1 符号

补集：A'

P_n^r：$_nP_r$

σ^2：$\mathrm{Var}(X) = \mathbb{E}[x^2] - [\mathbb{E}[x]]^2$

[statement]：$\mathbf{1}_{\{statement\}}$

2 公式

2.1 组合数

换系数：

$k\binom{n}{k} = n\binom{n-1}{k-1}$

组合数乘积：

$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$

$\sum_{i=0}^{m}\binom{n}{i}\binom{m}{m-i} = \binom{m+n}{m} \quad (n \geq m)$

$\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{2n}{n}$

组合数的带权和：

$\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$

$\sum_{i=0}^{n} i^2\binom{n}{i} = n(n+1)2^{n-2}$

杨辉三角列和：$\sum_{l=0}^{n}\binom{l}{k} = \binom{n+1}{k+1}$

组合数与斐波那契（Fibonacci）：

$\sum_{i=0}^{n}\binom{n-i}{i} = F_{n+1}$

卡特兰数（Catalan Numbers）：

$H_n = \sum_{i=1}^{n} H_{i-1}H_{n-i} = \frac{\binom{2n}{n}}{n+1} = \binom{2n}{n} - \binom{2n}{n-1}$

2.2 求和

$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$

$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}, r \neq 1$

$\sum_{k=1}^{n} kr^k = r\frac{1-(n+1)r^n+nr^{n+1}}{(1-r)^2}, r \neq 1$

2.3 Maclaurin Series

$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$\ln x = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x-1)^n = (x-1) - \frac{(x-1)^2}{2} - \cdots$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

2.4 数数

隔板法：$\sum_{i=1}^{k} x_i = n$

$x_i > 0$: $\binom{n-1}{k-1}$

$x_i \geq 0$: $\binom{n+k-1}{k-1}$

错位排列：$D_n = (n-1)(D_{n-1} + D_{n-2})$

容斥：$|\bigcup_{i=1}^{n} A_i| = \sum_{\emptyset \neq J \subseteq \{1,2,\ldots,n\}}(-1)^{|J|-1}\left|\bigcap_{j \in J} A_j\right|$

2.5 常见导数公式

$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$

$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$

$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$

$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x$

$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}}$

$\frac{\mathrm{d}}{\mathrm{d}x}\arccos x = -\frac{1}{\sqrt{1-x^2}}$

$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$

$z = f(x,y), \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t}$

2.6 积分（省略 $+C$）

$\int \frac{1}{x^2+\alpha^2}\mathrm{d}x = \frac{\arctan\frac{x}{\alpha}}{\alpha}$

$\int \frac{1}{\pm x^2 \mp \alpha^2}\mathrm{d}x = \frac{\ln\left(\frac{x \mp \alpha}{\pm x + \alpha}\right)}{2\alpha}$

$\int \frac{1}{ax^2+b}\mathrm{d}x = \frac{1}{\sqrt{ab}}\arctan\frac{\sqrt{a}x}{\sqrt{b}}$

$\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x = \arcsin\frac{x}{a} = -\arccos\frac{x}{a}$

$\int \sec^2 x\,\mathrm{d}x = \tan x$

$\int \csc^2 x\,\mathrm{d}x = -\cot x$

$\int \sec x \tan x\,\mathrm{d}x = \sec x$

$\int \csc x \cot x\,\mathrm{d}x = -\csc x$

$\int \tan x\,\mathrm{d}x = -\ln|\cos x| = \ln|\sec x|$

$\int \cot x\,\mathrm{d}x = \ln|\sin x|$

$\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x|$

$\int \csc x\,\mathrm{d}x = \ln|\csc x - \cot x| = \ln\left|\frac{\tan x - \sin x}{\sin x \tan x}\right|$

$\int x^n e^{ax}\,\mathrm{d}x = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,\mathrm{d}x$

$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}\,e^{\frac{b^2}{4a}+c}$

$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \iint_\Delta f(r\cos\varphi, r\sin\varphi)r\mathrm{d}r\mathrm{d}\varphi$

$\iiint_\Omega f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega_1} f(r\cos\theta \cdot \cos\varphi, r\sin\theta \cdot \cos\varphi, r\sin\varphi) \cdot r^2\cos\varphi\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta$

3 Distribution

3.1 Bernoulli Random Variable

概率：$p(1) = p, p(0) = 1 - p$

性质：$\mathbb{E}(X) = p, \mathrm{Var}(X) = p(1-p)$

3.2 Binomial Distribution

记号：$X \sim \mathrm{Bin}(n, p)$

概率：$p(i) = \binom{n}{i}p^i(1-p)^{n-i}$

性质：$\mathbb{E}(X) = np, \mathrm{Var}(X) = np(1-p)$

3.3 Poisson Distribution

记号：$X \sim \mathrm{Poisson}(\lambda)$

概率：$p(k) = \frac{\lambda^k}{k!}e^{-\lambda}$

本质上是一个 $n \to \infty$ 的二项分布，$\lambda = np$。

性质：$\mathbb{E}(X) = \lambda, \mathrm{Var}(X) = \lambda$

Approximate Bin: n large, p small $(n \geq 50, np \leq 5)$

3.4 Hypergeometric Distribution

记号：$X \sim \mathrm{Hypergeomet}(n, N, m)$

概率：$p(k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$

N 个球，m 个红球，不放回取出 n 个，有 k 红球。

$\mathbb{E}(X) = n \cdot \frac{m}{N}, \mathrm{Var}(X) = n \cdot \frac{m}{N}\left(1 - \frac{m}{N}\right)\left(1 - \frac{n-1}{N-1}\right)$

3.5 Uniform Distribution

记号：$X \sim \mathrm{Unif}(l, r)$

PDF：$f(x) = \frac{1}{r-l}$

CDF：$f(x) = \frac{x-l}{r-l}$

$\mathbb{E}(X) = \frac{a+b}{2}, \text{Var}(X) = \frac{(b-a)^2}{12}$

3.6 Normal Distribution

记号：$X \sim N\left(\mu, \sigma^2\right)$

PDF：$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\Phi(x) = \text{CDF of } N(0,1)$

To $N(0,1)$: $Z = \frac{X-\mu}{\sigma}$

Approximate Bin: $np(1-p) \geq 10$

$Z \sim N(0,1), \mathbb{E}(g'(Z)) = \mathbb{E}(Zg(Z))$, assuming that $\lim_{x\to\infty} \frac{g(x)}{e^{\frac{x^2}{2}}} = 0$. So $\mathbb{E}(Z^{n+1}) = n\mathbb{E}(Z^{n-1})$.

3.7 Exponential Distribution

$X \sim \text{Exp}(\lambda)$

PDF：$f(x) = \lambda e^{-\lambda x}, x \geq 0$

CDF：$F(X) = 1 - e^{-\lambda x}, x \geq 0$

$\Pr(X > x) = e^{-\lambda x}, x \geq 0$

$\mathbb{E}(X^n) = \frac{n}{\lambda}\mathbb{E}(X^{n-1}) = \frac{n!}{\lambda^n}$

$\mathbb{E}(X) = \frac{1}{\lambda}, \text{Var}(X) = \frac{1}{\lambda^2}$

3.8 Gamma Distribution

$X \sim \Gamma(\alpha, \lambda)$ or $X \sim \text{Gamma}(\alpha, \lambda)$

$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du, x > 0$

$\Gamma(x) = (x-1)\Gamma(x-1)$

$\Gamma(n) = (n-1)!$

PDF：$f(x) = \frac{\lambda^\alpha}{\Gamma(a)} x^{\alpha-1} e^{-\lambda x}, x > 0$

α：发生次数

$\mathbb{E}(X) = \frac{\alpha}{\lambda}, \text{Var}(X) = \frac{\alpha}{\lambda^2}$

$\mathbb{E}(X^n) = \frac{n+\alpha-1}{\lambda}\mathbb{E}(X^{n-1}) = \frac{\alpha^{\overline{n}}}{\lambda^n}$

3.9 Chi-Squared Distribution

$X \sim \chi^2(k)$

PDF：$f(x) = \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}$

k（自由度）个 $N(0,1)$ 所组成向量长度平方的分布。

$\mathbb{E}(X) = k$

4 Functions of a Random Variable

$Y = g(x)$, $g(x)$ increasing,

$f_Y(y) = \begin{cases} f_X\left(g^{-1}(y)\right) \cdot \left(\frac{d}{dy}g^{-1}(y)\right) & \exists x, y = g(x) \\ 0 & \forall x, y \neq g(x) \end{cases}$

5 Joint Distribution

5.1 Marginal CDF/PDF

$F_X(x) = \lim_{y\to+\infty} F(x,y)$

$f_X(x) = \int_{-\infty}^{+\infty} f(x,y) dy$

5.2 Sum of Random Variables

X, Y independent, $Z = X + Y$

$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x) dx$

$X_i \sim \Gamma(\alpha_i, \lambda) \to \sum X_i \sim \Gamma(\sum \alpha_i, \lambda)$

$X_i \sim N(\mu_i, \sigma_i^2) \to \sum X_i \sim N(\sum \mu_i, \sum \sigma_i^2)$

5.3 Order Statistics

$X_1, X_2, \cdots, X_n$ 独立，CDF 为 F，PDF 为 f.

令 U 为最大值，V 为最小值。

$F_U(u) = F^n(u), f_U(u) = nf(u)F^{n-1}(u)$

$F_V(v) = 1 - (1 - F(v))^n, f_V(v) = nf(v)[1 - F(v)]^{n-1}$

$F_{UV}(u,v) = \begin{cases} [F(u)]^n - [F(u) - F(v)]^n & u > v \\ [F(u)]^n & u \leq v \end{cases}$

$f_{UV}(u,v) = \begin{cases} n(n-1)f(u)f(v)[F(u) - F(v)]^{n-2} & u > v \\ 0 & u \leq v \end{cases}$

5.4 Covariance

$\mathbb{E}(g(X) \cdot h(Y)) = \mathbb{E}(g(X)) \cdot \mathbb{E}(h(Y))$

$\text{Cov}(X,Y) = \mathbb{E}((X - \mu_X)(Y - \mu_Y)) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

X, Y 独立 $\to \text{Cov}(X,Y) = 0$，但反过来不行

Cov 的线性性：

$\text{Cov}\left(a + \sum_{i=1}^n b_i X_i, c + \sum_{i=1}^m d_i Y_i\right) = \sum_{i=1}^n \sum_{j=1}^m b_i d_j \text{Cov}(X_i, Y_j)$

$\text{Var}\left(a + \sum_{i=1}^n b_i X_i\right) = \sum_{i=1}^n \sum_{j=1}^n b_i b_j \text{Cov}(X_i, Y_j)$

Correlation coefficient: $\rho = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$

柯西不等式：$-1 \leq \rho \leq 1$，$\rho = \pm 1$ 当且仅当 X, Y 线性相关

5.5 Sample Mean/Variance

$\overline{X} = \frac{1}{n} \sum_{i=1}^n X_i$

$S^2 = \frac{1}{n-1} \sum_{i=1}^n \left(X_i - \overline{X}\right)^2$

$\frac{1}{n-1}$ 的原因：为了对齐 σ^2，$\mathbb{E}(S^2) = \sigma^2$

5.6 Conditioning

$\Pr(E) = \int_{\mathbb{R}} \Pr(E \mid X = x) p(x) dx$

$\mathbb{E}(Y) = \mathbb{E}_X[\mathbb{E}_Y(Y \mid X)]$

$\text{Var}(Y \mid X) = \mathbb{E}(Y^2 \mid X) - [\mathbb{E}(Y \mid X)]^2$

$\text{Var}(Y) = \text{Var}[\mathbb{E}(Y \mid X)] + \mathbb{E}[\text{Var}(Y \mid X)]$

$T = \sum_{i=1}^N X_i$，X, N 随机

$Var(T) = (\mathbb{E}(X))^2 \text{Var}(N) + \mathbb{E}(N)\text{Var}(X)$

5.7 Central Limit Theorem

5.7.1 Inequalities

Markov's inequality: $\Pr\{X \geq t\} \leq \frac{\mathbb{E}(X)}{t}$，要求是 $X \geq 0, t > 0$

Chebyshev's Inequality：$\Pr\{|X - \mathbb{E}(X)| \geq t\} \leq \frac{\text{Var}(X)}{t^2}$

$\Pr\{|X - \mathbb{E}(X)| \geq k\sigma\} \leq \frac{1}{k^2}$

5.7.2 Law of Large Numbers

$\overline{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$

Weak: $\lim_{n\to\infty} \Pr\left\{\left|\overline{X}_n - \mu\right| > \epsilon\right\} = 0$

Strong: $\Pr\left\{\lim_{n\to\infty} \overline{X}_n = \mu\right\} = 1$

5.7.3 CLT

$S_n = \sum_{i=1}^n X_i$

$\lim_{n\to\infty} \Pr\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right\} = \Phi(x)$