On commutativity, total orders, and sorting

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In this talk, we study free monoids, free commutative monoids, and their connections with sorting and well-orders. Univalent foundations provides a rigorous framework for implementing these ideas, in the construction of free algebras, using higher inductive types and quotients, and reasoning upto equivalence using categorical universal properties. The main contributions are a framework for universal algebra (free algebras and their universal properties), various constructions of free monoids and free commutative monoids (with proofs of their universal properties), applications to proving combinatorial properties of these constructions, and finally an axiomatic understanding of sorting. Our work is formalized in Cubical Agda and available at: `https://github.com/pufferffish/agda-symmetries/`.

Background

First, we review the basics of universal algebra, free algebras and their universal property. We write Set for the category of hSets and functions. A signature σ is given by a type of operations with an arity function: $(\mathsf{op} : \mathsf{Set}) \times (\mathsf{ar} \colon \mathsf{op} \to \mathsf{Set})$. This gives a signature endofunctor $F_\sigma(X) :\equiv \sum_{(f \colon \mathsf{op})} X^{\mathsf{ar}(f)}$ on Set. A σ-structure $\mathfrak{X}$ is an F_σ-algebra: $(X : \mathsf{Set}) \times (\alpha_X \colon F_\sigma(X) \to X)$, with carrier set X, and a morphism of σ-structures is a F_σ-algebra morphism, giving the category of σ-algebras $\sigma\text{-}\mathsf{Alg}$.

The forgetful functor $U_\sigma \colon \sigma\text{-}\mathsf{Alg}$ to Set admits a left adjoint, giving the free σ-algebra construction on a carrier set. As is standard, this construction is given by an inductive type of trees $\mathsf{Tr}_\sigma(V)$, generated by two constructors, $\mathsf{leaf} \colon V \to \mathsf{Tr}_\sigma(V)$ and $\mathsf{node} \colon \Sigma_\sigma(\mathsf{Tr}_\sigma(V)) \to \mathsf{Tr}_\sigma(V)$. $\mathsf{Tr}_\sigma(V)$ is canonically a σ-algebra $\mathfrak{T}(V) = (\mathsf{Tr}_\sigma(V), \mathsf{node})$, with the universal map $\eta_V : V \to \mathsf{Tr}_\sigma(V)$ given by leaf. The universal property states that, given any σ-structure $\mathfrak{X}$, composition with η_V is an equivalence: $(-) \circ \eta_V \colon \sigma\text{-}\mathsf{Alg}(\mathfrak{T}(V), \mathfrak{X}) \xrightarrow{\sim} (V \to X)$. The inverse to this map is the extension operation $(-)^\sharp$, which extends a map $f \colon V \to X$ to a homomorphism $f^\sharp \colon \mathfrak{T}(V) \to \mathfrak{X}$.

An equational signature ε is given by a type of equations with an arity of free variables: $(\mathsf{eq} : \mathsf{Set}) \times (\mathsf{fv} \colon \mathsf{eq} \to \mathsf{Set})$. A system of equations (or a theory T) over (σ, ε) is given by a pair of trees on the set of free variables, for each equation: $\ell, r \colon (e : \mathsf{eq}) \to \mathsf{Tr}_\sigma(\mathsf{fv}(e))$. A σ-structure $\mathfrak{X}$ satisfies T, written $\mathfrak{X} \vDash T$, if, for each equation $e : \mathsf{eq}$ and $\rho \colon \mathsf{fv}(e) \to X$, $\rho^\sharp(\ell(e)) = \rho^\sharp(r(e))$. The full subcategory of $\sigma\text{-}\mathsf{Alg}$ given by σ-structures satisfying T is the variety of T-algebras in Set. Similarly, the forgetful functor to Set admits a left adjoint, which is classically constructed by quotienting the free σ-algebra by the congruence relation generated by T. However, we do not give the general construction for it, since it requires non-constructive principles [1], and instead consider the varieties of monoids and commutative monoids.

Monoids and commutativity

The signature for monoids σ_{Mon} is given by two operations (unit and multiplication) of arity 0 and 2, respectively, written as $(\mathsf{Fin}(2), \{0 \mapsto \mathsf{Fin}(0); 1 \mapsto \mathsf{Fin}(2)\})$. The equational signature for monoids $\varepsilon_{\mathsf{Mon}}$ is given by three equations (left unit, right unit, associativity) of free variable arity 1, 1, and 3, respectively, written as $(\mathsf{Fin}(3), \{0 \mapsto \mathsf{Fin}(1); 1 \mapsto \mathsf{Fin}(1); 2 \mapsto \mathsf{Fin}(3)\})$. The theory of monoids T_{Mon} is given by the pairs of left and right trees, using the free variables for each equation. Commutative monoids are given by the same signature of operations, but additionally include the commutativity equation, which uses 2 free variables.

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We study various constructions of free monoids and free commutative monoids, using HITs and quotients, and prove the universal property for each construction. We construct:

- $\mathsf{FreeMon}$ and $\mathsf{FreeCMon}$ HITs, given by generators for operations and higher generators for equations,
- List, SList, CList, given by cons-lists, cons-lists with adjacent swaps, cons-lists with a commutation relation, respectively. (see: [2, 3, 4])

Using quotients, we consider various commutativity relations on presentations of free monoids. Given a free monoid construction: $A \xrightarrow{\eta} \mathscr{L}(A)$, a commutativity relation is a binary relation $\approx$ on $\mathscr{L}(A)$ such that, $A \xrightarrow{\eta} \mathscr{L}(A) \xrightarrow{q} \mathscr{L}(A)/\approx$ is a free commutative monoid construction. From this we construct:

- PList, a quotient of List by various permutation relations,
- Bag, a quotient of $\mathsf{Array}(A) = (n : \mathbb{N}) \times (f : A^{\mathsf{Fin}(n)})$ by $(n, f) \sim (m, g) :\equiv (\sigma \colon \mathsf{Fin}(n) \simeq \mathsf{Fin}(m)) \times (f = g \circ \sigma)$.

Further, using the universal property we study various properties of these constructions:

- characterizations of the path spaces of each type,
- combinatorial properties, such as, $\mathscr{L}(A + B) \simeq \mathscr{L}(A) + \mathscr{L}(B)$, $\mathscr{M}(A + B) \simeq \mathscr{M}(A) \times \mathscr{M}(B)$,
- injectivity of $\mathsf{cons}_A(x, -) \colon \mathscr{L}(A) \to \mathscr{L}(A)$ and $\mathscr{M}(A) \to \mathscr{M}(A)$, for any $x : A$.

Total orders and Sorting

Finally, we use this framework to study sorting and total orders. It is commonly understood that lists are ordered lists and bags are unordered lists. Our aim is to give a conceptual explanation of this fact.

Given a total order on a set A, a sorting algorithm, informally, turns lists of A into sorted lists of A. Formally, this produces a well-behaved section to the canonical homomorphism from the free monoid to the free commutative monoid: $q : \mathscr{L}(X) \to \mathscr{M}(X)$.

Definition 1. *Given a section $s : SList\ X \to List\ X$ to the canonical map $List\ X \to SList\ X$, xs is said to be sorted if xs is in the image of s.*

We define the proposition $\textit{is-sorted} : List\ X \to \mathscr{U}$ to be $\lambda xs.\ \exists(ys : SList\ X).\ s(ys) = xs$. We also use the universal property of free monoid to define membership proofs for $List\ X$ and $SList\ X$. We do so by noting propositions form a commutative monoid under $\vee$, which allow us to define membership proof for an element x using the extension operation $(-)^\sharp$ by lifting the function $\lambda y.\ x = y$ from $X \to Prop$ to $List\ X \to Prop$ and $SList\ X \to Prop$ respectively.

Proposition 1. *Assume a total order on X. Then, there is a section $s : \mathscr{M}(X) \to \mathscr{L}(X)$ to q satisfying $\forall\, x\, y\, xs.\ \textit{is-sorted}(x :: xs) \to y \in x :: xs \to \textit{is-sorted}([x, y])$.*

We will prove the converse of this theorem.

Proposition 2. *A section $s : \mathscr{M}(X) \to \mathscr{L}(X)$ to q satisfying $\forall\, x\, y\, xs.\ \textit{is-sorted}(x :: xs) \to y \in x :: xs \to \textit{is-sorted}([x, y])$ implies a total order on X.*

To conclude, the framework of universal algebra can be generalised from sets to groupoids, using a system of coherences on top of system of equations. As an instance of this, we consider the construction free monoidal and free symmetric monoidal groupoids. This is currently work in progress, and we will mention the rudiments of the theory in the talk.

References

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