

1 符号

补集: A'

P_n^r : ${}_nP_r$

σ^2 : $\text{Var}(X) = \mathbb{E}[x^2] - [\mathbb{E}[x]]^2$

[statement]: $\mathbf{1}_{\{\text{statement}\}}$

2 公式

2.1 组合数

换系数:

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$

组合数乘积:

$$\binom{n}{r} \binom{r}{k} = \binom{n}{r-k} \binom{n-k}{r-k}$$

$$\sum_{i=0}^m \binom{n}{i} \binom{m}{m-i} = \binom{m+n}{m} \quad (n \geq m)$$

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}$$

组合数的带权和:

$$\sum_{i=0}^n i \binom{n}{i} = n 2^{n-1}$$

$$\sum_{i=0}^n i^2 \binom{n}{i} = n(n+1) 2^{n-2}$$

杨辉三角列和: $\sum_{l=0}^n \binom{l}{k} = \binom{n+1}{k+1}$

组合数与斐波那契 (Fibonacci):

$$\sum_{i=0}^n \binom{n-i}{i} = F_{n+1}$$

卡特兰数 (Catalan Numbers):

$$H_n = \sum_{i=1}^n H_{i-1} H_{n-i} = \frac{\binom{2n}{n}}{n+1} = \binom{2n}{n} - \binom{2n}{n-1}$$

2.2 求和

$$\sum_{k=1}^n k^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{k=1}^n k^3 = \frac{1}{4} n^2(n+1)^2$$

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}, r \neq 1$$

$$\sum_{k=1}^n k r^k = r \frac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2}, r \neq 1$$

2.3 Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = (x-1) - \frac{(x-1)^2}{2} + \dots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

2.4 数数

隔板法: $\sum_{i=1}^k x_i = n$

$x_i > 0$: $\binom{n-1}{k-1}$

$x_i \geq 0$: $\binom{n+k-1}{k-1}$

错位排列: $D_n = (n-1)(D_{n-1} + D_{n-2})$

容斥: $|\bigcup_{i=1}^n A_i| = \sum_{\emptyset \neq J \subseteq \{1,2,\dots,n\}} (-1)^{|J|-1} \left| \bigcap_{j \in J} A_j \right|$

2.5 常见导数公式

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

$$z = f(x, y), \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

2.6 积分 (省略 +C)

$$\int \frac{1}{x^2 + \alpha^2} dx = \frac{\arctan \frac{x}{\alpha}}{\alpha}$$

$$\int \frac{1}{\pm x^2 \mp \alpha^2} dx = \frac{\ln \left(\frac{x \mp \alpha}{x \pm \alpha} \right)}{2\alpha}$$

$$\int \frac{1}{ax^2 + b} dx = \frac{1}{\sqrt{ab}} \arctan \frac{\sqrt{a}x}{\sqrt{b}}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} = -\arccos \frac{x}{a}$$

$$\int \sec^2 x dx = \tan x$$

$$\int \csc^2 x dx = -\cot x$$

$$\int \sec x \tan x dx = \sec x$$

$$\int \csc x \cot x dx = -\csc x$$

$$\int \tan x dx = -\ln |\cos x| = \ln |\sec x|$$

$$\int \cot x dx = \ln |\sin x|$$

$$\int \sec x dx = \ln |\sec x + \tan x|$$

$$\int \csc x dx = \ln |\csc x - \cot x| = \ln \left| \frac{\tan x - \sin x}{\sin x \tan x} \right|$$

$$\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}$$

$$\iint_D f(x, y) dx dy = \iint_{\Delta} f(r \cos \varphi, r \sin \varphi) r dr d\varphi$$

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \iiint_{\Omega_1} f(r \cos \theta \cdot \cos \varphi, r \sin \theta \cdot \cos \varphi, r \sin \varphi) \cdot r^2 \cos \varphi dr d\varphi d\theta$$

3 Distribution

3.1 Bernoulli Random Variable

概率: $p(1) = p, p(0) = 1 - p$

性质: $\mathbb{E}(X) = p, \text{Var}(X) = p(1 - p)$

3.2 Binomial Distribution

记号: $X \sim \text{Bin}(n, p)$

概率: $p(i) = \binom{n}{i} p^i (1 - p)^{n-i}$

性质: $\mathbb{E}(X) = np, \text{Var}(X) = np(1 - p)$

3.3 Poisson Distribution

记号: $X \sim \text{Poisson}(\lambda)$

概率: $p(k) = \frac{\lambda^k}{k!} e^{-\lambda}$

本质上是一个 $n \rightarrow \infty$ 的二项分布, $\lambda = np$ 。

性质: $\mathbb{E}(X) = \lambda, \text{Var}(X) = \lambda$

Approximate Bin: n large, p small ($n \geq 50, np \leq 5$)

3.4 Hypergeometric Distribution

记号: $X \sim \text{Hypergeomet}(n, N, m)$

概率: $p(k) = \frac{\binom{m}{k} \binom{N-m}{n-k}}{\binom{N}{n}}$

N 个球, m 个红球, 不放回取出 n 个, 有 k 红球。

$\mathbb{E}(X) = n \cdot \frac{m}{N}, \text{Var}(X) = n \cdot \frac{m}{N} \left(1 - \frac{m}{N}\right) \left(1 - \frac{n-1}{N-1}\right)$

3.5 Uniform Distribution

记号: $X \sim \text{Unif}(l, r)$

PDF: $f(x) = \frac{1}{r-l}$

CDF: $f(x) = \frac{x-l}{r-l}$

$$\mathbb{E}(X) = \frac{a+b}{2}, \text{Var}(X) = \frac{(b-a)^2}{12}$$

3.6 Normal Distribution

记号: $X \sim N(\mu, \sigma^2)$

$$\text{PDF: } f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$\Phi(x)$ = CDF of $N(0, 1)$

To $N(0, 1)$: $Z = \frac{X-\mu}{\sigma}$

Approximate Bin: $np(1-p) \geq 10$

$Z \sim N(0, 1), \mathbb{E}(g'(Z)) = \mathbb{E}(Zg(Z))$, assuming that $\lim_{x \rightarrow \infty} \frac{g(x)}{e^{\frac{x^2}{2}}} = 0$. So $\mathbb{E}(Z^{n+1}) = n\mathbb{E}(Z^n)$.

3.7 Exponential Distribution

$X \sim \text{Exp}(\lambda)$

PDF: $f(x) = \lambda e^{-\lambda x}, x \geq 0$

CDF: $F(x) = 1 - e^{-\lambda x}, x \geq 0$

$\mathbb{P}(X > x) = e^{-\lambda x}, x \geq 0$

$\mathbb{E}(X^n) = \frac{n}{\lambda} \mathbb{E}(X^{n-1}) = \frac{n!}{\lambda^n}$

$\mathbb{E}(X) = \frac{1}{\lambda}, \text{Var}(X) = \frac{1}{\lambda^2}$

3.8 Gamma Distribution

$X \sim \Gamma(\alpha, \lambda)$ or $X \sim \text{Gamma}(\alpha, \lambda)$

$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du, x > 0$

$\Gamma(x) = (x-1)\Gamma(x-1)$

$\Gamma(n) = (n-1)!$

PDF: $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, x > 0$

α : 发生次数

$\mathbb{E}(X) = \frac{\alpha}{\lambda}, \text{Var}(X) = \frac{\alpha}{\lambda^2}$

$\mathbb{E}(X^n) = \frac{n+\alpha-1}{\lambda} \mathbb{E}(X^{n-1}) = \frac{\alpha^n}{\lambda^n}$

3.9 Chi-Squared Distribution

$X \sim \chi^2(k)$

PDF: $f(x) = \frac{1}{2^{\frac{k}{2}} \Gamma(\frac{k}{2})} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}$

k (自由度) 个 $N(0, 1)$ 所组成向量长度平方的分布。

$\mathbb{E}(X) = k$

4 Functions of a Random Variable

$Y = g(x)$, $g(x)$ increasing,

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left(\frac{d}{dy} g^{-1}(y)\right) & \exists x, y = g(x) \\ 0 & \forall x, y \neq g(x) \end{cases}$$

5 Joint Distribution

5.1 Marginal CDF/PDF

$F_X(x) = \lim_{y \rightarrow +\infty} F(x, y)$

$f_X(x) = \int_{-\infty}^{+\infty} f(x, y) dy$

5.2 Sum of Random Variables

X, Y independent, $Z = X + Y$

$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x) dx$

$X_i \sim \Gamma(\alpha_i, \lambda) \rightarrow \sum X_i \sim \Gamma(\sum \alpha_i, \lambda)$

$X_i \sim N(\mu_i, \sigma_i^2) \rightarrow \sum X_i \sim N(\sum \mu_i, \sum \sigma_i^2)$

5.3 Order Statistics

X_1, X_2, \dots, X_n 独立, CDF 为 F , PDF 为 f .

令 U 为最大值, V 为最小值。

$F_U(u) = F^n(u), f_U(u) = n f(u) F^{n-1}(u)$

$F_V(v) = 1 - (1 - F(v))^n, f_V(v) = n f(v) [1 - F(v)]^{n-1}$

$$F_{UV}(u, v) = \begin{cases} [F(u)]^n - [F(u) - F(v)]^n & u > v \\ [F(u)]^n & u \leq v \end{cases}$$

$$f_{UV}(u, v) = \begin{cases} n(n-1)f(u)f(v)[F(u) - F(v)]^{n-2} & u > v \\ 0 & u \leq v \end{cases}$$

5.4 Covariance

$\mathbb{E}(g(X) \cdot h(Y)) = \mathbb{E}(g(X)) \cdot \mathbb{E}(h(Y))$

$\text{Cov}(X, Y) = \mathbb{E}((X - \mu_X)(Y - \mu_Y)) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

X, Y 独立 $\rightarrow \text{Cov}(X, Y) = 0$, 但反过来不行

Cov 的线性性:

$$\text{Cov}(a + \sum_{i=1}^n b_i X_i, c + \sum_{i=1}^m d_i Y_i) = \sum_{i=1}^n \sum_{j=1}^m b_i d_j \text{Cov}(X_i, Y_j)$$

$$\text{Var}(a + \sum_{i=1}^n b_i X_i) = \sum_{i=1}^n \sum_{j=1}^n b_i b_j \text{Cov}(X_i, Y_j)$$

Correlation coefficient: $\rho = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$

柯西不等式: $-1 \leq \rho \leq 1$, $\rho = \pm 1$ 当且仅当 X, Y 线性相关

5.5 Sample Mean/Variance

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

$\frac{1}{n-1}$ 的原因: 为了对齐 σ^2 , $\mathbb{E}(S^2) = \sigma^2$

5.6 Conditioning

$\Pr(E) = \int_{\mathbb{R}} \Pr(E | X = x) p(x) dx$

$\mathbb{E}(Y) = \mathbb{E}_X[\mathbb{E}_Y(Y | X)]$

$\text{Var}(Y | X) = \mathbb{E}(Y^2 | X) - [\mathbb{E}(Y | X)]^2$

$\text{Var}(Y) = \text{Var}[\mathbb{E}(Y | X)] + \mathbb{E}[\text{Var}(Y | X)]$

$T = \sum_{i=1}^N X_i$, X, N 随机

$\text{Var}(T) = (\mathbb{E}(X))^2 \text{Var}(N) + \mathbb{E}(N) \text{Var}(X)$

5.7 Central Limit Theorem

5.7.1 Inequalities

Markov's inequality: $\Pr\{X \geq t\} \leq \frac{\mathbb{E}(X)}{t}$, 要求是 $X \geq 0, t > 0$

Chebyshev's Inequality: $\Pr\{|X - \mathbb{E}(X)| \geq t\} \leq \frac{\text{Var}(X)}{t^2}$
 $\Pr\{|X - \mathbb{E}(X)| \geq k\sigma\} \leq \frac{1}{k^2}$

5.7.2 Law of Large Numbers

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

Weak: $\lim_{n \rightarrow \infty} \Pr\{|\bar{X}_n - \mu| > \epsilon\} = 0$

Strong: $\Pr\{\lim_{n \rightarrow \infty} \bar{X}_n = \mu\} = 1$

5.7.3 CLT

$$S_n = \sum_{i=1}^n X_i$$

$$\lim_{n \rightarrow \infty} \Pr\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}}\right\} = \Phi(x)$$