JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 52, No. 3, MARCH 1987

Convex Programs with an Additional Reverse Convex Constraint

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Communicated by A. V. Fiacco

Abstract. A method is presented for solving a class of global optimization problems of the form (P): minimize $f(x)$, subject to $x \in D$, $g(x) \geq 0$, where D is a closed convex subset of R^n and f, g are convex finite functions R^n. Under suitable stability hypotheses, it is shown that a feasible point $\bar{x}$ is optimal if and only if $0 = \max\{g(x) : x \in D, f(x) \leq f(\bar{x})\}$. On the basis of this optimality criterion, the problem is reduced to a sequence of subproblems Q_k, $k = 1, 2, \ldots$, each of which consists in maximizing the convex function $g(x)$ over some polyhedron S_k. The method is similar to the outer approximation method for maximizing a convex function over a compact convex set.

Key Words. Reverse convex constraints, convex maximization, concave minimization, outer approximation methods.

1. Introduction

In this paper we shall be concerned with the following nonconvex optimization problem

$$\text{(P)} \quad \text{minimize } f(x),$$

$$\text{s.t.} \quad h_i(x) \leq 0, \qquad i = 1, 2, \ldots, m,$$

$$g(x) \geq 0,$$

where f, g, $h_i : R^n \to R$ are convex finite functions on R^n. Setting

$$h(x) = \max_{i=1,\ldots,m} h_i(x),$$

$$D = \{x: h(x) \leq 0\}, \qquad G = \{x: g(x) < 0\},$$

we note that the constraint set of this problem is a cavern of the form $D \setminus G$, where D is a closed convex set, G an open convex set.

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0022-3239/87/0300-0463$05.00/0 © 1987 Plenum Publishing Corporation

A simple example of this type of problems is furnished by the problem of minimizing the distance $f(x) = d(x, M)$ from a convex set M to a point $x \in R^n \setminus G$, where G is an open convex set containing M.

A large class of optimization problems, including convex minimization and convex maximization (or concave minimization) problems, can easily be cast in the form P. For instance, any problem

$$\min\{f(x) - g(x): x \in D\}, \tag{1}$$

where f, g are convex finite functions and D is a closed convex set, can be written as

$$\min\{f(x) - t: x \in D, g(x) - t \geq 0\}, \tag{2}$$

which is obviously a problem of the above type.

The main difficulty with problem P is connected with the presence of the reverse convex constraint $g(x) \geq 0$, which destroys the convexity and possibly even the connectivity of the feasible set. Optimization problems involving such reverse convex constraints were studied earlier by Rosen (Ref. 1), Avriel and Williams (Ref. 2), Mayer (Ref. 3), Ueing (Ref. 4), and more recently by Bansal and Jacobsen (Ref. 5), Hillestad and Jacobsen (Refs. 6 and 7), Tuy (Ref. 8), and Thuong (Ref. 9). Avriel and Williams (Ref. 2) showed that reverse convex constraints may occur in certain engineering design problems. Zaleesky (Ref. 10) argues that reverse convex constraints are likely to arise in many typical economic management applicatons. In an abstract setting, Singer (Ref. 11) related this type of nonconvex constraints to certain problems in approximation theory, when the set of approximation functions is the complement of a convex set.

It should be noted that, although the literature on nonconvex optimization has rapidly increased in recent years, most of the published papers either deal with the theoretical aspects of the problem or are concerned only with finding Kuhn-Tucker points or local solutions rather than global optima. A few papers (Refs. 6-9) have been devoted to the global minimization of a concave (in particular, linear) function under linear and reverse convex constraints, a problem closely related to, but not quite the same as P. Obviously, writing P in the form

$$\min\{t: f(x) \leq t,\ h(x) \leq 0,\ g(x) \geq 0\},$$

we shall convert it into a problem with a linear objective function. But to our knowledge, global optimization problems like P, where convex (nonlinear) and reverse convex constraints are copresent, have been little studied in the literature to date.

The present paper is an outgrowth of an earlier work (Ref. 12), where only the case $D = R^n$ was treated.