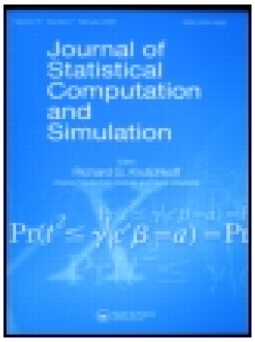

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Bayes estimation of Gompertz distribution parameters and acceleration factor under partially accelerated life tests with type-I censoring

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In this paper, the Bayesian approach is applied to the estimation problem in the case of step stress partially accelerated life tests with two stress levels and type-I censoring. Gompertz distribution is considered as a lifetime model. The posterior means and posterior variances are derived using the squared-error loss function. The Bayes estimates cannot be obtained in explicit forms. Approximate Bayes estimates are computed using the method of Lindley [D.V. Lindley, *Approximate Bayesian methods*, Trabajos Estadistica 31 (1980), pp. 223–237]. The advantage of this proposed method is shown. The approximate Bayes estimates obtained under the assumption of non-informative priors are compared with their maximum likelihood counterparts using Monte Carlo simulation.

Keywords: Bayesian estimation; step-stress test; acceleration factor; Gompertz distribution; maximum likelihood; squared-error loss function; non-informative priors; posterior mean; posterior variance; type-I censoring; Lindley's approximation; Monte Carlo simulation

MSC: 62N01; 62N05

Notations

n	number of step-stress test units (total sample size)
n_u, n_a	numbers of test units failed at use and accelerated conditions, respectively
n_c	number of censored units ($n_c = n - n_u - n_a$)
η	censoring time of a PALT
τ	stress change time in a step PALT ($\tau < \eta$)
T	lifetime of an item at normal use condition
Y	total lifetime of an item in a step PALT
y_i	observed value of the total lifetime Y_i of item i, $i = 1, \ldots, n$

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β	acceleration factor ($\beta > 1$)
α	shape parameter ($\alpha > 0$)
θ	scale parameter ($\theta > 0$)
$\wedge$	implies a maximum likelihood estimator
$\downarrow(\cdot)$	evaluated at $(\cdot)$
$y_{(1)} \leq \ldots \leq y(n_u) \leq \tau \leq y(n_{u+1}) \leq \ldots \leq y(n_{u+}n_a) \leq \eta$	ordered failure times

1. Introduction

A highly reliable component is often tested using accelerated life testing. Testing the component under its normal use condition can take a very long time. Hence, testing is conducted under conditions of higher stress to obtain failure information more quickly. However, to predict the performance of a component in the case of its normal use condition, the data must be extrapolated based on a certain model of acceleration.

As indicated by Pathak *et al.* [1], the model of acceleration is chosen so that the relationship between the parameters of the failure distribution and the accelerated stress conditions is known. These models are usually derived from an analysis of the physical mechanisms of failure of the component.

The tests performed under accelerated stress conditions are called fully accelerated life tests (FALT). Sometimes, such a relationship is not known and cannot be assumed. So, in this case, FALT cannot be used for reliability prediction. Instead, another type of tests called partially accelerated life tests (PALT) is used according to the proposed model by DeGroot and Goel [2].

In FALT, the component is run under accelerated stress conditions, but in PALT, it is run under normal use condition for a specified time. If it does not fail, the stress on it is raised until the component fails or the censoring time is reached. So, PALT combines both ordinary and accelerated life tests.

Most of literature performed on PALT considered the classical approach to estimate the parameters of interest [3–17].

From the Bayesian viewpoint, few studies have been considered on PALT. Gole [3] used the Bayesian approach for estimating the acceleration factor and the parameters in the case of step-stress PALT (SSPALT) with complete sampling for items having exponential and uniform distributions. DeGroot and Goel [2] investigated the optimal Bayesian design of a PALT in the case of the exponential distribution under complete sampling. Abdel-Ghani [10] considered the Bayesian approach to estimate the parameters of Weibull distribution in SSPALT with censoring. Recently, Ismail [13] obtained the Bayesian estimates of the Pareto distribution parameters under SSPALT with censored data.

In this paper, the two-parameter Gompertz distribution (GD) is considered under PALT and the main aim is to obtain the Bayes estimators (BEs) of the acceleration factor and the distribution parameters and compare them with the maximum likelihood estimators (MLEs) counterparts by Monte Carlo simulations. The squared-error loss function is considered and to make the comparison more meaningful, the non-informative priors (NIPs) on both the shape and scale parameters are assumed.

The rest of this paper is organized as follows. In Section 2, the model and test method are described. Approximate BEs for the parameters under consideration are derived in Section 3. In Section 4, BEs derived in Section 3 are obtained numerically, using Lindley's approximation and then compared with the MLEs. Finally, some concluding remarks and points for future work are introduced in Section 5.

2. The model and test method

2.1. *The GD as a lifetime model*

The GD plays an important role in modelling survival times, human mortality and actuarial tables. According to the literature, the GD was formulated by Gompertz [18] to fit mortality tables. Recently, many authors have contributed to the statistical methodology and characterization of this distribution. For example, Read [19], Gordon [20], Makany [21], Rao and Damaraju [22], Franses [23] and Wu and Lee [24]. Garg *et al.* [25] studied the properties of the GD and obtained the MLEs for the parameters. Chen [26] developed an exact confidence interval and an exact joint confidence region for the parameters of the GD under type-II censoring.

In this paper, the lifetimes of the test items are assumed to follow a GD with probability density function (pdf) as follows:

$$f(t;\theta,\alpha) = \theta \mathrm{e}^{\alpha t} \exp\left\{-\left(\frac{\theta}{\alpha}\right)(\mathrm{e}^{\alpha t}-1)\right\}, \quad t>0, \theta>0, \alpha>0. \tag{1}$$

This distribution does not seem to have received enough attention, possibly because of its complicated form [25]. It is worth noting that when $\alpha \to 0$, the GD will tend to an exponential distribution [27]. The two-parameter Gompertz model is a commonly used survival time distribution in actuarial science and reliability and life testing [28]. There are several forms for the GD given in the literature. Some of these are given in [29]. The pdf formula given above is the commonly used form and it is unimodal. It has positive skewness and an increasing hazard rate function. In addition, the GD can be interpreted as a truncated extreme value type-I distribution [29]. According to Jaheen [30], the GD has been used as a growth model, especially in epidemiological and biomedical studies.

The GD is a theoretical distribution of survival times. Gompertz [18] proposed a probability model for human mortality, based on the assumption that the 'average exhaustion of a man's power to avoid death to be such that at the end of equal infinitely small intervals of time he lost equal portions of his remaining power to oppose destruction, which he had at the commencement of these intervals' [31]. Also, according to Walker and Adham [32], the GD has many applications, particularly in medical and actuarial studies. However, there has been little recent work on the Gompertz in comparison with its early investigation. Osman [33] derived a compound Gompertz model by assuming that one of the parameters of the GD is a random variable following the gamma distribution. He studied the properties of compound GD and suggested its use for modelling lifetime data and analysing the survivals in heterogeneous populations.

The reliability function of the GD takes the form

$$R(t) = \exp\left\{-\left(\frac{\theta}{\alpha}\right)(\mathrm{e}^{\alpha t}-1)\right\}, \tag{2}$$

and the corresponding hazard rate is given by

$$h(t) = \theta \mathrm{e}^{\alpha t}. \tag{3}$$

Thus, the hazard rate increases exponentially over time.

2.2. *The test method*

Basic assumptions

(1) Two stress levels x_1 and x_2 (design and high) are used.

(2) For any level of stress, the life distribution of test unit is Gompertz.
(3) The total lifetime Y of an item is as follows:

$$Y = \begin{cases} T & \text{if } T \leq \tau, \\ \tau + \beta^{-1}(T - \tau) & \text{if } T > \tau, \end{cases} \tag{4}$$

where T is the lifetime of an item at normal use condition. This model is called the *tampered random variable* (TRV) model. It was proposed by DeGroot and Goel [2].
(4) The lifetimes $Y_1, \ldots, Y_n$ of the n test items are independent and identically distributed random variables (i.i.d. r.v.'s).

Test procedure

(1) Each of the n test items is first run at normal use condition.
(2) If it does not fail at normal use condition by a pre-specified time τ, then it is put on accelerated use condition and run until it fails or the censoring time is reached.

3. Bayes estimation of the model parameters

In this section, the squared-error loss function is considered. Under squared-error loss function, the BE of a parameter is its posterior expectation. The BEs cannot be expressed in explicit forms. Approximate BEs will be obtained under the assumption of NIPs using Lindley's approximation.

In many practical situations, the information about the parameters are available in an independent manner [34]. Thus, here it is assumed that the parameters are independent *a priori* and let the NIP for each parameter be represented by the limiting form of the appropriate natural conjugate prior.

It follows that a NIP for the acceleration factor β is given by

$$\pi_1(\beta) \propto \beta^{-1}, \quad \beta > 1.$$

Also, the NIP's for the scale parameter θ and the shape parameter α are, respectively, as

$$\pi_2(\theta) \propto \theta^{-1}, \;\; \theta > 0 \quad \text{and} \quad \pi_3(\alpha) \propto \alpha^{-1}, \;\; \alpha > 0.$$

Therefore, the joint NIP of the three parameters can be expressed by

$$\pi(\beta, \theta, \alpha) \propto (\beta\theta\alpha)^{-1}, \quad \beta > 1, \theta > 0, \alpha > 0. \tag{5}$$

Assuming that the lifetime of test unit is to follow GD(θ, α) with pdf in Equation (1). Therefore, the pdf of total lifetime Y of a unit tested under step-stress PALT is given by

$$f(y) = \begin{cases} f_1(y) & \text{if } 0 < y \leq \tau, \\ f_2(y) & \text{if } y > \tau, \end{cases}$$

where

$$f_1(y) = \theta \exp\{\alpha y - (\theta/\alpha)[\exp(\alpha y) - 1]\} \text{ as given in Equation (1),}$$

$$f_2(y) = \beta\theta \exp\{\alpha[\beta(y - \tau) + \tau] - (\theta/\alpha)[\exp(\alpha[\beta(y - \tau) + \tau]) - 1]\},$$

which is obtained by the transformation ariable technique using $f_1(y)$ and the model given in Equation (4).

The observed values of the total lifetime Y are given by

$$y_{(1)} \leq \ldots \leq y(n_u) \leq \tau \leq y(n_{u+1}) \leq \ldots \leq y(n_{u+}n_a) \leq \eta$$

Since the total lifetimes $Y_1, \ldots, Y_n$ of n units are i.i.d. r.v.'s, then the general form of the total likelihood function for them can be obtained as follows:

$$\begin{aligned} L(\beta, \theta, \alpha) \propto & \prod_{i=1}^{nu} \theta \exp\left\{\alpha y_i - \left(\frac{\theta}{\alpha}\right)[\exp(\alpha y_i) - 1]\right\} \\ & \times \prod_{i=1}^{na} \beta\theta \exp\left\{\alpha[\beta(y_i - \tau) + \tau] - \left(\frac{\theta}{\alpha}\right)[\exp(\alpha[\beta(y_i - \tau) + \tau]) - 1]\right\} \\ & \times \prod_{i=1}^{nc} \exp\left\{-\left(\frac{\theta}{\alpha}\right)[\exp(\alpha[\beta(\eta - \tau) + \tau]) - 1]\right\}. \end{aligned} \tag{6}$$

Forming the product of Equations (5) and (6), the joint posterior density function of β, θ and α given the data can be written as

$$\begin{aligned} \pi^*(\beta, \theta, \alpha|\text{data}) & \propto L(\beta, \theta, \alpha) \cdot \pi(\beta, \theta, \alpha) \\ & \propto \beta^{n_a-1}\theta^{n_u+n_a-1}\alpha^{-1} \prod_{i=1}^{nu} \exp\left\{\alpha y_i - \left(\frac{\theta}{\alpha}\right)[\exp(\alpha y_i) - 1]\right\} \\ & \times \prod_{i=1}^{na} \exp\left\{\alpha[\beta(y_i - \tau) + \tau] - \left(\frac{\theta}{\alpha}\right)[\exp(\alpha[\beta(y_i - \tau) + \tau]) - 1]\right\} \\ & \times \prod_{i=1}^{nc} \exp\left\{-\left(\frac{\theta}{\alpha}\right)[\exp(\alpha[\beta(\eta - \tau) + \tau]) - 1]\right\}. \end{aligned} \tag{7}$$

As mentioned earlier, under a squared-error loss function, the BE of a parameter is its posterior expectation. To obtain the posterior means and posterior variances of β, θ and α, non-tractable integrals will be confronted. It is not possible to compute them analytically. The marginal posteriors are somewhat unwieldy and require a numerical integration that may not converge. Instead, an approximation due to Lindley [35] via an asymptotic expansion of the ratio of two non-tractable integrals is used to obtain the approximate BEs. Lindley's approximation is evaluated at the MLEs of the model parameters.

Now, let Θ be a set of parameters $\{\Theta_1, \Theta_2, \ldots, \Theta_m\}$, where m is the number of parameters, then the posterior expectation of an arbitrary function $u(\Theta)$ can be asymptotically estimated by

$$\begin{aligned} E(u(\Theta)) &= \frac{\int_\Theta u(\Theta)\pi(\Theta)e^{\ln} L(y|\Theta)\, \mathrm{d}\Theta}{\int_\Theta \pi(\Theta)\mathrm{e}^{\ln}L(y|\Theta)\, \mathrm{d}\Theta} \\ &\approx \left[u + \left(\frac{1}{2}\right)\sum_{i,j}(u_{ij}^{(2)} + 2u_i^{(1)}\rho_j^{(1)})\sigma_{\mathrm{ij}} + \left(\frac{1}{2}\right)\sum_{i,j,k,s} L_{ijk}^{(3)}\sigma_{ij}\sigma_{ks}u_s^{(1)}\right] \downarrow \hat{\Theta}, \end{aligned} \tag{8}$$

which is the BE of $u(\Theta)$ under a squared-error loss function, where $\pi(\Theta)$ is the prior distribution of Θ, $u \equiv u(\Theta)$, $L \equiv L(\Theta)$ is the likelihood function, $\rho \equiv \rho(\Theta) = \log \pi(\Theta)$, σ_{ij} are the elements of the inverse of the asymptotic Fisher's information matrix of β, θ and α, and

$$u_i^{(1)} = \frac{\partial u}{\partial \Theta_i}, \quad u_{ij}^{(2)} = \frac{\partial^2 u}{\partial \Theta_i \partial \Theta_j}, \quad \rho_j^{(1)} = \frac{\partial \log \pi(\Theta)}{\partial \Theta_j} \quad \text{and} \quad L_{ijk}^{(3)} = \frac{\partial^3 \ln L(y|\Theta)}{\partial \Theta_i \partial \Theta_j \partial \Theta_k}.$$

Such an approximation is easy to use and does not require innovative programming and extensive computer time. According to Green [36], the linear BE in Equation (8) is a 'very good and operational approximation for the ratio of multi-dimension integrals'. As indicated by Sinha [37], it has led to many useful applications. However, if the domain of the parameters is a function of the parameters, BEs using Lindley's rule are not obtainable unless the MLEs exist. The derivation of posterior means and posterior variances is shown in the Appendix.

4. Simulation studies

In this section, numerical examples are provided to demonstrate the theoretical results given in this paper. The posterior means and posterior variances of the three parameters β, θ and α are derived assuming the NIP for each parameter under a squared-error loss function using type-I censored data. Since the BEs of the model parameters cannot be obtained analytically, approximate BEs are obtained numerically using the method of Lindley. The behaviour sampling of the approximate BEs is investigated and compared with the MLEs in terms of their variances and mean squared errors (MSEs) for different sample sizes and for different parameter values. The process is replicated 1000 times for each sample size and the average of estimates is computed. The results are listed in Tables 1 and 2.

Table 1. Average values of the MLEs and approximate BEs with associated estimated variances and MSEs when $\beta = 1.5$, $\theta = 0.8$, $\alpha = 0.5$, $\tau = 3$ and $\eta = 7$.

n	Parameter	Method	Estimate	Variance	MSE
25	β	ML	2.0151	0.2567	0.1953
		Bayes	1.9045	0.2113	0.1636
	θ	ML	1.4734	0.0472	0.0342
		Bayes	1.3145	0.0233	0.0298
	α	ML	0.8163	0.0382	0.0275
		Bayes	0.7982	0.0294	0.0222
50	β	ML	1.7532	0.1470	0.1371
		Bayes	1.6452	0.1083	0.1081
	θ	ML	1.0663	0.0264	0.0282
		Bayes	1.0131	0.0185	0.0261
	α	ML	0.7681	0.0249	0.0233
		Bayes	0.7035	0.0208	0.0191
75	β	ML	1.6621	0.0741	0.1143
		Bayes	1.5906	0.0635	0.0882
	θ	ML	0.9994	0.0172	0.0254
		Bayes	0.9113	0.0107	0.0242
	α	ML	0.6184	0.0065	0.0184
		Bayes	0.5735	0.0041	0.0155
100	β	ML	1.5476	0.0437	0.0853
		Bayes	1.5142	0.0428	0.0632
	θ	ML	0.8735	0.0116	0.0232
		Bayes	0.8453	0.0087	0.0229
	α	ML	0.5468	0.0022	0.0157
		Bayes	0.5292	0.0015	0.0131

Table 2. Average values of the MLEs and approximate BEs with associated estimated variances and MSEs when $\beta = 2.5$, $\theta = 2$, $\alpha = 1.5$, $\tau = 3$ and $\eta = 7$.

n	Parameter	Method	Estimate	Variance	MSE
25	β	ML	3.4381	0.3873	0.2461
		Bayes	3.0352	0.3391	0.2073
	θ	ML	2.7235	0.1432	0.1811
		Bayes	2.6167	0.0911	0.1589
	α	ML	2.2172	0.1923	0.1124
		Bayes	2.1456	0.1467	0.0793
50	β	ML	3.0107	0.2615	0.1844
		Bayes	2.9742	0.2246	0.1521
	θ	ML	2.4372	0.0812	0.1144
		Bayes	2.3420	0.0454	0.0921
	α	ML	2.0732	0.1172	0.0582
		Bayes	2.0164	0.0924	0.0274
75	β	ML	2.6377	0.1943	0.1343
		Bayes	2.5582	0.1672	0.1152
	θ	ML	2.2264	0.0111	0.0431
		Bayes	2.1563	0.0053	0.0364
	α	ML	1.7194	0.0491	0.0462
		Bayes	1.6932	0.0230	0.0236
100	β	ML	2.5291	0.1312	0.1061
		Bayes	2.4825	0.1153	0.0953
	θ	ML	2.0711	0.0041	0.0152
		Bayes	2.0433	0.0032	0.0117
	α	ML	1.5641	0.0063	0.0332
		Bayes	1.5422	0.0054	0.0210

Some of the points are quite clear from the numerical results. As expected, it is observed that the performances of both BEs and MLEs become better when the sample size increases. Also, it is observed that the approximate BEs approach the true values with increasing the sample size. When we compare the MLEs with the approximate BEs, using Lindley's technique in terms of their variances and MSEs, it is noted that the approximate BEs perform better than the MLEs. That is, the approximate BEs become with smaller variances and smaller MSEs as the sample size increases. These results coincide with the note of Achcar [38]. He said that the use of approximate Bayesian methods could be a good alternative for the usual asymptotically classical methods in accelerated life testing.

5. Conclusion

In this paper, the Bayes estimation of the parameters of GD and the acceleration factor was considered. The BEs were obtained under the assumptions of squared-error loss functions and NIPs. It was observed that the BEs cannot be obtained in explicit forms. Instead, Lindley's approximation was used to obtain the Bayesian estimates numerically. It was seen that the approximation works very well even for small sample sizes. It was observed that Lindley's method usually provides posterior variances smaller than the variances of the MLEs, which is an advantage of this method. It can be said that the intrinsic appeal of that method can be expressed in its being a sort of adjustment to the maximum likelihood approach to reduce variability. However, it was also observed for very large sample sizes that the Bayesian estimates and the MLEs become closer in terms of MSEs and variances. That is, for very large sample sizes, the performances are so far similar as expected.

But if we consider informative priors, then the performances of BEs will be much better than those of MLEs.

As a future work, the problem of Bayes estimation for the PALT model parameters can be extended to include informative priors and also to consider other techniques such as Markov Chain Monte Carlo (MCMC) technique to compute the approximate Bayes estimates, investigate their performances and compare them with those of the MLEs.

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Appendix

Here, there are three parameters in the model. That is, $m = 3$. Let the subscripts 1, 2 and 3 refer to β, θ and α, respectively. Therefore, the posterior means (BEs) of the three parameters can be expressed by

$$\beta^* = E(\beta|y) = \left[\beta - \left(\frac{\sigma_{11}}{\beta} + \frac{\sigma_{12}}{\theta} + \frac{\sigma_{13}}{\alpha}\right) + \left(\frac{1}{2}\right)(\sigma_{11}\Psi_1 + \sigma_{12}\Psi_2 + \sigma_{13}\Psi_3)\right] \downarrow \hat{\Theta}, \tag{A1}$$

$$\theta^* = E(\theta|y) = \left[\theta - \left(\frac{\sigma_{21}}{\beta} + \frac{\sigma_{22}}{\theta} + \frac{\sigma_{23}}{\alpha}\right) + \left(\frac{1}{2}\right)(\sigma_{21}\Psi_1 + \sigma_{22}\Psi_2 + \sigma_{23}\Psi_3)\right] \downarrow \hat{\Theta}, \tag{A2}$$

$$\alpha^* = E(\alpha|y) = \left[\alpha - \left(\frac{\sigma_{31}}{\beta} + \frac{\sigma_{32}}{\theta} + \frac{\sigma_{33}}{\alpha}\right) + \left(\frac{1}{2}\right)(\sigma_{31}\Psi_1 + \sigma_{32}\Psi_2 + \sigma_{33}\Psi_3)\right] \downarrow \hat{\Theta} \tag{A3}$$

Thus, the posterior variances can be obtained by

$$\mathrm{Var}(\beta|y) = E(\beta^2|y) - (\beta^*)^2 = \sigma_{11} - \left[\left(\frac{\sigma_{11}}{\beta} + \frac{\sigma_{12}}{\theta} + \frac{\sigma_{13}}{\alpha}\right) - \left(\frac{1}{2}\right)(\sigma_{11}\Psi_1 + \sigma_{12}\Psi_2 + \sigma_{13}\Psi_3)\right]^2 \downarrow \hat{\Theta}, \tag{A4}$$

$$\mathrm{Var}(\theta|y) = E(\theta^2|y) - (\theta^*)^2 = \sigma_{22} - \left[\left(\frac{\sigma_{21}}{\beta} + \frac{\sigma_{22}}{\theta} + \frac{\sigma_{23}}{\alpha}\right) - \left(\frac{1}{2}\right)(\sigma_{21}\Psi_1 + \sigma_{22}\Psi_2 + \sigma_{23}\Psi_3)\right]^2 \downarrow \hat{\Theta}, \tag{A5}$$

$$\mathrm{Var}(\alpha|y) = E(\alpha^2|y) - (\alpha^*)^2 = \sigma_{33} - \left[\left(\frac{\sigma_{31}}{\beta} + \frac{\sigma_{32}}{\theta} + \frac{\sigma_{33}}{\alpha}\right) - \left(\frac{1}{2}\right)(\sigma_{31}\Psi_1 + \sigma_{32}\Psi_2 + \sigma_{33}\Psi_3)\right]^2 \downarrow \hat{\Theta}, \tag{A6}$$

where

$$\Psi_1 = \sum_{i,j}\sigma_{ij}L^{(3)}_{ij1} \equiv \sigma_{11}L^{(3)}_{111} + 2\sigma_{12}L^{(3)}_{121} + 2\sigma_{13}L^{(3)}_{131} + \sigma_{22}L^{(3)}_{221} + 2\sigma_{23}L^{(3)}_{231} + \sigma_{33}L^{(3)}_{331},$$

$$\Psi_2 = \sum_{i,j}\sigma_{ij}L^{(3)}_{ij2} \equiv \sigma_{11}L^{(3)}_{112} + 2\sigma_{12}L^{(3)}_{122} + 2\sigma_{13}L^{(3)}_{132} + \sigma_{22}L^{(3)}_{222} + 2\sigma_{23}L^{(3)}_{232} + \sigma_{33}L^{(3)}_{332}$$

and

$$\Psi_3 = \sum_{i,j}\sigma_{ij}L^{(3)}_{ij3} \equiv \sigma_{11}L^{(3)}_{113} + 2\sigma_{12}L^{(3)}_{123} + 2\sigma_{13}L^{(3)}_{133} + \sigma_{22}L^{(3)}_{223} + 2\sigma_{23}L^{(3)}_{233} + \sigma_{33}L^{(3)}_{333} \quad \text{for } i, j = 1, 2, 3.$$

To compute the posterior means and the posterior variances of the three parameters β, θ and α, both second and third derivatives of the natural logarithm of the likelihood function must be obtained.

The likelihood function is shown in Equation (6). Its natural logarithm can be written as

$$\ln L = (n_u + n_a)\ln\theta + n_a \ln\beta + \alpha\left\{\sum_{i=1}^{n_u} y_i + \sum_{i=1}^{n_a}[\beta(y_i-\tau)+\tau]\right\}$$
$$-(\theta/\alpha)\left\{\sum_{i=1}^{n_u}[\exp(\alpha y_i)-1] + \sum_{i=1}^{n_a}[\exp(\alpha[\beta(y_i-\tau)+\tau])-1]\right.$$
$$\left. +_{n_c}[\exp(\alpha[\beta(\eta-\tau)+\tau])-1]\right\}.$$

The second derivatives of $\ln L$ with respect to β, θ and α are given by

$$\frac{\partial^2 \ln L}{\partial\beta^2} = -\frac{n_a}{\beta^2} - \theta\alpha\left\{\sum_{i=1}^{n_a}[(y_i-\tau)^2\exp(\alpha[\beta(y_i-\tau)+\tau])]\right.$$
$$\left. + n_c(\eta-\tau)^2\exp(\alpha[\beta(\eta-\tau)+\tau])\right\},$$
$$\frac{\partial^2 \ln L}{\partial\theta^2} = -\frac{n_u+n_a}{\theta^2},$$
$$\frac{\partial^2 \ln L}{\partial\alpha^2} = -\left(\frac{\theta}{\alpha^2}\right)[\alpha\omega_3 - \omega_2] + \left(\frac{\theta}{\alpha^4}\right)[\alpha^2\omega_2 - 2\alpha\omega_1],$$

where

$$\omega_1 = \sum_{i=1}^{n_u}[\exp(\alpha y_i)-1] + \sum_{i=1}^{n_a}[\exp(\alpha[\beta(y_i-\tau)+\tau])-1]$$
$$+_{n_c}[\exp(\alpha[\beta(\eta-\tau)+\tau])-1],$$
$$\omega_2 = \frac{\partial\omega_1}{\partial\alpha} = \sum_{i=1}^{n_u}[y_i\exp(\alpha y_i)] + \sum_{i=1}^{n_a}\{[\beta(y_i-\tau)+\tau]\exp(\alpha[\beta(y_i-\tau)+\tau])\}$$
$$+_{n_c}[\beta(\eta-\tau)+\tau][\exp(\alpha[\beta(\eta-\tau)+\tau])],$$
$$\omega_3 = \frac{\partial\omega_2}{\partial\alpha} = \sum_{i=1}^{n_u}[y_i^2\exp(\alpha y_i)] + \sum_{i=1}^{n_a}\{[\beta(y_i-\tau)+\tau]^2\exp(\alpha[\beta(y_i-\tau)+\tau])\}$$
$$+_{n_c}[\beta(\eta-\tau)+\tau]^2[\exp(\alpha[\beta(\eta-\tau)+\tau])],$$
$$\frac{\partial^2 \ln L}{\partial\beta\partial\theta} = -\sum_{i=1}^{n_a}\{(y_i-\tau)\exp(\alpha[\beta(y_i-\tau)+\tau])\}$$
$$+_{n_c}(\eta-\tau)\exp(\alpha[\beta(\eta-\tau)+\tau]).$$
$$\frac{\partial^2 \ln L}{\partial\beta\partial\alpha} = \sum_{i=1}^{n_a}(y_i-\tau) - \theta\left[\sum_{i=1}^{n_a}\{(y_i-\tau)[\beta(y_i-\tau)+\tau]\exp(\alpha[\beta(y_i-\tau)+\tau])\}\right.$$
$$\left. +_{n_c}(\eta-\tau)[\beta(\eta-\tau)+\tau]\exp(\alpha[\beta(\eta-\tau)+\tau])\right],$$
$$\frac{\partial^2 \ln L}{\partial\theta\partial\alpha} = -\left(\frac{1}{\alpha^2}\right)[\alpha\omega_2 - \omega_1].$$

Now, the third derivatives of $\ln L$ with respect to β, θ and α are as follows:

$$L_{111}^{(3)} = \frac{\partial^3 \ln L}{\partial\beta^3} = 2\frac{n_a}{\beta^3} - \theta\alpha\left\{\alpha\sum_{i=1}^{n_a}[(y_i-\tau)^3\exp(\alpha[\beta(y_i-\tau)+\tau])]\right.$$
$$\left. +_{n_c}\alpha(\eta-\tau)^3\exp(\alpha[\beta(\eta-\tau)+\tau])\right\},$$

$$L_{222}^{(3)} = \frac{\partial^3 \ln L}{\partial \theta^3} = \frac{2(n_u + n_a)}{\theta^3},$$

$$L_{333}^{(3)} = \frac{\partial^3 \ln L}{\partial \alpha^3} = \frac{2\theta}{\alpha^3}[\alpha\omega_3 - \omega_2] - \frac{\theta\omega_4}{\alpha} - \frac{4\theta}{\alpha^4}[\alpha\omega_2 - 2\omega_1] + \frac{\theta}{\alpha^4}[\alpha^2\omega_3 - 2\omega_1],$$

where

$$\omega_4 = \frac{\partial \omega_3}{\partial \alpha} = \sum_{i=1}^{n_u}[y_i^3 \exp(\alpha y_i)] + \sum_{i=1}^{n_a}\left\{[\beta(y_i - \tau) + \tau]^3 \exp(\alpha[\beta(y_i - \tau) + \tau])\right\}$$
$$+ n_c[\beta(\eta - \tau) + \tau]^3[\exp(\alpha[\beta(\eta - \tau) + \tau])].$$

$$L_{112}^{(3)} = L_{121}^{(3)} = L_{211}^{(3)} = \frac{\partial^3 \ln L}{\partial \beta^2 \partial \theta} = -\alpha \left\{ \sum_{i=1}^{n_a} [(y_i - \tau)^2 \exp(\alpha[\beta(y_i - \tau) + \tau])] \right.$$
$$\left. +_{n_c} \alpha(\eta - \tau)^2 \exp(\alpha[\beta(\eta - \tau) + \tau]) \right\},$$

$$L_{221}^{(3)} = L_{212}^{(3)} = L_{122}^{(3)} = \frac{\partial^3 \ln L}{\partial \theta^2 \partial \beta} = 0,$$

$$L_{123}^{(3)} = L_{213}^{(3)} = L_{132}^{(3)} = L_{213}^{(3)} = L_{231}^{(3)} = L_{312}^{(3)} = L_{321}^{(3)}$$
$$= \frac{\partial^3 \ln L}{\partial \beta \partial \theta \partial \alpha} = -\sum_{i=1}^{n_a} (y_i - \tau)[\beta(y_i - \tau) + \tau] \exp(\alpha[\beta(y_i - \tau) + \tau])$$
$$-_{n_c} (\eta - \tau)[\beta(\eta - \tau) + \tau] \exp(\alpha[\beta(\eta - \tau) + \tau]),$$

$$L_{331}^{(3)} = L_{313}^{(3)} = L_{133}^{(3)} = \frac{\partial^3 \ln L}{\partial \alpha^2 \partial \beta} = -\frac{\theta}{\alpha^2}[\alpha\omega_7 - \omega_6] + \frac{\theta}{\alpha^3}[\alpha\omega_6 - 2\omega_5],$$

where

$$\omega_5 = \frac{\partial \omega_1}{\partial \beta} = \alpha \sum_{i=1}^{n_a} (y_i - \tau) \exp(\alpha[\beta(y_i - \tau) + \tau])$$
$$+ \alpha\, n_c(\eta - \tau) \exp(\alpha[\beta(\eta - \tau) + \tau]),$$

$$\omega_6 = \frac{\partial \omega_2}{\partial \beta} = \alpha \sum_{i=1}^{n_a} (y_i - \tau)[\beta(y_i - \tau) + \tau] \exp(\alpha[\beta(y_i - \tau) + \tau])$$
$$+ \sum_{i=1}^{n_a} (y_i - \tau) \exp(\alpha[\beta(y_i - \tau) + \tau])$$
$$+ \alpha\, n_c(\eta - \tau)[\beta(\eta - \tau) + \tau][\exp(\alpha[\beta(\eta - \tau) + \tau])]$$
$$+_{n_c} (\eta - \tau) \exp(\alpha[\beta(\eta - \tau) + \tau])$$

and

$$\omega_7 = \frac{\partial \omega_3}{\partial \beta} = \alpha \sum_{i=1}^{n_a} (y_i - \tau)[\beta(y_i - \tau) + \tau]^2 \exp(\alpha[\beta(y_i - \tau) + \tau])$$
$$+ 2\sum_{i=1}^{n_a} (y_i - \tau)[\beta(y_i - \tau) + \tau] \exp(\alpha[\beta(y_i - \tau) + \tau])$$
$$+ \alpha\, n_c(\eta - \tau)[\beta(\eta - \tau) + \tau]^2[\exp(\alpha[\beta(\eta - \tau) + \tau])]$$
$$+ 2_{n_c}(\eta - \tau)[\beta(\eta - \tau) + \tau] \exp(\alpha[\beta(\eta - \tau) + \tau]).$$

$$L_{332}^{(3)} = L_{323}^{(3)} = L_{233}^{(3)} = \frac{\partial^3 \ln L}{\partial \alpha^2 \partial \theta} = -\frac{1}{\alpha^2}[\alpha\omega_3 - \omega_2] + \frac{1}{\alpha^3}[\alpha\omega_2 - 2\omega_1],$$

$$L_{223}^{(3)} = L_{232}^{(3)} = L_{322}^{(3)} = \frac{\partial^3 \ln L}{\partial \theta^2 \partial \alpha} = 0,$$

$$L_{113}^{(3)} = L_{131}^{(3)} = L_{311}^{(3)} = \frac{\partial^3 \ln L}{\partial \beta^2 \partial \alpha} = -\theta \left\{ \left[\sum_{i=1}^{n_a} (y_i - \tau)^2 \exp(\alpha[\beta(y_i - \tau) + \tau]) \right] \right.$$

$$\left. +_{n_c} (\eta - \tau)^2 \exp(\alpha[\beta(\eta - \tau) + \tau]) \right\}$$

$$- \theta\alpha \left\{ \sum_{i=1}^{n_a} (y_i - \tau)^2 [\beta(y_i - \tau) + \tau] \exp(\alpha[\beta(y_i - \tau) + \tau]) \right.$$

$$\left. +_{n_c} (\eta - \tau)^2 [\beta(\eta - \tau) + \tau][\exp(\alpha[\beta(\eta - \tau) + \tau])] \right\}.$$