ST102/ST109 Outline solutions to Exercise 6

1. The moment generating function, $M_X(t)$, for this distribution is:

$$M_X(t) = \mathrm{E}(\mathrm{e}^{tX}) = \sum_{x=0}^{1} \mathrm{e}^{tx} p(x) = \mathrm{e}^{t\times 0} p(0) + \mathrm{e}^{t\times 1} p(1) = (1-\pi) + \pi \mathrm{e}^t.$$

Hence $M_X'(t) = \pi \mathrm{e}^t$ and $M_X''(t) = \pi \mathrm{e}^t$. Therefore:

$$\mathrm{E}(X) = M_X'(0) = \pi \quad \text{and} \quad \mathrm{E}(X^2) = M_X''(0) = \pi.$$

Hence:

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \pi - \pi^2 = \pi(1-\pi).$$

2.* We have:

$$M_Y'(t) = cM_X'(t)\mathrm{e}^{c(M_X(t)-1)}$$

and:

$$M_Y''(t) = ((cM_X'(t))^2 + cM_X''(t))\mathrm{e}^{c(M_X(t)-1)}.$$

We know that:

$$M_X(0) = \mathrm{E}(\mathrm{e}^0) = \mathrm{E}(1) = 1, \quad M_X'(0) = \mu \quad \text{and} \quad M_X''(0) = \sigma^2 + \mu^2.$$

Therefore:

$$\mathrm{E}(Y) = M_Y'(0) = c\mu$$

and:

$$\mathrm{Var}(Y) = M_Y''(0) - (\mathrm{E}(Y))^2 = ((c\mu)^2 + c(\sigma^2+\mu^2)) - (c\mu)^2 = c(\sigma^2+\mu^2).$$

3. If X can take only a finite number of values $x_1, x_2, \ldots, x_k$ with probabilities $p_1, p_2, \ldots, p_k$, respectively, then the mgf of X will be:

$$M_X(t) = p_1\mathrm{e}^{tx_1} + p_2\mathrm{e}^{tx_2} + \cdots + p_k\mathrm{e}^{tx_k}.$$

By matching this expression for $M_X(t)$ with:

$$M_X(t) = \frac{2}{11}\mathrm{e}^t + \frac{4}{11}\mathrm{e}^{2t} + \frac{4}{11}\mathrm{e}^{4t} + \frac{1}{11}\mathrm{e}^{8t}$$

it can be seen that X can take only the four values 1, 2, 4 and 8 and hence:

$$p(x) = \begin{cases} 2/11 & \text{for } x = 1 \\ 4/11 & \text{for } x = 2 \text{ and } 4 \\ 1/11 & \text{for } x = 8 \\ 0 & \text{otherwise.} \end{cases}$$

4. (a) For the expression to be a valid density function, we need it to integrate to 1, so:

$$\int_0^1 (\alpha x + \beta x^2)\,\mathrm{d}x = \left[\frac{\alpha x^2}{2} + \frac{\beta x^3}{3}\right]_0^1 = 1.$$

Therefore:

$$\frac{\alpha}{2} + \frac{\beta}{3} = 1.$$

Also:

$$\mathrm{E}(X) = \int_0^1 x(\alpha x + \beta x^2)\,\mathrm{d}x = \left[\frac{\alpha x^3}{3} + \frac{\beta x^4}{4}\right]_0^1 = \frac{25}{36}.$$

Therefore:

$$\frac{\alpha}{3} + \frac{\beta}{4} = \frac{25}{36}.$$

Solving, we get $\alpha = 4/3$ and $\beta = 1$.

(b) It does not exist because $\mathrm{E}(1/X^2)$ does not exist as the integral:

$$\int_0^1 \left(\frac{4}{3x} + 1\right) \mathrm{d}x$$

is divergent.

(c) We have:

$$\begin{aligned}
P\left((2X-1)^2 < \frac{1}{4} \,\middle|\, X < \frac{1}{2}\right) &= P\left(\frac{1}{4} < X < \frac{3}{4} \,\middle|\, X < \frac{1}{2}\right) \\
&= \frac{P(\{1/4 < X < 3/4\} \cap \{X < 1/2\})}{P(X < 1/2)} \\
&= \frac{P(1/4 < X < 1/2)}{P(X < 1/2)}.
\end{aligned}$$

Now:

$$\begin{aligned}
P\left(\frac{1}{4} < X < \frac{1}{2}\right) &= \int_{1/4}^{1/2} \left(\frac{4}{3}x + x^2\right) \mathrm{d}x \\
&= \frac{2}{3}\left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2\right) + \frac{1}{3}\left(\left(\frac{1}{2}\right)^3 - \left(\frac{1}{4}\right)^3\right) \\
&= \frac{31}{192}.
\end{aligned}$$

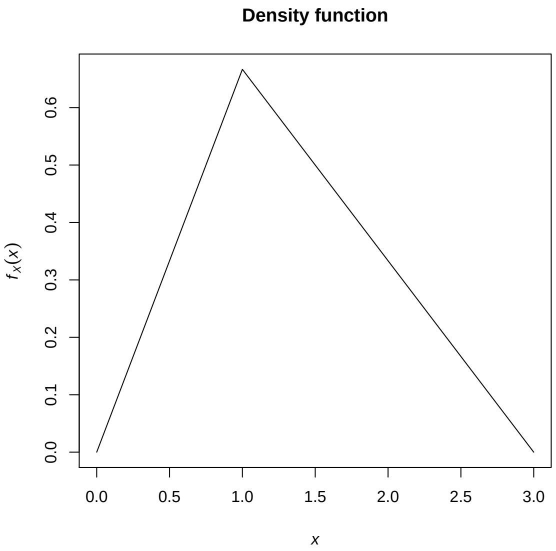
Also:

$$\begin{aligned}
P\left(X < \frac{1}{2}\right) &= \int_0^{1/2} \left(\frac{4}{3}x + x^2\right) \mathrm{d}x \\
&= \frac{2}{3}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 \\
&= \frac{5}{24}.
\end{aligned}$$

Therefore:

$$P\left((2X-1)^2 < \frac{1}{4} \,\middle|\, X < \frac{1}{2}\right) = \frac{31/192}{5/24} = \frac{31}{40} = 0.7750.$$

5.* (a) The pdf of X has the following form:



(b) We determine the cdf by integrating the pdf over the appropriate range, hence:

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^2/3 & \text{for } 0 \leq x < 1 \\ x - x^2/6 - 1/2 & \text{for } 1 \leq x \leq 3 \\ 1 & \text{for } x > 3. \end{cases}$$

This results from the following calculations. Firstly, for $x < 0$, we have:

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t = \int_{-\infty}^{x} 0\,\mathrm{d}t = 0.$$

For $0 \leq x < 1$, we have:

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t = \int_{-\infty}^{0} 0\,\mathrm{d}t + \int_{0}^{x} \frac{2t}{3}\,\mathrm{d}t = \left[\frac{2t^2}{6}\right]_0^x = \frac{x^2}{3}.$$

For $1 \leq x \leq 3$, we have:

$$\begin{aligned} F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t &= \int_{-\infty}^{0} 0\,\mathrm{d}t + \int_{0}^{1} \frac{2t}{3}\,\mathrm{d}t + \int_{1}^{x} \frac{3-t}{3}\,\mathrm{d}t \\ &= 0 + \frac{1}{3} + \left[t - \frac{t^2}{6}\right]_1^x \\ &= \frac{1}{3} + \left(x - \frac{x^2}{6}\right) - \left(1 - \frac{1}{6}\right) \\ &= x - \frac{x^2}{6} - \frac{1}{2}. \end{aligned}$$

(c) To find the mean we proceed as follows:

$$\begin{aligned}
\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} x\, f(x)\,\mathrm{d}x &= \int_0^1 \frac{2x^2}{3}\,\mathrm{d}x + \int_1^3 \frac{3x - x^2}{3}\,\mathrm{d}x \\
&= \left[\frac{2x^3}{9}\right]_0^1 + \left[\frac{x^2}{2} - \frac{x^3}{9}\right]_1^3 \\
&= \frac{2}{9} + \left(\frac{9}{2} - \frac{27}{9}\right) - \left(\frac{1}{2} - \frac{1}{9}\right) \\
&= \frac{4}{3}.
\end{aligned}$$

Similarly:

$$\begin{aligned}
\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x &= \int_0^1 \frac{2x^3}{3}\,\mathrm{d}x + \int_1^3 \frac{3x^2 - x^3}{3}\,\mathrm{d}x \\
&= \left[\frac{x^4}{6}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^4}{12}\right]_1^3 \\
&= \frac{1}{6} + \left(9 - \frac{27}{4}\right) - \left(\frac{1}{3} - \frac{1}{12}\right) \\
&= \frac{13}{6}.
\end{aligned}$$

Hence the variance is:

$$\sigma^2 = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \frac{13}{6} - \left(\frac{4}{3}\right)^2 = \frac{39}{18} - \frac{32}{18} = \frac{7}{18} \approx 0.3889.$$

Therefore, the standard deviation is $\sigma = \sqrt{0.3889} = 0.6236$.