ST102/ST109 Exercise 5

In this exercise you will practise aspects of discrete random variables. Question 1 requires the derivation of a probability function – take note of the hint. Question 2 asks for an expected value – again, take note of the hint. Question 3 involves checking that the necessary conditions of a probability function are satisfied for the function you obtain. Question 4 concerns the expectation of a function of a random variable. Finally, Question 5 requires a similar approach to Example 3.15 for the binomial distribution.

You should attempt these questions ahead of the corresponding class. It will be covered by your class teacher in your fifth class, which will take place in the week commencing Monday 4 November 2024.

1. Suppose that a box contains 13 red balls and 5 blue balls. If 8 balls are selected at random, without replacement, determine the probability function of X, the number of red balls which will be obtained.

   Hint: Begin by determining the sample space of X, then use combinations in a similar way as used in Exercise 3 Question 1.

2.* Suppose that on each play of a certain game James, a gambler, is equally likely to win or to lose. Suppose that when he wins, his fortune is doubled, and that when he loses, his fortune is cut in half. If James begins playing with a given fortune $c > 0$, what is the expected value of his fortune after n independent plays of the game?

   Hint: If $X_1, X_2, \ldots, X_n$ are independent random variables, then:

$$\mathrm{E}(X_1 X_2 \cdots X_n) = \mathrm{E}(X_1) \times \mathrm{E}(X_2) \times \cdots \times \mathrm{E}(X_n).$$

   That is, for independent random variables the 'expectation of the product' is the 'product of the expectations'. This will be introduced in Chapter 5: Multivariate random variables.

3. Consider a sequence of independent tosses of a fair coin. Let the random variable X denote the number of tosses needed to obtain the first head. Determine the probability function of X and verify it satisfies the necessary conditions for a valid probability function.

4. If a probability function of a random variable X is given by:

$$p(x) = \begin{cases} 1/2^x & \text{for } x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

   show that $\mathrm{E}(2^X)$ does not exist. For discussion, is there a link with Question 3?

5.* A discrete random variable X has possible values $0, 1, 2, \ldots$, and the probability function:

$$p(x) = \begin{cases} \mathrm{e}^{-\lambda} \lambda^x / x! & \text{for } x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

   where $\lambda > 0$ is a parameter. Show that $\mathrm{E}(X) = \lambda$ by determining $\sum x \, p(x)$.