ST202/ST206 – Michaelmas Term
Solutions to problem set 9

1. Using moment-generating functions, we have

$$
\begin{aligned}
M_Y(t) &= \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} \exp\{\lambda_i(e^t - 1)\} \\
&= \exp\left\{\sum_{i=1}^{n} \lambda_i(e^t - 1)\right\} = \exp\{\mu(e^t - 1)\}
\end{aligned}
$$

where $\mu = \sum_{i=1}^{n} \lambda_i$. We conclude that $Y \sim \text{Pois}\left(\sum_{i=1}^{n} \lambda_i\right)$.

2. We need to evaluate the convolution integral

$$
f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(u, z-u)dz\,.
$$

The tricky part is working out the correct region of integration. The support of the density corresponds to $0 < u < z - u < 1$. Solving these inequalities for u gives

$$
u > 0\,, \quad u > z - 1\,, \quad u < \frac{z}{2}\,, \quad u < z\,, \quad u < 1\,,
$$

that is, $\max\{0, z-1\} < u < \min\{1, \frac{z}{2}, z\}$. The upper bound is always $\frac{z}{2}$ (because $z < 2$), while the lower bound depends on whether or not z is less than 1. We have

$$
\begin{aligned}
f_Z(z) &= \int_{\max\{0,z-1\}}^{z/2} 2dz = 2\left(\frac{z}{2} - \max\{0, z-1\}\right) \\
&= \begin{cases} z & \text{if } 0 < z < 1, \\ 2 - z & \text{if } 1 < z < 2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

3. (a) Multiplying together the marginal and conditional gives us the joint mass/density function, and we can then integrate out Y. We have

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)dx \\
&= \int_0^{\infty} \frac{e^{-x}x^y}{y!} \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x} dx \\
&= \frac{\theta^\alpha}{y!\,\Gamma(\alpha)} \int_0^{\infty} x^{\alpha+y-1} e^{-(\theta+1)x} dx\,.
\end{aligned}
$$

We identify the integrand as part of a Gamma($\alpha+y, \theta+1$) density function; we just need to plug in the correct constant for it to integrate to 1, that is,

$$f_Y(y) = \frac{\theta^\alpha}{y!\,\Gamma(\alpha)} \frac{\Gamma(\alpha+y)}{(\theta+1)^{\alpha+y}} \underbrace{\int_0^\infty \frac{(\theta+1)^{\alpha+y}}{\Gamma(\alpha+y)} x^{\alpha+y-1} e^{-(\theta+1)x} dx}_{=1}$$

for $y = 0, 1, 2, \ldots$.

(b) We can work this out very easily by applying the law of iterated expectations. Alternatively, we can rearrange the expression to obtain

$$f_Y(y) = \frac{\Gamma(\alpha+y)}{y!\,\Gamma(\alpha)} \left(1 - \frac{\theta}{\theta+1}\right)^y \left(\frac{\theta}{\theta+1}\right)^\alpha \quad \text{for } y = 0, 1, 2, \ldots$$

which is a NegBin($\alpha, \frac{\theta}{\theta+1}$) mass function. Notice that α is not necessarily an integer, so we replace the factorials by Gamma functions, and the support of the distribution is the non-negative integers, i.e. we are counting the number of failures. From standard results, we know that $\mathbb{E}(Y) = \alpha\,\frac{\theta+1}{\theta} - \alpha = \alpha/\theta$.

4.* (a) The inverse transformation is

$$U = X \quad \text{and} \quad V = (\sqrt{1-\rho^2})^{-1}(Y - \rho X)\,,$$

which has Jacobian

$$J_{\mathbf{h}}(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\rho(\sqrt{1-\rho^2})^{-1} & (\sqrt{1-\rho^2})^{-1} \end{vmatrix}$$
$$= \frac{1}{\sqrt{1-\rho^2}}\,.$$

The joint density of U and V is

$$f_{U,V}(u,v) = f_U(u) f_V(v) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$$
$$= \frac{1}{2\pi} e^{-(u^2+v^2)/2}\,,$$

so the joint density of X and Y is

$$f_{X,Y}(x,y) = f_{U,V}\left(x, \frac{y-\rho x}{\sqrt{1-\rho^2}}\right) |J_{\mathbf{h}}(x,y)|$$
$$= \frac{1}{2\pi} e^{-\left\{x^2 + (y-\rho x)^2/(1-\rho^2)\right\}/2} \frac{1}{\sqrt{1-\rho^2}}$$
$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\} \quad \text{for } x, y \in \mathbb{R},$$

as required.

(b) The inverse of this transformation is

$$X = (X^* - \mu_X)/\sigma_X \quad \text{and} \quad Y = (Y^* - \mu_Y)/\sigma_Y,$$

which has Jacobian $(\sigma_X \sigma_Y)^{-1}$. The joint density of X^* and Y^* is, thus,

$$\begin{aligned}
&f_{X^*,Y^*}(x,y) \\
&= f_{X,Y}\left(\frac{x-\mu_X}{\sigma_X}, \frac{y-\mu_Y}{\sigma_Y}\right)\frac{1}{\sigma_X\sigma_Y} \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}\right.\right. \\
&\qquad\qquad \left.\left. -2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right\},
\end{aligned}$$

for $x, y \in \mathbb{R}$. This is the general form of the bivariate normal density.

(c) The marginal distribution of X is standard normal, so the conditional density is

$$\begin{aligned}
f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} &= \frac{\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\}}{\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}} \\
&= \frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left\{-\frac{x^2-2\rho xy+y^2-(1-\rho^2)x}{2(1-\rho^2)}\right\} \\
&= \frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left\{-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right\}.
\end{aligned}$$

This is a normal distribution with mean ρx and variance $1-\rho^2$.