ST102/ST109 Outline solutions to Exercise 5

1. Let the random variable X denote the number of red balls. As 8 balls are selected without replacement, the sample space of X is $S = \{3, 4, 5, 6, 7, 8\}$ because the maximum number of blue balls which could be obtained is 5 (all selected), hence a minimum of 3 red balls must be obtained, up to a maximum of 8 red balls. The number of possible combinations of 8 balls drawn from 18 is $\binom{18}{8}$. The x red balls chosen from 13 can occur in $\binom{13}{x}$ ways, and the $8 - x$ blue balls chosen from 5 can occur in $\binom{5}{8-x}$ ways. Hence, using classical probability:
$$p(x) = \frac{\binom{13}{x}\binom{5}{8-x}}{\binom{18}{8}}.$$
Therefore, the probability function is:
$$p(x) = \begin{cases} \binom{13}{x}\binom{5}{8-x}/\binom{18}{8} & \text{for } x = 3, 4, 5, 6, 7, 8 \\ 0 & \text{otherwise.} \end{cases}$$

2.* For $i = 1, 2, \ldots, n$, let $X_i = 2$ if James' fortune is doubled on the ith play of the game, and let $X_i = 1/2$ if his fortune is cut in half on the ith play. Hence:
$$\mathrm{E}(X_i) = 2 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{5}{4}.$$
After the first play of the game, James' fortune will be cX_1, after the second play it will be $(cX_1)X_2$, and by continuing in this way it is seen that after n plays James' fortune will be $cX_1X_2 \cdots X_n$. Since $X_1, X_2, \ldots, X_n$ are independent, and noting the hint:
$$\mathrm{E}(cX_1X_2 \cdots X_n) = c \times \mathrm{E}(X_1) \times \mathrm{E}(X_2) \times \cdots \times \mathrm{E}(X_n) = c\left(\frac{5}{4}\right)^n.$$

3. The sample space is clearly $S = \{1, 2, 3, \ldots\}$. If the first head appears on toss x, then the previous $x - 1$ tosses must have been tails. By independence of the tosses, and the fact it is a fair coin:
$$P(X = x) = \left(\frac{1}{2}\right)^{x-1} \times \frac{1}{2} = \left(\frac{1}{2}\right)^x.$$
Therefore, the probability function is:
$$p(x) = \begin{cases} 1/2^x & \text{for } x = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$
Clearly, $p(x) \geq 0$ for all x and:
$$\sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^x = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = \frac{1/2}{1 - 1/2} = 1$$
noting the sum to infinity of a geometric series with first term $a = 1/2$ and common ratio $r = 1/2$.

4. We have:

$$\mathrm{E}(2^X) = \sum_{x=1}^{\infty} 2^x p(x) = \sum_{x=1}^{\infty} 2^x \frac{1}{2^x} = \sum_{x=1}^{\infty} 1 = 1 + 1 + \cdots = \infty.$$

Note that this is the famous 'Petersburg paradox', according to which a player's expectation is infinite (i.e. does not exist) if they are to receive 2^X units of currency when, in a series of tosses of a fair coin, the first head appears on the xth toss, as seen in Question 3.

5.* We have:

$$\begin{aligned}\mathrm{E}(X) = \sum_{x=0}^{\infty} x\, p(x) = \sum_{x=0}^{\infty} x\, \frac{\mathrm{e}^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} x\, \frac{\mathrm{e}^{-\lambda}\lambda^x}{x!} = \lambda \sum_{x=1}^{\infty} \frac{\mathrm{e}^{-\lambda}\lambda^{x-1}}{(x-1)!} &= \lambda \sum_{y=0}^{\infty} \frac{\mathrm{e}^{-\lambda}\lambda^y}{y!} \\ &= \lambda \times 1 \\ &= \lambda\end{aligned}$$

where we replace $x - 1$ with y. The result follows from the fact that $\sum_{y=0}^{\infty} (\mathrm{e}^{-\lambda}\lambda^y)/y!$ is the sum of all non-zero values of a probability function of this form.

For completeness, we also give here a derivation of the variance of this distribution (another one, through the moment generating function, is given in Example 3.17). Consider first:

$$\begin{aligned}\mathrm{E}(X(X-1)) = \sum_{x=0}^{\infty} x(x-1)\, p(x) = \sum_{x=2}^{\infty} x(x-1)\, \frac{\mathrm{e}^{-\lambda}\lambda^x}{x!} = \lambda^2 \sum_{x=2}^{\infty} \frac{\mathrm{e}^{-\lambda}\lambda^{x-2}}{(x-2)!} &= \lambda^2 \sum_{y=0}^{\infty} \frac{\mathrm{e}^{-\lambda}\lambda^y}{y!} \\ &= \lambda^2 \times 1 \\ &= \lambda^2\end{aligned}$$

where $y = x - 2$. Also:

$$\begin{aligned}\mathrm{E}(X(X-1)) = \mathrm{E}(X^2 - X) = \sum_x (x^2 - x)\, p(x) &= \sum_x x^2 p(x) - \sum_x x\, p(x) \\ &= \mathrm{E}(X^2) - \mathrm{E}(X) \\ &= \mathrm{E}(X^2) - \lambda.\end{aligned}$$

Equating these and solving for $\mathrm{E}(X^2)$ we get $\mathrm{E}(X^2) = \lambda^2 + \lambda$. Therefore:

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$