ST102/ST109 Outline solutions to Exercise 8

1.* We have:

$$\begin{aligned}\text{E(profit)} &= \int_0^n (x - 0.5(n-x))\,\frac{1}{200}\,\mathrm{d}x + \int_n^{200} (n - 5(x-n))\,\frac{1}{200}\,\mathrm{d}x \\ &= \frac{1}{200}\left[\frac{x^2}{2} + \frac{(n-x)^2}{4}\right]_0^n + \frac{1}{200}\left[6nx - \frac{5x^2}{2}\right]_n^{200} \\ &= \frac{1}{200}(-3.25n^2 + 1{,}200n - 100{,}000).\end{aligned}$$

Differentiating with respect to n, we have:

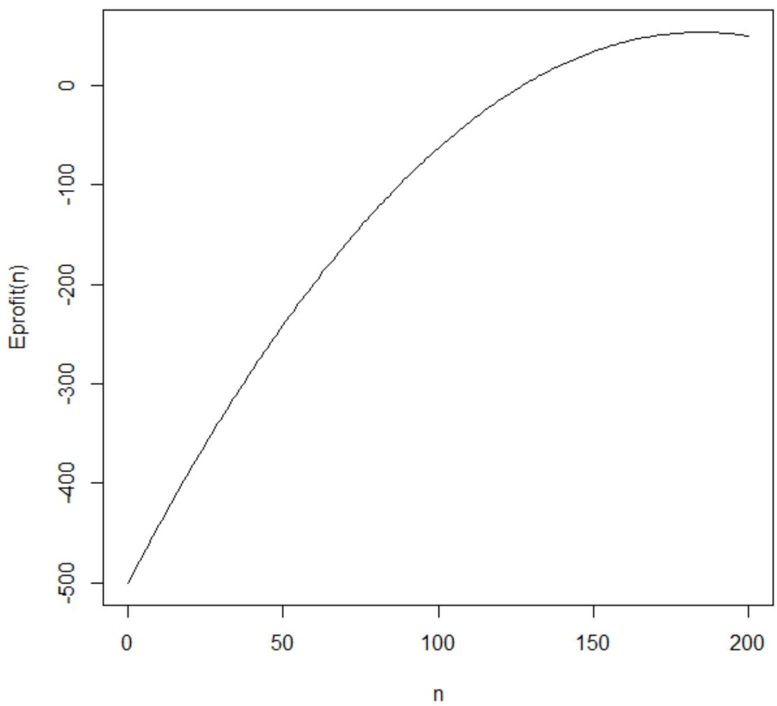
$$\frac{\mathrm{dE(profit)}}{\mathrm{d}n} = \frac{1}{200}(-6.5n + 1{,}200).$$

Equating to zero and solving, we have:

$$n = \frac{1{,}200}{6.5} \approx 185.$$

In R, we could calculate this using the following code:

```
> Eprofit<-function(n)(-3.25*n^2+1200*n-100000)/200
> n<-0:200
> plot(n,Eprofit(n),type="l")
> max(Eprofit(n))
[1] 53.84375
> n[Eprofit(n) == max(Eprofit(n))]
[1] 185
```



2. (a) The waiting time until the first call will follow an exponential distribution with $\lambda = 0.6$. Hence:

$$f(t) = \begin{cases} 0.6\mathrm{e}^{-0.6t} & \text{for } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b) The required probability can simply be calculated using integration:

$$P(T > 4) = \int_4^\infty 0.6\mathrm{e}^{-0.6t}\,\mathrm{d}t = -\left[\mathrm{e}^{-0.6t}\right]_4^\infty = \mathrm{e}^{-2.4} = 0.0907.$$

3. (a) The rate is given as 96 per hour, but it is convenient to work in numbers of minutes, so note that this is the same as $\lambda = 1.6$ arrivals per minute.

i. For two minutes, use $\lambda = 1.6 \times 2 = 3.2$. Hence:

$$P(X = 5) = \frac{\mathrm{e}^{-3.2}(3.2)^5}{5!} = 0.1140.$$

ii. For 45 seconds, $\lambda = 1.6 \times 0.75 = 1.2$. Hence:

$$\begin{aligned} P(X > 2) &= 1 - P(X \leq 2) \\ &= 1 - \sum_{x=0}^{2} \frac{\mathrm{e}^{-1.2}(1.2)^x}{x!} \\ &= 1 - \mathrm{e}^{-1.2}(1 + 1.2 + 0.72) \\ &= 0.1205. \end{aligned}$$

iii. The probability that the time to arrival of the next customer is less than one minute is $1 - P(\text{no arrivals in one minute}) = 1 - P(X = 0)$. For one minute we use $\lambda = 1.6$, hence:

$$1 - P(X = 0) = 1 - \frac{\mathrm{e}^{-1.6}(1.6)^0}{0!} = 1 - \mathrm{e}^{-1.6} = 1 - 0.2019 = 0.7981.$$

(b) The time to the next customer is more than t if there are no arrivals in the interval from 0 to t, which means that we need to use $\lambda = 1.6 \times t$. Now the conditional probability formula yields:

$$P(T > 2.4 \,|\, T > 1) = \frac{P(\{T > 2.4\} \cap \{T > 1\})}{P(T > 1)}$$

and, as in other instances, the two events $\{T > 2.4\}$ and $\{T > 1\}$ collapse to a single event, $\{T > 2.4\}$. Hence, using part iii. above for $P(T > 1) = P(X = 0)$, we have:

$$P(T > 2.4 \,|\, T > 1) = \frac{P(T > 2.4)}{P(T > 1)} = \frac{P(T > 2.4)}{0.2019}.$$

To calculate the numerator, use $\lambda = 1.6 \times 2.4 = 3.84$, hence (by the same method as in (iii.):

$$P(T > 2.4) = \frac{\mathrm{e}^{-3.84}(3.84)^0}{0!} = \mathrm{e}^{-3.84} = 0.0215.$$

Hence:

$$P(T > 2.4 \,|\, T > 1) = \frac{P(T > 2.4)}{P(T > 1)} = \frac{0.0215}{0.2019} = 0.1065.$$

4. (a) Yes, because the Poisson assumptions are probably satisfied – crashes are independent events and the crash rate is likely to remain constant.

   (b) Since $\lambda = 2.75$ crashes per year:

$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - \sum_{x=0}^{1} \frac{\mathrm{e}^{-2.75}(2.75)^x}{x!} = 0.7603.$$

   (c) Let Y = interval (in years) between the next two crashes. Therefore, we have $Y \sim \mathrm{Exp}(2.75)$. Since three months is 25% of a year, we have:

$$\begin{aligned} P(Y < 0.25) = \int_0^{0.25} 2.75\mathrm{e}^{-2.75y}\,\mathrm{d}y &= F(0.25) - F(0) \\ &= (1 - \mathrm{e}^{-2.75(0.25)}) - (1 - \mathrm{e}^{-2.75(0)}) \\ &= 1 - \mathrm{e}^{-0.6875} \\ &= 0.4972. \end{aligned}$$

5. Let X be the random variable representing the lifetime of a light bulb (in hours), so that for some value σ we have $X \sim N(185, \sigma^2)$. We want $P(X > 160) \geq 0.95$, such that:

$$P(X > 160) = P\left(Z > \frac{160 - 185}{\sigma}\right) = P\left(Z > -\frac{25}{\sigma}\right) \geq 0.95.$$

Note that this is the same as $P(Z > 25/\sigma) \leq 1 - 0.95 = 0.05$, so $25/\sigma = 1.645$, giving $\sigma = 15.20$ as the largest possible value for the standard deviation.