ST102/ST109 Exercise 2

In this exercise you will practise definitions and operations on sets. A starred '*' question is slightly more challenging. Set theory can seem somewhat abstract – however, remember the course material is cumulative in nature, so persevere as the effort will be worth it!

Your answers to this problem set should be submitted as a pdf file upload to Moodle. It will be covered by your class teacher in your second class, which will take place in the week commencing Monday 9 October 2023.

Your class teacher will confirm submission deadlines with you – these will vary slightly across class groups due to classes taking place on different days of the week.

1. Let A, B and C be events in a sample space, S. Using only the symbols $\cup$, $\cap$, () and ^{c}, find expressions for the following events:

   (a) *only* A occurs

   (b) *none* of the three events occurs

   (c) *exactly one* of the three events occurs

   (d) *at least two* of the three events occur

   (e) *exactly two* of the three events occur.

2. Let A and B be events in a sample space S. Use Venn diagrams to convince yourself that the two De Morgan's laws:

   $$(A \cap B)^c = A^c \cup B^c \tag{1}$$

   and:

   $$(A \cup B)^c = A^c \cap B^c \tag{2}$$

   are correct.

   For each of them, sketch two Venn diagrams – one for the expression on the left-hand side of the equation, and one for the right-hand side. Shade the areas corresponding to each expression, and hence show that for both (1) and (2) the left-hand and right-hand sides describe the same set.

3. Suppose that A and B are two subsets of a sample space S, and that:

   $$A^c = \{a, b, c\} \quad \text{and} \quad B^c = \{b, c, d\}.$$

   List all the elements belonging to the set $(A \cup B)^c$.

4. Suppose that a number x is to be selected from the real line, $\mathbb{R}$, and let A, B and C be the events represented by the following subsets of $\mathbb{R}$:

$$A = \{x \,|\, 1 \leq x \leq 5\}$$
$$B = \{x \,|\, 3 < x \leq 7\}$$
$$C = \{x \,|\, x \leq 0\}.$$

Describe each of the following events as a set of real numbers:

(a) A^c
(b) $A \cup B$
(c) $B \cap C^c$
(d) $A^c \cap B^c \cap C^c$
(e) $(A \cup B) \cap C$.

5.* (a) Use the rules of set operators (on page 29) to prove that the following represents a partition of set A:

$$A = (A \cap B) \cup (A \cap B^c). \tag{3}$$

In other words, prove that (3) is true, and also that $(A \cap B) \cap (A \cap B^c) = \emptyset$.

(b) The following are also partitions:

$$B = (B \cap A^c) \cup (B \cap A) \tag{4}$$
$$A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B). \tag{5}$$

Using the result that (3), (4) and (5) are partitions, prove the probability result:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$