathbb{R}_{\geq 0}$.

All of these symbols are interpreted in the concrete model of Appendix A by explicitly definable functions on sets of reals and finite sequences.
Relation symbols.
We use:

- Equality $=$ on each sort.
- Order $\leq$ and arithmetic relations on $\mathbb{R}$ (encoded in the usual way as sets in ZFC).

Terms are formed in the standard way from variables and function symbols; atomic formulas are equalities or inequalities between terms of appropriate sorts; formulas are built from atomic formulas using $\neg, \wedge, \vee, \to, \forall, \exists$.

B.2 Theory T: Axioms in $\mathcal{L}_T$

The theory T is a set of axioms in $\mathcal{L}_T$ capturing the DOC/MARI law book. It consists of:

- Algebraic axioms for $(\mathbf{D}, +_D, \cdot_D, 0_D, 1_D)$ as a finite commutative ring; $(\mathbf{I}, +_I, 0_I)$ as a finite cyclic group; $(\mathbf{V}, +_V, \cdot, \langle \cdot, \cdot \rangle_V, \|\cdot\|_V)$ as an inner-product space; $(\mathbf{X}, \langle \cdot, \cdot \rangle_X, \|\cdot\|_X)$ as a Hilbert space.
- Fejér axioms: for each integer parameter r, existence of a kernel symbol $F_r \in \mathbf{K}$ satisfying:
- Nonnegativity and unit mass: $\forall i \in \mathbf{I}\, \big(\mathrm{val}_K(F_r, i) \geq 0\big)$, $\quad \sum_{i \in \mathbf{I}} \mathrm{val}_K(F_r, i) = 1$.
- Symmetry: $\forall i \in \mathbf{I}\, \big(\mathrm{val}_K(F_r, i) = \mathrm{val}_K(F_r, -_I i)\big)$.
- Energy contraction for convolution with F_r (as in Corollary A.1.5).
- DOC axioms: legality of convex combinations of Fejér kernels, self-adjointness and contraction of T_W, projector properties of Ω, Π, and their commutation with convolution, as in Lemmas A.2.1–A.2.3.
- UFET axioms: consistency, stability, Γ, and equi-coercivity of (A_h) relative to A^*, expressed in first-order form using the symbols A, S, R, η, as in Definition A.3.2.

- UCL, KUEC, and GUM axioms: Only-Zero and sharp +1 axioms, sup-norm envelopes for Fejér kernels, and uniform operator bounds, as in Theorems A.4.1–A.4.2 and Lemmas A.4.3–A.4.4.
- USNE axioms: statements asserting that similarity via a bounded invertible operator preserves inequalities up to a fixed multiplicative factor (Lemma A.5.2).
- UAIP generic axioms: generic operator convergence under symbol convergence and UFET residual control (Lemma A.5.3). Problem-specific UAIP hypotheses (e.g. identifying a particular continuum operator with a DOC limit) are not included as axioms; they are stated explicitly as hypotheses in the body of the paper.

The precise list of axioms is finite up to schemas over integer parameters (for spans r and mesh scales h) and can be written as a recursive set of $\mathcal{L}_T$-sentences.

B.3 Analytic Fragment $\mathcal{C}$

We fix a syntactic fragment $\mathcal{C} \subset \mathrm{Sent}(\mathcal{L}_T)$ of closed sentences generated by the following grammar:

- Atomic formulas:
- Equalities $t_1 = t_2$ where t_1, t_2 are $\mathcal{L}_T$-terms of the same sort (including reals, vectors, kernels, indices).
- Inequalities $s_1 \leq s_2$, where s_1, s_2 are real-valued terms (built from inner products, norms, arithmetic combinations of reals, and evaluations of η).
- Formulas:
- Closed formulas built from atomic formulas using propositional connectives $(\neg, \wedge, \vee, \rightarrow)$ and first-order quantifiers over the sorts $\mathbf{D}, \mathbf{I}, \mathbf{V}, \mathbf{K}, \mathbf{X}, \mathbf{H}$.

In particular, $\mathcal{C}$ includes statements of the form: $\forall h \in \mathbf{H}\, \forall u \in \mathbf{X}_1 \Big(\|R(h, A(h, S(h,u))) - A^*(u)\|_X \leq \eta(h)\, \Phi(u) \Big)$, where $\mathbf{X}_1$ is a definable subspace of $\mathbf{X}$, and Φ is a term built from $\|\cdot\|_X$ and the inner product. More generally, $\mathcal{C}$ captures all the "norm/energy/residual" sentences stated in the main text.

$\mathcal{C}$ is thus a syntactically defined class of first-order sentences in $\mathcal{L}_T$; no higher-order quantification is allowed.

B.4 T as a Definitional Extension of a ZFC-Based Theory

Let $\mathcal{L}_0$ be the language of pure set theory with membership $\in$. In ZFC we can interpret:

- the reals $\mathbb{R}$,
- finite sets and functions,
- Hilbert spaces like $\ell^2(\mathbb{Z}/M\mathbb{Z})$ and $L^2(\Omega)$,
- all the concrete objects of Appendix A.

Define a base theory T_0 in a one-sorted language $\mathcal{L}_0 \cup \mathcal{L}_{\mathrm{anal}}$, where $\mathcal{L}_{\mathrm{anal}}$ contains symbols for the usual operations on reals, finite sequences, and L^2 functions. The axioms of T_0 are exactly the ZFC-axiomatized facts about these objects (ring/group structure, inner products, DFT, Fejér kernels, finite-difference Laplacians, etc.)—i.e. the content of Appendix A, expressed in $\mathcal{L}_0 \cup \mathcal{L}_{\mathrm{anal}}$.

The theory T in $\mathcal{L}_T$ is then obtained from T_0 by:

1. Introducing new sorts $\mathbf{D}, \mathbf{I}, \mathbf{V}, \mathbf{K}, \mathbf{X}, \mathbf{H}$.

2. Introducing new function symbols (e.g. $+_D, \cdot_D, F_r, \Omega, A, S, R, \eta$), each accompanied by a defining axiom that identifies it with a specific definable function in the underlying ZFC model (e.g., $+_D$ as addition modulo d, F_r as the particular finite array satisfying Lemma A.1.4, etc.).

These are purely definitional extensions: for each new function symbol f, there is a formula $\psi_f(\bar{x}, y)$ in the base language such that T contains the axiom $\forall \bar{x}\, \forall y \big(f(\bar{x}) = y \leftrightarrow \psi_f(\bar{x}, y)\big)$.

Lemma B.4.1 (Definitional extension). The theory T is a definitional extension of the base theory T_0: every $\mathcal{L}_T$-sentence φ can be translated to an $\mathcal{L}_0 \cup \mathcal{L}_{\text{anal}}$-sentence $\varphi^\sharp$ by replacing each new function symbol by its defining formula, and for all such φ, $T \vdash \varphi \iff T_0 \vdash \varphi^\sharp$.

Proof sketch. This is a standard fact about definitional extensions in first-order logic: see any logic text. The translation $\varphi \mapsto \varphi^\sharp$ is primitive recursive; soundness and completeness of first-order logic, together with the defining axioms, yield the equivalence of derivability. □

Since T_0 is itself a fragment of ZFC extended by conservative analytic definitions, we can regard $\varphi^\sharp$ as a ZFC-sentence (with parameters interpreted in the concrete model) and $T_0 \vdash \varphi^\sharp$ as "ZFC proves $\varphi^\sharp$."

B.5 Conservativity Theorem on $\mathcal{C}$

We now make precise the claim used in Section 5.

Theorem B.5.1 (Conservativity of T over ZFC on $\mathcal{C}$). Let $\varphi \in \mathcal{C}$ be a closed sentence in the analytic fragment. If $T \vdash \varphi$, then ZFC proves that φ holds in the concrete model constructed in Appendix A. Equivalently, if $\varphi^\sharp$ is the ZFC-translation of φ, then ZFC $\vdash \varphi^\sharp$.

Proof. Let $\varphi \in \mathcal{C}$ and suppose $T \vdash \varphi$. By Lemma B.4.1, $T_0 \vdash \varphi^\sharp$, where $\varphi^\sharp$ is an $\mathcal{L}_0 \cup \mathcal{L}_{\text{anal}}$-sentence obtained by expanding all defined symbols.

By construction, T_0 is axiomatized by sentences that are provable in ZFC about the concrete structures of Appendix A (finite rings, groups, Hilbert spaces, Fejér kernels, discrete operators, etc.). Thus, working inside ZFC, we can view T_0 as the theory of those structures and regard $T_0 \vdash \varphi^\sharp$ as asserting that $\varphi^\sharp$ is derivable from those ZFC-proven lemmas.

By the soundness theorem for first-order logic (itself provable in ZFC), if $T_0 \vdash \varphi^\sharp$, then every model of T_0 satisfies $\varphi^\sharp$, in particular the concrete model of Appendix A built in ZFC. Hence ZFC proves "the concrete model satisfies $\varphi^\sharp$," i.e. ZFC proves the corresponding analytic statement about norms, energies, and residuals.

Since $\varphi \in \mathcal{C}$ was arbitrary, this shows that T is conservative over ZFC on $\mathcal{C}$: no new analytic sentence in $\mathcal{C}$ provable in T escapes the reach of ZFC; each is already a ZFC theorem about the interpreted structures. □

B.6 Remarks on Relative Completeness (Optional Direction)

The reverse direction—showing that any ZFC theorem in the analytic domain can be recast as a T-theorem—is not needed for conservativity, but we note:

- The language $\mathcal{L}_T$ is essentially a structured re-packaging of the analytic symbols in $\mathcal{L}_0 \cup \mathcal{L}_{\text{anal}}$; each $\mathcal{L}_T$-symbol is a definable function or relation in ZFC.
- For any $\mathcal{C}$-sentence φ, the translations $\varphi \mapsto \varphi^\sharp \mapsto \varphi$ commute with proofs: if ZFC proves $\varphi^\sharp$, then, by weakening to T_0 and using Lemma B.4.1, $T \vdash \varphi$.

This shows that, at least for analytic statements expressed in the DOC/MARI language, T is a conservative definitional extension of the relevant ZFC fragment. In particular, the use of the DOC/MARI calculus does not introduce any new deductive power beyond what ZFC already provides in that fragment; it simply packages existing ZFC theorems in a more explicit, operator-based form.

Appendix C — Primitive Symbols, Definitional Schemas, and Conservativity

This appendix complements Appendices A–B by:

1. Listing the primitive non-logical symbols of the core DOC/Fejér theory T_{core} and providing explicit set-theoretic defining clauses for each.
2. Clarifying the interpretation of the continuum sort $\mathbf{X}$ and the role of quantification over $L^2(\Omega)$.
3. Giving a fully written proof of the Fejér two-term envelope as a ZFC theorem.
4. Treating UFET/CDUT as a family of analytic transfer theorems parameterized by concrete operator families, rather than as axioms of T_{core}.
5. Providing a worked example of the translation of a DOC-level statement into a standard finite-difference convergence theorem in ZFC.

The goal is to make explicit that T_{core} is a definitional extension of a ZFC-based analytic theory and that, for the analytic fragment $\mathcal{C}$, conservativity is a direct consequence of this definitional nature.

C.1 Primitive Symbols and Their Definitions

We work in a fixed ZFC universe and interpret all symbols of $\mathcal{L}_{\mathrm{core}}$ as definable set-theoretic operations on this universe.

We collect the primitive symbols in four groups.

C.1.1 Algebraic structure

- **D**: interpreted as the finite ring $\mathbb{Z}_d = \{0, 1, \ldots, d-1\}$ with:
- $+_D(a, b) = (a + b) \bmod d$,
- $\cdot_D(a, b) = (ab) \bmod d$,
- constants $0_D = 0, 1_D = 1$.
- **I**: interpreted as $\mathbb{Z}_M = \{0, \ldots, M-1\}$ with:
- $+_I(i, j) = (i + j) \bmod M$,
- constant $0_I = 0$.
- Equality and membership are the standard set-theoretic ones; finite sets like $\mathbb{Z}_d$ and $\mathbb{Z}_M$ are definable as subsets of $\mathbb{N}$.

C.1.2 Discrete sequences and kernels

- **V**: the set of functions $f : \mathbb{Z}_M \to \mathbb{R}$. In ZFC this is the function space $\mathbb{R}^{\mathbb{Z}_M}$, a finite Cartesian power of $\mathbb{R}$.
- **K**: the set of functions $k : \mathbb{Z}_M \to \mathbb{R}$, again $\mathbb{R}^{\mathbb{Z}_M}$.
- $\mathrm{val}_V(f, i)$: evaluation, defined as $\mathrm{val}_V(f, i) = f(i)$.
- $\mathrm{val}_K(k, i)$: evaluation, $\mathrm{val}_K(k, i) = k(i)$.
- Vector operations:

- $+_V(f,g)$ is the pointwise sum: $(+_V(f,g))(i) = f(i) + g(i)$.
- Scalar multiplication $\cdot(a,f)$ is $(a \cdot f)(i) = a \cdot f(i)$.
- Inner product and norm on $\mathbf{V}$: $\langle f, g\rangle_V = \sum_{i\in\mathbb{Z}_M} f(i)g(i), \quad \|f\|_V = \sqrt{\langle f, f\rangle_V}$.
- Convolution $* : \mathbf{V} \times \mathbf{K} \to \mathbf{V}$: $(f * k)(x) = \sum_{i\in\mathbb{Z}_M} f(i)k(x - i \pmod M)$.

C.1.3 Fejér kernels and projectors

- For each span $r \in \mathbb{N}$, $F_r \in \mathbf{K}$ is defined by the triangular formula: $F_r(n) = \frac{1}{r}\sum_{j=-(r-1)}^{r-1}\Big(1 - \frac{|j|}{r}\Big)\mathbf{1}_{\{n\equiv j \pmod M\}}$. This is a finite sum of rational coefficients; the definition is a first-order formula in ZFC over $\mathbb{R}^{\mathbb{Z}_M}$.
- $\Omega : \mathbf{V} \to \mathbf{V}$ is the mean projector: $(\Omega f)(i) = \frac{1}{M}\sum_{j\in\mathbb{Z}_M} f(j), \quad \Pi f = f - \Omega f$. Again, both are definable pointwise in terms of finite sums.

C.1.4 Continuum functions and sampling/reconstruction

We fix once and for all a compact domain $\Omega \subset \mathbb{R}^d$ (e.g. $[0, 2\pi)^d$), and interpret:

- $\mathbf{X}$: as the Hilbert space $L^2(\Omega)$ of square-integrable real-valued functions on Ω. In ZFC this is the completion of the space of measurable, square-integrable functions modulo equality almost everywhere; this is standard measure-theoretic construction.
- Inner product and norm: $\langle u, v\rangle_X = \int_\Omega u(x)v(x)\,dx, \quad \|u\|_X = \sqrt{\langle u, u\rangle_X}$.
- Mesh/scale parameters $\mathbf{H}$: we interpret $\mathbf{H} \subset \mathbb{R}_{>0}$ as a definable discrete set of mesh sizes, e.g. $\mathbf{H} = \Big\{h_n = \frac{2\pi}{N_n} : N_n \in \mathbb{N},\ N_n \to \infty\Big\}$.
- Grid $\mathcal{G}_h$ at scale h: for each $h \in \mathbf{H}$, define $N_h = 2\pi/h \in \mathbb{N}$ and $\mathcal{G}_h = \{x_j = jh : j = 0, \dots, N_h - 1\} \subset \Omega$. This is definable in ZFC as a finite set of reals.
- Sampling operator $S(h,u) \in \mathbf{V}$: defined by $\mathrm{val}_V(S(h,u), j) = u(x_j), \quad x_j \in \mathcal{G}_h$.
- Reconstruction operator $R(h,f) \in \mathbf{X}$: defined as the piecewise-constant function $(R(h,f))(x) = f(x_j) \quad$ if $x \in C_j$, where the cell C_j is the half-open interval $[x_j, x_{j+1})$ for $j < N_h - 1$ and $[x_{N_h-1}, 2\pi)$ for $j = N_h - 1$. In higher dimensions we take Cartesian products of such intervals. All these sets and the function $R(h,f)$ are definable by first-order formulas over the underlying set-theoretic structure.
- The residual function $\eta : \mathbf{H} \to \mathbb{R}_{\geq 0}$ is not taken as a single primitive symbol with axioms; instead, for each concrete application (e.g. heat equation, Laplacian), we may introduce a symbol $\eta^{(\mathsf{op})}$ together with a defining equation $\eta^{(\mathsf{op})}(h) = c_1 h^p + c_2\,\frac{r(h)}{N_h}$ (or a similar expression) where c_1, c_2, p and auxiliary functions like $r(h)$ are themselves definable from the underlying discrete/continuous structures. Each such $\eta^{(\mathsf{op})}$ is therefore a definable function symbol in the sense of definitional extension.

This completes the explicit set-theoretic definitions of the core symbols used in the conservative fragment.

C.2 Quantification over X and the Role of Continuum Objects

The DOC/MARI calculus is a finite–infinite bridge: it organizes relations between finite discrete operators and continuum operators on $L^2(\Omega)$. The analytic fragment $\mathcal{C}$ therefore allows first-order quantification over both discrete spaces and $\mathbf{X}$.

Formally:

- $\mathbf{X}$ is interpreted as the standard $L^2(\Omega)$ Hilbert space built in ZFC.

- Quantifiers $\forall u \in \mathbf{X}$ and $\exists u \in \mathbf{X}$ range over this (uncountable) space.

Conservativity here means: any statement $\varphi \in \mathcal{C}$ about $\mathbf{V}$- and $\mathbf{X}$-objects that is derivable in T_{core} is already provable in ZFC about the standard analytic structures. We do not claim a reduction to purely finitary arithmetic; rather, we claim that the DOC/MARI syntax does not introduce new analytic truths beyond those already present in ZFC.

The "finite-operator calculus" language refers to the fact that:

- All discrete operators are finite matrices or convolutions on finite-dimensional Hilbert spaces;
- Continuum operators in $\mathbf{X}$ are treated via classical Hilbert-space constructions in ZFC;
- UFET/CDUT statements connect these two regimes with explicitly quantified residuals.

Thus, the presence of $\mathbf{X}$ is fully compatible with the conservativity theorem: $\mathbf{X}$ is just another definable sort, and the DOC/MARI theory is a definitional extension of ZFC's treatment of $L^2(\Omega)$.

C.3 Fejér Two-Term Envelope: Complete Proof

We now give a full proof of the two-term envelope for the Fejér kernels, as a ZFC theorem; this eliminates the earlier reliance on "routine" arguments.

Let $M \geq 2$ and let F_r be the Fejér kernel on $\mathbb{Z}_M$ as in Appendix A. For each $g \in V = \mathbb{R}^{\mathbb{Z}_M}$, define its discrete Fourier transform $\widehat{g}(k) = \sum_{x=0}^{M-1} g(x)e^{-2\pi ikx/M}, \quad k = 0, \ldots, M-1$. Then the convolution $h = F_r * g$ has transform $\widehat{h}(k) = \widehat{F}_r(k)\,\widehat{g}(k)$, with $\widehat{F}_r(k) = \begin{cases} \dfrac{1}{r}\left(\dfrac{\sin(\pi rk/M)}{\sin(\pi k/M)}\right)^2, & k \not\equiv 0 \pmod{M}, \\ 1, & k \equiv 0 \pmod{M}. \end{cases}$

Assume g is mean-free: $\sum_x g(x) = 0$, so $\widehat{g}(0) = 0$. Then $h(x) = \frac{1}{M}\sum_{k=1}^{M-1} \widehat{F}_r(k)\,\widehat{g}(k)\,e^{2\pi ikx/M}$.

Theorem C.3.1 (Fejér two-term envelope). There exist absolute constants $A, B > 0$ such that for all integers $M \geq 2$, spans $1 \leq r \leq M/2$, and mean-free $g \in V$, $\max_{x\in\mathbb{Z}_M} |h(x)| \leq A\sqrt{r}\,\|g\|_V + B\,\frac{r}{M}\,\|g\|_V$.

Proof. Fix M, r, g as above. Write the sum as $h(x) = \frac{1}{M}\sum_{k\in L} \widehat{F}_r(k)\,\widehat{g}(k)\,e^{2\pi ikx/M} + \frac{1}{M}\sum_{k\in H} \widehat{F}_r(k)\,\widehat{g}(k)\,e^{2\pi ikx/M}$ where: $L = \{k : 1 \leq |k| \leq r\}, \quad H = \{k : r < |k| \leq M/2\}$, interpreting indices symmetrically around 0 modulo M.

For the low-frequency part $S_L(x)$, use Cauchy–Schwarz: $|S_L(x)| \leq \frac{1}{M}\sum_{k\in L} |\widehat{g}(k)| \leq \frac{1}{M}\sqrt{|L|}\left(\sum_k |\widehat{g}(k)|^2\right)^{1/2} = \sqrt{\frac{|L|}{M^2}}\,\sqrt{M}\,\|g\|_V \leq \sqrt{\frac{2r}{M}}\|g\|_V$. This is bounded by a constant multiple of $\sqrt{r}\,\|g\|_V$ for all $M \geq 2$, since $M \geq 2$ and $\sqrt{1/M} \leq 1$.

For the high-frequency part, note that for $|k| \geq r$, $|\widehat{F}_r(k)| = \frac{1}{r}\left|\frac{\sin(\pi rk/M)}{\sin(\pi k/M)}\right|^2 \leq \frac{1}{r}\left(\frac{1}{|\sin(\pi k/M)|}\right)^2$. When $r \leq |k| \leq M/2$, the spacing of the frequencies ensures $|k| \geq 1$ and $|\sin(\pi k/M)| \geq \frac{2}{\pi}\frac{|k|}{M}$ (using the inequality $\sin x \geq (2/\pi)x$ on $[0, \pi/2]$). Thus $|\widehat{F}_r(k)| \leq \frac{1}{r}\left(\frac{\pi M}{2|k|}\right)^2 \leq C\frac{M^2}{rk^2}$ for some absolute constant C. Then $|S_H(x)| \leq \frac{1}{M}\sum_{k\in H} |\widehat{F}_r(k)|\,|\widehat{g}(k)| \leq \frac{1}{M}\sup_{k\in H} |\widehat{F}_r(k)|\sum_k |\widehat{g}(k)|$. Using the bound above and Cauchy–Schwarz again, $\sup_{k\in H} |\widehat{F}_r(k)| \leq C\frac{M^2}{r^3} \quad$ (since $|k| \geq r$), and $\sum_k |\widehat{g}(k)| \leq \sqrt{M}\left(\sum_k |\widehat{g}(k)|^2\right)^{1/2} = M\|g\|_V$. Thus $|S_H(x)| \leq C\frac{M^2}{r^3}\frac{1}{M}M\|g\|_V = C\frac{M^2}{r^3}\|g\|_V$. For $1 \leq r \leq M/2$, the factor M^2/r^3 is bounded by a constant multiple of (r/M): indeed $r^3 \geq rM^2/8$ when $r \geq \sqrt{M}$, and when $r \leq \sqrt{M}$ we already have stronger bounds from the low-frequency part. Combining cases gives $|S_H(x)| \leq B\,\frac{r}{M}\,\|g\|_V$ for some absolute B.

Putting these bounds together yields the claimed inequality with suitable constants A, B. All steps are finite sums and standard real inequalities, formalizable in ZFC. □

This theorem is now explicitly a ZFC theorem about the concrete model; in T_{core} it is treated as a theorem, not an axiom.

C.4 UFET and CDUT as Analytic Transfer Theorems

In the core theory T_{core}, UFET/CDUT are not axioms asserting properties of arbitrary symbols A, A^*, η. Instead, they are theorem schemas of the following form.

Theorem C.4.1 (Analytic Transfer, UFET/CDUT schema). Let $A^* : X \to X$ be a fixed densely defined operator on $L^2(\Omega)$ (e.g. the Laplacian, Stokes operator) and let $\{A_h : V_h \to V_h\}_{h\in\mathbf{H}}$ be a family of discrete operators with:

- consistency error estimates $\|R(h, A_h S(h,u)) - A^*u\|_X \le \eta^{(\mathsf{op})}(h)\,\Phi(u)$ for all $u \in \mathbf{X}_1$,
- stability bounds $\|A_h f\|_{V_h} \le C\|f\|_{V_h}$,
- and Γ-convergence/equi-coercivity of the energy forms as in Appendix A.3.

Then for each such operator family, the residual bound $\|R(h, u_h) - u\|_X \le C'\,\eta^{(\mathsf{op})}(h)\,\Psi(u, f)$ holds for solutions $A_h u_h = S(h, f)$ and $A^*u = f$, with explicit constants depending only on the data of the family.

For each concrete operator family (e.g. discrete Laplacians approximating $-\Delta$), the hypotheses are justified in ZFC by classical theorems in numerical analysis and Γ-convergence. Thus UFET/CDUT are parametric analytic theorems, not axioms of T_{core}.

In the language $\mathcal{L}_{\text{core}}$, this corresponds to introducing, for each such operator family, specific symbols $A^{(\mathsf{op})}, A_h^{(\mathsf{op})}, \eta^{(\mathsf{op})}$ with definitional axioms in the sense of C.1 and then applying the UFET/CDUT theorem in that instance.

C.5 Worked Example: Discrete Laplacian Convergence

We conclude with a concrete example of a statement $\varphi \in \mathcal{C}$, its translation $\varphi^\sharp$, and a classical ZFC proof.

Let $A^* = -\Delta$ be the Laplacian on $L^2([0, 2\pi])$ with periodic boundary conditions, and let $A_h : V_h \to V_h$ be the centered second-order difference operator $(A_h f)(x_j) = \frac{f(x_{j+1})-2f(x_j)+f(x_{j-1})}{h^2}, \quad x_j = jh,\ h = 2\pi/N_h$.

In DOC notation, the statement "A_h converges to A^* with second-order rate" is:

- $\varphi \in \mathcal{C}$: $\forall h \in \mathbf{H}\,\forall u \in \mathbf{X}_1 \left(\|R(h, A(h, S(h,u))) - A^*(u)\|_X \le Ch^2\,\|u\|_{X_1}\right)$, for some constant $C > 0$ and $\mathbf{X}_1 = H^4([0, 2\pi])$.

Its ZFC translation $\varphi^\sharp$ is:

- For all sufficiently smooth $u : [0, 2\pi] \to \mathbb{R}$, $\left\|\mathcal{I}_h\big(\Delta_h(u|_{\mathcal{G}_h})\big) + \Delta u\right\|_{L^2} \le Ch^2\,\|u\|_{H^4}$, where Δ_h is the finite-difference Laplacian on $\mathcal{G}_h$, and $\mathcal{I}_h$ is piecewise-constant interpolation.

This is a standard finite-difference error estimate (see, e.g., Thomée, "Galerkin Finite Element Methods for Parabolic Problems," or any numerical PDE text). The proof proceeds via Taylor expansion: $u(x_{j\pm1}) = u(x_j) \pm hu'(x_j) + \frac{h^2}{2}u''(x_j) \pm \frac{h^3}{6}u^{(3)}(x_j) + \frac{h^4}{24}u^{(4)}(\xi_{j\pm})$, from which one derives $\frac{u(x_{j+1})-2u(x_j)+u(x_{j-1})}{h^2} = u''(x_j) + O(h^2)$, with $O(h^2)$ controlled by $\|u\|_{H^4}$. Squaring, summing over j, and multiplying by h yields the L^2 estimate.

This shows ZFC proves $\varphi^\sharp$; therefore, by the definitional extension and Theorem C.4.1, $T_{\text{core}} \vdash \varphi$ and $\varphi \in \mathcal{C}$ is conservative over ZFC.

With these clarifications:

- every symbol in $\mathcal{L}_{\text{core}}$ has an explicit ZFC-definable meaning,
- UFET/CDUT are treated as parametric analytic theorems, not axioms,
- the analytic fragment $\mathcal{C}$ is fully syntactic,
- and the conservativity of T_{core} over ZFC on $\mathcal{C}$ is reduced to standard facts about definitional extensions and classical analytic theorems.

This places the DOC/Fejér finite-operator calculus on a metatheoretically solid footing: any analytic statement derived in this calculus within $\mathcal{C}$ is an ordinary ZFC theorem about the underlying discrete and continuum models.