NT-4 — Finite Infinity and Residual Equivalence

A canonical residual budget for DOC/Superset regularisation

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Abstract

NT-1 established a finite analytic engine (DOC kernels, admissible operators, and finite-to-continuum transfer discipline). NT-2 developed Superset Theory on digital-root power tables (DRPTs), including splinter partitions, survivor fields, DOC-compatible window measurements, tilings, and a separation (Only–Zero) rigidity theorem. NT-3 defined a base-portable structural invariant alphabet and a deterministic selection calculus (SCFP/SCFP++) for canonically forcing dimensionless invariants without representation drift.

What remains is a canonical account of "infinite" analytic operations—limits, divergent series, and regularised values—expressed entirely inside the finitary architecture. Classical analysis typically treats such objects as limits taken in infinite-dimensional spaces and introduces multiple regularisation methods that can yield different "assigned values." The Marithmetics program requires the opposite: a finite operator language in which (i) all computations remain finite at every stage, (ii) any admissible refinement has an explicit residual law, and (iii) the regularised value is scheme-independent across a declared admissible class.

This paper formalizes that requirement. We define a *finite-infinity representation* of a formal object F as a pair $(\text{Reg}_{\text{DOC}}(F), \eta_F(h))$ consisting of a finite regularised value and an explicit residual law indexed by a scale parameter h (grid spacing, truncation index, or bandwidth). We then define *DOC-admissible regularisation schemes* as families of finite approximants and DOC-admissible windows/operators satisfying a small list of axioms: legality (positivity, mass preservation, spectral constraints), convergence compatibility for convergent objects, and a uniform residual budget. The main theorem (*Residual Equivalence*) proves that within this admissible class the regularised value is canonical and the residual law is unique up to a precisely controlled structural equivalence. Finally, we work three canonical examples—Zeno-type geometric series, Grandi's series, and the Dirichlet vs. Fejér treatment of Gibbs-type overshoot—to show how classical "paradoxes" are resolved as finite residual bookkeeping rather than philosophical tension.

Contents

1 Reader Contract

1.1 Proof standard

All mathematical statements in this paper are finite. Every construction is defined on explicit finite sets (finite grids, finite rings, finite tables, finite matrices), and every theorem is a statement about finite objects uniformly over a refinement parameter. Consequently, all arguments are formalizable in ZFC as bounded constructions.

1.2 Separation of claim types

This paper distinguishes:

1. **Mathematical claims:** definitions and theorems proved in-text.

2. **Evidence claims:** computed demonstrations (tables, logs, plots, run metadata).

Mathematical claims do not depend on computation. Evidence claims are cited only through the project's Authority-of-Record (AoR) archive.

1.3 Authority-of-Record citation surface (for evidence only)

AoR tag: `release-aor-20260125T043902Z`
AoR folder: `gum/authority_archive/AOR_20260125T043902Z_52befea`
Bundle sha256: `c299b1a7a8ef77f25c3ebb326cb73f060b3c7176b6ea9eb402c97273dc3cf66c`

Canonical artifacts:
Master bundle (zip):
https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T043902Z/gum/
authority_archive/AOR_20260125T043902Z_52befea/MARI_MASTER_RELEASE_20260125T043902Z_
52befea.zip
GUM report (v32):
https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T043902Z/gum/
authority_archive/AOR_20260125T043902Z_52befea/report/GUM_Report_v32_2026-01-25_
04-42-51Z.pdf
Claim ledger:
https://github.com/public-arch/Marithmetics/blob/release-aor-20260125T043902Z/gum/
authority_archive/AOR_20260125T043902Z_52befea/claim_ledger.jsonl

1.4 DOC baseline (methodological baseline document)

Deterministic Operator Calculus (DOC) is the baseline operator legality and transfer discipline for the suite and is treated as a master document. This paper uses DOC only to declare admissible kernels/windows/operators and does not assume any results beyond those stated explicitly in NT-1.

2 From Limits to Finite Infinity

Classical analysis organizes many objects through limits: infinite series as limits of partial sums, integrals as limits of Riemann sums, and Fourier expansions as limits of partial projections. It is also common to assign finite "regularised values" to divergent objects using methods (Cesàro, Abel, zeta regularisation, Borel, etc.) that are not always equivalent.

The Marithmetics requirement is different. The admissible universe is finite at every stage. Infinity is not an additional domain; it is a refinement parameter controlling a family of finite

approximants. The only meaningful "infinite" statement is therefore a statement about a residual law across refinement.

2.1 Finite infinity representation

Let $\mathcal{F}$ denote a class of formal objects that includes, for example:

- infinite series $\sum_{n \geq 0} a_n$, represented by finite partial sums;

- Fourier expansions, represented by finite spectral truncations;

- energies or quadratic functionals, represented by finite DOC kernels acting on grids;

- any construction where a scale parameter $h > 0$ indexes finite approximants in a controlled way.

For each $F \in \mathcal{F}$, a DOC-style representation provides:

- a refinement parameter $h \in (0, h_0]$ (or, equivalently, an integer index $N \to \infty$);

- a finite-dimensional real vector space V_h;

- an approximant $R_h(F) \in V_h$;

- and a scalar extraction map $\pi_h : V_h \to \mathbb{R}$ (often a mean, trace, or windowed measurement).

The "number" associated to the approximant at scale h is $\pi_h(R_h(F))$. Convergence, divergence, and regularisation are all properties of the behaviour of this scalar as $h \to 0$.

Definition 2.1 (Finite infinity representation). Let $F \in \mathcal{F}$. Let $\mathcal{R}$ be a DOC-admissible regularisation scheme (defined in Section 3). A *finite infinity representation* of F is a pair

$$\big(\mathrm{Reg}_{\mathrm{DOC}}(F),\ \eta_F(h)\big),$$

where:

1. $\mathrm{Reg}_{\mathrm{DOC}}(F) \in \mathbb{R}$ is a finite regularised value, and

2. $\eta_F(h)$ is a residual law such that for all sufficiently small h,

$$\pi_h(R_h(F)) \ = \ \mathrm{Reg}_{\mathrm{DOC}}(F) \ + \ \eta_F(h),$$

with $\eta_F(h) \to 0$ as $h \to 0$ in a controlled, explicitly bounded manner.

In this formulation, "infinity" is encoded entirely by the behaviour of $\eta_F(h)$ under refinement.

Remark 2.1 (What is and is not being claimed). Definition 2.1 does not assert that every formal object admits such a representation under every scheme. It defines what it means to have a finite infinity representation under a declared admissible scheme. The central theorem of this paper identifies conditions under which the regularised value is scheme-independent and the residual law is canonical up to a precisely defined equivalence.

2.2 Admissible regularisation schemes

Before giving axioms, we state the intended structure.

A DOC-admissible regularisation scheme $\mathcal{R}$ consists of:

1. A refinement index $h \mapsto V_h$ (finite-dimensional state spaces).

2. A mapping $F \mapsto R_h(F) \in V_h$ giving finite approximants.

3. A DOC-admissible operator family $T_h : V_h \to V_h$ (optional but typical), used to enforce smoothing/suppression or to implement a canonical measurement profile.

4. A scalar extraction $\pi_h : V_h \to \mathbb{R}$ yielding the reported value at scale h.

The DOC role is to constrain the operator families (and, when π_h is a windowed measurement, to constrain the windows) so that:

- the principal component (mean) is preserved;

- non-principal components cannot be sustained without explicit structural forcing;

- and admissible refinement (as $h \to 0$) has a uniform residual budget.

The rigorous definition is given in Section 3.4.

2.3 Residual Equivalence (informal statement)

The desired canonicity statement is:

Informal Claim (Residual Equivalence). Fix $F \in \mathcal{F}$. If $\mathcal{R}$ and $\mathcal{R}'$ are two DOC-admissible regularisation schemes applicable to F, then:

1. the DOC-regularised value is scheme-independent:

$$\text{Reg}_{\text{DOC}}^{\mathcal{R}}(F) \ = \ \text{Reg}_{\text{DOC}}^{\mathcal{R}'}(F);$$

2. the residual laws $\eta_F^{\mathcal{R}}(h)$ and $\eta_F^{\mathcal{R}'}(h)$ differ only by an admissible structural equivalence (precisely defined in Section 4), meaning that their difference is controlled by the universal residual budget associated to the admissible class.

This turns regularisation from a "method choice" into a theorem: within the admissible class, the value is canonical, and divergence is recorded as a controlled residual rather than a scheme-dependent ambiguity.

2.4 Scope and structure of the paper

Section 3 gives the axiomatic framework: what counts as a formal object, what counts as an approximant family, what residual laws are allowed, and what legality axioms define a DOC-admissible scheme.

Section 4 states and proves the Residual Equivalence Theorem. The proof is finite and relies on two ingredients: (i) contraction of non-principal components under DOC-admissible operators, and (ii) a uniform residual budget that bounds differences between admissible approximants at matched scale.

Section 5 defines the canonical outputs: the regularised value and the finite infinity equivalence class of residual laws, and it formalizes designed-fail detection for illegal schemes.

Section 6 works canonical examples, illustrating how classical paradoxes are resolved as residual bookkeeping.

3 Axioms for Finite Infinity

3.1 Formal objects

To avoid ambiguity, we define "formal object" at the level required for this paper.

A *formal object* F is specified by:

1. a refinement domain $\mathcal{H} \subset (0, h_0]$ with 0 as a limit point (or an equivalent cofinal integer index $N \to \infty$);

2. a family of finite-dimensional real vector spaces $\{V_h\}_{h \in \mathcal{H}}$;

3. an approximant map $R_h(F) \in V_h$ for each $h \in \mathcal{H}$;

4. a scalar extraction map $\pi_h : V_h \to \mathbb{R}$ for each $h \in \mathcal{H}$.

The extracted scalar sequence is

$$A_h(F) := \pi_h(R_h(F)).$$

Examples (canonical instances):

- **Series** $F = \sum_{n \geq 0} a_n$: take $h = 1/N$, $V_h = \mathbb{R}$, $R_h(F) = \sum_{n=0}^{N} a_n$, $\pi_h = \mathrm{id}$.

- **Fourier expansion**: take $h = 1/N$, $V_h = \mathbb{C}^{2N+1}$ (or $\mathbb{R}^{2N+1}$ for real signals), $R_h(F)$ the truncated coefficient vector, π_h a reconstruction functional evaluated at a point or integrated.

- **Energy on a grid**: take $h = 1/M$, $V_h = \mathbb{R}^{\mathbb{Z}_M}$, $R_h(F)$ a discrete field, π_h a DOC windowed measurement.

3.2 Schemes, Approximants, and Residual Laws

3.2.1 Approximants and admissible transforms

Let $F \in \mathcal{F}$ be a formal object with approximants $R_h(F) \in V_h$ and extraction $\pi_h : V_h \to \mathbb{R}$.

A regularisation scheme may optionally insert an admissible operator stage:

$$R_h(F) \;\mapsto\; T_h R_h(F),$$

and/or replace the extraction map by a windowed measurement π_h^W. Both mechanisms are finite and belong to the same general pattern: "regularise by applying a lawful operator and then measure."

Definition 3.1 (Scheme transform and extracted approximant)**.** A regularisation scheme $\mathcal{R}$ assigns, for each $h \in \mathcal{H}$,

1. a (possibly identity) linear map $T_h^{\mathcal{R}} : V_h \to V_h$, and

2. a (possibly original) linear functional $\pi_h^{\mathcal{R}} : V_h \to \mathbb{R}$.

The *extracted approximant scalar* under $\mathcal{R}$ is

$$A_h^{\mathcal{R}}(F) \;:=\; \pi_h^{\mathcal{R}}(T_h^{\mathcal{R}} R_h(F)).$$

Remark 3.1 (Why we allow both operator and measurement stages). In some settings the admissible structure is naturally an operator (e.g., a DOC kernel acting by convolution). In others it is naturally a measurement window (e.g., a probability weight on a finite index set). Both are linear, both are finite, and both are constrained by the same legality principles: positivity, mass preservation, and suppression of non-principal structure.

3.2.2 Regularised value and residual law

The extracted scalar sequence $A_h^{\mathcal{R}}(F)$ is finite for each h. To speak meaningfully about "infinity," we require a decomposition into a stable value and a residual error that is explicitly controlled.

Definition 3.2 (Residual law relative to a candidate value). Let $c \in \mathbb{R}$. The *residual law* of $\mathcal{R}$ on F relative to c is

$$\eta_{F,c}^{\mathcal{R}}(h) \ := \ A_h^{\mathcal{R}}(F) - c.$$

Definition 3.3 (Finite-infinity representation under a scheme). A scheme $\mathcal{R}$ yields a *finite-infinity representation* of F if there exists $c \in \mathbb{R}$ and a function $\eta(h)$ such that

$$A_h^{\mathcal{R}}(F) = c + \eta(h), \qquad \text{with} \qquad \eta(h) \to 0 \text{ as } h \to 0$$

and the decay of η is controlled by an explicit bound (defined below).

In that case we write

$$\mathrm{Reg}^{\mathcal{R}}(F) \ := \ c, \qquad \eta_F^{\mathcal{R}}(h) \ := \ \eta(h).$$

Remark 3.2 (The value is determined by the residual claim). In this paper "regularised value" is not a free label. It is the constant term in a refinement law. A scheme has not regularised F unless it has supplied both c and a vanishing residual $\eta(h)$ with an explicit budget.

3.2.3 Residual budgets

Residual control must be uniform across a declared admissible class. Otherwise a scheme could always claim success by asserting an object-dependent and scheme-dependent error bound after the fact.

Definition 3.4 (Residual budget). A *residual budget* is a function $\beta : \mathcal{H} \to \mathbb{R}_{\geq 0}$ such that $\beta(h) \to 0$ as $h \to 0$.

A residual law $\eta(h)$ is said to be *β-controlled* if

$$|\eta(h)| \ \leq \ \beta(h) \qquad \text{for all sufficiently small } h.$$

Definition 3.5 (Uniform residual budget for a scheme class). Let $\mathfrak{R}$ be a class of regularisation schemes applicable to a class $\mathcal{F}_0 \subseteq \mathcal{F}$. We say $\mathfrak{R}$ has *uniform residual budget* β on $\mathcal{F}_0$ if, for every $F \in \mathcal{F}_0$ and every $\mathcal{R} \in \mathfrak{R}$, there exists a value $\mathrm{Reg}^{\mathcal{R}}(F)$ such that

$$A_h^{\mathcal{R}}(F) = \mathrm{Reg}^{\mathcal{R}}(F) + \eta_F^{\mathcal{R}}(h), \qquad |\eta_F^{\mathcal{R}}(h)| \leq \beta(h)$$

for all sufficiently small h.

The existence of such a uniform budget is the mathematical content of "finite infinity." It converts "$h \to 0$" into a deterministic statement about a bounded error schedule.

3.3 Regularisation Axioms

We now state the axioms that define the admissible regularisation class used in Marithmetics. Each axiom is finitary and checkable on bounded objects.

Axiom 3.1 (Finiteness and determinism). For each F in the domain and each $h \in \mathcal{H}$, the extracted scalar $A_h^{\mathcal{R}}(F)$ is computed by a deterministic finite procedure from the declared data of $\mathcal{R}$ and F.

Axiom 3.2 (Positivity preservation). If V_h carries an order structure (e.g., coordinatewise order on $\mathbb{R}^n$), then for every h the operator stage and measurement stage preserve nonnegativity:

$$R_h(F) \geq 0 \ \Rightarrow \ T_h^{\mathcal{R}} R_h(F) \geq 0, \qquad u \geq 0 \ \Rightarrow \ \pi_h^{\mathcal{R}}(u) \geq 0.$$

Axiom 3.3 (Mass / mean preservation)**.** There is a distinguished "constant" element $\mathbf{1}_h \in V_h$ such that

$$T_h^{\mathcal{R}} \mathbf{1}_h = \mathbf{1}_h, \qquad \pi_h^{\mathcal{R}}(\mathbf{1}_h) = 1,$$

and $\pi_h^{\mathcal{R}}$ agrees with the natural mean on constants.

Axiom 3.4 (Non-principal suppression)**.** There is a canonical decomposition

$$V_h \;=\; \mathrm{span}\{\mathbf{1}_h\} \;\oplus\; V_h^0$$

into principal and mean-free subspaces such that $T_h^{\mathcal{R}}$ contracts the mean-free component uniformly:

$$\|T_h^{\mathcal{R}} u\| \;\leq\; \rho(h)\,\|u\| \qquad \text{for all } u \in V_h^0$$

for some $\rho(h) \in [0,1)$ bounded away from 1 on the admissible range, or such that the residual budget (Axiom 3.6) explicitly accounts for the resulting slower suppression.

Axiom 3.5 (Consistency on convergent objects)**.** If F is a formal object whose extracted scalars $A_h^{\mathrm{id}}(F) := \pi_h(R_h(F))$ converge to a classical value L as $h \to 0$, then every admissible scheme $\mathcal{R}$ applicable to F must satisfy

$$\mathrm{Reg}^{\mathcal{R}}(F) = L.$$

Axiom 3.6 (Uniform residual budget)**.** There exists a residual budget $\beta(h) \to 0$ such that for every F in the declared domain of the scheme,

$$A_h^{\mathcal{R}}(F) = \mathrm{Reg}^{\mathcal{R}}(F) + \eta_F^{\mathcal{R}}(h), \qquad |\eta_F^{\mathcal{R}}(h)| \leq \beta(h)$$

for all sufficiently small h.

Axiom 3.7 (Designed FAIL)**.** If any of R2–R6 are violated for a candidate scheme-object pair, the evaluation must not silently return a value. It must instead return a falsifier certificate identifying which axiom failed and exhibiting a finite counterexample at some scale h.

Remark 3.3 (Axiom set and the role of DOC)**.** Axioms R2–R4 are the DOC-aligned legality constraints. Axioms R5–R7 are the scheme-independence constraints. NT-1 provides the canonical model for R2–R4 on cyclic convolution operators; NT-2 provides the canonical model for positivity-preserving window measurements; NT-4 uses those legality classes to constrain admissible regularisation.

3.4 DOC-Admissible Regularisation Schemes

We now specialise the above axioms to the operator classes that are actually used throughout the suite.

3.4.1 DOC operator admissibility

In the core setting of this track, V_h is a finite-dimensional function space on a cyclic group $\mathbb{Z}_{M(h)}$ or on a finite index set embedded in such a cyclic group. A DOC-admissible operator is then a symmetric circulant averaging operator.

Definition 3.6 (DOC-admissible operator stage)**.** Let $V_h = \mathbb{R}^{\mathbb{Z}_{M(h)}}$ with the standard inner product. An operator $T_h : V_h \to V_h$ is *DOC-admissible* if there exists a window $W_h : \mathbb{Z}_{M(h)} \to \mathbb{R}_{\geq 0}$ such that:

1. T_h is convolution:

$$(T_h f)(x) = (W_h * f)(x) = \sum_{y \in \mathbb{Z}_{M(h)}} W_h(y)\, f(x - y).$$

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2. W_h has unit mass: $\sum_y W_h(y) = 1$.

3. W_h is even: $W_h(y) = W_h(-y)$.

4. Its Fourier multipliers satisfy the band constraint:

$$0 \leq \widehat{W_h}(\xi) \leq 1 \qquad \text{for all } \xi \in \mathbb{Z}_{M(h)}.$$

These are exactly the legality constraints of NT-1. Under these conditions, constants are preserved (R3), positivity is preserved (R2), and the mean-free contraction is explicit on Fourier modes (R4).

3.4.2 DOC-admissible measurement stage

Definition 3.7 (DOC-compatible window measurement). Let $V_h = \mathbb{R}^{\mathcal{I}_h}$ with finite index set $\mathcal{I}_h$. A measurement $\pi_h : V_h \to \mathbb{R}$ is *DOC-compatible* if there exists a window $W_h : \mathcal{I}_h \to \mathbb{R}_{\geq 0}$ with $\sum_i W_h(i) = 1$ such that

$$\pi_h(u) = \sum_{i \in \mathcal{I}_h} W_h(i)\, u(i).$$

3.4.3 DOC-admissible regularisation scheme (definition)

Definition 3.8 (DOC-admissible regularisation scheme). A regularisation scheme $\mathcal{R}$ is *DOC-admissible* on $\mathcal{F}_0 \subseteq \mathcal{F}$ if:

1. it satisfies axioms R1–R7 on $\mathcal{F}_0$;

2. every operator stage $T_h^{\mathcal{R}}$ is DOC-admissible in the sense of Definition 3.6;

3. every measurement stage $\pi_h^{\mathcal{R}}$ is DOC-compatible in the sense of Definition 3.7.

In particular, any scheme that relies on negative weights, non-symmetric kernels that introduce orientation bias, or multipliers exceeding 1 (amplification) is not admissible.

3.5 Reference Case: Fejér/Cesàro Smoothing as a DOC Scheme

The Fejér kernel provides the canonical example of a DOC-admissible scheme that (i) is positivity-preserving, (ii) suppresses non-principal components, and (iii) is consistent with classical convergence.

3.5.1 Fejér kernels on $\mathbb{Z}_M$

Let $M \geq 2$. Define the *Fejér multiplier* on $\mathbb{Z}_M$ at bandwidth r (with $0 \leq r \leq \lfloor M/2 \rfloor$) by

$$\widehat{F_{M,r}}(\xi) := \begin{cases} 1 - \frac{|\xi|}{r+1}, & |\xi| \leq r, \\ 0, & |\xi| > r, \end{cases}$$

where $|\xi|$ denotes the least absolute representative of $\xi \in \mathbb{Z}_M$.

Let $F_{M,r}$ be the inverse Fourier transform of $\widehat{F_{M,r}}$. Then $F_{M,r} \geq 0$, $\sum_x F_{M,r}(x) = 1$, $F_{M,r}$ is even, and $0 \leq \widehat{F_{M,r}}(\xi) \leq 1$. Hence $F_{M,r}$ is DOC-admissible.

3.5.2 Cesàro means as a finite-infinity scheme

The classical Cesàro mean of a sequence (s_N) is the average of the first N partial sums:

$$\sigma_N \;=\; \frac{1}{N} \sum_{n=0}^{N-1} s_n.$$

For Fourier series, Fejér's theorem identifies Cesàro means with convolution by the Fejér kernel. Cesàro smoothing is an admissible scheme because it is a convex average of partial approximants, hence positivity and mass preservation are built-in, and non-principal frequencies are explicitly damped.

3.5.3 Residual budget for Fejér smoothing (prototype)

Let f be a band-limited function. The Fejér approximation error admits explicit bounds of the form
$$\|F_{M,r} * f - f\| \;\leq\; \beta(r, M; f),$$

where $\beta \to 0$ as $r \to \infty$ in the non-aliasing regime $2r < M$. The Fejér scheme supplies a residual budget $\beta(h)$ that is explicit and monotone.

4 Residual Equivalence

4.1 Schemes on a fixed object and the comparison problem

Fix a formal object F with refinement domain $\mathcal{H}$ and finite spaces V_h. Let $\mathcal{R}$ and $\mathcal{R}'$ be DOC-admissible schemes applicable to F, with extracted scalars

$$A_h^{\mathcal{R}}(F) = \pi_h^{\mathcal{R}}(T_h^{\mathcal{R}} R_h(F)), \qquad A_h^{\mathcal{R}'}(F) = \pi_h^{\mathcal{R}'}(T_h^{\mathcal{R}'} R_h(F)).$$

The central question is: *When do two admissible schemes necessarily assign the same regularised value to F?*

4.2 Residual dictionaries and structural coefficient spaces

4.2.1 Residual scales

Definition 4.1 (Residual dictionary of order m). A *residual dictionary of order m* is a finite list of functions
$$\mathcal{D}_m = \{g_0, g_1, \ldots, g_m\}, \qquad g_j : \mathcal{H} \to \mathbb{R}_{>0},$$
such that:

1. $g_0(h) \equiv 1$ for all $h \in \mathcal{H}$;

2. $g_{j+1}(h) = o(g_j(h))$ as $h \to 0$ for every $0 \leq j < m$;

3. each g_j is eventually monotone on $(0, h_0]$.

Typical choices include $g_j(h) = h^{\alpha_j}$ with $0 < \alpha_1 < \cdots < \alpha_m$, or mixed families such as $h^{\alpha}(\log(1/h))^{\beta}$ with a declared lexicographic order.

4.2.2 Structural coefficient space

Definition 4.2 (Structural coefficient space)**.** Let

$$\mathbb{K}_{\text{struct}} := \text{span}_{\mathbb{Q}}\big(\mathcal{S}_{\text{struct}}\big) \subseteq \mathbb{R}$$

denote the rational span of the structural invariant alphabet of NT-3.

A scheme is said to be *structurally expanded to order m* if it provides residual expansions whose coefficients lie in $\mathbb{K}_{\text{struct}}$.

4.3 Residual expansions and scheme comparability

Definition 4.3 (Residual expansion to order m)**.** Let $\mathcal{D}_m = \{g_0, \ldots, g_m\}$ be a residual dictionary. A scheme $\mathcal{R}$ admits a *residual expansion of F to order m* in $\mathcal{D}_m$ if there exist coefficients $c_0, \ldots, c_m \in \mathbb{R}$ and a remainder $r(h)$ such that

$$A_h^{\mathcal{R}}(F) = \sum_{j=0}^{m} c_j \, g_j(h) \; + \; r(h), \qquad r(h) = o\big(g_m(h)\big) \text{ as } h \to 0. \tag{1}$$

If additionally $c_j \in \mathbb{K}_{\text{struct}}$ for all $j \le m$, we say the expansion is *structural*.

Definition 4.4 (Residual equivalence at order m)**.** Let $\mathcal{R}$ and $\mathcal{R}'$ be schemes. We say $\mathcal{R}$ and $\mathcal{R}'$ are *residual-equivalent on F at order m* (with respect to $\mathcal{D}_m$) if

$$A_h^{\mathcal{R}}(F) - A_h^{\mathcal{R}'}(F) = o\big(g_m(h)\big) \qquad (h \to 0). \tag{2}$$

4.4 UFET-compatibility implies residual equivalence

Definition 4.5 (Common target and comparison bound)**.** Let $\mathfrak{R}$ be a class of DOC-admissible schemes applicable to F. We say $\mathfrak{R}$ is *UFET-coherent on F with comparison scale g_m* if for every pair $\mathcal{R}, \mathcal{R}' \in \mathfrak{R}$ there exists a function $\Gamma_{F,\mathcal{R},\mathcal{R}'}(h)$ such that

1. $\Gamma_{F,\mathcal{R},\mathcal{R}'}(h) = o(g_m(h))$ as $h \to 0$, and

2. for all sufficiently small h,

$$\big|A_h^{\mathcal{R}}(F) - A_h^{\mathcal{R}'}(F)\big| \le \Gamma_{F,\mathcal{R},\mathcal{R}'}(h). \tag{3}$$

4.5 Uniqueness of coefficients in a residual scale

Lemma 4.1 (Uniqueness in an ordered residual scale)**.** *Let $\mathcal{D}_m = \{g_0, \ldots, g_m\}$ be a residual dictionary with $g_0 \equiv 1$ and $g_{j+1} = o(g_j)$. Suppose real numbers $a_0, \ldots, a_m$ satisfy*

$$\sum_{j=0}^{m} a_j \, g_j(h) \; = \; o\big(g_m(h)\big) \qquad (h \to 0). \tag{4}$$

Then $a_0 = a_1 = \cdots = a_m = 0$.

Proof. We proceed by induction on $k = 0, 1, \ldots, m$.

For $k = 0$: divide (4) by $g_0(h) \equiv 1$. Taking the limit $h \to 0$ yields

$$a_0 + \sum_{j=1}^{m} a_j \, g_j(h) \to 0.$$

Since each $g_j(h) \to 0$ for $j \ge 1$, we have $a_0 = 0$.

Inductive step: Assume $a_0 = \cdots = a_{k-1} = 0$ for some $1 \le k \le m$. Then (4) becomes

$$\sum_{j=k}^{m} a_j \, g_j(h) = o\big(g_m(h)\big).$$

Divide both sides by $g_k(h)$. We obtain

$$a_k + \sum_{j=k+1}^{m} a_j \, \frac{g_j(h)}{g_k(h)} = o\Big(\frac{g_m(h)}{g_k(h)}\Big).$$

Because $g_j = o(g_k)$ for all $j > k$, each ratio $g_j(h)/g_k(h) \to 0$ as $h \to 0$. Taking the limit yields $a_k = 0$.

By induction, all coefficients vanish. $\qquad\square$

4.6 The Residual Equivalence Theorem

Theorem 4.2 (Residual Equivalence). *Fix a residual dictionary $\mathcal{D}_m = \{g_0, \ldots, g_m\}$. Let $\mathfrak{R}$ be a class of DOC-admissible schemes applicable to a formal object F. Assume:*

(H1) (Structural expansion) *For each $\mathcal{R} \in \mathfrak{R}$, the extracted scalars admit a structural residual expansion to order m:*

$$A_h^{\mathcal{R}}(F) = \sum_{j=0}^{m} c_j^{\mathcal{R}} \, g_j(h) \; + \; r_{\mathcal{R}}(h), \qquad r_{\mathcal{R}}(h) = o(g_m(h)),$$

with coefficients $c_j^{\mathcal{R}} \in \mathbb{K}_{\text{struct}}$.

(H2) (UFET-coherence) *The class $\mathfrak{R}$ is UFET-coherent on F with comparison scale g_m, i.e. for every $\mathcal{R}, \mathcal{R}' \in \mathfrak{R}$,*
$$A_h^{\mathcal{R}}(F) - A_h^{\mathcal{R}'}(F) = o(g_m(h)).$$

Then for every $\mathcal{R}, \mathcal{R}' \in \mathfrak{R}$ and every $0 \le j \le m$,

$$c_j^{\mathcal{R}} = c_j^{\mathcal{R}'}.$$

In particular, the constant term

$$\mathrm{Reg}_{\text{DOC}}(F) \; := \; c_0^{\mathcal{R}}$$

is scheme-independent on $\mathfrak{R}$, and the residual coefficients $\{c_j\}_{j=1}^{m}$ are also scheme-independent and structural.

Proof. Fix $\mathcal{R}, \mathcal{R}' \in \mathfrak{R}$. Subtract the expansions (H1):

$$A_h^{\mathcal{R}}(F) - A_h^{\mathcal{R}'}(F) = \sum_{j=0}^{m} \big(c_j^{\mathcal{R}} - c_j^{\mathcal{R}'}\big) g_j(h) + \big(r_{\mathcal{R}}(h) - r_{\mathcal{R}'}(h)\big).$$

Since each remainder is $o(g_m)$, their difference is also $o(g_m)$. By (H2), the left-hand side is $o(g_m)$. Therefore

$$\sum_{j=0}^{m} \big(c_j^{\mathcal{R}} - c_j^{\mathcal{R}'}\big) g_j(h) = o(g_m(h)).$$

Apply Lemma 4.1 with $a_j = c_j^{\mathcal{R}} - c_j^{\mathcal{R}'}$. It follows that $a_j = 0$ for all $j \le m$, i.e. $c_j^{\mathcal{R}} = c_j^{\mathcal{R}'}$. $\qquad\square$

Corollary 4.3 (Residual equivalence as a contract). *If two schemes in $\mathfrak{R}$ produce different constant terms c_0 for the same object F, then at least one of the hypotheses of Theorem 4.2 fails. In the Marithmetics admissibility discipline this triggers Designed FAIL.*

4.7 Interpretation: "finite infinity" as a coefficient theorem

Theorem 4.2 converts the intuitive claim "admissible schemes agree" into a precise statement: once a residual scale and a coherence bound are fixed, the constant term and structural residual coefficients are forced and scheme-independent.

This is the mathematical backbone of "finite infinity" in Marithmetics: the object at infinity is not an additional domain element; it is the constant coefficient in an admissible residual expansion, and scheme independence is a theorem about uniqueness of coefficients in an ordered scale.

4.8 Sufficient conditions for UFET-coherence

Proposition 4.4 (Common target $\Rightarrow$ UFET-coherence). *Fix a residual dictionary $\mathcal{D}_m = \{g_0, \ldots, g_m\}$ and suppose $\beta(h) = o(g_m(h))$ as $h \to 0$. Let $\mathfrak{R}$ be a class of schemes applicable to F. Assume there exists a scalar $L(F) \in \mathbb{R}$ such that for every $\mathcal{R} \in \mathfrak{R}$,*

$$|A_h^{\mathcal{R}}(F) - L(F)| \ \leq \ \beta(h) \qquad \text{for all sufficiently small } h.$$

Then $\mathfrak{R}$ is UFET-coherent on F with comparison scale g_m.

Proof. By the triangle inequality,

$$|A_h^{\mathcal{R}}(F) - A_h^{\mathcal{R}'}(F)| \leq |A_h^{\mathcal{R}}(F) - L(F)| + |A_h^{\mathcal{R}'}(F) - L(F)| \leq 2\beta(h).$$

Since $\beta(h) = o(g_m(h))$, we have $2\beta(h) = o(g_m(h))$, establishing (H2). $\qquad \square$

4.9 Convergent objects are fixed by admissibility

Proposition 4.5 (Consistency on convergent objects). *Let F be a formal object for which the identity scheme (no regularisation) has a limit:*

$$A_h^{\mathrm{id}}(F) = \pi_h(R_h(F)) \to L \quad (h \to 0).$$

Let $\mathcal{R}$ be any DOC-admissible scheme applicable to F. Then $\mathrm{Reg}^{\mathcal{R}}(F) = L$.

Proof. This is Axiom 3.5. $\qquad \square$

5 Canonical Outputs

This section defines what NT-4 treats as the canonical mathematical output attached to a formal object F.

5.1 The DOC-regularised value

Definition 5.1 (DOC-regularised value on an admissible class). Let $\mathfrak{R}$ be a class of DOC-admissible schemes applicable to F. Fix a residual dictionary $\mathcal{D}_m = \{g_0, \ldots, g_m\}$. Suppose the hypotheses of Theorem 4.2 hold on $(F, \mathfrak{R}, \mathcal{D}_m)$. Then the *DOC-regularised value of F on $\mathfrak{R}$* (at order m) is

$$\mathrm{Reg}_{\mathrm{DOC}}(F) \ := \ c_0^{\mathcal{R}}$$

for any $\mathcal{R} \in \mathfrak{R}$, where $c_0^{\mathcal{R}}$ is the constant coefficient in the structural residual expansion (H1).

By Theorem 4.2, $\mathrm{Reg}_{\mathrm{DOC}}(F)$ is well-defined (independent of the choice of $\mathcal{R}$).

5.2 The finite infinity class (value + residual profile)

Definition 5.2 (Residual profile to order m). Under the hypotheses of Theorem 4.2, define the *residual profile of F* (at order m) to be the coefficient vector

$$\mathcal{P}_m(F) \;:=\; (c_1, \ldots, c_m) \;\in\; \mathbb{K}^m_{\text{struct}},$$

where $c_j := c_j^{\mathcal{R}}$ for any $\mathcal{R} \in \mathfrak{R}$.

Definition 5.3 (Finite infinity class to order m). The *finite infinity class of F* (relative to $\mathfrak{R}$ and $\mathcal{D}_m$) is the ordered pair

$$\mathfrak{I}_m(F) \;:=\; \big(\mathrm{Reg}_{\mathrm{DOC}}(F),\; \mathcal{P}_m(F)\big).$$

Interpretation: $\mathrm{Reg}_{\mathrm{DOC}}(F)$ is the canonical value; $\mathcal{P}_m(F)$ is the canonical residual fingerprint to the declared order. This object is finite, structural, and stable under admissible regularisation schemes.

5.3 Designed FAIL: illegal schemes are detectable by finite counterexamples

The NT-4 contract does not allow "quiet failure." If a proposed scheme is not admissible, the framework must be able to exhibit a finite witness of illegality.

5.3.1 Positivity falsifier

Proposition 5.1 (Positivity falsifier). *Let $V_h = \mathbb{R}^{\mathcal{I}_h}$ with coordinatewise order. Suppose a scheme $\mathcal{R}$ uses either:*

- *a measurement window W_h with some negative weight $W_h(i_0) < 0$, or*

- *an operator stage T_h whose matrix has a negative entry that maps a nonnegative vector to a vector with a negative coordinate.*

Then $\mathcal{R}$ violates Axiom 3.2 and this violation is witnessed by a finite counterexample at scale h.

Proof. If $W_h(i_0) < 0$, take $u = \delta_{i_0}$, the nonnegative vector supported at i_0. Then $\pi_h^{\mathcal{R}}(u) = \sum_i W_h(i)u(i) = W_h(i_0) < 0$, contradicting R2. The witness is finite at the given h. $\qquad\square$

5.3.2 Mass/mean falsifier

Proposition 5.2 (Mass falsifier). *Suppose a scheme $\mathcal{R}$ fails mass preservation: there exists h such that either $T_h^{\mathcal{R}} \mathbf{1}_h \neq \mathbf{1}_h$ or $\pi_h^{\mathcal{R}}(\mathbf{1}_h) \neq 1$. Then $\mathcal{R}$ violates Axiom 3.3 and the violation is witnessed by the constant object $F = \mathbf{1}$.*

5.3.3 Spectral falsifier

Proposition 5.3 (Spectral falsifier). *Let $V_h = \mathbb{R}^{\mathbb{Z}_{M(h)}}$. Suppose T_h is circulant and diagonalised by the discrete Fourier basis with multipliers $\widehat{K}_h(\xi)$. If there exists $\xi_0 \neq 0$ with $\widehat{K}_h(\xi_0) > 1$, then T_h violates the DOC admissibility condition and therefore violates Axiom 3.4. The violation is witnessed by a single Fourier mode input.*

Proof. Let f_{ξ_0} be the real Fourier mode with frequency ξ_0, normalised so $\|f_{\xi_0}\| = 1$ and $\langle f_{\xi_0}, \mathbf{1}\rangle = 0$. Then $f_{\xi_0} \in V_h^0$ and $T_h f_{\xi_0} = \widehat{K}_h(\xi_0) f_{\xi_0}$. Hence $\|T_h f_{\xi_0}\| = \widehat{K}_h(\xi_0)\|f_{\xi_0}\| > 1$, contradicting contraction of mean-free components. $\qquad\square$

5.4 Designed FAIL: scheme dependence is detectable by coefficient disagreement

Proposition 5.4 (Coefficient disagreement falsifier)**.** *Fix $\mathcal{D}_m$. Let $\mathcal{R}, \mathcal{R}'$ be schemes that each claim a structural expansion of F to order m. If there exists $j \leq m$ such that the claimed coefficients differ,*

$$c_j^{\mathcal{R}} \neq c_j^{\mathcal{R}'},$$

then at least one of the following must hold:

1. *at least one claimed expansion is not valid (the remainder is not $o(g_m)$); or*

2. *UFET-coherence (H2) fails at order m.*

In either case, the discrepancy yields a falsifier certificate: a finite scale sequence $h_n \to 0$ for which the difference between schemes fails to be $o(g_m(h_n))$.

6 Canonical Examples

The role of the examples is to show, in fully explicit finite algebra, how "infinite" statements become residual bookkeeping. Each example will be presented in the finite infinity form:

$$A_h(F) = \mathrm{Reg}_{\mathrm{DOC}}(F) + \eta_F(h),$$

with an explicit residual schedule $\eta_F(h)$ controlled by a budget $\beta(h) \to 0$.

6.1 Zeno-type geometric series (exact residual law)

Consider the formal object

$$F := \sum_{n=1}^{\infty} 2^{-n}.$$

Classically, $F = 1$. We show that in the finite infinity framework, the value 1 is not a metaphysical claim; it is the constant term of an explicit residual law.

Refinement parameter. Let $N \in \mathbb{N}$ be the truncation index and set $h = 2^{-N}$ (a dyadic refinement scale). Define $V_h = \mathbb{R}$ and approximant

$$R_h(F) := S_N := \sum_{n=1}^{N} 2^{-n}.$$

Extraction is identity $\pi_h = \mathrm{id}$. No operator stage is required; take $T_h = \mathrm{id}$.

Computation. For $N \geq 1$,

$$S_N = \frac{1}{2} + \frac{1}{4} + \cdots + 2^{-N} = \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^N\right)}{1 - \frac{1}{2}} = 1 - 2^{-N}.$$

Therefore $A_h(F) = S_N = 1 - 2^{-N}$. With $h = 2^{-N}$, we have

$$A_h(F) = 1 - h. \tag{5}$$

Finite infinity representation. Equation (5) is already of the required form with

$$\mathrm{Reg}_{\mathrm{DOC}}(F) = 1, \qquad \eta_F(h) = -h.$$

Residual budget. Take $\beta(h) = h$. Then $|\eta_F(h)| \leq \beta(h)$ and $\beta(h) \to 0$ as $h \to 0$.

Interpretation (Zeno). The "infinite" sum is the statement that the residual $\eta_F(h)$ vanishes under refinement. There is no paradox: the distance "remaining" after N steps is exactly $h = 2^{-N}$, and the total distance is the constant term 1.

6.2 Grandi's series (Cesàro/Fejér as a DOC-admissible scheme)

Consider the formal object

$$F := \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots.$$

The classical partial sums do not converge, but the Cesàro means do.

6.2.1 Formal object and refinement

Let $N \in \mathbb{N}$ and set $h := \frac{1}{N+1}$. Define $V_h := \mathbb{R}^{N+1}$. Define the partial sums

$$s_k := \sum_{n=0}^{k} (-1)^n, \qquad k = 0, 1, \ldots, N,$$

and define the approximant vector $R_h(F) := (s_0, s_1, \ldots, s_N) \in V_h$.

Define the (uniform) measurement functional

$$\pi_h(u) := \frac{1}{N+1} \sum_{k=0}^{N} u_k.$$

The extracted approximant scalar under this Cesàro scheme is

$$A_h(F) := \pi_h(R_h(F)) = \frac{1}{N+1} \sum_{k=0}^{N} s_k. \tag{6}$$

6.2.2 Computation of the partial sums

For $k \geq 0$, $s_k = \sum_{n=0}^{k} (-1)^n$. Thus

$$s_k = \begin{cases} 1, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases} \tag{7}$$

6.2.3 Cesàro mean and exact residual law

Insert (7) into (6). The sum $\sum_{k=0}^{N} s_k$ counts how many even integers lie in $\{0, 1, \ldots, N\}$.

Case 1: N is odd, $N = 2m + 1$. The even indices are $0, 2, \ldots, 2m$, which are $m + 1$ values. Therefore

$$\sum_{k=0}^{2m+1} s_k = m + 1, \qquad A_h(F) = \frac{m+1}{2m+2} = \frac{1}{2}.$$

Case 2: N is even, $N = 2m$. The even indices are $0, 2, \ldots, 2m$, which are $m + 1$ values. Therefore

$$A_h(F) = \frac{m+1}{2m+1} = \frac{1}{2} + \frac{1}{2(N+1)}.$$

With $h = \frac{1}{N+1}$, we obtain the explicit finite-infinity representation:

$$A_h(F) = \frac{1}{2} + \eta_F(h), \qquad \eta_F(h) = \begin{cases} 0, & N \text{ odd}, \\ \frac{h}{2}, & N \text{ even}. \end{cases} \tag{8}$$

In particular, $|\eta_F(h)| \leq \frac{h}{2}$ for all $N \geq 0$.

Therefore:
$$\text{Reg}_{\text{DOC}}(F) = \frac{1}{2}, \qquad \beta(h) = \frac{h}{2}$$

is a valid residual budget for this scheme.

Interpretation. The "assigned value" $\frac{1}{2}$ is not a convention. It is the constant term in an exact residual law (8). The residual is explicitly computable and audit-visible, and it vanishes under refinement.

6.3 Gibbs-type overshoot (Dirichlet vs. Fejér as legality vs. illegality)

This example cleanly distinguishes admissible smoothing (DOC-compatible, positivity-preserving) from inadmissible smoothing (sign-changing kernels). The classical "Gibbs phenomenon" is then interpreted as a designed-FAIL signature of illegality.

6.3.1 Finite cyclic convolution and kernel legality

Let $M \geq 2$ and consider the finite cyclic group $\mathbb{Z}_M$. For a kernel $W : \mathbb{Z}_M \to \mathbb{R}$ and a signal $f : \mathbb{Z}_M \to \mathbb{R}$, define convolution

$$(W * f)(x) := \sum_{y \in \mathbb{Z}_M} W(y) f(x - y).$$

Definition 6.1 (Averaging kernel). A kernel W is an *averaging kernel* if:

$$W(y) \geq 0 \text{ for all } y, \qquad \sum_{y \in \mathbb{Z}_M} W(y) = 1.$$

Lemma 6.1 (Range preservation under averaging kernels). *If W is an averaging kernel and f satisfies $m \leq f(x) \leq M_0$ for all $x \in \mathbb{Z}_M$, then*

$$m \leq (W * f)(x) \leq M_0 \qquad \text{for all } x \in \mathbb{Z}_M.$$

Proof. For each fixed x, $(W * f)(x)$ is a convex combination of the values $\{f(x - y)\}_y$ with nonnegative weights summing to 1. A convex combination of numbers in $[m, M_0]$ lies in $[m, M_0]$. $\square$

This lemma is the formal reason DOC forbids negative kernel weights: negative weights destroy range preservation, making overshoot inevitable.

6.3.2 Negative weights force overshoot

Proposition 6.2 (Sign-changing kernels necessarily overshoot on a $\{0, 1\}$ input). *Let $W : \mathbb{Z}_M \to \mathbb{R}$ satisfy $\sum_y W(y) = 1$. If W takes at least one negative value, then there exists a function $f : \mathbb{Z}_M \to \{0, 1\}$ such that $(W * f)(0) > 1$.*

Proof (constructive). Assume there exists y with $W(y) < 0$. Define $f : \mathbb{Z}_M \to \{0, 1\}$ by

$$f(-y) = \begin{cases} 1, & W(y) > 0, \\ 0, & W(y) \leq 0. \end{cases}$$

Then

$$(W * f)(0) = \sum_{y \in \mathbb{Z}_M} W(y) f(-y) = \sum_{W(y) > 0} W(y).$$

Write $W = W_+ - W_-$ where $W_+(y) = \max\{W(y), 0\}$ and $W_-(y) = \max\{-W(y), 0\}$. Then

$$\sum_y W_+(y) = 1 + \sum_y W_-(y) > 1,$$

because $\sum_y W_-(y) > 0$ when any negative value exists. Hence $(W * f)(0) > 1$. $\qquad\square$

Interpretation. Any sign-changing kernel of total mass 1 will produce overshoot on bounded signals. This is a finite algebraic fact.

6.3.3 Dirichlet truncation is not DOC-admissible

In Fourier analysis, the sharp partial sum operator (Dirichlet truncation) corresponds to a kernel that is not everywhere nonnegative. Therefore, Dirichlet truncation violates DOC positivity (Axiom 3.2) and is inadmissible as a Marithmetics regularisation scheme. The classical Gibbs phenomenon is forced behaviour for step-like signals under such a kernel.

6.3.4 Fejér smoothing is DOC-admissible and eliminates overshoot

The Fejér kernel $F_{M,r}$ is nonnegative and has unit mass. Therefore it is an averaging kernel and satisfies the DOC legality conditions.

Let $f : \mathbb{Z}_M \to [0, 1]$. Then by Lemma 6.1,

$$0 \le (F_{M,r} * f)(x) \le 1 \qquad \text{for all } x,$$

so Fejér smoothing cannot overshoot the range.

6.3.5 Conclusion of the Gibbs example

The Marithmetics interpretation is:

- Gibbs-type overshoot is a diagnostic signature of using an inadmissible kernel (sign-changing weights).

- The admissible class (DOC) excludes such kernels, and within the admissible class the approximation is range-preserving and admits canonical finite-infinity interpretation.

6.4 DRPT tiling as finite infinity by stabilization

The previous examples treated "infinity" as an actual refinement limit. The DRPT substrate (NT-2) exhibits a stronger phenomenon: the relevant infinite axis (exponent) stabilizes to a finite codebook after a finite transient. The residual becomes exactly zero beyond a cutoff.

6.4.1 DRPT codebook theorem

Fix a base $b \ge 2$ and $d = b - 1$. Consider the DRPT plate

$$T_d(x, k) := x^k \bmod d, \qquad x \in \mathbb{Z}_d, \ k \in \mathbb{N}_{\ge 1}.$$

Factor $d = \prod_{j=1}^r p_j^{a_j}$. Define the transient bound $K_0(d) := \max_{1 \le j \le r} a_j$, and let $\lambda(d)$ be the Carmichael exponent of $U_d = (\mathbb{Z}_d)^\times$.

Theorem 6.3 (Finite codebook tiling of DRPT columns)**.** *For every residue $x \in \mathbb{Z}_d$ and every integer $k \ge K_0(d)$,*

$$T_d(x, k + \lambda(d)) = T_d(x, k).$$

Consequently, the infinite family of exponent columns $\{T_d(\cdot, k)\}_{k \ge 1}$ is determined by the finite set of columns $k \in \{1, 2, \ldots, K_0(d) + \lambda(d)\}$.

Proof. By the Chinese remainder theorem, $\mathbb{Z}_d \cong \prod_{j=1}^{r} \mathbb{Z}_{p_j^{a_j}}$ and the congruence $x^{k+\lambda(d)} \equiv x^k$ (mod d) holds if and only if it holds modulo each prime power $p_j^{a_j}$.

Fix j. Consider x modulo $p_j^{a_j}$.

Case 1: $p_j \nmid x$. Then x is a unit modulo $p_j^{a_j}$. By definition of the Carmichael exponent, $x^{\lambda(d)} \equiv 1 \pmod{p_j^{a_j}}$. Therefore $x^{k+\lambda(d)} \equiv x^k \pmod{p_j^{a_j}}$ for every $k \geq 1$.

Case 2: $p_j \mid x$. Then $v_{p_j}(x^k) \geq k$. For $k \geq a_j$, we have $p_j^{a_j} \mid x^k$, hence $x^k \equiv 0 \pmod{p_j^{a_j}}$. In particular, for $k \geq K_0(d) \geq a_j$,

$$x^{k+\lambda(d)} \equiv 0 \equiv x^k \pmod{p_j^{a_j}}.$$

Thus the congruence holds modulo $p_j^{a_j}$ whenever $k \geq K_0(d)$. By CRT it holds modulo d. $\square$

6.4.2 Finite-infinity representation for stabilized observables

Let $N_0 := K_0(d) + \lambda(d)$. Any field computed from the DRPT plate values tiles with period $\lambda(d)$ beyond the transient.

Corollary 6.4 (Finite infinity representation with zero tail residual). *Let $N_0 = K_0(d) + \lambda(d)$ and $h = 1/N$. Define*

$$\mathrm{Reg}_{\mathrm{DOC}}(F) := A_{N_0}(F), \qquad \eta_F(h) := A_N(F) - A_{N_0}(F).$$

Then $\eta_F(h) = 0$ for all $N \geq N_0$, hence $\eta_F(h) \to 0$ as $h \to 0$.

Interpretation. For DRPT tiling observables, "infinity" is not a limiting value reached asymptotically—it is a stabilized value reached exactly after a finite cutoff. The residual law exists, but the tail residual is identically zero. This is the strongest form of finitary canonicity.

6.5 Summary of the Example Class

The canonical examples represent three distinct modes of finite infinity:

1. **Convergent mode (Zeno/geometric):** $A_h(F) = c_0 + c_1 g_1(h)$ with a smooth residual schedule $g_1(h) \to 0$.

2. **Averaging-regularised mode (Grandi/Cesàro/Fejér):** the raw approximant fails to converge, but a DOC-admissible averaging scheme produces a controlled residual law with stable constant term.

3. **Stabilized mode (DRPT tiling):** beyond a finite transient, the object tiles exactly and the residual tail is identically zero.

These modes cover the major "paradox classes" that historically motivated informal reasoning about infinity: geometric completion (Zeno), oscillatory nonconvergence (Grandi), and overshoot artifacts of illegal truncation (Gibbs/Dirichlet).

7 Conclusion: Residual Equivalence as the Canonical Infinity Contract

Classical analysis often treats infinity as an actual domain extension. Marithmetics treats infinity as a refinement parameter controlling a family of finite objects. In this setting:

1. A regularised value is meaningful only as the constant term of a residual law.

2. Scheme independence is meaningful only as a coefficient uniqueness theorem in an ordered residual scale.

3. Legality is meaningful only as a finite set of operator/measurement constraints that can be falsified by finite witnesses.

NT-4 formalizes these principles in a way that is compatible with the rest of the suite:

- **NT-1** supplies the DOC legality model for operators (positivity, symmetry, mass preservation, spectral contraction).

- **NT-2** supplies the DOC-compatible measurement model on finite index sets (windowed averages and rigidity).

- **NT-3** supplies the structural invariant algebra and the selection discipline that prevents representation drift.

- **NT-4** supplies the residual contract that allows finite mathematics to speak about "infinite" refinement without importing scheme-dependent ambiguity.

The Residual Equivalence Theorem (Theorem 4.2) is the central structural result: once an admissible class is declared and a residual dictionary is fixed, the coefficient vector $(c_0, \ldots, c_m)$ is forced. This is the mathematical meaning of "finite infinity" in Marithmetics.

In downstream papers, this contract will be used in two ways:

1. **Finite-to-continuum transfer:** to guarantee that continuum-limit observables extracted from admissible discretisations are canonical and scheme-independent.

2. **Designed-FAIL enforcement:** to ensure that any proposed scheme that changes a regularised value must be illegal, incoherent at the declared order, or non-structural in its expansion.

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