TODO

- Discription of the microstucture.
- Discription of the model
- Pictures
- macroscopic problem
- cell problem

Program structure

- General things
  - parameter file
- Microscopic problem
  - declare parameter
  - make grid
  - distribute levelset, material
  - assemble system
  - assemble rhs
  - boundary and interface conditions
  - solve
  - visualize solution
- Homogenized problem
  - Cell problem
    * make grid
    * levelset
    * assemble system
    * assemble rhs
    * periodic boundary condition, (interface condition?)
    * solve
    * visualize
  - Makroscopic Problem
    * compute tensor
    * make grid
    * assemble system
    * assemble rhs
    * boundary condition
    * solve
    * visualize solution
- Domain discription

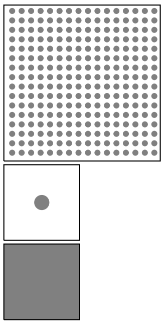

Abbildung 1: Microscopic problem

- PDE + Explaination and application
- trouble with microstructure
- assumption of periodicity
- homogenization process
- cell problem and macroscopic problem
- How does the equation change, interface and boundary condition
- Implementation details
  - unfitted mesh and level set function
  - xfem
  -
- three scale homogenization
- small cut problem
- ghost penalty

1 Homogenization

We consider a composite material in the domain $\Omega = S \cup P$, where S denotes the solid phase, and P denotes the pores.

We consider the following PDE on the microscopic domain.

$$\nabla \cdot (D_V \nabla u) = f(x) \qquad x \in S \tag{1}$$

$$D_V \nabla u \cdot \nu = k(u - \bar{u}) \qquad x \in \Gamma \tag{2}$$

The diffusion D_V is defined as

$$D_V(x) = \begin{cases} D_V = const, & x \in S \\ 0, & x \in P \end{cases}$$

To derive a homogenized problem we separate the scale in two parts.
We assume D_V to be constant in S.
Ansatz:

$$u_\varepsilon(x,y) = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \mathcal{O}(\varepsilon^3) \tag{3}$$

Chain rule for the gradient:

$$\nabla = (\nabla_x + \frac{1}{\varepsilon}\nabla_y) \tag{4}$$

The left hand side of (1) becomes:

$$\partial_t u_\varepsilon = \partial_t \left(u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y)\right) \tag{5}$$

The right hand side of (1) becomes:

$$\begin{aligned}
&\nabla \cdot (D_V \nabla u_\varepsilon) = \nabla \cdot (D_V \nabla(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)) \\
=&(\nabla_x + \frac{1}{\varepsilon}\nabla_y) \cdot (D_V(\nabla_x + \frac{1}{\varepsilon}\nabla_y)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)) \\
=&D_V(\varepsilon^{-2}(\nabla_y \cdot \nabla_y u_0) \\
&+ \varepsilon^{-1}(\nabla_x \cdot \nabla_y u_0 + \nabla_y \cdot \nabla_x u_0 + \nabla_y \cdot \nabla_y u_1) \\
&+ \varepsilon^{0}(\nabla_x \cdot \nabla_x u_0 + \nabla_x \cdot \nabla_y u_1 + \nabla_y \cdot \nabla_x u_1 + \nabla_y \cdot \nabla_y u_2) \\
&+ \varepsilon^{1}(\nabla_x \cdot \nabla_x u_1 + \nabla_x \cdot \nabla_y u_2 + \nabla_y \cdot \nabla_x u_2) \\
&+ \varepsilon^{2}(\nabla_x \cdot \nabla_x u_2))
\end{aligned}$$

Comparing the orders of ε we get for ε^{-2}:

$$\nabla_y \cdot \nabla_y u_0(x,y) = 0 \tag{6}$$

Because of the periodicity in y it is $u_0(x,y) = u_0(x)$.
Therefore we get from the ε^{-1}:

$$\nabla_x \cdot \nabla_y u_0(x) = \nabla_y \cdot \nabla_x u_0(x) = 0 \tag{7}$$

It follows:

$$\nabla_y \cdot \nabla_y u_1(x,y) = 0 \tag{8}$$

We write u_1 as linear combination of u_0

$$u_1(x,y) = \sum_{j=1}^{n} w_j(y) \partial_{x_j} u_0(x) \tag{9}$$

The y-gradient can be written as

$$\nabla_y u_1(x,y) = \nabla_y \sum_{j=1}^{n} w_j(y) \partial_{x_j} u_0(x) \tag{10}$$

$$= \sum_{j=1}^{n} \partial_{x_j} u_0(x) \nabla_y w_j(y) \tag{11}$$

$$= \sum_{j=1}^{n} \partial_{x_j} u_0(x) \sum_{i=1} e_i \partial_{y_i} w_j(y) \tag{12}$$

Or component-wise

$$\partial_{y_i} u_1(x, y) = \sum_{j=1}^{n} \partial_{x_j} u_0(x) \partial_{y_i} w_j(y)$$

It is obvious that $\nabla_x u_0(x)$ can be represented as:

$$\nabla_x u_0(x) = \sum_{i=1}^{n} e_i \partial_{x_i} u_0(x) \tag{13}$$

From the ε^0 term we get:

$$\partial_t u_0 = D_V(\nabla_x \cdot \nabla_x u_0 + \nabla_x \cdot \nabla_y u_1 + \nabla_y \cdot \nabla_x u_1 + \nabla_y \cdot \nabla_y u_2)$$

We integrate over the cell:

$$\int_{Y_S} \partial_t u_0 \,\mathrm{d}\, y = D_V(\int_{Y_S} \nabla_x \cdot (\nabla_x u_0 + \nabla_y u_1) \,\mathrm{d}\, y + \int_{Y_S} \nabla_y \cdot (\nabla_x u_1 + \nabla_y u_2) \,\mathrm{d}\, y)$$

Since $u_0(x)$ is independent of y it becomes:

$$\int_{Y_S} \partial_t u_0 \,\mathrm{d}\, y = \mid Y_S \mid \partial_t u_0$$

Dividing by $\mid Y_S \mid$ we get:

$$\partial_t u_0 = \frac{1}{\mid Y_S \mid} D_V(\int_{Y_S} \nabla_x \cdot (\nabla_x u_0 + \nabla_y u_1) \,\mathrm{d}\, y + \int_{Y_S} \nabla_y \cdot (\nabla_x u_1 + \nabla_y u_2) \,\mathrm{d}\, y)$$

The first integral then becomes:

$$\begin{aligned}
&D_V \int_{Y_S} \nabla_x \cdot (\nabla_x u_0(x) + \nabla_y u_1(x, y)) \,\mathrm{d}\, y \\
&\quad = \frac{1}{\mid Y_S \mid} D_V \int_{Y_S} \nabla_x \cdot (\sum_{i=1}^{n} e_i \partial_{x_i} u_0(x) + \sum_{j=1}^{n} \partial_{x_j} u_0(x) \sum_{i=1}^{n} e_i \partial_{y_i} w_j(y)) \,\mathrm{d}\, y \\
&\quad = \frac{1}{\mid Y_S \mid} D_V \int_{Y_S} \sum_{i=1}^{n} \partial_{x_i}^2 u_0(x) + \sum_{j=1}^{n} \sum_{i=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) \partial_{y_i} w_j(y) \,\mathrm{d}\, y \\
&\quad = \frac{1}{\mid Y_S \mid} D_V \int_{Y_S} \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} \partial_{x_i} \partial_{x_j} u_0(x) + \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) \partial_{y_i} w_j(y) \,\mathrm{d}\, y \\
&\quad = \frac{1}{\mid Y_S \mid} D_V \int_{Y_S} \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) (\delta_{ij} + \partial_{y_i} w_j(y)) \,\mathrm{d}\, y \\
&\quad = \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) \frac{1}{\mid Y_S \mid} D_V \int_{Y_S} (\delta_{ij} + \partial_{y_i} w_j(y)) \,\mathrm{d}\, y
\end{aligned}$$

We introduce now:

$$\hat{D}_{ij} = \frac{1}{\mid Y_S \mid} D_V \int_{Y_S} (\delta_{ij} + \partial_{y_i} w_j(y)) \,\mathrm{d}\, y \tag{14}$$

Then we get:

$$D_V \int_{Y_S} \nabla_x \cdot (\nabla_x u_0(x) + \nabla_y u_1(x,y)) \,\mathrm{d}\, y = \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) \hat{D}_{ij} \tag{15}$$

Consider the second integral and integrate by parts:

$$D_V \int_{Y_S} \nabla_y \cdot (\nabla_x u_1 + \nabla_y u_2) \,\mathrm{d}\, y = D_V \int_{\Gamma} (\nabla_x u_1 + \nabla_y u_2) \cdot \nu \,\mathrm{d}\, \Gamma(y) \tag{16}$$

Putting these two integrals together we get:

$$\int_{Y_S} \partial_t u_0(x) \,\mathrm{d}\, y = \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{x_i} \partial_{x_j} u_0(x) \hat{D}_{ij} + D_V \int_{\Gamma} (\nabla_x u_1 + \nabla_y u_2) \cdot \nu_y \,\mathrm{d}\, \Gamma(y) \tag{17}$$

Where ν_y is the normal vector in the y scale.
We consider now the boundary condition. Using the ansatz function we get:

$$\begin{aligned} k(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 - \bar{u}) = D_V (\varepsilon^{-1} \nabla_y u_0 & \qquad (18) \\ + \varepsilon^0 (\nabla_x u_0 + \nabla_y u_1) & \qquad (19) \\ + \varepsilon (\nabla_x u_1 + \nabla_y u_2) & \qquad (20) \\ + \varepsilon^2 \nabla_x u_2) \cdot \nu & \qquad (21) \end{aligned}$$

Where ν is the normal vector in the macroscale. ??? To apply the boundary condition, we need to express ν in terms of ν_y. Since the microstucture is represented in cell problem, we get the direction of the normal from ν_y but to get unit vector we have to scale it with $\frac{1}{\varepsilon}$. So it is $\nu = \frac{\nu_y}{\varepsilon}$
Therefore the boundary condition (18), becomes:

$$\begin{aligned} k(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 - \bar{u}) = D_V (\varepsilon^{-2} \nabla_y u_0 & \qquad (22) \\ + \varepsilon^{-1} (\nabla_x u_0 + \nabla_y u_1) & \\ + \varepsilon^0 (\nabla_x u_1 + \nabla_y u_2) & \qquad (23) \\ + \varepsilon \nabla_x u_2) \cdot \nu_y & \end{aligned}$$

Now we sort the terms by orders of ε and get:

$$\begin{aligned} \varepsilon^{-1} D_V (\nabla_y u_0(x)) \cdot \nu &= 0 \qquad (24) \\ D_V (\nabla_x u_0 + \nabla_y u_1) \cdot \nu &= 0 \\ \varepsilon D_V (\nabla_x u_1 + \nabla_y u_2) \cdot \nu &= k(u_0 - \bar{u}) \qquad (25) \\ \varepsilon^2 D_V (\nabla_x u_2) \cdot \nu &= \varepsilon k u_1 \\ 0 &= \varepsilon^2 k u_2 \end{aligned}$$

Applying the boundary condition to (17) and with the assumption $k \approx \varepsilon$ we get:

$$\int_{Y_S} \partial_t u_0(x)\,\mathrm{d}\,y = \sum_{i=1}^{n}\sum_{j=1}^{n} \partial_{x_i}\partial_{x_j} u_0(x)\hat{D}_{ij} + D_V \int_{\Gamma} (\nabla_x u_1 + \nabla_y u_2)\cdot \nu \,\mathrm{d}\,\Gamma(y) \tag{26}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \partial_{x_i}\partial_{x_j} u_0(x)\hat{D}_{ij} + \int_{\Gamma} u_0(x) - \bar{u}\,\mathrm{d}\,\Gamma(y) \tag{27}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \partial_{x_i}\partial_{x_j} u_0(x)\hat{D}_{ij} + \mid \Gamma \mid (u_0 - \bar{u}) \tag{28}$$