1 符号

补集：A'
P_n^r：${}_nP_r$
σ^2：$\mathrm{Var}(X)=\mathbb{E}[x^2]-[\mathbb{E}[x]]^2$

2 公式

2.0.1 组合数

换系数：
$k\binom{n}{k}=n\binom{n-1}{k-1}$
组合数乘积：
$\binom{n}{r}\binom{r}{k}=\binom{n}{k}\binom{n-k}{r-k}$
$\sum_{i=0}^{m}\binom{n}{i}\binom{m}{m-i}=\binom{m+n}{m}\quad(n\ge m)$
$\sum_{i=0}^{n}\binom{n}{i}^2=\binom{2n}{n}$
组合数的带权和：
$\sum_{i=0}^{n}i\binom{n}{i}=n2^{n-1}$
$\sum_{i=0}^{n}i^2\binom{n}{i}=n(n+1)2^{n-2}$
杨辉三角列和：$\sum_{l=0}^{n}\binom{l}{k}=\binom{n+1}{k+1}$
组合数与斐波那契（Fibonacci）：
$\sum_{i=0}^{n}\binom{n-i}{i}=F_{n+1}$
卡特兰数（Catalan Numbers）：
$H_n=\sum_{i=1}^{n}H_{i-1}H_{n-i}=\frac{\binom{2n}{n}}{n+1}=\binom{2n}{n}-\binom{2n}{n-1}$

2.1 求和

$\sum_{k=1}^{n}k^2=\frac{1}{6}n(n+1)(2n+1)$
$\sum_{k=1}^{n}k^3=\frac{1}{4}n^2(n+1)^2$
$\sum_{k=0}^{n-1}r^k=\frac{1-r^n}{1-r},r\ne 1$
$\sum_{k=1}^{n}kr^k=r\frac{1-(n+1)r^n+nr^{n+1}}{(1-r)^2},r\ne 1$

2.2 Maclaurin Series

$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$
$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$
$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$
$\ln x=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x-1)^n=(x-1)-\frac{(x-1)^2}{2}-\cdots$
$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots$

2.3 数数

隔板法：$\sum_{i=1}^{k}x_i=n$

- $x_i>0$: $\binom{n-1}{k-1}$
- $x_i\ge 0$: $\binom{n+k-1}{k-1}$

错位排列：$D_n=(n-1)\left(D_{n-1}+D_{n-2}\right)$

2.4 常见导数公式

$\frac{\mathrm{d}}{\mathrm{d}x}\tan x=\sec^2 x$
$\frac{\mathrm{d}}{\mathrm{d}x}\cot x=-\csc^2 x$
$\frac{\mathrm{d}}{\mathrm{d}x}\sec x=\sec x\tan x$
$\frac{\mathrm{d}}{\mathrm{d}x}\csc x=-\csc x\cot x$
$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x=\frac{1}{\sqrt{1-x^2}}$
$\frac{\mathrm{d}}{\mathrm{d}x}\arccos x=-\frac{1}{\sqrt{1-x^2}}$
$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x=\frac{1}{1+x^2}$

2.5 常见积分公式（都省略 $+C$）

$\int\frac{1}{x^2+\alpha^2}\mathrm{d}x=\frac{\arctan\frac{x}{\alpha}}{\alpha}$
$\int\frac{1}{\pm x^2\mp\alpha^2}\mathrm{d}x=\frac{\ln\left(\frac{x\mp\alpha}{\pm x+\alpha}\right)}{2\alpha}$
$\int\frac{1}{ax^2+b}\mathrm{d}x=\frac{1}{\sqrt{ab}}\arctan\frac{\sqrt{a}x}{\sqrt{b}}$
$\int\sqrt{a^2+x^2}\mathrm{d}x=\frac{1}{2}x\sqrt{a^2+x^2}+\frac{1}{2}a^2\ln\left(x+\sqrt{a^2+x^2}\right)$
$\int\frac{\sqrt{a^2+x^2}}{x}\mathrm{d}x=\sqrt{a^2+x^2}-a\ln\left(\frac{a+\sqrt{a^2+x^2}}{x}\right)$
$\int\frac{\sqrt{a^2+x^2}}{x^2}\mathrm{d}x=\ln\left(x+\sqrt{a^2+x^2}\right)-\frac{\sqrt{a^2+x^2}}{x}$
$\int\frac{1}{\sqrt{a^2+x^2}}\mathrm{d}x=\ln\left(x+\sqrt{a^2+x^2}\right)$
$\int\frac{1}{\sqrt{x^2-a^2}}\mathrm{d}x=\ln\left|x+\sqrt{x^2-a^2}\right|$
$\int\sqrt{a^2-x^2}\mathrm{d}x=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}$
$\int\frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x=\arcsin\frac{x}{a}=-\arccos\frac{x}{a}$
$\int\cos x\mathrm{d}x=\sin x$
$\int\sin x\mathrm{d}x=-\cos x$
$\int\sec^2 x\mathrm{d}x=\tan x$
$\int\csc^2 x\mathrm{d}x=-\cot x$
$\int\sec x\tan x\mathrm{d}x=\sec x$
$\int\csc x\cot x\mathrm{d}x=-\csc x$
$\int\tan x\mathrm{d}x=-\ln|\cos x|=\ln|\sec x|$
$\int\cot x\mathrm{d}x=\ln|\sin x|$
$\int\sec x\mathrm{d}x=\ln|\sec x+\tan x|$
$\int\csc x\mathrm{d}x=\ln|\csc x-\cot x|=\ln\left|\frac{\tan x-\sin x}{\sin x\tan x}\right|$
$\int\sin^n x\mathrm{d}x=-\frac{1}{n}\sin^{n-1}x\cos x+\frac{n-1}{n}\int\sin^{n-2}x\mathrm{d}x$
$\int\sin^2 x\mathrm{d}x=\frac{x}{2}-\frac{\sin 2x}{4}$
$\int\cos^n x\mathrm{d}x=\frac{1}{n}\cos^{n-1}x\sin x+\frac{n-1}{n}\int\cos^{n-2}x\mathrm{d}x$
$\int\cos^2 x\mathrm{d}x=\frac{x}{2}+\frac{\sin 2x}{4}$
$\int\tan^n x\mathrm{d}x=\frac{1}{n-1}\tan^{n-1}x-\int\tan^{n-2}x\mathrm{d}x$
$\int\tan^2 x\mathrm{d}x=\tan x-x$
$\int xe^{ax}\mathrm{d}x=\frac{1}{a^2}(ax-1)e^{ax}$
$\int x^n e^{ax}\mathrm{d}x=\frac{1}{a}x^n e^{ax}-\frac{n}{a}\int x^{n-1}e^{ax}\mathrm{d}x$
$\int e^{ax}\sin bx\mathrm{d}x=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)$
$\int e^{ax}\cos bx\mathrm{d}x=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)$
$\int x^n\ln x\mathrm{d}x=\frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x-1]$
$\int\frac{1}{x\ln x}\mathrm{d}x=\ln(\ln x)$
$\int_{-\infty}^{\infty}e^{-ax^2+bx+c}\,\mathrm{d}x=\sqrt{\frac{\pi}{a}}\,e^{\frac{b^2}{4a}+c}$
$\iint\limits_{D}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y=\oint_{L^+}(P\mathrm{d}x+Q\mathrm{d}y)$

3 Distribution

3.1 Bernoulli Random Variable

概率：$p(1)=p,p(0)=1-p$
性质：$\mathbb{E}(X)=p,\mathrm{Var}(X)=p(1-p)$

3.2 Binomial Distribution

记号：$X\sim\mathrm{Bin}(n,p)$
概率：$p(i)=\binom{n}{i}p^i(1-p)^{n-i}$
性质：$\mathbb{E}(X)=np,\mathrm{Var}(X)=np(1-p)$

3.3 Poisson Distribution

记号：$X \sim \text{Poisson}(\lambda)$
概率：$p(k) = \frac{\lambda^k}{k!}e^{-\lambda}$
本质上是一个 $n \to \infty$ 的二项分布，$\lambda = np$。
性质：$\mathbb{E}(X) = \lambda, \text{Var}(X) = \lambda$
Approximate Bin: n large, p small ($n \geq 50, np \leq 5$)

3.4 Hypergeometric Distribution

记号：$X \sim \text{Hypergeomet}(n, N, m)$
概率：$p(k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$
N 个球，m 个红球，不放回取出 n 个，有 k 红球。
性质:$\mathbb{E}(X) = n \cdot \frac{m}{N}, \text{Var}(X) = n \cdot \frac{m}{N}\left(1 - \frac{m}{N}\right)\left(1 - \frac{n-1}{N-1}\right)$

3.5 Uniform Distribution

记号：$X \sim \text{Unif}(l, r)$
PDF：$f(x) = \frac{1}{r-l}$
CDF：$f(x) = \frac{x-l}{r-l}$

3.6 Normal Distribution

记号：$X \sim N\left(\mu, \sigma^2\right)$
PDF：$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
$\Phi(x) = \text{CDF of } N(0,1)$
To $N(0,1)$: $Z = \frac{X-\mu}{\sigma}$
Approximate Bin: $np(1-p) \geq 10$

$Z \sim N(0,1), \mathbb{E}(g'(Z)) = \mathbb{E}(Zg(Z))$, assuming that $\lim_{x\to\infty} \frac{g(x)}{e^{\frac{x^2}{2}}} = 0$. So $\mathbb{E}(Z^{n+1}) = n\mathbb{E}(Z^{n-1})$.

3.7 Exponential Distribution

$X \sim \text{Exp}(\lambda)$
PDF：$f(x) = \lambda e^{-\lambda x}, x \geq 0$
CDF：$F(X) = 1 - e^{-\lambda x}, x \geq 0$
$\mathbb{P}r(X > x) = e^{-\lambda x}, x \geq 0$
$\mathbb{E}(X^n) = \frac{n}{\lambda}\mathbb{E}(X^{n-1}) = \frac{n!}{\lambda^n}$
$\mathbb{E}(X) = \frac{1}{\lambda}, \text{Var}(X) = \frac{1}{\lambda^2}$

3.8 Gamma Distribution

$X \sim \Gamma(\alpha, \lambda)$ or $X \sim \text{Gamma}(\alpha, \lambda)$
$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\mathrm{d}u, x > 0$
$\Gamma(x) = (x-1)\Gamma(x-1)$
$\Gamma(n) = (n-1)!$
PDF：$f(x) = \frac{\lambda^\alpha}{\Gamma(a)}x^{\alpha-1}e^{-\lambda x}, x > 0$
α：发生次数
$\mathbb{E}(X) = \frac{\alpha}{\lambda}, \text{Var}(X) = \frac{\alpha}{\lambda^2}$
$\mathbb{E}(X^n) = \frac{n+\alpha-1}{\lambda}\mathbb{E}(X^{n-1}) = \frac{\alpha^{\overline{n}}}{\lambda^n}$

3.9 Chi-Squared Distribution

$X \sim \chi^2(k)$
PDF：$f(x) = \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)}x^{\frac{k}{2}-1}e^{-\frac{x}{2}}$
k（自由度）个 $N(0,1)$ 所组成向量长度平方的分布。
$\mathbb{E}(X) = k$

4 Functions of a Random Variable

$Y = g(x)$, $g(x)$ increasing,

$$f_Y(y) = \begin{cases} f_X\left(g^{-1}(y)\right) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}y}g^{-1}(y)\right) & \exists x, y = g(x) \\ 0 & \forall x, y \neq g(x) \end{cases}$$

5 Joint Distribution

5.1 Marginal CDF/PDF

$F_X(x) = \lim_{y\to+\infty} F(x, y)$
$f_X(x) = \int_{-\infty}^{+\infty} f(x, y)\mathrm{d}y$

5.2 Independence

$\mathbb{P}r\{X \in A, Y \in B\} = \mathbb{P}r\{X \in A\} \cdot \mathbb{P}r\{Y \in B\}$
$\mathbb{P}r\left\{X_1 \in A_1, \ldots, X_n \in A_n\right\} = \prod \mathbb{P}r\{X_i \in A_i\}$
$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$
$F_{XY}(x, y) = F_X(x) \cdot F_Y(y)$

5.3 Sum of Random Variables

X, Y independent, $Z = X + Y$
$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x)f_Y(z-x)\mathrm{d}x$
convolution
$X \sim \Gamma(\alpha, \lambda), Y \sim \Gamma(\beta, \lambda) \to X + Y \sim \Gamma(\alpha + \beta, \lambda)$
$X \sim N(\mu_1, \sigma_1^2), Y \sim N(\mu_2, \sigma_2^2) \to X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$