1 公式

$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$

$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$

$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}, r \neq 1$

$\sum_{k=1}^{n} kr^k = r\frac{1-(n+1)r^n+nr^{n+1}}{(1-r)^2}, r \neq 1$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n = (x-1) - \frac{(x-1)^2}{2} - \cdots$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$

$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$

$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$

$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x$

$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}}$

$\frac{\mathrm{d}}{\mathrm{d}x}\arccos x = -\frac{1}{\sqrt{1-x^2}}$

$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$

$\int \frac{1}{x^2+\alpha^2}\mathrm{d}x = \frac{\arctan\frac{x}{\alpha}}{\alpha}$

$\int \frac{1}{\pm x^2 \mp \alpha^2}\mathrm{d}x = \frac{\ln\left(\frac{x\mp\alpha}{\pm x+\alpha}\right)}{2\alpha}$

$\int \frac{1}{ax^2+b}\mathrm{d}x = \frac{1}{\sqrt{ab}}\arctan\frac{\sqrt{a}x}{\sqrt{b}}$

$\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x = \arcsin\frac{x}{a} = -\arccos\frac{x}{a}$

$\int \sec^2 x \mathrm{d}x = \tan x$

$\int \csc^2 x \mathrm{d}x = -\cot x$

$\int \sec x \tan x \mathrm{d}x = \sec x$

$\int \csc x \cot x \mathrm{d}x = -\csc x$

$\int \tan x \mathrm{d}x = -\ln|\cos x| = \ln|\sec x|$

$\int \cot x \mathrm{d}x = \ln|\sin x|$

$\int \sec x \mathrm{d}x = \ln|\sec x + \tan x|$

$\int \csc x \mathrm{d}x = \ln|\csc x - \cot x| = \ln\left|\frac{\tan x - \sin x}{\sin x \tan x}\right|$

$\int x^n e^{ax}\mathrm{d}x = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\mathrm{d}x$

$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}\,e^{\frac{b^2}{4a}+c}$

2 Probability

2.1 Distribution

Poisson Distribution

本质上是一个 $n \to \infty$ 的二项分布，$\lambda = np$。

性质：$\mathbb{E}(X) = \lambda, \mathrm{Var}(X) = \lambda$

Approximate Bin: n large, p small ($n \geq 50, np \leq 5$)

Hypergeometric Distribution

记号：$X \sim \mathrm{Hypergeomet}(n, N, m)$

概率：$p(k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$

N 个球，m 个红球，不放回取出 n 个，有 k 红球。

$\mathbb{E}(X) = n \cdot \frac{m}{N}, \mathrm{Var}(X) = n \cdot \frac{m}{N}\left(1 - \frac{m}{N}\right)\left(1 - \frac{n-1}{N-1}\right)$

Normal Distribution

Approximate Bin: $np(1-p) \geq 10$

$Z \sim N(0,1), \mathbb{E}(g'(Z)) = \mathbb{E}(Zg(Z))$, assuming that $\lim_{x\to\infty}\frac{g(x)}{e^{\frac{x^2}{2}}} = 0$. So $\mathbb{E}(Z^{n+1}) = n\mathbb{E}(Z^{n-1})$.

Exponential Distribution

CDF：$F(X) = 1 - e^{-\lambda x}, x \geq 0$

$\mathbb{P}r(X > x) = e^{-\lambda x}, x \geq 0$

$\mathbb{E}(X^n) = \frac{n}{\lambda}\mathbb{E}(X^{n-1}) = \frac{n!}{\lambda^n}$

$\mathbb{E}(X) = \frac{1}{\lambda}, \mathrm{Var}(X) = \frac{1}{\lambda^2}$

Gamma Distribution

考试都用 $\Gamma(\alpha, \beta)$ 的形式

$\Gamma(x) = \int_0^{\infty} u^{x-1}e^{-u}\mathrm{d}u, x > 0$

$\Gamma(x) = (x-1)\Gamma(x-1)$

$\Gamma(n) = (n-1)!$

α：发生次数

$\mathbb{E}(X) = \alpha\beta, \mathrm{Var}(X) = \alpha\beta^2$

$\mathbb{E}(X^n) = (n+\alpha-1)\beta \cdot \mathbb{E}(X^{n-1}) = \alpha^{\overline{n}}\beta^n$

Chi-Squared Distribution

$X \sim \chi^2(k)$

$\mathbb{E}(X) = k$

$\mathcal{N}(0,1)^2 \sim \chi_1^2$

$\chi_n^2 \sim \Gamma\left(\frac{n}{2}, 2\right)$

$\frac{1}{\sigma^2}\sum(X_i - \mu)^2 \sim \chi_n^2$

$\frac{1}{\sigma^2}\sum(X_i - \overline{X})^2 \sim \chi_{n-1}^2$

t-Distribution

$T_k = \frac{C}{\sqrt{D/k}}, C \sim \mathcal{N}(0,1), D = \chi_k^2$

$\frac{\sqrt{n}(\overline{X}-\mu)}{S} \sim t_{n-1}$ for $X_i \sim \mathcal{N}(\mu, \sigma^2)$

$\sigma^2 = \frac{v}{v-2}$ for $v > 2$, ∞ for $1 < v \leq 2$

$f(t) = \frac{\Gamma[(n+1)/2]}{\sqrt{n\pi}\Gamma(n/2)}\left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}$

F-Distribution

$F(m,n) = \frac{U/m}{V/n}, U \sim \chi_m^2, V \sim \chi_n^2$

$f(w) = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \cdot \left(\frac{m}{n}\right)^{\frac{m}{2}} \cdot w^{\frac{m}{2}-1} \cdot \left(1 + \frac{m}{n}w\right)^{-\frac{m+n}{2}}$

$\mu = \frac{n}{n-2}$ for $n > 2$

$\sigma^2 = \frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}$ for $n > 4$

2.2 MGF

$M(X) = \mathbb{E}\left[e^{tX}\right]$

$M^{(m)}(X) = \mathbb{E}\left[X^m\right]$

Distribution	MGF	PMF/PDF
Bernoulli(p)	$pe^t + 1 - p$	$p(1) = p$
Binomial(n, p)	$(1 - p + pe^t)^n$	$\binom{n}{k}p^k(1-p)^{n-k}$
Poisson(λ)	$e^{\lambda(e^t-1)}$	$p(k) = \frac{\lambda^k}{k!}e^{-\lambda}$
Geo(p)	$\frac{pe^t}{1-(1-p)e^t}$	$(1-p)^{k-1}p$
$\mathcal{N}(\mu, \sigma^2)$	$e^{t\mu + \frac{1}{2}\sigma^2 t^2}$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
Exp(λ)	$\frac{\lambda}{\lambda - t}$	$\lambda e^{-\lambda x}$
$\Gamma(\alpha, \beta)$	$(1 - \beta t)^{-\alpha}$	$\frac{1}{\beta^k\Gamma(k)}x^{k-1}e^{-\frac{x}{\beta}}$
χ_k^2	$(1-2t)^{-\frac{k}{2}}$	$\frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)}x^{\frac{k}{2}-1}e^{-\frac{x}{2}}$

2.3 Central Limit Theorem

Markov's inequality: $\mathbb{P}\mathrm{r}\{X \geq t\} \leq \frac{\mathbb{E}(X)}{t}$，要求是 $X \geq 0, t > 0$

Chebyshev's Inequality: $\mathbb{P}\mathrm{r}\{|X - \mathbb{E}(X)| \geq t\} \leq \frac{\mathrm{Var}(X)}{t^2}$

$\mathbb{P}\mathrm{r}\{|X - \mathbb{E}(X)| \geq k\sigma\} \leq \frac{1}{k^2}$

Weak LLN: $\lim_{n\to\infty}\mathbb{P}\mathrm{r}\left\{\left|\overline{X}_n - \mu\right| > \epsilon\right\} = 0$

Strong LLN: $\mathbb{P}\mathrm{r}\left\{\lim_{n\to\infty}\overline{X}_n = \mu\right\} = 1$

CLT: $\lim_{n\to\infty}\mathbb{P}\mathrm{r}\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right\} = \Phi(x)$

3 Estimation

3.1 Method of Moments (MME)

$\mathbb{E}(X_1^j) = \mu_j$
$\mu_j = g_j(\boldsymbol{\theta})$
$\theta_k = h_k(\boldsymbol{\mu})$

3.2 Maximum Likelihood (MLE)

$L(\boldsymbol{x} \mid \boldsymbol{\theta}) = \prod_{i=1}^n f(x_i \mid \boldsymbol{\theta})$
Standard conditions:

1. $L(\theta) > 0$ for all $\theta \in (a, b)$
2. $\frac{\partial L(\theta)}{\partial \theta}$ exists for all $\theta \in (a, b)$
3. $\lim_{\theta \to a^+} L(\theta) = \lim_{\theta \to b^-} L(\theta) = 0$

3.3 Estimate an Estimator

Bias$(\hat{\theta}) = \mathbb{E}_\theta(\hat{\theta}) - \theta$
unbiased: Bias$(\hat{\theta}) = 0$
Standard Error: $\text{SE}(\hat{\theta}) = \sqrt{\text{Var}(\hat{\theta})}$
Rule of thumb: 如果 sample 足够大，$\theta \in [\hat{\theta} - \text{SE}(\hat{\theta}), \hat{\theta} + \text{SE}(\hat{\theta})]$ 是 70%，$\theta \in [\hat{\theta} - 2 \cdot \text{SE}(\hat{\theta}), \hat{\theta} + 2 \cdot \text{SE}(\hat{\theta})]$ 是 95%
Mean Squared Error: $\text{MSE}(\hat{\theta}) = \mathbb{E}\left[\left(\hat{\theta} - \theta\right)^2\right]$
$\text{MSE}(\hat{\theta}) = \text{Bias}(\hat{\theta})^2 + \text{Var}(\hat{\theta}) = \text{Bias}(\hat{\theta})^2 + \text{SE}(\hat{\theta})^2$
Bias 是要准，SE 是要快，MSE 是成年人我都要
如何走向人生巅峰：$\widehat{\theta'_n} = \frac{\theta}{\mathbb{E}(\widehat{\theta_n})}\widehat{\theta_n}$
Consistent: $\forall \varepsilon > 0 \Rightarrow \lim_{n\to\infty} \mathbb{P}(|\widehat{\theta_n} - \theta| > \varepsilon) = 0$
咋证：$\lim_{n\to\infty} \text{Bias}(\widehat{\theta_n}) = 0 \wedge \lim_{n\to\infty} \text{Var}(\widehat{\theta_n}) = 0$
MME 只要是 h 连续一定是 consistent 嘞

3.4 Fisher Information

Log-likelihood function: $\ell = \log L$
Score function: $V(\boldsymbol{X} \mid \theta) = \frac{\partial}{\partial \theta}\ell(\boldsymbol{X} \mid \theta)$
Fisher Information: $I_{\boldsymbol{X}}(\theta) = \mathbb{E}\left[V(\boldsymbol{X} \mid \theta)^2\right]$
Condition (*) （离散同理，换成 $\sum$）：
$\int_{-\infty}^{\infty} \frac{\partial}{\partial \theta} f(x \mid \theta)\mathrm{d}x = \frac{\partial}{\partial \theta}\int_{-\infty}^{\infty} f(x \mid \theta)\mathrm{d}x = 0$
满足 (*) 这个可以推：$\mathbb{E}(V) = 0 \wedge \text{Var}(V) = I$
Fisher Info Alternative Formula
$I_{\boldsymbol{X}}(\theta) = nI_{X_1}(\theta) = -n \cdot \mathbb{E}[\frac{\partial^2}{\partial \theta^2}\log L(X_1 \mid \theta)]$
证明是首先需要注意到
$\frac{\partial}{\partial \theta} f(x \mid \theta) = \left[\frac{\partial}{\partial \theta}\log f(x \mid \theta)\right] f(x \mid \theta)$
然后把 $0 = \frac{\partial}{\partial \theta}\int f(x \mid \theta)\mathrm{d}x = \int \left[\frac{\partial}{\partial \theta}\log f(x \mid \theta)\right] f(x \mid \theta)\mathrm{d}x$ 两边再求个偏导

3.5 Cramér-Rao Lower Bound (CRLB)

条件：$\frac{\partial}{\partial \theta}\left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(\boldsymbol{x}) f(\boldsymbol{x} \mid \theta)\mathrm{d}x_1 \cdots \mathrm{d}x_n\right] = \left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(\boldsymbol{x}) \frac{\partial}{\partial \theta} f(\boldsymbol{x} \mid \theta)\mathrm{d}x_1 \cdots \mathrm{d}x_n\right]$, For all h with $\mathbb{E}(|h(\boldsymbol{X})|) < \infty$
$\text{Var}(\hat{\theta}) \geq \frac{\left[\frac{\partial}{\partial \theta}\mathbb{E}(T)\right]^2}{\mathbb{E}\left[\left(\frac{\partial}{\partial \theta}\log f(\boldsymbol{X} \mid \theta)\right)^2\right]}$
如果 $\hat{\theta}$ 是 unbiased 的，那我们有 $\text{Var}(\hat{\theta}) \geq \frac{1}{nI_{X_1}(\theta)}$
efficient 1. unbiased, 2. $\text{Var}(\hat{\theta}) = \frac{1}{nI_{X_1}(\theta)}$

3.6 Asymptotic Normality

Regularity Conditions ($\hat{\theta}$ 是 MLE)

1. $\frac{\partial^3}{\partial \theta^3} f(x \mid \theta)$ exists and continuous
2. $\exists (a, b) \subseteq S, \theta_0 \in (a, b)$
3. (support) $x \in \mathbb{R} : f(x \mid \theta) > 0$ is the same $\forall \theta$

不满足：Bernoulli(p): 找不到区间；$U(0, b)$: 不 support
$\sqrt{nI_{X_1}(\theta_0)}(\hat{\theta} - \theta_0) \sim \mathcal{N}(0, 1)$

3.7 Confidence Intervals

upper percentage point: z_α, $\mathbb{P}(X > z_\alpha) = \alpha$
$\forall \theta_0 \in S, \mathbb{P}(L \leq \theta_0 \leq U) = 1 - \alpha$
exact $100(1-\alpha)\%$ confidence interval for θ
$1 - \alpha$: **confidence level**
需要注意的是 L 和 R 才是尊 Random Var
Pivotal Quantity: The **distribution** of $Q(\boldsymbol{X}, \theta)$ does not depend on any unknown parameter
如果 $X_i \sim N(\mu, 1)$，$\overline{X}$ 是不可以的（因为 μ 不知道），但是 $\sqrt{n}(\overline{X} - \mu)$ 是可以的
How to prove: PDF/CDF/MGF 都可以
asymptotically pivotal quantity: $Q_{n\to\infty} \to \Psi$
approximate / large sample confidence intervals
$\left[\hat{\theta} - \frac{z_{\alpha/2}}{\sqrt{nI(\hat{\theta})}}, \hat{\theta} + \frac{z_{\alpha/2}}{\sqrt{nI(\hat{\theta})}}\right]$ to approx $100(1-\alpha)\%$

4 Hypothesis Testing

如果 H_0 发生了，那 $\boldsymbol{X}$ 发生的概率有多小
p-value $p = \mathbb{P}(T(\boldsymbol{X}) \geq s \mid H_0)$
significance level: α, critical value: $t(\alpha)$
$p < \alpha \Leftrightarrow s > t(\alpha) \Leftrightarrow$ reject H_0
算 significance level：
T 越大越 against H_0：$\max_s(\mathbb{P}(T(\boldsymbol{X}) \geq s \mid H_0) \geq \alpha)$
T 越小越 against H_0：$\min_s(\mathbb{P}(T(\boldsymbol{X}) \leq s \mid H_0) \geq \alpha)$
Neyman-Pearson tests
Parameter Space: $\Omega = \Omega_0 \cup \Omega_1$
$H_0 : \theta \in \Omega_0, H_1 : \theta \in \Omega_1$
如果 $|\Omega_0| = 1$ 叫 simple hypothesis，否则 composite
我其实想要的是 H_1：

1. rejecting the H_0 in favor of the H_1.
2. there is not enough evidence to support the H_1.

Type I error (α): H_0 本不该被 reject，却 reject 了，FP
Type I error (β): H_0 该被 rej，却没 rej, FN
rejection region $R \subseteq \mathbb{R}^n$: 样本在这里就选 H_1
size α 所有 H_0 成立的情况下，rej 的最大可能性
$\sup_{\theta \in \Omega_0} \mathbb{P}(H_0 \text{ is rejected} \mid \theta) = \alpha$
level H_0 成立，rej 的可能性小于等于他 $\sup \leq \alpha$
power H_1 成立，有多大可能性拒绝 H_0
$\text{Power}(\theta) = \mathbb{P}(H_0 \text{ rejected} \mid \theta \in \Omega_1) = 1 - \beta(\theta)$
Neyman-Pearson Lemma
simple vs simple
Likelihood ratio $\Lambda(\boldsymbol{x}) = \frac{L(\boldsymbol{x} \mid \theta_0)}{L(\boldsymbol{x} \mid \theta_1)}$
rejection region $R = \{\boldsymbol{x} \in \mathbb{R}^n : \Lambda \leq t\}$
Monotone Likelihood Ratio (MLR)
对于 $\theta < \theta'$，存在 $T(\boldsymbol{x})$ 使得 $\frac{p_{\theta'}(x)}{p_\theta(x)}$ 对 $T(\boldsymbol{x})$ 是不降的
simple vs composite

uniformly most powerful (UMP)
满足 MLR，UMP test 存在，sup 在 $\theta = \theta_0$ 时候取到

如果 (L, U) 是一个 $100(1-\alpha)\%$ 的 confidence interval，那么 reject $H_0 : 0 = \theta_0 \Leftrightarrow \theta_0 \notin (L, U)$ 是 α size 的

反过来也可以说