$$\partial_t u = \nabla \cdot (D_V \nabla u) \qquad x \in Y_S \tag{1}$$
$$D_V \nabla u \cdot \nu = k(u - \bar{u}) \qquad x \in \Gamma \tag{2}$$

We assume D_V to be constant in Y_S.
Ansatz:

$$u_\varepsilon(x,y) = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \mathcal{O}(\varepsilon^3) \tag{3}$$

Chain rule for the gradient:

$$\nabla = (\nabla_x + \frac{1}{\varepsilon}\nabla_y) \tag{4}$$

The left hand side of (1) becomes:

$$\partial_t u_\varepsilon = \partial_t \left(u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) \right) \tag{5}$$

The right hand side of (1) becomes:

$$\begin{aligned}
&\nabla \cdot (D_V \nabla u_\varepsilon) = \nabla \cdot (D_V \nabla (u_0 + \varepsilon u_1 + \varepsilon^2 u_2))\\
=&(\nabla_x + \frac{1}{\varepsilon}\nabla_y) \cdot (D_V(\nabla_x + \frac{1}{\varepsilon}\nabla_y)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2))\\
=&D_V(\varepsilon^{-2}(\nabla_y \cdot \nabla_y u_0)\\
&+\varepsilon^{-1}(\nabla_x \cdot \nabla_y u_0 + \nabla_y \cdot \nabla_x u_0 + \nabla_y \cdot \nabla_y u_1)\\
&+\varepsilon^{0}(\nabla_x \cdot \nabla_x u_0 + \nabla_x \cdot \nabla_y u_1 + \nabla_y \cdot \nabla_x u_1 + \nabla_y \cdot \nabla_y u_2)\\
&+\varepsilon^{1}(\nabla_x \cdot \nabla_x u_1 + \nabla_x \cdot \nabla_y u_2 + \nabla_y \cdot \nabla_x u_2)\\
&+\varepsilon^{2}(\nabla_x \cdot \nabla_x u_2))
\end{aligned}$$

Comparing the orders of ε we get for ε^{-2}:

$$\nabla_y \cdot \nabla_y u_0(x,y) = 0 \tag{6}$$

Because of the periodicity in y it is $u_0(x,y) = u_0(x)$.
Therefore we get from the ε^{-1}:

$$\nabla_x \cdot \nabla_y u_0(x) = \nabla_y \cdot \nabla_x u_0(x) = 0 \tag{7}$$

It follows:

$$\nabla_y \cdot \nabla_y u_1(x,y) = 0 \tag{8}$$

We write u_1 as linear combination of u_0

$$u_1(x,y) = \sum_{j=1}^{n} w_j(y) \partial_{x_j} u_0(x) \tag{9}$$

The y-gradient can be written as

$$\nabla_y u_1(x,y) = \nabla_y \sum_{j=1}^{n} w_j(y)\partial_{x_j} u_0(x) \tag{10}$$

$$= \sum_{j=1}^{n} \partial_{x_j} u_0(x) \nabla_y w_j(y) \tag{11}$$

$$= \sum_{j=1}^{n} \partial_{x_j} u_0(x) \sum_{i=1} e_i \partial_{y_i} w_j(y) \tag{12}$$

Or component-wise

$$\partial_{y_i} u_1(x,y) = \sum_{j=1}^{n} \partial_{x_j} u_0(x) \partial_{y_i} w_j(y)$$

It is obvious that $\nabla_x u_0(x)$ can be represented as:

$$\nabla_x u_0(x) = \sum_{i=1}^{n} e_i \partial_{x_i} u_0(x) \tag{13}$$

From the ε^0 term we get:

$$\partial_t u_0 = D_V(\nabla_x \cdot \nabla_x u_0 + \nabla_x \cdot \nabla_y u_1 + \nabla_y \cdot \nabla_x u_1 + \nabla_y \cdot \nabla_y u_2)$$

We integrate over the cell:

$$\int_{Y_S} \partial_t u_0 \,\mathrm{d}\, y = D_V(\int_{Y_S} \nabla_x \cdot (\nabla_x u_0 + \nabla_y u_1) \,\mathrm{d}\, y + \int_{Y_S} \nabla_y \cdot (\nabla_x u_1 + \nabla_y u_2) \,\mathrm{d}\, y)$$

The first integral then becomes:

$$\begin{aligned}
&D_V \int_{Y_S} \nabla_x \cdot (\nabla_x u_0(x) + \nabla_y u_1(x,y)) \,\mathrm{d}\, y \\
&= D_V \int_{Y_S} \nabla_x \cdot (\sum_{i=1}^{n} e_i \partial_{x_i} u_0(x) + \sum_{j=1}^{n} \partial_{x_j} u_0(x) \sum_{i=1}^{n} e_i \partial_{y_i} w_j(y)) \,\mathrm{d}\, y \\
&= D_V \int_{Y_s} \sum_{i=1}^{n} \partial_{x_i}^2 u_0(x) + \sum_{j=1}^{n} \sum_{i=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) \partial_{y_i} w_j(y) \,\mathrm{d}\, y \\
&= D_V \int_{Y_S} \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} \partial_{x_i} \partial_{x_j} u_0(x) + \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) \partial_{y_i} w_j(y) \,\mathrm{d}\, y \\
&= D_V \int_{Y_S} \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) (\delta_{ij} + \partial_{y_i} w_j(y)) \,\mathrm{d}\, y \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) D_V \int_{Y_S} (\delta_{ij} + \partial_{y_i} w_j(y)) \,\mathrm{d}\, y
\end{aligned}$$

We introduce now:

$$\hat{D}_{ij} = D_V \int_{Y_S} (\delta_{ij} + \partial_{y_i} w_j(y)) \,\mathrm{d}\, y \tag{14}$$

Then we get:

$$D_V \int_{Y_S} \nabla_x \cdot (\nabla_x u_0(x) + \nabla_y u_1(x,y)) \,\mathrm{d}\, y = \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) \hat{D}_{ij} \tag{15}$$

Consider the second integral and integrate by parts:

$$D_V \int_{Y_S} \nabla_y \cdot (\nabla_x u_1 + \nabla_y u_2) \,\mathrm{d}\, y = D_V \int_{\Gamma} (\nabla_x u_1 + \nabla_y u_2) \cdot \nu \,\mathrm{d}\, \Gamma(y) \tag{16}$$

Putting these two integrals together we get:

$$\int_{Y_S} \partial_t u_0(x) \,\mathrm{d}\, y = \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) \hat{D}_{ij} + D_V \int_{\Gamma} (\nabla_x u_1 + \nabla_y u_2) \cdot \nu \,\mathrm{d}\, \Gamma(y) \tag{17}$$

We consider now the boundary condition. Using the ansatz function we get:

$$\begin{aligned} k(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 - \bar{u}) = D_V (&\varepsilon^{-1} \nabla_y u_0 \\ &+ \varepsilon^0 (\nabla_x u_0 + \nabla_y u_1) \\ &+ \varepsilon (\nabla_x u_1 + \nabla_y u_2) \\ &+ \varepsilon^2 \nabla_x u_2) \cdot \nu \end{aligned}$$

Assume $k = \mathcal{O}(\varepsilon)$:

$$\begin{aligned} \varepsilon^{-1} D_V (\nabla_y u_0(x)) \cdot \nu &= 0 && (18) \\ D_V (\nabla_x u_0 + \nabla_y u_1) \cdot \nu &= 0 && (19) \\ \varepsilon D_V (\nabla_x u_1 + \nabla_y u_2) \cdot \nu &= k(u_0 - \bar{u}) && (20) \\ \varepsilon^2 D_V (\nabla_x u_2) \cdot \nu &= \varepsilon k u_1 && (21) \\ 0 &= \varepsilon^2 k u_2 && (22) \end{aligned}$$

Applying the boundary condition to (17) and with the assumption $k \approx \varepsilon$ we get:

$$\int_{Y_S} \partial_t u_0(x) \,\mathrm{d}\, y = \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) \hat{D}_{ij} + D_V \int_{\Gamma} (\nabla_x u_1 + \nabla_y u_2) \cdot \nu \,\mathrm{d}\, \Gamma(y) \tag{23}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) \hat{D}_{ij} + \int_{\Gamma} u_0(x) - \bar{u} \,\mathrm{d}\, \Gamma(y) \tag{24}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) \hat{D}_{ij} + \mid \Gamma \mid (u_0 - \bar{u}) \tag{25}$$

With this assumption for the boundary condition, we get:
Assume k is independent of ε:

$$\varepsilon^{-1} D_V(\nabla_y u_0(x)) \cdot \nu = 0 \tag{26}$$

$$D_V(\nabla_x u_0 + \nabla_y u_1) \cdot \nu = k(u_0 - \bar{u}) \tag{27}$$

$$\varepsilon D_V(\nabla_x u_1 + \nabla_y u_2) \cdot \nu = \varepsilon k u_1 \tag{28}$$

$$\varepsilon^2 D_V(\nabla_x u_2) \cdot \nu = \varepsilon^2 k u_2 \tag{29}$$

With this assumption we get:

$$\int_{Y_S} \partial_t u_0(x) \,\mathrm{d}\, y = \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) \hat{D}_{ij} + D_V \int_\Gamma (\nabla_x u_1 + \nabla_y u_2) \cdot \nu \,\mathrm{d}\,\Gamma(y) \tag{30}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) \hat{D}_{ij} + \int_\Gamma k u_1(x, y) \,\mathrm{d}\,\Gamma(y) \tag{31}$$

$$\tag{32}$$