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The pdf of a quotient of two independent normal distributed variables:

Let X and Y be independent continuous random variables, with pdfs $f_X(x)$ and $f_Y(y)$, respectively. Assume that X is zero for at most a set of isolated points. Let W=Y/X. Then:

$$f_W(w) = \int_{-\infty}^{\infty} |x| \, f_X(x) \, f_Y(wx) \, dx \qquad (1)$$

Larsen (2012)

Knowing that the pdfs of X and Y are:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2} \qquad (2)$$

Replacing in (1):

$$f_W(w) = \int_{-\infty}^{\infty} |x| \left[\frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}\right]\left[\frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{1}{2}\left(\frac{wx-\mu_Y}{\sigma_Y}\right)^2}\right] dx$$

$$= \left(\frac{1}{2\pi\sigma_X\sigma_Y}\right)\int_{-\infty}^{\infty} |x| e^{-\underbrace{\left[\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \frac{1}{2}\left(\frac{wx-\mu_Y}{\sigma_Y}\right)^2\right]}_{u(x)}} dx \qquad (3)$$

Defining:

$$u(x) = \frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \frac{1}{2}\left(\frac{wx-\mu_Y}{\sigma_Y}\right)^2 \qquad (4)$$

Expanding the quadratic expressions:

$$u(x) = \frac{x^2 - 2x\mu_X + \mu_X^2}{2\sigma_X^2} + \frac{(wx)^2 - 2wx\mu_Y + \mu_Y^2}{2\sigma_Y^2}$$

$$= \frac{\left(\sigma_Y^2 + \sigma_X^2 w^2\right)x^2 - 2\left(\sigma_Y^2\mu_X + \sigma_X^2 w\mu_Y\right)x + \left(\sigma_Y^2\mu_X^2 + \sigma_X^2\mu_Y^2\right)}{2\sigma_X^2\sigma_Y^2} \qquad (5)$$

Considering the integral is respecting to x, we have the following constants:

$$a = \frac{\sigma_Y^2 + \sigma_X^2 w^2}{2\sigma_X^2\sigma_Y^2}, \quad b = \frac{\sigma_Y^2\mu_X + \sigma_X^2 w\mu_Y}{\sigma_X^2\sigma_Y^2}, \quad c = \frac{\sigma_Y^2\mu_X^2 + \sigma_X^2\mu_Y^2}{2\sigma_X^2\sigma_Y^2} \qquad (6)$$

Replacing (6) in (5):

$$u(x) = ax^2 - bx + c \tag{7}$$

Replacing $u(x)$ from (7) in (3):

$$f(w) = \left(\frac{1}{2\pi\sigma_X\sigma_Y} \right) e^{-c} \int_{-\infty}^{\infty} |x| e^{-\left(ax^2 - bx\right)} dx = \left(\frac{1}{2\pi\sigma_X\sigma_Y} \right) e^{-c} \int_{-\infty}^{\infty} |x| e^{-ax^2 + bx} dx \tag{8}$$

Defining:

$$K = \frac{1}{2\pi\sigma_X\sigma_Y} e^{-c} \tag{9}$$

Replacing K from (9) in (8):

$$f_W(w) = K \int_{-\infty}^{\infty} |x| e^{-ax^2 + bx} dx \tag{10}$$

Splitting the integral in two sides ($-\infty < x < 0$) and ($0 < x < \infty$) in order to decompose the absolute function |x|:

$$\begin{aligned} f_W(w) &= K \left(-\int_{-\infty}^{0} x e^{-ax^2 + bx} dx + \int_{0}^{\infty} x e^{-ax^2 + bx} dx \right) \\ &= K \left(\underbrace{\int_{0}^{\infty} x e^{-ax^2 - bx} dx}_{g(w)} + \underbrace{\int_{0}^{\infty} x e^{-ax^2 + bx} dx}_{h(w)} \right) \end{aligned} \tag{11}$$

Both $g(w)$ and $h(w)$ definite integrals are function of w because a and b are constant regarding to x, but function regarding to w (6). Now, we will proceed solving each integral.

We will solve these integrals by using an online tool for calculating indefinite integrals as a reference (integral-calculator.com).

Solving $g(w)$ (11):

$$g(w) = \int_{0}^{\infty} x e^{-ax^2 - bx} dx \tag{12}$$

Substituting x in the non-exponential factor:

$$x = \frac{1}{2a}\left(2ax + b\right) - \frac{b}{2a}$$

Then:

$$g(w)=\int_0^{\infty}\left(\frac{1}{2a}(2ax+b)-\frac{b}{2a}\right)e^{-ax^2-bx}dx=\frac{1}{2a}\underbrace{\int_0^{\infty}(2ax+b)e^{-ax^2-bx}dx}_{\alpha(w)}-\frac{b}{2a}\underbrace{\int_0^{\infty}e^{-ax^2-bx}dx}_{\beta(w)} \tag{13}$$

Solving $\alpha(w)$:

$$\alpha(w)=\int_0^{\infty}(2ax+b)e^{-ax^2-bx}dx \tag{14}$$

Defining:

$$u=-ax^2-bx \Rightarrow \frac{du}{dx}=-2ax-b \Rightarrow dx=-\frac{1}{2ax+b}du$$

Applying in α(w) (14):

$$\alpha(w)=\int_0^{\infty}(2ax+b)e^{u}\left(-\frac{1}{2ax+b}\right)du=-\int_0^{\infty}e^{u}du=-e^{u}\Big|_0^{\infty}$$

Replacing u:

$$a>0 \quad always \Rightarrow \alpha(w)=-e^{-ax^2-bx}\Big|_0^{\infty}=-(-1)=1 \tag{15}$$

Solving $\beta(w)$:

$$\beta(w)=\int_0^{\infty}e^{-ax^2-bx}dx \tag{16}$$

Completing the square in the exponent:

$$\beta(w)=\int_0^{\infty}e^{-\left[\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2-\frac{b^2}{4a}\right]}dx=e^{\frac{b^2}{4a}}\int_0^{\infty}e^{-\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2}dx \tag{17}$$

Defining:

$$u=\sqrt{a}x+\frac{b}{2\sqrt{a}} \Rightarrow \frac{du}{dx}=\sqrt{a} \Rightarrow dx=\frac{1}{\sqrt{a}}du \tag{18}$$

Applying in β(w):

$$\beta(w)=e^{\frac{b^2}{4a}}\int_{x=0}^{x\to\infty}\frac{e^{-u^2}}{\sqrt{a}}du \tag{19}$$

In this last integral we remember that the limits are defined in terms of x, and not for u. To put the limits in terms of u, just apply the limits of x in u. Thus:

$$u(x=0)=\frac{b}{2\sqrt{a}}, \qquad u(x\to\infty)\to\infty$$

Replacing limits in $\beta(w)$:

$$\beta(w)=e^{\frac{b^2}{4a}}\int_{\frac{b}{2\sqrt{a}}}^{\infty}\frac{e^{-u^2}}{\sqrt{a}}du \tag{20}$$

Making some changes that do not alter the equation:

$$\beta(w)=\left(\frac{\sqrt{\pi}}{2\sqrt{a}}e^{\frac{b^2}{4a}}\right)\underbrace{\frac{2}{\sqrt{\pi}}\int_{\frac{b}{2\sqrt{a}}}^{\infty}e^{-u^2}du}_{erfc\left(\frac{b}{2\sqrt{a}}\right)} \tag{21}$$

The function $erfc(z)$ is a special function of statistical mathematics called *the complementary error function*, and it is related to the error function $erf(z)$:

$$erfc(z)=1-erf(z)$$

$$erf(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt, \quad erfc(z)=\frac{2}{\sqrt{\pi}}\int_z^{\infty}e^{-t^2}dt$$

Thus:

$$\beta(w)=\left(\frac{\sqrt{\pi}}{2\sqrt{a}}e^{\frac{b^2}{4a}}\right)\left[1-erf\left(\frac{b}{2\sqrt{a}}\right)\right] \tag{22}$$

Applying the solutions of $\alpha(w)$ (15), $\beta(w)$ (22) in $g(w)$ (13):

$$g(w)=\frac{1}{2a}(1)-\frac{b}{2a}\left\{\left(\frac{\sqrt{\pi}}{2\sqrt{a}}e^{\frac{b^2}{4a}}\right)\left[1-erf\left(\frac{b}{2\sqrt{a}}\right)\right]\right\}$$

$$=\frac{1}{2a}-\frac{b\sqrt{\pi}e^{\frac{b^2}{4a}}}{4a^{\frac{3}{2}}}\left[1-erf\left(\frac{b}{2\sqrt{a}}\right)\right] \tag{23}$$

Solving $h(w)$ (11):

$$h(w)=\int_0^{\infty}xe^{-ax^2+bx}dx \tag{24}$$

Substituting x in the non-exponential factor:

$$x=\frac{1}{2a}\left(2ax-b\right)+\frac{b}{2a},$$

Then:

$$h(w)=\int_0^{\infty}\left[\frac{1}{2a}\left(2ax-b\right)+\frac{b}{2a}\right]e^{-ax^2+bx}dx=\frac{1}{2a}\underbrace{\int_0^{\infty}\left(2ax-b\right)e^{-ax^2+bx}dx}_{\eta(w)}+\frac{b}{2a}\underbrace{\int_0^{\infty}e^{-ax^2+bx}dx}_{\chi(w)}$$

(25)

Solving η(x) (ec.B.25):

$$\eta(w)=\int_0^{\infty}\left(2ax-b\right)e^{-ax^2+bx}dx \tag{26}$$

Defining:

$$u=bx-ax^2\Rightarrow\frac{du}{dx}=b-2ax\Rightarrow dx=\frac{1}{b-2ax}du$$

Applying in η(w):

$$\eta(w)=\int_0^{\infty}\left(2ax-b\right)e^{u}\,\frac{1}{b-2ax}du=-\int_0^{\infty}e^{u}du=-e^{u}\Big|_0^{\infty}$$

Replacing u:

$$\eta(w)=-e^{bx-ax^2}\Big|_0^{\infty}=-(-1)=1 \tag{27}$$

Solving χ(x) (25):

$$\chi(w)=\int_0^{\infty}e^{-ax^2+bx}dx \tag{28}$$

Completing the square on the exponent:

$$\chi(w)=\int_0^{\infty}e^{-\left[\left(\sqrt{a}x-\frac{b}{2\sqrt{a}}\right)^2-\frac{b^2}{4a}\right]}dx=e^{\frac{b^2}{4a}}\int_0^{\infty}e^{-\left(\sqrt{a}x-\frac{b}{2\sqrt{a}}\right)^2}dx \tag{29}$$

Defining:

$$u=\sqrt{a}x-\frac{b}{2\sqrt{a}}\Rightarrow\frac{du}{dx}=\sqrt{a}\Rightarrow dx=\frac{1}{\sqrt{a}}du$$

Applying in χ(w):

$$\chi(w)=e^{\frac{b^2}{4a}}\int_{x=0}^{x\to\infty}\frac{e^{-u^2}}{\sqrt{a}}du \tag{30}$$

In this last integral the limits are defined in terms of x. To put the limits in terms of u, we must apply the limits of x in u. Thus:

$$u(x=0)=-\frac{b}{2\sqrt{a}},\qquad u(x\to\infty)\to\infty$$

Replacing in χ(w) (ec.B.30):

$$\chi(w)=e^{\frac{b^2}{4a}}\int_{-\frac{b}{2\sqrt{a}}}^{\infty}\frac{e^{-u^2}}{\sqrt{a}}du \tag{31}$$

Making some changes that do not alter the equation:

$$\chi(w)=\left(\frac{\sqrt{\pi}}{2\sqrt{a}}e^{\frac{b^2}{4a}}\right)\underbrace{\frac{2}{\sqrt{\pi}}\int_{-\frac{b}{2\sqrt{a}}}^{\infty}e^{-u^2}du}_{erfc\left(\frac{-b}{2\sqrt{a}}\right)} \tag{32}$$

This solution also involves the *complementary error function* $erfc(z)$, so the final expression for χ(w), in terms $erf(z)$ could be:

$$\chi(w)=\left(\frac{\sqrt{\pi}}{2\sqrt{a}}e^{\frac{b^2}{4a}}\right)\left[1-erf\left(\frac{-b}{2\sqrt{a}}\right)\right] \tag{33}$$

However, as we shall see, to achieve a simpler final solution for $f(w)$, we must make a slight modification to this result. For this, we note that $erf(z)$ is an even function:

$$erf(-z)=-erf(z)$$

Then:

$$\chi(w)=\left(\frac{\sqrt{\pi}}{2\sqrt{a}}e^{\frac{b^2}{4a}}\right)\left[1+erf\left(\frac{b}{2\sqrt{a}}\right)\right] \tag{34}$$

Applying the solutions of $\eta(w)$ (27), $\chi(w)$ (34) in $h(w)$ (25):

$$h(w)=\int_0^{\infty}\left[\frac{1}{2a}\left(2ax-b\right)+\frac{b}{2a}\right]e^{-ax^2+bx}dx=\frac{1}{2a}\int_0^{\infty}\left(2ax-b\right)e^{-ax^2+bx}dx+\frac{b}{2a}\int_0^{\infty}e^{-ax^2+bx}dx$$

$$h(w)=\frac{1}{2a}(1)+\frac{b}{2a}\left\{\left(\frac{\sqrt{\pi}}{2\sqrt{a}}e^{\frac{b^2}{4a}}\right)\left[1+erf\left(\frac{b}{2\sqrt{a}}\right)\right]\right\}$$

$$=\frac{1}{2a}+\frac{b\sqrt{\pi}e^{\frac{b^2}{4a}}}{4a^{\frac{3}{2}}}\left[1+erf\left(\frac{b}{2\sqrt{a}}\right)\right] \qquad (35)$$

Final solution of $f(w)$:

Applying the solutions of $g(w)$ (23) y $h(w)$ (35) in $f(w)$ (11):

$$f(w)=K\left(\int_0^{\infty}xe^{-ax^2-bx}dx+\int_0^{\infty}xe^{-ax^2+bx}dx\right)=K\left(g(w)+h(w)\right)$$

$$=K\left(\frac{1}{2a}-\frac{b\sqrt{\pi}e^{\frac{b^2}{4a}}}{4a^{\frac{3}{2}}}\left[1-erf\left(\frac{b}{2\sqrt{a}}\right)\right]+\frac{1}{2a}+\frac{b\sqrt{\pi}e^{\frac{b^2}{4a}}}{4a^{\frac{3}{2}}}\left[1+erf\left(\frac{b}{2\sqrt{a}}\right)\right]\right)$$

$$=K\left(\frac{1}{a}+\frac{b\sqrt{\pi}e^{\frac{b^2}{4a}}}{2a^{\frac{3}{2}}}erf\left(\frac{b}{2\sqrt{a}}\right)\right) \qquad (36)$$

Remembering:

$$a=\frac{\sigma_Y^2+\sigma_X^2w^2}{2\sigma_X^2\sigma_Y^2},\qquad b=\frac{\sigma_Y^2\mu_X+\sigma_X^2w\mu_Y}{\sigma_X^2\sigma_Y^2},\qquad c=\frac{\sigma_Y^2\mu_X^2+\sigma_X^2\mu_Y^2}{2\sigma_X^2\sigma_Y^2} \qquad \text{from (6)}$$

$$K=\frac{1}{2\pi\sigma_X\sigma_Y}e^{-c} \qquad \text{from (9)}$$

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Finally, applying in (36):

$$f(w)=\frac{1}{2\pi\sigma_X\sigma_Y}e^{-\frac{\sigma_Y^2\mu_X^2+\sigma_X^2\mu_Y^2}{2\sigma_X^2\sigma_Y^2}}\left(\frac{1}{\frac{\sigma_Y^2+\sigma_X^2w^2}{2\sigma_X^2\sigma_Y^2}}+\frac{\left(\frac{\sigma_Y^2\mu_X+\sigma_X^2w\mu_Y}{\sigma_X^2\sigma_Y^2}\right)\sqrt{\pi}e^{\frac{\left(\frac{\sigma_Y^2\mu_X+\sigma_X^2w\mu_Y}{\sigma_X^2\sigma_Y^2}\right)^2}{4\left(\frac{\sigma_Y^2+\sigma_X^2w^2}{2\sigma_X^2\sigma_Y^2}\right)}}}{2\left(\frac{\sigma_Y^2+\sigma_X^2w^2}{2\sigma_X^2\sigma_Y^2}\right)^{\frac{3}{2}}}erf\left(\frac{\frac{\sigma_Y^2\mu_X+\sigma_X^2w\mu_Y}{\sigma_X^2\sigma_Y^2}}{2\sqrt{\frac{\sigma_Y^2+\sigma_X^2w^2}{2\sigma_X^2\sigma_Y^2}}}\right)\right)$$

With $w=y/x$ (37)

In Figure 1 we show some of the graphs of the probability density functions that can be obtained with different parameters of means and standard deviations for the variables x and y.

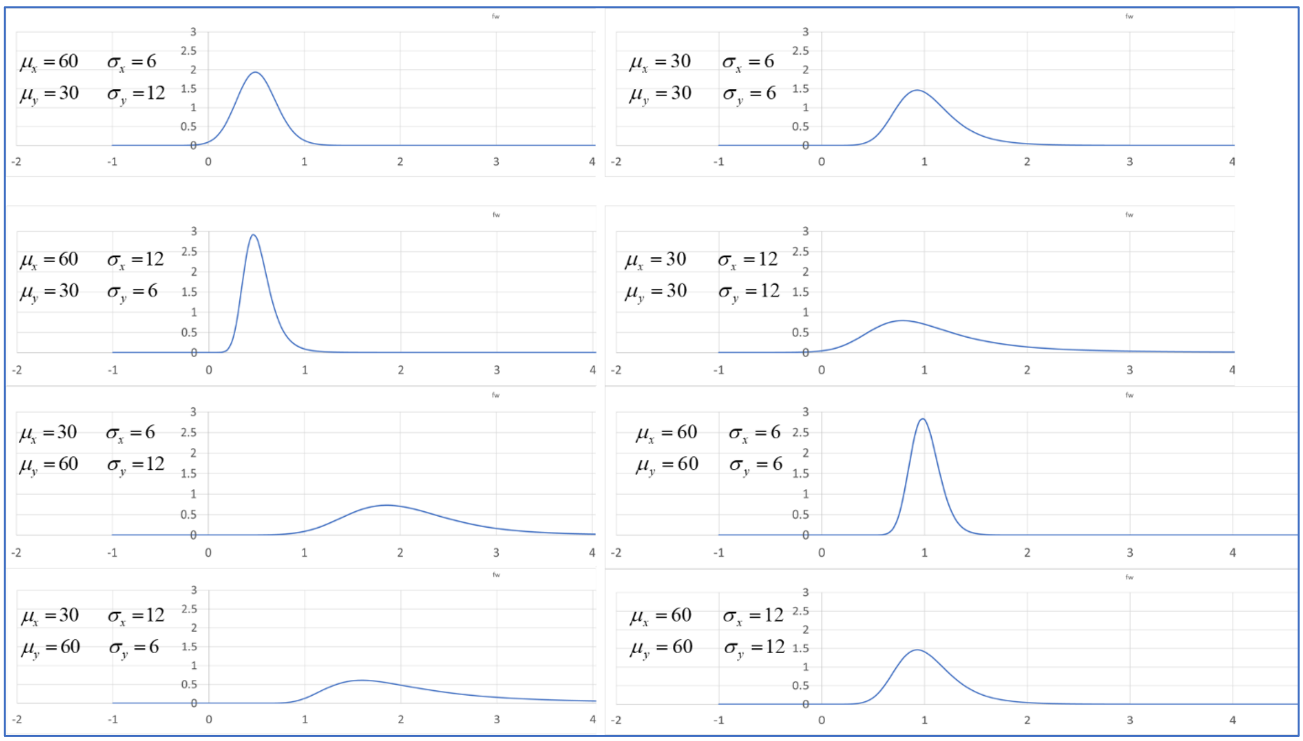

Figure 1