ST202/ST206 – Autumn Term

Solutions to problem set 2

1. The possible values for r are $\{0, 1, 2, 3, 4\}$.

   If the six people stand in a circle, the five possible values are equally likely. To see this, imagine that you fix the position of A. B is equally likely to be in any of the five other places in the circle, so the probability is $1/5$. This generalises easily to n people; r takes values in $\{0, 1, \ldots, n-2\}$, each with probability $1/(n-1)$.

   If the six people stand in a line, it is easiest to draw the sample space. We number the positions in the row from 1 to 6, so there are 30 (equally likely) outcomes. The sample space looks like this:

		A					
		1	2	3	4	5	6
	1		$\circ$	$\circ$	$\circ$	$\circ$	$\circ$
	2	$\circ$		$\circ$	$\circ$	$\circ$	$\circ$
B	3	$\circ$	$\circ$		$\circ$	$\circ$	$\circ$
	4	$\circ$	$\circ$	$\circ$		$\circ$	$\circ$
	5	$\circ$	$\circ$	$\circ$	$\circ$		$\circ$
	6	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	

   Now we can mark each outcome with the number of people between A and B.

		A					
		1	2	3	4	5	6
	1		0	1	2	3	4
	2	0		0	1	2	3
B	3	1	0		0	1	2
	4	2	1	0		0	1
	5	3	2	1	0		0
	6	4	3	2	1	0	

   Let C_r be the event that there are exactly r people between A and B.

Counting the number of outcomes for each case gives

$$
\begin{aligned}
P(C_0) &= 10/30 = 1/3 \\
P(C_1) &= 8/30 = 4/15 \\
P(C_2) &= 6/30 = 1/5 \\
P(C_3) &= 4/30 = 2/15 \\
P(C_4) &= 2/30 = 1/15.
\end{aligned}
$$

Now it is easy to generalize to the case of n positions. There are $n^2 - n = n(n-1)$ equally likely positions for A nd B, and $2(n-r-1)$ of these have r people between A and B. So

$$P(r \text{ people between } A \text{ and } B) = \frac{2(n-r-1)}{n(n-1)}$$

for $r = 0, 1, 2, \ldots, n-2$.

It is often true that arrangements on a circle have simpler properties than arrangements on a line.

2. Let U_k, $k = 1, \ldots, n$ be the event that the k^{th} urn is picked, and B_i, $i = 1, 2$, the event that the i^{th} ball picked is black.

   (a) Each urn contains the same number of balls, so the first ball is equally likely to be any of the $n(n-1)$ total balls. Of these, half are black and half are red (verify this!), so $P(B_1) = 1/2$.

   (b) Imagine that you draw the two balls simultaneously, one with each hand, which is the same as drawing without replacement. If you consider the ball in your left hand to be the first ball drawn, then $P(B_1) = 1/2$. If, instead, you consider the ball in your *right* hand to be the first one drawn, then clearly $P(B_2) = 1/2$ (the ball in your left hand).

   (c) We have

   $$P(B_2|B_1) = \frac{P(B_1 \cap B_2)}{P(B_1)}.$$

To compute the numerator, condition on the urn chosen:

$$\begin{aligned}
P(B_1 \cap B_2) &= \sum_{k=1}^{n} P(B_1 \cap B_2 | U_k) P(U_k) \\
&= \sum_{k=1}^{n-1} \frac{n-k}{n-1} \frac{n-k-1}{n-2} \frac{1}{n} \\
&= \frac{1}{n(n-1)(n-2)} \left[\sum_{k=1}^{n-1} (n-k)^2 - \sum_{k=1}^{n-1} (n-k) \right] \\
&= \frac{1}{n(n-1)(n-2)} \left[\sum_{j=1}^{n-1} j^2 - \sum_{j=1}^{n-1} j \right] \\
&= \frac{1}{n(n-1)(n-2)} \left[\frac{n(n-1)(2n-1)}{6} - \frac{n(n-1)}{2} \right] \\
&= \frac{1}{(n-2)} \left[\frac{(2n-1)}{6} - \frac{1}{2} \right] = \frac{2(n-2)}{6(n-2)} = \frac{1}{3}
\end{aligned}$$

3. A and B are clearly independent events. We can also see that $P(C) = P(\{HH, TT\}) = 1/2$ and $P(A \cap C) = P(\{HH\}) = 1/4 = P(A)P(C)$, so A and C are independent. Similarly, B and C are independent. We deduce that $\{A, B, C\}$ are pairwise independent.

   To show mutual independence, we need to check two conditions:

   - Any subset of two or more events are (mutually) independent.
   - $P(A \cap B \cap C) = P(A)P(B)P(C)$

   We have already proved that the first condition holds (it's just pairwise independence), but the second condition does not hold:

   $$P(A \cap B \cap C) = P(\{HH\}) = 1/4\,, \text{ but } P(A)P(B)P(C) = (1/2)^3 = 1/8\,.$$

   We deduce that the events are not mutually independent.

4. (a) We can use a sample space of eight equally likely outcomes if we take into account birth order. Using M and F for boys and girls, we can write

   $$\Omega = \{MMM, MMF, MFM, FMM, MFF, FMF, FFM, FFF\}$$

   Counting the outcomes in the events gives

   $$\begin{aligned}
   P(A) &= P(\{MMF, MFM, FMM, MFF, FMF, FFM\}) \\
   &= 6/8 = 3/4 \\
   P(B) &= P(\{MMM, MMF, MFM, FMM\}) = 4/8 = 1/2 \\
   P(A \cap B) &= P(\{MMF, MFM, FMM\}) = 3/8
   \end{aligned}$$

(b) For families of four children there are 16 equally likely outcomes. Just two of these have children all of the same gender, so $P(A) = 14/16 = 7/8$. There is one outcome with all boys and $\binom{4}{1} = 4$ outcomes with one girl. All the other families have more than one girl, so $P(B) = 5/16$. There are four families with children of both genders and no more than one girl, so $P(A \cap B) = 4/16 = 1/4$.

(c) In (a) we have $P(A \cap B) = P(A)P(B)$, so $A \perp B$. It is hard to see this independence intuitively. One needs to verify it to be sure.

In (b) we have $P(A \cap B) = \frac{1}{4} \neq \frac{7}{8} \times \frac{5}{16} = P(A)P(B)$, so the events are not independent.

(d) Let D_k be the event that a family has exactly k boys, for $k = 0, 1, 2, \ldots$. We can work out $P(D_k)$ by conditioning on the total number of children in the family. If a family has j children, there are 2^j possible arrangements of girls and boys, of which $\binom{j}{k}$ have exactly k boys and $j-k$ girls. We write

$$P(D_k) = \sum_{j=0}^{\infty} P(D_k|C_j)P(C_j) = \sum_{j=k}^{\infty} \frac{\binom{j}{k}}{2^j} \frac{1}{2^{j+1}} = \sum_{j=k}^{\infty} \binom{j}{k} 2^{-2j-1}$$

To evaluate this sum we need to use the formula for the negative binomial series. One form of this is

$$(x+1)^{-n} = \sum_{i=0}^{\infty} \binom{n+i-1}{i} (-x)^i \,,$$

where n is a positive integer. Setting $i = j - k$ in our summation gives

$$\begin{aligned} P(D_k) &= \sum_{i=0}^{\infty} \binom{k+i}{k} 2^{-2(k+i)-1} \\ &= \sum_{i=0}^{\infty} \binom{n+i-1}{n-1} 2^{-2(n-1+i)-1} \qquad \text{(set } k = n-1\text{)} \\ &= 2^{-(2n-1)} \sum_{i=0}^{\infty} \binom{n+i-1}{n-1} 2^{-2i} \,. \end{aligned}$$

Because $\binom{n+i-1}{n-1} = \binom{n+i-1}{i}$, the summation is the negative binomial series for $x = -2^{-2} = -1/4$. The final result is

$$P(D_k) = 2^{-(2n-1)}(-1/4+1)^{-n} = \frac{1}{2^{2(k-1)+1}} \left(\frac{3}{4}\right)^{-k-1} = \frac{2}{3^{k+1}} \,.$$