PH-4 — Ω–⊮ Controller and Observer Dynamics on a Finite Substrate: Admissible Control, Measurement, and Transfer as Explicit Operators

(Authority-of-Record Edition)

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Authority-of-Record tag: `release-aor-20260125T043902Z`
AoR folder: `gum/authority_archive/AOR_20260125T043902Z_52befea`
Bundle sha256:
`c299b1a7a8ef77f25c3ebb326cb73f060b3c7176b6ea9eb402c97273dc3cf66c`

Authority-of-Record Evidence URLs (canonical; stable)

Core artifacts

- **Master archive (zip):** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T gum/authority_archive/AOR_20260125T043902Z_52befea/MARI_MASTER_RELEASE_20260125T043902Z_52befea.zip
- **AoR report (v32 PDF):** https://github.com/public-arch/Marithmetics/blob/release-aor-2026012 gum/authority_archive/AOR_20260125T043902Z_52befea/report/GUM_Report_v32_2026-01-25_04-42-51Z.pdf
- **AoR report manifest:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T gum/authority_archive/AOR_20260125T043902Z_52befea/report/GUM_Report_v32_2026-01-25_04-42-51Z.pdf.manifest.json
- **Bundle seal (bundle_sha256.txt):** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T043902Z/gum/authority_archive/AOR_20260125T043902Z_52befea/GUM_BUNDLE_v30_20260125T043902Z/bundle_sha256.txt

Ledgers and indices

- **claim_ledger.jsonl:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T04 gum/authority_archive/AOR_20260125T043902Z_52befea/claim_ledger.jsonl
- **run_metadata.json:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T0 gum/authority_archive/AOR_20260125T043902Z_52befea/run_metadata.json
- **demo_index.csv:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T043 gum/authority_archive/AOR_20260125T043902Z_52befea/GUM_BUNDLE_v30_20260125T043902Z/tables/demo_index.csv
- **falsification_matrix.csv:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260 gum/authority_archive/AOR_20260125T043902Z_52befea/GUM_BUNDLE_v30_20260125T043902Z/tables/falsification_matrix.csv

- **run_reproducibility.csv:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260 gum/authority_archive/AOR_20260125T043902Z_52befea/GUM_BUNDLE_v30_20260125T043902Z/ tables/run_reproducibility.csv

Primary evidence surfaces used in this paper

- **DEMO-69 stdout** (OATB: operator admissibility + transfer + paradox pack + Ω reuse): https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T043902Z/ gum/authority_archive/AOR_20260125T043902Z_52befea/GUM_BUNDLE_v30_20260125T043902Z/ logs/controllers__demo-69-oatb-operator-admissibility-transfer-bridge.out.txt
- **DEMO-69 stderr:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T043 gum/authority_archive/AOR_20260125T043902Z_52befea/GUM_BUNDLE_v30_20260125T043902Z/ logs/controllers__demo-69-oatb-operator-admissibility-transfer-bridge.err.txt
- **DEMO-56 stdout** (DOC vs FD operator behavior; convergence/ringing controls): https:// github.com/public-arch/Marithmetics/blob/release-aor-20260125T043902Z/gum/authority_ archive/AOR_20260125T043902Z_52befea/GUM_BUNDLE_v30_20260125T043902Z/logs/controllers_ _demo-56-deterministic-operator-calculus-vs-fd.out.txt
- **DEMO-56 stderr:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T043 gum/authority_archive/AOR_20260125T043902Z_52befea/GUM_BUNDLE_v30_20260125T043902Z/ logs/controllers__demo-56-deterministic-operator-calculus-vs-fd.err.txt
- **DEMO-59 stdout** (electromagnetism suite as admissibility stress test): https://github. com/public-arch/Marithmetics/blob/release-aor-20260125T043902Z/gum/authority_archive/ AOR_20260125T043902Z_52befea/GUM_BUNDLE_v30_20260125T043902Z/logs/controllers__demo-59-elec out.txt
- **DEMO-59 stderr:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T043 gum/authority_archive/AOR_20260125T043902Z_52befea/GUM_BUNDLE_v30_20260125T043902Z/ logs/controllers__demo-59-electromagnetism.err.txt
- **DEMO-34 stdout** (Ω integration witness inside SM-class pipeline): https://github.com/ public-arch/Marithmetics/blob/release-aor-20260125T043902Z/gum/authority_archive/ AOR_20260125T043902Z_52befea/GUM_BUNDLE_v30_20260125T043902Z/logs/bridge__demo-34-omega-sm- out.txt
- **DEMO-34 stderr:** https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T043 gum/authority_archive/AOR_20260125T043902Z_52befea/GUM_BUNDLE_v30_20260125T043902Z/ logs/bridge__demo-34-omega-sm-master-flagship-v1.err.txt

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Abstract

This paper formalizes the controller/measurement layer used in the Marithmetics physics track as an explicit finite operator theory with audit-grade admissibility constraints. The central construction is Ω–$\nVdash$: a uniquely determined principal projector Ω (forced by an observer-invariance contract) together with a one-parameter contraction of the complementary, mean-free subspace. We prove stability and commutation properties that make "observe then control" equivalent to "control then observe" under the declared observer group, and we define a legality boundary for smoothing and measurement windows (positivity, unit mass, and non-expansiveness). The Authority-of-Record evidence closes the scientific loop: admissible operators must pass invariant-preservation and boundedness gates across deterministic budgets, while explicitly constructed illegal operators (sharp truncation and signed high-frequency injection) must fail with reproducible signatures. These pass/fail separations are recorded in DEMO-69 (OATB flagship), cross-checked in DEMO-56 (operator calculus vs finite differences) and DEMO-59 (electromagnetism stress test), and shown to integrate non-interveningly with the Standard-Model pipeline in DEMO-34.

Keywords: finite operator calculus; admissible control; measurement windows; Fejér kernel; designed-fail falsification; reproducible computation; audit-grade evidence; Ω-projection.

Contents

1 Introduction: Why "Control" Must Be a Lawful Operator

1.1 The scientific hazard

Any finite computational pipeline that claims to produce structure-level quantities (constants, closures, or cross-domain invariants) must answer an immediate question: what prevents the act of stabilization, smoothing, or measurement from becoming a hidden tuning knob?

In conventional numerical work, "filtering" is often treated as an implementation detail. In foundational work it cannot be: a non-admissible filter can match a target metric while violating constraints that are later interpreted as physical (positivity, boundedness, conservation, symmetry). If this degree of freedom is not regulated, the pipeline may be internally consistent and still be scientifically ambiguous.

PH-4's thesis is that the ambiguity is removable at finite size, without appealing to asymptotic rhetoric:

- the controller/measurement layer must be expressed as explicit operators on a finite state space;
- those operators must satisfy a conservative admissibility contract (positivity, unit mass, non-expansiveness, symmetry/commutation);
- the contract must be falsifiable by designed-fail substitutions; and
- the pass/fail outcomes must be bound to cryptographically pinned, replayable evidence artifacts.

1.2 Relationship to DOC and to the physics track

PH-4 sits above the Deterministic Operator Calculus (DOC) baseline: it assumes that the underlying computational substrate is deterministic and that operator composition is treated as a mathematically meaningful act, not a numerically informal one.

However, PH-4 is not a restatement of DOC. Its role is narrower and more operational:

- to specify what it means to "observe," "measure," and "control" within a finite substrate;
- to prove the minimal structure (existence/uniqueness, commutation, contraction) that makes those acts lawful; and
- to tie the lawfulness boundary to AoR demos that separate admissible operators from illegal operators under designed-fail tests.

The domain papers (PH-1–PH-3) rely on this layer whenever a computation uses smoothing, transfer, stabilization, windowed measurement, or coarse-graining. PH-4 provides the legal and evidentiary foundation that makes those usages interpretable.

1.3 Authority-of-Record posture for this paper

PH-4 is an Authority-of-Record edition. This means:

- every quantitative statement taken from execution is traceable to a pinned AoR demo log (see the Evidence URLs at the top of this paper);
- every "must fail" statement refers to an explicit falsifier class enumerated in the falsification matrix (URL above); and
- reproducibility is treated as part of the claim boundary (run reproducibility table URL above), not as an external promise.

When this paper says "admissible passes and illegal fails," it means "the AoR records PASS/FAIL outcomes under fixed manifests," not "the authors believe this should be true."

1.4 Scope and non-claims

PH-4 does not claim global existence or uniqueness theorems for nonlinear PDEs, and it does not claim that Ω–$\nVdash$ "solves physics." The claims are strictly about operator legality and its consequences:

- how to define a principal (observer-fixed) channel on a finite substrate;
- how to design a controller that is forced to preserve that channel while suppressing the complement in a contractive way;
- why admissible smoothing is required for boundedness/positivity preservation in transfer and measurement; and
- how to audit these statements through designed-fail controls and deterministic artifacts.

2 Finite Observer Space and the Unique Principal Projector Ω

This section constructs the principal projector Ω from first principles. The construction is deliberately conservative: it assumes only a finite state space, an observer group acting on that state space, and a calibration requirement that fixes the meaning of "principal."

2.1 Finite state space

Fix an integer $M \geq 2$ and define the index set

$$I_M = \{0, 1, \ldots, M-1\}.$$

Let $H_M = \mathbb{R}^M$ with inner product

$$\langle x, y \rangle = \sum_{n \in I_M} x_n y_n,$$

and Euclidean norm $\|x\|_2 = \sqrt{\langle x, x \rangle}$.

Let $\vec{1} \in \mathbb{R}^M$ denote the all-ones vector $\vec{1} = (1, 1, \ldots, 1)^T$.

2.2 Observer group (translation gauge) and fixed space

Define the cyclic shift P_1 by

$$(P_1 x)_n = x_{(n-1) \bmod M}.$$

For $k \in I_M$, set $P_k = (P_1)^k$, and define the observer group

$$G_M = \{P_k : k \in I_M\} \cong \mathbb{Z}/M\mathbb{Z}.$$

The group acts orthogonally on H_M. The fixed space is:

Lemma 2.1 (Fixed space of the shift group)**.**

$$\mathrm{Fix}(G_M) = \{x \in H_M : P_k x = x \textit{ for all } k\} = \mathrm{span}\{\vec{1}\}.$$

Proof. If $P_1 x = x$ then $x_{n-1} = x_n$ for all n, so all components are equal and $x = c\vec{1}$. Conversely, any $c\vec{1}$ is fixed by every shift. □

Interpretation. If a quantity is declared "observer-independent under translation," then it must be compatible with this fixed space. The smallest such content is the constant (mean) channel.

2.3 Admissible principal projector: axioms

We now state the operator contract that defines "principal measurement" in this setting.

Definition 2.2 (Admissible principal projector). A linear map $\Omega : H_M \to H_M$ is *admissible* if it satisfies:

(A1) **Idempotence:** $\Omega^2 = \Omega$.

(A2) **Orthogonality:** $\Omega^T = \Omega$.

(A3) **Observer commutation:** $\Omega P_k = P_k \Omega$ for all $k \in I_M$.

(A4) **Calibration:** $\Omega \vec{1} = \vec{1}$.

Define $\Pi = I - \Omega$ (the complementary projector).

Axioms (A1)–(A2) ensure Ω is an orthogonal projector; (A3) ensures Ω is compatible with the observer group; (A4) fixes the physical meaning of "principal" by requiring that the unit-mass/mean channel is preserved.

2.4 Existence and uniqueness of Ω

Theorem 2.3 (Existence and uniqueness). *There exists a unique admissible principal projector Ω, given by*

$$\Omega = \frac{1}{M}\vec{1}\vec{1}^T,$$

equivalently, for any $x \in H_M$,

$$(\Omega x)_n = \frac{1}{M}\sum_{m \in I_M} x_m \quad \textit{for all } n.$$

Proof. **Existence:** direct verification.

- $\Omega^2 = \Omega$ and $\Omega^T = \Omega$ follow from the rank-one form $(1/M)\vec{1}\vec{1}^T$.
- Ω commutes with every P_k because $P_k\vec{1} = \vec{1}$ and $P_k^T\vec{1} = \vec{1}$, hence $P_k\Omega = \Omega = \Omega P_k$.
- $\Omega\vec{1} = \vec{1}$ holds by construction.

Uniqueness: Let Ω be admissible. By (A3), Ω commutes with P_1, hence with all of G_M. Any operator commuting with cyclic shifts is circulant, hence diagonalizable in the discrete Fourier basis. By (A1)–(A2), its eigenvalues lie in $\{0, 1\}$. Calibration (A4) forces eigenvalue 1 on the constant mode. Lemma 2.1 shows the shift-fixed space is one-dimensional; therefore Ω has rank 1 and equals the orthogonal projector onto span$\{\vec{1}\}$, which is exactly $(1/M)\vec{1}\vec{1}^T$. $\square$

Corollary 2.4 (Orthogonal split). *Every $x \in H_M$ decomposes uniquely as*

$$x = \Omega x + \Pi x,$$

with $\Omega x \perp \Pi x$ and

$$\|x\|_2^2 = \|\Omega x\|_2^2 + \|\Pi x\|_2^2.$$

This decomposition is the mathematical backbone of the controller layer: Ω isolates the observer-fixed content; Π isolates the mean-free (non-principal) content where high-frequency injection, ringing, and constraint violations typically live.

3 Ω–$\nVdash$: The Universal Controller as Principal Preservation + Complement Contraction

Section 2 fixed the principal projector Ω uniquely from an observer-invariance contract. The Ω–$\nVdash$ layer is the corresponding controller schema: it is the most conservative, symmetry-compatible way to "stabilize" or "measure" a finite state without creating an unregulated degree of freedom.

3.1 Definition and geometric meaning

Let Ω be the unique admissible principal projector on H_M, and let $\Pi = I - \Omega$ be the orthogonal complement projector (Section 2.4).

Definition 3.1 (Ω–κ controller family)**.** For $\kappa \in [0, 1]$, define the controller

$$\Omega_\kappa := \Omega + \kappa\Pi = \kappa I + (1 - \kappa)\Omega.$$

Interpretation. Ω_κ interpolates between two extremal operators:

- $\kappa = 1$: $\Omega_1 = I$ (no control; identity).
- $\kappa = 0$: $\Omega_0 = \Omega$ (full collapse to the principal channel).

For intermediate κ, Ω_κ preserves the principal content exactly and suppresses the non-principal content by a scalar factor κ.

3.2 Spectral structure and stability

Because H_M splits orthogonally as $\mathrm{Im}(\Omega) \oplus \mathrm{Im}(\Pi)$, the action of Ω_κ is diagonal with respect to this decomposition:

- On $\mathrm{Im}(\Omega)$: Ω_κ acts as eigenvalue 1.
- On $\mathrm{Im}(\Pi)$: Ω_κ acts as eigenvalue κ.

Proposition 3.2 (Self-adjointness, commutation, and contraction)**.** *For every $\kappa \in [0, 1]$:*

1. *Ω_κ is self-adjoint: $\Omega_\kappa^T = \Omega_\kappa$.*
2. *Ω_κ commutes with the observer group: $\Omega_\kappa P_k = P_k \Omega_\kappa$ for all k.*
3. *Ω_κ is non-expansive in ℓ^2: $\|\Omega_\kappa x\|_2 \le \|x\|_2$ for all $x \in H_M$.*
4. *The energy contraction is exact:*

$$\|\Omega_\kappa x\|_2^2 = \|\Omega x\|_2^2 + \kappa^2 \|\Pi x\|_2^2.$$

Proof. (1) follows from $\Omega^T = \Omega$ and $\Pi^T = \Pi$. (2) follows from Ω and Π commuting with each P_k, hence any affine combination does. (3)–(4) follow from orthogonality (Corollary 2.4): $\Omega x \perp \Pi x$ and $\Omega_\kappa x = \Omega x + \kappa \Pi x$. □

This proposition is the core mathematical claim of the controller layer: under an observer-invariance contract, Ω_κ is a lawful stabilization that cannot amplify non-principal energy, cannot distort the principal content, and cannot select an observer frame.

3.3 Generalized Ω control: admissible smoothing on the complement

The Ω–κ family is the minimal controller. In practice, pipelines often require additional structure: smoothing or transfer operators (e.g., Fejér/MSF) that attenuate specific high-frequency modes while preserving invariants. The general admissible version is "Ω plus an admissible operator on the complement."

Definition 3.3 (Ω–S controller schema)**.** Let $S : H_M \to H_M$ be a linear operator such that:

(S1) $S\Omega = \Omega S = \Omega$ (principal channel fixed),

(S2) $S(\mathrm{Im}(\Pi)) \subset \mathrm{Im}(\Pi)$ (no leakage from complement to principal),

(S3) $\|Sx\|_2 \le \|x\|_2$ for all $x \in \mathrm{Im}(\Pi)$ (contraction on the complement),

(S4) $SP_k = P_k S$ for all k (observer compatibility).

Define $\Omega_S := \Omega + S\Pi$.

Remark 3.4. Ω_κ is the special case Ω_S with $S = \kappa I$ on $\mathrm{Im}(\Pi)$.

Proposition 3.5 (Stability of Ω–S)**.** *If S satisfies (S1)–(S4), then:*

1. *Ω_S preserves the principal channel: $\Omega\Omega_S = \Omega_S\Omega = \Omega$.*
2. *Ω_S is non-expansive: $\|\Omega_S x\|_2 \le \|x\|_2$ for all x.*
3. *Ω_S commutes with the observer group: $\Omega_S P_k = P_k \Omega_S$.*

Proof. (1) follows from (S1). (2) follows from orthogonality and (S3): $\Omega_S x = \Omega x + S\Pi x$ with $\Omega x \perp S\Pi x$. (3) follows from Ω and S commuting with each P_k. □

3.4 What Ω control is (and is not)

Ω control is an operator-level instrument with a narrow purpose:

- It is a symmetry-forced separation of the observer-fixed channel from its orthogonal complement.
- It is a lawful contraction of the complement that cannot "manufacture" agreement by amplifying energy, introducing signed mass, or selecting a preferred observer origin.

It is *not* a substitute for the coupled selection laws of the invariant channel (PH-2/PH-1). In the AoR, this non-substitution boundary is enforced by ablation tests: if the upstream coupled constraints are removed, the pipeline "explodes" with non-target outputs, and Ω cannot repair that failure.

4 DOC-Admissible Smoothing and Measurement: Legality Criteria for Transfer Operators

Ω_κ constrains "how much" of the non-principal sector survives. A second and independent question is "what form" suppression or measurement is permitted to take. PH-4 treats smoothing kernels and measurement windows as operators that must satisfy a conservative admissibility contract, so they cannot win by violating physical-type constraints such as positivity or conservation.

4.1 DOC-admissible smoothing on the cyclic torus

We work on the cyclic group $\mathbb{Z}/M\mathbb{Z}$, identified with I_M. Let $k : I_M \to \mathbb{R}$ be a kernel, and define the cyclic convolution operator $C_k : H_M \to H_M$ by

$$(C_k x)_n := \sum_{j \in I_M} k_j x_{(n-j) \bmod M}.$$

Definition 4.1 (DOC-admissible smoothing kernel)**.** A kernel k is *DOC-admissible* if it satisfies:

(K1) **Nonnegativity:** $k_j \geq 0$ for all j.

(K2) **Unit mass:** $\sum_j k_j = 1$.

(K3) **Even symmetry:** $k_j = k_{(-j) \bmod M}$.

(K4) **Spectral attenuation:** its discrete Fourier multipliers satisfy $0 \leq \hat{k}(\ell) \leq 1$ for all ℓ (within declared numerical tolerance in implementation).

Items (K1)–(K2) enforce "lawful averaging" (convex combination). Item (K3) is the minimal symmetry that prevents directional bias on a cyclic domain. Item (K4) is the operator-norm and PSD proxy: it forbids gain and forbids signed cancellation disguised as smoothing.

Definition 4.2 (DAO-admissibility; observer commutation)**.** A DOC-admissible convolution operator is *DAO-admissible* if it commutes with the observer group G_M (equivalently, with P_1). For cyclic convolution this is automatic: every convolution operator is circulant and hence commutes with shifts.

4.2 Why nonnegativity and unit mass are non-negotiable

Lemma 4.3 (Convex-hull preservation)**.** *If k satisfies (K1)–(K2) and x satisfies $a \leq x_n \leq b$ for all n, then $y = C_k x$ satisfies $a \leq y_n \leq b$ for all n.*

Proof. Each y_n is a convex combination of entries of x with weights $k_j \geq 0$ summing to 1. □

Corollary 4.4 (Positivity preservation)**.** *If $x_n \geq 0$ for all n, then $(C_k x)_n \geq 0$ for all n.*

These statements are elementary, but they are the legality boundary for transfer and measurement. Without (K1)–(K2), a pipeline can "improve" a score by creating negative lobes, overshoot, or signed partitions that violate the conservation/positivity constraints later interpreted as physical.

4.3 Illegal controls: sharp truncation and signed high-frequency injection

PH-4 requires explicit illegal controls, not to "compare methods," but to prove that legality constraints are discriminative (a pipeline that passes regardless of legality is not credible).

Illegal control A (sharp spectral truncation / Dirichlet). Define a multiplier $W_{\text{sharp}}(\ell) = 1$ for $|\ell| \leq R$ and 0 otherwise. The corresponding real-space kernel has negative lobes (discrete Gibbs phenomenon), violating (K1).

Illegal control B (signed injection). Define $W_{\text{signed}}(\ell)$ that flips sign or preserves large high-frequency weight beyond R. This violates (K4) and typically violates (K1) in real space as well.

In the AoR, these illegal controls are required to fail boundedness/positivity gates and to exhibit measurable HF-retention signatures. The canonical audit surface is DEMO-69 (OATB), with corroborating failures in DEMO-56 and DEMO-59.

4.4 Canonical admissible family: discrete Fejér smoothing

The Fejér family is the canonical positive control for smoothing legality because it is simultaneously:

- nonnegative in real space,
- unit mass,
- shift-commuting (circulant), and
- non-amplifying in frequency space.

One discrete characterization that makes nonnegativity explicit is the squared Dirichlet-sum form.

Lemma 4.5 (Discrete Fejér kernel is nonnegative)**.** *Let $r \geq 0$. Define*

$$F_r(n) := \frac{1}{r+1} \left| \sum_{k=0}^{r} \exp(2\pi i k n / M) \right|^2,$$

for $n \in I_M$. Then $F_r(n) \geq 0$ for all n, and $\sum_n F_r(n) = 1$ after the standard DFT normalization (the normalization convention used by the implementation is recorded in the AoR capsule manifests).

Consequently, convolution with F_r satisfies the convex-hull lemma (Lemma 4.3) and is a lawful transfer operator for boundedness/positivity-sensitive computations.

4.5 Measurement windows and coarse-graining are operators

In Marithmetics, "measurement" is not treated as a narrative act ("we read off a value"). It is treated as an explicit operator on a finite state, because anything else leaves a loophole: a reader cannot tell whether a reported value is a property of the state or a property of a signed/biased measurement window.

Definition 4.6 (Admissible measurement window)**.** A window is a weight vector $w \in \mathbb{R}^M$. The windowed measurement of a state $x \in H_M$ is

$$\mathsf{M}_w(x) := \langle w, x \rangle = \sum_{n \in I_M} w_n x_n.$$

The window w is *admissible* if:

(W1) $w_n \geq 0$ for all n (no signed mass),

(W2) $\sum_n w_n = 1$ (unit mass / probability normalization),

(W3) w is declared (fixed) by the manifest (no implicit tuning),

(W4) for shift-invariant measurements, the family of windows is shift-equivariant (observer neutrality).

Lemma 4.7 (Measurement boundedness)**.** *If w satisfies (W1)–(W2) and x satisfies $a \leq x_n \leq b$ for all n, then*

$$a \leq \mathsf{M}_w(x) \leq b.$$

Proof. $\mathsf{M}_w(x)$ is a convex combination of the entries of x when $w_n \geq 0$ and $\sum w_n = 1$. □

Corollary 4.8 (No "negative readings" from nonnegative states). *If $x_n \geq 0$ for all n and w is admissible, then $\mathsf{M}_w(x) \geq 0$.*

Definition 4.9 (Admissible coarse-graining partition). A coarse-graining is a collection of windows $\{w^{(j)}\}_{j=1}^J$ such that, for every index n,

$$\sum_{j=1}^{J} w_n^{(j)} = 1, \qquad w_n^{(j)} \geq 0.$$

The corresponding coarse-grained measurements are $\mathsf{M}_{w^{(j)}}(x)$. The partition is admissible if each window is admissible and the partition-of-unity condition holds.

Proposition 4.10 (Mass conservation under admissible coarse-graining). *If $\{w^{(j)}\}$ is an admissible coarse-graining partition and $x \in H_M$, then*

$$\sum_{j=1}^{J} \mathsf{M}_{w^{(j)}}(x) = \sum_{n \in I_M} x_n.$$

Proof.

$$\sum_j \mathsf{M}_{w^{(j)}}(x) = \sum_j \sum_n w_n^{(j)} x_n = \sum_n \left(\sum_j w_n^{(j)} \right) x_n = \sum_n x_n.$$

□

This is the legality condition that prevents a measurement stage from silently injecting or removing mass: mass may be redistributed across windows, but it cannot be created or destroyed.

4.6 Designed-fail measurement: signed windows are illegal

A signed window can artificially reduce variance, cancel high-frequency structure, or enforce agreement by subtracting from one region what it adds to another. This is exactly why such windows are disallowed for "measurement-typed" claims.

Definition 4.11 (Signed window falsifier). A window w is a *signed falsifier* if $\sum_n w_n = 1$ but there exists n with $w_n < 0$. A minimal witness of illegality is the negativity floor

$$w_{\min} := \min_n w_n < 0.$$

Designed-fail requirement. If a pipeline substitutes a signed window into a measurement gate that is required to be admissible, the run must FAIL and must emit a stable witness: either a direct sign-lobe report ($w_{\min} < 0$) or a boundedness/positivity violation that cannot occur under admissible windows.

4.7 Summary: what PH-4 means by "admissible transfer and measurement"

At this point the legality boundary is explicit:

- **Admissible smoothing kernels (DOC/DAO):** nonnegative, unit mass, even/shift-neutral, and non-amplifying in spectral multipliers.
- **Admissible measurement windows:** nonnegative, unit mass, declared by manifest, and observer-neutral when required.
- **Illegal controls:** sharp truncation and signed injection (kernels or windows) are required to fail, not merely to perform worse.

5 Residual-Budget Inheritance: Control Cannot Create Convergence

The preceding sections regulated the operator class. This section regulates the claims that may be made when those operators are used.

PH-4's central scientific safeguard is *residual-budget inheritance*:

> If a discrete operator K_h approximates a target operator L only up to a declared residual η_h, then applying any admissible observer or admissible controller cannot reduce the true approximation error below what η_h allows.

This prevents a common rhetorical move: showing that a controlled output "looks right," and then implying that the controller created correctness.

5.1 Abstract setup: a target operator, an approximation, and a declared residual

Let $\mathcal{H}$ be a finite Hilbert space and let $L : \mathcal{H} \to \mathcal{H}$ be a target operator (the "ideal" action under the declared model). Let $K_h : \mathcal{H} \to \mathcal{H}$ be a finite approximation indexed by a refinement parameter h (or budget K).

A *declared residual-budget bound* is a statement of the form

$$\|K_h x - Lx\|_2 \le \eta_h \|x\|_2 \qquad \text{for all } x \in \mathcal{H},$$

with $\eta_h \to 0$ under refinement, and with η_h treated as an explicit, auditable envelope (not a hidden constant).

5.2 Observers (orthogonal symmetries) preserve residual bounds exactly

Let U be an orthogonal operator ($U^T U = I$), representing an admissible observer action (e.g., a shift P_k).

Lemma 5.1 (Observer invariance of the residual bound). *If $\|K_h x - Lx\|_2 \le \eta_h \|x\|_2$ holds, then*

$$\|UK_h x - ULx\|_2 \le \eta_h \|x\|_2$$

holds with the same η_h.

Proof. $\|UK_h x - ULx\|_2 = \|U(K_h x - Lx)\|_2 = \|K_h x - Lx\|_2$ because U is norm-preserving. □

Interpretation. Observer actions cannot "improve" approximation quality; they can only re-express it.

5.3 Controllers (non-expansive maps) inherit residual bounds conservatively

Let $C : \mathcal{H} \to \mathcal{H}$ be a controller with operator norm $\|C\|_{2\to 2} \le 1$ (non-expansive), which includes Ω_κ for $\kappa \in [0, 1]$ (Proposition 3.2) and DOC-admissible smoothing operators (Section 4).

Lemma 5.2 (Controller inheritance of the residual bound). *If $\|K_h x - Lx\|_2 \le \eta_h \|x\|_2$ holds and $\|C\| \le 1$, then*

$$\|CK_h x - CLx\|_2 \le \eta_h \|x\|_2.$$

Proof. $\|CK_h x - CLx\|_2 = \|C(K_h x - Lx)\|_2 \le \|C\| \cdot \|K_h x - Lx\|_2 \le \eta_h \|x\|_2$. □

Corollary 5.3 (Control cannot beat the declared residual). *If a controlled pipeline reports improvement in a downstream diagnostic, the improvement must be interpreted as stability within the residual envelope, not as creation of new accuracy beyond η_h.*

This corollary is the precise sense in which Ω is "non-intervening" with respect to truth: it may suppress unstable components, but it cannot manufacture correctness outside the declared approximation class.

5.4 Residual additivity under composition: where "teeth" comes from

Pipelines compose many operators. The residual envelope must therefore compose as well.

Lemma 5.4 (Telescoping residual bound for product operators). *Let A_j and B_j be bounded operators with $\|A_j\| \le 1$, $\|B_j\| \le 1$. Define products*

$$T_A = A_m A_{m-1} \cdots A_1, \qquad T_B = B_m B_{m-1} \cdots B_1.$$

Then

$$\|T_A - T_B\| \le \sum_{j=1}^{m} \|A_j - B_j\|.$$

Interpretation. If each step has an audited residual contribution, then the overall pipeline has an audited residual budget given by the sum. This is exactly the mathematical form of "residual accounting."

In the AoR, this accounting is not only asserted; it is enforced by *counterfactual teeth*: reducing a lawful budget must degrade a score by a declared margin. The canonical teeth surface for the controller/transfer layer is DEMO-69 (OATB), with corroborating budget-sensitivity evidence in DEMO-56 and DEMO-59.

6 Authority-of-Record: DEMO-69 (OATB) as the Primary Controller/Transfer Audit Surface

DEMO-69 ("OATB MASTER FLAGSHIP: Operator Admissibility Transfer Bridge") is the canonical evidence surface for PH-4 because it performs, in one deterministic run, the exact separation PH-4 requires:

- admissible vs illegal operators (Fejér vs sharp truncation vs signed injection),
- boundedness/positivity preservation under transfer,
- high-frequency retention witnesses for illegal controls,
- counterfactual teeth under lawful budget reduction, and
- reuse of the same controller schema across multiple PDE-class adapters.

6.1 Determinism identity: why stdout is evidence, not narrative

A log becomes evidence only if it is bound to an identity:

- spec identity (the declared test plan),
- determinism identity (replay stability), and
- verified verdict (pass/fail with total gate weight).

DEMO-69 prints these identities in its run transcript. PH-4 treats the presence of these identities as the minimal condition for interpreting any controller result as scientific.

6.2 Stage A: admissible vs illegal kernel separation (sign-lobe + HF witness)

PH-4 reads the kernel separation in DEMO-69 as a legality certificate, not as a performance comparison:

- Fejér must be nonnegative in real space (to numerical tolerance) and must not retain high-frequency weight beyond the declared cutoff.
- sharp truncation must exhibit negative lobes (Gibbs boundary) and therefore fail the admissibility gate.
- signed injection must exhibit both negative lobes and a large high-frequency retention witness.

These outcomes are required by the falsification matrix and are recorded by DEMO-69 as explicit gates with PASS/FAIL.

6.3 STAGE 4 (DEMO-69): Sharp-Transfer Witness (Gibbs vs Fejér) + Deterministic Teeth

PH-4 makes a strict distinction between (i) a metric being numerically small and (ii) an operator being lawful. STAGE 4 of DEMO-69 is the canonical witness of that distinction: it shows that an illegal operator can score well on a bulk metric while violating the boundedness/positivity contract that makes the computation physically interpretable.

DEMO-69 prints, for three transfer variants (Fejér admissible; sharp truncation illegal; signed injection illegal), three diagnostics:

- "distance vs truth" (relative L2 distance to a declared reference),
- minimum value of the transferred field (as a nonnegativity/undershoot witness),
- overshoot relative to the $[0, 1]$ admissible envelope (as a Gibbs witness).

Recorded values (DEMO-69 STAGE 4):

- Distances vs truth (rel L2): Fejér$= 0.140531$, sharp$= 0.091217$, signed$= 0.239279$
- $\min(y)$: Fejér$= 7.361219 \times 10^{-3}$, sharp$= -8.100846 \times 10^{-2}$, signed$= -1.620169 \times 10^{-1}$
- Overshoot: Fejér$= -0.044278$, sharp$= 0.101205$, signed$= 0.202410$

The gates imposed in STAGE 4 are explicitly tied to the deterministic tolerance $\varepsilon = 1/\sqrt{q_2}$, derived from the certified triple (Stage 1). With $q_2 = 30$, DEMO-69 reports $\varepsilon = 0.18257419$ and uses $\varepsilon^2 \approx 0.033$ as the floor for "macroscopic" overshoot/undershoot.

The resulting pass/fail separations are:

- **Gate T1** (admissible accuracy within deterministic envelope): Fejér distance vs truth $\leq \varepsilon$ (PASS).
- **Gate T2** (Gibbs witness): illegal filters must exhibit overshoot $\geq \varepsilon^2$ while Fejér does not (PASS).
- **Gate T3** (positivity/boundedness): Fejér preserves nonnegativity (PASS).
- **Gate T4** (illegal undershoot): illegal kernels create negative density below $-\varepsilon^2$ (PASS).

A crucial interpretive point follows directly from the numbers: the sharp truncation has a smaller L2 distance than Fejér ($0.091217 < 0.140531$), yet it produces negative density (min $= -0.08100846$) and large overshoot (0.101205). This is exactly the scenario PH-4 is designed to eliminate: a pipeline cannot claim physical meaning by optimizing a bulk error metric while violating legality constraints.

Deterministic teeth (counterfactual budget reduction).

DEMO-69 then reduces the admissible budget via a counterfactual triple and requires a predictable degradation by a factor at least $1 + \varepsilon$. It records:

- Counterfactual: $q_3^{cf} = 51 \Rightarrow r_{cf} = 5$ with $\mathrm{dist}_{cf} = 0.318089$ and degrade=True
- **Gate CF1:** budget reduction degrades by $1 + \varepsilon$ (PASS), with $\mathrm{dist}_P = 0.1405$, $\mathrm{dist}_{CF} = 0.3181$, $1 + \varepsilon = 1.183$

6.4 STAGE 5 (DEMO-69): Paradox Pack (Finite↔Continuum + Measure + Quantum Collapse)

STAGE 5 operationalizes the philosophical claim that "paradox is a transfer/measurement bug" into a falsifiable software instrument. PH-4 treats this stage as a unified audit of:

- partial summation (Zeno-type limits),
- summation methods (Grandi/Cesàro),
- Gibbs phenomena as a legality boundary (Dirichlet vs Fejér),
- measurement/coarse-graining legality (positive partitions vs signed partitions),
- quantum-typed collapse legality (admissible vs sharp, high-frequency injection).

The stage records the following concrete witnesses:

Zeno and Grandi (finite partial sums with declared convergence rules):

- **Gate Z1:** Zeno partial sum close to 1: sum$= 0.999999999069$, err$= 9.313 \times 10^{-10}$ (PASS)
- **Gate G1:** Grandi Cesàro close to 1/2: cesaro$= 0.500000$, err$= 0.000 \times 10^{0}$ (PASS)

Gibbs legality boundary (Dirichlet vs Fejér, same phenomenon as transfer stage):

- Gibbs overshoot: Dirichlet$= 0.091747$, Fejér$= -0.011686$
- Undershoot(min): Dirichlet$= -0.086785$, Fejér$= 0.001998$
- **Gate Gi1:** Dirichlet overshoot $\geq \varepsilon^2$ (PASS)
- **Gate Gi2:** Fejér eliminates overshoot (PASS; overshoot is nonpositive)
- **Gate Gi3:** Fejér preserves nonnegativity (PASS)
- **Gate Gi4:** Dirichlet creates negative undershoot below $-\varepsilon^2$ (PASS)

Measurement and partitions (mass preservation vs illegal signed mass):

- **Gate H1:** Hilbert-hotel shifts preserve total mass: mass$= 0.250000$ (PASS)
- **Gate H2:** positive partitions preserve mass: $\Delta = 0.000 \times 10^{0}$ (PASS)
- **Gate H3:** signed partitions generate illegal negative mass: signed_mass$= -0.234848$ (PASS)

Quantum-typed witnesses (interference + collapse legality):

- **Gate Q1:** interference present (coherent differs from incoherent): rel_L2= 0.7284 (PASS)
- **Gate C2:** sharp collapse injects vastly more HF than OATB: ratio= 9.28×10^{26} (PASS)
- **Gate QT:** reducing admissible budget degrades localization by $1 + \varepsilon$: $r = 32 \to 16$ miss= $0.000963 \to 0.002507$, $\varepsilon = 0.183$ (PASS)

Interpretation.

STAGE 5 supplies a single organizing principle for multiple "paradox families": the observed contradiction arises when one uses a non-admissible transfer or a non-admissible measurement (signed mass, sharp truncation) and then interprets the result as physically meaningful. Under the DOC/DAO admissibility boundary (nonnegative, unit mass, non-amplifying), the contradictions collapse into routine inequalities (boundedness, positivity, mass conservation) that are already enforced at finite size.

6.5 STAGE 6 (DEMO-69): Ω Reuse Across PDEs (Heat 3D/4D + 4D Vector Field)

PH-4's "universal controller" claim is deliberately limited: Ω is universal not because it encodes domain-specific physics, but because it is forced by observer invariance and therefore can be reused as a lawful stabilizer whenever a domain adapter decomposes a finite state into principal vs non-principal components.

STAGE 6 provides a domain-portability audit: the same Ω-typed mechanism is applied across three PDE-class adapters, while enforcing mass conservation and measuring improvements in tracking and high-frequency suppression.

DEMO-69 records:

3D heat:

- **Gate Ω3D-M:** mass conserved (uncontrolled + controlled): $\Delta\text{mass}_{\text{ctrl}} = 0.000 \times 10^{0}$ (PASS)
- **Gate Ω3D-T:** tracking improves ($\text{err}_{\text{un}}/\text{err}_{\text{ctrl}} \geq 1.3$): factor= 1.66 (PASS)
- **Gate Ω3D-HF:** HF error suppressed (hf_ratio ≤ 0.85): ratio= 0.809 (PASS)

4D heat:

- **Gate Ω4D-M:** mass conserved: $\Delta\text{mass}_{\text{ctrl}} = -2.910 \times 10^{-11}$ (PASS)
- **Gate Ω4D-T:** tracking improves (≥ 1.2): factor= 1.29 (PASS)
- **Gate Ω4D-HF:** HF error suppressed (≤ 0.75): ratio= 0.715 (PASS)

4D vector field (incompressibility and kinetic energy suppression):

- **Gate ΩV-DIV:** incompressibility improved (div_ratio ≤ 0.7): ratio= 2.739×10^{-19} (PASS)
- **Gate ΩV-KE:** energy damped (ke_ratio ≤ 0.7): ratio= 2.334×10^{-4} (PASS)
- **Gate ΩV-HF:** HF KE damped (hf_ratio ≤ 0.9): ratio= 3.005×10^{-36} (PASS)

Interpretation.

The stage establishes three claims simultaneously:

1. Ω control does not violate conservation constraints (mass preserved to tight tolerances).
2. Ω control improves tracking and reduces high-frequency error energy (without tuning).
3. The mechanism is portable across PDE-class adapters (heat and vector field), consistent with Ω being a symmetry-forced operator rather than a domain-specific parameterization.

6.6 STAGE 7 (DEMO-69): Cross-Base Invariance + Rigidity Audit

Although cross-base invariance is developed formally in PH-2, PH-4 must cite it because a controller/measurement layer is otherwise vulnerable to a representational loophole: if legality or selection changes under base change, then the "controller" could be a disguised formatting rule.

STAGE 7 performs two complementary tests:

1. Base invariance of the coupled selector (positive control).
2. Non-invariance of a digit-dependent selector (designed-fail negative control).

Base invariance (positive control):
DEMO-69 enumerates bases $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16\}$ and reports:

- decoded_triple = $(137, 107, 103)$ for every tested base, with pools_match=True
- **Gate B1:** base encode/decode round-trip holds (PASS)
- **Gate B2:** primary triple invariant across tested bases (PASS)

Digit dependence (designed-fail):
DEMO-69 then runs a digit-dependent selector and records distinct outputs across bases (including None outcomes):

- distinct digit-selector outputs across the same base set: distinct$= 5$
- **Gate DF:** digit-dependent selector is NOT invariant (PASS)

Rigidity scan (non-genericity audit):
DEMO-69 further tests a neighborhood of variants and records:

- Variants tested: 36288
- Variants with a triple: 2387 (6.58%)
- Distinct triples seen: 837
- Primary hits: 4 (0.01%)

With gates:

- **Gate R1:** nontrivial scan (some variants yield a triple) (PASS)
- **Gate R2:** not generic (most variants yield no triple): any_frac$= 0.066$ (PASS)
- **Gate R3:** primary not ubiquitous: hit_frac$= 0.000$ (PASS)
- **Gate R4:** primary appears at least once: primary_hits$= 4$ (PASS)

6.7 STAGE 8 (DEMO-69): Integration Ledger and Determinism Closure

DEMO-69 ends by explicitly stating what the flagship unifies:

- transfer legality (Fejér vs sharp/signed),
- paradox resolution (Zeno/Grandi/Gibbs + measurement + quantum collapse),
- PDE control portability (Ω reuse),
- base invariance vs digit hacks,

- rigidity (non-genericity).

This ledger is then cryptographically closed by:

- spec_sha256: `f88eb6ef2f6c2ce1a11c4cd6664880b49a22c85486e91c5e4a1dec081383da1e`
- determinism_sha256: `aef5bfbfa1efc26072e7d8cd2e2ffd92bedfbca5323c8f4a794cd55f7215af76`
- final verdict: VERIFIED with passed_weight= 32.00/32.00

7 Independent AoR Cross-Checks: DEMO-56 and DEMO-59 Reproduce the Same Legality Separations

DEMO-69 is the primary audit surface for PH-4 because it unifies admissibility, transfer, paradox, Ω reuse, and base-invariance in a single run. A scientific instrument, however, must also be robust under independent instantiations. This section records two such instantiations:

- **DEMO-56:** DOC vs classical finite differences (operator-calculus cross-check).
- **DEMO-59:** electromagnetism (electrostatics + Maxwell-class operators).

These demos are not "supporting narratives." They are independent AoR artifacts that reproduce the same legality boundary:

- Fejér (admissible) preserves boundedness/positivity and has controlled total variation;
- sharp truncation and signed controls can exhibit lower bulk error metrics in narrow settings, but they must fail the legality gates by negative lobes, overshoot, and/or HF retention;
- counterfactual teeth (budget reduction) must degrade performance predictably.

7.1 DEMO-56: kernel admissibility is discriminative and replayable

DEMO-56 repeats the kernel admissibility audit, reporting explicit real-space minima:

- Fejér $k_{\min} = 1.4845867057 \times 10^{-7}$ (nonnegative within tiny numerical slack)
- sharp truncation $k_{\min} = -0.1026225707$ (negative lobes)
- signed control $k_{\min} = -0.1299921965$ (negative lobes)

The demo treats these as hard gates. This is a direct replication (with different run context) of the DEMO-69 kernel sign-lobe separation, and it is evidence that the legality boundary is not a one-demo artifact.

7.2 DEMO-56: the ringing/TV audit reproduces the "metric vs legality" separation

DEMO-56 then runs linear advection of a step and reports three diagnostics at three times ($t = 0.13, 0.37, 0.71$):

- L2 distance (bulk error metric),
- overshoot_mass (boundedness violation witness),
- total variation TV (ringing/oscillation witness).

Representative snapshot ($t = 0.37$)**:**

- L2: Fejér = 0.0571049942, sharp = 0.0409367022, signed = 0.0935972757
- overshoot_mass: Fejér = 0, sharp = 0.0053648904, signed = 0.0176566422
- TV: Fejér = 1.9866737196, sharp = 5.2993519735, signed = 7.7082171866

Interpretation. Sharp truncation can reduce L2 while producing non-admissible ringing and boundedness violations (nonzero overshoot_mass and large TV). Fejér is lawful precisely because it preserves boundedness and suppresses oscillatory artifacts in a controlled way.

7.3 DEMO-56: nonlinear and 3D flow suites show admissibility is not a toy restriction

DEMO-56 runs:

E3 (Fisher–KPP reaction–diffusion, $N = 512,\ K = 60$**):**

- L2 vs internal reference: FD = 0.0312776258, Fejér = $1.8214154224 \times 10^{-6}$, sharp = 0.0312775903, signed = 0.0312775903
- boundedness (Fejér): min = 0.6107950512, max = 0.6107950512
- gates: admissible method competitive vs FD, non-admissible worse than admissible, boundedness preserved.

E4 (3D Navier–Stokes Taylor–Green, smoke tier $N = 64$**):**

- errors vs truth (normalized L2): Fejér = 0.5451459885, sharp = 1.1530512571, signed = 1.1530512571
- divergence L2: Fejér = $1.1224013038 \times 10^{-11}$, sharp = $1.0267235950 \times 10^{-9}$, signed = $8.2762613429 \times 10^{-10}$
- HF energy fraction ($> K_{\text{primary}}$): Fejér = $1.9915956710 \times 10^{-22}$, sharp = $5.0365775909 \times 10^{-16}$, signed = $1.2301059546 \times 10^{-16}$
- gates: incompressibility holds, admissible closer to truth than non-admissible controls, and non-admissible controls inject far more HF energy than the admissible method.

7.4 DEMO-59: electromagnetism reproduces admissibility gates and adds physical scaling structure

DEMO-59 repeats the kernel sign-lobe separation in two filter families (electrostatics and Maxwell):

Electrostatics family ($N = 64,\ K = 15$)**:**

- Fejér $k_{\min} = 0.000 \times 10^0$
- sharp $k_{\min} = -0.105335$
- signed $k_{\min} = -0.318054$
- HF weight energy fraction ($> K$): fejer = 0.000000, sharp = 0.000000, signed = 1.000000

Maxwell family ($N = 128,\ K = 31$)**:**

- Fejér $k_{\min} = 0.000 \times 10^0$
- sharp $k_{\min} = -0.105911$

- signed $k_{\min} = -0.318246$
- HF weight energy fraction ($> K$): fejer = 0.000000, sharp = 0.000000, signed = 1.000000

This is the same legality separation as DEMO-69 and DEMO-56, now observed in an electromagnetism-typed operator suite.

7.5 DEMO-59: Coulomb/Gauss scaling + curvature distinguishes lawful smoothing from illegal controls

DEMO-59 then tests electrostatics scaling using a radial list $r = [4, 6, 8, 10, 12]$ and reports:

Slope $\log |E|$ vs $\log r$ (expected ~ -2):

- truth $= -1.906371$
- admissible $= -1.791637$
- sharp $= -1.766929$
- signed $= -2.686532$

Ringing curvature (mean $|d2|$):

- truth $= 0.002242$
- admissible $= 0.002522$
- sharp $= 0.029511$
- signed $= 0.003761$

These are precisely the discriminative properties PH-4 requires: the admissible operator behaves like a lawful transfer, while illegal controls are detected by HF retention and curvature/oscillation witnesses.

8 What These Cross-Checks Establish for PH-4

Together, DEMO-56 and DEMO-59 establish three facts required for PH-4's program-level legitimacy:

1. The admissibility boundary is reproducible across independent demos: Fejér is nonnegative; sharp/signed controls are not.
2. The boundary is discriminative: illegal controls can sometimes look "better" on a bulk metric, but they are rejected by boundedness/positivity/TV/HF witnesses that cannot be satisfied simultaneously by non-admissible operators.
3. The controller layer has teeth: counterfactual budget reduction produces predictable, gated degradation, which prevents the method from being compatible with arbitrary tuning.

The next section connects the controller layer back into the Standard-Model pipeline, showing that Ω and admissible transfer appear as non-intervening legal infrastructure rather than post-hoc patching.

9 Integration Boundary: Ω as Non-Intervening Legal Infrastructure in the $\Omega \to$SM Master Flagship (DEMO-34)

PH-4 is not a Standard-Model paper, and it does not re-derive the Standard-Model closures developed elsewhere in the physics track. However, PH-4 must establish a specific integration claim:

> The Ω–$\nVdash$ controller and DOC-admissible transfer layer appear in the flagship SM-class pipeline as a declared legality scaffold, not as post-hoc patching; and the resulting constants are tied to the Ω certificate rather than to representation or tuning degrees of freedom.

The Authority-of-Record surface for this integration claim is DEMO-34 ("OMEGA→SM MASTER FLAGSHIP (v1)").

9.1 Stage 1A (DEMO-34): UFET PDE slope gate (diffusion toy)

DEMO-34 begins with a minimal PDE sanity check: a diffusion toy run with a refinement list

$$h = [0.03125, 0.015625, 0.0078125, 0.00390625]$$

and recorded errors

$$e = [0.5843837844318345, 0.14970499885409186, 0.03765597272675875, 0.009428416695397962].$$

It then reports:

- $\text{slope}(\log e \text{ vs } \log h) = 1.9852 \to$ [OK],

together with an energy-decay gate $\to$ [OK].

Interpretation. This is not presented as a proof of global PDE correctness. It is a sanity gate consistent with a second-order decay regime (slope near 2), confirming that the operator layer used in the pipeline is not behaving as an arbitrary smoothing knob at the lowest level.

9.2 Stage 1B (DEMO-34): Fejér / Ω controller sanity (legal structure, not tuning)

DEMO-34 then runs a Fejér/Ω controller sanity suite, explicitly checking:

- legality ($0 \leq F \leq 1$ and $F[0] = 1$): [OK]
- symmetry ($F[k] \approx F[M-k]$): [OK]
- low-pass ordering (low > high): [OK]
- average multiplier in $(0, 1)$: [OK]

These are operator-structure constraints: they confirm that the smoothing/control multipliers are (i) bounded, (ii) symmetric, and (iii) low-pass in the sense required for admissible averaging. They are not fitness metrics relative to a target dataset.

9.3 Stage 2A–2C (DEMO-34): Core2 window certificate and Tier-A_1 Ω joint-triple certificate

PH-4 requires one additional fact about the Ω pipeline: that the Ω constants used downstream are not selected by a pliable criterion that the controller could "help." DEMO-34 provides this by (i) producing a narrow survivor set in a local window and (ii) certifying uniqueness under a coupled lawbook up to a declared band.

Definition 9.1 (Core2 window mini-certificate)**.** A "Core2 window mini-certificate" is a constrained search over a small w-band, producing survivor lists for three channels (alpha/U(1), SU(2), and a third channel denoted pc2), together with necessity tests indicating which gates are required to prevent explosion of survivors.

Definition 9.2 (Tier-A_1 joint-triple Ω certificate)**.** A "Tier-A_1 joint-triple certificate" is the claim that, under the coupled lawbook constraints, there exists a unique triple (w_U, s_2, s_3) in a declared band $[80, B]$ (here $B = 1{,}000{,}000$ in DEMO-34), and that ablation of certain coupling gates causes explosion (many triples), demonstrating necessity of those gates.

Stage 2A (DEMO-34) reports:

- alpha survivors: [137]
- su2 survivors: [107]
- pc2 survivors: [103]

and derived Ω constants:

$$\{\mathrm{wU} : 137, \mathrm{s2} : 107, \mathrm{s3} : 103, \mathrm{q3} : 17, \mathrm{q2} : 30, \mathrm{r3} : 3\}.$$

Stage 2C (DEMO-34) then certifies:

- target triple: $(137, 107, 103)$
- constructed triple: $(137, 107, 103)$ (matches)
- triples_found: $[(137, 107, 103)]$
- unique target: [OK]

on the band 80..1,000,000.

Most importantly for PH-4's non-tuning posture, DEMO-34 performs ablation necessity tests and records explicit explosion counts when key couplings are removed:

- drop_u1_lock: count = 290, explodes = True
- drop_q2_coupling: count = 1089, explodes = True
- drop_pc2_lock: count = 78476, explodes = True
- drop_su2_sanity: count = 1, explodes = False (redundant under coupling)

Interpretation. This is the precise boundary PH-4 needs: the Ω triple is certified as unique under coupled constraints, and the couplings that make it unique are necessity-tested. Therefore the controller layer cannot plausibly be accused of "choosing" the triple, because the triple is fixed upstream by a coupling contract whose ablations are explicitly audited.

9.4 Stage 4A (DEMO-34): Rational anchor block tied to Ω constants

DEMO-34 then constructs a "rational anchor block" explicitly tied to the certified Ω constants and checks agreement within a conservative 2% envelope.

It reports:

- alpha_em anchor = 0.007299270, ref = 0.007297353, $\Delta\% = 0.026\%$
- alpha_s anchor = 0.117647059, ref = 0.117900000, $\Delta\% = -0.215\%$
- sin2W anchor = 0.233333333, ref = 0.231220000, $\Delta\% = 0.914\%$
- anchor gate (all within 2%): [OK]

9.5 Stage 4B (DEMO-34): SM snapshot overlay and the "non-intervening" controller claim

DEMO-34 concludes with a Standard-Model snapshot overlay (explicitly stating that PDG values are used only for Δ% overlay; pipeline outputs are the "pred" column). It reports:

- alpha_em: pred 0.007299 vs ref 0.007297 (Δ% 0.023%)
- $\sin^2(\theta_W)$: pred 0.233333 vs ref 0.231220 (Δ% 0.914%)
- alpha_s: pred 0.117647 vs ref 0.117900 (Δ% −0.215%)
- M_W/M_Z: pred 0.875595 vs ref 0.881469 (Δ% −0.666%)
- Γ_Z/M_Z: pred 0.027075 vs ref 0.027363 (Δ% −1.053%)

and a summary tally:

- greens$\leq$ 10%: 9
- yellows$\leq$ 30%: 2
- reds$>$ 30%: 0
- RMS($|\Delta|$): 6.500%
- Monte Carlo sanity: RMS percentile (lower is better): 0.00%

Interpretation (PH-4's point). PH-4 does not claim that Ω produces these observables. PH-4 claims something narrower and testable:

- the controller/transfer layer enters the pipeline through legality gates (Fejér/Ω sanity) and designed-fail separations,
- the Ω constants used by the pipeline are certified upstream under coupled lawbook constraints and ablation necessity,
- and the resulting SM-class snapshot is therefore not plausibly attributable to a hidden, pliable smoothing or measurement rule.

In other words, the controller layer is *non-intervening* with respect to "choosing the outcome": it is constrained to preserve the principal channel and enforce admissible transfer, while the discrete selection certificate and its ablations bear the burden of fixing the constants.

10 Generalization: Ω as the Trivial-Representation Projector for Any Finite Observer Group

Sections 2–3 presented Ω and Ω–$\nVdash$ in the minimal observer setting (cyclic shifts on a finite grid). That setting is sufficient for the controller/transfer audits reported in DEMO-69, DEMO-56, and DEMO-59, and it is the correct setting for many discretized PDE adapters. However, the program uses observer actions beyond cyclic shifts (e.g., finite residue structures and character decompositions). PH-4 therefore records the general form of Ω: it is the orthogonal projector onto the fixed subspace of a finite group action, i.e., onto the trivial representation.

10.1 Finite group actions and the fixed subspace

Let G be a finite group and let $U : G \to \mathrm{O}(\mathcal{H})$ be an orthogonal (unitary) representation of G on a finite real Hilbert space $\mathcal{H}$. For $g \in G$, write U_g for the action.

Define the fixed subspace

$$\mathrm{Fix}(G) := \{x \in \mathcal{H} : U_g x = x \text{ for all } g \in G\}.$$

Definition 10.1 (Observer neutrality)**.** A map $T : \mathcal{H} \to \mathcal{H}$ is *observer-neutral* if $TU_g = U_g T$ for all $g \in G$. A scalar observable $I : \mathcal{H} \to \mathbb{R}$ is observer-neutral if $I(U_g x) = I(x)$ for all $g \in G$.

10.2 The group-average projector

The principal projector Ω is forced by the same axioms as in Section 2 (idempotence, orthogonality, commutation with observer group), but it now takes an explicit universal form.

Theorem 10.2 (Principal projector as group average)**.** *Define*

$$\Omega := \frac{1}{|G|} \sum_{g \in G} U_g.$$

Then Ω is the orthogonal projector onto $\mathrm{Fix}(G)$*. In particular:*

1. *Ω is idempotent and self-adjoint: $\Omega^2 = \Omega$, $\Omega^T = \Omega$.*
2. *Ω commutes with every group action: $\Omega U_g = U_g \Omega$ for all g.*
3. $\mathrm{Im}(\Omega) = \mathrm{Fix}(G)$ *and* $\ker(\Omega) = (\mathrm{Fix}(G))^{\perp}$.

Proof sketch. (2) follows by reindexing the group sum. Self-adjointness follows since each U_g is orthogonal and $U_g^T = U_{g^{-1}}$, and the set $\{g^{-1}\}$ equals G. Idempotence follows because averaging twice equals averaging once. Finally, Ω fixes exactly the vectors invariant under all U_g, hence projects onto $\mathrm{Fix}(G)$. □

Remark 10.3. Section 2 is recovered by taking G to be the cyclic shift group G_M.

10.3 Uniqueness under calibration

In Section 2, calibration was expressed as $\Omega\vec{1} = \vec{1}$. In the general setting, calibration must refer to the specific "unit" or "mass" vector that defines the trivial representation in a given domain.

Definition 10.4 (Calibration vector)**.** A vector $c \in \mathcal{H}$ is a *calibration vector* if $U_g c = c$ for all $g \in G$, i.e., $c \in \mathrm{Fix}(G)$, and it represents the unit normalization (e.g., constant density, conserved mass mode, or identity channel).

Proposition 10.5 (Uniqueness of Ω on a 1-dimensional fixed space). *If* $\dim(\mathrm{Fix}(G)) = 1$ *and* c *is a calibration vector spanning* $\mathrm{Fix}(G)$*, then* Ω *is uniquely determined by:*

(A1) $\Omega^2 = \Omega$,

(A2) $\Omega^T = \Omega$,

(A3) $\Omega U_g = U_g \Omega$ *for all* g*, and*

(A4) $\Omega c = c$.

In that case, Ω *is the rank-one projector* $\Omega = \frac{\langle \cdot, c\rangle}{\langle c,c\rangle} c$*, which matches the group-average formula.*

10.4 Connection to character decompositions and "principal vs non-principal" channels

Many program adapters decompose a finite state into irreducible components under a finite group action. The principal channel is precisely the trivial representation; the non-principal channels are the orthogonal complement.

- In cyclic PDE grids, this is "mean (DC mode)" vs "mean-free modes."
- In residue-group/character settings, this is "principal character sector" vs "non-principal character sectors."

PH-4 does not re-develop the character theory; it records the controller consequence:

> Once a trivial-representation projector Ω is fixed, any admissible Ω-controller must preserve Ω exactly and contract the orthogonal complement.

This is the group-theoretic version of Proposition 3.2.

11 Constraint Compatibility: When Ω-Control Preserves the Laws It Is Supposed to Respect

A controller that suppresses instability by violating constraints is not admissible in the sense required for the physics track. PH-4 therefore records a second, independent admissibility boundary:

> A controller must preserve the declared constraint projectors (mass, divergence-free constraints, gauge constraints) to the extent those constraints are part of the modeled law.

11.1 Constraint projectors and commutation

Let $\mathcal{H}$ be a finite Hilbert space and let $P : \mathcal{H} \to \mathcal{H}$ be an orthogonal projector representing a constraint subspace (e.g., divergence-free vector fields, mean-zero constraint, or gauge-fixed subspace). A state x is constraint-satisfying if $Px = x$.

Proposition 11.1 (Constraint preservation under commuting controllers). *Let* $C : \mathcal{H} \to \mathcal{H}$ *be any linear operator satisfying* $CP = PC$*. If* x *satisfies the constraint* $Px = x$*, then* Cx *also satisfies the constraint:*

$$P(Cx) = C(Px) = Cx.$$

Corollary 11.2 (Constraint preservation for Ω-controllers). *If* $P\Omega = \Omega P$*, then every* Ω*–*κ *controller* $\Omega_\kappa = \Omega + \kappa(I - \Omega)$ *commutes with* P *and therefore preserves the constraint subspace.*

11.2 When does P commute with Ω?

In the cyclic-grid case, Ω is a circulant projector onto the DC mode. Any constraint projector built from translation-invariant operators (e.g., circulant differential stencils, spectral projectors that depend only on wavenumber magnitude, or convolution-typed admissible smoothing) lies in the commutant of the shift group and therefore commutes with Ω.

In the finite-group case (Section 10), Ω is the projector onto $\mathrm{Fix}(G)$. Any constraint projector P that is observer-neutral (commutes with U_g for all g) will commute with Ω because both live in the commutant of G.

This is the structural reason PH-4 treats "observer neutrality" and "constraint neutrality" as the same mathematical posture: both are commutation requirements with the observer group action.

11.3 Mass conservation: Ω preserves the calibrated mass channel

In the cyclic setting, the conserved "mass" observable is the sum of entries, equivalently the inner product with the calibration vector $\vec{1}$. Since Ω fixes $\vec{1}$, it preserves the mean channel exactly.

Lemma 11.3 (Mean preservation). *For $x \in \mathbb{R}^M$, define the mean $\mu(x) = \frac{1}{M}\langle \vec{1}, x\rangle$. Then $\mu(\Omega x) = \mu(x)$ and $\mu(\Omega_\kappa x) = \mu(x)$ for all κ.*

Proof. Ω acts as the identity on $\mathrm{span}\{\vec{1}\}$ and annihilates its orthogonal complement, so it cannot change the mean. The same is true for Ω_κ because Ω_κ differs from identity only on the mean-free subspace. □

11.4 Summary: the admissible controller must be compatible with both observer symmetry and constraints

PH-4's admissibility posture is therefore two-axis:

- **Symmetry axis:** commutation with the observer group (so no hidden frame-selection).
- **Constraint axis:** commutation with declared constraint projectors (so stability is not purchased by law violation).

In the finite operator setting used by the program, these axes can be audited explicitly: the operator either commutes (and preserves the constraint) or it does not, and conservation/constraint witnesses can be recorded as deterministic gates in AoR runs.

12 Citation, Verification, and Chain-of-Custody Protocol for PH-4 Claims

PH-4 is written in two layers:

- The mathematical layer (definitions, lemmas, theorems) is self-contained and does not depend on empirical calibration.
- The evidence layer (AoR demos and artifacts) is cryptographically identified and must be cited when PH-4 makes an operational claim ("this gate passed," "this control failed," "this separation was observed").

This section specifies how the second layer is to be cited and verified.

12.1 Canonical AoR identity for this rewrite

All AoR citations in PH-4 are anchored to a single canonical release tag and archive folder:

- **AoR release tag:** `release-aor-20260125T043902Z`
- **AoR archive folder:** `gum/authority_archive/AOR_20260125T043902Z_52befea`
- **Bundle sha256 (AoR citation identity):** `c299b1a7a8ef77f25c3ebb326cb73f060b3c7176b6ea9eb402c`

12.2 What counts as a citable evidence object in PH-4

PH-4 treats the following as citable evidence objects:

(E1) A demo stdout/stderr pair under the AoR tag (the primary narrative of deterministic gates).

(E2) A vendored artifact tied to a demo (plots, JSON results, sha manifests), when present.

(E3) A ledger/table in the bundle (falsification matrix, reproducibility table, indices).

(E4) A run-level transcript/metadata object (runner transcript, run metadata) establishing custody.

In contrast, a derived paragraph in this paper is *not* evidence unless it links to at least one of (E1)–(E4) for each operational claim.

12.3 Minimal verification procedure for any operational claim

When PH-4 says “DEMO-X establishes Y,” the minimum verification path is:

Step V1 (locate demo identity). Open demo_index.csv and locate DEMO-X and its file basenames.

Step V2 (read the demo transcript). Open the demo stdout. Confirm the header, the spec_sha256 (when printed), and the final verdict line.

Step V3 (confirm determinism identity). Read the determinism_sha256 printed at the end of stdout and confirm it matches the AoR’s reproducibility record.

Step V4 (confirm falsifiers were not silently omitted). Locate the relevant designed-fail class in falsification_matrix.csv and confirm that the demo contains the corresponding fail witness and PASS/FAIL classification as required.

Step V5 (confirm chain-of-custody context). For any claim that depends on the global run posture (time, platform, mode, manifest), consult runner_transcript.txt and run_metadata.json.

12.4 Audit principle: “show the fail”

PH-4’s claims are designed to be adversarially verifiable. For every “passes when lawful” claim, there must exist a paired “fails when unlawful” witness:

- Fejér kernel nonnegativity vs sharp/signed negative lobes (DEMO-69 Stage 3).
- Fejér bounded step transfer vs sharp/signed overshoot and negativity (DEMO-69 Stage 4; DEMO-59 Stage 5).
- Base-invariant selector vs digit-dependent selector non-invariance (DEMO-69 Stage 7).
- Counterfactual budget reduction degrades by $(1+\varepsilon)$ (DEMO-69 Stage 4; DEMO-56 E1/E4; DEMO-59 Stage 4B/5).

Any rewrite that retains the pass but omits the fail is considered incomplete.

13 Scope, Claims, and Non-Claims of PH-4

PH-4 is intentionally narrow in what it asserts as "proved," and intentionally strict in what it asserts as "observed."

13.1 What PH-4 proves (mathematical layer)

PH-4 proves, within finite-dimensional real Hilbert spaces:

- the characterization of Ω as the unique observer-invariant principal projector in the cyclic shift setting (Sections 2–3),
- the generalization of Ω as the group-average projector onto the fixed subspace for any finite observer group (Section 10),
- admissibility conditions for transfer kernels and measurement windows sufficient to guarantee boundedness/positivity/mass preservation properties (Sections 4–5 and 11), and
- residual-budget inheritance inequalities: non-expansive controllers cannot beat a declared approximation residual, and composition budgets telescope (Section 5).

These results are not claims about nature. They are claims about what is mathematically forced when one insists on symmetry, admissibility, and conservation in finite operator pipelines.

13.2 What PH-4 records as observed (AoR layer)

PH-4 records the following as AoR-observed facts, each tied to explicit demo logs:

- Kernel admissibility separations (Fejér vs sharp/signed) and their deterministic gates (DEMO-69, DEMO-56, DEMO-59).
- Paradox-pack unification under admissibility (Zeno/Grandi/Gibbs/partitions/collapse) as explicit gated outcomes (DEMO-69 Stage 5).
- Ω reuse across PDE adapters with conservation and constraint witnesses (DEMO-69 Stage 6).
- Representation/base invariance of the coupled selector and non-invariance of digit-dependent selectors (DEMO-69 Stage 7; DEMO-64).
- Integration of legality infrastructure into the $\Omega \to$SM flagship in a way that is upstream-certified (unique joint triple + ablation necessity) rather than post-hoc tuned (DEMO-34).

13.3 What PH-4 does not claim

PH-4 does *not* claim:

- that admissibility alone identifies the Standard Model, cosmology, or any specific empirical constant;
- that Fejér filtering is "the physics," rather than a lawful transport/measurement instrument;
- that Ω is a universal causal mechanism in nature, rather than a symmetry-forced projector in finite observer group actions;
- that a small bulk error metric implies physical interpretability without legality gates.

PH-4 instead asserts a *prerequisite*: if a program intends to interpret finite computations as physically meaningful, then it must (i) declare its admissible operator class, (ii) show designed-fail controls, and (iii) close the chain-of-custody.

14 Placement in the Physics Track: What PH-4 Provides to the Rest of the Suite

PH-4 is the controller and admissibility spine of the physics track. It supplies three ingredients that the remaining physics papers may treat as infrastructure rather than re-proving each time:

1. A symmetry-forced principal channel (Ω) and a non-principal contraction (Ω–$\nVdash$) whose role is stabilizing without intervention on the principal invariant.
2. A lawful transfer and measurement class (DOC/DAO admissibility) that preserves boundedness/positivity/mass and rejects sharp/signed corruptions by deterministic gates.
3. A teeth posture: counterfactual budgets must degrade by $(1+\varepsilon)$, and digit-dependent / non-injective transports must fail.

15 Conclusion: PH-4 as the Lawful Controller/Measurement Layer of the Marithmetics Architecture

PH-4 establishes a single, conservative principle and then proves that it can be made audit-grade:

> If a finite-substrate program intends to interpret computed quantities as observables, then control, transfer, and measurement must be explicit operators constrained by an admissibility contract, and those constraints must be enforced by designed-fail falsification under a reproducible chain-of-custody.

Within that program posture, PH-4 contributes four results.

15.1 Symmetry forces the principal projector Ω

Given a finite observer group action and a calibration vector defining the "unit" channel, the principal projector Ω is forced: it is the orthogonal projector onto the fixed subspace (the trivial representation). In the cyclic setting, Ω is the mean projector (rank-one onto the DC mode). This removes ambiguity from "what is principal": it is not chosen; it is fixed by invariance and calibration.

15.2 The Ω–$\nVdash$ controller is the minimally non-intervening stabilizer

Once Ω is fixed, Ω–κ is the minimal stabilizer that:

- preserves the principal channel exactly,
- contracts the complementary channel by a declared factor κ,
- commutes with observer actions, and
- is non-expansive in ℓ^2.

This controller is "universal" only in the strict operator sense: it is the same symmetry-forced pattern across any domain where a finite observer group defines a trivial representation.

15.3 Transfer and measurement legality prevents hidden tuning

PH-4 formalizes conservative admissibility contracts for:

- smoothing kernels (nonnegative, unit mass, non-amplifying multipliers),
- measurement windows (nonnegative weights, unit mass, manifest-declared),
- coarse-graining partitions (partition-of-unity and mass conservation).

These contracts enforce boundedness/positivity properties that are elementary mathematically but decisive scientifically: they prevent a pipeline from "winning" by signed cancellation or by sharp truncation artifacts.

15.4 Evidence closes the loop: admissible passes; illegal fails; budgets have teeth

PH-4's claims are not left at the level of "should." The AoR provides explicit, deterministic separations:

- Fejér admissibility ($k_{\min} \geq 0$ to tolerance) vs sharp/signed negative lobes, replicated across DEMO-69, DEMO-56, DEMO-59.
- Gibbs-type overshoot/undershoot and negative density as designed-fail witnesses of illegality (DEMO-69 Stage 4; DEMO-59 step suite).
- Signed partitions generate illegal negative mass, providing an explicit measurement falsifier (DEMO-69 paradox pack).
- Counterfactual budget reduction degrades performance by $\geq (1 + \varepsilon)$ in $\geq 3/4$ cases, preventing arbitrary tuning (DEMO-69, DEMO-56, DEMO-59).
- Base-invariance of the coupled selector vs non-invariance of digit-dependent selectors (DEMO-69, DEMO-64).
- Integration boundary: Ω constants are fixed upstream by a coupled joint-triple certificate with ablation necessity; the controller layer enters as legality scaffold rather than as a selection knob (DEMO-34).

15.5 What a reader should now be able to do

After PH-4, a reader should be able to:

- recognize when a later paper uses control, smoothing, or measurement, and identify whether it is inside the admissible operator class;
- verify that any "stability improvement" is not achieved by illegality (negative mass, overshoot, HF injection);
- independently reproduce the pass/fail boundaries via AoR logs, without reliance on interpretive prose; and
- separate claims about "lawful operators" (proved) from claims about "domain closures" (proved elsewhere, evidenced by other AoR surfaces).

PH-4's role in the suite is therefore foundational: it converts "controller" and "measurement" from informal computational habits into explicit, testable, and audit-grade operator statements.

A Proof Details for the Cyclic Shift Commutant (Circulant Operators)

Let P_1 be the cyclic shift on $\mathbb{R}^M$, $(P_1x)_n = x_{n-1}$. Let C be an $M \times M$ matrix.

Claim. C commutes with P_1 ($CP_1 = P_1C$) if and only if C is circulant.

Proof. Consider matrix entries:

$$(CP_1)_{n,m} = \sum_j C_{n,j}(P_1)_{j,m} = C_{n,m+1}.$$

$$(P_1C)_{n,m} = \sum_j (P_1)_{n,j}C_{j,m} = C_{n-1,m}.$$

Thus $CP_1 = P_1C$ implies $C_{n,m+1} = C_{n-1,m}$ for all n, m. Iterating yields $C_{n,m}$ depends only on $(n - m) \bmod M$. Hence C is circulant.

Conversely, if $C_{n,m} = c_{n-m}$, then shifting both indices preserves the difference, and the same identity holds, so $CP_1 = P_1C$. □

B Discrete Fejér Kernel Nonnegativity Witness

Let $r \geq 0$ and define

$$F_r(n) = \frac{1}{r+1}\left|\sum_{k=0}^{r} \exp(2\pi ikn/M)\right|^2.$$

Then $F_r(n) \geq 0$ for all n and defines a nonnegative kernel. Under the program's DFT normalization (recorded in capsule manifests for each demo that uses Fejér), F_r has unit mass and produces multipliers in $[0, 1]$. This provides the canonical positive control for admissible transfer.

C Telescoping Inequality for Residual Composition

Let A_j and B_j be bounded operators with norms ≤ 1. Then

$$T_A - T_B = \sum_{j=1}^{m} A_m \cdots A_{j+1}(A_j - B_j)B_{j-1} \cdots B_1$$

and taking norms yields

$$\|T_A - T_B\| \leq \sum_j \|A_j - B_j\|.$$

This is the operator basis for additive residual accounting in composed pipelines.