ST102 Class 16 – Solutions to Additional exercises

1. The significance level is:

$$\alpha = P(\text{reject } \mathrm{H}_0 \,|\, \mathrm{H}_0 \text{ is true}) = P(X \geq 0.90 \,|\, \theta = 1) = \int_{0.90}^{1} 2x \, \mathrm{d}x = \left[x^2\right]_{0.90}^{1} = 0.19.$$

2. Let $\{X_1, X_2, \ldots, X_{400}\}$, taking values either 1 or 0, be the outcomes of an experiment of tossing a coin 400 times, where:

$$P(X_i = 1) = \pi = 1 - P(X_i = 0)$$

for $\pi \in (0, 1)$, and 0 otherwise. We are interested in testing:

$$\mathrm{H}_0 : \pi = 0.50 \quad \text{vs.} \quad \mathrm{H}_1 : \pi \neq 0.50.$$

Let $T = \sum_{i=1}^{400} X_i$. Under H_0, then $T \sim \mathrm{Bin}(400, 0.50) \approx N(200, 100)$, using the normal approximation of the binomial distribution, where $\mu = n\pi_0 = 400 \times 0.50 = 200$ and $\sigma^2 = n\pi_0(1 - \pi_0) = 400 \times 0.50 \times 0.50 = 100$. We observe $t = 217$, hence (using the continuity correction):

$$P(T \geq 216.5) = P\left(Z \geq \frac{216.5 - 200}{\sqrt{100}}\right) = P(Z \geq 1.65) = 0.0495.$$

Therefore, the p-value is:

$$2 \times P(Z \geq 1.65) = 0.0990$$

which is far larger than $\alpha = 0.05$, hence we do not reject H_0 and conclude that there is no evidence to suggest that the coin is not fair at the 5% significance level.

(Note that the test would be significant if we set $\mathrm{H}_1 : \pi > 0.50$, as the p-value would be 0.0495 which is less than 0.05 (just). However, we have no *a priori* reason to perform an upper-tailed test – we should not determine our hypotheses by observing the sample data, rather the hypotheses should be set *before* any data are observed.)

Alternatively, one could apply the central limit theorem such that under H_0 we have:

$$\bar{X} \sim N\left(\pi, \frac{\pi(1 - \pi)}{n}\right) = N(0.50, 0.000625)$$

approximately, since $n = 400$. We observe $\bar{x} = 217/400 = 0.5425$, hence:

$$P(\bar{X} \geq 0.5425) = P\left(Z \geq \frac{0.5425 - 0.50}{\sqrt{0.000625}}\right) = P(Z \geq 1.70) = 0.0446.$$

Therefore, the p-value is:

$$2 \times P(Z \geq 1.70) = 2 \times 0.0446 = 0.0892$$

leading to the same conclusion.

3. (a) We have:

$$\alpha = P(\text{reject H}_0 \,|\, \text{H}_0) = P(X \leq 3 \,|\, \pi = 0.75) = \sum_{x=0}^{3} \binom{7}{x} (0.75)^x (0.25)^{7-x} = 0.0706.$$

(b) We have:

π	β
0.75	0.0706
0.65	0.1998
0.55	0.3917
0.45	0.6083
0.35	0.8002
0.25	0.9294
0.15	0.9879