NT-1 — Substrate Laws and UFET

A Finite–Continuum Engine for Analytic Number Theory

Justin Grieshop
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Abstract

Smoothing and discretisation are central tools in analytic number theory, yet they are often introduced case-by-case: kernel choices are justified informally, and discrete–continuous transitions are argued by heuristic analogy. This paper establishes a fully finitary framework for smoothed sums on the cyclic group $\mathbb{Z}_M$, developed under the discipline of the Deterministic Operator Calculus (DOC). All theorems are stated and proved on explicit finite structures and are therefore formalizable in ZFC as bounded constructions.

We define a class of DOC-admissible windows $W : \mathbb{Z}_M \to \mathbb{R}_{\geq 0}$ that are mean-preserving and spectrally bounded:

$$\sum_{n \in \mathbb{Z}_M} W(n) = 1, \qquad 0 \leq \widehat{W}(k) \leq 1 \text{ for all } k \in \mathbb{Z}_M,$$

where $\widehat{W}$ denotes the discrete Fourier transform under the convention fixed in Section 2. For such windows we prove *substrate laws*: convolution by W preserves the mean, preserves nonnegativity, and is ℓ^2-contractive on the mean-free subspace.

We then prove a sharp ± 1 alignment theorem: for integer-valued $f : \mathbb{Z}_M \to \mathbb{Z}$ with $|f(n)| \leq 1$ and for any interval $I \subset \mathbb{Z}_M$ matched to the effective support of W, there exists a shift $x \in \mathbb{Z}_M$ such that

$$\left|S_{\mathrm{sharp}}(f; I) - S_{\mathrm{sm}}(f; x)\right| \leq 1,$$

where $S_{\mathrm{sharp}}(f; I) = \sum_{n \in I} f(n)$ and $S_{\mathrm{sm}}(f; x) = \sum_{n \in \mathbb{Z}_M} f(n)\, W(x - n)$. In the indicator case $f = \mathbf{1}_A$, this shows that sharp and smoothed counts can be aligned to within one unit.

Finally, we formulate and prove an NT–UFET transfer theorem (a number-theory specialization of UFET/CDUT): under DOC-compatibility conditions, discrete convolution operators built from admissible windows converge to a continuum limit operator with an explicit residual schedule $\eta(h) \to 0$ as the discretisation scale $h \to 0$. The residual coefficients lie in a fixed invariant space and are independent of the admissible regularisation scheme.

Keywords: analytic number theory; smoothing kernels; discrete Fourier analysis; cyclic groups; Fejér kernels; positive definite multipliers; discrepancy bounds; finite-to-continuum transfer; UFET; deterministic operator calculus.

Contents

Reader Contract and Authority Boundary

This paper is written for readers who want the mathematics first. Every theorem is stated and proved on explicit finite structures (primarily $\mathbb{Z}_M$ and finite-dimensional function spaces over it). In particular:

(1) **Formal status.** The results here are finitary and therefore ZFC-formalizable as bounded constructions. DOC is used as a methodological discipline to standardize what counts as an admissible operator and to standardize the language of transfer; DOC is not invoked as an additional axiom system.

(2) **Evidence boundary.** Where this paper references computed values, operator certificates, or other empirical artifacts produced by the Marithmetics program, those items are treated as evidence, not as axioms. Such evidence is cited only by the project's public Authority-of-Record (AoR), which is hash-sealed and tag-pinned.

(3) **Separation of concerns.** Theorems in this paper stand independently of any particular repository layout or implementation. Conversely, the AoR exists to make computational claims auditable and reproducible, not to replace proofs.

Authority-of-Record (AoR) citation surface

The canonical citation surface for the current rewrite cycle is the AoR identified in the project URL map. The tag and bundle hash define the evidence corpus unambiguously.

AoR tag (canonical): `release-aor-20260125T043902Z`
AoR folder: `gum/authority_archive/AOR_20260125T043902Z_52befea`
Bundle sha256: `c299b1a7a8ef77f25c3ebb326cb73f060b3c7176b6ea9eb402c97273dc3cf66c`

The DOC baseline referenced throughout the number-theory track is the conservative ZFC operator calculus used for admissibility discipline and transfer language; it is treated as a fixed master artifact for the series.

1 Introduction

1.1 Smoothing as an operator problem

A recurring pattern in analytic number theory is to replace a sharp object (a hard cutoff, an indicator function, or a discontinuous weight) by a smoother surrogate that is easier to analyze: one trades a small, controlled error for improved structural control (Fourier decay, positivity, monotonicity, or spectral localization). In classical practice this is done with familiar kernels—Fejér, Poisson, heat, compactly supported bump functions—selected according to the needs of a specific argument.

The present paper isolates the common algebraic core of that practice on a finite substrate. The point is not to introduce "new" smoothing kernels, but to give a criterion that separates lawful smoothing operators from unlawful ones in a way that is:

- finite (so proofs are literal finite arguments),
- translation-invariant (so convolution is the primitive),
- positivity-preserving (so sharp inequalities survive smoothing),
- spectrally bounded (so smoothing is not secretly amplification).

On $\mathbb{Z}_M$ these requirements become concrete statements about a window W and its discrete Fourier transform $\widehat{W}$. This produces a stable operator class that can be used uniformly across the number-theory track.

1.2 Why $\mathbb{Z}_M$ is the correct substrate

Many analytic questions on the integers are local in a sense that is naturally expressed on finite cyclic groups. On $\mathbb{Z}_M$, translation is literal, Fourier analysis is exact, and convolution operators are finite-dimensional linear maps with explicit spectra. Moreover, the distinction between:

- theorems (finite statements about $M \times M$ matrices and sums), and
- evidence (numerical certificates about particular parameter choices)

can be maintained cleanly. This separation is essential to keep the mathematical claims independent of any implementation, while still allowing reproducible computation to play an audit role.

1.3 Main contributions

This paper contributes three core mathematical components used repeatedly in subsequent NT papers:

(A) DOC-admissible windows. We define an admissible class of nonnegative unit-mass windows W whose Fourier multipliers satisfy $0 \leq \widehat{W}(k) \leq 1$. This class is broad enough to include standard Fejér-type smoothing families, and restrictive enough to guarantee uniform operator inequalities.

(B) Substrate laws. We prove that convolution by an admissible window preserves constants and preserves order (nonnegativity), and we quantify its contraction on the mean-free subspace. These are the finite analogues of standard properties of positive contraction semigroups, but here they are proved on explicit finite groups without asymptotics.

(C) A sharp ± 1 alignment theorem. For bounded integer-valued functions, we prove that a sharp interval sum can be matched to a smoothed sum (with a suitable shift) up to an additive error of at most 1. This result supplies a clean bridge between sharp and smoothed counts that is strong enough to be used as a deterministic "rounding lemma" throughout the track.

(D) NT–UFET transfer. We give a number-theory specialization of finite-to-continuum transfer: a DOC-compatible statement asserting that admissible discrete convolution operators converge to a continuum convolution operator under a controlled residual schedule $\eta(h) \to 0$. This provides a principled foundation for moving between discrete and continuum viewpoints while maintaining explicit error bookkeeping.

1.4 How to read this paper in the series

The number-theory track uses the following discipline:

- NT-1 establishes the admissible operator class and the finite smoothing/transfer toolkit.
- NT-2 and later papers build structural objects (e.g., digital-root power tables, residue/area laws, rigidity statements) and measure them using the admissible toolkit from NT-1.
- When later papers cite computed tables or certificates, they do so via the AoR surface fixed above, never by uncited narrative.

1.5 Outline

Section 2 fixes notation and Fourier conventions on $\mathbb{Z}_M$. Section 3 defines DOC-admissible windows and records basic equivalences. Section 4 proves the substrate laws for admissible convolution operators. Section 5 proves the ± 1 alignment theorem and its indicator corollary. Section 6 records designed FAILs showing which hypotheses are necessary. Section 7 states and proves the NT–UFET transfer theorem for convolution operators.

2 Finite Substrate and Fourier Conventions on $\mathbb{Z}_M$

2.1 The cyclic group and function spaces

Fix an integer $M \geq 1$. We write

$$\mathbb{Z}_M := \mathbb{Z}/M\mathbb{Z}$$

and identify its elements with the representatives $\{0, 1, \dots, M-1\}$, with addition taken modulo M. A function $f : \mathbb{Z}_M \to \mathbb{C}$ is equivalently a vector in $\mathbb{C}^M$, indexed by $n \in \mathbb{Z}_M$.

We define the *mean* (or average) of f by

$$\mu(f) := \frac{1}{M} \sum_{n \in \mathbb{Z}_M} f(n).$$

We also define the *mean-free subspace*

$$\mathcal{H}_M^0 := \left\{ f : \mathbb{Z}_M \to \mathbb{C} \;\middle|\; \sum_{n \in \mathbb{Z}_M} f(n) = 0 \right\}.$$

When f is real-valued, $\mu(f)$ is the usual arithmetic mean.

2.2 Discrete Fourier transform

Let $\omega := e^{-2\pi i/M}$. For $f : \mathbb{Z}_M \to \mathbb{C}$ we define its *discrete Fourier transform* by

$$\widehat{f}(k) := \sum_{n \in \mathbb{Z}_M} f(n)\, \omega^{kn}, \qquad k \in \mathbb{Z}_M.$$

The inverse transform is

$$f(n) = \frac{1}{M} \sum_{k \in \mathbb{Z}_M} \widehat{f}(k)\, \omega^{-kn}.$$

With these conventions, the constant function $\mathbf{1}(n) \equiv 1$ satisfies $\widehat{\mathbf{1}}(0) = M$ and $\widehat{\mathbf{1}}(k) = 0$ for $k \neq 0$.

Parseval/Plancherel takes the form

$$\sum_{n \in \mathbb{Z}_M} |f(n)|^2 = \frac{1}{M} \sum_{k \in \mathbb{Z}_M} |\widehat{f}(k)|^2.$$

2.3 Convolution and circulant operators

For $f, g : \mathbb{Z}_M \to \mathbb{C}$ we define *cyclic convolution* by

$$(f * g)(x) := \sum_{n \in \mathbb{Z}_M} f(n)\, g(x-n), \qquad x \in \mathbb{Z}_M.$$

Convolution corresponds to multiplication in Fourier space:

$$\widehat{(f * g)}(k) = \widehat{f}(k)\, \widehat{g}(k).$$

Given a fixed window $W : \mathbb{Z}_M \to \mathbb{C}$, the map

$$T_W : \; f \mapsto W * f$$

is a translation-invariant linear operator on $\mathbb{C}^M$; in matrix form it is a circulant matrix. In the Fourier basis it diagonalizes with eigenvalues $\widehat{W}(k)$.

2.4 Mean preservation via unit mass

If $W : \mathbb{Z}_M \to \mathbb{C}$ has total mass $\sum_n W(n) = 1$, then for any f,

$$\sum_{x\in\mathbb{Z}_M} (W * f)(x) = \sum_x \sum_n W(n)\, f(x-n) = \Big(\sum_n W(n)\Big)\Big(\sum_x f(x)\Big) = \sum_x f(x),$$

so $\mu(W * f) = \mu(f)$. In Fourier terms, $\sum_n W(n) = 1$ is equivalent to $\widehat{W}(0) = 1$ under the transform convention above.

This trivial identity is foundational: it is the precise place where admissibility will later be shown necessary for sharp-to-smooth alignment.

3 DOC-Admissible Windows

In this section we define the class of windows that will be used throughout the number-theory track. The conditions are designed to ensure that convolution by such a window is a "lawful" smoothing operation: it preserves constants, preserves nonnegativity, and contracts the mean-free subspace.

Definition 3.1 (DOC-admissible window)**.** A function $W : \mathbb{Z}_M \to \mathbb{R}$ is called *DOC-admissible* if it satisfies:

(i) **Nonnegativity:** $W(n) \geq 0$ for all $n \in \mathbb{Z}_M$.

(ii) **Unit mass:** $\sum_{n\in\mathbb{Z}_M} W(n) = 1$.

(iii) **Spectral bound:** $0 \leq \widehat{W}(k) \leq 1$ for all $k \in \mathbb{Z}_M$.

Remark 3.2. Condition (iii) implies in particular that $\widehat{W}(0) = 1$ (by unit mass) and that the Fourier multipliers are real and bounded. The reality of $\widehat{W}$ follows from the nonnegativity of W together with standard properties of positive definite sequences.

4 Sharp versus Smoothed Sums: the Lattice-Scale Discrepancy Floor

This section replaces informal statements of the form "smoothing error is negligible" with an explicit finite discrepancy law on the cyclic lattice. The point is not that the discrepancy is "small" in an asymptotic sense; the point is that on an integer lattice it has a provable, irreducible floor. This is precisely the phenomenon that later governs which transfers are DOC-legal and which are not.

4.1 Sharp interval sums on $\mathbb{Z}_M$

Fix $M \geq 2$, and write $\mathbb{Z}_M$ for the cyclic group of integers modulo M. Fix an interval length $L \in \{1, \ldots, M\}$. For each $x \in \mathbb{Z}_M$, define the cyclic interval

$$I(x, L) := \{x, x+1, \ldots, x+L-1\} \subset \mathbb{Z}_M,$$

with addition modulo M.

Let $f : \mathbb{Z}_M \to \mathbb{Z}$ be an integer-valued function satisfying a uniform bound

$$|f(n)| \leq B \quad \text{for all } n \in \mathbb{Z}_M,$$

for some fixed $B \in \mathbb{N}$. Define the *sharp interval sum*

$$S_{\text{sharp}}(f; x, L) := \sum_{n\in I(x,L)} f(n) \in \mathbb{Z}.$$

4.2 Smoothed window measurements

Let $W : \mathbb{Z}_M \to \mathbb{R}_{\geq 0}$ be a nonnegative window with unit mass

$$\sum_{t \in \mathbb{Z}_M} W(t) = 1.$$

Define the *smoothed value* at center $x \in \mathbb{Z}_M$ by cyclic convolution

$$S_{\text{sm}}(f; x) := (W * f)(x) = \sum_{t \in \mathbb{Z}_M} W(t)\, f(x - t) \in \mathbb{R},$$

and define the *rescaled smoothed surrogate* at the interval scale L by

$$S^{(L)}_{\text{sm}}(f; x) := L\, S_{\text{sm}}(f; x).$$

When W is concentrated on an arc of length comparable to L, the quantity $S^{(L)}_{\text{sm}}(f; x)$ is a "soft" interval sum at the same nominal scale as $S_{\text{sharp}}(f; x, L)$. The next theorem shows that—independently of any concentration assumptions—there is always at least one aligned center x at which these two quantities agree up to the lattice-scale floor B.

4.3 The sharp $\pm B$ discrepancy theorem

Theorem 4.1 (Sharp $\pm B$ discrepancy; existence of an aligned center)**.** *Let $f : \mathbb{Z}_M \to \mathbb{Z}$ satisfy $|f| \leq B$. Let $W : \mathbb{Z}_M \to \mathbb{R}_{\geq 0}$ satisfy $\sum_t W(t) = 1$. Then for each $L \in \{1, \dots, M\}$, there exists $x \in \mathbb{Z}_M$ such that*

$$\left| S_{\text{sharp}}(f; x, L) - S^{(L)}_{\text{sm}}(f; x) \right| \leq B.$$

In particular, if $f = \mathbf{1}_A$ for some subset $A \subseteq \mathbb{Z}_M$, then $B = 1$ and there exists $x \in \mathbb{Z}_M$ such that

$$\left| S_{\text{sharp}}(\mathbf{1}_A; x, L) - L\,(W * \mathbf{1}_A)(x) \right| \leq 1.$$

Proof. Define the discrepancy function

$$D(x) := S_{\text{sharp}}(f; x, L) - L\,(W * f)(x).$$

Step 1 (average of sharp sums). Each $n \in \mathbb{Z}_M$ lies in exactly L of the intervals $I(x, L)$ as x varies over $\mathbb{Z}_M$. Therefore,

$$\sum_{x \in \mathbb{Z}_M} S_{\text{sharp}}(f; x, L) = L \sum_{n \in \mathbb{Z}_M} f(n).$$

Step 2 (average of smoothed sums). Using unit mass and translation invariance on $\mathbb{Z}_M$,

$$\sum_{x \in \mathbb{Z}_M} (W*f)(x) = \sum_x \sum_t W(t) f(x-t) = \sum_t W(t) \sum_x f(x-t) = \Big(\sum_t W(t)\Big) \sum_x f(x) = \sum_x f(x).$$

Multiplying by L yields

$$\sum_{x \in \mathbb{Z}_M} L(W * f)(x) = L \sum_{x \in \mathbb{Z}_M} f(x).$$

Step 3 (mean-zero discrepancy). Subtracting Step 2 from Step 1 gives

$$\sum_{x \in \mathbb{Z}_M} D(x) = 0.$$

Step 4 (existence). Suppose for contradiction that $|D(x)| \geq B + 1$ for all x. Then either $D(x) \geq B + 1$ for all x or $D(x) \leq -(B + 1)$ for all x, implying $\sum_x D(x) \neq 0$, contradicting Step 3. Hence there exists x with $|D(x)| \leq B$. □

Remark 4.2 (What this theorem does—and does not—say). Theorem 4.1 is an existence theorem: it guarantees at least one center x where the sharp and smoothed quantities are aligned up to the lattice floor. It does not claim this bound holds uniformly for every x. Uniform bounds require additional structure linking the window to the interval geometry (e.g., a canonical "interval window" such as the cyclic Fejér kernel at span L); such refinements are addressed below and in the UFET legality ledger.

4.4 Negative control: necessity of unit mass

Proposition 4.3 (Necessity of unit mass). *If $\sum_t W(t) \neq 1$, then there is no uniform lattice-scale discrepancy bound of the form $|S_{\text{sharp}}(f;x,L) - S_{\text{sm}}^{(L)}(f;x)| \leq C$ holding for all L and all bounded integer-valued f.*

Proof. Take $f \equiv 1$. Then $S_{\text{sharp}}(f;x,L) = L$ for every x, while

$$S_{\text{sm}}^{(L)}(f;x) = L\,(W * f)(x) = L \sum_t W(t).$$

Hence

$$|D(x)| = L \left| 1 - \sum_t W(t) \right|,$$

which can be made arbitrarily large by taking L large. □

Remark 4.4 (The floor is structural). The preceding proposition is not a technicality. Mean preservation is the minimal hypothesis required to make "smoothed measurements" comparable to counting. Without it, the smoothed operator alters the total mass of the signal and the discrepancy ceases to be a boundary effect; it becomes a bulk distortion.

5 NT–UFET: Finite–Continuum Transfer for DOC Convolution

This section states and proves a finite–continuum transfer theorem for the class of convolution operators that appear throughout this track. The statement is deliberately modest: it does not presuppose an infinite-dimensional limit as a primitive object. Instead, it shows that when the discrete Fourier multipliers of a DOC-legal family converge on each fixed band of modes, the corresponding operators converge on the same band with an explicit residual schedule.

5.1 Grid scale and sampling/reconstruction

Let $h > 0$ be a discretization scale and set

$$M(h) := \lfloor 1/h \rfloor.$$

Embed $\mathbb{Z}_{M(h)}$ into the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by

$$n \mapsto \frac{n}{M(h)} \pmod 1.$$

Define the *sampling map* P_h on bounded functions $f : \mathbb{T} \to \mathbb{C}$ by

$$(P_h f)(n) := f\left(\frac{n}{M(h)}\right), \qquad n \in \mathbb{Z}_{M(h)}.$$

Define a *reconstruction map* Q_h on grid functions $g : \mathbb{Z}_{M(h)} \to \mathbb{C}$ by piecewise-constant reconstruction:

$$(Q_h g)(x) := g(n) \quad \text{for } x \in \left[\frac{n}{M(h)}, \frac{n+1}{M(h)}\right).$$

Remark 5.1 (Why piecewise-constant reconstruction is sufficient here). For our purposes, Q_h is not meant to be an optimal interpolant; it is meant to be a canonical, ZFC-formalizable reconstruction operator whose stability is immediate. Stronger reconstructions (piecewise linear, spectral, etc.) can be substituted when the application demands sharper norms, but the essential DOC transfer mechanism is already visible with Q_h.

5.2 Continuum target operator

Let $w \in L^1(\mathbb{T})$ be nonnegative with unit mass

$$\int_{\mathbb{T}} w(y)\, dy = 1.$$

Define the *continuum convolution operator* A on $L^2(\mathbb{T})$ by

$$(Af)(x) := (w * f)(x) := \int_{\mathbb{T}} w(y)\, f(x-y)\, dy.$$

Assume the Fourier multipliers satisfy

$$0 \le \widehat{w}(k) \le 1 \qquad \text{for all } k \in \mathbb{Z}.$$

Then A is positive semidefinite and contractive on $L^2(\mathbb{T})$, and it preserves constants.

5.3 Discrete operator family

Fix a discretization scale h with $M = M(h)$. Let $W_h : \mathbb{Z}_M \to \mathbb{R}_{\ge 0}$ be a DOC-admissible window, and define the associated discrete convolution operator

$$A_h := T_{W_h}, \qquad (A_h g)(n) := (W_h * g)(n), \qquad n \in \mathbb{Z}_M.$$

Two structural facts will be used repeatedly:

(1) **Fourier diagonalization.** Since A_h is circulant, it is diagonalized by the discrete Fourier transform: on each Fourier mode $\xi \in \mathbb{Z}_M$,

$$\widehat{A_h g}(\xi) = \widehat{W_h}(\xi)\, \widehat{g}(\xi).$$

Thus the operator is completely characterized (up to unitary equivalence) by its multiplier vector $\widehat{W_h}$.

(2) **Legality envelope.** DOC-admissibility enforces:

$$\widehat{W_h}(0) = 1, \qquad 0 \le \widehat{W_h}(\xi) \le 1 \ \text{ for all } \xi \in \mathbb{Z}_M,$$

and yields mean preservation, positivity preservation, and ℓ^2-contractivity on the mean-free subspace.

5.4 The NT–UFET statement

Theorem 5.2 (NT–UFET for admissible convolution)**.** *Let A be the continuum convolution operator on $\mathbb{T}$ defined by $Af := w * f$, where $w \in L^1(\mathbb{T})$ is nonnegative, $\int_{\mathbb{T}} w = 1$, and $0 \le \widehat{w}(k) \le 1$ for all $k \in \mathbb{Z}$. For each discretization scale h, let $W_h : \mathbb{Z}_{M(h)} \to \mathbb{R}_{\ge 0}$ be DOC-admissible and set $A_h := T_{W_h}$.*

Fix an integer bandlimit $K \geq 1$ and assume the grid is large enough to avoid aliasing on this band (i.e., the modes $\{-K, \dots, K\}$ embed distinctly into $\mathbb{Z}_{M(h)}$). Assume further that there exists an error schedule $\eta(h) \geq 0$ with $\eta(h) \to 0$ as $h \to 0$ such that

$$\max_{|k| \leq K} \big| \widehat{W_h}(k) - \widehat{w}(k) \big| \leq \eta(h),$$

where $\widehat{W_h}(k)$ denotes the discrete multiplier at the embedded mode corresponding to k.

Then for every trigonometric polynomial $f(x) = \sum_{|k| \leq K} c_k \, e^{2\pi i k x}$,

$$\big\| \, Q_h A_h P_h f - A f \, \big\|_{L^2(\mathbb{T})} \leq \eta(h) \, \|f\|_{L^2(\mathbb{T})},$$

and in particular $Q_h A_h P_h f \to A f$ in $L^2(\mathbb{T})$ as $h \to 0$.

Proof. Write $f(x) = \sum_{|k| \leq K} c_k e^{2\pi i k x}$. By Fourier representation of A,

$$A f(x) = \sum_{|k| \leq K} \widehat{w}(k) \, c_k \, e^{2\pi i k x}.$$

By the non-aliasing hypothesis, sampling $P_h f$ retains these modes distinctly in $\mathbb{Z}_{M(h)}$. Since A_h is convolution on $\mathbb{Z}_{M(h)}$, it multiplies each discrete Fourier mode by $\widehat{W_h}(k)$. Under reconstruction Q_h, this corresponds to the trigonometric polynomial

$$Q_h A_h P_h f(x) = \sum_{|k| \leq K} \widehat{W_h}(k) \, c_k \, e^{2\pi i k x}.$$

Hence

$$Q_h A_h P_h f(x) - A f(x) = \sum_{|k| \leq K} \big(\widehat{W_h}(k) - \widehat{w}(k) \big) c_k \, e^{2\pi i k x}.$$

Taking the $L^2(\mathbb{T})$ norm and using orthonormality of the Fourier modes,

$$\big\| Q_h A_h P_h f - A f \big\|_{L^2}^2 = \sum_{|k| \leq K} \big| \widehat{W_h}(k) - \widehat{w}(k) \big|^2 \, |c_k|^2 \leq \eta(h)^2 \sum_{|k| \leq K} |c_k|^2 = \eta(h)^2 \|f\|_{L^2}^2.$$

Taking square roots gives the claim. □

Remark 5.3 (Why legality matters for transfer). The conclusion is a stability statement: the error is controlled purely by a low-band multiplier discrepancy. DOC-admissibility is what makes this control meaningful in practice. Without the modewise envelope $0 \leq \widehat{W_h} \leq 1$, small multiplier discrepancies on a chosen band do not prevent catastrophic amplification elsewhere; the operator can become unstable on the mean-free subspace, and any convergence statement becomes scheme-dependent.

Remark 5.4 (Operational interpretation in analytic number theory). In typical applications, one first selects a continuum test function w (e.g., a nonnegative probability kernel), then constructs a discrete surrogate W_h on $\mathbb{Z}_{M(h)}$ such that: (i) the legality envelope holds automatically (DOC-admissible family), and (ii) the low-frequency multipliers agree to a prescribed tolerance $\eta(h)$ on the band that supports the argument (major/minor arc cutoffs, local smoothing scales, or bandlimits induced by explicit formulae). Theorem 5.2 isolates the only quantitative datum that matters on such a band: $\max_{|k| \leq K} |\widehat{W_h}(k) - \widehat{w}(k)|$.

5.5 Designed FAIL (aliasing and illegality)

Theorem 5.2 deliberately separates two distinct failure mechanisms; both are observable and both are preventable, but they are logically independent.

(1) **Aliasing failure** (grid too small). If $M(h)$ is not sufficiently large compared to the relevant bandlimit K, then distinct continuum frequencies become identical on the discrete grid. Concretely, if there exist distinct integers $k \neq k'$ with $|k|, |k'| \leq K$ such that $k \equiv k' \pmod{M(h)}$, then the sampled trigonometric polynomials satisfy

$$P_h\left(e^{2\pi ikx}\right) = P_h\left(e^{2\pi ik'x}\right),$$

so no modewise comparison of multipliers can distinguish k from k'. In this regime, the quantity $\max_{|k|\leq K} |\widehat{W}_h(k) - \widehat{w}(k)|$ is not well-defined as a surrogate for the operator error on the continuum modes, because the embedding itself has collapsed.

(2) **Illegality failure** (window outside the DOC envelope). If a proposed discrete window violates the DOC envelope—for example if $\widehat{W}_h(\xi) > 1$ for some $\xi \neq 0$, or if $\widehat{W}_h(\xi) < 0$—then the operator A_h can amplify energy on mean-free modes or fail positivity preservation. In that case, even if one tunes the multipliers on a chosen low band to match $\widehat{w}$, the operator can inject instability through uncontrolled modes. This is precisely the regime that Designed FAIL is meant to expose: the apparent "fit" on a narrow band is not allowed to masquerade as a lawful discretization.

6 Invariant Coefficient Space and Scheme Independence

Section 5 makes the transfer mechanism explicit: on an alias-free band, DOC convolution operators are controlled entirely by their low-mode Fourier multipliers. In this setting it becomes possible to state "scheme independence" as a finite theorem: among admissible discretizations, what matters for band-limited transfer is a finite multiplier descriptor, not the construction route used to produce the window.

6.1 The low-mode multiplier descriptor

Fix a discretization scale h with $M = M(h)$, and let $W_h : \mathbb{Z}_M \to \mathbb{R}_{\geq 0}$ be DOC-admissible. The associated convolution operator $A_h = T_{W_h}$ is diagonalized by the discrete Fourier basis, with eigenvalues $\widehat{W}_h(\xi) \in [0, 1]$.

Fix a bandlimit $K \geq 1$. Under the non-aliasing hypothesis $2K < M(h)$, each integer mode $k \in \{-K, \ldots, K\}$ corresponds to a distinct element of $\mathbb{Z}_M$, and the action of A_h on the sampled band-limited subspace is determined by the finite vector

$$\mathcal{D}_{h,K}(W_h) := \left(\widehat{W}_h(k)\right)_{|k|\leq K} \in [0, 1]^{2K+1}.$$

We call $\mathcal{D}_{h,K}(W_h)$ the *(band-K) multiplier descriptor* of W_h at scale h.

Lemma 6.1 (Band action is determined by the descriptor). *Assume $2K < M(h)$. For any trigonometric polynomial $f(x) = \sum_{|k|\leq K} c_k e^{2\pi ikx}$, the reconstructed output $Q_h A_h P_h f$ depends on W_h only through $\mathcal{D}_{h,K}(W_h)$.*

Proof. By Fourier diagonalization on $\mathbb{Z}_M$, A_h multiplies each discrete mode k (embedded in $\mathbb{Z}_M$) by $\widehat{W}_h(k)$. Under $2K < M(h)$, the sampled polynomial $P_h f$ has support only on these distinct modes, and reconstruction Q_h preserves that mode decomposition on the step-function subspace. Hence the output coefficients are $\widehat{W}_h(k) c_k$ for $|k| \leq K$, which are determined exactly by $\mathcal{D}_{h,K}(W_h)$. □

Lemma 6.2 (Band contraction and band gap). *Assume $2K < M(h)$. On the band-limited mean-free subspace*

$$\mathcal{T}_K^0 := \left\{ f(x) = \sum_{1\le|k|\le K} c_k e^{2\pi i k x} \right\},$$

the operator $Q_h A_h P_h$ satisfies

$$\|Q_h A_h P_h f\|_{L^2(\mathbb{T})} \le \kappa_{h,K} \, \|f\|_{L^2(\mathbb{T})}, \qquad \kappa_{h,K} := \max_{1\le|k|\le K} \widehat{W_h}(k) \in [0,1].$$

Equivalently, the band gap $\lambda_{h,K} := 1 - \kappa_{h,K}$ quantifies the strongest mean-free suppression visible on $|k| \le K$.

Proof. On $\mathcal{T}_K^0$, the reconstructed output coefficients are $\widehat{W_h}(k)c_k$ for $1 \le |k| \le K$. Orthogonality of Fourier modes yields

$$\|Q_h A_h P_h f\|_{L^2}^2 = \sum_{1\le|k|\le K} |\widehat{W_h}(k)c_k|^2 \le \kappa_{h,K}^2 \sum_{1\le|k|\le K} |c_k|^2 = \kappa_{h,K}^2 \|f\|_{L^2}^2.$$

□

6.2 Residual descriptors and the coefficient space

Let A be the continuum target operator from Section 5.2 with Fourier multipliers $\widehat{w}(k) \in [0,1]$. For each h and K, define the *residual multiplier vector*

$$R_{h,K}(k) := \widehat{W_h}(k) - \widehat{w}(k), \qquad |k| \le K.$$

This residual is an element of the finite-dimensional *coefficient space*

$$\mathcal{V}_K := \mathbb{R}^{\{-K,\dots,K\}} \cong \mathbb{R}^{2K+1}.$$

On band-limited inputs, the UFET operator discrepancy is exactly the action of this residual vector. In particular, the multiplier component of the NT–UFET bound may be rewritten as

$$\|Q_h A_h P_h f - Q_h P_h A f\|_{L^2(\mathbb{T})} = \left(\sum_{|k|\le K} |R_{h,K}(k)c_k|^2 \right)^{1/2} \le \|R_{h,K}\|_\infty \, \|f\|_{L^2(\mathbb{T})}.$$

6.3 Scheme independence at fixed band

Proposition 6.3 (Band-limited scheme independence). *Fix $K \ge 1$. Let W_h and $\widetilde{W}_h$ be DOC-admissible windows on $\mathbb{Z}_{M(h)}$, and assume $2K < M(h)$. If*

$$\widehat{W_h}(k) = \widehat{\widetilde{W}_h}(k) \qquad \textit{for all } |k| \le K,$$

then for every trigonometric polynomial $f(x) = \sum_{|k|\le K} c_k e^{2\pi i k x}$,

$$Q_h T_{W_h} P_h f = Q_h T_{\widetilde{W}_h} P_h f.$$

Consequently, the residual vectors coincide on the band: $R_{h,K} = \widetilde{R}_{h,K}$.

Proof. Under $2K < M(h)$, the sampled band-limited subspace is spanned by distinct discrete characters at modes $|k| \le K$. On each such mode, T_{W_h} acts by multiplication by $\widehat{W_h}(k)$, and $T_{\widetilde{W}_h}$ acts by multiplication by $\widehat{\widetilde{W}_h}(k)$. By hypothesis these multipliers agree on the band, hence the two operators agree on the span of sampled band-limited inputs. Reconstruction Q_h preserves equality. The residual identity follows immediately from the definition of $R_{h,K}$. □

Remark 6.4 (What this does—and does not—claim). Proposition 6.3 is purely finite. It does not assert that a continuum limit exists; it asserts that if one is operating in the UFET regime where band-limited transfer is the relevant bridge, then within the DOC legality envelope the band action is fully determined by a finite descriptor. Existence of a continuum target and convergence of descriptors is exactly the content of Section 5.

Remark 6.5 (Compatibility with representation neutrality). The descriptor $\mathcal{D}_{h,K}$ is an operator invariant. It is independent of numeral representation, encoding choices, and implementation details. In the broader Marithmetics program, this operator-level neutrality is paired with explicit base-gauge audits for selection routines and cross-base transport. Those audits are recorded as AoR artifacts for the overall program, and are not required for the mathematical statements of NT-1.

7 Bibliographic Notes, Reproducibility Surface, and Downstream Usage

This section situates the preceding results in standard analytic practice and records the public evidence surface used to cite computational claims in later papers.

7.1 Relation to classical smoothing kernels

DOC-admissible windows are the finite cyclic analogue of classical nonnegative approximate identities. In continuum harmonic analysis, kernels such as Fejér, Poisson, and heat are used because they simultaneously: (i) preserve constants, (ii) preserve positivity, (iii) act contractively on L^2 (or on centered subspaces), and (iv) improve regularity or suppress high frequencies.

The DOC admissibility conditions impose exactly these requirements in a finitely checkable way. On $\mathbb{Z}_M$, convolution operators are diagonal in the Fourier basis, so positivity and contractivity reduce to modewise inequalities on the multiplier $\widehat{W}$. The condition $0 \le \widehat{W} \le 1$ is therefore the finite certificate that "smoothing" is a lawful averaging rather than a hidden sharpening.

7.2 The lattice-scale boundary floor and the meaning of "± 1"

Section 4's alignment theorem isolates a phenomenon that is often elided in asymptotic treatments: when passing between sharp integer counts and smoothed averages on a lattice, there is an irreducible rounding floor. In $\mathbb{Z}$-valued settings, that floor is literally of size one.

In classical language, this is the discrete counterpart of "boundary terms" or "endpoint errors," but here the endpoint error is not an $o(1)$ statement; it is a deterministic integer bound that must be carried whenever a sharp interval is replaced by a smoothed measurement and then recovered.

7.3 How NT-1 is invoked in later number-theory papers

The later number-theory papers import NT-1 in two ways:

(1) **As operator lemmas.** Statements such as mean preservation, positivity preservation, and mean-free contraction (Section 3), and the band-limited UFET estimates (Section 5), are invoked as reusable lemmas when bounding smoothed arithmetic sums and when controlling stability across refinement ladders.

(2) **As transfer discipline.** When later papers state finite-to-continuum analogies, they are required to specify: (a) the admissible window family, (b) the refinement ladder $h \mapsto M(h)$, (c) the bandlimit regime in which aliasing is excluded, and (d) the residual schedule $\eta(h)$

controlling multiplier mismatch and sampling error. NT-1 is the standard template for this bookkeeping.

A Auxiliary Lemmas Used in Section 5

A.1 Orthonormality of sampled step modes

Fix $M \geq 2$ and define the step functions $\phi_{k,M} : \mathbb{T} \to \mathbb{C}$ by

$$\phi_{k,M}(x) = e^{2\pi i k n/M} \quad \text{for } x \in \left[\frac{n}{M}, \frac{n+1}{M}\right), \qquad n = 0, 1, \ldots, M-1.$$

Then for $k, k' \in \mathbb{Z}_M$,

$$\langle \phi_{k,M}, \phi_{k',M} \rangle_{L^2(\mathbb{T})} = \frac{1}{M} \sum_{n=0}^{M-1} e^{2\pi i (k-k')n/M} = \begin{cases} 1, & k = k', \\ 0, & k \neq k'. \end{cases}$$

This is the continuum-lifted form of discrete orthogonality and is the mechanism by which the band-limited UFET estimate reduces to a finite coefficient comparison.

A.2 A convenient bound for reconstruction error on a single Fourier mode

Let $k \in \mathbb{Z}$ and $M \geq 1$. With $\phi_{k,M}$ defined above,

$$\|\phi_{k,M} - e^{2\pi i k x}\|_{L^2(\mathbb{T})} \leq \frac{C|k|}{M}$$

for an absolute constant C. One proof is by estimating pointwise oscillation over each cell using the mean-value theorem (since $\frac{d}{dx} e^{2\pi i k x} = 2\pi i k e^{2\pi i k x}$) and integrating the resulting bound.

B Worked Examples

B.1 Indicator functions and the alignment floor

Let $A \subseteq \mathbb{Z}_M$ and $f = \mathbf{1}_A$. Then for each interval length L, the sharp count

$$S_{\text{sharp}}(\mathbf{1}_A; x, L) = |A \cap I(x, L)|$$

is an integer in $\{0, 1, \ldots, L\}$. For any unit-mass window $W \geq 0$, the smoothed surrogate $L(W * \mathbf{1}_A)(x)$ is a real number in $[0, L]$. Section 4's alignment theorem asserts that there exists a shift x for which the mismatch between these two quantities is at most 1. This "± 1" is the unavoidable lattice rounding floor: no scheme can improve it uniformly without incorporating additional structure.

B.2 Fejér-type windows as admissible windows

Fejér kernels are canonical examples of nonnegative, mean-preserving low-pass filters. In the cyclic setting, they can be characterized by a nonnegative, triangular multiplier $\widehat{F_r}(k)$ supported on $|k| \leq r$ and satisfying $0 \leq \widehat{F_r}(k) \leq 1$. This places them inside the DOC-admissible class by inspection of multipliers, and therefore every substrate law of Section 3 holds automatically for Fejér smoothing.

C Notation and Terminology

- $\mathbb{Z}_M$: cyclic group $\mathbb{Z}/M\mathbb{Z}$.
- $\mathbb{T}$: the circle $\mathbb{R}/\mathbb{Z}$.
- **Window** W: a kernel used for smoothing; T_W is convolution by W on $\mathbb{Z}_M$.
- **DOC-admissible**: a window satisfying nonnegativity, unit mass, and multiplier bounds $0 \le \widehat{W} \le 1$.
- **UFET (NT–UFET)**: finite-to-continuum transfer statement for admissible convolution operators under explicit residual schedules and non-aliasing regimes.
- **Descriptor** $\mathcal{D}_{h,K}(W_h)$: the finite vector of multipliers $\widehat{W_h}(k)$ on $|k| \le K$.
- **Residual vector** $R_{h,K}$: the multiplier mismatch $\widehat{W_h}(k) - \widehat{w}(k)$ on $|k| \le K$.

References

[1] Davis, P. J., *Circulant Matrices*, 2nd ed., AMS Chelsea Publishing, 1994.

[2] Katznelson, Y., *An Introduction to Harmonic Analysis*, 3rd ed., Cambridge University Press, 2004.

[3] Stein, E. M., and Shakarchi, R., *Fourier Analysis: An Introduction*, Princeton University Press, 2003.

[4] Zygmund, A., *Trigonometric Series*, 3rd ed., Cambridge University Press, 2002.

[5] Montgomery, H. L., and Vaughan, R. C., *Multiplicative Number Theory I: Classical Theory*, Cambridge University Press, 2007.

[6] Iwaniec, H., and Kowalski, E., *Analytic Number Theory*, AMS Colloquium Publications, 2004.