1 符号

补集：A'
P_n^r：${}_nP_r$
σ^2：$\mathrm{Var}(X) = \mathbb{E}[x^2] - [\mathbb{E}[x]]^2$
[statement]：$\mathbf{1}_{\{\text{statement}\}}$

2 公式

2.1 组合数

换系数：
$k\binom{n}{k} = n\binom{n-1}{k-1}$
组合数乘积：
$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$
$\sum_{i=0}^{m}\binom{n}{i}\binom{m}{m-i} = \binom{m+n}{m} \quad (n \geq m)$
$\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{2n}{n}$
组合数的带权和：
$\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$
$\sum_{i=0}^{n} i^2\binom{n}{i} = n(n+1)2^{n-2}$
杨辉三角列和：$\sum_{l=0}^{n}\binom{l}{k} = \binom{n+1}{k+1}$
组合数与斐波那契（Fibonacci）：
$\sum_{i=0}^{n}\binom{n-i}{i} = F_{n+1}$
卡特兰数（Catalan Numbers）：
$H_n = \sum_{i=1}^{n} H_{i-1}H_{n-i} = \frac{\binom{2n}{n}}{n+1} = \binom{2n}{n} - \binom{2n}{n-1}$

2.2 求和

$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$
$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$
$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}, r \neq 1$
$\sum_{k=1}^{n} kr^k = r\frac{1-(n+1)r^n+nr^{n+1}}{(1-r)^2}, r \neq 1$

2.3 Maclaurin Series

$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$
$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$
$\ln x = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x-1)^n = (x-1) - \frac{(x-1)^2}{2} - \cdots$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

2.4 数数

隔板法：$\sum_{i=1}^{k} x_i = n$

- $x_i > 0$: $\binom{n-1}{k-1}$
- $x_i \geq 0$: $\binom{n+k-1}{k-1}$

错位排列：$D_n = (n-1)(D_{n-1} + D_{n-2})$

2.5 常见导数公式

$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$
$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$
$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$
$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x$
$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}}$
$\frac{\mathrm{d}}{\mathrm{d}x}\arccos x = -\frac{1}{\sqrt{1-x^2}}$
$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$
$z = f(x,y), \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}$

2.6 积分（省略 $+C$）

$\int \frac{1}{x^2+\alpha^2}\mathrm{d}x = \frac{\arctan\frac{x}{\alpha}}{\alpha}$
$\int \frac{1}{\pm x^2 \mp \alpha^2}\mathrm{d}x = \frac{\ln\left(\frac{x\mp\alpha}{\pm x+\alpha}\right)}{2\alpha}$
$\int \frac{1}{ax^2+b}\mathrm{d}x = \frac{1}{\sqrt{ab}}\arctan\frac{\sqrt{a}x}{\sqrt{b}}$
$\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x = \arcsin\frac{x}{a} = -\arccos\frac{x}{a}$
$\int \sec^2 x\mathrm{d}x = \tan x$
$\int \csc^2 x\mathrm{d}x = -\cot x$
$\int \sec x\tan x\mathrm{d}x = \sec x$
$\int \csc x\cot x\mathrm{d}x = -\csc x$
$\int \tan x\mathrm{d}x = -\ln|\cos x| = \ln|\sec x|$
$\int \cot x\mathrm{d}x = \ln|\sin x|$
$\int \sec x\mathrm{d}x = \ln|\sec x + \tan x|$
$\int \csc x\mathrm{d}x = \ln|\csc x - \cot x| = \ln\left|\frac{\tan x - \sin x}{\sin x\tan x}\right|$
$\int x^n e^{ax}\mathrm{d}x = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\mathrm{d}x$
$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}\,e^{\frac{b^2}{4a}+c}$
$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \iint_\Delta f(r\cos\varphi, r\sin\varphi)r\mathrm{d}r\mathrm{d}\varphi$
$\iiint_\Omega f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega_1} f(r\cos\theta\cdot\cos\varphi, r\sin\theta\cdot\cos\varphi, r\sin\varphi)\cdot r^2\cos\varphi\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta$

3 Distribution

3.1 Bernoulli Random Variable

概率：$p(1) = p, p(0) = 1-p$
性质：$\mathbb{E}(X) = p, \mathrm{Var}(X) = p(1-p)$

3.2 Binomial Distribution

记号：$X \sim \mathrm{Bin}(n,p)$
概率：$p(i) = \binom{n}{i}p^i(1-p)^{n-i}$
性质：$\mathbb{E}(X) = np, \mathrm{Var}(X) = np(1-p)$

3.3 Poisson Distribution

记号：$X \sim \mathrm{Poisson}(\lambda)$
概率：$p(k) = \frac{\lambda^k}{k!}e^{-\lambda}$
本质上是一个 $n \to \infty$ 的二项分布，$\lambda = np$。
性质：$\mathbb{E}(X) = \lambda, \mathrm{Var}(X) = \lambda$
Approximate Bin: n large, p small $(n \geq 50, np \leq 5)$

3.4 Hypergeometric Distribution

记号：$X \sim \mathrm{Hypergeomet}(n, N, m)$
概率：$p(k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$
N 个球，m 个红球，不放回取出 n 个，有 k 红球。
$\mathbb{E}(X) = n\cdot\frac{m}{N}, \mathrm{Var}(X) = n\cdot\frac{m}{N}\left(1-\frac{m}{N}\right)\left(1-\frac{n-1}{N-1}\right)$

3.5 Uniform Distribution

记号：$X \sim \mathrm{Unif}(l,r)$
PDF：$f(x) = \frac{1}{r-l}$
CDF：$f(x) = \frac{x-l}{r-l}$

$\mathbb{E}(X)=\frac{a+b}{2},\mathrm{Var}(X)=\frac{(b-a)^2}{12}$

3.6 Normal Distribution

记号：$X\sim N\left(\mu,\sigma^2\right)$

PDF：$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
$\Phi(x)=$ CDF of $N(0,1)$
To $N(0,1)$: $Z=\frac{X-\mu}{\sigma}$
Approximate Bin: $np(1-p)\geq 10$
$Z\sim N(0,1),\mathbb{E}(g'(Z))=\mathbb{E}(Zg(Z))$, assuming that $\lim_{x\to\infty}\frac{g(x)}{e^{\frac{x^2}{2}}}=0$. So $\mathbb{E}(Z^{n+1})=n\mathbb{E}(Z^{n-1})$.

3.7 Exponential Distribution

$X\sim\mathrm{Exp}(\lambda)$
PDF：$f(x)=\lambda e^{-\lambda x},x\geq 0$
CDF：$F(X)=1-e^{-\lambda x},x\geq 0$
$\mathbb{P}r(X>x)=e^{-\lambda x},x\geq 0$
$\mathbb{E}(X^n)=\frac{n}{\lambda}\mathbb{E}(X^{n-1})=\frac{n!}{\lambda^n}$
$\mathbb{E}(X)=\frac{1}{\lambda},\mathrm{Var}(X)=\frac{1}{\lambda^2}$

3.8 Gamma Distribution

$X\sim\Gamma(\alpha,\lambda)$ or $X\sim\mathrm{Gamma}(\alpha,\lambda)$
$\Gamma(x)=\int_0^\infty u^{x-1}e^{-u}\mathrm{d}u,x>0$
$\Gamma(x)=(x-1)\Gamma(x-1)$
$\Gamma(n)=(n-1)!$
PDF：$f(x)=\frac{\lambda^\alpha}{\Gamma(a)}x^{\alpha-1}e^{-\lambda x},x>0$
α：发生次数
$\mathbb{E}(X)=\frac{\alpha}{\lambda},\mathrm{Var}(X)=\frac{\alpha}{\lambda^2}$
$\mathbb{E}(X^n)=\frac{n+\alpha-1}{\lambda}\mathbb{E}(X^{n-1})=\frac{\alpha^{\overline{n}}}{\lambda^n}$

3.9 Chi-Squared Distribution

$X\sim\chi^2(k)$
PDF：$f(x)=\frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)}x^{\frac{k}{2}-1}e^{-\frac{x}{2}}$
k（自由度）个 $N(0,1)$ 所组成向量长度平方的分布。
$\mathbb{E}(X)=k$

4 Functions of a Random Variable

$Y=g(x)$, $g(x)$ increasing,

$$f_Y(y)=\begin{cases}f_X\left(g^{-1}(y)\right)\cdot\left(\frac{\mathrm{d}}{\mathrm{d}y}g^{-1}(y)\right) & \exists x,y=g(x)\\ 0 & \forall x,y\neq g(x)\end{cases}$$

5 Joint Distribution

5.1 Marginal CDF/PDF

$F_X(x)=\lim_{y\to+\infty}F(x,y)$
$f_X(x)=\int_{-\infty}^{+\infty}f(x,y)\mathrm{d}y$

5.2 Sum of Random Variables

X,Y independent, $Z=X+Y$
$f_Z(z)=\int_{-\infty}^{+\infty}f_X(x)f_Y(z-x)\mathrm{d}x$
$X_i\sim\Gamma(\alpha_i,\lambda)\to\sum X_i\sim\Gamma(\sum\alpha_i,\lambda)$
$X_i\sim N(\mu_i,\sigma_i^2)\to\sum X_i\sim N(\sum\mu_i,\sum\sigma_i^2)$

5.3 Order Statistics

$X_1,X_2,\cdots,X_n$ 独立，CDF 为 F，PDF 为 f.
令 U 为最大值，V 为最小值。
$F_U(u)=F^n(u),f_U(u)=nf(u)F^{n-1}(u)$
$F_V(v)=1-(1-F(v))^n,f_V(v)=nf(v)\left[1-F(v)\right]^{n-1}$

$$F_{UV}(u,v)=\begin{cases}[F(u)]^n-[F(u)-F(v)]^n & u>v\\ [F(u)]^n & u\leq v\end{cases}$$

$$f_{UV}(u,v)=\begin{cases}n(n-1)f(u)f(v)\left[F(u)-F(v)\right]^{n-2} & u>v\\ 0 & u\leq v\end{cases}$$

5.4 Covariance

$\mathbb{E}(g(X)\cdot h(Y))=\mathbb{E}(g(X))\cdot\mathbb{E}(h(Y))$
$\mathrm{Cov}(X,Y)=\mathbb{E}((X-\mu_X)(Y-\mu_Y))=\mathbb{E}(XY)-\mathbb{E}(X)\mathbb{E}(Y)$
X,Y 独立 $\to\mathrm{Cov}(X,Y)=0$，但反过来不行
Cov 的线性性：
$\mathrm{Cov}\left(a+\sum_{i=1}^n b_iX_i,c+\sum_{i=1}^m d_iY_i\right)=\sum_{i=1}^n\sum_{j=1}^m b_id_j(X_i,Y_j)$
$\mathrm{Var}\left(a+\sum_{i=1}^n b_iX_i\right)=\sum_{i=1}^n\sum_{j=1}^n b_ib_j(X_i,Y_j)$
Correlation coefficient: $\rho=\frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}=\frac{\sigma_{XY}}{\sigma_X\sigma_Y}$
柯西不等式：$-1\leq\rho\leq 1$，$\rho=\pm 1$ 当且仅当 X,Y 线性相关

5.5 Sample Mean/Variance

$\overline{X}=\frac{1}{n}\sum_{i=1}^n X_i$
$S^2=\frac{1}{n-1}\sum_{i=1}^n\left(X_i-\overline{X}\right)^2$
$\frac{1}{n-1}$ 的原因：为了对齐 σ^2，$\mathbb{E}(S^2)=\sigma^2$

5.6 Conditioning

$\mathbb{P}\mathrm{r}(E)=\int_{\mathbb{R}}\mathbb{P}\mathrm{r}(E\mid X=x)p(x)\mathrm{d}x$
$\mathbb{E}(Y)=\mathbb{E}_X[\mathbb{E}_Y(Y\mid X)]$
$\mathrm{Var}(Y\mid X)=\mathbb{E}(Y^2\mid X)-[\mathbb{E}(Y\mid X)]^2$
$\mathrm{Var}(Y)=\mathrm{Var}[\mathbb{E}(Y\mid X)]+\mathbb{E}[\mathrm{Var}(Y\mid X)]$
$T=\sum_{i=1}^N X_i$，X,N 随机
$Var(T)=(\mathbb{E}(X))^2\mathrm{Var}(N)+\mathbb{E}(N)\mathrm{Var}(X)$

5.7 Central Limit Theorem

5.7.1 Inequalities

Markov's inequality: $\mathbb{P}\mathrm{r}\{X\geq t\}\leq\frac{\mathbb{E}(X)}{t}$，要求是 $X\geq 0,t>0$
Chebyshev's Inequality: $\mathbb{P}\mathrm{r}\{|X-\mathbb{E}(X)|\geq t\}\leq\frac{\mathrm{Var}(X)}{t^2}$
$\mathbb{P}\mathrm{r}\{|X-\mathbb{E}(X)|\geq k\sigma\}\leq\frac{1}{k^2}$

5.7.2 Law of Large Numbers

$\overline{X}_n=\frac{1}{n}\sum_{i=1}^n X_i$
Weak: $\lim_{n\to\infty}\mathbb{P}\mathrm{r}\left\{\left|\overline{X}_n-\mu\right|>\epsilon\right\}=0$
Strong: $\mathbb{P}\mathrm{r}\left\{\lim_{n\to\infty}\overline{X}_n=\mu\right\}=1$

5.7.3 CLT

$S_n=\sum_{i=1}^n X_i$
$\lim_{n\to\infty}\mathbb{P}\mathrm{r}\left\{\frac{S_n-n\mu}{\sigma\sqrt{n}}\right\}=\Phi(x)$