$$\partial_t u = \nabla \cdot (D_V \nabla u) \qquad x \in \Omega_1 \tag{1}$$

$$D_V \nabla u \cdot \nu = k(u - \bar{u}) \qquad x \in \Gamma \tag{2}$$

We assume D_V to be constant in Ω_1.

$$D(x) = \chi_\varepsilon(x) D_V$$

Ansatz:

$$u_\varepsilon(x,y) = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \mathcal{O}(\varepsilon^3) \tag{3}$$

Chain rule for the gradient:

$$\nabla = (\nabla_x + \frac{1}{\varepsilon}\nabla_y) \tag{4}$$

The left hand side of (1) becomes:

$$\partial_t u_\varepsilon = \partial_t \left(u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) \right) \tag{5}$$

The right hand side of (1) becomes:

$$\begin{aligned}
\nabla \cdot (D_V \nabla u_\varepsilon) =& \nabla \cdot (D_V \nabla (u_0 + \varepsilon u_1 + \varepsilon^2 u_2)) \\
=& (\nabla_x + \frac{1}{\varepsilon}\nabla_y) \cdot (D_V (\nabla_x + \frac{1}{\varepsilon}\nabla_y)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)) \\
=& D_V (\varepsilon^{-2} (\nabla_y \cdot \nabla_y u_0) \\
&+ \varepsilon^{-1} (\nabla_x \cdot \nabla_y u_0 + \nabla_y \cdot \nabla_x u_0 + \nabla_y \cdot \nabla_y u_1) \\
&+ \varepsilon^{0} (\nabla_x \cdot \nabla_x u_0 + \nabla_x \cdot \nabla_y u_1 + \nabla_y \cdot \nabla_x u_1 + \nabla_y \cdot \nabla_y u_2) \\
&+ \varepsilon^{1} (\nabla_x \cdot \nabla_x u_1 + \nabla_x \cdot \nabla_y u_2 + \nabla_y \cdot \nabla_x u_2) \\
&+ \varepsilon^{2} (\nabla_x \cdot \nabla_x u_2))
\end{aligned}$$

Comparing the orders of ε we get for ε^{-2}:

$$\nabla_y \cdot \nabla_y u_0(x,y) = 0 \tag{6}$$

Because of the periodicity in y it is $u_0(x,y) = u_0(x)$.

Therefore we get from the ε^{-1}:

$$\nabla_x \cdot \nabla_y u_0(x) = \nabla_y \cdot \nabla_x u_0(x) = 0 \tag{7}$$

It follows:

$$\nabla_y \cdot \nabla_y u_1(x,y) = 0 \tag{8}$$

We write u_1 as linear combination of u_0

$$u_1(x,y) = \sum_{j=1}^{n} w_j(y) \partial_{x_j} u_0(x) \tag{9}$$

Kann man das wirklich machen, wenn $D_V = const$?

From the ε^0 term we get:

$$\partial_t u_0 = D_V(\nabla_x \cdot \nabla_x u_0 + \nabla_x \cdot \nabla_y u_1 + \nabla_y \cdot \nabla_x u_1 + \nabla_y \cdot \nabla_y u_2)$$

We integrate over the cell:

$$\begin{aligned}\int_{Y_S} \partial_t u_0 \, \mathrm{d}\, y = D_V(&\int_{Y_S} \nabla_x \cdot \nabla_x u_0 \, \mathrm{d}\, y \\ &+ \int_{Y_S} \nabla_x \cdot \nabla_y u_1 \, \mathrm{d}\, y \\ &+ \int_{Y_S} \nabla_y \cdot (\nabla_x u_1 + \nabla_y u_2) \, \mathrm{d}\, y)\end{aligned}$$

Consider the last integral and integrate by parts:

$$D_V \int_{Y_S} \nabla_y \cdot (\nabla_x u_1 + \nabla_y u_2) \, \mathrm{d}\, y = D_V \int_\Gamma (\nabla_x u_1 + \nabla_y u_2) \cdot \nu \, \mathrm{d}\, \Gamma(y)$$

Applying the boundary condition we get:

$$D_V \int_{Y_S} \nabla_y \cdot (\nabla_x u_1 + \nabla_y u_2) \, \mathrm{d}\, y = k \int_\Gamma u_1 \cdot \nu \, \mathrm{d}\, \Gamma(y)$$

The first two integrals together with the representation of u_1 as linear combination in (9) become:

$$\begin{aligned}&D_V(\int_{Y_S} \nabla_x \cdot \nabla_x u_0 \, \mathrm{d}\, y + \int_{Y_S} \nabla_x \cdot \nabla_y u_1 \, \mathrm{d}\, y) \\ =&D_V(\nabla_x \cdot \nabla_x u_0 \int_{Y_S} \mathrm{d}\, y + \int_{Y_S} \nabla_x \cdot \nabla_y \sum_{j=1}^n w_j(y) \partial_{x_j} u_0(x) \, \mathrm{d}\, y)\end{aligned}$$

From the boundary condition we get:

$$\begin{aligned}k(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 - \bar{u}) = D_V(&\varepsilon^{-1} \nabla_y u_0 \\ &+\varepsilon^0(\nabla_x u_0 + \nabla_y u_1) \\ &+\varepsilon(\nabla_x u_1 + \nabla_y u_2) \\ &+\varepsilon^2 \nabla_x u_2) \cdot \nu\end{aligned}$$

$$0 = D_V \varepsilon^{-1} \nabla_y u_0(x,y) \cdot \nu \tag{10}$$

$$\Rightarrow u_0(x,y) = u_0(x) \text{ or } \nabla_y u_0(x,y) \perp \nu \tag{11}$$