Homogenization

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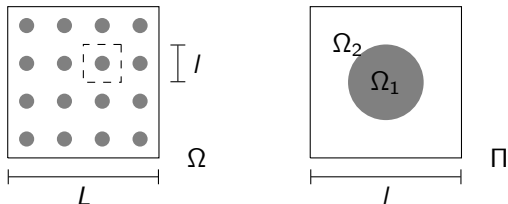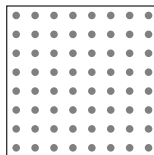
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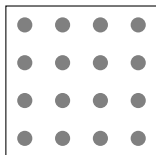Homogenized
Problem

Two-Scale
Convergence

Given a composite material

- Ω - domain with macroscale L
- Π - periodic cell with microscale l
- Assume $\varepsilon = \frac{l}{L} \ll 1$
- Solve with FEM too expensive (very fine mesh)

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▶ Homogenization: $\varepsilon \to 0$

Example (Diffusion Problem)

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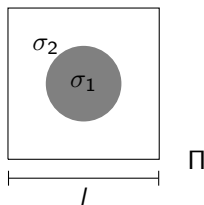
$$-\operatorname{div}(\sigma(x)\nabla u) = g(x), \qquad x \in \Omega$$
$$u = 0, \qquad x \in \partial\Omega$$

Diffusion Tensor rapidly oscillates

$$\sigma(x) = \begin{cases} \sigma_1, & x \in \Omega_1 \\ \sigma_2, & x \in \Omega_2 \end{cases}$$

Example (Diffusion Problem)

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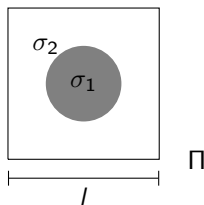
$$-\operatorname{div}(\sigma(x/\varepsilon)\nabla u_\varepsilon) = g(x), \qquad x \in \Omega$$
$$u_\varepsilon = 0, \qquad x \in \partial\Omega$$

Diffusion Tensor rapidly oscillates

$$\sigma(x) = \begin{cases} \sigma_1, & x \in \Omega_1 \\ \sigma_2, & x \in \Omega_2 \end{cases}$$

Formal Asymptotic Expansion

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Ansatz:

$$u_\varepsilon(x) = u_0(x, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + ...$$

- with u_ε is Π-periodic in $y = \frac{x}{\varepsilon}$
- x - slow variable and y - fast variable
- Consider x and y as independent variables

With chain rule for ∇:

$$\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y$$

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$$-\operatorname{div}(\hat{\sigma}\nabla u_0) = g(x), \qquad\qquad x \in \Omega$$
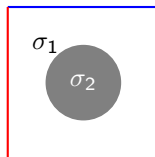$$u_0 = 0, \qquad\qquad x \in \partial\Omega$$

▶ $\hat{\sigma}$ - homogenized diffusion tensor

$$\hat{\sigma}_{i,j} = \frac{1}{|\Pi|} \int_\Pi (\sigma(y)(\nabla\chi_i + \boldsymbol{e}_i)) \cdot (\nabla\chi_j + \boldsymbol{e}_j)\, \mathrm{d}\, y$$

▶ $\boldsymbol{e}_i$ - unit vector
▶ χ_i - solution of the cell problem

$$-\operatorname{div}[\sigma(y)[\nabla\chi_i + \boldsymbol{e}_i]] = 0, \quad y \in \Pi$$

Cell Problem

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Homogenized Problem

Two-Scale Convergence

$$-\operatorname{div}\left[\sigma(y)[\nabla\chi_i + \boldsymbol{e}_i]\right] = 0, \quad y \in \Pi$$

σ_1

σ_2

▶ periodic boundary conditions

Convergence

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Theorem Let $u_\varepsilon(x)$ be the solution of:

$$-\operatorname{div}(\sigma(\frac{x}{\varepsilon})\nabla u_\varepsilon) = g(x), \qquad x \in \Omega$$
$$u_\varepsilon = 0, \qquad x \in \partial\Omega$$

in a bounded domain Ω. Then

$$u_\varepsilon \rightharpoonup u_0 \qquad \text{weakly in } H_0^1(\Omega)$$
$$\sigma_\varepsilon \nabla u_\varepsilon \rightharpoonup \hat{\sigma}\nabla u_0 \qquad \text{weakly in } L^2(\Omega)$$

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Two-Scale Convergence

Definition Let $\{u_\varepsilon\}$ be a sequence of functions from $L^2(\Omega)$. We say that u_ε two-scale converges to a limit $u_0(x, y) \in L^2(\Omega \times \Pi)$, and write

$$u_\varepsilon(x) \rightharpoonup u_0(x, y)$$

if $\{u_\varepsilon\}$ is bounded in $L^2(\Omega)$, and ...

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \Psi(x, \frac{x}{\varepsilon}) \, \mathrm{d}\, x = \int_\Omega \int_\Pi u_0(x, y) \Psi(x, y) \, \mathrm{d}\, y \, \mathrm{d}\, x$$

▶ version of weak convergence
▶ strong convergence $\Rightarrow$ Two-scale convergence
▶ two-scale convergence $\Rightarrow$ weak convergence (but limits may differ)