Article

Title

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Abstract: To be completed.

Keywords: Gauge equation, spectral sequence, KV-cohomology, hessian manifold, statistical manifold.

1. Introduction

1.1. Notations and writing conventions

All manifolds are assumed to be smooth. Thorough this document, the next writing conventions are applied: M is a smooth manifold. For a vector bundle $E \xrightarrow{\pi} M$, the notation $\Gamma(U; E)$ with $U \subset M$ an open subset of the manifold M stands for the $C^\infty(M)$-module of smooth sections over U. The functor $U \mapsto \Gamma(U; E)$ defines a sheaf denoted by Γ_E. Finally, $\Gamma(E)$ is a shorthand notation for $\Gamma(M; E)$. Lowercase letters are used for sections, uppercase ones for tangent vectors.

2. The gauge equation

Let $E \xrightarrow{\pi} M$ be a vector bundle. An affine connection ∇ is a $\mathbb{R}$-linear mapping [?]:

$$\nabla : \Gamma(E) \to \Gamma(T^\star M \otimes E) \tag{1}$$

such that for any $f \in C^\infty(M)$, $\nabla_X fs = df \otimes s + f \nabla_X s$. Let $E^\star \xrightarrow{\pi^\star} M$ be the bundle obtained by dualizing E fiberwise. A section $\theta \in \Gamma(E^* \otimes E)$, that is a $(1,1)$-tensor, defines two bundle morphisms:

$$\begin{tikzcd} E \arrow[r, "\theta"] \arrow[dr, "\pi"'] & E \arrow[d, "\pi"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} E^\star \arrow[r, "\theta^t"] \arrow[dr, "\pi^\star"'] & E^\star \arrow[d, "\pi^\star"] \\ & M \end{tikzcd} \tag{2}$$

where θ^t is such that for any $p \in M$, $X \in T_p M$, $\xi \in T_p^\star M$:

$$\left(\theta_p^t \xi\right)(X) = \xi(\theta_p X) \tag{3}$$

Definition 2.1. Let (∇_1, ∇_2) be a couple of affine connections. A $(1,1)$-tensor θ is said to be a solution of the gauge equation if for any $s \in \Gamma(E)$:

$$\nabla_1 \theta s = \theta \nabla_2 s \tag{4}$$

Citation: Lastname, F.; Lastname, F.; Lastname, F. Title. *Mathematics* **2024**, *1*, 0. https://doi.org/

Received:
Revised:
Accepted:
Published:

or equivalently if the next diagram commutes:

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{\nabla_2} & \Gamma(T^\star M \otimes E) \\ \big\downarrow{\scriptstyle \theta} & & \big\downarrow{\scriptstyle Id \otimes \theta} \\ \Gamma(E) & \xrightarrow{\nabla_1} & \Gamma(T^\star M \otimes E) \end{array} \tag{5}$$

Definition ?? can be made local, giving rise to diagrams:

$$\begin{array}{ccc} \Gamma(U;E) & \xrightarrow{\nabla_2} & \Gamma(U;T^\star M \otimes E) \\ \big\downarrow{\scriptstyle \theta_U} & & \big\downarrow{\scriptstyle Id \otimes \theta_U} \\ \Gamma(U;E) & \xrightarrow{\nabla_1} & \Gamma(U;T^\star M \otimes E) \end{array} \tag{6}$$

with U an open subset of M and $\theta_U \in \Gamma(U; E^\star \otimes E)$.

Definition 2.2. Let be ∇ be an affine connection. Its dual is the affine connection:

$$\nabla^\star \colon \Gamma(E^\star) \to \Gamma(T^\star M \otimes E^\star) \tag{7}$$

defined by the relation:

$$(\nabla^\star \xi)(s) = \xi(s) - \xi(\nabla s) \tag{8}$$

Proposition 2.1. *If θ is a solution of the gauge equation with connections (∇_1, ∇_2), then $\theta^\star$ is a solution of the gauge equation with connections $\left(\nabla_2^\star, \nabla_1^\star\right)$*

Proof. For $s \in \Gamma(E)$, $\xi \in \Gamma(E^\star)$:

$$(\nabla_2^\star(\theta^\star\xi))(s) = (\theta^\star\xi)(s) - (\theta^\star\xi)\nabla_2 s = \xi(\theta s) - \xi(\theta\nabla_2 s) \tag{9}$$

$$= \xi(\theta s) - \xi(\nabla_1 \theta s) = (\theta^\star \nabla_1^\star \xi)(s) \tag{10}$$

□

Given a couple of connections $(\nabla_1 \nabla_2)$, the difference $D_{1,2} = \nabla_1 - \nabla_2$ is a

3. KV cohomology

4. A spectral sequence

5. Application to statistical manifolds

Author Contributions: The authors have contributed equally to this work.

Funding: This research received no external funding

Acknowledgments: In this section you can acknowledge any support given which is not covered by the author contribution or funding sections. This may include administrative and technical support, or donations in kind (e.g., materials used for experiments).

Conflicts of Interest: The authors declare no conflicts of interest.

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