1 公式

1.1 求和

$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$
$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}, r \neq 1$
$\sum_{k=1}^{n} kr^k = r\frac{1-(n+1)r^n+nr^{n+1}}{(1-r)^2}, r \neq 1$

1.2 Maclaurin Series

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$
$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$
$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n = (x-1) - \frac{(x-1)^2}{2} - \cdots$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

1.3 常见导数公式

$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$
$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$
$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x \tan x$
$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x$
$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}}$
$\frac{\mathrm{d}}{\mathrm{d}x}\arccos x = -\frac{1}{\sqrt{1-x^2}}$
$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$

1.4 积分（省略 $+C$）

$\int \frac{1}{x^2+\alpha^2}\mathrm{d}x = \frac{\arctan\frac{x}{\alpha}}{\alpha}$
$\int \frac{1}{\pm x^2 \mp \alpha^2}\mathrm{d}x = \frac{\ln\left(\frac{x\mp\alpha}{\pm x+\alpha}\right)}{2\alpha}$
$\int \frac{1}{ax^2+b}\mathrm{d}x = \frac{1}{\sqrt{ab}}\arctan\frac{\sqrt{a}x}{\sqrt{b}}$
$\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x = \arcsin\frac{x}{a} = -\arccos\frac{x}{a}$
$\int \sec^2 x\mathrm{d}x = \tan x$
$\int \csc^2 x\mathrm{d}x = -\cot x$
$\int \sec x \tan x\mathrm{d}x = \sec x$
$\int \csc x \cot x\mathrm{d}x = -\csc x$
$\int \tan x\mathrm{d}x = -\ln|\cos x| = \ln|\sec x|$
$\int \cot x\mathrm{d}x = \ln|\sin x|$
$\int \sec x\mathrm{d}x = \ln|\sec x + \tan x|$
$\int \csc x\mathrm{d}x = \ln|\csc x - \cot x| = \ln\left|\frac{\tan x - \sin x}{\sin x \tan x}\right|$
$\int x^n e^{ax}\mathrm{d}x = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\mathrm{d}x$
$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}\,e^{\frac{b^2}{4a}+c}$

2 Probability

2.1 Distribution

2.1.1 Poisson Distribution

记号：$X \sim \mathrm{Poisson}(\lambda)$
概率：$p(k) = \frac{\lambda^k}{k!}e^{-\lambda}$
本质上是一个 $n \to \infty$ 的二项分布，$\lambda = np$。
性质：$\mathbb{E}(X) = \lambda, \mathrm{Var}(X) = \lambda$
Approximate Bin: n large, p small $(n \geq 50, np \leq 5)$

2.1.2 Hypergeometric Distribution

记号：$X \sim \mathrm{Hypergeomet}(n, N, m)$
概率：$p(k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$
N 个球，m 个红球，不放回取出 n 个，有 k 红球。
$\mathbb{E}(X) = n \cdot \frac{m}{N}, \mathrm{Var}(X) = n \cdot \frac{m}{N}\left(1 - \frac{m}{N}\right)\left(1 - \frac{n-1}{N-1}\right)$

2.1.3 Normal Distribution

PDF：$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
To $N(0,1)$: $Z = \frac{X-\mu}{\sigma}$
Approximate Bin: $np(1-p) \geq 10$
$Z \sim N(0,1), \mathbb{E}(g'(Z)) = \mathbb{E}(Zg(Z))$, assuming that $\lim_{x\to\infty} \frac{g(x)}{e^{\frac{x^2}{2}}} = 0$. So $\mathbb{E}(Z^{n+1}) = n\mathbb{E}(Z^{n-1})$.

2.1.4 Exponential Distribution

$X \sim \mathrm{Exp}(\lambda)$
PDF：$f(x) = \lambda e^{-\lambda x}, x \geq 0$
CDF：$F(X) = 1 - e^{-\lambda x}, x \geq 0$
$\mathbb{P}r(X > x) = e^{-\lambda x}, x \geq 0$
$\mathbb{E}(X^n) = \frac{n}{\lambda}\mathbb{E}(X^{n-1}) = \frac{n!}{\lambda^n}$
$\mathbb{E}(X) = \frac{1}{\lambda}, \mathrm{Var}(X) = \frac{1}{\lambda^2}$

2.1.5 Gamma Distribution

考试都用 $\Gamma(\alpha, \beta)$ 的形式
$\Gamma(x) = \int_0^{\infty} u^{x-1}e^{-u}\mathrm{d}u, x > 0$
$\Gamma(x) = (x-1)\Gamma(x-1)$
$\Gamma(n) = (n-1)!$
PDF：$f(x) = \frac{\lambda^\alpha}{\Gamma(a)}x^{\alpha-1}e^{-\lambda x}, x > 0$
α：发生次数
$\mathbb{E}(X) = \frac{\alpha}{\lambda}, \mathrm{Var}(X) = \frac{\alpha}{\lambda^2}$
$\mathbb{E}(X^n) = \frac{n+\alpha-1}{\lambda}\mathbb{E}(X^{n-1}) = \frac{\alpha^{\overline{n}}}{\lambda^n}$

2.1.6 Chi-Squared Distribution

$X \sim \chi^2(k)$
PDF：$f(x) = \frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)}x^{\frac{k}{2}-1}e^{-\frac{x}{2}}$
k（自由度）个 $N(0,1)$ 所组成向量长度平方的分布。
$\mathbb{E}(X) = k$

2.2 MGF

$M(X) = \mathbb{E}\left[e^{tX}\right]$
$M^{(m)}(X) = \mathbb{E}\left[X^m\right]$

2.3 Covariance

$\mathbb{E}(g(X) \cdot h(Y)) = \mathbb{E}(g(X)) \cdot \mathbb{E}(h(Y))$
$\mathrm{Cov}(X,Y) = \mathbb{E}((X-\mu_X)(Y-\mu_Y)) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$
X, Y 独立 $\to \mathrm{Cov}(X,Y) = 0$，但反过来不行
Cov 的线性性：
$\mathrm{Cov}\left(a + \sum_{i=1}^{n} b_i X_i, c + \sum_{i=1}^{m} d_i Y_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} b_i d_j \mathrm{Cov}(X_i, Y_j)$
$\mathrm{Var}\left(a + \sum_{i=1}^{n} b_i X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_i b_j \mathrm{Cov}(X_i, Y_j)$
Correlation coefficient: $\rho = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y}$
柯西不等式：$-1 \leq \rho \leq 1$，$\rho = \pm 1$ 当且仅当 X, Y 线性相关

2.4 Sample Mean/Variance

$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$
$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2$
$\frac{1}{n-1}$ 的原因：为了对齐 σ^2，$\mathbb{E}(S^2) = \sigma^2$

2.5 Conditioning

$\Pr(E) = \int_{\mathbb{R}} \Pr(E \mid X = x) p(x) \mathrm{d}x$
$\mathbb{E}(Y) = \mathbb{E}_X[\mathbb{E}_Y(Y \mid X)]$
$\mathrm{Var}(Y \mid X) = \mathbb{E}(Y^2 \mid X) - [\mathbb{E}(Y \mid X)]^2$
$\mathrm{Var}(Y) = \mathrm{Var}[\mathbb{E}(Y \mid X)] + \mathbb{E}[\mathrm{Var}(Y \mid X)]$
$T = \sum_{i=1}^{N} X_i$，X, N 随机
$Var(T) = (\mathbb{E}(X))^2 \mathrm{Var}(N) + \mathbb{E}(N) \mathrm{Var}(X)$

2.6 Central Limit Theorem

2.6.1 Inequalities

Markov's inequality: $\Pr\{X \geq t\} \leq \frac{\mathbb{E}(X)}{t}$，要求是 $X \geq 0, t > 0$
Chebyshev's Inequality：$\Pr\{|X - \mathbb{E}(X)| \geq t\} \leq \frac{\mathrm{Var}(X)}{t^2}$
$\Pr\{|X - \mathbb{E}(X)| \geq k\sigma\} \leq \frac{1}{k^2}$

2.6.2 Law of Large Numbers

$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$
Weak: $\lim_{n\to\infty} \Pr\left\{\left|\overline{X}_n - \mu\right| > \epsilon\right\} = 0$
Strong: $\Pr\left\{\lim_{n\to\infty} \overline{X}_n = \mu\right\} = 1$

2.6.3 CLT

$S_n = \sum_{i=1}^{n} X_i$
$\lim_{n\to\infty} \Pr\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right\} = \Phi(x)$