Approximation Algorithms:
The Primal-Dual Method

My T. Thai

Overview of the Primal-Dual Method

Consider the following primal program, called $\mathcal{P}$:

$$\begin{array}{lll} \min & \sum_{j=1}^{n} c_j x_j & \\ \text{st} & \sum_{j=1}^{n} a_{ij} x_j \geq b_i & i = 1, \ldots, m \\ & x_j \geq 0 & j = 1, \ldots, n \end{array}$$

Then the dual program $\mathcal{D}$ is:

$$\begin{array}{lll} \max & \sum_{i=1}^{m} b_i y_i & \\ \text{st} & \sum_{i=1}^{m} a_{ij} y_i \leq c_j & j = 1, \ldots, n \\ & y_i \geq 0 & i = 1, \ldots, n \end{array}$$

Recall the **Complementary slackness conditions**:

- **Primal complementary slackness conditions:**
  For each $1 \leq j \leq n$*: either* $x_j = 0$ *or* $\sum_{i=1}^{m} a_{ij} y_i = c_j$*; and*
- **Dual complementary slackness conditions:**
  For each $1 \leq i \leq m$*: either* $y_i = 0$ *or* $\sum_{j=1}^{n} a_{ij} x_j = b_i$

Given an optimization problem (NP-hard), formulate this problem as an IP and relax it to obtain and LP. In the rounding techniques, we round the optimal solution x^* of the LP according to some fashions to obtain the integral solution. In the primal-dual method, we find a feasible integral solution to the LP (thus to the IP) from the scratch (instead of solving the LP) using the Dual $\mathcal{D}$ as our guiding. More specifically:

Instead of ensuring both conditions of complementary slackness conditions, in the approximation design, we do either:

1. Ensure the primal conditions and suitably relax the dual conditions:
   For each $1 \le i \le m$*: either* $y_i = 0$ *or* $b_i \le \sum_{j=1}^{n} a_{ij}x_j \le \beta b_i$ *where* $\beta > 1$
2. Ensure the dual conditions and suitably relax the primal conditions:
   For each $1 \le j \le n$*: either* $x_j = 0$ *or* $c_j/\alpha \le \sum_{i=1}^{m} a_{ij}y_i \le c_j$ *where* $\alpha > 1$

If we use the first way, that is, ensure the primal conditions and relax the dual conditions, we have:

Lemma 1 *If x and y are the feasible solutions of $\mathcal{P}$ and $\mathcal{D}$ respectively satisfying the conditions in the first way, then*

$$\sum_{j=1}^{n} c_j x_j \le \beta \sum_{i=1}^{m} b_i y_i$$

Proof.

$$\begin{aligned} \textstyle\sum_{j=1}^{n} c_j x_j &= \textstyle\sum_{j=1}^{n}(\sum_{i=1}^{m} a_{ij}y_i)x_j \\ &= \textstyle\sum_{i=1}^{m}(\sum_{j=1}^{n} a_{ij}x_j)y_i \\ &\le \beta \textstyle\sum_{i=1}^{m} b_i y_i \end{aligned}$$

More generally, let $\alpha = 1$ if the primal conditions are ensured and $\beta = 1$ if the dual conditions are ensure, we have:

Primal complementary slackness conditions

Let $\alpha \ge 1$

For each $1 \le j \le n$*: either* $x_j = 0$ *or* $c_j/\alpha \le \sum_{i=1}^{m} a_{ij}y_i \le c_j$

Dual complementary slackness conditions

Let $\beta \ge 1$

For each $1 \le i \le m$*: either* $y_i = 0$ *or* $b_i \le \sum_{j=1}^{n} a_{ij}x_j \le \beta b_i$

Lemma 2 *If x and y are the feasible solutions of $\mathcal{P}$ and $\mathcal{D}$ respectively satisfying the conditions stated above then:*

$$\sum_{j=1}^{n} c_j x_j \le \alpha \cdot \beta \sum_{i=1}^{m} b_i y_i$$

Primal-Dual based Approximation Algorithms:

1. Formulate a given problem as an IP. Relax the variable constrains to obtain the primal LP $\mathcal{P}$, then find the dual $\mathcal{D}$

2. Starts with a primal infeasible solution x and a dual feasible solution y, usually $x = 0$ and $y = 0$.

3. Until x is feasible, do

   (a) Increase the value y_i in some fashion until some dual constraints go tight, i.e, $\sum_{i=1}^{m} a_{ij} y_i = \alpha c_j$ for some j while always maintaining the feasibility of y. In other words, iteratively improves the optimality of the dual solution.

   (b) Select some subset of tight dual constraints and increase the values of primal variables corresponding to these constraints by an *integral* amounts. (Why by integral amounts? This ensures that the final solution x is integral.)

4. The cost of the dual solution is used as a lower bound on OPT. Note that the approximation guarantee of the algorithm is $\alpha\beta$.

Weighted Vertex Cover via Primal-Dual Method

WEIGHTED VERTEX COVER (WVC): Given an undirected graph $G = (V, E)$ where $|V| = n$ and $|E| = m$ and a cost function on vertices $c : V \to \mathbb{Q}^+$, find a subset $C \subseteq V$ such that every edge $e \in E$ has at least one endpoint in C and C has a minimum cost.

Formulate VC as the following IP:

For each vertex $i \in V$ ($V = \{1, 2, \ldots, n\}$), let $x_i \in \{0, 1\}$ be variables such that $x_i = 1$ if $i \in C$; otherwise, $x_i = 0$. We have:

$$\begin{array}{ll} \min & \sum_{i=1}^{n} c_i x_i \\ \text{st} & x_i + x_j \geq 1 \qquad \forall (i,j) \in E \\ & x_i \in \{0,1\} \qquad \forall i \in V \end{array} \tag{1}$$

The corresponding primal LP $\mathcal{P}$ of the above IP is as follows:

$$\begin{array}{ll} \min & \sum_{i=1}^{n} c_i x_i \\ \text{st} & x_i + x_j \geq 1 \qquad \forall (i,j) \in E \\ & x_i \geq 0 \qquad \forall i \in V \end{array} \tag{2}$$

Assign the dual variable y_{ij} to the constraint $x_i + x_j \geq 1$. We have the corresponding dual $\mathcal{D}$:

$$\begin{array}{ll} \max & \sum_{(i,j) \in E} y_{ij} \\ \text{st} & \sum_{j:(i,j) \in E} y_{ij} \leq c_i \qquad \forall i \in V \\ & y_{ij} \geq 0 \qquad \forall (i,j) \in E \end{array} \tag{3}$$

Let choose $\alpha = 1$ and $\beta > 1$, that is to ensure the primal conditions and suitably relax the dual conditions:
For each vertex $i \in E$: either $x_i = 0$ or $\sum_{j:(i,j) \in E} y_{ij} = c_i$
For each edge $(i,j) \in E$: either $y_{ij} = 0$ or $1 \leq x_i + x_j \leq \beta$ where $\beta > 1$

Therefore, when $x_i \neq 0$, then $\sum_{j:(i,j) \in E} y_{ij} = c_i$. When $\sum_{j:i \in (i,j)} y_{ij} = c_i$ for some i, we said this constraint goes *tight*.

Algorithm 1 Primal-Dual Algorithm for WVC

Initialize $x = 0$ and $y = 0$
while $E \neq \emptyset$ **do**
  Select nonempty set $E' \subseteq E$
  Raise y_{ij} for each edge $(i,j) \in E'$ until some dual constraints go tight. That is $\sum_{j:i\in(i,j)} y_{ij} = c_i$ for some i
  Let S be the set of vertices i corresponding to dual constraints that just went tight
  For each $i \in S$, set $x_i = 1$ and delete all edges (i,j) from E, that is, delete all edges incident to vertices in S
end while
Return set $C = \{i \mid x_i = 1\}$

Lemma 3 *Let x and y the the solutions obtained from the above algorithm, then x is primal feasible and y is dual feasible.*

Proof. Note that each edge (i,j) removed from E is incident on some vertex i such that $x_i = 1$. Additionally, the loop is terminated when all edges have been removed. Therefore, $x_i + x_j \geq 1$ for all $(i,j) \in E$, that is, x is feasible to $\mathcal{P}$.

Likewise, once the constraint goes tight for some i, that is $\sum_{j:(i,j)\in E} y_{ij} = c_i$, the algorithm removes these edges. Therefore, none of the value y_{ij} exceeding the value c_i. Hence, y is feasible to $\mathcal{D}$.

□

Theorem 1 *The above algorithm produces a vertex cover C with an approximation ratio 2.*

Proof. Let OPT be the cost of an optimal vertex cover. We have:

$$\text{cost}(C) = \sum_{i\in C} c_i x_i = \sum_{i\in C} c_i = \sum_{i\in C} \left(\sum_{j:(i,j)\in E} y_{ij} \right)$$

The last equality follows from the fact that we set $x_i = 1$ only for vertices i corresponding to tight dual constraints, i.e., $i \in C$ implies $\sum_{j:(i,j)\in E} y_{ij} = c_i$.

Also, note that:

$$\sum_{i\in C} \left(\sum_{j:(i,j)\in E} y_{ij} \right) = \sum_{(i,j)\in E} \left(\sum_{i\in C\cap\{i,j\}} 1 \right) y_{ij} \leq 2 \sum_{(i,j)\in E} y_{ij}$$

The last inequality follows from the fact that $|(i,j)| = 2$ for all edge (i,j), i.e., each edge has two endpoints, so we may count y_{ij} twice for each edge (i,j).

Therefore, we conclude that:

$$\text{cost}(C) \leq 2 \sum_{(i,j) \in E} y_{ij} \leq 2OPT(\mathcal{D}) \leq 2OPT$$

$\square$

Weighted Set Cover via Primal-Dual Method

WEIGHTED SET COVER: Given a universe $U = \{1, \dots, m\}$, a collection $\mathcal{S}$ of subsets of U, $\mathcal{S} = \{S_1, \dots, S_n\}$, and a cost function $c : \mathcal{S} \to \mathbb{Q}^+$, find a minimum cost sub-collection $\mathcal{C} = \{S_j \mid 1 \leq j \leq n\}$ such that $\mathcal{C}$ covers all elements of U.

Let c_j be the weight of subset S_j. Let x_j be a binary variables such that $x_j = 1$ if $S_j \in \mathcal{C}$; otherwise, $x_j = 0$. We have the corresponding IP:

$$\begin{array}{ll} \min & \sum_{j=1}^{n} c_j x_j \\ \text{st} & \sum_{i \in S_j} x_j \geq 1 \qquad \forall i \in \{1, ..., m\} \\ & x_j \in \{0, 1\} \qquad \forall j \in \{1, \dots, n\} \end{array} \tag{4}$$

The corresponding primal LP $\mathcal{P}$:

$$\begin{array}{ll} \min & \sum_{j=1}^{n} c_j x_j \\ \text{st} & \sum_{j : i \in S_j} x_j \geq 1 \qquad \forall i \in \{1, \dots, m\} \\ & x_j \geq 0 \qquad \forall j \in \{1, \dots, n\} \end{array} \tag{5}$$

Let y_i corresponding to the constraint $\sum_{i \in S_j} x_j \geq 1$, we have the following dual $\mathcal{D}$:

$$\begin{array}{ll} \max & \sum_{i=1}^{m} y_i \\ \text{st} & \sum_{i \in S_j} y_i \leq c_j \qquad \forall j \in \{1, \dots, n\} \\ & y_i \geq 0 \qquad \forall i \in \{1, \dots, m\} \end{array} \tag{6}$$

Recall that we could obtain an f-approximation algorithm using the LP rounding technique where $f = \max_i |\{S_j \mid i \in S_j\}|$. Assume that we also want to find an f-approximation algorithm using the primal-dual method. Let set $\alpha = 1$ and $\beta = f$, that is, to ensure the primal conditions while relaxing the dual conditions:

For each $1 \leq j \leq n$*: either* $x_j = 0$ *or* $\sum_{i \in S_j} y_i = c_j$
For each $1 \leq i \leq m$*: either* $y_i = 0$ *or* $\sum_{j: i \in S_j} x_j \leq f$

Remarks: By definition of f, the second condition is always satisfied no matter how we set x_j.

For some j, if the dual constraint $\sum_{i \in S_j} y_i = c_j$, we say that this constraint goes tight and the corresponding S_j is called tight. We have the following algorithm:

Algorithm 2 Weighted Set Cover via Primal-Dual

```
Initialize x = 0 and y = 0
while U ≠ ∅ do
    Choose an uncovered element, say i, and raise y_i until some set in 𝒮 goes tight, say S_j
    Choose all these sets S_j and set x_j = 1
    Remove all the elements in these sets S_j from U
    Remove all these sets S_j from the collections 𝒮
end while
Return 𝒞 = {S_j |x_j = 1}
```

Theorem 2 *The above algorithm achieves an approximation factor of* f

Proof. The proof is straightforward. We only need to show that x and y obtained from the above algorithm are feasible (to the primal and dual respectively). Since we choose $\alpha = 1$ and $\beta = f$, it follows that the approximation ratio is f.

$\square$

GENERAL COVER

Definition 1 *Given a collection of multisets of U (rather than a collection of sets). A* multiset *contains a specified number of copies of each element. Let* a_{ij} *denote the multiplicity of element i in multiset* S_j*. Find a sub-collection* $\mathcal{C}$ *such that the weight of* $\mathcal{C}$ *is minimum and* $\mathcal{C}$ *covers each element i* b_i *times. (rather than just one time)*

The corresponding IP:

$$\begin{array}{ll} \min & \sum_{j=1}^{n} w_j x_j \\ \text{st} & a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i \quad \forall i \in \{1, ..., m\} \\ & x_j \in \{0, 1\} \quad \forall j \in \{1, ..., n\} \end{array} \tag{7}$$

The corresponding LP:

$$\begin{array}{ll} \min & \sum_{j=1}^{n} w_j x_j \\ \text{st} & a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i \quad \forall i \in \{1, ..., m\} \\ & 0 \leq x_j \leq 1 \quad \forall j \in \{1, ..., n\} \end{array} \tag{8}$$

Question: Can we use the technique in the WSC problem to obtain an f-approximation algorithm for the General Cover problem where $f = \max_i \sum_j a_{ij}$?

Minimum Steiner Forest via Primal-Dual

In this lecture, we study two new techniques of the primal-dual method:

1. Increasing simultaneously multiple dual variables
2. Refining the feasible solutions using the reverse delete step

Let's learn these through the Minimum Steiner Forest (MSF) problem, defined as follows:
Minimum Steiner Forest (MSF): Given an undirected graph $G = (V, E)$, a cost function $c : E \to \mathbb{Q}^+$, and a collection of disjoint subsets of V, $S_1, \ldots, S_k$, find a minimum cost subgraph of G in which each pair of vertices belonging to the same S_i is connected. Such a subgraph is cycle-free and is called a Steiner Forest.

Formulate the MSF problem as an IP

For any $X \subseteq V$, we set $f(X) = 1$ iff there exists a $u \in X$ and $v \in \bar{X}$ such that u and v belong to some set S_i. Otherwise, $f(X) = 0$. In other words, we say X separates S_i.

Let $\delta(X)$ be the set of edges with exactly one endpoint in X. In other words, $\delta(X)$ is the set of edges that cross the cut $(X, \bar{X})$: $\delta(X) = [X, \bar{X}]$.

Let a binary variable x_e indicate whether the edge e is chosen in the subgraph (min cost forest) for each edge $e \in E$.

Main Observation: Given a set X that separates S_i, any Steiner tree connecting S_i must use at least one edge in $\delta(X)$. So, we formulate MSF as the following IP:

$$\begin{array}{ll} \min & \sum_{e \in E} c_e x_e \\ \text{st.} & \sum_{e : e \in \delta(X)} x_e \geq f(X) \quad \forall X \subseteq V \\ & x_e \in \{0, 1\} \end{array} \tag{9}$$

Then the corresponding LP $\mathcal{P}$ is:

$$
\begin{array}{ll}
\min & \sum_{e \in E} c_e x_e \\
\text{st.} & \sum_{e: e \in \delta(X)} x_e \geq f(X) \quad \forall X \subseteq V \\
& x_e \geq 0
\end{array} \tag{10}
$$

Assign the dual variable y_X for the constraint $\sum_{e:e\in\delta(X)} x_e \geq f(X) \quad \forall X \subseteq V$, we have the following dual $\mathcal{D}$.

$$
\begin{array}{ll}
\max & \sum_{X \subseteq V} \quad f_X y_X \\
\text{st.} & \sum_{X: e \in \delta(X)} y_X \leq c_e \quad \forall e \in E \\
& y_X \geq 0
\end{array} \tag{11}
$$

A 2-Approximation Algorithm

Let choose $\alpha = 1$ and $\beta = 2$, that is:

Primal Conditions: For each $e \in E$, $x_e \neq 0 \Rightarrow \sum_{i:e\in\delta(X)} y_X = c_e$.

Relaxed Dual Conditions: For each subset $X \subseteq V$, $y_X \neq 0 \Rightarrow \sum_{e:e\in\delta(X)} x_e \leq 2f(X)$.

Due to the objective function of $\mathcal{D}$, we just consider X such that $f(X) = 1$. Let's first introduce some terminology before describing the algorithm.

A set X is *unsatisfied* if there is no picked edge in $\delta(X)$ and $f(X) = 1$. A set X is *active* if X is unsatisfied and minimal. Let F denote the subgraph the algorithm is constructing. Based on our observation, F is a MSF iff there is no unsatisfied X.
Basic Ideas: At each iteration, *simultaneously* increase y_X of all active set X's at the same rate until some edge e goes tight, that is $\sum_{X:e\in\delta(X)} y_X = c_e$. Then add this tight edge e to F. Note that F may contains some *redundant* edges, that is edges whose deletion still makes F feasible. Therefore, at the last step, after constructing F, we will run a *reverse delete step.* The algorithm is described as follows:

To understand this algorithm, let's run this algorithm on a nontrivial example. I will use an example in the book, page 201 and walk through it in

Algorithm 3 2-Approximation Algorithm for FST via Dual-Primal

Initialize $F \leftarrow \emptyset$; $y \leftarrow 0$; $j \leftarrow 0$
while There exists an unsatisfied set X (or F is infeasible) **do**
  Let $\mathcal{X}$ be the set of all active X
  Simultaneously increase y_X at the same rate for each active set $X \in \mathcal{X}$ until some edge e goes tight
  Refer to e as e_j
  $F \leftarrow F \cup \{e_j\}$
  $j++$;
end while
Reverse Delete Step: For $j = |F|$ to 1 do:
if $F - \{e_j\}$ is a primal feasible, then $F \leftarrow F - \{e_j\}$
Refer this set F (after reverse delete step) as F'
Return F'

class.

Before proving the correctness and algorithm ratio of the algorithm, let's first show it's not difficult to find all the active sets in an iteration. As mentioned, you can think of active set as a connected components of the current F. Hence, it's sufficient to consider all the connected components of the current F.

Lemma 4 *Set X is active at iteration i iff it is a connected component of F (at iteration i) and $f(X) = 1$*

Proof. ($\Rightarrow$) It's clear by the definition.

($\Leftarrow$) Suppose X is a connected component of F and $f(X) = 1$. Then X is a minimal unsatisfied set, otherwise, there is a larger connected component of F contains X. □

Lemma 5 *F' and y obtained from the above algorithm is feasible solutions (primal and dual respectively)*

Proof. First, we show F' is feasible primal solution. Let F be the result obtained right before the Reverse Delete Step. It's easy to see that F is feasible since F interconnecting all T_i and F is a acyclic. To show F is a acyclic, we note that only the dual variables of connected components are increased. Thus none of the edges in a connected component will become tight in later iterations. In the reverse delete step, we do not delete any edge along the unique path from u to v where $u, v \in T_i$. Hence, F' is still feasible.

Now, we show that y is feasible dual solution, that is, $\sum_{X:e\in\delta(X)} y_X \leq c_e, \forall e \in E$. Suppose edge e_j becomes tight at an iteration j, that is, $\sum_{X:e_j\in\delta(X)} y_X = c_{e_j}$. Then all the sets X such that $e_j \in \delta(X)$ with $f(X) = 1$ becomes satisfied and none will be active in later iterations. Consequently, all the dual variables y_X of these sets will never be increased again in the later iterations. Hence, y_X is feasible.

□

Lemma 6 *Let $deg_{F'}(X)$ denote the number of edges of F' crossing the cut $[X, \bar{X}]$, we have:*

$$\sum_{e\in F'} c_e \leq 2 \sum_{X\subseteq V} y_X$$

Proof. Since every picked edge is tight, we have:

$$\begin{aligned}
\sum_{e\in F'} c_e &= \sum_{e\in F'} \sum_{e\in\delta(X)} y_X \\
&= \sum_{X\subseteq V} \sum_{e\in\delta(X)\cap F'} y_X \\
&= \sum_{X\subseteq V} deg_{F'}(X) \cdot y_X
\end{aligned}$$

Now, we need to prove that

$$\sum_{X\subseteq V} deg_{F'}(X) \cdot y_X \leq 2 \sum_{X\subseteq V} y_X$$

Instead of showing this, we will prove a stronger result: that in each iteration, the increase in the lhs of this inequality is bounded by the increase in the rhs of the inequality.

Consider an iteration. Let Δ be the increase in the dual variables of the active sets in this iteration. (You can consider that Δ is the average of the active sets with respect to F'). Thus we want to show that:

$$\Delta \left(\sum_{X\in\mathcal{X}} deg_{F'}(X) \right) \leq 2\Delta|\mathcal{X}|$$

or

$$\left(\sum_{X \in \mathcal{X}} deg_{F'}(X) \right) \leq 2|\mathcal{X}|$$

Consider a graph F' (which is a forest). Contract all the nodes in each active set X to a single node, called *red* nodes. Delete all the singletons and call this new graph H. Let R denote the red nodes in H and B denotes the rest of nodes (not red) in H. Then the degree of the red node in H is $deg_{F'}(X)$ where X is a corresponding active set. Hence we need to prove:

$$\sum_{v \in R} deg_H(v) \leq 2|R|$$

We have:

$$\begin{aligned} \textstyle\sum_{v \in R} deg_H(v) &= 2|E(H)| - \textstyle\sum_{v \in B} deg_H(v) \\ &\leq 2(|R| + |B|) - \textstyle\sum_{v \in B} deg_H(v) \\ &\leq 2(|R| + |B|) - 2|B| = 2|R| \end{aligned}$$

The last inequality dues to the fact that because no blue vertex has degree one, otherwise, X is not an active set. (Indeed, we can prove this claim as follows)

Lemma 7 *At iteration i, let C be a connected component with respect to F. If $f(C) = 0$ then $deg_{F'}(C) \neq 1$*

Suppose that $deg_{F'}(C) = 1$, there there is a unique edge e of F' that crosses the cut $[C, \bar{C}]$. Since F' does not contain redundant edges, there must be a u and v which belong to some S_i such that $u \in C$ and $v \in \bar{C}$. This implies that $f(C) = 1$, contracting to our assumption.

□

Theorem 3 *Our algorithm obtains a steiner forest with an approximation ratio 2*

METRIC UNCAPACITATED FACILITY LOCATION (FL)

In this lecture, we will study a new algorithm which differs from the previous in two respects:

1. The primal and dual pairs have negative coefficients
2. We will relax the primal complimentary slackness conditions.

Definition 2 *Given a set F of facilities and a set C of cities. Let f_i be the cost of opening facility $i \in F$, and c_{ij} be the cost of connecting city j to (opened) facility i where c_{ij} satisfies the triangle inequality. The problem is to find a subset $O \subseteq F$ of facilities that should be opened, and an assignment function $\theta : C \to O$ assigning every city j to open facilities $\theta(j)$ in such a way that the total cost of opening facilities and connecting cities to open facilities is minimized, that is, to minimize the objective function:*

$$\sum_{i \in O} f_i + \sum_{j \in C} c_{\theta(j)j}$$

Let x_i be an indicator variable denoting whether facility i is opened and y_{ij} be an indicator variable denoting whether city j is assigned to facility i. We have the following IP:

$$\begin{array}{ll} \min & \sum_{i \in F} f_i x_i + \sum_{i \in F, j \in C} c_{ij} y_{ij} \\ \text{st} & \sum_{i \in F} y_{ij} \geq 1 \quad j \in C \\ & x_i - y_{ij} \geq 0 \quad i \in F, j \in C \\ & x_i, y_{ij} \in \{0, 1\} \quad i \in F, j \in C \end{array} \tag{12}$$

Remarks:

- The first constraint ensures that each city is connected to a facility
- The second constraint ensures this facility must be opened

We have the corresponding primal LP $\mathcal{P}$ as follows:

$$
\begin{array}{ll}
\min & \sum_{i\in F} f_i x_i + \sum_{i\in F, j\in C} c_{ij} y_{ij} \\
\text{st} & \sum_{i\in F} y_{ij} \geq 1 \quad j \in C \\
& x_i - y_{ij} \geq 0 \quad i \in F, j \in C \\
& x_i \geq 0 \quad i \in F \\
& y_{ij} \geq 0 \quad i \in F, j \in C
\end{array}
\tag{13}
$$

Then the dual $\mathcal{D}$ is:

$$
\begin{array}{ll}
\max & \sum_{j\in C} s_j \\
\text{st} & \sum_{j\in C} t_{ij} \leq f_i \quad i \in F \\
& s_j - t_{ij} \leq c_{ij} \quad i \in F, j \in C \\
& s_j \geq 0 \quad j \in C \\
& t_{ij} \geq 0 \quad i \in F, j \in C
\end{array}
\tag{14}
$$

The complementary slackness conditions of (13) and (14) are as follows:

Primal conditions:

- For all $i \in F$, if $x_i > 0$ then $\sum_{j\in C} t_{ij} = f_i$
- For all $i \in F$, $j \in C$, if $y_{ij} > 0$ then $s_j - t_{ij} = c_{ij}$

Dual conditions:

- For all $j \in C$, if $s_j > 0$ then $\sum_{i\in F} y_{ij} = 1$
- For all $i \in F$, $j \in C$, if $t_{ij} > 0$, then $x_i - y_{ij} = 0$

Observations: Let O be the set of open facilities and θ be the assignment function. That is, $x_i = 1$ iff $i \in O$ and $y_{ij} = 1$ iff $\theta(j) = i$, we have:

- If $x_i = 1$, then $\sum_{j\in C} t_{ij} = f_i$. That is the sum of all dual variables t_{ij} must fully pay the cost of opening that facility i.
- If $x_i = 1$ and $\theta(j) \neq i$, thus $y_{ij} = 0$, then $t_{ij} = 0$. That is dual variable t_{ij} does not pay the cost of opening the facilities that it (city j) does not connect to.
- If $\theta(j) = i$, thus $y_{ij} = 1$, then $s_j - t_{ij} = c_{ij}$. Rewrite this, we have $s_j = t_{ij} + c_{ij}$. So in order for a city j to connect to the facility i, it pays

the cost of connecting c_{ij} (using the edge (i,j)) and the cost of opening the facility (t_{ij}).

From these observations, we are going to relax the primal conditions (and ensure the dual conditions) as follows:

- For all $i \in O$, if $x_i > 0$ then $f_i/\alpha \leq \sum_{j \in C} t_{ij} \leq f_i$
- For all $j \in C$, if $y_{\theta(j)j} > 0$ then $(c_{\theta(j)j}/\alpha) \leq s_j - t_{\theta(j)j} \leq c_{\theta(j)j}$

The algorithm consists of two phases as follows:

Phase 1

$O' \leftarrow \emptyset$ //set of temporarily open facilities
$J \leftarrow \emptyset$ //set of all connected cities so far
$s \leftarrow 0$, $t \leftarrow 0$ //initilize s and t
while $J \neq C$ **do**
  Increase uniformly all s_j, $j \in C - J$, that is each s_j will be increased by 1 in unit time
  After a while, some edge (i,j) becomes *tight*, that is $s_j - t_{ij} = c_{ij}$.
  if $i \in O'$ **then**
    Add j into J and declare i the "connection witness" for j
  end if
  Increase uniformly t_{ij} (Note that the edge with $t_{ij} > 0$ is called "special")
  After a while, there is some i such that $\sum_{j \in C} t_{ij} = f_i$
  for each such i in any order **do**
    $O' \leftarrow O' \cup \{i\}$
    for each tight edge (i,j) with $j \notin J$ **do**
      $J \leftarrow J \cup \{j\}$ //i is called a "connection witness" for j
    end for
  end for
end while

Phase 2

Let $H = (O', C)$ be the bipartite graph containing only special edges, that is $t_{ij} > 0$
Let O be a maximal subset of O' such that there is no path of length 2 in H between any two vertices in O.
Declare all facilities in O open
for each $j \in C$ **do**
  if There exists $i \in O$ such that (i, j) is special **then**
    $\theta(j) \leftarrow i$ // j directly connected to i
  else
    Let i be the connection witness for j
    if $i \in O$ **then**
      $\theta(j) \leftarrow i$ // j directly connected to i
    else
      There must be some city $i' \in O$ within H-distance 2 from i
      $\theta(j) \leftarrow i'$ // j indirectly connected to i'
    end if
  end if
end for

Theorem 4 *The above algorithm gives an approximation ratio 3*

Proof. Let s and t be the returned dual-feasible solution. To prove the approximation ratio, we need to compare the cost of the approximated solution, which is $\sum_{i \in O} f_i + \sum_{j \in C} c_{\theta(j)j}$, to the cost of the dual-feasible solution (s, t), which is $\sum_{j \in C} s_j$.

Let $s_j = s_j^f + s_j^c$ where s_j^f denotes the cost of opening the facility and s_j^c denotes the cost of connecting j to i as discussed in the observation part. In particular, if j is directly connected to $i = \theta(j)$, then set $s_j^f = t_{ij}$ and $s_j^c = c_{ij}$. If j is indirectly connected to $i = \theta(j)$, then set $s_j^f = 0$ and $s_j^c = s_j$.

Note that if $i \in O$ and (i, j) is special, then j is directly connected to i. Hence we have:

$$\sum_{i \in O} f_i = \sum_{i \in O} \sum_{j:(i,j) special} t_{ij} = \sum_{j \in C} s_j^f$$

Now consider the case where j is indirectly connected to $i = \theta(j)$. We claim that $c_{ij} \leq 3s_j^c$. (Recall that if j is directly connected to i, then $c_{ij} = s_j^c$). When j is indirectly connect to i, there must be a city $j' \in C$ and i' which is the connection witness for j such that (i, j') and (i', j') are special. By the triangle inequality, if we can prove that all $c_{i'j}, c_{i'j'}$, and $c_{ij'}$ are at most s_j, then the claim is proven.

Since edge (i', j) is tight (i' is a connection witness for j), $c_{i'j} \leq s_j$. To prove the other two inequalities, it is sufficient to prove $s_{j'} \leq s_j$ (because $c_{ij'} \leq s_{j'}$ and $c_{i'j'} \leq s_{j'}$) . Since i' is a connection witness for j, s_j must be increased until right at or after the time i' became temporarily open. Since both edges (i', j') and (i, j') are special, $s_{j'}$ couldn't have increased after the time i' became temporarily open. Thus we have $s_{j'} \leq s_j$.

Therefore,

$$\sum_{i \in O} f_i + \sum_{j \in C} c_{\theta(j)j} \leq \sum_{j \in C} \left(s_j^f + 3s_j^c \right) \leq 3opt$$

where *opt* is the cost of the optimal solution.

□