PH-4 — Ω–$\mathbb{1}$ Layer and Observer Dynamics on a Finite Substrate

Controller/Measurement as Explicit Operators, Commutation, and a Unified Residual Budget

(Authority-of-Record Edition)

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Abstract

PH-1 fixes the deterministic operator calculus (DOC) and establishes lawful discrete dynamics on a finite substrate. PH-2 fixes a representation-independent constants slice and a cross-base integrity contract (“base as gauge,” designed-fail). PH-3 closes a cosmology lawbook and audits it with spectrum-level bridges and deterministic teeth.

PH-4 introduces a fourth structural layer: an explicit finite controller/observation layer that sits above the DOC dynamics and composes lawfully with the lifting pipeline. The paper formalizes two objects that are often implicit in physical reasoning:

1. Ω–$\mathbb{1}$ **(Omega-one) controller.** A universal projection/suppression schema that decomposes any finite state into a principal component and a non-principal component, and then contracts the non-principal component by a lawful amount without violating DOC admissibility constraints (mean preservation, positivity where required, and spectral legality).
2. **Observer group.** A finite group of representation transforms acting on the measurement surface rather than on the underlying invariants. The observer layer is defined so that admissible observers preserve the principal (invariant) content while only relabeling or coarse-graining the representation.

The central result is a composition law: when Ω–$\mathbb{1}$ control and observer transforms are restricted to the DOC-admissible commutant, controlled observations inherit a single residual budget consistent with the universal residual law,

$$\eta_h \;=\; c_1 h^2 \;+\; c_2 \frac{r(h)}{M_y},$$

where h is the refinement scale, $r(h)$ is an admissible Fejér/MSF span, and M_y is the finite window size of the lifted domain. PH-4 thereby makes measurement and control explicit and auditable: they are no longer hidden degrees of freedom, but finite operators with certificates, commutation conditions, and designed-fail boundaries.

1 Reader Contract

1.1 Purpose and scope

This paper does not propose new constants and does not modify the selection rules established upstream. Its purpose is to close a structural gap that typically remains implicit:

- **Control** (what is suppressed, gated, stabilized) and

- **Observation** (what is measured, coarse-grained, represented)

must be expressed as explicit finite operators if the program is to remain falsifiable and non-circular.

PH-4 supplies those operators and the admissibility contracts under which they may be composed with DOC dynamics and with the finite-to-continuum lift.

1.2 Claim tiering

PH-4 contains two kinds of content:

- **Tier A (operator-form theorems).** Finite linear-algebraic and harmonic-analytic statements proved within the paper (boundedness, commutation, invariance of principal content, residual budget inheritance). These do not require external data.

- **AoR-anchored evidence (implementation audits).** Where the paper references executed operator certificates, designed-fail outcomes, or controller/transfer demonstrations, those are treated as evidence claims and are cited by AoR logs and ledgers.

1.3 Non-circularity rule

Ω–$\nVdash$ control and observer transforms are permitted only as downstream operators once the invariant content has been fixed. They may not be used to tune selection or to retroactively choose templates. This rule is the controller/observer analogue of the PH-2 non-circularity contract.

Evidence Capsule (AoR)

AoR tag (canonical): `aor-20260209T040755Z`
AoR folder: `gum/authority_archive/AOR_20260209T040755Z_0fc79a0`
Bundle sha256 (v30): `c299b1a7a8ef77f25c3ebb326cb73f060b3c7176b6ea9eb402c97273dc3cf66c`

Core indices and falsification surfaces: `demo_index.csv`, `falsification_matrix.csv`, `run_reproducibility.csv`

Primary AoR demos referenced in PH-4:
- DEMO-56 (DOC vs finite-difference controller comparison)
- DEMO-69 (OATB: operator admissibility transfer bridge)
- DEMO-34 (Ω-SM controller bridge flagship v1)
- DEMO-59 (electromagnetism controller demo)

This paper treats the AoR as the authoritative execution record. The present manuscript provides the mathematical definitions, contracts, and proofs needed to interpret those records without ambiguity.

2 Ω–$\nVdash$ Controller and Observer Layer

2.1 Finite state space and principal/non-principal decomposition

Fix $M \in \mathbb{N}$, $M \geq 2$. Let $\mathcal{H}_M$ denote $\mathbb{R}^M$ or $\mathbb{C}^M$ with inner product

$$\langle x, y \rangle := \sum_{n=0}^{M-1} x_n \overline{y_n}, \qquad \|x\|_2 := \sqrt{\langle x, x \rangle}.$$

Let $\mathbf{1} \in \mathcal{H}_M$ denote the constant vector $(1, 1, \ldots, 1)^\top$ and define the normalized constant vector

$$\mathbf{e}_{\nVdash} \;:=\; \frac{1}{\sqrt{M}}\mathbf{1}.$$

Define the *principal subspace*

$$\mathcal{P} \;:=\; \operatorname{span}\{\mathbf{e}_{\nVdash}\},$$

and its orthogonal complement

$$\mathcal{P}^\perp \;:=\; \{x \in \mathcal{H}_M : \langle x, \mathbf{e}_{\nVdash} \rangle = 0\}.$$

The corresponding orthogonal projectors are

$$\Pi_{\nVdash}(x) \;:=\; \langle x, \mathbf{e}_{\nVdash} \rangle\, \mathbf{e}_{\nVdash}, \qquad \Pi_\Omega(x) \;:=\; x - \Pi_{\nVdash}(x).$$

Thus every state decomposes uniquely as

$$x \;=\; x_{\nVdash} + x_\Omega, \qquad x_{\nVdash} := \Pi_{\nVdash} x, \;\; x_\Omega := \Pi_\Omega x.$$

Interpretation. In physical terms, $x_{\nVdash}$ is the "mean / principal" component (the part that cannot be removed by any mean-preserving smoothing), and x_Ω is the "mean-free / non-principal" component (the part on which DOC-admissible contractive operators act as genuine suppressors). In arithmetic-domain adapters, $\mathcal{P}$ corresponds to the principal character sector and $\mathcal{P}^\perp$ to the non-principal sector; PH-4 uses the present Hilbert-space formulation because it is basis-independent and makes the commutation constraints transparent.

2.2 Definition of the Ω–$\nVdash$ controller

Let $\kappa \in [0, 1]$. Define the Ω–$\nVdash$ *controller operator*

$$\Omega_\kappa \;:=\; \Pi_{\nVdash} + \kappa\, \Pi_\Omega.$$

Equivalently,

$$\Omega_\kappa(x) \;=\; x_{\nVdash} + \kappa\, x_\Omega.$$

This operator has three immediate properties that capture the intended control semantics.

Proposition 2.1 (Mean preservation and boundedness)**.** *For any $x \in \mathcal{H}_M$ and $\kappa \in [0, 1]$,*

1. *$\Pi_{\nVdash}\Omega_\kappa(x) = \Pi_{\nVdash}(x)$ (principal content is preserved), and*
2. *$\|\Omega_\kappa(x)\|_2 \le \|x\|_2$ (non-expansive in ℓ^2).*

Proof. (1) Since $\Pi_{\nVdash}\Pi_\Omega = 0$ and $\Pi_{\nVdash}^2 = \Pi_{\nVdash}$,

$$\Pi_{\nVdash}\Omega_\kappa = \Pi_{\nVdash}(\Pi_{\nVdash} + \kappa\Pi_\Omega) = \Pi_{\nVdash}.$$

(2) Orthogonality gives $\|x\|_2^2 = \|x_{\nVdash}\|_2^2 + \|x_\Omega\|_2^2$ and

$$\|\Omega_\kappa(x)\|_2^2 = \|x_{\nVdash}\|_2^2 + \kappa^2\|x_\Omega\|_2^2 \le \|x_{\nVdash}\|_2^2 + \|x_\Omega\|_2^2 = \|x\|_2^2. \qquad \square$$

The Ω–$\nVdash$ controller therefore does exactly one thing: it contracts the non-principal mass while leaving the principal mass untouched. For control-theoretic interpretation, the control strength is the scalar κ. For falsifiability, κ must be fixed by a lawbook rule or by a declared controller protocol; it may not be tuned post-hoc.

2.3 DOC admissibility and why Ω–$\mathbb{K}$ is only half the controller

Ω–$\mathbb{K}$ is a projector-level controller. In physical pipelines, it is composed with smoothing or transfer operators that must satisfy DOC admissibility constraints (positivity, unit mass, spectral multipliers in $[0,1]$ where required). The generic controlled step therefore has the form

$$x \;\mapsto\; \Omega_\kappa \, \mathcal{K} \, x,$$

where $\mathcal{K}$ is a DOC-admissible operator (typically a convolution kernel or a multiscale Fejér/MSF operator).

PH-4's role is to make explicit what is otherwise an implicit degree of freedom: if $\mathcal{K}$ fails DOC admissibility, then any downstream "stability" or "measurement" statement is no longer a claim about the finite substrate; it is a claim about an illegal operator. For this reason, PH-4 ties Ω–$\mathbb{K}$ to operator-admissibility transfer audits (DEMO-69) and to designed-fail boundaries in the falsification matrix.

2.4 Observer group: representation transforms as explicit finite actions

Fix M as above and define the cyclic shift operator P_1 by

$$(P_1 x)_n \;:=\; x_{(n-1) \bmod M}.$$

For $k \in \{0, 1, \dots, M-1\}$, define $P_k := P_1^k$. Each P_k is unitary (orthogonal in the real case): $P_k^* P_k = I$. The set

$$G_M \;:=\; \{P_k : k = 0, \dots, M-1\}$$

is a finite group under composition, isomorphic to $\mathbb{Z}/M\mathbb{Z}$.

Definition 2.2 (Admissible observer)**.** *An* admissible observer transformation *is any $O \in G_M$. The observer acts on the measurement surface by $x \mapsto Ox$.*

The physical interpretation is deliberately conservative: admissible observers are those that only relabel indices (change coordinates, shift origin, permute measurement bins) without introducing amplitude distortions or digit injections.

Proposition 2.3 (Observer neutrality of the principal content)**.** *For all k, $P_k \mathbf{1} = \mathbf{1}$. Consequently, for any x,*

$$\Pi_{\mathbb{K}}(P_k x) = \Pi_{\mathbb{K}}(x).$$

Proof. The shift permutes coordinates, so it preserves constants: $P_k \mathbf{1} = \mathbf{1}$. Then

$$\langle P_k x, \mathbf{e}_{\mathbb{K}} \rangle = \langle x, P_k^* \mathbf{e}_{\mathbb{K}} \rangle = \langle x, \mathbf{e}_{\mathbb{K}} \rangle,$$

and thus $\Pi_{\mathbb{K}}(P_k x) = \Pi_{\mathbb{K}}(x)$. □

This is the first precise statement of "observer neutrality" in its minimal form: admissible observers do not change the principal content of a state.

2.5 The commutant: why admissible smoothing commutes with admissible observers

The DOC-admissible kernels used throughout Marithmetics are translation-invariant on the relevant finite domains (convolutions / circulant operators). The essential structural reason is:

Theorem 2.4 (Commutant of G_M equals the circulants)**.** *A linear operator $C \in \mathbb{R}^{M \times M}$ commutes with all $P_k \in G_M$ if and only if C is circulant. In particular, any convolution-type smoothing operator commutes with all admissible observers.*

Sketch of proof. If C is circulant, it is translation-invariant and commutes with shifts. Conversely, if $CP_1 = P_1C$, a standard index calculation shows $C_{n,m}$ depends only on $(n-m)$ mod M, which is precisely the circulant condition. □

Consequently, any DOC-admissible smoothing $\mathcal{K}$ implemented as a circulant convolution lies in the commutant of the admissible observer group. This commutation is the operational content of "measurement should not depend on relabeling."

2.6 Ω–observer commutation

Ω–$\nVdash$ commutes with every admissible observer:

Proposition 2.5. *For all k and $\kappa \in [0,1]$,*

$$P_k\,\Omega_\kappa \;=\; \Omega_\kappa\,P_k.$$

Proof. Since P_k preserves $\mathbf{e}_{\nVdash}$, we have $P_k\Pi_{\nVdash} = \Pi_{\nVdash}P_k$. Then $P_k\Pi_\Omega = \Pi_\Omega P_k$ follows from $\Pi_\Omega = I - \Pi_{\nVdash}$. Therefore $P_k(\Pi_{\nVdash} + \kappa\Pi_\Omega) = (\Pi_{\nVdash} + \kappa\Pi_\Omega)P_k$. □

This is the algebraic core of the PH-4 architecture: admissible observation (shifts/permutations) commutes with admissible control (Ω–$\nVdash$) and with admissible smoothing (circulant kernels). As a result, "control then observe" equals "observe then control," provided all operators are within the declared admissible class.

3 DOC-Admissible Control: Kernel Class, Controller Class, and Illegal Controls

PH-4 makes a strict separation between (i) what the dynamics are (DOC: lawful finite operators) and (ii) how we stabilize / observe them (Ω–$\nVdash$ + observers). The stabilizer/observer layer is permitted only if it lives inside a declared admissible operator class with explicit certificates.

3.1 The admissible kernel class on a finite torus

Fix $M \geq 2$ and work on the cyclic group $\mathbb{Z}_M$. A (real) kernel $k : \mathbb{Z}_M \to \mathbb{R}$ defines a circulant convolution operator $\mathcal{K}$ by

$$(\mathcal{K}x)_n \;=\; (k * x)_n \;:=\; \sum_{m\in\mathbb{Z}_M} k_{n-m}\,x_m.$$

Definition 3.1 (DOC-admissible smoothing operator)**.** *A circulant operator $\mathcal{K}$ is* DOC-admissible *if its kernel k satisfies:*

1. ***Nonnegativity:*** *$k_n \geq 0$ for all $n \in \mathbb{Z}_M$.*
2. ***Unit mass:*** *$\sum_{n\in\mathbb{Z}_M} k_n = 1$.*
3. ***Spectral admissibility:*** *the discrete Fourier multiplier $\widehat{k}(\ell)$ satisfies*

$$0 \;\leq\; \widehat{k}(\ell) \;\leq\; 1 \quad \textit{for all} \quad \ell \in \mathbb{Z}_M.$$

Condition (3) is the operational PSD/energy condition for the smoothing family: it implies the operator is a convex contraction in Fourier space and, in particular, is non-expansive in ℓ^2.

Proposition 3.2 (Immediate consequences)**.** *If $\mathcal{K}$ is DOC-admissible, then:*

(a) $\mathcal{K}\mathbf{1} = \mathbf{1}$ *(mean preservation).*

(b) $\|\mathcal{K}\|_{2\to 2} \le 1$ *(non-expansive).*

(c) $\mathcal{K}$ *commutes with all shifts* P_k *(hence with all admissible observers in §2).*

Proof. (a) Unit mass implies convolution with k preserves constants. (b) Parseval gives $\|\mathcal{K}x\|_2^2 = \sum_\ell |\widehat{k}(\ell)|^2 |\widehat{x}(\ell)|^2 \le \sum_\ell |\widehat{x}(\ell)|^2 = \|x\|_2^2$. (c) Circulant operators commute with shifts by construction. □

Remark 3.3 (Fejér/MSF as canonical admissible kernels)**.** *The Fejér kernel* F_r *(periodized triangular weight) and the multi-scale Fejér (MSF) convex hull are the project's canonical admissible family: they satisfy nonnegativity, unit mass, and* $0 \le \widehat{F_r}(\ell) \le 1$*. In later sections, when we say "admissible smoothing," we mean "Fejér/MSF-admissible unless explicitly stated otherwise."*

3.2 The admissible controller class

The Ω–$\nVdash$ map from §2 is not, by itself, a smoothing kernel; it is a projector-level contraction of the mean-free subspace. We therefore define the admissible controlled operator class as the composition of Ω–$\nVdash$ with an admissible smoothing operator.

Definition 3.4 (DOC-admissible controlled operator)**.** *Fix* $\kappa \in [0, 1]$*. A* controlled operator *is any operator of the form*

$$\mathcal{C}_{\kappa,\mathcal{K}} \;:=\; \Omega_\kappa\,\mathcal{K},$$

where $\mathcal{K}$ *is DOC-admissible.*

Proposition 3.5 (Controller admissibility)**.** *For any* $\kappa \in [0, 1]$ *and any DOC-admissible* $\mathcal{K}$*,*

(a) $\mathcal{C}_{\kappa,\mathcal{K}}\mathbf{1} = \mathbf{1}$.

(b) $\|\mathcal{C}_{\kappa,\mathcal{K}}\|_{2\to 2} \le 1$.

(c) $\mathcal{C}_{\kappa,\mathcal{K}}$ *commutes with all admissible observers* P_k*.*

(d) If $x \ge 0$ *componentwise, then* $\mathcal{K}x \ge 0$*. Consequently, any sign-creation in* $\mathcal{C}_{\kappa,\mathcal{K}}x$ *can only occur through the mean-free projection* Π_Ω *(and is therefore auditable and not a hidden kernel artifact).*

Proof. (a) Since $\mathcal{K}\mathbf{1} = \mathbf{1}$ and $\Omega_\kappa\mathbf{1} = \mathbf{1}$, we have $\mathcal{C}_{\kappa,\mathcal{K}}\mathbf{1} = \mathbf{1}$. (b) $\|\Omega_\kappa\|_{2\to 2} \le 1$ by Proposition 2.1 and $\|\mathcal{K}\|_{2\to 2} \le 1$ by Proposition 3.2, so $\|\Omega_\kappa\mathcal{K}\| \le 1$. (c) Both Ω_κ and $\mathcal{K}$ commute with all shifts; hence so does their product. (d) Convolution with a nonnegative kernel preserves componentwise nonnegativity. □

The purpose of Proposition 3.5(d) is interpretive: if a domain adapter requires a notion of "positivity" (energy density, probability density, etc.), then illegal kernel-induced negativity is ruled out by admissibility. Any negative lobes that do appear must be attributable either to an explicitly declared mean-free projection or to an explicitly declared signed control (which is then illegal unless the domain contract permits it).

3.3 Illegal controls as designed-fail objects

A controller layer that cannot fail is not falsifiable. PH-4 therefore makes certain illegal operator classes explicit and uses them only as negative controls.

Definition 3.6 (Two canonical illegal controls)**.** *1.* ***Sharp cutoff (Dirichlet truncation).*** *Replace an admissible Fejér/MSF weight with a hard spectral projector:*

$$\widehat{k}_{\mathrm{sharp}}(\ell) = \mathbf{1}_{\{|\ell| \le R\}}.$$

This violates the kernel admissibility contract in real space: the inverse DFT kernel has oscillatory negative lobes (discrete Gibbs phenomenon), so $k_{\mathrm{sharp}} \not\geq 0$.

2. ***Signed injection.*** *Use a signed multiplier to force high-frequency energy:*

$$\widehat{k}_{\mathrm{signed}}(\ell) = \begin{cases} +1, & |\ell| \le R, \\ -1, & |\ell| > R, \end{cases}$$

which is not a PSD multiplier and defeats energy and positivity constraints.

Both classes are prohibited for Tier-A structural steps. They are used only as designed-fail controls to demonstrate that operator admissibility constraints are nontrivial and that the audit surface detects violations.

3.4 Why we require these contracts

Without Definition 3.1 and Definition 3.4, the phrase "controller" becomes ambiguous: one can always "stabilize" a calculation by injecting dissipation, truncation, or signed high-frequency cancellation. PH-4's purpose is to remove that freedom. A controller is only permitted if it is:

- mean-preserving,
- non-expansive in ℓ^2,
- observer-commuting, and
- supported by explicit operator certificates (Section 5).

4 Residual Budget Inheritance Under Observation and Control

PH-4's central quantitative claim is not that $\Omega\text{–}\nVdash$ creates correctness; it is that $\Omega\text{–}\nVdash$ does not destroy correctness when correctness is already controlled by the finite-to-continuum residual budget. In other words, control and observation must preserve the residual law, not override it.

4.1 The residual law as the universal error currency

Let $h \to 0$ be a refinement scale and let M_y denote the finite substrate window size used in the lift (for example, a primorial window or another explicitly declared finite torus size). Let $r(h)$ denote the Fejér/MSF span used to suppress aliasing at scale h.

The universal residual budget used throughout the lift pipeline can be written (in one canonical form) as

$$\eta_h \;=\; c_1\, h^2 \;+\; c_2\, \frac{r(h)}{M_y}, \qquad (h \to 0,\; r(h) \to \infty,\; r(h)/M_y \to 0),$$

where:

- the h^2 term is a low-band symbol/discretization accuracy contribution, and
- the $r(h)/M_y$ term is the high-band alias (Fejér/MSF tail) contribution.

PH-4 uses this law as an abstract budget that is imported from the lifting apparatus: any domain adapter is permitted to claim "continuum behavior" only inside an explicit η_h envelope.

4.2 Observation does not change the residual budget

Let $O \in G_M$ be an admissible observer (a unitary shift/permutation). For any operator discrepancy E,

$$\|OEO^{-1}\|_{2\to 2} \;=\; \|E\|_{2\to 2}.$$

Lemma 4.1 (Observer invariance of operator error). *If $\|K-L\|_{2\to 2} \le \eta$, then for any admissible observer O,*

$$\|OKO^{-1} - OLO^{-1}\|_{2\to 2} \le \eta.$$

Proof. Since O is unitary, $\|OEO^{-1}\| = \|E\|$ for the operator norm. Take $E = K - L$. $\square$

Interpretation. A referee is entitled to change coordinates on the measurement surface. If the error budget is real, it cannot depend on that coordinate choice.

4.3 Ω–$\nVdash$ control does not enlarge the residual budget

Lemma 4.2 (Ω–$\nVdash$ contraction of discrepancies). *Let $\kappa \in [0,1]$. If $\|K-L\|_{2\to 2} \le \eta$, then*

$$\|\Omega_\kappa K - \Omega_\kappa L\|_{2\to 2} \;\le\; \eta.$$

Proof. Ω_κ has operator norm at most 1 (Proposition 2.1), hence

$$\|\Omega_\kappa(K-L)\| \le \|\Omega_\kappa\|\,\|K-L\| \le 1\cdot\eta. \qquad \square$$

Corollary 4.3 (Controlled lifts preserve the same η_h). *Suppose K_h is a DOC-admissible discrete operator intended to approximate a continuum operator L under the lift, and suppose the lift guarantee provides*

$$\|K_h - L_h\|_{2\to 2} \le \eta_h.$$

Then for any $\kappa \in [0,1]$,

$$\|\Omega_\kappa K_h - \Omega_\kappa L_h\|_{2\to 2} \le \eta_h,$$

and for any admissible observer O,

$$\|O\Omega_\kappa K_h O^{-1} - O\Omega_\kappa L_h O^{-1}\|_{2\to 2} \le \eta_h.$$

This is the precise sense in which PH-4 treats observation and control as lawful overlays: they may restrict or stabilize, but they may not fabricate convergence by consuming the error budget.

4.4 Composition rule for multi-step controlled pipelines

Controlled pipelines are compositions of admissible operators:

$$\mathcal{T}_h \;=\; \Omega_{\kappa_m}\mathcal{K}_h^{(m)}\cdots\Omega_{\kappa_2}\mathcal{K}_h^{(2)}\,\Omega_{\kappa_1}\mathcal{K}_h^{(1)}.$$

Lemma 4.4 (Residual additivity under non-expansive composition)**.** *Assume each $\mathcal{K}_h^{(j)}$ is DOC-admissible and each $\kappa_j \in [0,1]$. Let $L^{(j)}$ be the intended continuum counterparts. If*

$$\|\mathcal{K}_h^{(j)} - L_h^{(j)}\|_{2\to 2} \le \eta_h^{(j)} \quad \textit{for each } j,$$

then the composed discrepancy obeys

$$\|\mathcal{T}_h - \mathcal{T}\|_{2\to 2} \le \sum_{j=1}^{m} \eta_h^{(j)},$$

where $\mathcal{T}$ is the corresponding continuum composition.

Proof. Expand by telescoping:

$$\begin{aligned}\mathcal{T}_h - \mathcal{T} = \sum_{j=1}^{m} &\left(\Omega_{\kappa_m}\mathcal{K}_h^{(m)} \cdots \Omega_{\kappa_{j+1}}\mathcal{K}_h^{(j+1)}\right) \Omega_{\kappa_j}(\mathcal{K}_h^{(j)} - L_h^{(j)}) \\ &\times \left(\Omega_{\kappa_{j-1}} L_h^{(j-1)} \cdots \Omega_{\kappa_1} L_h^{(1)}\right),\end{aligned}$$

then take norms. Every prefactor is non-expansive in ℓ^2 by admissibility, so each term is bounded by $\eta_h^{(j)}$, and summing yields the claim. □

This lemma is the bookkeeping rule used implicitly throughout PH-4: once operators are constrained to be non-expansive, residual budgets compose by straightforward addition.

5 OATB in the AoR: Operator Admissibility Transfer as an Audited Bridge

The Operator Admissibility Transfer Bridge (OATB) is the AoR's mechanism for preventing "silent illegality" in controllers and measurement operators. Its role is to produce an explicit PASS/FAIL record that:

- the kernels used in smoothing/transfer are nonnegative and unit mass,
- their multipliers satisfy $0 \le \widehat{k} \le 1$,
- the induced real-space kernels have no negative lobes beyond declared numerical tolerance, and
- the commutation contracts required by §2 hold at the declared error scale.

The canonical AoR surfaces for OATB are DEMO-69 stdout/stderr.

5.1 What OATB must certify (paper-level statement)

A controller/transfer operator $\mathcal{K}$ is *OATB-certified* if and only if the audit records:

1. **Kernel certificate:** $k \ge 0$, $\sum k = 1$, and a recorded numeric lower bound $k_{\min} \ge -\varepsilon_{\text{num}}$ with ε_{num} declared.
2. **Spectral certificate:** $\min_\ell \widehat{k}(\ell) \ge -\varepsilon_{\text{num}}$ and $\max_\ell \widehat{k}(\ell) \le 1 + \varepsilon_{\text{num}}$.
3. **Observer commutation:** $\|[\mathcal{K}, P_1]\|_{2\to 2} \le \varepsilon_{\text{num}}$ (equivalently, circulant structure).
4. **Principal preservation:** $\|\mathcal{K}\mathbf{1} - \mathbf{1}\|_2 \le \varepsilon_{\text{num}}$.

5. **Designed-fail separation:** the illegal controls of §3 produce unambiguous failures (negative lobes, spectral violations, HF injection), demonstrating that the audit is not vacuous.

In the AoR, these certificates are recorded as printed metrics and PASS/FAIL flags in the OATB run output. The purpose of PH-4 is to define precisely what those metrics mean and why they are necessary; the AoR is the execution record that the metrics were actually computed and logged.

5.2 AoR execution record: DEMO-69 (OATB) as a domain-spanning admissibility certificate

This section records (and interprets) the canonical OATB execution artifact. The OATB flagship is explicitly designed to be domain-spanning: it tests kernel admissibility, sharp-boundary failure modes, paradox resolution, Ω-reuse across PDEs, and cross-base/rigidity constraints in a single audit surface. The paper-level role of this section is therefore twofold:

1. To show that the admissible operator class of §3 is non-empty and algorithmically testable (Fejér passes).
2. To show that illegal controls are not cosmetic—they produce detectable, quantifiable failures (sharp/signed fail in multiple independent ways).

All numerical claims in §5 are printed in the AoR stdout for DEMO-69.

5.2.1 Stage 1: deterministic selection and the invariant scale ε

DEMO-69 begins with the same triple selection used throughout the AoR, yielding the primary triple

$$(w_U, s_2, s_3) = (137, 107, 103),$$

with four counterfactual triples captured outside the primary window. The derived invariants printed in DEMO-69 are

$$q_2 = 30, \qquad q_3 = 17, \qquad v_{2U} = 3, \qquad \varepsilon = \frac{1}{\sqrt{q_2}} \approx 0.18257419.$$

This ε is used as an auditable, non-tunable tolerance and "teeth margin" throughout the rest of the run (e.g., ε^2 floors for overshoot/undershoot, and $(1+\varepsilon)$ degradation requirements under counterfactual budgets).

5.2.2 Stage 2: UFET triangle multipliers and a near-constant witness

OATB includes a UFET-style triangle multiplier test at multiple spans $r \in \{8, 16, 32\}$. The audit records the minimum multiplier $H_{\min} \approx 1/(r+1)$, the DC multiplier $H_{\max} = 1$, and the corresponding near-constant witness $K(r)$, with a spread bound and an explicit closeness-to-2/3 bound:

$$K(8) = 0.670782, \qquad K(16) = 0.667820, \qquad K(32) = 0.666973,$$

$$\mathrm{spread}(K(r)) \le 1\% \quad (\text{observed: } 0.570\%), \qquad |K - 2/3| \le 2\% \quad (\text{observed: } 0.001858).$$

This stage is not intended to "derive" admissibility; it is intended to verify that the triangle family used in the admissible operator palette exhibits the expected band-span behavior and does not drift under refinement in a way that would reintroduce hidden tuning degrees of freedom.

5.2.3 Stage 3: kernel admissibility audit (real-space minima and HF weight)

With $N = 2048$ and $r = 16$, DEMO-69 reports real-space minimum values for three filters:

- **Fejér (admissible):**
$$\min k_{\mathrm{Fej}} = 5.176815 \times 10^{-10}, \qquad \mathrm{HF\_weight\_frac}(> r) = 0.$$
- **Sharp cutoff (illegal control):**
$$\min k_{\mathrm{sharp}} = -3.510996 \times 10^{-3}, \qquad \mathrm{HF\_weight\_frac}(> r) = 0.$$
- **Signed injection (illegal control):**
$$\min k_{\mathrm{signed}} = -9.677734 \times 10^{-1}, \qquad \mathrm{HF\_weight\_frac}(> r) = 0.983887.$$

The corresponding gates are (i) Fejér is nonnegative within declared numerical tolerance, (ii) sharp and signed exhibit negative lobes, and (iii) signed retains "large HF weight" with an explicit floor (reported as 0.250 in this run).

The mathematical meaning, relative to §3, is direct: the Fejér family behaves like a lawful DOC-admissible smoothing operator; sharp and signed fail the nonnegativity/PSD-style constraints in distinct, measurable ways.

5.2.4 Stage 4: sharp-transfer witness and counterfactual "teeth"

Stage 4 in DEMO-69 audits transport behavior by comparing a "truth" reference to variants filtered by Fejér, sharp, and signed. The audit prints three diagnostics that are specifically aligned to the paper-level contracts:

1. A **distance-to-truth** diagnostic (relative L^2 distance):
$$d_{\mathrm{Fej}} = 0.140531, \quad d_{\mathrm{sharp}} = 0.091217, \quad d_{\mathrm{signed}} = 0.239279,$$
with an admissible gate requiring $d_{\mathrm{Fej}} \le \varepsilon$ (observed: $0.140531 \le 0.182574$).
2. A **nonnegativity/undershoot** diagnostic via the minimum of the transported density y:
$$\min y_{\mathrm{Fej}} = 7.361219\times10^{-3}, \quad \min y_{\mathrm{sharp}} = -8.100846\times10^{-2}, \quad \min y_{\mathrm{signed}} = -1.620169\times10^{-1},$$
with the illegal controls required to cross a negative floor at scale $-\varepsilon^2$.
3. A **Gibbs/overshoot** diagnostic:
$$\mathrm{overshoot}_{\mathrm{Fej}} = -0.044278, \quad \mathrm{overshoot}_{\mathrm{sharp}} = 0.101205, \quad \mathrm{overshoot}_{\mathrm{signed}} = 0.202410,$$
with the illegal controls required to exceed a floor at scale ε^2.

Crucially, this stage also includes a **counterfactual-budget degradation check**. Under a budget reduction induced by a counterfactual triple, the run records

$$r_{\mathrm{cf}} = 5, \qquad d_{\mathrm{cf}} = 0.318089,$$

and enforces a "teeth" inequality of the form

$$d_{\mathrm{cf}} \;\ge\; (1+\varepsilon)\, d_{\mathrm{Fej}},$$

which is satisfied in the AoR execution (and printed as a passing gate).

In the language of PH-4, this is the operational meaning of "control is not a free knob": when the admissible budget is reduced, the performance metric must worsen in a predictable direction, and the AoR enforces that asymmetry.

5.2.5 Stage 5: paradox pack as an admissibility unifier (finite↔continuum + measure + collapse)

DEMO-69 includes a "paradox pack" that treats several classical pathologies as instances of the same boundary: admissible summation/transfer vs illegal truncation/partition. The AoR reports, among other gates:

- **Zeno partial sum close to 1:**

$$\sum_{n=0}^{N} 2^{-n} = 0.999999999069, \qquad \mathrm{err} = 9.313 \times 10^{-10}.$$

- **Grandi Cesàro sum close to 1/2:**

$$\sigma_{\text{Cesàro}} = 0.500000.$$

- **Gibbs comparison (Dirichlet vs Fejér):** overshoot and undershoot:

$$\mathrm{ov}_{\mathrm{Dir}} = 0.091747, \quad \mathrm{ov}_{\mathrm{Fej}} = -0.011686,$$

$$\min_{\mathrm{Dir}} = -0.086785, \quad \min_{\mathrm{Fej}} = 0.001998,$$

  with explicit floors at ε^2 and nonnegativity tolerances for Fejér.

The run also includes measure-conservation tests (Hilbert-hotel mass preservation and positive partitions) and an explicit illegal signed-partition mass witness (negative mass). Finally, it includes a quantum interference/collapse segment, including an extreme HF injection ratio printed for "sharp collapse," and a budget-reduction degradation gate for localization when the admissible budget r is reduced (e.g., $r = 32 \to 16$).

For PH-4, these paradox-pack tests serve one purpose: they demonstrate that the admissibility conditions placed on observation/control operators are not ad hoc. They are the same conditions that separate convergent, physically interpretable limits from oscillatory, sign-violating, or mass-violating pathologies.

5.2.6 Stage 6: Ω reuse across PDEs as a controller-layer portability test

DEMO-69 explicitly evaluates Ω-style control in multiple PDE settings (3D heat, 4D heat, and a 4D vector field diagnostic) and records a family of gates:

- Mass conservation holds under control (up to tiny numeric drift).
- Tracking improves under control by specified factors (e.g., 1.66 in the 3D heat tracking gate; 1.29 in the 4D heat tracking gate).
- High-frequency error/energy is suppressed (ratios printed in the run, e.g. 0.809, 0.715).
- Vector-field incompressibility improves dramatically (a divergence ratio reported as 2.739×10^{-19}) and kinetic energy is damped (2.334×10^{-4}).

This stage provides the portability evidence that motivates PH-4's architectural claim: $\Omega\text{–}\nvdash$ is not a domain-specific trick. It is a finite operator that behaves consistently under multiple operator/field contracts when constrained by the admissible class of §3.

5.2.7 Stage 7: cross-base invariance and rigidity

DEMO-69 includes a cross-base encode/decode check over multiple bases $(2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16)$, printing that the decoded triple remains $(137, 107, 103)$ and that the lane survivor pools match under the declared transform discipline. Two gates are explicitly enforced:

- Base round-trip holds and the primary triple is invariant across tested bases.
- A digit-dependent selector is not invariant, exhibiting multiple distinct outputs across bases (a designed-fail with "distinct=5" in this run).

The run then performs a rigidity scan (neighborhood variant test), reporting:

- Variants tested: 36,288.
- Variants yielding a triple: 2,387 (6.58%).
- Distinct triples seen: 837.
- Primary hits: 4 (0.01%).

and enforces gates that the scan is nontrivial, that the triple is not generic, that the primary is not ubiquitous, and that the primary appears at least once.

For PH-4, this stage closes the loop with §2: admissible observers and admissible base-transforms must not be exploitable degrees of freedom. DEMO-69 records the invariant behavior for the lawful selector and records explicit failure for digit-based hacks.

5.2.8 Determinism hash and final verdict

The AoR prints a determinism hash for DEMO-69 and records the final verdict as VERIFIED with a full-weight pass ledger. The determinism hash printed in the run is

determinism_sha256 = `aef5bfbfa1efc26072e7d8cd2e2ffd92bedfbca5323c8f4a794cd55f7215af76`,

and the run records "DEMO-69 VERIFIED."

5.3 Interpretation: what OATB contributes to PH-4

The role of OATB in the PH-4 architecture is structural, not decorative:

1. It defines a certificate boundary for when a controller/observer operator may be used without undermining falsifiability.
2. It provides a unified failure narrative across domains: the same illegal kernels that create Gibbs-style artifacts also create negative density in transport, violate mass partitions, and inject HF components in collapse-style tests.
3. It demonstrates portability: Ω control improves tracking and suppresses HF error across multiple PDE classes while maintaining mass constraints.
4. It re-asserts cross-base integrity: invariance is audited (not assumed), and digit-dependent hacks are forced to fail.

With OATB in place, PH-4's "controller/observer layer" is not an unregulated interpretive freedom; it is a lawful operator layer constrained by auditable conditions and designed-fail boundaries.

6 DOC vs Classical Finite Differences: Why Admissibility is a Numerical Boundary (DEMO-56)

PH-4's operator definitions would be incomplete if they remained purely abstract. A referee's primary objection would be practical: "Even if your operators are lawful, do they behave numerically in ways that justify treating them as the correct stabilizers/observers for finite-to-continuum lifts?"

DEMO-56 is included in the AoR precisely to answer that objection in a minimal, falsifiable way: it compares DOC-admissible spectral operators (with Fejér admissibility discipline) against classical finite differences and against illegal spectral controls (sharp/signed).

All numerical claims in §6 are printed in the AoR stdout for DEMO-56.

6.1 Kernel admissibility (again) is the first gate

DEMO-56 begins by re-confirming the kernel sign boundary:

- Fejér kernel minimum: $k_{\min} = 1.4845867057 \times 10^{-7}$ (nonnegative within tiny slack).
- Sharp truncation kernel minimum: $k_{\min} = -0.1026225707$ (negative lobes).
- Signed control kernel minimum: $k_{\min} = -0.1299921965$ (negative lobes).

This repetition is intentional: PH-4 treats admissibility as a cross-domain "legal boundary," so it must be visible across multiple audit surfaces, not only OATB.

6.2 Linear advection of a step: ringing and total variation as an illegal-kernel witness

A discontinuous step profile under advection is a classical setting in which illegal spectral truncations create Gibbs-style overshoot and excessive total variation.

DEMO-56 reports, at multiple times (e.g., $t = 0.13, 0.37, 0.71$):

- Fejér has zero overshoot mass (solution remains within $[0, 1]$ under the declared boundedness proxy).
- Sharp and signed have nonzero overshoot mass (ringing).
- Total variation is dramatically larger under sharp/signed than under Fejér.

For example, at $t = 0.37$ the run reports:

$$\mathrm{TV}_{\mathrm{Fej}} \approx 1.9867, \qquad \mathrm{TV}_{\mathrm{sharp}} \approx 5.2994, \qquad \mathrm{TV}_{\mathrm{signed}} \approx 7.7082,$$

with overshoot mass 0 for Fejér and strictly positive for sharp/signed (values printed in the run).

This stage's gates match PH-4's controller contract: admissible smoothing is permitted to stabilize and suppress HF artifacts, but only if the stabilizer is itself lawful and does not introduce sign violations or unbounded ringing.

6.3 Convergence under refinement without tuning

DEMO-56 records a convergence sweep for the Fejér-admissible method at increasing budgets $K \in \{30, 60, 120\}$, reporting a non-increasing error trend:

$$e_{K=30} = 0.0806730691, \quad e_{K=60} = 0.0571049942, \quad e_{K=120} = 0.0393776920.$$

The gate is explicitly phrased as "no tuning": error must not worsen as resolution/budget is increased.

This is an operational restatement of the residual-budget posture in §4: refinement should contract error in the direction predicted by admissible smoothing and finite window constraints.

6.4 Teeth: counterfactual budgets degrade performance

DEMO-56 includes counterfactual triples that change the derived budget via $q_3 \mapsto K$ while holding the physical test fixed. In the advection test, counterfactual budgets (reported as $K = 20$ for the counterfactuals in this run) yield a larger error (e.g., 0.0981630451), and the run enforces a degradation inequality by $(1 + \varepsilon)$ in at least 3/4 cases (achieving 4/4 in this execution).

This is the same teeth principle used in OATB and is the central anti-tuning constraint for PH-4: controller strength/budget is not an aesthetic choice; it has a predictable performance signature.

6.5 Smooth Poisson: spectral vs 2nd-order FD as a controlled baseline

For a smooth manufactured solution of Poisson's equation, DEMO-56 reports:

- 2nd-order FD error: $7.9380355830 \times 10^{-5}$.
- Spectral (DOC-admissible inverse) error: $2.7752322819 \times 10^{-16}$.
- Ratio: $3.4961197299 \times 10^{-12}$, with a decisive pass gate phrased against an ε^6 reference scale.

This test is included to show that the DOC spectral operator is not merely "stable"; it is correct in the smooth regime to a degree that classical low-order methods cannot match at fixed N.

6.6 Reaction–diffusion (Fisher–KPP): boundedness and illegal controls

For Fisher–KPP in 1D at fixed $N = 512$, DEMO-56 reports:

- L^2 vs internal reference: FD = 0.0312776258, Fejér = $1.8214154224 \times 10^{-6}$.
- Sharp and signed controls are much worse (errors printed as ≈ 0.0312775903).
- Boundedness for the admissible method is preserved (min/max printed as equal in the run for the recorded snapshot, indicating no ringing/overshoot under the test configuration).

The role in PH-4 is direct: admissible smoothing is an explicit stability contract that is visible in nonlinear settings; illegal truncations do not merely "look worse," they violate the intended boundedness behavior.

6.7 3D Navier–Stokes (Taylor–Green): incompressibility, HF injection, and teeth

In a 3D Taylor–Green vortex test (smoke-tier $N = 64$), DEMO-56 reports errors vs an admissible truth run:

- Errors: Fejér = 0.5451459885, sharp = 1.1530512571, signed = 1.1530512571.
- Divergence L^2: all variants remain small (values printed, with Fejér at 1.1224×10^{-11} and illegal variants larger but still under the declared incompressibility gate).
- HF energy fraction $> K_{\text{primary}}$: Fejér $\approx 1.99 \times 10^{-22}$ while sharp/signed are $\sim 10^{-16}$, and the run enforces a "non-admissible controls inject high-frequency energy" gate.
- Counterfactual budgets degrade strongly: with $K = 5$ the counterfactual error is 0.9730648994, satisfying the $(1 + \varepsilon)$ degradation gate in 4/4 cases.

This test connects to PH-4's Ω claims by emphasizing the same underlying logic: admissibility suppresses HF injection, and budget reduction produces predictable degradation. These are the numerical analogues of the operator-level non-expansiveness and residual-budget inheritance claims proved in §4.

6.8 Interpretation: what DEMO-56 establishes (and what it does not)

The role of DEMO-56 inside PH-4 is not to claim that any one numerical method is "the" correct method. Rather, DEMO-56 is used to substantiate a narrower and more structural proposition:

Proposition 6.1 (Numerical boundary is an admissibility boundary)**.** *When comparing solvers at fixed discretization scale, the decisive distinction for PH-4 is not "spectral vs finite-difference," but admissible vs non-admissible. In particular, kernels that violate the DOC admissibility contract (e.g., sharp truncations or signed high-frequency injection) systematically manifest failure modes that are independent of the specific PDE: negative lobes, overshoot/ringing, and high-frequency energy injection. Conversely, Fejér/MSF-admissible operators provide stable suppression without breaking mean or mass constraints, and they support monotone improvement under refinement, subject to the residual schedule inherited from the lift.*

The observable content of this proposition is recorded in DEMO-56 through paired positive and negative controls:

- **Positive controls:** admissible Fejér/MSF operators preserve the intended invariants (e.g., boundedness proxies, incompressibility gates, monotone refinement improvement) and satisfy "teeth" degradation under budget reduction.
- **Negative controls:** illegal kernels visibly violate the same invariants and degrade in a way that is not compatible with "mere parameter choice."

Equally important is what DEMO-56 does not establish. DEMO-56 does not prove global existence for any PDE; it does not imply that the continuum limit is solved by "spectral methods." Its role is to demonstrate that, for the controller/observer layer, legality is a measurable property (auditable by kernels and commutators), not a rhetorical property.

6.9 Consequence for PH-4: controllers cannot be "numerical tricks"

In the PH-4 architecture, a controller is permitted only if it can be expressed as a DOC-admissible operator (Definition 3.4) and if its effect on observables is accountable within a declared residual budget (Section 4). DEMO-56 contributes a practical corollary:

Corollary 6.2 (Controller legitimacy requires designed-fail separation)**.** *A stabilization layer is scientifically interpretable as "control" only if the same pipeline can be made to fail under narrowly specified illegal substitutions (sharp truncation, signed injection, non-unit mass windows), and those failures produce robust, repeatable signatures. Otherwise "stability" is compatible with hidden tuning and cannot be interpreted as structure.*

This corollary is one of the reasons PH-4 treats designed-fail tests as first-class AoR artifacts (Section 5).

7 Case Studies in Observation and Control as Lawful Operations

PH-4's abstract operator contracts are intended to be read alongside concrete case studies. This section packages three such case studies, each designed to be minimal in assumptions and maximal in auditability.

7.1 Case study I: observer commutation and the "safe" conserved channel

Recall the observer group $G_M = \{P_k\}$ acting by cyclic shifts on $\mathcal{H}_M$ (Section 2). PH-4's controller layer is safe to compose with observation only when it is compatible with this symmetry.

The minimal compatibility requirement is commutation:

$$[\mathcal{C}, P_k] = 0 \quad \text{for all } k \in \mathbb{Z}_M,$$

at least up to declared numerical tolerance when implemented.

For circulant operators (including Fejér/MSF smoothing), this commutation is exact. For the Ω–$\nVdash$ controller family,

$$\Omega_\kappa \;=\; \Omega + \kappa(I - \Omega), \qquad \mathcal{C}_{\kappa,\mathcal{K}} \;=\; \Omega_\kappa \mathcal{K},$$

commutation is inherited because both Ω_κ and $\mathcal{K}$ lie in the commutant of G_M (Sections 2–3).

Operational check (AoR posture). Rather than asking a referee to accept "commutes by symmetry," the AoR treats commutation as an auditable diagnostic: either the commutator norm is near machine precision (PASS), or it is not (FAIL). In the program architecture this is not a cosmetic check; it is the difference between "observer changes representation" and "observer changes content."

7.2 Case study II: quantum interference as a measurement-window boundary

The simplest finite model of "measurement" in the PH-4 sense is a nonnegative, unit-mass window functional W acting on a nonnegative density ρ:

$$\mathcal{M}_W(\rho) \;=\; \sum_{n \in \mathbb{Z}_M} W_n \, \rho_n, \qquad W_n \geq 0, \quad \sum_n W_n = 1.$$

This is the discrete analogue of integrating a probability density against a normalized test function.

In the quantum setting, the density is $\rho = |\psi|^2$ and the interference boundary is the familiar non-additivity:

$$|\psi_{A \cup B}|^2 \;\neq\; |\psi_A|^2 + |\psi_B|^2 \quad \text{(in general)}.$$

PH-4's contribution is not to re-derive quantum theory, but to specify the legality boundary for "measurement operators" on the finite substrate:

- If $W \geq 0$ and $\sum W = 1$, then $\mathcal{M}_W(\rho) \in [0, 1]$ whenever ρ is a probability density.
- If W is signed (negative weights) or not unit mass, then $\mathcal{M}_W$ can produce negative "probabilities" or violate mass conservation, and this is classified as an illegal measurement operator (designed-fail).

This is exactly the kind of designed-fail separation demonstrated in the OATB paradox pack: the same negative-weight pathologies that cause Gibbs ringing also violate probability and partition-mass invariants when measurement windows are allowed to be signed.

7.3 Case study III: coarse-graining and partitions of unity

A common objection to finite-substrate work is that "coarse-graining can hide errors." PH-4 answers this by requiring coarse-graining to be expressed as a partition of unity with nonnegative weights.

Let $\{W^{(j)}\}_{j=1}^{J}$ be windows on $\mathbb{Z}_M$ such that

$$W_n^{(j)} \geq 0, \qquad \sum_n W_n^{(j)} = 1 \text{ for each } j, \qquad \sum_{j=1}^{J} W_n^{(j)} = 1 \text{ for each } n.$$

For a density $u \geq 0$, define coarse-grained masses

$$M_j \;=\; \sum_n W_n^{(j)}\, u_n.$$

Then mass conservation is exact:

$$\sum_{j=1}^{J} M_j \;=\; \sum_n u_n.$$

The designed-fail is immediate: if weights are signed or $\sum_j W^{(j)} \neq 1$, then either negative masses appear or total mass is not preserved. These are finite, auditable failures that do not depend on any continuum asymptotic reasoning.

7.4 Unifying mechanism across case studies: admissibility is a conservation principle

The three case studies above are intentionally elementary. Their purpose is to make one point legible:

Admissibility is a conservation principle. Whether the object is a controller (Ω–$\not K$), a measurement window, or a coarse-graining partition, the admissible class is characterized by preservation laws (mean, mass, positivity, non-expansiveness). Illegal controls fail by violating those same laws in ways that are visible at finite size.

This is the conceptual bridge that makes the Ω/observer layer usable in the rest of the physics track: it ensures that "control" and "observation" are not interpretive degrees of freedom, but lawful operators with explicit invariants and explicit failure modes.

8 Electromagnetism as an Operator-Admissibility Stress Test (DEMO-59)

The purpose of this section is to document a canonical example in which "controller legality" is not a stylistic preference but an experimentally visible boundary. The Maxwell and electrostatics suites provide observables with well-known scaling and boundedness expectations; admissible controllers respect those expectations under the AoR budgets, while illegal controls exhibit reproducible violations.

All quantitative statements in this section are printed in the AoR stdout for DEMO-59.

8.1 Budgets, scales, and admissible parameterization

DEMO-59 begins with the deterministic primary triple

$$(w_U, s_2, s_3) = (137, 107, 103),$$

and the derived invariants

$$q_2 = 30, \qquad q_3 = 17, \qquad v_{2U} = 3, \qquad \varepsilon = \frac{1}{\sqrt{q_2}} \approx 0.1825741858.$$

These invariants determine the admissible budget scales used in the electromagnetic testbeds:

Electrostatics (3D):

$$N_3 = 64, \qquad K_{3,\text{primary}} = 15, \qquad K_{3,\text{truth}} = 31.$$

Maxwell operators (2D):

$$N_2 = 128, \qquad K_{2,\text{primary}} = 31, \qquad K_{2,\text{truth}} = 63.$$

The key posture is consistent with Sections 3–4: the controller is permitted to stabilize or transfer only within a declared budget, and performance must degrade in a predictable direction under counterfactual budget reductions.

8.2 Kernel legality audit: Fejér vs illegal controls

Before any physics observable is evaluated, DEMO-59 audits the kernel legality boundary for the filter families used in both electrostatics and Maxwell suites. Two non-admissible controls are included:

- **Sharp cutoff** (Dirichlet truncation), illegal by real-space sign violations.
- **Signed control** (HF injection), illegal by both sign violations and HF retention.

For electrostatics (1D diagnostic at $N = 64$, $K = 15$) the audit reports:

$$k_{\min}^{\text{Fej}} = 0.000000, \qquad k_{\min}^{\text{sharp}} = -0.105335, \qquad k_{\min}^{\text{signed}} = -0.318054,$$

and HF-weight energy fraction beyond K:

$$\text{HFfrac}(> K) = 0 \ (\text{Fej}), \quad 0 \ (\text{sharp}), \quad 1 \ (\text{signed}).$$

For the Maxwell filter family ($N = 128$, $K = 31$) the same boundary is reproduced:

$$k_{\min}^{\text{Fej}} = 0.000000, \qquad k_{\min}^{\text{sharp}} = -0.105911, \qquad k_{\min}^{\text{signed}} = -0.318246,$$

with the same HF-fraction pattern $0, 0, 1$.

This is the operational instantiation of Definition 3.1: Fejér lies inside the admissible class; sharp and signed are explicit illegals.

8.3 Electrostatics (3D): Coulomb/Gauss scaling as an admissibility witness

The electrostatics test evaluates whether the electric field magnitude follows the expected Coulomb/Gauss radial scaling. In a standard normalization, one expects

$$|E(r)| \ \propto \ r^{-2} \quad \Longleftrightarrow \quad \log|E| \ = \ c - 2\log r,$$

so the slope of $\log|E|$ vs $\log r$ should be close to -2.

DEMO-59 uses the radius list

$$r \in \{4, 6, 8, 10, 12\},$$

and reports the fitted slopes:

- truth: -1.906371
- admissible (Fejér): -1.791637
- sharp: -1.766929

- signed: -2.686532

It also reports two auxiliary diagnostics, each designed to discriminate lawful smoothing from illegal ringing:

(1) the spread of $r^2\langle|E|\rangle$ (standard deviation; lower is better):

$$\text{spread}: \quad \text{truth} = 0.002664,\ \text{Fej} = 0.005186,\ \text{sharp} = 0.011280,\ \text{signed} = 0.006484,$$

and (2) a ringing-curvature proxy (mean $|d^2|$; higher indicates oscillatory artifacts):

$$\text{curvature}: \quad \text{truth} = 0.002242,\ \text{Fej} = 0.002522,\ \text{sharp} = 0.029511,\ \text{signed} = 0.003761.$$

The gates enforced in the AoR are exactly aligned with PH-4's controller thesis:

- truth slope must be near -2 within a tolerance derived from ε;
- admissible slope must also be near -2;
- signed must retain HF beyond K (operator falsifier);
- at least one non-admissible control must exhibit stronger ringing curvature than the admissible run.

The scientific point is not that the admissible run is numerically identical to "truth." The point is that the admissible operator class respects the correct scaling behavior while the illegal controls produce distinct and interpretable failure signatures (HF retention and ringing).

8.4 Counterfactual teeth in electrostatics

To prevent post-hoc budget selection, the electrostatics suite is repeated under counterfactual triples. In DEMO-59, the counterfactuals all share $q_3 = 51$, inducing a reduced electrostatic budget $K_3 = 5$. The run prints a counterfactual score and enforces the teeth condition: counterfactual performance must degrade by at least $(1+\varepsilon)$ in a majority threshold (here 4/4).

This is the same principle as in Sections 4 and 6: budgets are structural outputs of the triple, not free parameters, and "control" must reflect that constraint.

8.5 Maxwell-class operator suite (2D): boundedness and Gibbs-type falsifiers

The Maxwell suite in DEMO-59 contains two distinct tests: a step-front reconstruction falsifier and a smooth broadband deformation test with teeth.

Step reconstruction (Gibbs/overshoot proxy). The boundedness expectation is that an admissible reconstruction should not create values outside $[0,1]$ when reconstructing a discontinuous indicator-type front. DEMO-59 reports:

$$\text{overshoot (max outside } [0,1]): \quad \text{Fej} = 0,\ \text{sharp} = 0.06843775,\ \text{signed} = 0.25000000,$$

$$\text{TV}_x \text{ (avg over } y): \quad \text{Fej} = 1.960251,\ \text{sharp} = 4.168921,\ \text{signed} = 4.168921.$$

The enforced gates are:

- Fejér reconstruction is bounded (overshoot $= 0$);
- sharp truncation exhibits Gibbs overshoot above a floor tied to ε^2.

Smooth broadband field: counterfactual teeth. A second test evaluates a smooth Gaussian bump under filtering, reporting a primary distortion score and counterfactual scores at reduced budget $K_2 = 10$. In DEMO-59:

$$\text{Primary score} = 0.0108850, \qquad \text{Counterfactual score} = 0.0313388 \text{ (all four counterfactuals)},$$

and the teeth gate again requires degradation by $(1+\varepsilon)$ in $\geq 3/4$ cases, achieved as 4/4.

8.6 Determinism hash and audit closure

DEMO-59 prints its determinism hash:

$$\texttt{determinism\_sha256} = \texttt{5a74664c0e5719d72eb2b5f8345829aa8531175b1fdb5b39da45264d68d77530},$$

and records the final verdict as VERIFIED.

In the PH-4 architecture, this determinism hash is not administrative. It is the cryptographic closure that binds the controller/observer legality boundary to a reproducible execution, preventing retroactive modification of the evidence surface.

9 Ω–$\nVdash$ as a Cross-Domain Control Schema: Bridge to SM-Class Outputs (DEMO-34)

The electromagnetic case study above establishes the operational boundary: admissible operators respect scaling and boundedness constraints; illegal operators violate them and do so in quantifiable ways.

DEMO-34 is included for a different role: it demonstrates that the same Ω–Fejér admissible operator palette can be nested inside the "Ω-certificate" pipeline that produces SM-class outputs from the coupled lawbook. This does not replace the SM papers (PH-1/PH-2/PH-3); rather, it records how the controller/observer layer plugs into them without creating new degrees of freedom.

All quantitative statements in this section are printed in the AoR stdout for DEMO-34.

9.1 UFET diffusion slope: a minimal finite-to-continuum sanity check

DEMO-34 begins with a toy diffusion test that reports a refinement slope (log error vs log h) of 1.9852, together with an energy decay gate. The slope being near 2 is the expected signature of a second-order refinement regime under the chosen discretization model. In PH-4 terms, this is the minimal sanity check that "admissible smoothing + lawful dynamics" yields the correct asymptotic trend in an environment where illegal truncations would typically distort the convergence behavior.

9.2 Ω certificate and ablation necessity in a joint band

DEMO-34 then constructs the joint triple $(137, 107, 103)$ and audits ablation necessity at band scale (up to 10^6 in this run). The run records that dropping key coupled constraints causes "explosions" (large counts of non-target triples), while the coupled lawbook yields the unique target triple in the band:

- constructed triple matches target: $(137, 107, 103)$;
- joint triple is unique in the tested band;
- ablations such as "drop u_1 lock," "drop q_2 coupling," and "drop pc_2 lock" explode with explicit witness samples.

For PH-4, this matters because controller/observer overlays must not be allowed to "repair" failures of the coupled selection. DEMO-34 shows the coupled constraints are doing real work; the controller layer is not substituting for them.

9.3 Rational anchor block tied to Ω invariants

DEMO-34 includes an explicit rational anchor block tied to Ω-derived invariants:

$$\alpha_{\mathrm{em}}^{\mathrm{anchor}} = 0.007299270, \qquad \alpha_s^{\mathrm{anchor}} = 0.117647059, \qquad \sin^2\theta_W^{\mathrm{anchor}} = 0.233333333,$$

and prints reference overlays and relative deviations, with a gate requiring all to be within 2%. The deviations printed in the run are 0.026%, −0.215%, and 0.914%, respectively, passing the anchor gate.

In the PH-4 narrative, this anchor block is an instructive boundary: the controller/observer layer can be used to transport and stabilize the computation, but the anchors themselves are audit-visible rational objects tied to the Ω-certificate and not adjustable by representation.

9.4 The controller/observer layer as a non-intervening overlay

DEMO-34 is significant for PH-4 because it displays, in a single audit surface, the separation that this paper insists on:

- **Upstream:** coupled constraints and templates determine the invariant outputs.
- **Downstream:** admissible control/transfer operators may be used to compute and transport those outputs without creating an additional degree of freedom.

In particular, DEMO-34's ablation block demonstrates that the coupled constraint system is not optional: when key couplings are removed, the system admits many non-target triples ("explodes"), and the audit prints witness samples. This is the correct behavior under the PH-4 contract: the controller/observer layer is not permitted to "repair" or "retrofit" selection failures. It operates only after the invariant channel has been fixed and therefore cannot substitute for the coupled lawbook constraints that create uniqueness.

9.5 Interpretation boundary for the anchor block

The anchor block in DEMO-34 is included here as a controller-layer integration witness, not as a substitute for PH-2's constants work. The anchor values are explicitly rational and are treated as representation-independent objects when paired with the cross-base integrity contract (PH-2).

PH-4's role is narrower: to show that, when the pipeline is run with admissible control/transfer operators, the anchor values are obtained in a way that is:

- deterministic (AoR transcript + determinism hash),
- coupled (ablation necessity), and
- portable (eligible for observer commutation and admissible transport audits).

The full interpretation and comparison of these anchors to experimental constants belongs to the PH-2/PH-1 authority record and the dedicated constants papers. PH-4 cites DEMO-34 only to establish that the Ω/controller layer composes with the invariant channel without compromising the falsification boundary.

10 Ω–$\nVdash$ as a Universal Projection Schema on Finite Harmonic Analysis

Sections 2–4 defined Ω–$\nVdash$ using the constant vector $\mathbf{1}$ and its orthogonal complement in a Euclidean Hilbert space. That presentation is intentionally accessible, but it can make Ω–$\nVdash$ appear "special" to cyclic index sets.

This section formalizes Ω–$\nVdash$ as a universal construction on any finite group (and, in particular, on the finite groups that arise in Marithmetics: translation groups for PDE lifts and Dirichlet-character groups for residue-class arithmetic). The payoff is conceptual and structural: Ω–$\nVdash$ is not a heuristic damping trick; it is the unique, symmetry-respecting way to separate the trivial representation from the non-trivial sector and then suppress the non-trivial sector by a lawful amount.

10.1 Finite group setting and the trivial representation projector

Let G be a finite group and let $\mathcal{H}(G)$ be the Hilbert space of complex-valued functions on G with inner product

$$\langle f, g\rangle \;:=\; \sum_{x\in G} f(x)\,\overline{g(x)}.$$

Define the constant function $\nVdash_G(x) \equiv 1$. The trivial representation subspace is

$$\mathcal{P}(G) \;:=\; \operatorname{span}\{\nVdash_G\},$$

and its orthogonal complement

$$\mathcal{P}(G)^{\perp} \;:=\; \{f \in \mathcal{H}(G) : \langle f, \nVdash_G\rangle = 0\}.$$

Define the orthogonal projector onto $\mathcal{P}(G)$ by

$$(\Pi_{\nVdash} f)(x) \;:=\; \frac{1}{|G|}\left(\sum_{y\in G} f(y)\right)\nVdash_G(x) \;=\; \frac{1}{|G|}\sum_{y\in G} f(y).$$

Thus $\Pi_{\nVdash} f$ is the group average of f, viewed as a constant function. The complementary projector is $\Pi_\Omega := I - \Pi_{\nVdash}$.

Proposition 10.1 (Symmetry and uniqueness of the trivial projector)**.** $\Pi_{\nVdash}$ *is the unique rank-one orthogonal projector that is invariant under the left regular action of* G *on* $\mathcal{H}(G)$*.*

Sketch. The left regular action $L_g f(x) = f(g^{-1}x)$ preserves the uniform measure on G. The only functions fixed by all L_g are constants. Therefore the fixed space is $\operatorname{span}\{\nVdash_G\}$, and the orthogonal projector onto this fixed space is unique. □

This is the general form of "observer neutrality" in representation-theoretic language: admissible observers act through the group, and the trivial sector is their fixed content.

10.2 Universal Ω–$\nVdash$ controller on $\mathcal{H}(G)$

Definition 10.2 (Ω–$\nVdash$ controller on a finite group)**.** *For* $\kappa \in [0, 1]$*, define*

$$\Omega_\kappa^{(G)} \;:=\; \Pi_{\nVdash} + \kappa\,\Pi_\Omega.$$

Equivalently,

$$\Omega_\kappa^{(G)}(f) = \Pi_{\nVdash} f + \kappa\,(f - \Pi_{\nVdash} f).$$

Proposition 10.3 (Group-level stability and invariance)**.** *For every finite group* G *and every* $\kappa \in [0, 1]$*:*

1. $\Omega_\kappa^{(G)}$ *is non-expansive:* $\|\Omega_\kappa^{(G)} f\|_2 \le \|f\|_2$*.*
2. $\Omega_\kappa^{(G)}$ *preserves the trivial content:* $\Pi_{\nVdash}\Omega_\kappa^{(G)} = \Pi_{\nVdash}$*.*

3. $\Omega_\kappa^{(G)}$ *commutes with the left regular action:* $L_g\Omega_\kappa^{(G)} = \Omega_\kappa^{(G)} L_g$ *for all* $g \in G$.

Proof. (1) Orthogonal decomposition $\mathcal{H}(G) = \mathcal{P}(G) \oplus \mathcal{P}(G)^\perp$ gives $\|f\|_2^2 = \|f_{\nVdash}\|_2^2 + \|f_\Omega\|_2^2$ and $\|\Omega_\kappa f\|_2^2 = \|f_{\nVdash}\|_2^2 + \kappa^2\|f_\Omega\|_2^2 \le \|f\|_2^2$. (2) follows from $\Pi_{\nVdash}\Pi_\Omega = 0$. (3) follows from Proposition 10.1: $\Pi_{\nVdash}$ is G-invariant and therefore commutes with L_g; so does Π_Ω. □

This proposition is the abstract mathematical content behind the project's claim that Ω–$\nVdash$ is "universal in operator form." It is universal because it is the unique symmetry-respecting way to separate the trivial sector and damp everything else without enlarging norms.

10.3 Specialization I: translation group on the finite torus (PDE lift setting)

Let $G = \mathbb{Z}_M$ under addition. Then $\mathcal{H}(G) \cong \mathbb{C}^M$, the left regular action is the cyclic shift P_k, and the character basis is the discrete Fourier basis $e^{2\pi i\ell n/M}$. The trivial character is $\ell = 0$, i.e., constants.

Thus the construction of Sections 2–3 is exactly the specialization of Definition 10.2 to $G = \mathbb{Z}_M$. The commutant theorem for circulants (Section 2) is the corresponding statement that translation-invariant operators are precisely those diagonal in the character basis.

10.4 Specialization II: Dirichlet character sector (residue-class arithmetic setting)

Let $m \in \mathbb{N}$, and let $G = (\mathbb{Z}/m\mathbb{Z})^\times$ be the multiplicative unit group. Dirichlet characters modulo m are precisely the homomorphisms

$$\chi : G \to \mathbb{C}^\times$$

extended by $\chi(x) = 0$ when x is not a unit (standard extension). The principal character χ_0 is the trivial character on G (equal to 1 on units).

For functions $f : G \to \mathbb{C}$, the character expansion is an orthonormal decomposition into irreducible representations (since G is abelian). The trivial sector is exactly the span of χ_0. Therefore, the Ω–$\nVdash$ controller on $\mathcal{H}(G)$ is a literal "principal vs non-principal" separator in the Dirichlet character basis:

- $\Pi_{\nVdash}$ extracts the principal (trivial) character component (the average over G).
- Π_Ω extracts the non-principal character content (mean-zero sector).
- Ω_κ contracts the non-principal sector while preserving the principal sector.

This is the structural bridge that allows PH-4 to speak consistently across tracks: the Ω–$\nVdash$ controller used in PDE lifts and the Ω–$\nVdash$ controller used in residue/character pipelines are the same operator schema instantiated on different finite groups.

10.5 Why this matters for falsifiability

A common failure mode in "controller narratives" is that the controller is selected because it produces a desired result. The group-level formalization above blocks that failure mode in two ways:

1. **Symmetry forcing.** The trivial projector $\Pi_{\nVdash}$ is uniquely forced by the observer symmetry (Proposition 10.1). If a proposed "principal" sector is not the fixed space of the declared observers, it is not an observer-invariant invariant and is therefore disallowed for structural claims.

2. **Controlled freedom.** The only remaining degree of freedom in Ω–$\nVdash$ is a single scalar κ that multiplies the non-principal sector. In Marithmetics this scalar is not tuned arbitrarily; it is tied to declared invariants and residual budgets, and it is audited by teeth constraints under counterfactual budgets (Sections 4–6).

This is the sense in which Ω–$\nVdash$ is a constitutional controller: it is a symmetry-forced projector plus a single controlled contraction parameter, not a high-dimensional stabilization knob.

11 Ω Reuse Across PDE Classes: Adapter Contracts and Measurable "Control Wins"

The Ω–$\nVdash$ layer was introduced as a structural projector that separates principal (observer-fixed) mass from non-principal mass, and as a lawful contraction on the latter. Up to this point, the paper has established (i) the mathematical admissibility of the operator class, and (ii) the necessity of admissible smoothing in transfer contexts (Sections 3–5). We now record the key operational claim that makes Ω relevant beyond a single demonstration:

Ω is reusable across PDE classes, provided each PDE is equipped with an adapter contract that defines (a) what Ω acts on, and (b) which error and constraint metrics must improve (or fail) deterministically.

This section formalizes that contract and logs the AoR's flagship reuse evidence.

11.1 Adapter contract: what must be declared for Ω to be meaningful

An Ω claim is only meaningful when the following items are explicit.

(1) State representation. The domain must specify a finite state vector $x \in \mathcal{H}_M$ (or a product space $\mathcal{H}_M^d$ for vector fields) on which Ω acts linearly. In PDE settings this is typically a Fourier-spectral state or a circulant state induced by a periodic grid (so that shift-commutation and DFT diagonalization hold as in Section 2).

(2) Constraint projector(s). If the domain has constraints (e.g., incompressibility, Gauss law, gauge fixing), it must specify a projector P and an admissibility criterion that ensures either:

- exact commutation $[P, \Omega] = 0$ (ideal spectral case), or
- controlled commutator in the residual budget sense (Section 4).

(3) Score metrics and their direction. The domain must declare which observables are audited, what "improvement" means, and what floor values constitute a pass. In the AoR, the recurring pattern is:

- conservation checks (mass, normalization) must remain within a tight tolerance;
- tracking error to a reference must improve by a fixed factor bound;
- high-frequency (HF) content must contract;
- constraint violations (divergence, negativity) must contract dramatically.

(4) Negative controls. At least one designed-fail kernel must be included in the same evaluation harness, so that the pipeline can demonstrate "pass under admissibility" and "fail under non-admissibility." For Ω-reuse claims, the AoR consistently pairs the Fejér kernel with illegal sharp and signed controls (Sections 5–6).

11.2 AoR flagship evidence: Ω reuse across heat flow and vector-field constraints

The AoR's operator-admissibility flagship explicitly reuses the same Ω operator across multiple PDE targets in a single audit run, with conservation checks and improvement ratios recorded as gates.

Heat flow (3D and 4D). In the AoR run, Ω is applied to a 3D heat evolution and separately to a 4D heat evolution. In both cases:

- mass conservation is verified after control; and
- tracking error to the reference improves (the uncontrolled error divided by the controlled error exceeds a fixed factor); and
- HF error is suppressed (the controlled HF error is a strict fraction of the uncontrolled HF error).

The recorded gates are:

3D heat:

- mass conserved: $\Delta\text{mass}_{\text{ctrl}} = 0.000 \times 10^0$,
- tracking improvement factor $= 1.66$,
- HF suppression ratio $= 0.809$.

4D heat:

- mass conserved: $\Delta\text{mass}_{\text{ctrl}} = -2.910 \times 10^{-11}$,
- tracking improvement factor $= 1.29$,
- HF suppression ratio $= 0.715$.

These numbers are not presented as "best possible" outcomes; they are presented as deterministic, reproducible certificates that the same Ω construction—without retuning the controller—induces measurable and directionally correct wins across dimensionalities under the same admissibility discipline.

Vector-field constraint enforcement (4D). The AoR further applies Ω to a 4D vector-field setting with incompressibility/constraint metrics, logging three gates:

- incompressibility improves: divergence ratio $= 2.739 \times 10^{-19}$,
- kinetic energy is damped: energy ratio $= 2.334 \times 10^{-4}$,
- HF kinetic energy is damped: HF ratio $= 3.005 \times 10^{-36}$.

The magnitude of these ratios is operationally important: it demonstrates that, for this adapter, Ω is not merely "smoothing"; it functions as a decisive suppressor of the monitored illegal content (constraint violation and HF energy) while maintaining the conservation gate.

11.3 Interpretation: what is (and is not) being claimed by "Ω reuse"

The AoR evidence supports the following scoped statement.

Proposition 11.1 (Schema-level universality, adapter-level falsifiability)**.** *Given (i) a finite state representation on which Ω is admissible, (ii) an explicit metric suite with conservation and constraint gates, and (iii) paired illegal controls, Ω reuse is a falsifiable claim: either the gates pass deterministically under admissible kernels and fail under illegal controls, or the reuse claim is rejected for that adapter.*

What is *not* claimed here:

- that Ω by itself "solves" a PDE class in the sense of providing a full existence/uniqueness theory;
- that Ω eliminates the need for the residual-budget schedule (Section 4) in refinement limits;
- that Ω can be applied without specifying the state space and constraint projectors.

Instead, Ω reuse is claimed as a portable, auditable mechanism: it supplies a universal operator form whose validity is decided by explicit adapter contracts and AoR-grade PASS/FAIL evidence.

12 Constraint Compatibility: Why Ω Does Not "Smuggle In" Constraint Violations Under Admissible Lifts

A recurring skeptical objection is that any projector strong enough to suppress HF components might also violate constraints (e.g., incompressibility, Gauss law) by acting "off-manifold." This section records the mathematical reason Ω does not have that failure mode in the admissible setting, and how the AoR audits the claim.

12.1 Circulant commutation: Ω and Fejér smoothing commute with shift-defined projectors

In the finite periodic setting used throughout Marithmetics, the admissible kernels (Fejér) and the admissible observer projector Ω are circulant (equivalently, they commute with the cyclic shift group G_M). By Theorem 2.4 and the DFT diagonalization of circulants:

- Ω is diagonal in the DFT basis with eigenvalue 1 on the $k = 0$ (constant) mode and 0 on $k \neq 0$ modes;
- Fejér smoothing S_r is also diagonal in the DFT basis with multipliers $\widehat{F_r}(k) \in [0, 1]$;
- therefore $\Omega S_r = S_r \Omega$, and similarly $\Pi S_r = S_r \Pi$.

Now consider any constraint projector P that is itself circulant (or more generally diagonal in the DFT basis), which is the standard form of spectral Helmholtz projectors, spectral divergence-free projectors, and many gauge-compatible projectors. Such a P commutes with all circulants, hence:

$$[P, \Omega] = 0, \qquad [P, S_r] = 0, \qquad [P, \Omega_{\kappa_\Omega}] = 0.$$

Therefore the Ω action preserves the constraint subspace exactly in the ideal spectral case: applying Ω cannot introduce constraint violations that were not already present in the state representation.

12.2 AoR audit posture: constraints are checked as first-class gates

The AoR does not rely solely on commutation reasoning; it audits constraint metrics directly. In the Ω reuse stage of the OATB flagship, incompressibility improvement is asserted as an explicit gate, and the divergence ratio is recorded as the measured witness, along with energy and HF energy ratios.

This auditing style is structurally important: it ensures that Ω is not treated as "mathematically safe" by assumption, but is treated as "safe because it passes explicit constraint gates under admissible operators and fails under illegal controls," which is the only standard compatible with the AoR's constitutional posture.

13 Cryptographic and Reproducibility Closure for Controller Claims

PH-4 is not solely a mathematical paper. It is a paper about when a mathematical operator can be treated as a scientific instrument. For controllers this distinction is decisive: an operator can be "right" on paper and still be unusable as an evidentiary mechanism if its outputs drift with execution environment, FFT plans, thread scheduling, or implicit trig rounding choices.

Accordingly, the controller layer in Marithmetics is required to satisfy a reproducibility closure:

A controller claim is valid only if the operator is (i) fully specified, (ii) hash-identified, and (iii) reproducible under the AoR manifest discipline, with explicit negative controls that detect and reject nondeterministic or non-admissible execution paths.

This section formalizes that closure and records the AoR's reproducibility surfaces relevant to PH-4.

13.1 Manifest discipline: what must be fixed for equality to be meaningful

Let T denote any finite operator used in a controller role (e.g., Ω, Ω_κ, Fejér smoothing S_r, composite controllers such as $\Omega_\kappa S_r \Omega_\kappa$, or projector-controlled resolvents). A claim of the form

$$T(x) = y \quad \text{(as an auditable equality claim)}$$

is meaningless unless the entire evaluation context is fixed. In Marithmetics, the minimal context for such a claim is a manifest tuple

$$\mathsf{M} \;=\; (\mathsf{code\_hash},\ \mathsf{env\_hash},\ \mathsf{params},\ \mathsf{data\_hash},\ \mathsf{result\_hash},\ \mathsf{rng\_seed},\ \mathsf{ulps\_budget}),$$

with the interpretation:

- code_hash: cryptographic hash of the operator and projector implementation sources (including FFT conventions, window definitions, and quantization rules, if any);
- env_hash: cryptographic hash identifying the runtime environment (container image, lockfile, library versions);
- params: the full declared integer parameter set (grid sizes, spans, cutoffs, residual schedule parameters, and any controller scalars such as κ);
- data_hash: hash of input data (test vectors, initial states, reference solutions);
- result_hash: hash of the produced outputs (including intermediate artifacts when used for equality claims);
- rng_seed: the declared seed when randomized test vectors or sampling are used;
- ulps_budget: a measured floating-point deviation budget, reported explicitly.

Equality protocol. A controller equality claim is accepted only as an equality under manifest:

$$(\mathsf{M}, x) \;\mapsto\; y,$$

and any change in M constitutes a different claim. This is the cryptographic meaning of "no hidden knobs": the claim is tied not merely to the formula for T, but to the computable instantiation of T.

13.2 Cross-language parity and rationalized kernels: defeating "implementation numerology"

For smoothing-based controllers, one specific source of illegitimate drift is trig rounding (e.g., evaluating $\sin(\cdot)$ in different library implementations). To prevent this, the controller layer adopts a parity posture:

- if a kernel multiplier can be made exact (e.g., by rationalization and serialization of the multiplier table), it is made exact in audit mode;
- parity across independent language stacks is treated as an adversarial test, not as a convenience.

The operational goal is not to claim that "floating point is irrelevant," but to require that floating-point deviations are bounded and dominated by the theoretical residual η_h when a residual schedule is in force. In particular, for claims that depend on contraction and commutation (Sections 3–5), it is essential that any implementation drift be separable from the mathematical signal.

13.3 AoR reproducibility surface: run-reproducibility table

The AoR includes a dedicated reproducibility ledger that records per-demo stability across runs under controlled settings, with failures explicitly captured and classified. This is the correct evidentiary posture for controllers: rather than asserting determinism as an axiom, the AoR records determinism (or its failure modes) as a first-class artifact, tied to the same bundle seal as the rest of the project.

Interpretation. PH-4 treats the existence of this ledger as a structural precondition for any controller-based claim: a controller is not admissible for scientific use unless it is admissible for reproducible execution.

13.4 Why PH-4 requires cryptographic closure (and why physics papers inherit it)

The physics papers (PH-1–PH-3) cite constants and invariants as outputs of the pipeline. Those outputs must be portable across bases (PH-2) and stable under admissible transformations (Foundations). But in practice, the pipeline is implemented, executed, and audited by code. Therefore, any physics claim that depends on controller stabilization inherits a second obligation:

- the controller must be described mathematically (Sections 2–4), and
- the controller must be described operationally (manifest + determinism) so that the mathematical description is not undermined by implementation drift.

PH-4 supplies the second half of this obligation.

14 Designed-Fail Controls Specific to Controllers and Observers

A controller layer can fail in ways that do not occur in purely algebraic derivations. In particular:

- a controller can "work" while silently violating invariance principles (e.g., by smuggling in an observer-dependent bias), and
- a controller can "work" by exploiting implementation nondeterminism (e.g., FFT plan entropy), thereby breaking scientific reproducibility while appearing numerically stable on a single machine.

PH-4 therefore requires controller-specific designed-fail controls in addition to the cross-base designed-fail controls formalized in PH-2.

14.1 Negative control class IV: illegal kernels that violate admissibility axioms

The admissible smoothing operators used throughout Marithmetics obey at minimum:

- positivity (PSD),
- unit mass (mean preservation),
- symmetry (self-adjointness), and
- spectral attenuation $\widehat{K}(k) \in [0, 1]$ with a controlled tail.

Illegal kernel controls are constructed by violating one of these axioms while leaving the code path otherwise similar. Examples include:

- sharp cutoffs that introduce Gibbs artifacts while preserving some norms,
- signed kernels that break positivity, and
- non-normalized kernels that violate mean preservation.

Designed-fail requirement. When substituted into the controller path, these illegal kernels must trigger explicit FAIL signatures under the AoR falsification protocol (e.g., mass drift beyond tolerance, instability of constraint metrics, or failure of cross-language parity gates). A controller layer that continues to "pass" under these substitutions is rejected as non-discriminative.

14.2 Negative control class V: non-commuting observers

In Sections 2 and 10, admissible observers were formalized as projectors onto the fixed space of a declared symmetry action (translation, group action, or base-gauge action). Illegal observers are then constructed by violating this requirement:

- selecting a "principal component" that is not the fixed space of the declared observer group, or
- applying a non-unitary (or not symmetry-equivariant) transformation and treating it as an observer.

Designed-fail requirement. When such an observer is used in place of Ω, one must observe either:

- loss of invariance (outputs become representation-dependent), or
- collapse of the commutation identities that guarantee constraint compatibility (Section 12),

together with an explicit FAIL classification in the AoR falsification catalog.

14.3 Negative control class VI: nondeterminism as a falsifier

A distinct designed-fail class for controllers is execution nondeterminism. This is not a mathematical violation; it is a scientific violation: if two runs of the "same" operator produce different outputs under uncontrolled FFT planning or thread scheduling, then equality claims are invalid and any "instrument" interpretation is disallowed.

Designed-fail requirement. The AoR records nondeterminism signatures as FAILs (with explicit remediation steps such as locked FFT planning and single-thread execution). The purpose is to ensure that determinism is not merely asserted but is tested and that the system refuses to issue equality-typed claims when determinism is not satisfied.

14.4 AoR falsification surface for controller claims

The falsification catalog in the AoR enumerates negative controls across categories (representation, arithmetic transport, CRT hygiene, digit drift/injection, and controller-layer violations). PH-4 uses this catalog as the authoritative surface for controller designed-fail claims: the paper asserts only the existence and necessity of the falsifiers and cites the AoR for their enumerated outcomes.

15 Scope and Limitations (PH-4 claim boundary)

PH-4 fixes three things and explicitly declines to fix others.

Fixed by PH-4

1. **Observer legality.** $\Omega\text{-}\nVdash$ is the unique symmetry-respecting projector onto the fixed subspace of the declared observer group (Sections 2, 10).

2. **Controller legality.** Admissible smoothing and Ω contraction preserve principal content, suppress non-principal content, and (under commutation hypotheses) preserve constraint subspaces (Sections 3, 12).

3. **Instrument legality.** Controller claims are not valid without cryptographic closure and reproducibility closure (Section 13), and they are not credible without designed-fail controls (Section 14).

Not fixed by PH-4

1. **Complete PDE theory.** PH-4 does not claim global existence/uniqueness theorems for nonlinear PDEs. It provides a lawful operator layer used inside finite-to-continuum bridges, with explicit scope control.

2. **Physical interpretation of constants.** PH-4 does not interpret Φ-channel outputs as experimental constants; it supplies the controller and transfer legality required for those interpretations elsewhere.

3. **Universality beyond declared symmetry.** The universality claim is schema-level (Definition 10.2) and depends on explicit adapter contracts; it is not a claim that any arbitrary smoothing or damping rule is admissible.

This boundary is the paper's principal defensive posture against overclaiming: it is more important that the operator layer be lawful and auditable than that it be rhetorically universal.

16 Operational Definitions of AoR Metrics Used in PH-4

PH-4 cites AoR runs that report numerical "gates" and "scores." To prevent any ambiguity about what those gates mean, this section defines the metrics in a paper-level, representation-independent way. When an AoR demo uses a variant of a metric (e.g., normalization, windowing, or dimension-averaging), that variant must be stated explicitly in the demo output; the definitions below give the canonical forms.

16.1 Kernel minimum and the sign-lobe test

Let $k : \mathbb{Z}_M \to \mathbb{R}$ be a real convolution kernel defining $(\mathcal{K}x) = k * x$. The *kernel minimum* is

$$k_{\min} := \min_{n \in \mathbb{Z}_M} k_n.$$

The primary admissibility sign gate is:

- **admissible:** $k_{\min} \geq -\varepsilon_{\text{num}}$,
- **illegal control:** $k_{\min} \leq -\delta$ for a declared macroscopic floor $\delta \gg \varepsilon_{\text{num}}$.

The AoR typically uses ε_{num} at or below 10^{-12} and δ at or above 10^{-6}, with run-specific values printed in stdout (e.g., DEMO-69, DEMO-56, DEMO-59).

16.2 Spectral multiplier bounds and PSD admissibility

Let $\widehat{k}(\ell)$ denote the DFT multiplier of k. The canonical spectral admissibility contract is:

$$0 \leq \widehat{k}(\ell) \leq 1 \quad \text{for all} \quad \ell \in \mathbb{Z}_M,$$

again interpreted with a declared numerical tolerance ε_{num} when computed in floating arithmetic.

This contract is the operational PSD proxy for smoothing: it enforces non-expansiveness and prevents signed high-frequency cancellation from being disguised as smoothing.

16.3 HF weight fraction and HF power fraction

Two related quantities appear in PH-4-cited runs:

(A) HF weight fraction (filter-only). If $W(\ell)$ is a frequency-domain weight (multiplier), define

$$\text{HF\_weight\_frac}(> R) := \frac{\sum_{|\ell|>R} |W(\ell)|^2}{\sum_{\ell} |W(\ell)|^2}.$$

For signed injection controls, HF_weight_frac is expected to be large (often near 1).

(B) HF power fraction (signal-weighted). If a spectrum $P(\ell) \geq 0$ is present (as in PH-3 tilt/amplitude bridges or PDE field diagnostics), define

$$\text{HF}(> R) := \frac{\sum_{|\ell|>R} P(\ell)\, |W(\ell)|^2}{\sum_{\ell \neq 0} P(\ell)\, |W(\ell)|^2}.$$

This fraction is used as an injection witness: illegal signed controls must satisfy an HF floor tied to $\max(10\,\text{HF}_{\text{Fej}}, \varepsilon^2)$ in the AoR's standard gate family.

16.4 Overshoot and undershoot (boundedness diagnostics)

Let u be a scalar field on $\mathbb{Z}_M$ or $\mathbb{Z}_M^d$ which is expected (by contract) to lie in an interval $[a, b]$. Define:

- **Overshoot amplitude:**
$$\text{ov}(u) := \max\{0,\ \max u - b\}.$$
- **Undershoot amplitude:**
$$\text{un}(u) := \max\{0,\ a - \min u\}.$$

Some AoR tests also report *overshoot mass*, defined by summing exceedances (discrete integral):

$$\text{ov\_mass}(u) \;\coloneqq\; \sum_x \max\{0,\; u(x) - b\}.$$

Boundedness gates are stated explicitly per demo; the archetypal designed-fail pattern is:

- admissible Fejér smoothing yields $\text{ov}(u) \approx 0$ (and $\text{un}(u) \approx 0$ when positivity is expected),
- sharp/signed illegal controls yield $\text{ov}(u)$ and/or $\text{un}(u)$ above a macroscopic floor (often tied to ε^2).

16.5 Total variation (ringing diagnostics)

For a 1D periodic field $u : \mathbb{Z}_M \to \mathbb{R}$, define total variation

$$\text{TV}(u) \;\coloneqq\; \sum_{n \in \mathbb{Z}_M} |u_{n+1} - u_n|,$$

with indices interpreted modulo M.

For a 2D field $u : \mathbb{Z}_{M_x} \times \mathbb{Z}_{M_y} \to \mathbb{R}$, a common diagnostic is the mean-in-y x-variation

$$\text{TV}_x(u) \;\coloneqq\; \frac{1}{M_y} \sum_{j \in \mathbb{Z}_{M_y}} \sum_{i \in \mathbb{Z}_{M_x}} |u_{i+1,j} - u_{i,j}|.$$

Large increases in TV under sharp/signed controls serve as a quantitative witness of oscillatory artifacts (Gibbs-type ringing).

16.6 Tracking error and improvement ratio

Let u be a computed field and $u^\star$ a reference (truth or higher-budget run). The canonical relative L^2 tracking error is

$$e(u; u^\star) \;\coloneqq\; \frac{\|u - u^\star\|_2}{\|u^\star\|_2}.$$

If u_{base} is an uncontrolled baseline and u_{ctrl} a controlled run (under the same test), the improvement ratio is

$$\text{impr} \;\coloneqq\; \frac{e(u_{\text{base}}; u^\star)}{e(u_{\text{ctrl}}; u^\star)}.$$

A controller “win” is operationally recorded by $\text{impr} > 1$ together with conservation gates (mass/mean) and HF suppression gates.

16.7 Constraint diagnostics for vector fields

For a vector field v on a periodic grid, define:

- **Divergence norm:** $\|\nabla \cdot v\|_2$ computed by the declared discrete divergence operator (spectral or finite difference, as specified).
- **Incompressibility ratio:**

$$\text{div\_ratio} \;\coloneqq\; \frac{\|\nabla \cdot v\|_2}{\|v\|_2}.$$

- **Kinetic energy:** $E(v) \coloneqq \|v\|_2^2$ (up to normalization conventions).
- **Energy ratio:** $E(v_{\text{ctrl}})/E(v_{\text{base}})$ or $E(v)/E(v^\star)$, as specified.

Constraint gates require that admissible control does not degrade conservation beyond tolerance and that constraint violations contract (often dramatically).

16.8 "Teeth" under counterfactual budgets

A designed-fail teeth check is a deterministic asymmetry test: under a counterfactual triple, a derived budget (e.g., K or span r) changes, and a score must worsen by a declared margin.

Let S_{primary} denote a score (tracking error, curvature, HF fraction, or a composite) under the primary triple/budget, and let S_{cf} denote the corresponding score under a counterfactual triple/budget. The canonical teeth requirement is:

$$S_{\text{cf}} \;\geq\; (1+\varepsilon)\, S_{\text{primary}} \quad \text{in at least} \quad \frac{3}{4}\,\text{counterfactuals.}$$

The tolerance ε is itself derived deterministically (in the AoR flagship family, $\varepsilon = 1/\sqrt{q_2}$). Teeth gates appear prominently in OATB (DEMO-69), in DOC vs FD comparisons (DEMO-56), and in electromagnetic suites (DEMO-59).

17 Conclusion

PH-4 makes control and observation explicit within the Marithmetics architecture.

Mathematically, it introduces a universal finite controller, $\Omega\text{–}\nVdash$, defined as the symmetry-forced projector onto the trivial (observer-fixed) sector plus a single scalar contraction of the orthogonal sector. It then defines admissible observer actions and proves the commutation structure that makes "observe then control" equivalent to "control then observe" when all operators lie inside the DOC-admissible commutant.

Operationally, PH-4 enforces the central constitutional constraint of the project: controllers are not free knobs. A controller is permitted only if it is (i) DOC-admissible, (ii) auditable by explicit certificates, (iii) falsifiable by designed-fail substitutions, and (iv) cryptographically reproducible under AoR manifest discipline. These requirements are not philosophical; they are implemented and recorded in the AoR's operator admissibility and controller demos (DEMO-69, DEMO-56, DEMO-59, DEMO-34).

Scientifically, PH-4's contribution is the removal of an ambiguity that otherwise undermines finite-substrate work: the ambiguity of what it means to "stabilize" a computation. Under PH-4, stabilization is not a narrative move; it is a lawful operator statement with a measurable boundary. The same illegal controls that generate Gibbs-type pathologies in harmonic analysis also generate sign violations and HF injection in transport and field tests, and those violations are logged as explicit failures under the falsification protocol.

PH-4 therefore completes the foundational operator stack required by the rest of the physics track. With DOC (PH-1), representation-independent constants and cross-base integrity (PH-2), cosmology closure and bridges (PH-3), and now explicit controller/observer legality (PH-4), subsequent domain papers can treat control and observation as auditable operators rather than hidden degrees of freedom.

18 Data, Code, and Evidence Availability

All evidence cited in PH-4 is contained in the AoR tagged release `aor-20260209T040755Z` , under the canonical folder:

`gum/authority_archive/AOR_20260209T040755Z_0fc79a0` .

The bundle seal and the principal report artifacts are:

- **bundle sha256 (GUM_BUNDLE v30):** `c299b1a7a8ef77f25c3ebb326cb73f060b3c7176b6ea9eb402` (see `bundle_sha256.txt`).
- **master AoR zip (all artifacts):** `MARI_MASTER_RELEASE_20260209T040755Z_0fc79a0 .zip`.

- **GUM report (PDF) and manifest.**
- **claim ledger** (`claim_ledger.jsonl`) and **run metadata** (`run_metadata.json`).

For PH-4 specifically, the primary executable evidence surfaces are:

- **OATB** (operator admissibility + paradox pack + Ω reuse): DEMO-69 stdout/stderr.
- **DOC-admissible operators vs finite differences:** DEMO-56 stdout/stderr.
- **Electromagnetism operator suite:** DEMO-59 stdout/stderr.
- $\Omega \to$**SM bridge integration witness:** DEMO-34 stdout/stderr.
- **falsification catalog** (controller-relevant designed-fail outcomes enumerated): `falsification_matrix.c`
- **reproducibility ledger:** `run_reproducibility.csv`.

19 Citation Format (Recommended)

When citing PH-4 for a specific operator claim, the citation must include:

1. This paper (PH-4) and its version/date;
2. The AoR tag and bundle sha256; and
3. The specific AoR demo log(s) that contain the numerical gates referenced by the claim.

A canonical citation string is:

Justin Grieshop, "PH-4 — Ω–$\nVdash$ Layer and Observer Dynamics on a Finite Substrate: Controller/Measurement as Explicit Operators, Commutation, and a Unified Residual Budget (Authority-of-Record Edition)," Marithmetics AoR tag `aor-20260209T040755Z`, bundle sha256 `c299b1a7a8ef77f25c3ebb326cb73f060b3c7176b6ea9eb402c97273dc3cf66c`, with evidence surfaces: DEMO-69 (OATB), DEMO-56 (DOC vs FD), DEMO-59 (EM), DEMO-34 ($\Omega \to$SM bridge).

For claim-level citations, add the relevant AoR stdout URLs explicitly (e.g., DEMO-69 stdout for admissibility gates; DEMO-56 stdout for ringing/convergence gates; DEMO-59 stdout for Coulomb scaling gates).

20 Referee Verification Checklist (PH-4)

This checklist is designed for a hostile referee who wishes to verify PH-4 with minimal interpretive discretion.

20.1 AoR identity and integrity

1. Verify the AoR tag and folder structure.
2. Open `bundle_sha256.txt` and record the bundle sha256.
3. Optionally download the master AoR zip and verify its hash against the bundle seal.

20.2 Operator admissibility and designed-fail separation

1. Open DEMO-69 stdout and locate the kernel minima for Fejér vs sharp vs signed. Confirm: Fejér nonnegative, sharp and signed negative. Confirm signed HF injection witness is large.

2. In the same demo, locate the transport overshoot/undershoot gates and verify: admissible passes, illegal controls fail. Confirm the counterfactual teeth gate is satisfied.

3. In the paradox pack portion, locate the Dirichlet vs Fejér Gibbs witnesses and confirm the sign/overshoot separation.

20.3 Numerical boundary: admissible vs non-admissible in PDE testbeds

1. Open DEMO-56 stdout and verify:

   - advection step test: sharp/signed have larger TV and overshoot;
   - convergence under refinement is monotone (reported errors decrease with K);
   - Poisson test: spectral error is near machine precision relative to 2nd-order FD;
   - counterfactual budget teeth degrade in the declared direction.

20.4 Electromagnetism as scaling/boundedness witness

1. Open DEMO-59 stdout and verify:

   - kernel sign boundary (Fejér vs sharp vs signed);
   - electrostatic Coulomb slope is near -2 for truth/admissible and deviates strongly for signed;
   - ringing curvature and boundedness witnesses separate admissible from illegal controls;
   - counterfactual teeth degrade as declared.

20.5 Integration boundary: controller does not substitute for coupled selection

1. Open DEMO-34 stdout and verify:

   - the coupled triple is produced as stated;
   - ablations "explode" with explicit witnesses, demonstrating the coupled constraints do real work;
   - anchor block passes within the declared tolerance and is printed as part of the run record.

20.6 Determinism and reproducibility

1. Confirm that each cited demo prints a determinism hash and a VERIFIED verdict.

2. Consult `run_reproducibility.csv` to confirm that determinism (or declared nondeterminism) is recorded as a first-class artifact for the relevant demos.

A referee who completes this checklist has verified PH-4's claim boundary: controller and observation operators are lawful (admissible), discriminative (designed-fail), portable across adapters, and bound to a reproducible, cryptographically sealed evidence surface.

A Proof of Theorem 2.4 (Commutant of the Shift Group Equals the Circulants)

Let $M \geq 2$. Let P_1 be the cyclic shift on $\mathbb{C}^M$, $(P_1x)_n = x_{n-1}$ (indices modulo M). Let $C \in \mathbb{C}^{M\times M}$.

Claim. $CP_1 = P_1C$ if and only if C is circulant, i.e., there exists $c \in \mathbb{C}^M$ such that

$$C_{n,m} = c_{n-m \pmod M}.$$

Proof. Suppose $CP_1 = P_1C$. Consider matrix entries:

$$(CP_1)_{n,m} = \sum_j C_{n,j}(P_1)_{j,m} = C_{n,m+1},$$

because P_1 has ones on the subdiagonal modulo wrap-around, so it maps column m to column $m+1$. Similarly,

$$(P_1C)_{n,m} = \sum_j (P_1)_{n,j}C_{j,m} = C_{n-1,m}.$$

Thus $CP_1 = P_1C$ implies

$$C_{n,m+1} = C_{n-1,m} \quad \text{for all} \quad n, m.$$

Iterating this relation k times gives

$$C_{n,m} = C_{n-k,m-k} \quad \text{for all} \quad k.$$

In particular, $C_{n,m}$ depends only on $n - m \pmod M$. Define $c_d := C_{d,0}$ for $d \in \mathbb{Z}_M$. Then

$$C_{n,m} = C_{n-m,0} = c_{n-m},$$

which is the circulant form.

Conversely, if C is circulant, then C is translation-invariant and therefore commutes with shifts: convolution operators commute with translations by construction, hence $CP_1 = P_1C$. This establishes the equivalence.

Since $G_M = \{P_k\}$ is generated by P_1, commutation with P_1 implies commutation with every P_k, and the commutant of G_M is precisely the set of circulant matrices. □

B Fejér Multiplier and a Nonnegativity Witness on the Discrete Torus

Fix $M \geq 2$ and choose an integer span r with $0 \leq r \leq \lfloor (M-1)/2 \rfloor$. Define the triangle multiplier

$$H_r(\ell) := \begin{cases} 1 - \dfrac{|\ell|}{r+1}, & |\ell| \leq r, \\ 0, & |\ell| > r, \end{cases} \qquad \ell \in \mathbb{Z}_M,$$

with $|\ell|$ interpreted as the minimal representative magnitude on the symmetric frequency set.

Then:

1. $0 \leq H_r(\ell) \leq 1$ for all ℓ, and $H_r(0) = 1$.
2. The inverse DFT kernel $F_r = \mathcal{F}^{-1}[H_r]$ satisfies $\sum_n F_r(n) = 1$ (unit mass).

3. A nonnegativity witness is given by the squared Dirichlet-sum form

$$F_r(n) = \frac{1}{r+1} \left| \sum_{k=0}^{r} e^{2\pi i k n/M} \right|^2 \;\geq\; 0,$$

which is the discrete specialization of the classical Fejér kernel identity.

Therefore the Fejér kernel (in this discrete form) satisfies the DOC-admissible smoothing contract of Definition 3.1.

C Telescoping Proof of Residual Additivity (Lemma 4.4)

Let A_j and B_j be bounded operators with $\|A_j\| \leq 1$, $\|B_j\| \leq 1$. Consider products

$$T_A \;=\; A_m A_{m-1} \cdots A_1, \qquad T_B \;=\; B_m B_{m-1} \cdots B_1.$$

Then

$$T_A - T_B = \sum_{j=1}^{m} A_m \cdots A_{j+1} \, (A_j - B_j) \, B_{j-1} \cdots B_1.$$

Taking norms and using submultiplicativity with the non-expansive bounds gives

$$\|T_A - T_B\| \leq \sum_{j=1}^{m} \|A_j - B_j\|.$$

Specializing to $A_j = \Omega_{\kappa_j} \mathcal{K}_h^{(j)}$ and $B_j = \Omega_{\kappa_j} L_h^{(j)}$, and using $\|\Omega_{\kappa_j}\| \leq 1$, yields Lemma 4.4. $\square$