Homogenization

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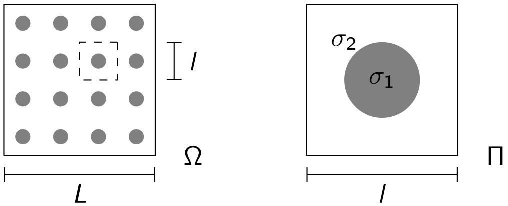
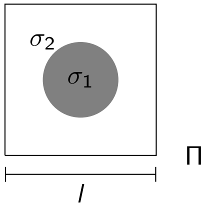
Introduction

Given a composite material



- Ω - domain with scale L
- Π - periodic cell with scale l
- $\varepsilon = \frac{l}{L} \ll 1$

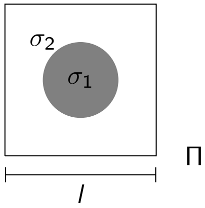
Example (Conductivity Problem)



$$
\begin{aligned}
-\operatorname{div}(\sigma(x)\nabla u) &= g(x), && x \in \Omega \\
u &= 0, && x \in \partial\Omega
\end{aligned}
$$

$$
\sigma(x) = \begin{cases} \sigma_1, x \in \Omega_1 \\ \sigma_2, x \in \Omega_2 \end{cases}
$$

Example (Conductivity Problem)



$$
\begin{aligned}
-\operatorname{div}(\sigma(x/\varepsilon)\nabla u_\varepsilon) &= g(x), && x \in \Omega \\
u_\varepsilon &= 0, && x \in \partial\Omega
\end{aligned}
$$

$$
\sigma(x) = \begin{cases} \sigma_1, x \in \Omega_1 \\ \sigma_2, x \in \Omega_2 \end{cases}
$$

Formal Asymptotic Expansion

Ansatz:

$$u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \ldots$$

Formal Asymptotic Expansion

Ansatz:

$$u(x,y) = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \ldots$$

with $y = \frac{x}{\varepsilon}$
Consider x (slow) and y (fast) as independent variables

Two-scale problem

$$\varepsilon = \frac{l}{L} \ll 1$$

$$\varepsilon \to 0$$

$$\begin{aligned} -\operatorname{div}(\sigma(x/\varepsilon)\nabla u_\varepsilon) &= g(x), & x \in \Omega \\ u_\varepsilon &= 0, & \partial\Omega \end{aligned}$$

$$-\operatorname{div}(\hat{\sigma}\nabla u_0) = g(x), \qquad x \in \Omega \tag{1}$$

$$u_0 = 0, \qquad x \in \partial\Omega \tag{2}$$

$$-\operatorname{div}(\sigma(\frac{x}{\varepsilon})\nabla u_\varepsilon) = g(x), \qquad x \in \Omega$$

$$u_\varepsilon = 0, \qquad x \in \partial\Omega$$

$$u_\varepsilon \rightharpoonup u_0 \qquad \text{weakly in } H_0^1(\Omega)$$

$$\sigma_\varepsilon \nabla u_\varepsilon \rightharpoonup \hat{\sigma} \nabla u_0 \qquad \text{weakly in } L^2(\Omega)$$

Effective Conductivity Tensor

$$\hat{\sigma}_{i,j} = \frac{1}{|\Pi|} \int_{\Pi} (\sigma(y)(\nabla \chi_i + \boldsymbol{e}_i)) \cdot (\nabla \chi_j + \boldsymbol{e}_j) \, \mathrm{d}\, y$$

Cell problem

$$-\operatorname{div}\left[\sigma(y)[\nabla\chi_i + \boldsymbol{e}_i]\right] = 0, \quad y \in \Pi \tag{3}$$

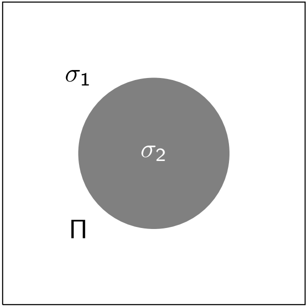


Abbildung: Unit cell

Two-Scale Convergence

Weak Convergence
Two-Scale convergence

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\Psi(x, \frac{x}{\varepsilon})\,\mathrm{d}\,x = \int_\Omega \int_\Pi u_0(x,y)\Psi(x,y)\,\mathrm{d}\,y\,\mathrm{d}\,x$$