Homogenization

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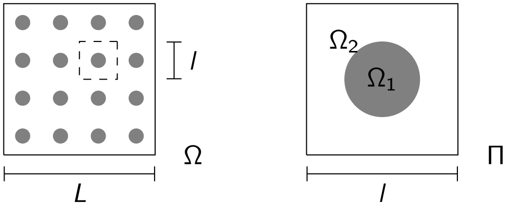
Introduction

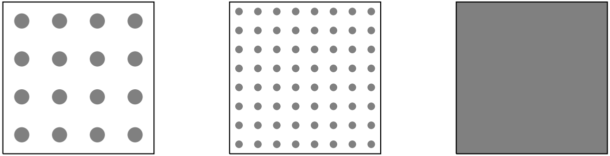
Given a composite material



- Ω - domain with macroscale L
- Π - periodic cell with microscale l
- Assume $\varepsilon = \frac{l}{L} \ll 1$
- Solve with FEM too expensive (very fine mesh)

Homogenization

- Homogenization: $\varepsilon \to 0$



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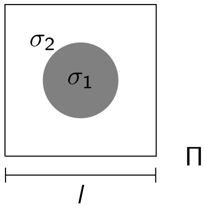
Example (Diffusion Problem)

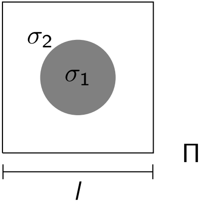


$$
\begin{aligned}
-\operatorname{div}(\sigma(x)\nabla u) &= g(x), && x \in \Omega \\
u &= 0, && x \in \partial\Omega
\end{aligned}
$$

Diffusion Tensor rapidly oscillates

$$
\sigma(x) = \begin{cases} \sigma_1, & x \in \Omega_1 \\ \sigma_2, & x \in \Omega_2 \end{cases}
$$

Example (Diffusion Problem)



$$
\begin{aligned}
-\operatorname{div}(\sigma(x/\varepsilon)\nabla u_\varepsilon) &= g(x), && x \in \Omega \\
u_\varepsilon &= 0, && x \in \partial\Omega
\end{aligned}
$$

Diffusion Tensor rapidly oscillates

$$
\sigma(x) = \begin{cases} \sigma_1, & x \in \Omega_1 \\ \sigma_2, & x \in \Omega_2 \end{cases}
$$

Formal Asymptotic Expansion

Ansatz:

$$u_\varepsilon(x) = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \dots$$

- with u_ε is Π-periodic in $y = \frac{x}{\varepsilon}$
- x - slow variable and y - fast variable
- Consider x and y as independent variables

With chain rule for ∇:

$$\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y$$

Homogenized Problem

$$-\operatorname{div}(\hat{\sigma}\nabla u_0) = g(x), \qquad x \in \Omega$$
$$u_0 = 0, \qquad x \in \partial\Omega$$

- $\hat{\sigma}$ - homogenized diffusion tensor

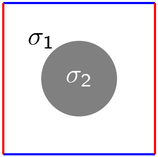
$$\hat{\sigma}_{i,j} = \frac{1}{|\Pi|}\int_\Pi (\sigma(y)(\nabla\chi_i + \boldsymbol{e}_i)) \cdot (\nabla\chi_j + \boldsymbol{e}_j)\,\mathrm{d}\,y$$

- $\boldsymbol{e}_i$ - unit vector
- χ_i - solution of the cell problem

$$-\operatorname{div}\left[\sigma(y)[\nabla\chi_i + \boldsymbol{e}_i]\right] = 0, \quad y \in \Pi$$

Cell Problem

$$-\operatorname{div}\left[\sigma(y)[\nabla\chi_i + \boldsymbol{e}_i]\right] = 0, \quad y \in \Pi$$



- periodic boundary conditions

Convergence

Theorem Let $u_\varepsilon(x)$ be the solution of:

$$
\begin{aligned}
-\operatorname{div}(\sigma(\frac{x}{\varepsilon})\nabla u_\varepsilon) &= g(x), && x \in \Omega \\
u_\varepsilon &= 0, && x \in \partial\Omega
\end{aligned}
$$

in a bounded domain Ω. Then

$$
\begin{aligned}
u_\varepsilon &\rightharpoonup u_0 && \text{weakly in } H_0^1(\Omega) \\
\sigma_\varepsilon \nabla u_\varepsilon &\rightharpoonup \hat{\sigma}\nabla u_0 && \text{weakly in } L^2(\Omega)
\end{aligned}
$$

Two-Scale Convergence

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Definition

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \Psi(x, \frac{x}{\varepsilon}) \, \mathrm{d}\, x = \int_\Omega \int_\Pi u_0(x, y) \Psi(x, y) \, \mathrm{d}\, y \, \mathrm{d}\, x$$

- version of weak convergence
- strong convergence $\Rightarrow$ Two-scale convergence
- two-scale convergence $\Rightarrow$ weak convergence (but limits may differ)