ST202/ST206 – Michaelmas Term

Solutions to problem set 7

1. (a) The density must integrate to 1 over the support,

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dxdy = \int_0^2\int_0^1 kxy\ dxdy \\
&= \int_0^2 \left[\frac{k}{2}x^2y\right]_{x=0}^{x=1} dy = \int_0^2 \frac{k}{2}y\ dy = \left[\frac{k}{4}y^2\right]_0^2 = k \\
\Rightarrow k &= 1\,.
\end{aligned}
$$

(b) To evaluate each marginal, integrate out the other variable.

$$
f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_0^2 xy\ dy = \left[\frac{xy^2}{2}\right]_0^2 = 2x\,,
$$

$$
\Rightarrow f_X(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}
$$

$$
f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx = \int_0^1 xy\ dx = \left[\frac{x^2y}{2}\right]_0^1 = \frac{y}{2}\,,
$$

$$
\Rightarrow f_Y(y) = \begin{cases} y/2 & \text{if } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}
$$

(c) Using the marginal densities,

$$
\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} xf_X(x)\ dx = \int_0^1 2x^2\ dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3} \\
\mathbb{E}(Y) &= \int_{-\infty}^{\infty} yf_Y(y)\ dy = \int_0^2 \frac{y^2}{2}\ dy = \left[\frac{y^3}{6}\right]_0^2 = \frac{4}{3} \\
\mathbb{E}(Y^2) &= \int_{-\infty}^{\infty} y^2f_Y(y)\ dy = \int_0^2 \frac{y^3}{2}\ dy = \left[\frac{y^4}{8}\right]_0^2 = 2 \\
\text{Var}(Y) &= \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = \frac{2}{9}\,.
\end{aligned}
$$

(d) We want the expectation of a function of both X and Y, so we need to use the joint density.

$$\begin{aligned}
\mathbb{E}[9(X-1)Y^2] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 9(x-1)y^2 f_{X,Y}(x,y)\ dxdy \\
&= 9\int_0^2\int_0^1 x(x-1)y^3\ dxdy \\
&= 9\int_0^2 \left[\left(\frac{x^3}{3}-\frac{x^2}{2}\right)y^3\right]_{x=0}^{x=1} dy \\
&= 9\int_0^2 -\frac{1}{6}y^3\ dy = 9\left[-\frac{y^4}{24}\right]_0^2 = -6\,.
\end{aligned}$$

2. (a) The density needs to be non-negative for all (x,y), i.e. $k \geq 0$. We also need it to integrate to 1 over the support, so

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dxdy = \int_0^1\int_0^y k(x^2+y^2)dxdy \\
&= \int_0^1 k\left[\frac{x^3}{3}+xy^2\right]_{x=0}^{x=y} dy = \int_0^1 k\ \frac{4}{3}y^3 dy = k\left[\frac{y^4}{3}\right]_0^1 = \frac{k}{3}
\end{aligned}$$

$\Rightarrow k = 3\,.$

(b) We compute $P(X<Y^2)$ by evaluating $\int\int_B f_{X,Y}(x,y)dxdy$, where $B = \{(x,y)\in\mathbb{R}^2 : 0<x<y^2\,,\ 0<y<1\}$. A diagram makes this clearer. We have

$$\begin{aligned}
P(X<Y^2) &= \int_0^1\int_0^{y^2} 3(x^2+y^2)dxdy = \int_0^1 \left[x^3+3y^2x\right]_{x=0}^{x=y^2} dy \\
&= \int_0^1 (y^6+3y^4)dy = \left[\frac{y^7}{7}+\frac{3y^5}{5}\right]_0^1 = \frac{26}{35}.
\end{aligned}$$

3. First evaluate the marginal for X,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_0^x 8xydy = \left[4xy^2\right]_{y=0}^{y=x} = 4x^3$$

$$\Rightarrow f_X(x) = \begin{cases} 4x^3 & \text{if } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

We can use the marginal to find the moments of X,

$$\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} xf_X(x)dx = \int_0^1 4x^4dx = \left[\frac{4x^5}{5}\right]_0^1 = \frac{4}{5}\,, \\
\mathbb{E}(X^2) &= \int_{-\infty}^{\infty} x^2f_X(x)dx = \int_0^1 4x^5dx = \left[\frac{4x^6}{6}\right]_0^1 = \frac{2}{3}\,, \\
\mathrm{Var}(X) &= \mathbb{E}(X^2)-\mathbb{E}(X)^2 = \frac{2}{3}-\left(\frac{4}{5}\right)^2 = 2/75
\end{aligned}$$

Similarly, the marginal density of Y is

$$f_Y(y) = \begin{cases} 4y(1-y^2) & \text{if } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

which we use to find $\mathbb{E}(Y) = \frac{8}{15}$ and $\mathrm{Var}(Y) = 11/225$.

We also need

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\, dydx = \int_0^1 \int_0^x 8x^2y^2\, dydx \\ &= \int_0^1 \left[\frac{8x^2y^3}{3}\right]_{y=0}^{y=x} dx = \int_0^1 \frac{8x^5}{3} dx = \left[\frac{8x^6}{18}\right]_0^1 = \frac{4}{9}\,. \end{aligned}$$

Putting everything together, we find

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{4}{225}\,,$$

and

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{4}{\sqrt{66}} \approx 0.492\,.$$

5.* (a) This cannot work because (using a slight abuse of notation)

$$G(\infty,\infty) = F_X(\infty) + F_Y(\infty) = 1 + 1 = 2\,.$$

This is inconsistent with $G(\infty,\infty)$ being a probability.

(b) This one satisfies all the properties of a joint CDF. In fact, if we define $F_{X,Y}(x,y) = G(x,y)$, we have

$$\begin{aligned} F_{X,Y}(x,y) &= P(X \le x, Y \le y) = P(X \le x)P(Y \le y) \\ &= F_X(x)F_Y(y)\,, \end{aligned}$$

which implies that $X \le x$ and $Y \le y$ are independent events. This corresponds to the case where X and Y are independent random variables (more on this later).

(c) This one does not work because

$$G(-\infty,\infty) = \max[F_X(-\infty), F_Y(\infty)] = \max(0,1) = 1\,,$$

but this should be 0.

(d) Consider the joint distribution for which X and Y always take exactly the same value (this is a *singular* distribution), and for which the marginal distribution function of X is $F_X(x)$. A typical value for (X,Y) is (x,x). The support of this distribution in the

XY plane is the line $y = x$. If G is the joint distribution function we have

$$\begin{aligned} G(x, y) &= P(X \leq x, Y \leq y) \\ &= P(X \leq x, X \leq y) \\ &= \min[F_X(x), F_X(y)]. \end{aligned}$$

So we have shown that G is a valid joint CDF, though for a singular distribution. A more difficult exercise would be to show that G is a valid joint CDF in the general case; we can do this by working through the standard properties.