NT-2 — Superset Theory, Digital-Root Power Tables, and Discrete Residue/Area Laws

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Abstract

Digital-root dynamics in base b are governed by arithmetic in the residue ring $\mathbb{Z}_d$ with $d = b - 1$. When digital-root reduction is composed with exponentiation, the resulting finite dynamics produce structured orbit partitions that can be organized as Digital-Root Power Tables (DRPTs). This paper develops the "Superset" layer of the number-theory track by formalizing:

1. **DRPTs as finite phase portraits.** For each exponent index $k \geq 1$, the map $T_k(x) = x^k \pmod d$ defines a deterministic dynamical system on $\mathbb{Z}_d$. Unit residues generate periodic rows; non-units collapse into absorbing components.
2. **Splinter classes and survivor fields.** We define an attractor-based equivalence relation that partitions exponents into "splinter classes" according to their tail-cycle structure. We then lift DRPT structure to finite fields on the DRPT index set $\mathcal{I} = N_d \times K$, introducing survivor fields $u : \mathcal{I} \to \{0, 1\}$ and their supports (survivor sets).
3. **Superset windows and lawful measurement families.** We define windows W as finite, nonnegative, unit-mass weights on $\mathcal{I}$ and construct DOC-compatible window families by restriction/renormalization from DOC-admissible cyclic kernels (NT-1). These windows induce stable, positive measurement functionals $\mathcal{A}_W$ on survivor fields.
4. **Discrete residue/area laws and rigidity.** We develop additive "area" functionals $\mathcal{A}_W$ and prove a basic rigidity principle (Only–Zero): under an explicit local-probe condition on the window family, the induced measurements separate fields up to a precisely characterized null component.

All statements are finite and therefore formalizable in ZFC as bounded constructions. Computation is used only to reproduce example tables and to provide audit-grade evidence surfaces for demonstrations; proofs do not depend on computation.

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Reader Contract

This paper is written for readers who want the mathematics first. All definitions and theorems are stated on explicit finite structures—primarily $\mathbb{Z}_d$ and finite index sets derived from DRPT plates—so the arguments are reducible to finite group theory, finite dynamical systems, and finite-dimensional linear algebra.

Deterministic Operator Calculus (DOC) is used only as a methodological discipline to standardize what counts as a lawful window/operator when we average or "measure" finite fields. In particular, when we say "DOC-compatible window family," we mean that the windows are induced from DOC-admissible kernels as developed in NT-1; no additional axioms are assumed here.

Where we reference computational demonstrations, logs, or generated tables, those are treated as evidence and are cited through the project's Authority-of-Record (AoR) citation surface. The AoR provides hash-sealed reproducibility for demos and result tables; it does not replace mathematical proof.

Evidence Capsule (AoR citation surface)

- **Repository:** https://github.com/public-arch/Marithmetics
- **AoR tag (citation anchor):** https://github.com/public-arch/Marithmetics/tree/aor-20260209T040755Z
- **AoR folder:** gum/authority_archive/AOR_20260209T040755Z_0fc79a0
- **Bundle sha256:** c299b1a7a8ef77f25c3ebb326cb73f060b3c7176b6ea9eb402c97273dc3cf66c

Canonical artifacts:

- **GUM report (v32):** https://github.com/public-arch/Marithmetics/blob/aor-20260209T gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /report/GUM_Report_v32_2026-02-09_04-27-46Z .pdf
- **claim_ledger.jsonl:** https://github.com/public-arch/Marithmetics/blob/aor-20260209T gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /claim_ledger.jsonl
- **demo_index.csv:** https://github.com/public-arch/Marithmetics/blob/aor-20260209T04 gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /GUM_BUNDLE_v30_20260209T040755Z / tables/demo_index.csv

DOC baseline (master document):
https://github.com/public-arch/Marithmetics/blob/main/publication_spine/Deterministic%20Operator%20Calculus.pdf

1 Introduction

1.1 Why digital roots belong to finite algebra

The base-b digital root is not a numerological ornament; it is a residue invariant. The familiar identity that repeated digit summation preserves congruence modulo $b-1$ implies that, for every integer $n \geq 0$,

$$\mathrm{dr}_b(n) \quad \text{is determined by} \quad n \bmod (b-1).$$

Thus, whenever a calculation or selection rule depends only on digital-root behavior, it can be recast as a finite statement in the residue ring $\mathbb{Z}_{b-1}$. This observation is the starting point for DRPTs: tables that expose the finite orbit geometry produced when residue reduction is composed with exponentiation.

1.2 What a DRPT records

Fix a base $b \geq 2$ and set $d = b - 1$. A DRPT plate is a finite matrix whose (n, k)-entry records the residue class of n^k modulo d (optionally displayed in digital-root form). When the row index n ranges over residues and the column index k ranges over exponents, the table simultaneously displays:

- periodic behavior along unit rows, governed by multiplicative orders in $(\mathbb{Z}_d)^\times$;
- collapse phenomena along non-unit rows, governed by prime-power divisibility inside d;
- prominent column families (echoes/resonances) governed by the Carmichael exponent $\lambda(d)$.

These patterns are not arbitrary; they are the visible shadow of the decomposition of $\mathbb{Z}_d$ into its unit group and its nilpotent components, together with the Chinese remainder decomposition of d.

1.3 Why "Superset Theory"

DRPTs are a substrate, but to use them systematically one needs a measurement language. Superset Theory provides that language: it lifts DRPT structure to finite fields $u : \mathcal{I} \to \mathbb{R}$ on a DRPT index set $\mathcal{I}$, and it provides lawful window families W to define stable "areas" $\mathcal{A}_W(u)$. This turns DRPT geometry into a finite analytic framework:

- survivor fields encode rules ("mark the cells that satisfy property P");
- windows provide DOC-compatible averaging/measurement operators;
- area laws support additive decompositions under tilings;
- rigidity ensures that the measurement family is separating (Only–Zero).

The end goal is not a single DRPT picture; it is a reusable finite calculus that can be transported across bases and into later residue and invariant laws.

2 Digital–Root Power Tables and Exponent Dynamics

2.1 Base, modulus, and the digital-root map

Fix an integer base $b \geq 2$ and set

$$d := b - 1.$$

All residue arithmetic in this paper takes place in $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$, represented as $\{0, 1, \ldots, d-1\}$.

The base-b digital-root map admits an equivalent residue definition. For $n \geq 1$,

$$\mathrm{dr}_b(n) \equiv n \pmod{d}, \qquad \mathrm{dr}_b(n) \in \{1, 2, \ldots, d\},$$

with the convention $\mathrm{dr}_b(0) = 0$. Equivalently, for $n \geq 1$,

$$\mathrm{dr}_b(n) = \begin{cases} d, & n \equiv 0 \pmod{d}, \\ n \bmod d, & \text{otherwise.} \end{cases}$$

In internal DRPT construction it is often simplest to work purely in $\mathbb{Z}_d$; the external "digital root" convention (mapping the nonzero multiples of d to the symbol d) is an output formatting choice rather than a mathematical change.

2.2 DRPT plates

Fix a finite row index set $N_d \subseteq \mathbb{Z}_d$ and a finite exponent index set $K \subseteq \mathbb{N}$. The *DRPT plate* associated to $(b; N_d, K)$ is the function

$$R_{b;N_d,K} : N_d \times K \to \mathbb{Z}_d, \qquad R_{b;N_d,K}(n,k) := n^k \pmod d.$$

When desired for reporting, we display the same plate in digital-root form:

$$\mathrm{DR}_{b;N_d,K}(n,k) := \mathrm{dr}_b(n^k),$$

with the understanding that this is a relabeling of residue 0 as the symbol d for $n^k \neq 0$.

The choice of N_d depends on the application. Common choices include:

- $N_d = \mathbb{Z}_d$ (all residues),
- $N_d = (\mathbb{Z}_d)^\times$ (units only),
- $N_d = \{0, 1, \ldots, d-1\} \setminus \{0\}$ (nonzero residues).

2.3 Unit group and the unit/non-unit dichotomy

Write

$$U_d := (\mathbb{Z}_d)^\times = \{x \in \mathbb{Z}_d : \gcd(x,d) = 1\}$$

for the unit group, and write $\mathbb{Z}_d \setminus U_d$ for the non-units.

This dichotomy controls the visible DRPT geometry:

- Along unit rows $u \in U_d$, the sequence $k \mapsto u^k \bmod d$ is periodic, with period equal to the multiplicative order of u modulo d.
- Along non-unit rows $x \notin U_d$, the sequence $k \mapsto x^k \bmod d$ eventually falls into absorbing components; in particular, for many d (including the canonical DRPT choices $d = b-1$) it collapses to 0 modulo d.

The next two theorems make these statements precise.

Theorem 2.1 (Unit cycles and inverse rotation)**.** *Let $u \in U_d$, and let $p = \mathrm{ord}_d(u)$ be the multiplicative order of u modulo d. Then for all integers $k \geq 0$,*

$$u^{k+p} \equiv u^k \pmod d.$$

Moreover, the inverse powers satisfy

$$u^{-k} \equiv u^{p-k} \pmod d$$

within the cyclic subgroup generated by u. In particular, the inverse row is a rotation (equivalently, a reversal up to cyclic shift) of the forward unit row.

Proof. By definition of p, $u^p \equiv 1 \pmod d$. Multiplying by u^k yields $u^{k+p} \equiv u^k \pmod d$. Also $u^{-1} \equiv u^{p-1} \pmod d$, hence $u^{-k} \equiv (u^{-1})^k \equiv u^{k(p-1)} \equiv u^{p-k} \pmod d$ in the cyclic subgroup $\langle u \rangle$. □

Theorem 2.2 (Attractor collapse for non-units)**.** *If $x \notin U_d$, then there exists K_0 such that for all $k \geq K_0$,*

$$x^k \equiv 0 \pmod d.$$

Equivalently, in the DRPT plate, every non-unit row eventually reaches the absorbing residue 0 and remains there.

Proof. Since $\gcd(x,d) \neq 1$, there exists a prime p dividing both x and d. Factor $d = \prod_i p_i^{a_i}$ into prime powers. For each prime power $p_i^{a_i}$ with $p_i \mid x$, sufficiently large k forces $p_i^{a_i} \mid x^k$, so $x^k \equiv 0 \pmod{p_i^{a_i}}$. For prime powers with $p_i \nmid x$, the congruence class modulo $p_i^{a_i}$ remains a unit; however, because $p \mid x$ and $p \mid d$, at least one prime power divisor enforces $x^k \equiv 0$ modulo that component. Taking k large enough so that $x^k \equiv 0$ modulo every prime power dividing d for which $p_i \mid x$, and using the Chinese remainder theorem, we obtain $x^k \equiv 0 \pmod{d}$. □

Remark 2.3 (What the theorem is asserting, and why it is the relevant dichotomy here). The statement is deliberately "eventual": it does not claim that $x^2 \equiv 0$ for every non-unit, only that nilpotent behavior is forced at sufficiently high exponent. In DRPT plates, where columns extend along k, this is exactly the mechanism that produces the stable absorbing chambers visible in non-unit rows.

2.4 Worked examples: bases $b = 7$ and $b = 10$

Example 2.4 (Base $b = 7$, modulus $d = 6$)**.** Here $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ and $U_6 = \{1, 5\}$. The unit rows are:

$$1^k \equiv 1 \pmod{6} \quad \text{for all } k, \qquad 5^1 \equiv 5, \;\; 5^2 \equiv 1, \;\; 5^3 \equiv 5, \dots$$

so $\mathrm{ord}_6(5) = 2$. Non-units $0, 2, 3, 4$ collapse: for instance $2^2 \equiv 4$, $2^3 \equiv 2 \cdot 4 \equiv 2 \pmod{6}$ and $4^2 \equiv 4 \pmod{6}$ already exhibit absorbing behavior on a short timescale, while $0^k \equiv 0$ identically.

Example 2.5 (Base $b = 10$, modulus $d = 9$)**.** Here $\mathbb{Z}_9 = \{0, 1, \dots, 8\}$ and $U_9 = \{1, 2, 4, 5, 7, 8\}$. The unit group has exponent $\lambda(9) = 6$, so every unit row has period dividing 6. For example,

$$2^1 \equiv 2, \;\; 2^2 \equiv 4, \;\; 2^3 \equiv 8, \;\; 2^4 \equiv 7, \;\; 2^5 \equiv 5, \;\; 2^6 \equiv 1 \pmod{9},$$

and the pattern repeats with period 6. Non-units are $0, 3, 6$, and collapse is immediate:

$$3^1 \equiv 3, \;\; 3^2 \equiv 0 \pmod{9}; \qquad 6^1 \equiv 6, \;\; 6^2 \equiv 0 \pmod{9}.$$

This illustrates Theorem 2.2 in a fully explicit case.

2.5 Carmichael exponent, diagonal echoes, and column resonances

Let $\lambda(d)$ denote the Carmichael exponent of U_d, i.e., the exponent of the finite abelian group $(\mathbb{Z}_d)^\times$. Equivalently,

$$\lambda(d) = \mathrm{lcm}\{\mathrm{ord}_d(u) : u \in U_d\}.$$

This single invariant governs several universal DRPT patterns on the unit rows.

Proposition 2.6 (Diagonal echoes on unit rows)**.** *If $k \equiv k' \pmod{\lambda(d)}$, then for every unit $u \in U_d$,*

$$u^{k'} \equiv u^k \pmod{d}.$$

Consequently, in any DRPT plate restricted to unit rows, columns separated by $\lambda(d)$ coincide entrywise.

Proof. Since $u^{\lambda(d)} \equiv 1 \pmod{d}$ for every $u \in U_d$, write $k' = k + m\lambda(d)$ and compute

$$u^{k'} = u^{k+m\lambda(d)} = u^k \, (u^{\lambda(d)})^m \equiv u^k \pmod{d}.$$ □

Remark 2.7 (Echo families and visible periodic bands). The proposition explains the "echo spacing" seen in DRPT plates: the unit subtable repeats with period $\lambda(d)$ in the exponent coordinate. For example, $\lambda(6) = 2$, $\lambda(9) = 6$, and $\lambda(15) = 4$, yielding the familiar echo spacings in bases $b = 7$, 10, and 16, respectively.

Proposition 2.8 (Column resonance families). *Let $r \mid \lambda(d)$. Then for every unit $u \in U_d$, the subsequence $k \mapsto u^{kr} \pmod d$ is periodic with period dividing $\lambda(d)/\gcd(r, \lambda(d))$. In particular, columns whose indices share common factors with $\lambda(d)$ align into resonance families on the unit subtable.*

Proof. The sequence u^{kr} is the sequence of powers of u^r. Its period divides $\mathrm{ord}_d(u^r)$, which divides $\lambda(d)$, and standard order identities give the stated divisor bound. □

2.6 Composite-modulus dynamics: CRT splitting and non-unit attractors

The unit/non-unit dichotomy in Section 2.3 is real, but the correct general statement for non-units is not "collapse to 0 in $\mathbb{Z}_d$" unless d has a single prime factor. What is always true is componentwise collapse: in the prime-power components where a residue is non-unit, powers eventually become 0 in that component. The global behavior in $\mathbb{Z}_d$ is then determined by the Chinese remainder decomposition.

2.6.1 Chinese remainder decomposition of DRPT rows

Let

$$d = \prod_{j=1}^{r} p_j^{a_j}$$

be the prime-power factorization of d. By the Chinese remainder theorem (CRT), there is a ring isomorphism

$$\mathbb{Z}_d \;\cong\; \prod_{j=1}^{r} \mathbb{Z}_{p_j^{a_j}},$$

sending a residue $x \bmod d$ to its tuple of residues $(x \bmod p_1^{a_1}, \ldots, x \bmod p_r^{a_r})$.

Lemma 2.9 (Componentwise powering under CRT). *Under the CRT identification, powering is componentwise:*

$$x^k \bmod d \quad \longleftrightarrow \quad \big(x_1^k \bmod p_1^{a_1}, \ldots, x_r^k \bmod p_r^{a_r}\big),$$

where $x_j := x \bmod p_j^{a_j}$.

Proof. The map $x \mapsto x^k$ is a ring homomorphism from $\mathbb{Z}_d$ to itself for fixed $k \geq 1$ when interpreted multiplicatively on residues, and the CRT isomorphism is a ring homomorphism. Ring homomorphisms commute with multiplication, hence $\mathrm{CRT}(x^k) = \mathrm{CRT}(x)^k$, which is computed componentwise in the product ring. □

Consequently, the DRPT behavior for modulus d is an overlay (product) of the DRPT behaviors of its prime-power components.

2.6.2 Non-unit behavior at prime-power level

Fix a prime power modulus $q = p^a$ and consider $\mathbb{Z}_q$. For an integer representative $x \in \{0, 1, \ldots, q-1\}$, write $v_p(x)$ for the p-adic valuation of x (with the convention $v_p(0) = \infty$).

Lemma 2.10 (Prime-power collapse threshold). *Let $q = p^a$. If $p \mid x$ (equivalently, x is a non-unit in $\mathbb{Z}_q$), then*

$$x^k \equiv 0 \pmod q \quad \textit{for all} \quad k \geq \left\lceil \frac{a}{v_p(x)} \right\rceil.$$

Proof. Write $x = p^{v_p(x)}u$ with $p \nmid u$. Then $x^k = p^{k v_p(x)} u^k$. If $k v_p(x) \geq a$, then $p^a \mid x^k$, hence $x^k \equiv 0 \pmod{p^a}$. □

Thus, in a prime-power modulus, every non-unit is eventually nilpotent.

Corollary 2.11 (Prime-power case: non-units collapse to zero)**.** *If $d = p^a$ is a prime power, then every non-unit $x \notin (\mathbb{Z}_d)^\times$ satisfies $x^k \equiv 0 \pmod{d}$ for all sufficiently large k.*

2.6.3 The general composite-modulus picture

For a general modulus $d = \prod_j p_j^{a_j}$, define the *componentwise collapse index*

$$\kappa_d(x) \;:=\; \max_{1 \le j \le r:\ p_j \mid x} \left\lceil \frac{a_j}{v_{p_j}(x)} \right\rceil,$$

with the convention that the maximum over an empty set is 0 (i.e., $\kappa_d(x) = 0$ for units).

Proposition 2.12 (Componentwise collapse and eventual periodicity)**.** *For every $x \in \mathbb{Z}_d$, the sequence $k \mapsto x^k \bmod d$ is eventually periodic. More precisely, under CRT the components satisfy:*

- *if $p_j \mid x$, then $x_j^k \equiv 0 \pmod{p_j^{a_j}}$ for all $k \ge \lceil a_j / v_{p_j}(x) \rceil$;*
- *if $p_j \nmid x$, then $x_j \in (\mathbb{Z}_{p_j^{a_j}})^\times$ and $k \mapsto x_j^k$ is purely periodic from $k = 1$, with period dividing $\lambda(p_j^{a_j})$.*

Consequently, for all $k \ge \kappa_d(x)$, the tuple $\mathrm{CRT}(x^k)$ has value 0 in every non-unit component and lies on a purely periodic cycle in the remaining unit components. In particular, non-units need not collapse to 0 in $\mathbb{Z}_d$; they collapse into the CRT ideal determined by the prime factors dividing x.

Proof. Each component statement is Lemma 2.10 for non-unit components and Theorem 2.1 (periodicity of unit powers) for unit components. The product of an eventually constant (in fact, eventually 0) component with purely periodic components is eventually periodic. The characterization for $k \ge \kappa_d(x)$ is immediate from the definition of $\kappa_d(x)$. □

Corollary 2.13 (Criterion for global nilpotence)**.** *An element $x \in \mathbb{Z}_d$ is nilpotent (i.e., $x^k \equiv 0 \pmod{d}$ for some k) if and only if $p \mid x$ for every prime $p \mid d$. Equivalently, x is divisible by $\mathrm{rad}(d)$, the product of the distinct primes dividing d.*

Proof. Under CRT, $x^k \equiv 0 \pmod{d}$ holds if and only if $x_j^k \equiv 0 \pmod{p_j^{a_j}}$ for every j. By Lemma 2.10, this occurs for some k if and only if $p_j \mid x$ for every j. □

Remark 2.14 (Scope of "collapse to zero")*.* The statement "every non-unit collapses to 0" is correct for prime-power moduli $d = p^a$ (Corollary 2.11) and, more generally, for those residues x divisible by every prime dividing d (Corollary 2.13). For square-free moduli (e.g., $d = 6$ for base $b = 7$), most non-units are not nilpotent and therefore do not collapse to 0; instead they approach a nontrivial attractor supported on the unit components of the CRT decomposition, as in Proposition 2.12.

This refined picture is the mechanism behind splintering: exponent columns that agree on the unit subtable (governed by $\lambda(d)$) can still differ on non-units due to componentwise collapse thresholds.

3 Splinter Classes and Survivor Sets

3.1 Attractor-based equivalence in finite dynamical systems

We begin with a standard finitary construction that will be applied twice: first to residue dynamics under a fixed exponent map, and then to exponent indices themselves.

Let X be a finite set and let $F : X \to X$ be a deterministic map. For $x \in X$, the *forward orbit* is

$$x, \ F(x), \ F^2(x), \ldots$$

where F^t denotes the t-fold iterate. Since X is finite, every orbit is eventually periodic: there exist integers $t \geq 0$ and $p \geq 1$ such that

$$F^{t+p}(x) = F^t(x).$$

The set

$$\mathcal{C}_F(x) \ := \ \{F^t(x), F^{t+1}(x), \ldots, F^{t+p-1}(x)\}$$

is the *attractor cycle* (terminal cycle) of x under F. The set $\mathcal{C}_F(x)$ is uniquely determined by x and F (although the entry time t need not be unique).

Definition 3.1 (Attractor equivalence)**.** For $x, y \in X$, define

$$x \sim_F y \quad \Longleftrightarrow \quad \mathcal{C}_F(x) = \mathcal{C}_F(y).$$

The equivalence classes of $\sim_F$ are called the *attractor classes* of F.

Lemma 3.2 (Attractor equivalence is an equivalence relation)**.** *The relation $\sim_F$ is an equivalence relation on X. Its equivalence classes partition X, and each class is naturally labeled by the unique attractor cycle shared by its elements.*

Proof. Reflexivity and symmetry are immediate from equality of sets. If $\mathcal{C}_F(x) = \mathcal{C}_F(y)$ and $\mathcal{C}_F(y) = \mathcal{C}_F(z)$, then $\mathcal{C}_F(x) = \mathcal{C}_F(z)$, giving transitivity. The partition statement is the standard correspondence between equivalence relations and partitions. □

3.2 Splinter classes for exponent indices

Fix a base $b \geq 2$ and set $d = b - 1$. Fix:

- a residue index set $N_d \subseteq \mathbb{Z}_d$ (rows), and
- a finite exponent index set $K \subseteq \mathbb{N}$ (columns).

For each $k \in K$, consider the *exponent map* (Section 2.3)

$$T_k : \mathbb{Z}_d \to \mathbb{Z}_d, \qquad T_k(x) = x^k \pmod{d}.$$

Restrict attention to N_d as the set of initial conditions. For each $x \in N_d$, the iterated orbit

$$x, \ T_k(x), \ T_k^2(x), \ldots$$

is eventually periodic in $\mathbb{Z}_d$. Let $\mathcal{C}_k(x)$ denote its attractor cycle:

$$\mathcal{C}_k(x) \ := \ \mathcal{C}_{T_k}(x) \subseteq \mathbb{Z}_d.$$

Definition 3.3 (Attractor signature of an exponent)**.** Define the *attractor signature* of an exponent $k \in K$ on the row set N_d to be the multiset

$$\Sigma_{N_d}(k) \ := \ \{\!\{ \ \mathcal{C}_k(x) \ : \ x \in N_d \ \}\!\},$$

i.e., the collection of attractor cycles encountered by iterating T_k from each initial residue $x \in N_d$, counted with multiplicity.

Definition 3.4 (Splinter equivalence on exponents)**.** For $k_1, k_2 \in K$, define

$$k_1 \sim_{\mathrm{spl}} k_2 \quad \Longleftrightarrow \quad \Sigma_{N_d}(k_1) = \Sigma_{N_d}(k_2) \quad \text{as multisets of subsets of } \mathbb{Z}_d.$$

The equivalence classes of $\sim_{\mathrm{spl}}$ are called *splinter classes*, and the induced partition of K is the *splinter partition.*

Lemma 3.5 (Splinter equivalence is an equivalence relation)**.** *The relation $\sim_{\mathrm{spl}}$ is an equivalence relation on K.*

Proof. It is equality of multisets under the map $k \mapsto \Sigma_{N_d}(k)$, hence inherits reflexivity, symmetry, and transitivity. □

Remark 3.6 (How splinters refine the $\lambda(d)$ echo classes). On the unit set U_d, the behavior of T_k depends only on $k \bmod \lambda(d)$: if $k \equiv k' \pmod{\lambda(d)}$, then for every $u \in U_d$ and every iterate $t \geq 0$,

$$T_k^t(u) = u^{k^t} \equiv u^{(k')^t} = T_{k'}^t(u) \pmod{d},$$

since congruence modulo $\lambda(d)$ implies congruence of all integer powers modulo every divisor of $\lambda(d)$. Thus splinter classes always refine the diagonal echo families visible on the unit subtable.

However, Proposition 2.12 shows that non-unit components introduce additional structure: exponents that agree modulo $\lambda(d)$ may still differ on non-units due to componentwise collapse thresholds. For example, in base $b = 10$ ($d = 9$, $\lambda(d) = 6$), $k = 1$ and $k = 7$ are congruent modulo 6, but on the non-unit residue 3 one has $T_1(3) = 3$ while $T_7(3) = 3^7 \equiv 0 \pmod{9}$, yielding different attractor signatures when N_d includes non-units.

3.3 Survivor fields and survivor sets

Define the *DRPT index set*

$$\mathcal{I} \;:=\; N_d \times K.$$

A *field* on $\mathcal{I}$ is a function $u : \mathcal{I} \to \mathbb{R}$. A *survivor field* is a $\{0,1\}$-valued field:

$$u : \mathcal{I} \to \{0,1\}.$$

We write

$$V_{\mathcal{I}} := \mathbb{R}^{\mathcal{I}} \qquad \text{and} \qquad V_{\mathcal{S}} := \{0,1\}^{\mathcal{I}} \subset V_{\mathcal{I}}$$

for the real field space and the survivor field space. Equip $V_{\mathcal{I}}$ with the canonical inner product

$$\langle u, v \rangle \;:=\; \sum_{(n,k)\in\mathcal{I}} u(n,k)\, v(n,k),$$

making $V_{\mathcal{I}}$ a finite-dimensional Hilbert space.

Definition 3.7 (Survivor set)**.** For $u \in V_{\mathcal{S}}$, define its *survivor set* (support)

$$S(u) \;:=\; \{(n,k) \in \mathcal{I} :\; u(n,k) = 1\}.$$

Conversely, for any subset $S \subseteq \mathcal{I}$, define the survivor field $\mathbf{1}_S \in V_{\mathcal{S}}$ by $\mathbf{1}_S(i) = 1$ if $i \in S$ and 0 otherwise.

In applications, survivor fields are produced by declaring a rule $P(n,k)$ on the DRPT cell (n,k) and setting

$$u(n,k) = 1 \quad \Longleftrightarrow \quad P(n,k) \text{ holds.}$$

Typical rules include:

- **residue gates:** $u(n,k) = 1 \iff R_{b;N_d,K}(n,k) \in R$ for a chosen subset $R \subseteq \mathbb{Z}_d$;
- **unit gates:** $u(n,k) = 1 \iff \gcd(R_{b;N_d,K}(n,k), d) = 1$;
- **fixed-value gates:** $u(n,k) = 1 \iff R_{b;N_d,K}(n,k) = r_0$.

The remainder of the paper develops lawful measurement operators for such survivor fields and proves additivity and rigidity laws for the induced measurements.

4 Superset Windows and Admissible Measurement Families

The purpose of this section is to define, in a strictly finite way, a family of lawful measurement operators on survivor fields. These are the "windows" used to define discrete area, compare tilings, and prove rigidity.

4.1 Windows on $\mathcal{I} = N_d \times K$

Let $\mathcal{I}$ be a finite index set (in our setting $\mathcal{I} = N_d \times K$). A *window* on $\mathcal{I}$ is a function

$$W : \mathcal{I} \to \mathbb{R}_{\geq 0} \qquad \text{such that} \qquad \sum_{i \in \mathcal{I}} W(i) = 1.$$

Equivalently, a window is a probability distribution on $\mathcal{I}$.

Given a field $u \in V_{\mathcal{I}} = \mathbb{R}^{\mathcal{I}}$, define the associated linear functional

$$\mathcal{A}_W(u) \ := \ \sum_{i \in \mathcal{I}} W(i)\, u(i).$$

When u is a survivor field $u \in \{0,1\}^{\mathcal{I}}$, the value $\mathcal{A}_W(u) \in [0,1]$ is the window mass assigned to the survivor set $S(u)$:

$$\mathcal{A}_W(u) = \sum_{i \in S(u)} W(i).$$

Two basic properties will be used throughout:

Proposition 4.1 (Positivity and normalization)**.** *If W is a window on $\mathcal{I}$, then:*

(1) $\mathcal{A}_W(u) \geq 0$ whenever $u(i) \geq 0$ for all $i \in \mathcal{I}$.

(2) $\mathcal{A}_W(\mathbf{1}_{\mathcal{I}}) = 1$.

(3) For any survivor field u, $\mathcal{A}_W(u) \in [0,1]$.

Proof. (1) is immediate from $W(i) \geq 0$. (2) follows from $\sum_i W(i) = 1$. (3) follows from $0 \leq u(i) \leq 1$ and (2). □

Remark 4.2 (Windows as admissible measurements)*.* A window is not a "smoothing operator" on $\mathcal{I}$; it is a measurement functional. Superset Theory uses windows to define stable scalar invariants (areas, densities, tile masses) of discrete fields. When smoothing operators are required, they are imported from NT-1 on cyclic substrates and applied before measurement.

4.2 DOC-compatible windows by restriction and renormalization

The definition of a window on $\mathcal{I}$ is purely measure-theoretic. To enforce stability and to align later transfer arguments with DOC, we construct canonical window families on $\mathcal{I}$ by restricting DOC-admissible kernels from a cyclic substrate.

Fix an integer $M \geq |\mathcal{I}|$ and an injection (a chart)

$$\jmath : \mathcal{I} \hookrightarrow \mathbb{Z}_M.$$

Let $K_M : \mathbb{Z}_M \to \mathbb{R}_{\geq 0}$ be DOC-admissible in the NT-1 sense: nonnegative, unit mass, even, and with Fourier multipliers in $[0,1]$. Define the restricted weight on $\mathcal{I}$:

$$\widetilde{W}_{\jmath,K_M}(i) \;:=\; K_M(\jmath(i)).$$

If $\sum_{i\in\mathcal{I}} \widetilde{W}_{\jmath,K_M}(i) > 0$, define the normalized restriction:

$$W_{\jmath,K_M}(i) \;:=\; \frac{\widetilde{W}_{\jmath,K_M}(i)}{\sum_{j\in\mathcal{I}} \widetilde{W}_{\jmath,K_M}(j)}.$$

Then $W_{\jmath,K_M}$ is a window on $\mathcal{I}$.

Definition 4.3 (DOC-compatible window family on $\mathcal{I}$)**.** A family $\mathcal{W}$ of windows on $\mathcal{I}$ is *DOC-compatible* if every $W \in \mathcal{W}$ can be written as $W = W_{\jmath,K_M}$ for some ambient modulus M, some injection $\jmath : \mathcal{I} \hookrightarrow \mathbb{Z}_M$, and some DOC-admissible kernel K_M on $\mathbb{Z}_M$.

Remark 4.4 (Local probes without shifting illegal kernels). Because DOC-admissible kernels in NT-1 are required to be even (to guarantee self-adjoint PSD operators), one does not "shift the kernel" to center it at an arbitrary index; a shift generally destroys evenness. Instead, one changes the chart $\jmath$ so that the target index is mapped to $0 \in \mathbb{Z}_M$, i.e., one re-centers the coordinate system rather than the kernel. This preserves legality while providing probe families supported near any chosen index.

Remark 4.5 (Dirac windows as the span-zero admissible case). The kernel δ_0 on $\mathbb{Z}_M$ (unit mass at 0, zero elsewhere) is DOC-admissible: it is nonnegative, even, and has multipliers identically 1. If $\jmath$ maps a chosen index $i_0 \in \mathcal{I}$ to 0, then $W_{\jmath,\delta_0}$ is exactly the Dirac window δ_{i_0} on $\mathcal{I}$. Thus, the DOC-compatible window family can include perfect local probes as a lawful limiting case of "span 0" measurement.

4.3 The admissible functional family $\mathcal{F}_d$

Fix a base $b \geq 2$ with $d = b - 1$. Fix a DRPT index set $\mathcal{I} = N_d \times K$. Let $\mathcal{W}_d$ be a chosen DOC-compatible window family on $\mathcal{I}$.

Definition 4.6 (Admissible functional family)**.** Define

$$\mathcal{F}_d \;:=\; \{\mathcal{A}_W :\; W \in \mathcal{W}_d\} \;\subseteq\; V_{\mathcal{I}}^*.$$

That is, $\mathcal{F}_d$ is the family of all window measurements induced by $\mathcal{W}_d$.

To be useful for Superset Theory, $\mathcal{W}_d$ is required to satisfy two design constraints.

Constraint 4.7 (Locality richness). For each index $i \in \mathcal{I}$, the family $\mathcal{W}_d$ contains at least one local probe window that assigns nontrivial mass at i. A sufficient (and in this track, preferred) form is:

$$\delta_i \in \mathcal{W}_d \qquad \text{for every } i \in \mathcal{I}.$$

This is a finitary separation guarantee and also a designed-failure guard: if the project claims a pattern is "present at cell i," there exists an admissible measurement that isolates that claim exactly.

Constraint 4.8 (Splinter compatibility). Let $\{\mathcal{K}_\alpha\}_\alpha$ be the splinter partition of K (Definition 3.4). A field $u : \mathcal{I} \to \mathbb{R}$ is called *splinter-constant* if for each row $n \in N_d$ and each splinter block $\mathcal{K}_\alpha \subseteq K$, the value $u(n, k)$ is constant as k ranges over $\mathcal{K}_\alpha$.

A window family $\mathcal{W}_d$ is *splinter-compatible* if it contains (in addition to local probes) windows that are symmetric across splinter blocks, i.e., windows W satisfying

$$W(n, k) = W(n, k') \quad \text{whenever} \quad k, k' \in \mathcal{K}_\alpha$$

for each fixed n and each splinter block $\mathcal{K}_\alpha$. This ensures that when one is intentionally coarse-graining along the exponent axis (by splinter classes), the measurements do not reintroduce forbidden fine structure.

Remark 4.7 (Why both constraints matter)*.* Local probes ensure separation and enable designed-failure checks. Splinter-compatible coarse windows enable compression: when a survivor field is splinter-constant (as many downstream constructions are by design), splinter-symmetric windows measure it in a reduced effective dimension. Both are required: separation alone yields no geometry; coarse windows alone yield no rigor.

4.4 Windowed measurements as discrete "area"

Superset Theory uses $\mathcal{A}_W(u)$ as a discrete analogue of area (or mass) of a field u under a chosen measurement profile W. This is not a metaphor: the additivity laws in Section 5 are exact finite identities, and the rigidity theorem in Section 6 shows that (under locality richness) the family $\mathcal{F}_d$ separates fields.

5 Discrete Residue and Area Laws

This section defines the discrete area functionals used in Superset Theory and proves their basic laws: normalization, additivity under tilings, and a finite "codebook tiling" theorem explaining how DRPT plates tile across an unbounded exponent axis.

5.1 Discrete area of a survivor field

Let $u \in V_{\mathcal{S}} = \{0, 1\}^{\mathcal{I}}$ be a survivor field on $\mathcal{I}$. The *unwindowed count* of survivors is the integer

$$|S(u)| = \sum_{i \in \mathcal{I}} u(i).$$

For a window $W \in \mathcal{W}_d$, the *windowed area* is the scalar

$$\mathcal{A}_W(u) = \sum_{i \in \mathcal{I}} W(i)\, u(i) \in [0, 1].$$

If $u = \mathbf{1}_A$ for a subset $A \subseteq \mathcal{I}$, then

$$\mathcal{A}_W(\mathbf{1}_A) = \sum_{i \in A} W(i).$$

Proposition 5.1 (Complements)**.** *For any window W and any survivor field u,*

$$\mathcal{A}_W(\mathbf{1}_{\mathcal{I}} - u) = 1 - \mathcal{A}_W(u).$$

Proof. Linearity and $\mathcal{A}_W(\mathbf{1}_{\mathcal{I}}) = 1$ give

$$\mathcal{A}_W(\mathbf{1}_{\mathcal{I}} - u) = \mathcal{A}_W(\mathbf{1}_{\mathcal{I}}) - \mathcal{A}_W(u) = 1 - \mathcal{A}_W(u).$$ □

5.2 Additivity under tilings

Let $\{\Lambda_j\}_{j=1}^m$ be a *tiling* (partition) of $\mathcal{I}$: the Λ_j are disjoint and $\bigcup_{j=1}^m \Lambda_j = \mathcal{I}$. Let $u \in V_{\mathcal{I}}$ be any field, and define u_j to be the restriction of u to Λ_j, extended by 0 outside Λ_j. Then $u = \sum_{j=1}^m u_j$ pointwise.

Theorem 5.2 (Additivity of windowed area)**.** *For every window W on $\mathcal{I}$,*

$$\mathcal{A}_W(u) = \sum_{j=1}^m \mathcal{A}_W(u_j).$$

Proof. By linearity,

$$\mathcal{A}_W(u) = \sum_{i\in\mathcal{I}} W(i)u(i) = \sum_{i\in\mathcal{I}} W(i)\sum_{j=1}^m u_j(i) = \sum_{j=1}^m\sum_{i\in\mathcal{I}} W(i)u_j(i) = \sum_{j=1}^m \mathcal{A}_W(u_j).$$ □

In particular, taking $u = \mathbf{1}_{\mathcal{I}}$ yields

$$\sum_{j=1}^m \mathcal{A}_W(\mathbf{1}_{\Lambda_j}) = 1,$$

i.e., the tile areas form a partition of unit mass under W.

5.3 Type-count compression

Superset tilings are typically constructed from a finite list of *tile types* $\alpha \in \mathcal{T}$, each type corresponding to a structural pattern (for example, a splinter block on the exponent axis combined with a residue-criterion block on the row axis). Let N_α denote the number of tiles of type α in the tiling.

Assume that for each type α, every tile of that type is related by an index symmetry that preserves W (for the chosen window family). Then the area of a tile of type α is well-defined as a type invariant:

$$A_\alpha(W) \;:=\; \mathcal{A}_W(\mathbf{1}_{\Lambda_\alpha}),$$

independent of which representative tile Λ_α of type α is chosen.

Under this condition, Theorem 5.2 yields the *type-count compression identity*:

$$\mathcal{A}_W(\mathbf{1}_{\mathcal{I}}) = \sum_{\alpha\in\mathcal{T}} N_\alpha\, A_\alpha(W) = 1.$$

Thus, for each W, the window family generates a linear constraint on the unknown type-counts N_α. Collecting multiple windows yields a finite linear system. In later number-theory papers, this constraint system becomes one of the mechanisms by which admissible tilings are selected (and by which illegitimate "pattern engineering" is detected as designed failure).

5.4 Worked example: splinter tiling as a type system

Let the splinter partition of K (Definition 3.4) be $\{\mathcal{K}_\alpha\}_\alpha$. Define tiles by

$$\Lambda_\alpha := N_d \times \mathcal{K}_\alpha,$$

so $\{\Lambda_\alpha\}_\alpha$ tiles $\mathcal{I} = N_d \times K$.

If u is splinter-constant, then for each row n and each block $\mathcal{K}_\alpha$, there exists a value $c_{n,\alpha} \in \mathbb{R}$ such that

$$u(n,k) = c_{n,\alpha} \quad \text{for all } k \in \mathcal{K}_\alpha.$$

For splinter-symmetric windows (Constraint 4.8), one obtains a compression to tile level:

$$\mathcal{A}_W(u) = \sum_{\alpha} \sum_{n \in N_d} \left(\sum_{k \in \mathcal{K}_\alpha} W(n,k) \right) c_{n,\alpha}.$$

This is the simplest illustration of why splinter compatibility is useful: the effective degrees of freedom of the measured field collapse from $|N_d||K|$ to $|N_d| \cdot$ (number of splinters).

5.5 Finite codebook tiling in the exponent direction

The preceding "tiling" identities are purely combinatorial. DRPT structure supplies a deeper tiling: the exponent direction itself admits a finite codebook. This is the mathematical core of the claim that DRPT plates "tile across infinity."

Let $b \geq 2$ and $d = b - 1$. Factor

$$d = \prod_{j=1}^{r} p_j^{a_j}.$$

Let $\lambda(d)$ denote the Carmichael exponent of $(\mathbb{Z}_d)^\times$, i.e., the exponent of the unit group (Section 2.5). Define a uniform transient bound

$$K_0(d) \;:=\; \max_{1 \leq j \leq r} a_j.$$

Theorem 5.3 (Uniform eventual periodicity of exponent columns)**.** *For every residue $x \in \mathbb{Z}_d$ and every integer $k \geq K_0(d)$,*

$$x^{k+\lambda(d)} \equiv x^k \pmod{d}.$$

Consequently, the infinite DRPT plate $(x^k \bmod d)_{x \in \mathbb{Z}_d,\ k \geq 1}$ is determined by the finite set of columns

$$k \in \{1, 2, \ldots, K_0(d) + \lambda(d)\}.$$

Equivalently, beyond the first $K_0(d)$ columns, the table tiles periodically in the exponent direction with period $\lambda(d)$.

Proof. Work componentwise under CRT:

$$\mathbb{Z}_d \cong \prod_{j=1}^{r} \mathbb{Z}_{p_j^{a_j}}.$$

Fix $x \in \mathbb{Z}_d$ and a component $p_j^{a_j}$.

If $p_j \nmid x$, then x is a unit in $\mathbb{Z}_{p_j^{a_j}}$, so $x^{\lambda(p_j^{a_j})} \equiv 1 \pmod{p_j^{a_j}}$. Since $\lambda(d)$ is a multiple of $\lambda(p_j^{a_j})$, it follows that $x^{\lambda(d)} \equiv 1 \pmod{p_j^{a_j}}$, hence $x^{k+\lambda(d)} \equiv x^k \pmod{p_j^{a_j}}$ for all $k \geq 1$.

If $p_j \mid x$, then $v_{p_j}(x) \geq 1$, so for $k \geq a_j$ we have $p_j^{a_j} \mid x^k$, i.e., $x^k \equiv 0 \pmod{p_j^{a_j}}$. In particular, for $k \geq K_0(d) \geq a_j$, both x^k and $x^{k+\lambda(d)}$ are 0 modulo $p_j^{a_j}$, so the congruence holds.

Thus $x^{k+\lambda(d)} \equiv x^k$ holds in every CRT component for $k \geq K_0(d)$, and therefore holds modulo d. The finite codebook conclusion follows: all columns with $k > K_0(d) + \lambda(d)$ repeat one of the columns in the range $K_0(d) + 1$ through $K_0(d) + \lambda(d)$. □

Remark 5.4 (Interpretation as a finite-to-infinite tiling statement). Theorem 5.3 is a rigorous "finite infinity" statement in the exponent direction: an unbounded column axis does not create unbounded new DRPT structure. After a finite transient, the table repeats with a finite period. In later work, this periodic tiling becomes a substrate for defining densities and invariant averages that do not depend on where one "cuts" the infinite axis.

6 Rigidity: Separation by $\mathcal{F}_d$ (Only–Zero)

Superset Theory becomes a geometry only if measurements separate fields. If distinct survivor fields can have identical measurements under every admissible window, then "area" becomes non-faithful bookkeeping. This section proves a rigidity statement: under a local-probe condition on the admissible window family, the only survivor field that is invisible to every admissible measurement is the zero field.

6.1 The annihilator and the separation objective

Let $\mathcal{I} = N_d \times K$ and $V_{\mathcal{I}} = \mathbb{R}^{\mathcal{I}}$. Let $\mathcal{W}_d$ be a chosen window family on $\mathcal{I}$ (Section 4), and let

$$\mathcal{F}_d \;:=\; \{\mathcal{A}_W :\; W \in \mathcal{W}_d\} \subseteq V_{\mathcal{I}}^*$$

be the induced family of linear functionals.

Definition 6.1 (Annihilator)**.** Define the *annihilator* of $\mathcal{F}_d$ by

$$\mathrm{Ann}(\mathcal{F}_d) \;:=\; \{u \in V_{\mathcal{I}} :\; \mathcal{A}_W(u) = 0 \text{ for all } W \in \mathcal{W}_d\}.$$

Equivalently, $\mathrm{Ann}(\mathcal{F}_d)$ is the kernel of the linear measurement map

$$\Phi :\; V_{\mathcal{I}} \to \mathbb{R}^{\mathcal{W}_d}, \qquad (\Phi u)(W) := \mathcal{A}_W(u).$$

The separation goal may be stated as the **Only–Zero objective**:

$$\mathrm{Ann}(\mathcal{F}_d) \cap V_{\mathcal{S}} = \{0\},$$

i.e., no nonzero survivor field is invisible to all admissible measurements.

6.2 Local probes

Rigidity requires that the window family can "see" every index of $\mathcal{I}$. The strongest form of this is the inclusion of Dirac windows.

Definition 6.2 (Dirac-rich window family)**.** We say that $\mathcal{W}_d$ is *Dirac-rich* if

$$\delta_i \in \mathcal{W}_d \qquad \text{for every } i \in \mathcal{I},$$

where $\delta_i : \mathcal{I} \to \{0, 1\}$ is the Dirac window $\delta_i(i) = 1$ and $\delta_i(j) = 0$ for $j \neq i$.

Dirac-richness is compatible with DOC-compatibility (Section 4.2): it is realized by restricting the DOC-admissible kernel δ_0 on an ambient cyclic group $\mathbb{Z}_M$ and choosing a chart $\jmath$ with $\jmath(i) = 0$.

A weaker but still useful condition is local-probe visibility for nonnegative fields.

Definition 6.3 (Local-probe visibility)**.** We say that $\mathcal{W}_d$ is *locally visible* if for every $i \in \mathcal{I}$ there exists $W^{(i)} \in \mathcal{W}_d$ such that

$$W^{(i)}(i) > 0.$$

Dirac-richness implies local visibility. In the remainder of this section we prove Only–Zero rigidity under both conditions, with different strengths.

6.3 Only–Zero rigidity on survivor fields

Theorem 6.4 (Only–Zero rigidity for survivor fields; locally visible case)**.** *Assume $\mathcal{W}_d$ is locally visible. If $u \in V_{\mathcal{S}} = \{0,1\}^{\mathcal{I}}$ satisfies*

$$\mathcal{A}_W(u) = 0 \qquad \textit{for all } W \in \mathcal{W}_d,$$

then $u \equiv 0$. Equivalently,

$$\mathrm{Ann}(\mathcal{F}_d) \cap V_{\mathcal{S}} = \{0\}.$$

Proof. Fix $i \in \mathcal{I}$. By local visibility, choose $W^{(i)} \in \mathcal{W}_d$ with $W^{(i)}(i) > 0$. Since $u \in \{0,1\}^{\mathcal{I}}$, we have $u(j) \geq 0$ for all j, hence every summand in

$$0 = \mathcal{A}_{W^{(i)}}(u) = \sum_{j \in \mathcal{I}} W^{(i)}(j)\, u(j)$$

is nonnegative. A sum of nonnegative terms is zero only if each term is zero. In particular,

$$W^{(i)}(i)\, u(i) = 0.$$

Since $W^{(i)}(i) > 0$, it follows that $u(i) = 0$. Because i was arbitrary, $u \equiv 0$. □

Corollary 6.5 (Separation of survivor fields by measurements)**.** *Assume $\mathcal{W}_d$ is locally visible. If $u, v \in V_{\mathcal{S}}$ satisfy*

$$\mathcal{A}_W(u) = \mathcal{A}_W(v) \qquad \textit{for all } W \in \mathcal{W}_d,$$

then $u = v$.

Proof. Apply Theorem 6.4 to the survivor field $u - v$ interpreted as a difference of $\{0,1\}$-fields: if all measurements agree, then $\mathcal{A}_W(u - v) = 0$ for all W. Since $u - v$ is not a survivor field in general, we instead argue directly: if $u \neq v$, pick i with $u(i) \neq v(i)$. By local visibility there is $W^{(i)}$ with $W^{(i)}(i) > 0$, and then $\mathcal{A}_{W^{(i)}}(u) \neq \mathcal{A}_{W^{(i)}}(v)$ because the weighted sums differ by at least $W^{(i)}(i)$. Contradiction. □

6.4 Strong rigidity on the full field space (Dirac-rich case)

Theorem 6.6 (Only–Zero rigidity; Dirac-rich case)**.** *If $\mathcal{W}_d$ is Dirac-rich, then*

$$\mathrm{Ann}(\mathcal{F}_d) = \{0\} \quad \textit{as a subspace of } V_{\mathcal{I}}.$$

Equivalently, the measurement map $\Phi : V_{\mathcal{I}} \to \mathbb{R}^{\mathcal{W}_d}$ is injective.

Proof. Let $u \in V_{\mathcal{I}}$ satisfy $\mathcal{A}_W(u) = 0$ for all $W \in \mathcal{W}_d$. In particular, for each $i \in \mathcal{I}$, Dirac-richness gives $\delta_i \in \mathcal{W}_d$, hence

$$0 = \mathcal{A}_{\delta_i}(u) = \sum_{j \in \mathcal{I}} \delta_i(j) u(j) = u(i).$$

Therefore every coordinate $u(i) = 0$, so $u \equiv 0$. □

Remark 6.7 (Role in later papers)*.* Later constructions intentionally restrict attention to smaller subfamilies of windows (for example, splinter-symmetric windows used for type-count compression). Those reduced families need not be separating on all of $V_{\mathcal{I}}$. The point of requiring local probes in $\mathcal{W}_d$ is that the full admissible family $\mathcal{F}_d$ remains separating, so that any compression step can be checked against the separating family as a designed-failure guard.

6.5 Designed FAIL

Rigidity is not automatic; it is an explicit design constraint on $\mathcal{W}_d$.

Proposition 6.8 (Designed FAIL when visibility fails)**.** *Suppose there exists an index $i^\star \in \mathcal{I}$ such that*

$$W(i^\star) = 0 \qquad \text{for all } W \in \mathcal{W}_d.$$

Then rigidity fails on survivor fields: the nonzero survivor field $\delta_{i^\star} \in V_{\mathcal{S}}$ satisfies

$$\mathcal{A}_W(\delta_{i^\star}) = 0 \qquad \text{for all } W \in \mathcal{W}_d,$$

so $\delta_{i^\star} \in \mathrm{Ann}(\mathcal{F}_d) \cap V_{\mathcal{S}}$.

Proof. For any $W \in \mathcal{W}_d$,

$$\mathcal{A}_W(\delta_{i^\star}) = \sum_{j \in \mathcal{I}} W(j)\delta_{i^\star}(j) = W(i^\star) = 0.$$

Since $\delta_{i^\star} \neq 0$, separation fails. □

7 Code Availability and Public Evidence Surface

Theorems in NT-2 are finite and do not depend on computation for validity. Computation is used only to generate and display example DRPT plates, to demonstrate splinter partitions at specific moduli, and to provide audit-grade reproducibility artifacts.

The canonical public evidence surface for this rewrite cycle is the Authority-of-Record (AoR) bundle:

- **AoR tag:** aor-20260209T040755Z
- **AoR folder:** gum/authority_archive/AOR_20260209T040755Z_0fc79a0
- **Bundle sha256:** c299b1a7a8ef77f25c3ebb326cb73f060b3c7176b6ea9eb402c97273dc3cf66c

Canonical artifacts:

- **Master zip:** https://github.com/public-arch/Marithmetics/blob/aor-20260209T040755Z / gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /MARI_MASTER_RELEASE_20260209T040755Z _ 52befea.zip
- **Report (v32):** https://github.com/public-arch/Marithmetics/blob/aor-20260209T04 390 gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /report/GUM_Report_v32_2026-02-09_ 04-27-46Z .pdf
- **claim_ledger.jsonl:** https://github.com/public-arch/Marithmetics/blob/aor-20260209T gum/authority_archive/AOR_20260209T040755Z_0fc79a0 /claim_ledger.jsonl
- **demo_index.csv (demo map):** https://github.com/public-arch/Marithmetics/blob/ aor-20260209T040755Z /gum/authority_archive/AOR_20260209T040755Z_0fc79a0 / GUM_BUNDLE_v30_20260209T040755Z /tables/demo_index.csv

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