

1 公式

1.0.1 组合数

换系数:

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$

组合数乘积:

$$\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k}$$

$$\sum_{i=0}^m \binom{m}{i} \binom{n}{m-i} = \binom{m+n}{m} \quad (n \geq m)$$

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}$$

组合数的带权和:

$$\sum_{i=0}^n i \binom{n}{i} = n 2^{n-1}$$

$$\sum_{i=0}^n i^2 \binom{n}{i} = n(n+1) 2^{n-2}$$

$$\text{杨辉三角列和: } \sum_{l=0}^n \binom{l}{k} = \binom{n+1}{k+1}$$

组合数与斐波那契 (Fibonacci):

$$\sum_{i=0}^n \binom{n-i}{i} = F_{n+1}$$

卡特兰数 (Catalan Numbers):

$$H_n = \sum_{i=1}^n H_{i-1} H_{n-i} = \frac{\binom{2n}{n}}{n+1} = \binom{2n}{n} - \binom{2n}{n-1}$$

1.1 求和

$$\sum_{k=1}^n k^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{k=1}^n k^3 = \frac{1}{4} n^2(n+1)^2$$

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}, r \neq 1$$

$$\sum_{k=1}^n k r^k = r \frac{1-(n+1)r^n + n r^{n+1}}{(1-r)^2}, r \neq 1$$

1.2 Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = (x-1) - \frac{(x-1)^2}{2} - \dots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

1.3 数数

隔板法: $\sum_{i=1}^k x_i = n$

$$\bullet x_i > 0: \binom{n-1}{k-1}$$

$$\bullet x_i \geq 0: \binom{n+k-1}{k-1}$$

错位排列: $D_n = (n-1)(D_{n-1} + D_{n-2})$

1.4 常见导数公式

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

1.5 常见积分公式 (都省略 +C)

$$\int \frac{1}{x^2+\alpha^2} dx = \frac{\arctan \frac{x}{\alpha}}{\alpha}$$

$$\int \frac{1}{\pm x^2 \mp \alpha^2} dx = \frac{\ln \left(\frac{x \mp \alpha}{x \pm \alpha} \right)}{2\alpha}$$

$$\int \frac{1}{ax^2+b} dx = \frac{1}{\sqrt{ab}} \arctan \frac{\sqrt{a}x}{\sqrt{b}}$$

$$\int \sqrt{a^2+x^2} dx = \frac{1}{2} x \sqrt{a^2+x^2} + \frac{1}{2} a^2 \ln(x + \sqrt{a^2+x^2})$$

$$\int \frac{\sqrt{a^2+x^2}}{x} dx = \sqrt{a^2+x^2} - a \ln \left(\frac{a+\sqrt{a^2+x^2}}{x} \right)$$

$$\int \frac{\sqrt{a^2+x^2}}{x^2} dx = \ln(x + \sqrt{a^2+x^2}) - \frac{\sqrt{a^2+x^2}}{x}$$

$$\int \frac{1}{\sqrt{a^2+x^2}} dx = \ln(x + \sqrt{a^2+x^2})$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln|x + \sqrt{x^2-a^2}|$$

$$\int \sqrt{a^2-x^2} dx = \frac{1}{2} x \sqrt{a^2-x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin \frac{x}{a} = -\arccos \frac{x}{a}$$

$$\int \cos x dx = \sin x$$

$$\int \sin x dx = -\cos x$$

$$\int \sec^2 x dx = \tan x$$

$$\int \csc^2 x dx = -\cot x$$

$$\int \sec x \tan x dx = \sec x$$

$$\int \csc x \cot x dx = -\csc x$$

$$\int \tan x dx = -\ln|\cos x| = \ln|\sec x|$$

$$\int \cot x dx = \ln|\sin x|$$

$$\int \sec x dx = \ln|\sec x + \tan x|$$

$$\int \csc x dx = \ln|\csc x - \cot x| = \ln \left| \frac{\tan x - \sin x}{\sin x \tan x} \right|$$

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

$$\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4}$$

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

$$\int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4}$$

$$\int \tan^n x dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx$$

$$\int \tan^2 x dx = \tan x - x$$

$$\int x e^{ax} dx = \frac{1}{a^2} (ax-1) e^{ax}$$

$$\int x^n e^{ax} dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

$$\int e^{ax} \sin bxdx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$$

$$\int e^{ax} \cos bxdx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$$

$$\int x^n \ln x dx = \frac{x^{n+1}}{(n+1)^2} [(n+1) \ln x - 1]$$

$$\int \frac{1}{x \ln x} dx = \ln(\ln x)$$

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_{L^+} (P dx + Q dy)$$

2 Distribution

2.1 Bernoulli Random Variable

概率: $p(1) = p, p(0) = 1-p$

性质: $\mathbb{E}(X) = p, \text{Var}(X) = p(1-p)$

2.2 Binomial Distribution

记号: $X \sim \text{Bin}(n, p)$

概率: $p(i) = \binom{n}{i} p^i (1-p)^{n-i}$

性质: $\mathbb{E}(X) = np, \text{Var}(X) = np(1-p)$

2.3 Poisson Distribution

记号: $X \sim \text{Poisson}(\lambda)$

概率: $p(k) = \frac{\lambda^k}{k!} e^{-\lambda}$

本质上是一个 $n \rightarrow \infty$ 的二项分布, $\lambda = np$.

性质: $\mathbb{E}(X) = \lambda, \text{Var}(X) = \lambda$

Approximate Bin: n large, p small ($n \geq 50, np \leq 5$)

2.4 Hypergeometric Distribution

记号: $X \sim \text{Hypergeomet}(n, N, m)$

概率: $p(k) = \frac{\binom{m}{k} \binom{N-m}{n-k}}{\binom{N}{n}}$

N 个球, m 个红球, 不放回取出 n 个, 有 k 红球。

性质: $\mathbb{E}(X) = n \cdot \frac{m}{N}$, $\text{Var}(X) = n \cdot \frac{m}{N} \left(1 - \frac{m}{N}\right) \left(1 - \frac{n-1}{N-1}\right)$

2.5 Uniform Distribution

记号: $X \sim \text{Unif}(l, r)$

PDF: $f(x) = \frac{1}{r-l}$

CDF: $f(x) = \frac{x-l}{r-l}$

2.6 Normal Distribution

记号: $X \sim N(\mu, \sigma^2)$

PDF: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\Phi(x) = \text{CDF of } N(0, 1)$

To $N(0, 1)$: $Z = \frac{X-\mu}{\sigma}$

Approximate Bin: $np(1-p) \geq 10$

$Z \sim N(0, 1)$, $\mathbb{E}(g'(Z)) = \mathbb{E}(Zg(Z))$, assuming that

$\lim_{x \rightarrow \infty} \frac{g(x)}{e^{\frac{x^2}{2}}} = 0$. So $\mathbb{E}(Z^{n+1}) = n\mathbb{E}(Z^{n-1})$.

2.7 Exponential Distribution

$X \sim \text{Exp}(\lambda)$

PDF: $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$

CDF: $F(X) = 1 - e^{-\lambda x}$, $x \geq 0$

$\Pr(X > x) = e^{-\lambda x}$, $x \geq 0$

$\mathbb{E}(X^n) = \frac{n}{\lambda} \mathbb{E}(X^{n-1}) = \frac{n!}{\lambda^n}$

$\mathbb{E}(X) = \frac{1}{\lambda}$, $\text{Var}(X) = \frac{1}{\lambda^2}$

2.8 Gamma Distribution

$X \sim \Gamma(\alpha, \lambda)$ or $X \sim \text{Gamma}(\alpha, \lambda)$

$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$, $x > 0$

$\Gamma(x) = (x-1)\Gamma(x-1)$

$\Gamma(n) = (n-1)!$

PDF: $f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$, $x > 0$

α : 发生次数

$\mathbb{E}(X) = \frac{\alpha}{\lambda}$, $\text{Var}(X) = \frac{\alpha}{\lambda^2}$

$\mathbb{E}(X^n) = \frac{n+\alpha-1}{\lambda} \mathbb{E}(X^{n-1}) = \frac{\alpha^n}{\lambda^n}$

2.9 Chi-Squared Distribution

$X \sim \chi^2(k)$

PDF: $f(x) = \frac{1}{2^{\frac{k}{2}} \Gamma(\frac{k}{2})} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}$

k (自由度) 个 $N(0, 1)$ 所组成向量长度平方的分布。

$\mathbb{E}(X) = k$

3 Functions of a Random Variable

$Y = g(X)$, $g(X)$ increasing,

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left(\frac{d}{dy} g^{-1}(y)\right) & \exists x, y = g(x) \\ 0 & \forall x, y \neq g(x) \end{cases}$$

4 Joint Distribution

4.1 Marginal CDF/PDF

$F_X(x) = \lim_{y \rightarrow +\infty} F(x, y)$

$f_X(x) = \int_{-\infty}^{+\infty} f(x, y) dy$

4.2 Independence

$\Pr\{X \in A, Y \in B\} = \Pr\{X \in A\} \cdot \Pr\{Y \in B\}$

$\Pr\{X_1 \in A_1, \dots, X_n \in A_n\} = \prod \Pr\{X_i \in A_i\}$

$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$

$F_{XY}(x, y) = F_X(x) \cdot F_Y(y)$

4.3 Sum of Random Variables

X, Y independent, $Z = X + Y$

$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x) dx$

convolution

$X \sim \Gamma(\alpha, \lambda), Y \sim \Gamma(\beta, \lambda) \rightarrow X + Y \sim \Gamma(\alpha + \beta, \lambda)$

$X \sim N(\mu_1, \sigma_1^2), Y \sim N(\mu_2, \sigma_2^2) \rightarrow X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$