Contents

Problem Set 1

Problem 1. [24 points]

Translate the following sentences from English to predicate logic. The domain that you are working over is X, the set of people. You may use the functions $S(x)$, meaning that "x has been a student of 6.042," $A(x)$, meaning that "x has gotten an 'A' in 6.042," $T(x)$, meaning that "x is a TA of 6.042," and $E(x, y)$, meaning that "x and y are the same person."

(a) [6 pts] There are people who have taken 6.042 and have gotten A's in 6.042

$$\exists x \in X, S(x) \ and \ A(x).$$

(b) [6 pts] All people who are 6.042 TA's and have taken 6.042 got A's in 6.042

$$\forall x \in X, T(x) \ and \ S(x) \implies A(x).$$

(c) [6 pts] There are no people who are 6.042 TA's who did not get A's in 6.042.

$$\forall x \in X, T(x) \iff A(x).$$

(d) [6 pts] There are at least three people who are TA's in 6.042 and have not taken 6.042

$$\exists x, y, z \in X, x \neq y \ and \ y \neq z \ and \ x \neq z \ and \ T(x) \ and \ T(y) \ and \ T(z) \ and \ \neg S(x) \ and \ \neg S(y) \ and \ \neg S(z)$$

Problem 2. [24 points]

Use a truth table to prove or disprove the following statements:

(a) [12 pts]

$$\neg(P \vee (Q \wedge R)) = (\neg P) \wedge (\neg Q \vee \neg R)$$

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$\neg(P \vee (Q \wedge R))$	$\neg P$	$\neg Q$	$\neg R$	$\neg Q \vee \neg R$	$(\neg P) \wedge (\neg Q \vee \neg R)$
T	T	T	T	T	F	F	F	F	F	F
T	T	F	F	T	F	F	F	T	T	F
T	F	T	F	T	F	F	T	F	T	F
T	F	F	F	T	F	F	T	T	T	F
F	T	T	T	T	F	T	F	F	F	F
F	T	F	F	F	T	T	F	T	T	T
F	F	T	F	F	T	T	T	F	T	T
F	F	F	F	F	T	T	T	T	T	T

According to the truth table above

$$\neg(P \vee (Q \wedge R)) = (\neg P) \vee (\neg Q \vee \neg R)$$

(b) [12 pts]

$$\neg(P \wedge (Q \vee R)) = \neg P \vee (\neg Q \vee \neg R)$$

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$\neg(P \vee (Q \wedge R))$	$\neg P$	$\neg Q$	$\neg R$	$\neg Q \vee \neg R$	$\neg P \vee (\neg Q \vee \neg R)$
T	T	T	T	T	F	F	F	F	F	F
T	T	F	F	T	F	F	F	T	T	T
T	F	T	F	T	F	F	T	F	T	T
T	F	F	F	T	F	F	T	T	T	T
F	T	T	T	T	F	T	F	F	F	T
F	T	F	F	F	T	T	F	T	T	T
F	F	T	F	F	T	T	T	F	T	T
F	F	F	F	F	T	T	T	T	T	T

According to the truth table above

$$\neg(P \vee (Q \wedge R)) \neq \neg P \vee (\neg Q \vee \neg R)$$

Problem 3. [24 points]

The binary logical connectives $\wedge$ (and), $\vee$ (or), and $\implies$ (implies) appear often in not only computer programs, but also everyday speech. In computer chip designs, however, it is considerably easier to construct these out of another operation, *nand*, which is simpler to represent in a circuit. Here is the truth table for *nand*:

P	Q	P *nand* Q
true	*true*	*false*
true	*false*	*true*
false	*true*	*true*
false	*false*	*true*

a) [12 pts] For each of the following expressions, find an equivalent expression using only nand and $\neg$ (not), as well as grouping parentheses to specify the order in which the operations apply. You may use A, B, and the operators any number of times.

(i) $A \wedge B$

A	B	$A \wedge B$	A *nand* B	$\neg(A$ *nand* $B)$
T	T	T	F	T
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

$$A \wedge B = \neg(A \text{ } nand \text{ } B)$$

(ii) $A \vee B$

A	B	$A \vee B$	$\neg A$	$\neg B$	$\neg A$ *nand* $\neg B$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	T	T	F	T
F	F	F	T	T	F

$$A \vee B = \neg A \text{ } nand \text{ } \neg B$$

(iii) $A \implies B$

A	B	$A \implies B$	$\neg A$	$\neg A \vee B$	$\neg B$	A *nand* $\neg B$
T	T	T	F	T	F	T
T	F	F	F	F	T	F
F	T	T	T	T	F	T
F	F	T	T	T	T	T

$$A \implies B = A \text{ } nand \text{ } \neg B$$

b) [4 pts] It is actually possible to express each of the above using only nand, without needing to use $\neg$. Find an equivalent expression for $(\neg A)$ using only nand and grouping parentheses.

A	$\neg A$	A *nand* A
T	F	F
F	T	T

$$\neg A = A \ nand \ A$$

(c) [8 pts] The constants $true$ and $false$ themselves may be expressed using only $nand$. Construct an expression using an arbitrary statement A and $nand$ that evaluates to $true$ regardless of whether A is $true$ or $false$. Construct a second expression that always evaluates to $false$. Do not use the constants $true$ and $false$ themselves in your statements.

First let's devise a tautology i.e an expression that is always true regardless of its component.

One example of tautology is $A \lor \neg A$

We previously saw that:

$$A \lor B = \neg A \ nand \ \neg B$$

$$A \lor \neg A = \neg A \ nand \ A \ with \ B = \neg A \ and \ \neg\neg A = A$$

$$A \lor \neg A = (A \ nand \ A) \ nand \ A \ as \ \neg A = A \ nand \ A$$

So, $(A \ nand \ A) \ nand \ A$ is an expression that always evaluate to $true$ whether A is $true$ or $false$.

On the other hand, a contradiction is a statement that is always $false$, regardless of the truth values of its individual components.

For example, the statement $A \land \neg A$ is a contradiction because it is always $false$, no matter the truth value of A.

As we previously saw,

$$A \land B = \neg(A \ nand \ B)$$

With $B = \neg A$

$$A \land \neg A = \neg(A \ nand \ \neg A)$$

With $\neg A = A \ nand \ A$

$$A \land \neg A = (A \ nand \ (A \ nand \ A)) \ nand \ (A \ nand \ (A \ nand \ A))$$

We can conclude that $(A \ nand \ (A \ nand \ A)) \ nand \ (A \ nand \ (A \ nand \ A))$ is an expression that is always $_{\text{false}}$ regardless of the truth value of A.