PH-1 — One-Action Field Dynamics on a Finite Substrate

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Abstract

We define a single finite action functional $S_{\mathrm{DOC}}[\Phi]$ on an explicit discrete substrate and derive its discrete Euler–Lagrange equations as lawful finite dynamics. The construction is conservative: it assumes (and does not modify) the mathematics-track infrastructure—Deterministic Operator Calculus (DOC) admissible operators, the DRPT/Superset arithmetic substrate, symmetry-constrained fixed-point selection (SCFP/SCFP++), and the finite-to-continuum transfer discipline (UFET / Residual Equivalence). The objective of PH-1 is to show that one unified action can generate a family of deterministic, stable, finite equations whose continuum facades (under admissible refinement) align with the operator forms used throughout later physics papers: diffusion, wave propagation, gauge-type transport, and dissipative/SPDE-like evolution.

The action uses three ingredients only: (i) finite fields Φ defined on explicit lattices X_h, (ii) DOC-admissible kernels (and tensor products thereof) as lawful locality and smoothing primitives, and (iii) a finite invariant coefficient alphabet $\mathcal{S}_{\mathrm{struct}}$ supplying structural coefficients without observational tuning in PH-1. The resulting discrete dynamics preserve DOC legality invariants (mean preservation; positivity preservation where applicable; contractive suppression on mean-free components for admissible smoothing) and admit a quantified finite-to-continuum statement: for observables computed from $S_{\mathrm{DOC}}[\Phi]$, there exists an admissible residual schedule $\eta(h) \to 0$ controlling the discrepancy between the discrete observable and its continuum facade.

Where this paper references computed values or certificates produced by the program, those items are treated as evidence and cited via the canonical Authority-of-Record (AoR) bundle. Representative exact outputs include the Φ-channel rationals $\alpha = 1/137$, $\alpha_s = 2/17$, and $\sin^2\theta_W = 7/30$ (see the AoR constants tables).

Keywords: finite action principle; discrete Euler–Lagrange; admissible smoothing; Fejér kernels; lawful operators; finite-to-continuum transfer; residual equivalence; gauge-covariant differences

0. Reader Contract

0.1 Audience and intent

This paper is written for physicists and mathematically literate readers who are willing to treat standard continuum PDE/field theory as a facade of a more primitive finite substrate. The paper's intent is not to propose a new continuum Lagrangian. Its intent is to (i) define a single action on a finite substrate, (ii) derive finite equations of motion from that action, and (iii) specify the admissible refinement discipline under which those finite dynamics admit continuum facades.

0.2 Dependency boundary

PH-1 assumes the following have already been fixed (and are not revised here):

1. The arithmetic substrate on residue rings $\mathbb{Z}_d$ and its DRPT/Superset geometry layer.
2. DOC admissibility: the legality class of deterministic operators on finite ℓ^2-spaces over cyclic groups, including the admissible kernel families used for smoothing/locality.
3. The structural coefficient alphabet $\mathcal{S}_{\text{struct}}$ (a finite, auditable set of rational/invariant tokens) and the rule that coefficients are selected from this alphabet, not tuned continuously.
4. UFET / Residual Equivalence: the finite-to-continuum transfer discipline, including explicit residual schedules $\eta(h)$ and scheme-independence inside the admissible class.

Consequently, PH-1 should be read as an application paper in the strict sense: it constructs dynamics without altering the legality lawbook.

0.3 Claim boundary (what PH-1 does and does not claim)

PH-1 claims:

- **Existence:** there exists an explicit finite action $S_{\text{DOC}}[\Phi]$ on an explicit finite lattice X_h, built only from DOC-admissible locality primitives and coefficients from $\mathcal{S}_{\text{struct}}$.
- **Derivation:** the discrete Euler–Lagrange equations derived from $S_{\text{DOC}}[\Phi]$ yield well-defined finite stationarity conditions and (when embedded into standard time-stepping forms) yield stable deterministic update dynamics within the DOC admissible boundary.
- **Transfer:** under admissible refinement, the resulting finite dynamics admit continuum facades in the UFET sense with an explicit residual schedule $\eta(h) \to 0$.

PH-1 does not claim:

- That any particular choice of sector coefficients is uniquely "the Standard Model" or "the Universe." Sector identification and parameter closure are the scope of PH-2 and PH-3, and are evidence-backed via the AoR.
- That continuum physics is fundamental. Continuum objects appear only as facades attached to finite constructions under a disciplined limit.
- That numerical matches alone establish truth. All empirical-facing claims are fenced by AoR evidence, and all mathematical-facing claims are fenced by DOC admissibility and UFET residual bounds.

Evidence Capsule (Canonical AoR Citation Surface)

Repository: https://github.com/public-arch/Marithmetics
AoR tag (canonical citation anchor): `aor-20260209T040755Z`
AoR folder: `gum/authority_archive/AOR_20260209T040755Z_0fc79a0`
Bundle sha256: `c299b1a7a8ef77f25c3ebb326cb73f060b3c7176b6ea9eb402c97273dc3cf66c`

Canonical artifacts:

- **Master bundle (zip):** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /MARI_MASTER_RELEASE_20260209T040755Z_ 0fc79a0 .zip

- **GUM Report v32:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T043 gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /report/GUM_Report_v32_2026-01-25_ 04-27-46Z .pdf
- **Report manifest:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T0439 gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /report/GUM_Report_v32_2026-01-25_ 04-27-46Z .pdf.manifest.json
- **Claim ledger:** https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /claim_ledger.jsonl
- **AoR summary:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T043902 gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /SUMMARY.md
- **Run transcript:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T04390 gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /runner_transcript.txt
- **Run metadata:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T043902 gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /run_metadata.json
- **Constants tables:**
  https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z / gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /GUM_BUNDLE_v30_20260209T040755Z / tables/constants_master.csv
  https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z / gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /GUM_BUNDLE_v30_20260209T040755Z / tables/constants_master.json
- **Bundle seal file:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T04390 gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /GUM_BUNDLE_v30_20260209T040755Z / bundle_sha256.txt

Baseline mathematics artifact (unchanged master reference):

- **Deterministic Operator Calculus (DOC):** https://github.com/public-arch/Marithmetics/blob/main/publication_spine/Deterministic%20Operator%20Calculus.pdf

1 Introduction and Scope

1.1 Why an action on a finite substrate

The standard action principle in physics is typically stated on continuum manifolds and produces continuum Euler–Lagrange equations. PH-1 inverts the ontological priority: the primitive object is a finite substrate with explicit operators and explicit legal constraints. An "action" is therefore not assumed to be an integral over a manifold; it is defined as a finite sum over a finite domain using only operators whose stability and legality are auditable in the DOC sense.

This inversion has two consequences that determine the paper's structure:

- First, locality is not asserted verbally; it is implemented by explicitly local difference operators and by DOC-admissible kernels whose Fourier multipliers lie in a legal band (in particular, admissible smoothing is mean-preserving and contractive on mean-free components).
- Second, continuum PDEs appear only as facades, attached after the fact via an admissible refinement schedule $h \to 0$ with explicit residual budgets $\eta(h)$.

1.2 The finite analytic architecture assumed (compressed map)

PH-1 assumes an established pipeline, summarized here only to fix terminology:

- **DRPT/Superset layer:** an arithmetic substrate over residue rings $\mathbb{Z}_d$ supplying structured periodic families, index geometry tokens, and cross-base invariants.
- **DOC layer:** deterministic, auditable operator calculus on finite $\ell^2(\mathbb{Z}_M)$ spaces; admissible kernels provide stable smoothing/locality primitives.
- **Lawbook selection (SCFP/SCFP++):** bounded elimination dynamics selecting distinguished integers/invariants that seed structural coefficients.
- **Structural alphabet $\mathcal{S}_{\text{struct}}$:** a fixed finite set of coefficient tokens (rational combinations of selected invariants and admissible operator invariants), used as the coefficient source for physics sector reductions.
- **UFET / Residual Equivalence:** a quantified discipline for finite-to-continuum transfer, defining "continuum facade" with explicit residual schedules and scheme-independence within admissible classes.

PH-1 uses these layers as axioms of practice: it does not restate their proofs in full, but it does restate the minimal definitions needed to make the action construction and its variations fully explicit.

1.3 What "lawful" means in this paper

Because PH-1 is explicitly finite, "lawful" is not a metaphor. In this paper, an operator or update rule is lawful if it is built from the DOC admissible class and therefore inherits the DOC stability invariants relevant to the construction at hand (e.g., mean preservation and contractivity on mean-free components for admissible smoothing operators). When we later refer to "Designed FAIL," we mean a concrete violation of DOC admissibility (for example, a kernel whose Fourier multiplier exceeds 1 on some mode), which produces an observable finite instability and is therefore excluded from the claim boundary.

1.4 Organization of the paper

Section 2 defines the substrate objects used by the action: the discrete domain, field bundle, admissible differences, and admissible smoothing/locality primitives. Section 3 defines the single action $S_{\text{DOC}}[\Phi]$ and derives discrete Euler–Lagrange equations, first for a scalar sector and then in the gauge-covariant form needed for later papers. Section 4 records sector reductions (diffusive, wave, transport, dissipative) as explicit specializations of the one-action. Section 5 formalizes the admissible refinement and residual budgets used to attach continuum facades. Section 6 provides worked micro-examples intended to make the mechanics legible without relying on any empirical matching.

2 Substrate, Lattice, and Field Content

PH-1 is written to be compatible with two complementary "substrates":

1. A purely arithmetic substrate (finite, residue-class based), used to define canonical invariants and symmetry-constrained structures that are demonstrably stable across base representation; and

2. A physical lattice substrate (finite, periodic or bounded), used to define discrete fields, projectors, and admissible operators whose continuum limits are controlled by a residual law and whose numerical behavior is captured by deterministic manifests.

The aim of this section is to specify the objects that will appear in the action functional of §4, and to state the admissibility constraints that make the subsequent physics well-posed and audit-grade.

2.1 Arithmetic substrate and the Superset layer (finite residue geometry)

PH-1 assumes the existence of a finite residue geometry generated by a fixed modulus and its structured families.

Let $M \in \mathbb{N}$, $M \geq 2$, and let $\mathbb{Z}_M := \mathbb{Z}/M\mathbb{Z}$. The arithmetic substrate used throughout the MARI program is built from:

- a cyclic indexing set $I_M = \{0, 1, \ldots, M-1\}$,
- a finite Hilbert space $\mathcal{H}_M := \mathbb{R}^M$ (or $\mathbb{C}^M$ when convenient), and
- the cyclic shift (observer) group acting on $\mathcal{H}_M$ by permutation operators P_k (see Foundations for the canonical definitions and invariance properties).

The Superset layer is the constructive device that produces structured, base-portable families of arithmetic objects (tables, tilings, invariants, and their symmetry classes) from residue arithmetic. In its minimal form—sufficient for PH-1—it provides:

1. A canonical finite domain on which invariants are defined (typically $\mathbb{Z}_M$ or products $\mathbb{Z}_{M_1} \times \cdots \times \mathbb{Z}_{M_d}$).
2. A natural symmetry group (shifts, reflections, and admissible conjugations) under which the invariants are stable.
3. A finite spectral calculus (Fourier diagonalization on cyclic groups) that allows one to state and verify operator admissibility in a basis-independent way.

PH-1 does not require the reader to accept any specific interpretive identification of this arithmetic substrate with spacetime. Instead, the role of the substrate here is precise and limited: it supplies a finite, symmetry-controlled arena in which operators and constraints can be stated exactly and then transferred to physical lattices with an explicit residual budget (§5).

Audit note (AoR anchoring). The canonical AoR surface for the current release enumerates the master bundle, indices, and reproducible run transcripts. In particular, the bundle indices and the reproducibility tables (including demo and falsification matrices) are pinned under the tag aor-20260209T040755Z in:

- https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z/gum/authority_archive/AOR_20260209T040755Z_0fc79a0/SUMMARY.md
- https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z/gum/authority_archive/AOR_20260209T040755Z_0fc79a0/GUM_BUNDLE_v30_20260209T040755Z/tables/demo_index.csv
- https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z/gum/authority_archive/AOR_20260209T040755Z_0fc79a0/MARI_MASTER_RELEASE_20260209T040755Z_0fc79a0.zip

These indices are the citation surface for all "what ran / what was produced / what was falsified" questions, independent of any narrative in this manuscript.

2.2 Physical lattice substrate $\mathcal{G}_h$: domains, fields, and inner products

For physics statements, PH-1 works on a finite lattice $\mathcal{G}_h$ equipped with an ℓ^2 inner product and the discrete analogues of the differential operators relevant to the domain (EM, fluid, linearized GR, etc.). The default setting for the flagship demonstrations is the periodic torus (for clean spectral diagonalization), but the formalism is compatible with bounded domains provided the boundary projector is specified (see §3).

2.2.1 Periodic grids (torus model)

Fix spatial dimension $d \in \{1, 2, 3\}$ and a mesh parameter $h > 0$. The periodic grid is represented as a finite abelian group:

$$\mathcal{G}_h \cong \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_d}, \qquad N_i \in \mathbb{N},\ N_i \geq 2,$$

with the understanding that physical length scales are encoded by the map $x \mapsto hx$ when desired. The finite Fourier transform diagonalizes all circulant (convolution) operators on $\mathcal{G}_h$.

Define the discrete inner product for scalar fields $u, v : \mathcal{G}_h \to \mathbb{R}$ by

$$\langle u, v\rangle_{\ell^2} := \sum_{x\in\mathcal{G}_h} u(x)v(x), \qquad \|u\|_{\ell^2}^2 := \langle u, u\rangle_{\ell^2}.$$

Vector- and tensor-valued fields use the componentwise extension.

2.2.2 Bounded grids (domain model)

For bounded domains $\Omega \subset \mathbb{R}^d$, PH-1 assumes:

- a discrete sampling S_h onto a mesh $\mathcal{G}_h \subset \Omega$, and
- an explicitly specified boundary projector $\Pi_{B,h}$ that enforces the desired boundary constraints (Dirichlet/Neumann/mixed or domain-specific constraints).

The admissibility requirements on $\Pi_{B,h}$ are the same as for all PH-1 projectors: idempotence (or quantified near-idempotence under refinement) and compatibility with the operator class used in the action (§3).

2.3 Field content Φ: kinematics first, interpretation second

PH-1 uses a field-first formulation. That is: we specify the algebraic type and constraint class of fields, and only later attach the domain interpretation (EM, YM, GR, fluids) by choosing the projector and operator symbols.

A PH-1 state is a tuple

$$\Phi := (\phi,\ A,\ u,\ h,\ \dots)$$

with components assigned to the appropriate discrete form-degree (in the sense of discrete exterior calculus when used) or to an equivalent primal/dual staggering. The minimal suite required for the flagship bridge demonstrations is:

- Scalar fields $\phi : \mathcal{G}_h \to \mathbb{R}$ (0-forms),
- Vector potentials / velocities $A, u : \mathcal{G}_h \to \mathbb{R}^d$ (1-forms or edge-fields),
- Derived curls / vorticities / fluxes $B = \nabla_h \times A$ (2-forms or face-fields in 3D),
- Transverse-traceless tensor fields $h : \mathcal{G}_h \to \mathbb{R}^{d\times d}$ for linearized gravitational-wave sectors, with TT projector $\Pi_{\mathrm{TT},h}$.

The key design choice is that constraints are not enforced by hand: they are enforced by projectors (Helmholtz, TT, Hodge, domain-boundary) that appear explicitly in the action and in the update operators. This has two consequences that matter for rigor:

1. Constraint satisfaction becomes a theorem (up to residual), not an implementation detail.
2. Failures can be engineered and detected reproducibly (e.g., omitting a projector produces a predictable constraint-RMS signature).

2.4 Structural coefficients and admissible operator families

PH-1 distinguishes two categories of "inputs":

1. **Structural inputs** (finite, discrete, admissibility-constrained): lattice sizes, bandlimits, window spans, projector types, and operator symbol families. These are permitted inputs to the theory.
2. **Empirical anchors** (dimensioned scalings): used only when mapping the dimensionless model back to SI-interpreted quantities. These are not permitted to alter the discrete invariants or the admissibility class; they appear as external calibration in later papers, not as degrees of freedom inside PH-1.

The operators used in PH-1 updates are taken from a universal class of admissible Fourier multipliers on periodic grids and their admissible analogues on bounded grids. Concretely, on $\mathcal{G}_h \cong \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_d}$, an admissible linear operator K_h is assumed to be diagonalizable by the discrete Fourier transform with symbol $\widehat{K}_h(k)$ satisfying, at minimum:

- Reality / symmetry appropriate to the field type (scalar/vector/tensor; parity constraints),
- Non-pathological low-mode behavior (finite and stable near $k = 0$ when required), and
- Admissibility constraints (the DOC constraints) guaranteeing:
- stable energy behavior,
- controlled aliasing under smoothing, and
- a residual law under refinement.

This is the point at which the deterministic operator calculus (DOC) enters PH-1: DOC is the admissibility contract that makes the finite engine defensible as a scientific instrument rather than as an arbitrary discretization. The physics of PH-1 is therefore not "choose a scheme and hope"; it is "choose from an admissible class whose error and failure modes are part of the specification."

The AoR evidence surface explicitly tracks the DOC and EM bridge demonstrations as reproducible runs (see, e.g., the AoR stdout/stderr logs for the deterministic operator calculus comparison and the electromagnetism demo):

- https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z/gum/authority_archive/AOR_20260209T040755Z_0fc79a0/GUM_BUNDLE_v30_20260209T040755Z/logs/controllers__demo-56-deterministic-operator-calculus-vs-fd.out.txt
- https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z/gum/authority_archive/AOR_20260209T040755Z_0fc79a0/GUM_BUNDLE_v30_20260209T040755Z/logs/controllers__demo-59-electromagnetism.out.txt

In §3–§5, we make the admissibility constraints explicit (projectors, locality primitives, and residual budgets) and then state the single-action formulation that generates the bridge-consistent update operators used across domains.

3 The One-Action Functional $S_{\mathrm{DOC}}[\Phi]$

This section defines a single finite action functional on the DOC lattice façade and derives the corresponding discrete Euler–Lagrange equations. The construction is intentionally modular: one action yields multiple sector reductions by (i) selecting which fields are active and (ii) enabling subsets of coupling coefficients drawn from the fixed invariant alphabet $\mathcal{S}_{\mathrm{struct}}$ (defined in the number-theory track).

3.1 Discrete integration and action density

Let

$$X \;=\; \mathbb{Z}_{M_t} \times \mathbb{Z}_{M_x}^d, \qquad x = (t, \mathbf{n}),$$

with periodic boundary conditions in all coordinates. Fix lattice steps $h_t > 0$ and $h_x > 0$ (for notational simplicity we take a uniform spatial step; anisotropic steps are admissible but not needed here).

For any scalar density $\mathcal{L} : X \to \mathbb{R}$, define the discrete integral

$$\sum_{x \in X} \mathcal{L}(x)\, h_t\, h_x^d,$$

which is the finite analogue of $\int \mathcal{L}\, dt\, d^d x$.

We define an action functional by specifying a Lagrangian density $\mathcal{L}(\Phi; x)$ that depends on fields Φ and their DOC-local differences at x, and then summing over the lattice:

$$S_{\mathrm{DOC}}[\Phi] \;\coloneqq\; \sum_{x \in X} \mathcal{L}(\Phi; x)\, h_t\, h_x^d.$$

3.2 Field bundle and coefficient alphabet

Let the full field bundle be

$$\Phi \equiv (\phi, \psi, U, \mathfrak{g}),$$

where:

- $\phi : X \to \mathbb{R}$ is a real scalar façade;
- $\psi : X \to \mathbb{C}^{N_s}$ is a spinor-like façade (finite-dimensional, with N_s fixed by the sector);
- $U_\mu(x) \in G_{\mathrm{eff}}$ are link variables (a finite gauge façade, discussed below);
- $\mathfrak{g}$ denotes finite geometric data supplied externally (for example: a Superset-derived collar/area token and its DOC-smoothed derivatives), treated as a background in PH-1.

The non-negotiable constraint in PH-1 is that every coupling coefficient is drawn from the fixed finite alphabet $\mathcal{S}_{\mathrm{struct}}$. Concretely: each coefficient is either (i) a rational expression in the invariant tuple $(w_U, s_2, s_3, q_2, q_3, v, \Theta)$, (ii) a DOC-admissible operator invariant (spectral envelope, contraction factor, etc.), or (iii) a Superset invariant (tile area ratios, residue-weighted area measures, etc.). No parameter in PH-1 is tuned against observational targets.

We denote couplings generically by

$$\kappa_0, \kappa_1, \kappa_2, \dots \in \mathrm{span}_{\mathbb{Q}}(\mathcal{S}_{\mathrm{struct}}),$$

with the understanding that later sector papers (PH-2 onward) identify specific values and corresponding evidence artifacts.

3.3 Lagrangian density: matter + gauge + geometry + dissipation

We define the one-action density as a sum of four finite, local terms:

$$\mathcal{L}(\Phi; x) = \mathcal{L}_{\mathrm{m}}(\phi, \psi, U; x) + \mathcal{L}_{\mathrm{g}}(U; x) + \mathcal{L}_{\mathrm{geom}}(\phi, \mathfrak{g}; x) + \mathcal{L}_{\mathrm{diss}}(\phi; x).$$

Each term below is written to be: (i) DOC-local (built from admissible difference/smoothing primitives), (ii) finite on every lattice, and (iii) stable under admissible refinement.

(A) Matter term $\mathcal{L}_{\mathrm{m}}$

Scalar kinetic and potential. Let ∇_t and ∇_i denote DOC-local (first-order) differences in time and the i-th spatial direction, respectively. Define

$$\mathcal{L}_{\phi}(x) = \frac{\kappa_\phi}{2} \left(\nabla_t \phi(x)\right)^2 - \frac{\kappa_\phi c_\phi^2}{2} \sum_{i=1}^{d} (\nabla_i \phi(x))^2 - \kappa_V \, V(\phi(x)),$$

where $c_\phi \in \mathrm{span}_{\mathbb{Q}}(\mathcal{S}_{\mathrm{struct}})$ is a finite wave-speed façade parameter and V is a chosen finite polynomial potential, e.g.

$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4, \qquad m^2, \lambda \in \mathrm{span}_{\mathbb{Q}}(\mathcal{S}_{\mathrm{struct}}).$$

Spinor kinetic in a gauge background. Let D be the DOC-local discrete Dirac operator built from gauge-covariant differences (defined once link variables are fixed) and a fixed set of matrices γ^μ. Define

$$\mathcal{L}_{\psi}(x) = \kappa_\psi \, \Re\Big(\overline{\psi(x)}^{\top} (D\psi)(x)\Big) - \kappa_m \, \overline{\psi(x)}^{\top} \psi(x).$$

Yukawa-like interaction. Define the matter density as

$$\mathcal{L}_{\mathrm{m}}(x) = \mathcal{L}_{\phi}(x) + \mathcal{L}_{\psi}(x) + \kappa_{\mathrm{Y}} \, \phi(x) \, \overline{\psi(x)}^{\top} \psi(x),$$

with $\kappa_{\mathrm{Y}} \in \mathrm{span}_{\mathbb{Q}}(\mathcal{S}_{\mathrm{struct}})$.

(B) Gauge term $\mathcal{L}_{\mathrm{g}}$

Fix a finite effective gauge group G_{eff} and a representation with trace Tr. For each oriented plaquette (μ, ν), define the plaquette holonomy

$$U_{\mu\nu}(x) := U_\mu(x) \, U_\nu(x + \hat{e}_\mu) \, U_\mu(x + \hat{e}_\nu)^{-1} \, U_\nu(x)^{-1}.$$

Define a gauge curvature scalar

$$\mathcal{F}_{\mu\nu}(x) := \mathrm{Tr}\big(I - U_{\mu\nu}(x)\big),$$

and set

$$\mathcal{L}_{\mathrm{g}}(x) = \frac{\kappa_g}{2} \sum_{\mu<\nu} |\mathcal{F}_{\mu\nu}(x)|^2 .$$

This term is finite, local, and gauge-invariant under lattice gauge transformations.

(C) Geometry coupling $\mathcal{L}_{\mathrm{geom}}$

Geometry in PH-1 is encoded by a bounded finite scalar field $\kappa(x)$ derived from the Superset collar/area layer and DOC smoothing. The intent is not to assume a continuum metric tensor; rather, $\kappa(x)$ acts as a discrete curvature façade at the chosen scale.

Define a minimal curvature coupling:

$$\mathcal{L}_{\mathrm{geom}}(x) = \kappa_{\mathrm{R}}\, \kappa(x)\, \phi(x)^2 + \kappa_{\nabla\kappa} \sum_{i=1}^{d} (\nabla_i \kappa(x))(\nabla_i \phi(x))\, \phi(x).$$

Both couplings are finite and DOC-local. They serve as the bridge point to later curvature-driven sector reductions (including the GR façade treated explicitly in later physics papers).

(D) Dissipation $\mathcal{L}_{\mathrm{diss}}$

DOC admissibility isolates a stable dissipative mechanism: suppress mean-free modes while preserving the mean. In action form, dissipation can be represented by a quadratic penalty on a DOC-smoothed residual.

Fix a DOC-admissible smoothing operator S (for example, Fejér smoothing tensorized across spatial dimensions) and define the mean projector Ω on each time slice. Let

$$r(x) := \big((I-\Omega)\phi\big)(x) - \big(S(I-\Omega)\phi\big)(x),$$

and set

$$\mathcal{L}_{\mathrm{diss}}(x) = \frac{\kappa_{\mathrm{diss}}}{2}\, r(x)^2.$$

Because S is contractive on mean-free components, this term penalizes unsuppressed high-mode energy in a DOC-lawful way. This is the canonical “stability governor” that later papers specialize into the elimination and transport façades.

3.4 Discrete Euler–Lagrange equations

We now derive the discrete Euler–Lagrange equations for the scalar field ϕ. The spinor and gauge variations follow the same pattern with covariant differences and the appropriate group variation; the scalar case is the cleanest representative of the DOC mechanism.

Let $\phi \mapsto \phi + \epsilon\, \delta\phi$ with $\delta\phi$ arbitrary and periodic. Differentiating $S_{\mathrm{DOC}}[\Phi]$ at $\epsilon = 0$ and applying discrete summation by parts yields the stationarity condition

$$\frac{\delta S_{\mathrm{DOC}}}{\delta\phi}(x) = 0 \qquad \text{for all } x \in X.$$

Discrete integration by parts. For the kinetic term, the standard finite identity (periodic boundaries) gives

$$\sum_x (\nabla_t \phi)(\nabla_t \delta\phi)\, h_t h_x^d = -\sum_x (\nabla_t^* \nabla_t \phi)\, \delta\phi\, h_t h_x^d,$$

and similarly for each spatial direction:

$$\sum_x (\nabla_i \phi)(\nabla_i \delta\phi)\, h_t h_x^d = -\sum_x (\nabla_i^* \nabla_i \phi)\, \delta\phi\, h_t h_x^d.$$

Collecting terms. The potential contributes $\kappa_V V'(\phi)\, \delta\phi$. The geometry term contributes a finite DOC-local expression; we denote by $\mathcal{G}(x)$ the explicit variation of the mixed $(\nabla\kappa)(\nabla\phi)\phi$ coupling (a finite combination of κ, ϕ, and their discrete differences). The dissipation term yields a DOC-lawful linear operator supported on mean-free components; we denote this contribution by $\mathcal{D}(x)$.

Therefore, the Euler–Lagrange equation for ϕ can be written as

$$\kappa_\phi \left(\nabla_t^* \nabla_t \phi(x) \;-\; c_\phi^2 \sum_{i=1}^{d} \nabla_i^* \nabla_i \phi(x) \right) + \kappa_V \, V'(\phi(x)) - \kappa_{\mathrm{R}} \, \kappa(x) \, \phi(x) - \kappa_{\nabla\kappa} \, \mathcal{G}(x) + \kappa_{\mathrm{diss}} \, \mathcal{D}(x) = 0. \tag{1}$$

A canonical reduction. In the sector $\kappa_{\nabla\kappa} = 0$ and $\kappa_{\mathrm{diss}} = 0$, the scalar equation reduces to a standard discrete Klein–Gordon / wave façade:

$$\nabla_t^* \nabla_t \phi \;-\; c_\phi^2 \Delta \phi \;+\; \frac{\kappa_V}{\kappa_\phi} V'(\phi) \;-\; \frac{\kappa_{\mathrm{R}}}{\kappa_\phi} \kappa \, \phi = 0,$$

with $\Delta \phi := \sum_{i=1}^{d} \nabla_i^* \nabla_i \phi$. The point is not that this is "assumed," but that it is a controlled sector reduction of one DOC-lawful action.

3.5 Lawfulness of update dynamics

PH-1 uses the Euler–Lagrange equations in two complementary ways:

1. **Stationary solutions:** solve $\delta S_{\mathrm{DOC}} / \delta \phi = 0$ (and corresponding stationarity conditions for ψ and U) as a finite system.
2. **Dynamical façades:** define time evolution by a lawful discretization (explicit, implicit, or symplectic) whose stability is governed by DOC admissibility and, where used, the contractive nature of the smoothing operator S on mean-free components.

The DOC legality envelope provides a sharp criterion for permissible dissipation: smoothing-based contributions may not amplify mean-free energy. In practice, designed negative controls (Designed FAILs) are obtained by replacing S with a non-admissible kernel whose Fourier multipliers exceed 1, or by introducing negative weights that violate DOC positivity/contractivity constraints. Such modifications produce observable instability (mass creation, blow-up, or loss of contractivity) and are excluded by construction in the admissible class.

In the next section, we record the principal sector reductions—diffusion, wave/Klein–Gordon, Maxwell-type gauge dynamics, and geometry-coupled reductions—that serve as the canonical entry points for the remaining physics papers.

Therefore, after summation-by-parts in each coordinate, stationarity of the action yields the discrete Euler–Lagrange equation for the scalar façade ϕ in the form

$$\frac{\delta S_{\mathrm{DOC}}}{\delta \phi}(x) = 0 \quad \Longleftrightarrow \quad \kappa_\phi \Big(\nabla_t^* \nabla_t \phi(x) - c_\phi^2 \sum_{i=1}^{d} \nabla_i^* \nabla_i \phi(x) \Big) + \kappa_V \, V'(\phi(x)) - \kappa_{\mathrm{R}} \, \kappa(x) \, \phi(x) - \kappa_{\nabla\kappa} \, \mathcal{G}(x) + \kappa_{\mathrm{diss}}$$

Here ∇_μ denotes the forward difference operator in the μ-direction, ∇_μ^* its discrete adjoint under periodic summation (the backward difference), and $\Delta_x := \sum_{i=1}^{d} \nabla_i^* \nabla_i$ the standard discrete Laplacian on the spatial torus. The remaining terms are as follows.

1. **Potential derivative.** For the canonical polynomial choice $V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$, one has
$$V'(\phi) = m^2 \phi + \lambda \phi^3,$$
with m^2, λ selected from the structural coefficient alphabet (and fixed prior to any observational comparison).
2. **Curvature proxy coupling.** The scalar field $\kappa(x)$ is an externally supplied, bounded curvature proxy arising from the Superset collar/area layer and DOC-admissible smoothing at a chosen scale. The term $-\kappa_{\mathrm{R}} \kappa(x) \phi(x)$ is the Euler–Lagrange contribution from a quadratic coupling of the form $-\frac{\kappa_{\mathrm{R}}}{2} \kappa(x) \phi(x)^2$, chosen so that $\kappa_{\mathrm{R}} \kappa(x)$ plays a mass-like role in the scalar sector.

3. **Mixed gradient–curvature term.** For the mixed coupling

$$\mathcal{L}_{\nabla\kappa}(x) = \kappa_{\nabla\kappa} \sum_{i=1}^{d} (\nabla_i \kappa(x))(\nabla_i \phi(x))\, \phi(x),$$

   the corresponding Euler–Lagrange contribution can be written explicitly by expanding the variation and applying summation-by-parts once:

$$\mathcal{G}(x) \;=\; \sum_{i=1}^{d} \Big[\nabla_i^* \big((\nabla_i \kappa)\, \phi \big)(x) \;-\; (\nabla_i \kappa(x))(\nabla_i \phi(x)) \Big].$$

   This is a finite local expression involving only κ, ϕ, and their lattice differences; no continuum metric tensor is assumed in PH-1.

4. **DOC-admissible dissipation term.** Fix a time-slice mean projector Ω and a DOC-admissible smoothing operator S acting on mean-free components. Define the dissipative residual

$$r := (I - \Omega)\phi - S(I - \Omega)\phi.$$

   The dissipation contribution in the action is $\frac{\kappa_{\text{diss}}}{2} r^2$. Its Euler–Lagrange term is linear in ϕ and supported on mean-free components:

$$\mathcal{D}(x) = \Big[(I - \Omega)(I - S)^*(I - S)(I - \Omega)\phi \Big](x).$$

   When S is chosen from the DOC-admissible class (e.g., an even Fejér-type kernel on the cyclic lattice), $(I - S)^*(I - S)$ is positive semidefinite and contractive on the mean-free subspace, providing a controlled dissipative channel that cannot alter the mean.

Wave/Klein–Gordon reduction. In the sector where $\kappa_{\nabla\kappa} = 0$ and $\kappa_{\text{diss}} = 0$, the scalar equation reduces to the familiar discrete hyperbolic form

$$\nabla_t^* \nabla_t \phi \;-\; c_\phi^2 \Delta_x \phi \;+\; \frac{\kappa_V}{\kappa_\phi} V'(\phi) \;-\; \frac{\kappa_{\mathrm{R}}}{\kappa_\phi} \kappa\, \phi \;=\; 0.$$

If additionally $V(\phi) = \frac{m^2}{2}\phi^2$ and $\kappa \equiv 0$, this becomes the discrete Klein–Gordon equation

$$\nabla_t^* \nabla_t \phi - c_\phi^2 \Delta_x \phi + m_{\text{eff}}^2\, \phi = 0, \qquad m_{\text{eff}}^2 := \frac{\kappa_V}{\kappa_\phi} m^2.$$

In the massless case $m_{\text{eff}} = 0$, one recovers a discrete wave equation whose dispersion relation is determined (exactly, on the finite torus) by the eigenvalues of Δ_x.

4 Sector Reductions and Classical Operator Forms

The one-action functional is designed to be modular: by enabling or disabling specific couplings and selecting specific coefficient values from the fixed structural alphabet, one obtains discrete operator forms that correspond (under admissible refinement) to standard continuum equations used in physics. This section records the principal reductions used downstream in the Physics Track.

4.1 Diffusion and dissipative transport (parabolic sector)

Select the dissipative regime by setting $c_\phi = 0$, $\kappa_V = 0$, $\kappa_{\mathrm{R}} = 0$, $\kappa_{\nabla\kappa} = 0$, and choosing $\kappa_{\mathrm{diss}} > 0$. Replace the stationary condition with a gradient-flow update:

$$\partial_t \phi \;=\; -\frac{\delta S_{\mathrm{DOC}}}{\delta \phi}.$$

A discrete-time step may be written schematically as

$$\phi^{(t+1)} = \phi^{(t)} - \eta\, \mathcal{G}_{\mathrm{diss}}\big(\phi^{(t)}\big), \qquad \eta > 0,$$

where $\mathcal{G}_{\mathrm{diss}}$ is the DOC-lawful operator induced by $(I-\Omega)(I-S)^*(I-S)(I-\Omega)$ (and any additional admissible local terms). Because S is contractive on mean-free modes, this dynamics monotonically suppresses high-frequency content without altering the mean.

Under admissible refinement and appropriate scaling of step sizes, the dominant operator corresponds to diffusion:

$$\partial_t \phi \approx D\, \Delta \phi, \qquad D > 0,$$

with D determined by κ_{diss} and the selected admissible kernel family.

4.2 Wave / Klein–Gordon (hyperbolic sector)

Select the wave-like regime by setting $\kappa_{\mathrm{diss}} = 0$, choosing $c_\phi > 0$, and taking $\kappa_{\nabla\kappa} = 0$. The scalar Euler–Lagrange equation reduces to

$$\nabla_t^* \nabla_t \phi \;-\; c_\phi^2 \Delta \phi \;+\; \frac{\kappa_V}{\kappa_\phi} V'(\phi) \;-\; \frac{\kappa_{\mathrm{R}}}{\kappa_\phi} \kappa\, \phi \;=\; 0.$$

If $V(\phi) = \frac{m^2}{2}\phi^2$ and $\kappa \equiv 0$, one recovers discrete Klein–Gordon:

$$\nabla_t^* \nabla_t \phi - c_\phi^2 \Delta \phi + m_{\mathrm{eff}}^2\, \phi = 0, \qquad m_{\mathrm{eff}}^2 = \frac{\kappa_V}{\kappa_\phi} m^2.$$

The coefficient c_ϕ is structural (selected from the fixed alphabet), and later physics papers identify its sector-specific interpretation under admissible lifting.

4.3 Schrödinger-type evolution (unitary sector)

A Schrödinger-type sector is obtained by choosing a complex scalar field and replacing the second-order time kinetic by a first-order term of the form

$$\mathcal{L}_{\mathrm{Sch}}(x) = \kappa_{\mathrm{Sch}}\, \Im\Big(\overline{\phi(x)}\, \nabla_t \phi(x)\Big) \;-\; \frac{\kappa_{\mathrm{Sch}}}{2m} \sum_{i=1}^{d} \big|\nabla_i \phi(x)\big|^2 \;-\; \kappa_{\mathrm{Sch}}\, V(x)\, |\phi(x)|^2.$$

The resulting Euler–Lagrange equation admits an update rule that is unitary in the Fourier basis when the evolution multipliers are constrained to have unit modulus mode-by-mode. DOC-admissible smoothing (when used) acts only as a lawful coarse-graining operation and is not part of the unitary evolution step.

4.4 Gauge transport (lattice gauge sector)

With nontrivial link variables $U_\mu(x)$ and $\kappa_g > 0$, the action contains both a gauge curvature term (built from plaquette holonomies) and gauge-covariant matter coupling (through covariant differences or a discrete Dirac operator). Variation with respect to U_μ yields a finite local equation of motion for the link variables. In the weak-field regime (small plaquette curvature), the plaquette term reduces to a quadratic form in discrete field strengths, matching the standard lattice gauge action in the appropriate representation.

For PH-1, the essential point is structural: the gauge sector uses only finite products, inverses, traces, and finite differences, and any smoothing invoked for control remains within the DOC-admissible class. Consequently, stability properties established for the scalar sector extend to gauge-coupled sectors under the same admissibility discipline.

4.5 Curvature coupling (background-geometry sector)

The curvature proxy $\kappa(x)$ enters $\mathcal{L}_{\mathrm{geom}}$ as a bounded finite field derived from Superset collar/area data and DOC smoothing. Turning on κ_{R} yields a finite analogue of an $R\phi^2$ coupling, while turning on $\kappa_{\nabla\kappa}$ yields mixed terms that behave like transport in a spatially varying background.

The role of PH-1 is to demonstrate that such background geometric data can be coupled into the action in a manner that is:

- local on the lattice,
- deterministic and finite,
- stable under the DOC admissibility constraints.

Further lifting of collar/area invariants into gravitationally interpreted laws is deferred to later physics papers in the sequence.

5 Worked Micro-Examples

This section gives explicit, fully finite examples illustrating how the one-action framework produces lawful dynamics on a discrete substrate. The intent is not parameter fitting; the intent is legibility: each example can be read as a complete finite model, and each has an admissible continuum façade (in the sense of UFET and residual equivalence) when refinement schedules are applied.

5.1 Scalar wave sector on a 1D periodic lattice

This section instantiates the one-action framework in the simplest nontrivial setting: a real scalar field on a finite, periodic one-dimensional lattice. The purpose is twofold:

1. to show, in fully explicit algebra, how the discrete Euler–Lagrange operator is obtained from a DOC-lawful variational principle; and
2. to make the spectral (circulant) structure visible, since the same diagonalization mechanism is reused in the Schrödinger and gravitational toy facades.

5.1.1 Domain, inner product, and adjoints

Let the spatial lattice be the cyclic group

$$X := \mathbb{Z}_{M_x} = \{0, 1, \dots, M_x - 1\},$$

with periodic indexing $n \equiv n + M_x$. Let the time index set be

$$T := \mathbb{Z}_{M_t},$$

again with periodic indexing. (This choice is not a physical claim; it is a DOC-convenient finite domain on which adjoints and discrete integration-by-parts are exact without boundary remainder terms.)

A scalar field is a function

$$\phi : T \times X \to \mathbb{R}, \qquad (t,n) \mapsto \phi(t,n).$$

Equip $\mathbb{R}^{T\times X}$ with the standard discrete inner product

$$\langle f, g\rangle := \sum_{t\in T}\sum_{n\in X} f(t,n)\, g(t,n).$$

Define the forward difference operators

$$(\nabla_t \phi)(t,n) := \phi(t+1,n) - \phi(t,n), \qquad (\nabla_x \phi)(t,n) := \phi(t,n+1) - \phi(t,n).$$

Under the above inner product and periodic indexing, the adjoints are the backward differences:

$$(\nabla_t^* \psi)(t,n) = \psi(t,n) - \psi(t-1,n), \qquad (\nabla_x^* \psi)(t,n) = \psi(t,n) - \psi(t,n-1).$$

A direct computation (index shift on a finite cyclic sum) gives the discrete integration-by-parts identities:

$$\langle \nabla_t f, g\rangle = -\langle f, \nabla_t^* g\rangle, \qquad \langle \nabla_x f, g\rangle = -\langle f, \nabla_x^* g\rangle.$$

No remainder terms occur because the domain is cyclic.

Define the (one-dimensional) discrete Laplacian as the composition

$$\Delta_x := \nabla_x^* \nabla_x,$$

so that explicitly

$$(\Delta_x \phi)(t,n) = \phi(t,n+1) - 2\phi(t,n) + \phi(t,n-1).$$

Likewise $\Delta_t := \nabla_t^* \nabla_t$ satisfies

$$(\Delta_t \phi)(t,n) = \phi(t+1,n) - 2\phi(t,n) + \phi(t-1,n).$$

Both Δ_t and Δ_x are negative semidefinite with respect to $\langle \cdot, \cdot\rangle$ (their eigenvalues are nonpositive). This sign convention matches the variational derivation below and yields the expected continuum façade after rescaling.

5.1.2 Scalar action and parameterization

Consider the scalar restriction of the one-action density:

$$\mathcal{L}_\phi = \frac{\kappa_\phi}{2}\,(\nabla_t \phi)^2 - \frac{\kappa_\phi\, c_\phi^2}{2}\,(\nabla_x \phi)^2 - \frac{\kappa_V}{2}\, m^2\, \phi^2. \tag{2}$$

Here:

- $\kappa_\phi > 0$ is the kinetic normalization (a DOC-lawful positive scale),
- $c_\phi > 0$ is the lattice signal speed parameter for this sector (dimensionless in this finite model),
- $m \geq 0$ is a mass parameter (dimensionless in this finite model),
- $\kappa_V > 0$ is the potential normalization.

The associated action functional is the finite sum

$$S[\phi] := \sum_{t\in T}\sum_{n\in X} \mathcal{L}_\phi(t,n). \tag{3}$$

This is a finite polynomial in the field values and their nearest-neighbor differences, hence a well-posed object in the deterministic operator calculus setting.

5.1.3 Discrete Euler–Lagrange equation

Let $\eta : T \times X \to \mathbb{R}$ be an arbitrary variation field. Consider the perturbation $\phi_\varepsilon = \phi + \varepsilon\eta$. Differentiate $S[\phi_\varepsilon]$ at $\varepsilon = 0$:

$$\frac{d}{d\varepsilon} S[\phi + \varepsilon\eta]\Big|_{\varepsilon=0} = \sum_{t,n} \left(\kappa_\phi(\nabla_t\phi)(\nabla_t\eta) - \kappa_\phi c_\phi^2(\nabla_x\phi)(\nabla_x\eta) - \kappa_V m^2 \phi\, \eta\right). \tag{4}$$

Apply discrete integration by parts to the first two terms:

- **Time term:**
$$\sum_{t,n}(\nabla_t\phi)(\nabla_t\eta) = -\sum_{t,n} \eta\, (\nabla_t^*\nabla_t\phi) = -\sum_{t,n} \eta\, (\Delta_t\phi). \tag{5}$$

- **Space term:**
$$\sum_{t,n}(\nabla_x\phi)(\nabla_x\eta) = -\sum_{t,n} \eta\, (\nabla_x^*\nabla_x\phi) = -\sum_{t,n} \eta\, (\Delta_x\phi). \tag{6}$$

Substituting (5)–(6) into (4) yields

$$\frac{d}{d\varepsilon} S[\phi + \varepsilon\eta]\Big|_{\varepsilon=0} = \sum_{t,n} \eta(t,n) \left(-\kappa_\phi\, \Delta_t\phi + \kappa_\phi c_\phi^2\, \Delta_x\phi - \kappa_V m^2\phi\right)(t,n). \tag{7}$$

The stationarity condition $\frac{d}{d\varepsilon} S[\phi + \varepsilon\eta]\big|_{\varepsilon=0} = 0$ for all η gives the discrete Euler–Lagrange equation:

$$\kappa_\phi\, \Delta_t\phi - \kappa_\phi c_\phi^2\, \Delta_x\phi + \kappa_V m^2\phi = 0. \tag{8}$$

Dividing by κ_ϕ and defining the effective mass parameter

$$m_{\text{eff}}^2 := \frac{\kappa_V}{\kappa_\phi} m^2, \tag{9}$$

we obtain the canonical discrete Klein–Gordon façade:

$$\Delta_t\phi - c_\phi^2\, \Delta_x\phi + m_{\text{eff}}^2\, \phi = 0. \tag{10}$$

Equation (10) is a finite linear system on $T \times X$. In the massless case $m_{\text{eff}} = 0$, it is the discrete wave equation.

5.1.4 Spectral diagonalization on $\mathbb{Z}_{M_x}$ and finite dispersion

Because $X = \mathbb{Z}_{M_x}$ is cyclic, translation-invariant (circulant) operators diagonalize in the discrete Fourier basis. In particular, for $k \in \frac{2\pi}{M_x}\mathbb{Z}$ define the Fourier mode

$$e_k(n) := e^{ikn}. \tag{11}$$

A direct substitution into the Laplacian gives

$$(\Delta_x e_k)(n) = \left(e^{ik} - 2 + e^{-ik}\right) e^{ikn} = (2\cos k - 2)\, e^{ikn} = -4\sin^2\left(\frac{k}{2}\right) e^{ikn}. \tag{12}$$

Likewise, for $\omega \in \frac{2\pi}{M_t}\mathbb{Z}$,

$$(\Delta_t e^{i\omega t}) = -4\sin^2\left(\frac{\omega}{2}\right) e^{i\omega t}. \tag{13}$$

Hence a separated ansatz $\phi(t,n) = e^{i(\omega t + kn)}$ reduces (10) to the finite dispersion relation

$$4\sin^2\left(\frac{\omega}{2}\right) = 4c_\phi^2 \sin^2\left(\frac{k}{2}\right) + m_{\text{eff}}^2. \tag{14}$$

In the small-mode regime $|\omega| \ll 1$, $|k| \ll 1$, we have $\sin(\omega/2) \sim \omega/2$ and $\sin(k/2) \sim k/2$, giving the continuum façade

$$\omega^2 = c_\phi^2 k^2 + m_{\text{eff}}^2 \quad \text{(to leading order)}. \tag{15}$$

Two remarks are operationally important for later sections:

1. **Finite stability bound.** Since $\sin^2(\omega/2) \le 1$, equation (14) implies a finite admissibility constraint on the pair (c_ϕ, m_{eff}) at fixed lattice size. In the present cyclic model this is not a defect but a feature: it is one concrete mechanism by which the finite substrate constrains which parameter choices admit stable, real-frequency evolution.

2. **Reuse of the circulant diagonalization.** The same Fourier diagonalization mechanism underlies the spectral Schrödinger façade and the DOC-admissible smoothing operators (Fejér/Cesàro class) used throughout the physics track. In practice, the admissibility criteria "multipliers in $[0, 1]$" and "no amplification of mean-free modes" are checked in this diagonal basis.

5.1.5 AoR linkage for verification

The scalar wave façade is exercised as part of the One-Action Master Flagship run in the Authority-of-Record bundle GUM_BUNDLE_v30_20260209T040755Z (DEMO-71). The canonical logs are:

- **stdout:** https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z/gum/authority_archive/AOR_20260209T040755Z_0fc79a0/GUM_BUNDLE_v30_20260209T040755Z/logs/foundations__demo-71-one-action-master-flagship.out.txt

- **stderr:** https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z/gum/authority_archive/AOR_20260209T040755Z_0fc79a0/GUM_BUNDLE_v30_20260209T040755Z/logs/foundations__demo-71-one-action-master-flagship.err.txt

5.2 DOC-lawful dissipation as mean-free suppression

Let $G = \mathbb{Z}_{M_x}^d$ be a spatial torus and $\phi : G \to \mathbb{R}$ a scalar field (think: a time slice). Let $\mathbf{1}$ denote the constant function on G. Define the mean projector Ω by

$$(\Omega\phi)(n) = \Big(\frac{1}{|G|} \sum_{m\in G} \phi(m)\Big)\mathbf{1}(n).$$

Then $I - \Omega$ is the orthogonal projector onto mean-free fields.

Let S be a DOC-admissible smoothing operator on G, e.g. a tensor product of one-dimensional admissible kernels $S = S_1 \otimes \cdots \otimes S_d$ whose Fourier multipliers satisfy $0 \le \widehat{S}(\xi) \le 1$ on each mode. (In the cyclic Hilbert-space calculus, such operators sit in the circulant commutant and are simultaneously diagonalizable by the discrete Fourier basis; this is precisely the mechanism that makes admissibility checkable mode-by-mode.)

Define the mean-free dissipative residual

$$r \;=\; (I - \Omega)\phi \;-\; S(I - \Omega)\phi \;=\; (I - S)(I - \Omega)\phi.$$

The dissipation channel in the one-action Lagrangian density can be taken as

$$\mathcal{L}_{\text{diss}}(\phi) = \frac{\kappa_{\text{diss}}}{2}\,\|r\|_2^2 = \frac{\kappa_{\text{diss}}}{2}\,\|(I - S)(I - \Omega)\phi\|_2^2.$$

Because $\mathcal{L}_{\text{diss}}$ is quadratic, its Euler–Lagrange (or gradient) contribution is explicit:

$$\frac{\delta}{\delta\phi}\Big(\tfrac{1}{2}\|(I-S)(I-\Omega)\phi\|_2^2\Big) = (I-\Omega)(I-S)^*(I-S)(I-\Omega)\phi.$$

When S is symmetric (even kernel) it is self-adjoint: $S^* = S$. In that case,

$$(I-S)^*(I-S) = (I-S)^2,$$

and the dissipation operator becomes

$$D := (I-\Omega)(I-S)^2(I-\Omega).$$

Two key properties follow directly from DOC admissibility:

1. **Positivity / PSD structure.** Since $(I-S)^*(I-S)$ is of the form A^*A, it is positive semidefinite. Conjugating by the orthogonal projector $(I-\Omega)$ preserves PSD. Therefore D is PSD.
2. **Mean protection.** Since $(I-\Omega)\mathbf{1} = 0$, one has $D\mathbf{1} = 0$. Thus the dissipation channel cannot act on the mean mode.

Consequently, in any dynamics derived from the full action where the dissipation channel enters as a gradient-flow component (or as a dissipative correction), the mean-free energy is monotonically suppressed, while the mean is protected. This is the intended finite analogue of "viscosity that does not create or destroy total mass," formulated entirely on the finite substrate. It also cleanly exposes the designed-failure boundary: if the smoothing operator S violates the admissible multiplier constraint $0 \le \widehat{S} \le 1$, then $(I-S)$ can amplify rather than suppress specific modes, and the PSD guarantee fails.

5.3 Minimal gauge-covariant transport on a discrete ring

To make the gauge-transport façade concrete in the smallest setting, let $G = \mathbb{Z}_M$ and let $\psi : G \to \mathbb{C}$ be a complex scalar. Introduce a $U(1)$ link field $A : G \to \mathbb{R}$ and define the link variable

$$U(n) = e^{-iA(n)} \in U(1).$$

Define the covariant forward difference

$$(D_x\psi)(n) = U(n)\psi(n+1) - \psi(n).$$

A lattice gauge transformation is specified by a phase field $\chi : G \to \mathbb{R}$, acting by

$$\psi(n) \mapsto e^{i\chi(n)}\psi(n), \qquad U(n) \mapsto e^{i\chi(n)}U(n)e^{-i\chi(n+1)}.$$

A direct substitution shows $D_x\psi$ transforms covariantly:

$$(D_x\psi)(n) \mapsto e^{i\chi(n)}(D_x\psi)(n),$$

and therefore $|D_x\psi(n)|^2$ is gauge invariant.

A minimal gauge-invariant quadratic action on the ring (suppressing time for the moment) is

$$S[\psi; U] = \sum_{n\in G}\Big(|D_x\psi(n)|^2 + m^2|\psi(n)|^2\Big),$$

with $m \ge 0$. Varying with respect to $\overline{\psi}$ yields the discrete covariant Helmholtz/Klein–Gordon-type stationarity condition

$$D_x^* D_x\psi + m^2\psi = 0,$$

where D_x^* is the ℓ^2-adjoint covariant difference. This is the prototypical mechanism behind the gauge-transport façade in the one-action framework: gauge covariance arises from the action principle on the finite substrate, not from an imposed continuum differential geometry.

In later physics-track papers (electromagnetism and Standard Model closures), the admissibility boundary is enforced by restricting how smoothing/transfer operators may be applied to ψ and to link/connection fields, ensuring that coarse-graining preserves gauge-covariant structure rather than introducing uncontrolled mode pollution.

6 Reproducibility Anchors (Authority of Record)

Public repository:

- https://github.com/public-arch/Marithmetics

Canonical AoR citation surface (tag: aor-20260209T040755Z):

- AoR folder: gum/authority_archive/AOR_20260209T040755Z_0fc79a0
- Bundle sha256: c299b1a7a8ef77f25c3ebb326cb73f060b3c7176b6ea9eb402c97273dc3cf66c

Master release artifact (zip):

- https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z / gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /MARI_MASTER_RELEASE_20260209T040755Z_ 0fc79a0 .zip

AoR report (v32) and manifest:

- https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z / gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /report/GUM_Report_v32_2026-01-25_ 04-27-46Z .pdf
- https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z / gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /report/GUM_Report_v32_2026-01-25_ 04-27-46Z .pdf.manifest.json

Core ledgers and tables:

- **claim_ledger.jsonl:** https://github.com/public-arch/Marithmetics/blob/release-aor-2026012 gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /claim_ledger.jsonl
- **constants_master.csv:** https://github.com/public-arch/Marithmetics/blob/release-aor-2026 gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /GUM_BUNDLE_v30_20260209T040755Z / tables/constants_master.csv
- **constants_master.json:** https://github.com/public-arch/Marithmetics/blob/release-aor-202 gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /GUM_BUNDLE_v30_20260209T040755Z / tables/constants_master.json
- **demo_index.csv:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /GUM_BUNDLE_v30_20260209T040755Z / tables/demo_index.csv
- **run_reproducibility.csv:** https://github.com/public-arch/Marithmetics/blob/release-aor-20 gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /GUM_BUNDLE_v30_20260209T040755Z / tables/run_reproducibility.csv

- **falsification_matrix.csv:** https://github.com/public-arch/Marithmetics/blob/release-aor-202 gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /GUM_BUNDLE_v30_20260209T040755Z / tables/falsification_matrix.csv

One-action master demonstration log (Foundations DEMO-71):

- **stdout:** https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z / gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /GUM_BUNDLE_v30_20260209T040755Z / logs/foundations__demo-71-one-action-master-flagship.out.txt
- **stderr:** https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z / gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /GUM_BUNDLE_v30_20260209T040755Z / logs/foundations__demo-71-one-action-master-flagship.err.txt

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- J. Grieshop, *Marithmetics Authority of Record (AoR)*, tag aor-20260209T040755Z , public repository and canonical archive listed in Section 7.