Homogenization

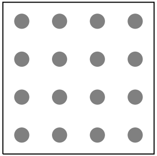
Janna Puderbach

December 22, 2022

Content

Introduction

Given a composite material



microscopic scale

- fine scale of inhomogeneous part
- too expensive to solve with FEM

macroscopic scale

- Consider the material as homogeneous
- easy to solve

Example

$$
\begin{aligned}
-\operatorname{div}(\sigma(x)\nabla u) &= g(x), & x &\in \Omega \\
u &= U, & x &\in \partial\Omega
\end{aligned}
$$

$$\varepsilon = \frac{l}{L} \ll 1$$

$$
\begin{aligned}
-\operatorname{div}(\sigma(\tfrac{x}{\varepsilon})\nabla u_\varepsilon) &= g(x), && x \in \Omega \\
u_\varepsilon &= 0, && x \in \partial\Omega
\end{aligned}
$$

$$
\begin{aligned}
u_\varepsilon &\rightharpoonup u_0 && \text{weakly in } H_0^1(\Omega) \\
\sigma_\varepsilon \nabla u_\varepsilon &\rightharpoonup \hat{\sigma} \nabla u_0 && \text{weakly in } L^2(\Omega)
\end{aligned}
$$

Effective Conductivity Tensor

$$\hat{\sigma}_{i,j} = \frac{1}{|\Pi|} \int_{\Pi} (\sigma(y)(\nabla \chi_i + \boldsymbol{e}_i)) \cdot (\nabla \chi_j + \boldsymbol{e}_j) \, \mathrm{d}\, y$$

Cell problem

$$-\operatorname{div}\left[\sigma(y)[\nabla\chi_i + \boldsymbol{e}_i]\right] = 0, \quad y \in \Pi \tag{1}$$

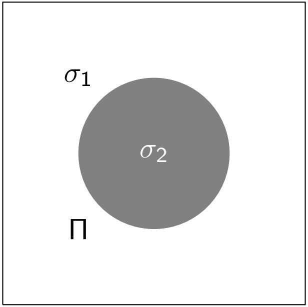


Figure: Unit cell

Formal Asymptotic Expansion

slow and fast variables independent

$$u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \ldots \qquad (2)$$

Two-Scale Convergence

Weak Convergence
Two-Scale convergence

$$\lim_{\varepsilon \to 0} \int_{\Omega} u_{\varepsilon}(x)\Psi(x, \frac{x}{\varepsilon})\,\mathrm{d}\,x = \int_{\Omega}\int_{\Pi} u_0(x,y)\Psi(x,y)\,\mathrm{d}\,y\,\mathrm{d}\,x$$