Homogenization

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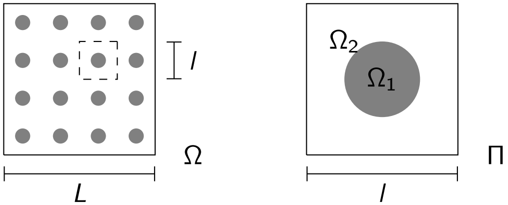
Content

Introduction

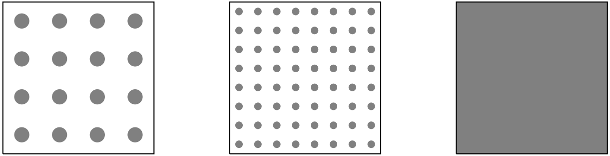
Given a composite material



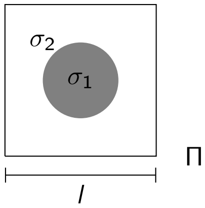
- Ω - domain with macroscale L
- Π - periodic cell with microscale l
- Assume $\varepsilon = \frac{l}{L} \ll 1$
- Solve with FEM too expensive (very fine mesh)

Homogenization

- Homogenization: $\varepsilon \to 0$



Example (Diffusion Problem)

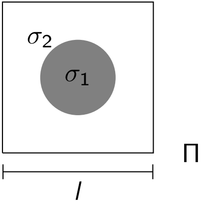


$$
\begin{aligned}
-\operatorname{div}(\sigma(x)\nabla u) &= g(x), && x \in \Omega \\
u &= 0, && x \in \partial\Omega
\end{aligned}
$$

Diffusion Tensor rapidly oscillates

$$
\sigma(x) = \begin{cases} \sigma_1, & x \in \Omega_1 \\ \sigma_2, & x \in \Omega_2 \end{cases}
$$

Example (Diffusion Problem)



$$
\begin{aligned}
-\operatorname{div}(\sigma(x/\varepsilon)\nabla u_\varepsilon) &= g(x), && x \in \Omega \\
u_\varepsilon &= 0, && x \in \partial\Omega
\end{aligned}
$$

Diffusion Tensor rapidly oscillates

$$
\sigma(x) = \begin{cases} \sigma_1, & x \in \Omega_1 \\ \sigma_2, & x \in \Omega_2 \end{cases}
$$

Formal Asymptotic Expansion

Ansatz:

$$u_\varepsilon(x) = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \ldots$$

- with u_ε is Π-periodic in $y = \frac{x}{\varepsilon}$
- x - slow variable and y - fast variable
- Consider x and y as independent variables

With chain rule for ∇:

$$\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y$$

$$
\begin{aligned}
&-\varepsilon^{-2}\operatorname{div}_y(\sigma(y)\nabla_y u_0(x,y))\\
&-\varepsilon^{-1}[\operatorname{div}_y(\sigma(y)\nabla_y u_1(x,y)) + \operatorname{div}_y(\sigma(y)\nabla_x u_0(x,y)\\
&+\operatorname{div}_x(\sigma(y)\nabla_y u_0(x,y))]\\
&-\varepsilon^{0}[\operatorname{div}_y(\sigma(y)\nabla_y u_2(x,y)\sigma(y)\nabla_x u_1(x,y))\\
&+\operatorname{div}_x(\sigma(y)\nabla_y u_1(x,y) + \operatorname{div}_x(\sigma(y)\nabla_x u_0(x,y)]\\
&-\varepsilon^{1}\ldots = g(x)
\end{aligned}
$$

Homogenized Problem

$$
\begin{aligned}
-\operatorname{div}(\hat{\sigma}\nabla u_0) &= g(x), & x &\in \Omega \\
u_0 &= 0, & x &\in \partial\Omega
\end{aligned}
$$

- $\hat{\sigma}$ - homogenized diffusion tensor

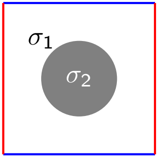
$$\hat{\sigma}_{i,j} = \frac{1}{|\Pi|}\int_\Pi (\sigma(y)(\nabla\chi_i + \boldsymbol{e}_i)) \cdot (\nabla\chi_j + \boldsymbol{e}_j)\,\mathrm{d}\,y$$

- $\boldsymbol{e}_i$ - unit vector
- χ_i - solution of the cell problem

$$-\operatorname{div}\left[\sigma(y)[\nabla\chi_i + \boldsymbol{e}_i]\right] = 0, \quad y \in \Pi$$

Cell Problem

$$-\operatorname{div}\left[\sigma(y)[\nabla\chi_i + \boldsymbol{e}_i]\right] = 0, \quad y \in \Pi$$



- periodic boundary conditions

Convergence

Theorem Let $u_\varepsilon(x)$ be the solution of:

$$\begin{aligned} -\operatorname{div}(\sigma(\frac{x}{\varepsilon})\nabla u_\varepsilon) &= g(x), && x \in \Omega \\ u_\varepsilon &= 0, && x \in \partial\Omega \end{aligned}$$

in a bounded domain Ω. Then

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0 && \text{weakly in } H_0^1(\Omega) \\ \sigma_\varepsilon \nabla u_\varepsilon &\rightharpoonup \hat{\sigma}\nabla u_0 && \text{weakly in } L^2(\Omega) \end{aligned}$$

Two-Scale Convergence

Definition

- Let $\{u_\varepsilon\}$ be a sequence of functions from $L^2(\Omega)$
- u_ε two-scale converges to a limit $u_0(x, y) \in L^2(\Omega \times \Pi)$
- if $\{u_\varepsilon\}$ is bounded in $L^2(\Omega)$ and
- for any function $\Psi(x, y) \in D(\Omega, C^\infty_{per}(\Pi))$ the following equality holds:

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\Psi(x, \frac{x}{\varepsilon})\,\mathrm{d}\,x = \int_\Omega \int_\Pi u_0(x, y)\Psi(x, y)\,\mathrm{d}\,y\,\mathrm{d}\,x$$

- version of weak convergence
- strong convergence $\Rightarrow$ Two-scale convergence
- two-scale convergence $\Rightarrow$ weak convergence (but limits may differ)