Understanding quantum information and computation

By John Watrous

Lesson 2

Multiple systems

IBM

Classical states

Suppose that we have two systems:

- X is a system having classical state set Σ.
- Y is a system having classical state set Γ.

Imagine that X and Y are placed side-by-side, with X on the left and Y on the right, and viewed together as if they form a single system.

We denote this new compound system by (X, Y) or XY.

Question

What are the classical states of (X, Y)?

Answer

The classical state set of (X, Y) is the *Cartesian product*

$$\Sigma \times \Gamma = \{(a, b) : a \in \Sigma \text{ and } b \in \Gamma\}$$

Classical states

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$$\Sigma \times \Gamma = \{(a, b) : a \in \Sigma \text{ and } b \in \Gamma\}$$

Example

If $\Sigma = \{0, 1\}$ and $\Gamma = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$, then

$$\Sigma \times \Gamma = \{(0, \clubsuit), (0, \diamondsuit), (0, \heartsuit), (0, \spadesuit), (1, \clubsuit), (1, \diamondsuit), (1, \heartsuit), (1, \spadesuit)\}$$

Classical states

This description generalizes to more than two systems in a natural way.

Suppose $X_1, \ldots, X_n$ are systems having classical state sets $\Sigma_1, \ldots, \Sigma_n$, respectively.

The classical state set of the n-tuple $(X_1, \ldots, X_n)$, viewed as a single compound system, is the Cartesian product

$$\Sigma_1 \times \cdots \times \Sigma_n = \{(a_1, \ldots, a_n) : a_1 \in \Sigma_1, \ \ldots, \ a_n \in \Sigma_n\}$$

Example

If $\Sigma_1 = \Sigma_2 = \Sigma_3 = \{0, 1\}$, then the classical state set of (X_1, X_2, X_3) is

$$\begin{aligned}\Sigma_1 \times \Sigma_2 \times \Sigma_3 = \{&(0,0,0),(0,0,1),(0,1,0),(0,1,1),\\ &(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}\end{aligned}$$

Classical states

An n-tuple $(a_1, \ldots, a_n)$ may also be written as a *string* $a_1 \cdots a_n$.

Example

Suppose $\mathsf{X}_1, \ldots, \mathsf{X}_{10}$ are bits, so their classical state sets are all the same:

$$\Sigma_1 = \Sigma_2 = \cdots = \Sigma_{10} = \{0, 1\}$$

The classical state set of $(\mathsf{X}_1, \ldots, \mathsf{X}_{10})$ is the Cartesian product

$$\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_{10} = \{0, 1\}^{10}$$

Classical states

An n-tuple $(a_1, \ldots, a_n)$ may also be written as a *string* $a_1 \cdots a_n$.

Example

The classical state set of $(X_1, \ldots, X_{10})$ is the Cartesian product

$$\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_{10} = \{0, 1\}^{10}$$

Written as strings, these classical states look like this:

$$\begin{array}{c} 0000000000 \\ 0000000001 \\ 0000000010 \\ 0000000011 \\ \vdots \\ 1111111111 \end{array}$$

Classical states

Convention

Cartesian products of classical state sets are ordered *lexicographically* (i.e., dictionary ordering):

- We assume the individual classical state sets are already ordered.
- Significance decreases from left to right.

Example

The Cartesian product $\{1, 2, 3\} \times \{0, 1\}$ is ordered like this:

$$(1,0),\ (1,1),\ (2,0),\ (2,1),\ (3,0),\ (3,1)$$

When n-tuples are written as strings and ordered in this way, we observe familiar patterns, such as $\{0, 1\} \times \{0, 1\}$ being ordered as $00, 01, 10, 11$.

Probabilistic states

Probabilistic states of compound systems associate probabilities with the Cartesian product of the classical state sets of the individual systems.

Example

This is a probabilistic state of a pair of bits (X, Y):

$$
\begin{aligned}
\Pr\big((X,Y) = (0,0)\big) &= \tfrac{1}{2} \\
\Pr\big((X,Y) = (0,1)\big) &= 0 \\
\Pr\big((X,Y) = (1,0)\big) &= 0 \\
\Pr\big((X,Y) = (1,1)\big) &= \tfrac{1}{2}
\end{aligned}
$$

Probabilistic states

Probabilistic states of compound systems associate probabilities with the Cartesian product of the classical state sets of the individual systems.

Example

This is a probabilistic state of a pair of bits (X,Y):

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{array}{l} \leftarrow \text{probability associated with state } 00 \\ \leftarrow \text{probability associated with state } 01 \\ \leftarrow \text{probability associated with state } 10 \\ \leftarrow \text{probability associated with state } 11 \end{array}$$

Probabilistic states

Definition

For a given probabilistic state of (X, Y), we say that X and Y are *independent* if

$$\Pr((X,Y) = (a,b)) = \Pr(X = a)\Pr(Y = b)$$

for all $a \in \Sigma$ and $b \in \Gamma$.

Suppose that a probabilistic state of (X, Y) is expressed as a vector:

$$|\pi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} p_{ab}|ab\rangle$$

The systems X and Y are independent if there exist probability vectors

$$|\phi\rangle = \sum_{a\in\Sigma} q_a|a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b\in\Gamma} r_b|b\rangle$$

such that $p_{ab} = q_a r_b$ for all $a \in \Sigma$ and $b \in \Gamma$.

Probabilistic states

Example

The probabilistic state of a pair of bits (X, Y) represented by the vector

$$|\pi\rangle = \frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle$$

is one in which X and Y are independent. The required condition is true for these probability vectors:

$$|\phi\rangle = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle \quad \text{and} \quad |\psi\rangle = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle$$

Probabilistic states

Example

For the probabilistic state

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle$$

of two bits (X, Y), we have that X and Y are not independent.

If they were, we would have numbers q_0, q_1, r_0, r_1 such that

$$\begin{aligned} q_0 r_0 &= \tfrac{1}{2} \\ q_0 r_1 &= 0 \\ q_1 r_0 &= 0 \\ q_1 r_1 &= \tfrac{1}{2} \end{aligned}$$

But if $q_0 r_1 = 0$, then either $q_0 = 0$ or $r_1 = 0$ (or both), contradicting either the first or last equality.

Tensor products of vectors

Definition

The *tensor product* of two vectors

$$|\phi\rangle = \sum_{a\in\Sigma} \alpha_a |a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b\in\Gamma} \beta_b |b\rangle$$

is the vector

$$|\phi\rangle \otimes |\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_a \beta_b |ab\rangle$$

Equivalently, the vector $|\pi\rangle = |\phi\rangle \otimes |\psi\rangle$ is defined by this condition:

$$\langle ab|\pi\rangle = \langle a|\phi\rangle\langle b|\psi\rangle \qquad (\text{for all } a \in \Sigma \text{ and } b \in \Gamma)$$

Tensor products of vectors

Definition

$$|\phi\rangle = \sum_{a\in\Sigma} \alpha_a |a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b\in\Gamma} \beta_b |b\rangle$$

$$|\phi\rangle \otimes |\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_a \beta_b |ab\rangle$$

Example

$$|\phi\rangle = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle \quad \text{and} \quad |\psi\rangle = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle$$

$$|\phi\rangle \otimes |\psi\rangle = \frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle$$

Tensor products of vectors

Definition

$$|\phi\rangle = \sum_{a\in\Sigma} \alpha_a |a\rangle \quad \text{and} \quad |\psi\rangle = \sum_{b\in\Gamma} \beta_b |b\rangle$$

$$|\phi\rangle \otimes |\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_a \beta_b |ab\rangle$$

Alternative notation for tensor products:

$$|\phi\rangle|\psi\rangle = |\phi\rangle \otimes |\psi\rangle$$

$$|\phi \otimes \psi\rangle = |\phi\rangle \otimes |\psi\rangle$$

Tensor products of vectors

Following our convention for ordering the elements of Cartesian product sets, we obtain this specification for the tensor product of two column vectors:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} \alpha_1 \beta_1 \\ \vdots \\ \alpha_1 \beta_k \\ \alpha_2 \beta_1 \\ \vdots \\ \alpha_2 \beta_k \\ \vdots \\ \alpha_m \beta_1 \\ \vdots \\ \alpha_m \beta_k \end{pmatrix}$$

Tensor products of vectors

Example

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \alpha_1\beta_3 \\ \alpha_1\beta_4 \\ \alpha_2\beta_1 \\ \alpha_2\beta_2 \\ \alpha_2\beta_3 \\ \alpha_2\beta_4 \\ \alpha_3\beta_1 \\ \alpha_3\beta_2 \\ \alpha_3\beta_3 \\ \alpha_3\beta_4 \end{pmatrix}$$

Tensor products of vectors

Observe the following expression for tensor products of standard basis vectors:

$$|a\rangle \otimes |b\rangle = |a\rangle|b\rangle = |ab\rangle$$

Alternatively, writing (a, b) as an ordered pair rather than a string, we could write

$$|a\rangle \otimes |b\rangle = |(a, b)\rangle$$

but it is more common to write

$$|a\rangle \otimes |b\rangle = |a, b\rangle$$

(It is a standard convention in mathematics to eliminate parentheses when they do not serve to add clarity or remove ambiguity.)

Tensor products of vectors

Important property of tensor products

The tensor product of two vectors is *bilinear.*

1. Linearity in the first argument:

$$\big(|\phi_1\rangle + |\phi_2\rangle\big) \otimes |\psi\rangle = |\phi_1\rangle \otimes |\psi\rangle + |\phi_2\rangle \otimes |\psi\rangle$$

$$\big(\alpha|\phi\rangle\big) \otimes |\psi\rangle = \alpha\big(|\phi\rangle \otimes |\psi\rangle\big)$$

2. Linearity in the second argument:

$$|\phi\rangle \otimes \big(|\psi_1\rangle + |\psi_2\rangle\big) = |\phi\rangle \otimes |\psi_1\rangle + |\phi\rangle \otimes |\psi_2\rangle$$

$$|\phi\rangle \otimes \big(\alpha|\psi\rangle\big) = \alpha\big(|\phi\rangle \otimes |\psi\rangle\big)$$

Notice that scalars "float freely" within tensor products:

$$\big(\alpha|\phi\rangle\big) \otimes |\psi\rangle = |\phi\rangle \otimes \big(\alpha|\psi\rangle\big) = \alpha\big(|\phi\rangle \otimes |\psi\rangle\big) = \alpha|\phi\rangle \otimes |\psi\rangle$$

Tensor products of vectors

Tensor products generalize to three or more systems.

If $|\phi_1\rangle, \ldots, |\phi_n\rangle$ are vectors, then the tensor product

$$|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$$

is defined by the equation

$$\langle a_1 \cdots a_n | \psi \rangle = \langle a_1 | \phi_1 \rangle \cdots \langle a_n | \phi_n \rangle$$

Equivalently, the tensor product of three or more vectors can be defined recursively:

$$|\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle = \big(|\phi_1\rangle \otimes \cdots \otimes |\phi_{n-1}\rangle\big) \otimes |\phi_n\rangle$$

The tensor product of three or more vectors is *multilinear.*

Measurements of probabilistic states

Measurements of compound systems work in the same way as measurements of single systems — provided that all of the systems are measured.

Example

Suppose that two bits (X, Y) are in the probabilistic state

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle$$

Measuring both bits yields the outcome 00 with probability 1/2 and the outcome 11 with probability 1/2.

Measurements of probabilistic states

Question

Suppose two systems (X, Y) are together in some probabilistic state. What happens when we measure X and do nothing to Y?

Answer

1. The probability to observe a particular classical state $a \in \Sigma$ when just X is measured is

$$\Pr(\mathsf{X} = a) = \sum_{b \in \Gamma} \Pr\bigl((\mathsf{X}, \mathsf{Y}) = (a, b)\bigr)$$

2. There may still exist uncertainty about the classical state of Y, depending on the outcome of the measurement:

$$\Pr(\mathsf{Y} = b \,|\, \mathsf{X} = a) = \frac{\Pr\bigl((\mathsf{X}, \mathsf{Y}) = (a, b)\bigr)}{\Pr(\mathsf{X} = a)}$$

Measurements of probabilistic states

These formulas can be expressed using the Dirac notation as follows.

Suppose that (X, Y) is in some arbitrary probabilistic state:

$$\sum_{(a,b)\in\Sigma\times\Gamma} p_{ab}|ab\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} p_{ab}|a\rangle \otimes |b\rangle = \sum_{a\in\Sigma}|a\rangle \otimes \Biggl(\sum_{b\in\Gamma} p_{ab}|b\rangle\Biggr)$$

1. The probability that a measurement of X yields an outcome $a \in \Sigma$ is

$$\Pr(\mathsf{X} = a) = \sum_{b\in\Gamma} p_{ab}$$

2. Conditioned on the outcome $a \in \Sigma$, the probabilistic state of Y becomes

$$\frac{\sum_{b\in\Gamma} p_{ab}|b\rangle}{\sum_{c\in\Gamma} p_{ac}}$$

Measurements of probabilistic states

Example

Suppose (X, Y) is in the probabilistic state

$$\frac{1}{12}\vert 00\rangle + \frac{1}{4}\vert 01\rangle + \frac{1}{3}\vert 10\rangle + \frac{1}{3}\vert 11\rangle$$

We write this vector as follows:

$$\vert 0\rangle \otimes \left(\frac{1}{12}\vert 0\rangle + \frac{1}{4}\vert 1\rangle\right) + \vert 1\rangle \otimes \left(\frac{1}{3}\vert 0\rangle + \frac{1}{3}\vert 1\rangle\right)$$

Measurements of probabilistic states

Example

Suppose (X, Y) is in the probabilistic state

$$|0\rangle \otimes \left(\frac{1}{12}|0\rangle + \frac{1}{4}|1\rangle\right) + |1\rangle \otimes \left(\frac{1}{3}|0\rangle + \frac{1}{3}|1\rangle\right)$$

Case 1: the measurement outcome is 0.

$$\Pr(\text{outcome is } 0) = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$$

Conditioned on this outcome, the probabilistic state of Y becomes

$$\frac{\frac{1}{12}|0\rangle + \frac{1}{4}|1\rangle}{\frac{1}{3}} = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle$$

Measurements of probabilistic states

Example

Suppose (X,Y) is in the probabilistic state

$$|0\rangle \otimes \left(\frac{1}{12}|0\rangle + \frac{1}{4}|1\rangle\right) + |1\rangle \otimes \left(\frac{1}{3}|0\rangle + \frac{1}{3}|1\rangle\right)$$

Case 2: the measurement outcome is 1.

$$\Pr(\text{outcome is } 1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Conditioned on this outcome, the probabilistic state of Y becomes

$$\frac{\frac{1}{3}|0\rangle + \frac{1}{3}|1\rangle}{\frac{2}{3}} = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle$$

Measurements of probabilistic states

The same method can be used when Y is measured rather than X. Suppose that (X,Y) is in some arbitrary probabilistic state:

$$\sum_{(a,b)\in\Sigma\times\Gamma} p_{ab}|ab\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} p_{ab}|a\rangle\otimes|b\rangle = \sum_{b\in\Gamma}\left(\sum_{a\in\Sigma} p_{ab}|a\rangle\right)\otimes|b\rangle$$

1. The probability that a measurement of Y yields an outcome $a \in \Sigma$ is

$$\Pr(\mathsf{Y}=b) = \sum_{a\in\Sigma} p_{ab}$$

2. Conditioned on the outcome $b \in \Gamma$, the probabilistic state of X becomes

$$\frac{\sum_{a\in\Sigma} p_{ab}|a\rangle}{\sum_{c\in\Sigma} p_{c,b}}$$

Operations on probabilistic states

Probabilistic operations on compound systems are represented by stochastic matrices having rows and columns that correspond to the Cartesian product of the individual systems' classical state sets.

Example

A *controlled-NOT* operation on two bits X and Y:

> *If* $\mathsf{X} = 1$, *then perform a NOT operation on* Y, *otherwise do nothing.*

X is the *control bit* that determines whether or not a NOT operation is applied to the *target bit* Y.

Operations on probabilistic states

Example

A *controlled-NOT* operation on two bits X and Y:

> *If* $\mathsf{X} = 1$*, then perform a NOT operation on* Y*, otherwise do nothing.*

X is the *control bit* that determines whether or not a NOT operation is applied to the *target bit* Y.

Action on standard basis

$$
\begin{aligned}
|00\rangle &\mapsto |00\rangle\\
|01\rangle &\mapsto |01\rangle\\
|10\rangle &\mapsto |11\rangle\\
|11\rangle &\mapsto |10\rangle
\end{aligned}
$$

Matrix representation

$$
\begin{pmatrix}
1 & 0 & 0 & 0\\
0 & 1 & 0 & 0\\
0 & 0 & 0 & 1\\
0 & 0 & 1 & 0
\end{pmatrix}
$$

Operations on probabilistic states

Example

Here is a different operation on two bits (X, Y):

> *With probability 1/2, set* Y *to be equal to* X*, otherwise set* X *to be equal to* Y*.*

The matrix representation of this operation is as follows:

$$\begin{pmatrix} 1 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Operations on probabilistic states

Question

Suppose we have two probabilistic operations, each on its own system, described by stochastic matrices:

1. M is an operation on X.
2. N is an operation on Y.

If we *simultaneously* perform the two operations, how do we describe the effect on the compound system (X, Y)?

Tensor products of matrices

Definition

The *tensor product* of two matrices

$$M = \sum_{a,b\in\Sigma} \alpha_{ab} |a\rangle\langle b| \quad \text{and} \quad N = \sum_{c,d\in\Gamma} \beta_{cd} |c\rangle\langle d|$$

is the matrix

$$M \otimes N = \sum_{a,b\in\Sigma} \sum_{c,d\in\Gamma} \alpha_{ab} \beta_{cd} |ac\rangle\langle bd|$$

Equivalently, $M \otimes N$ is defined by this condition:

$$\langle ac|M \otimes N|bd\rangle = \langle a|M|b\rangle\langle c|N|d\rangle \quad \text{(for all } a, b \in \Sigma \text{ and } b, d \in \Gamma\text{)}$$

Tensor products of matrices

Definition

$$M = \sum_{a,b\in\Sigma} \alpha_{ab} |a\rangle\langle b| \qquad \text{and} \qquad N = \sum_{c,d\in\Gamma} \beta_{cd} |c\rangle\langle d|$$

$$M \otimes N = \sum_{a,b\in\Sigma} \sum_{c,d\in\Gamma} \alpha_{ab} \beta_{cd} |ac\rangle\langle bd|$$

An alternative, but equivalent, way to define $M \otimes N$ is that it is the unique matrix that satisfies the equation

$$(M \otimes N)|\phi \otimes \psi\rangle = M|\phi\rangle \otimes N|\psi\rangle$$

for every choice of vectors $|\phi\rangle$ and $|\psi\rangle$.

Tensor products of matrices

$$
\begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1m} \\
\vdots & \ddots & \vdots \\
\alpha_{m1} & \cdots & \alpha_{mm}
\end{pmatrix}
\otimes
\begin{pmatrix}
\beta_{11} & \cdots & \beta_{1k} \\
\vdots & \ddots & \vdots \\
\beta_{k1} & \cdots & \beta_{kk}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\begin{matrix}
\alpha_{11}\beta_{11} & \cdots & \alpha_{11}\beta_{1k} \\
\vdots & \ddots & \vdots \\
\alpha_{11}\beta_{k1} & \cdots & \alpha_{11}\beta_{kk}
\end{matrix} & \cdots &
\begin{matrix}
\alpha_{1m}\beta_{11} & \cdots & \alpha_{1m}\beta_{1k} \\
\vdots & \ddots & \vdots \\
\alpha_{1m}\beta_{k1} & \cdots & \alpha_{1m}\beta_{kk}
\end{matrix} \\
\vdots & \ddots & \vdots \\
\begin{matrix}
\alpha_{m1}\beta_{11} & \cdots & \alpha_{m1}\beta_{1k} \\
\vdots & \ddots & \vdots \\
\alpha_{m1}\beta_{k1} & \cdots & \alpha_{m1}\beta_{kk}
\end{matrix} & \cdots &
\begin{matrix}
\alpha_{mm}\beta_{11} & \cdots & \alpha_{mm}\beta_{1k} \\
\vdots & \ddots & \vdots \\
\alpha_{mm}\beta_{k1} & \cdots & \alpha_{mm}\beta_{kk}
\end{matrix}
\end{pmatrix}
$$

Tensor products of matrices

Example

$$\begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} \otimes \begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_{00}\beta_{00} & \alpha_{00}\beta_{01} & \alpha_{01}\beta_{00} & \alpha_{01}\beta_{01} \\ \alpha_{00}\beta_{10} & \alpha_{00}\beta_{11} & \alpha_{01}\beta_{10} & \alpha_{01}\beta_{11} \\ \alpha_{10}\beta_{00} & \alpha_{10}\beta_{01} & \alpha_{11}\beta_{00} & \alpha_{11}\beta_{01} \\ \alpha_{10}\beta_{10} & \alpha_{10}\beta_{11} & \alpha_{11}\beta_{10} & \alpha_{11}\beta_{11} \end{pmatrix}$$

Tensor products of matrices

Tensor products of three or more matrices are defined in an analogous way.

If $M_1, \ldots, M_n$ are matrices, then the tensor product $M_1 \otimes \cdots \otimes M_n$ is defined by the condition

$$\langle a_1 \cdots a_n | M_1 \otimes \cdots \otimes M_n | b_1 \cdots b_n \rangle = \langle a_1 | M_1 | b_1 \rangle \cdots \langle a_n | M_n | b_n \rangle$$

Alternatively, the tensor product of three or more matrices can be defined recursively, similar to what we observed for vectors.

The tensor product of matrices is *multiplicative:*

$$(M_1 \otimes \cdots \otimes M_n)(N_1 \otimes \cdots \otimes N_n) = (M_1 N_1) \otimes \cdots \otimes (M_n N_n)$$

Operations on probabilistic states

Question

Suppose we have two probabilistic operations, each on its own system, described by stochastic matrices:

1. M is an operation on X.
2. N is an operation on Y.

If we *simultaneously* perform the two operations, how do we describe the effect on the compound system (X, Y)?

Answer

The action is described by the tensor product $M \otimes N$.

Tensor products represent *independence* — this time between operations.

Operations on probabilistic states

Example

Recall this operation from Lesson 1:

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

Suppose this operation is performed on a bit X, and a NOT operation is (independently) performed on a second bit Y.

The combined operation on the compound system (X, Y) then has this matrix representation:

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Operations on probabilistic states

A common situation that we encounter is one in which one operation is performed on one system and *nothing* is done to another system.

The same prescription is followed, noting that doing nothing is represented by the *identity matrix.*

Example

Resetting a bit X to the 0 state and doing nothing to a bit Y yields this operation on (X, Y):

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Quantum states

Quantum state vectors of multiple systems are represent by column vectors whose indices correspond to the Cartesian product of the individual systems' classical state sets.

Example

If X and Y are qubits, the classical state set for the pair (X, Y) is

$$\{0,1\} \times \{0,1\} = \{00, 01, 10, 11\}$$

These are examples of quantum state vectors of the pair (X, Y):

$$\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle$$

$$\frac{3}{5}|00\rangle - \frac{4}{5}|11\rangle$$

$$|01\rangle$$

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These are examples of quantum state vectors of the pair (X, Y):

$$\frac{1}{2}|0\rangle|0\rangle + \frac{i}{2}|0\rangle|1\rangle - \frac{1}{2}|1\rangle|0\rangle - \frac{i}{2}|1\rangle|1\rangle$$

$$\frac{3}{5}|0\rangle|0\rangle - \frac{4}{5}|1\rangle|1\rangle$$

$$|0\rangle|1\rangle$$

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These are examples of quantum state vectors of the pair (X, Y):

$$\frac{1}{2}|0\rangle \otimes |0\rangle + \frac{i}{2}|0\rangle \otimes |1\rangle - \frac{1}{2}|1\rangle \otimes |0\rangle - \frac{i}{2}|1\rangle \otimes |1\rangle$$

$$\frac{3}{5}|0\rangle \otimes |0\rangle - \frac{4}{5}|1\rangle \otimes |1\rangle$$

$$|0\rangle \otimes |1\rangle$$

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Example

If X and Y are qubits, the classical state set for the pair (X, Y) is

$$\{0, 1\} \times \{0, 1\} = \{00, 01, 10, 11\}$$

These are examples of quantum state vectors of the pair (X, Y):

$$\frac{1}{2}|0\rangle_{\mathsf{X}}|0\rangle_{\mathsf{Y}} + \frac{i}{2}|0\rangle_{\mathsf{X}}|1\rangle_{\mathsf{Y}} - \frac{1}{2}|1\rangle_{\mathsf{X}}|0\rangle_{\mathsf{Y}} - \frac{i}{2}|1\rangle_{\mathsf{X}}|1\rangle_{\mathsf{Y}}$$

$$\frac{3}{5}|0\rangle_{\mathsf{X}}|0\rangle_{\mathsf{Y}} - \frac{4}{5}|1\rangle_{\mathsf{X}}|1\rangle_{\mathsf{Y}}$$

$$|0\rangle_{\mathsf{X}}|1\rangle_{\mathsf{Y}}$$

Quantum states

Quantum state vectors of multiple systems are represent by column vectors whose indices correspond to the Cartesian product of the individual systems' classical state sets.

Example

If X and Y are qubits, the classical state set for the pair (X, Y) is

$$\{0,1\} \times \{0,1\} = \{00, 01, 10, 11\}$$

These are examples of quantum state vectors of the pair (X, Y):

$$\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle$$

$$\frac{3}{5}|00\rangle - \frac{4}{5}|11\rangle$$

$$|01\rangle$$

Quantum states

Tensor products of quantum state vectors are also quantum state vectors.

Let $|\phi\rangle$ be a quantum state vector of a system X and let $|\psi\rangle$ be a quantum state vector of a system Y. The tensor product

$$|\phi\rangle \otimes |\psi\rangle$$

is then a quantum state vector of the system (X, Y).

States of this form are called *product states.* They represent *independence* between the systems X and Y.

More generally, if $|\psi_1\rangle, \ldots, |\psi_n\rangle$ are quantum state vectors of systems $X_1, \ldots, X_n$, then

$$|\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$$

is a quantum state vector representing a product state of the compound system $(X_1, \ldots, X_n)$.

Quantum states

Example

The quantum state vector

$$\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle$$

is an example of a product state:

$$\begin{aligned}
&\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle \\
&\qquad = \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right)
\end{aligned}$$

Quantum states

Example

The quantum state vector

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

of two qubits is not a product state.

Quantum states

Suppose it were possible to write

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = |\phi\rangle \otimes |\psi\rangle$$

It would then follow that

$$\langle 0|\phi\rangle\langle 1|\psi\rangle = \langle 01|\phi \otimes \psi\rangle = 0$$

implying that

$$\langle 0|\phi\rangle = 0 \text{ or } \langle 1|\psi\rangle = 0 \text{ (or both)}$$

This contradicts these equalities:

$$\langle 0|\phi\rangle\langle 0|\psi\rangle = \langle 00|\phi \otimes \psi\rangle = \frac{1}{\sqrt{2}}$$

$$\langle 1|\phi\rangle\langle 1|\psi\rangle = \langle 11|\phi \otimes \psi\rangle = \frac{1}{\sqrt{2}}$$

Example

The quantum state vector

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

of two qubits is not a product state.

Quantum states

The previous example of a quantum state vector is one of the four *Bell states,* which collectively form the *Bell basis.*

The Bell basis

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

Quantum states

Here are a couple of well-known examples of quantum state vectors for three-qubits.

GHZ state

$$\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$$

W state

$$\frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle$$

Measurements

Measurements of compound systems work in the same way measurements of single systems — provided that all of the systems are measured.

If $|\psi\rangle$ a quantum state of a system $(X_1, \ldots, X_n)$, and every one of the systems is measured, then each n-tuple

$$(a_1, \ldots, a_n) \in \Sigma_1 \times \cdots \times \Sigma_n$$

(or string $a_1 \cdots a_n$) is obtained with probability

$$\bigl|\langle a_1 \cdots a_n | \psi \rangle\bigr|^2$$

Measurements

Measurements of compound systems work in the same way measurements of single systems — provided that all of the systems are measured.

Example

If the pair (X, Y) is in the quantum state

$$\frac{3}{5}|0\rangle|\heartsuit\rangle - \frac{4i}{5}|1\rangle|\spadesuit\rangle$$

then measuring both systems yields the outcome $(0, \heartsuit)$ with probability $9/25$ and the outcome $(1, \spadesuit)$ with probability $16/25$.

Measurements

Question

Suppose two systems (X, Y) are together in some *quantum* state.
What happens when we measure X and do nothing to Y?

A quantum state vector of (X, Y) takes the form

$$|\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_{ab}|ab\rangle$$

If both X and Y are measured, then each outcome $(a, b) \in \Sigma \times \Gamma$ appears with probability

$$\big|\langle ab|\psi\rangle\big|^2 = |\alpha_{ab}|^2$$

Measurements

Question

Suppose two systems (X, Y) are together in some *quantum* state.
What happens when we measure X and do nothing to Y?

A quantum state vector of (X, Y) takes the form

$$|\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_{ab}|ab\rangle$$

If just X is measured, the probability for each outcome $a \in \Sigma$ to appear must therefore be equal to

$$\Pr(\text{outcome is } a) = \sum_{b\in\Gamma} \left|\langle ab|\psi\rangle\right|^2 = \sum_{b\in\Gamma} |\alpha_{ab}|^2$$

Similar to the probabilistic setting, the quantum state of Y changes as a result...

Measurements

A quantum state vector of (X, Y) takes the form

$$|\psi\rangle = \sum_{(a,b)\in\Sigma\times\Gamma} \alpha_{ab}|ab\rangle$$

We can express the vector $|\psi\rangle$ as

$$|\psi\rangle = \sum_{a\in\Sigma} |a\rangle \otimes |\phi_a\rangle$$

where

$$|\phi_a\rangle = \sum_{b\in\Gamma} \alpha_{ab}|b\rangle$$

for each $a \in \Sigma$.

Measurements

A quantum state vector of (X,Y) takes the form

$$|\psi\rangle = \sum_{a\in\Sigma} |a\rangle \otimes |\phi_a\rangle \qquad \text{where} \qquad |\phi_a\rangle = \sum_{b\in\Gamma} \alpha_{ab}|b\rangle$$

1. The probability to obtain each outcome $a \in \Sigma$ is

$$\operatorname{Pr}(\text{outcome is } a) = \sum_{b\in\Gamma} |\alpha_{ab}|^2 = \big\| |\phi_a\rangle \big\|^2$$

2. As a result of the standard basis measurement of X giving the outcome a, the quantum state of (X,Y) becomes

$$|a\rangle \otimes \frac{|\phi_a\rangle}{\big\| |\phi_a\rangle \big\|}$$

Measurements

Example

Suppose that (X, Y) is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{i}{2\sqrt{2}}|10\rangle - \frac{1}{2\sqrt{2}}|11\rangle$$

and X is measured.

We begin by writing

$$|\psi\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle\right) + |1\rangle \otimes \left(\frac{i}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right)$$

Measurements

Example

Suppose that (X, Y) is in the state

$$|\psi\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle \right) + |1\rangle \otimes \left(\frac{i}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle \right)$$

and X is measured.

The probability for the measurement to result in the outcome 0 is

$$\left\| \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle \right\|^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

in which case the state of (X, Y) becomes

$$|0\rangle \otimes \frac{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle}{\sqrt{\frac{3}{4}}} = |0\rangle \otimes \left(\sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle \right)$$

Measurements

Example

Suppose that (X,Y) is in the state

$$|\psi\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|1\rangle\right) + |1\rangle \otimes \left(\frac{i}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right)$$

and X is measured.

The probability for the measurement to result in the outcome 1 is

$$\left\| \frac{i}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle \right\|^2 = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

in which case the state of (X,Y) becomes

$$|1\rangle \otimes \frac{\frac{i}{2\sqrt{2}}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle}{\sqrt{\frac{1}{4}}} = |1\rangle \otimes \left(\frac{i}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

Measurements

Example

Suppose that (X, Y) is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{i}{2\sqrt{2}}|10\rangle - \frac{1}{2\sqrt{2}}|11\rangle$$

and Y is measured.

We begin by writing

$$|\psi\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{2\sqrt{2}}|1\rangle\right) \otimes |0\rangle + \left(\frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right) \otimes |1\rangle$$

Measurements

Example

Suppose that (X, Y) is in the state

$$|\psi\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{2\sqrt{2}}|1\rangle\right) \otimes |0\rangle + \left(\frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right) \otimes |1\rangle$$

and Y is measured.

The probability for the measurement to result in the outcome 0 is

$$\left\| \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{2\sqrt{2}}|1\rangle \right\|^2 = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$$

in which case the state of (X, Y) becomes

$$\frac{\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{2\sqrt{2}}|1\rangle}{\sqrt{\frac{5}{8}}} = \left(\sqrt{\frac{4}{5}}|0\rangle + \frac{i}{\sqrt{5}}|1\rangle\right) \otimes |0\rangle$$

Measurements

Example

Suppose that (X, Y) is in the state

$$|\psi\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{2\sqrt{2}}|1\rangle\right) \otimes |0\rangle + \left(\frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle\right) \otimes |1\rangle$$

and Y is measured.

The probability for the measurement to result in the outcome 1 is

$$\left\| \frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle \right\|^2 = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

in which case the state of (X, Y) becomes

$$\frac{\frac{1}{2}|0\rangle - \frac{1}{2\sqrt{2}}|1\rangle}{\sqrt{\frac{3}{8}}} = \left(\sqrt{\frac{2}{3}}|0\rangle - \frac{1}{\sqrt{3}}|1\rangle\right) \otimes |1\rangle$$

Unitary operations

Quantum operations on compound systems are represented by unitary matrices whose rows and columns correspond to the Cartesian product of the classical state sets of the individual systems.

Example

Suppose X has classical state set $\{1,2,3\}$ and Y has classical state set $\{0,1\}$. This unitary matrix represents an operation on (X,Y):

$$U = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{i}{2} & -\frac{1}{2} & 0 & 0 & -\frac{i}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} & 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Unitary operations

The combined action of a collection of unitary operations applied independently to a collection of systems is represented by the *tensor product* of the unitary matrices.

That is, if $\mathsf{X}_1, \ldots, \mathsf{X}_n$ are quantum systems, $U_1, \ldots, U_n$ are unitary matrices representing operations on these systems, and the operations are performed independently on the systems, the combined action on $(\mathsf{X}_1, \ldots, \mathsf{X}_n)$ is represented by the matrix

$$U_1 \otimes \cdots \otimes U_n$$

In particular, if we perform a unitary operation U on a system X and do nothing to a system Y, the operation on (X, Y) we obtain is represented by the unitary matrix

$$U \otimes \mathbb{1} \quad \text{or alternatively} \quad U \otimes \mathbb{1}_{\mathsf{Y}}$$

Unitary operations

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That is, if $\mathsf{X}_1, \ldots, \mathsf{X}_n$ are quantum systems, $U_1, \ldots, U_n$ are unitary matrices representing operations on these systems, and the operations are performed independently on the systems, the combined action on $(\mathsf{X}_1, \ldots, \mathsf{X}_n)$ is represented by the matrix

$$U_1 \otimes \cdots \otimes U_n$$

In particular, if we perform a unitary operation V on a system Y and do nothing to a system X, the operation on (X, Y) we obtain is represented by the unitary matrix

$$\mathbb{1} \otimes V \quad \text{or alternatively} \quad \mathbb{1}_{\mathsf{X}} \otimes V$$

Unitary operations

Example

Suppose X and Y are qubits.

Performing a Hadamard operation on X and doing nothing to Y is equivalent to performing this unitary operation on (X,Y):

$$H \otimes \mathbb{1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Unitary operations

Example

Suppose X and Y are qubits.

Performing a Hadamard operation on Y and doing nothing to X is equivalent to performing this unitary operation on (X, Y):

$$\mathbb{1} \otimes H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Unitary operations

Not every unitary operation on a compound system can be expressed as a tensor product of unitary operations.

Example

Suppose that X and Y are systems that share the same classical state set Σ. The *swap operation* on the pair (X, Y) exchange the contents of the two systems:

$$\mathrm{SWAP}|\phi \otimes \psi\rangle = |\psi \otimes \phi\rangle$$

It can be expressed using the Dirac notation as follows:

$$\mathrm{SWAP} = \sum_{a,b\in\Sigma} |a\rangle\langle b| \otimes |b\rangle\langle a|$$

Unitary operations

Not every unitary operation on a compound system can be expressed as a tensor product of unitary operations.

Example

The swap operation can be expressed using the Dirac notation as follows:

$$\mathrm{SWAP} = \sum_{a,b\in\Sigma} |a\rangle\langle b| \otimes |b\rangle\langle a|$$

For instance, when X and Y are qubits, we find that

$$\mathrm{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Unitary operations

Not every unitary operation on a compound system can be expressed as a tensor product of unitary operations.

Example

$$\mathrm{SWAP}|\phi^+\rangle = |\phi^+\rangle$$
$$\mathrm{SWAP}|\phi^-\rangle = |\phi^-\rangle$$
$$\mathrm{SWAP}|\psi^+\rangle = |\psi^+\rangle$$
$$\mathrm{SWAP}|\psi^-\rangle = -|\psi^-\rangle$$

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$
$$|\phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$$
$$|\psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$
$$|\psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

Unitary operations

Suppose that X is a qubit and Y is an arbitrary system.

For every unitary operation U on Y, a *controlled-U* operation is a unitary operation on the pair (X, Y) defined as follows:

$$|0\rangle\langle 0| \otimes \mathbb{1}_{\mathsf{Y}} + |1\rangle\langle 1| \otimes U = \begin{pmatrix} \mathbb{1}_{\mathsf{Y}} & 0 \\ 0 & U \end{pmatrix}$$

Example

A controlled-NOT operation (where the first qubit is the control):

$$|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Unitary operations

Suppose that X is a qubit and Y is an arbitrary system.

For every unitary operation U on Y, a *controlled-U* operation is a unitary operation on the pair (X, Y) defined as follows:

$$|0\rangle\langle 0| \otimes \mathbb{1}_{\mathsf{Y}} + |1\rangle\langle 1| \otimes U = \begin{pmatrix} \mathbb{1}_{\mathsf{Y}} & 0 \\ 0 & U \end{pmatrix}$$

Example

A controlled-NOT operation (where the second qubit is the control):

$$\mathbb{1} \otimes |0\rangle\langle 0| + \sigma_x \otimes |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Unitary operations

Suppose that X is a qubit and Y is an arbitrary system.

For every unitary operation U on Y, a *controlled-*U operation is a unitary operation on the pair (X, Y) defined as follows:

$$|0\rangle\langle 0| \otimes \mathbb{1}_{\mathsf{Y}} + |1\rangle\langle 1| \otimes \mathrm{U} = \begin{pmatrix} \mathbb{1}_{\mathsf{Y}} & 0 \\ 0 & \mathrm{U} \end{pmatrix}$$

Example

A controlled-σ_z (or controlled-Z) operation:

$$|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Unitary operations

Suppose that X is a qubit and Y is an arbitrary system.

For every unitary operation U on Y, a *controlled-U* operation is a unitary operation on the pair (X, Y) defined as follows:

$$|0\rangle\langle 0| \otimes \mathbb{1}_{\mathsf{Y}} + |1\rangle\langle 1| \otimes \mathsf{U} = \begin{pmatrix} \mathbb{1}_{\mathsf{Y}} & 0 \\ 0 & \mathsf{U} \end{pmatrix}$$

Example

A controlled-σ_z (or controlled-Z) operation:

$$\mathbb{1} \otimes |0\rangle\langle 0| + \sigma_z \otimes |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Unitary operations

Example

A controlled-SWAP operation (on three qubits):

$$|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \mathrm{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This operation is also known as a *Fredkin operation* (or Fredkin gate).

Unitary operations

Example

A controlled-controlled-NOT operation (on three qubits):

$$
\begin{aligned}
&|0\rangle\langle 0| \otimes \mathbb{1} \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \bigl(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \sigma_x \bigr) \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{aligned}
$$

This operation is better known as a *Toffoli operation* (or Toffoli gate).