ST102 Outline solutions to Exercise 12

1. By the central limit theorem we have:

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

approximately, as $n \to \infty$. Hence:

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{\sqrt{n}(\bar{X}_n - 6)}{3} \to Z \sim N(0, 1).$$

Therefore:

$$P(|\bar{X}_n - \mu| < 0.3) \approx P(|Z| < 0.1\sqrt{n}) = 2 \times \Phi(0.1\sqrt{n}) - 1.$$

However, $2 \times \Phi(0.1\sqrt{n}) - 1 \geq 0.90$ if and only if:

$$\Phi(0.1\sqrt{n}) \geq \frac{1 + 0.90}{2} = 0.95$$

which is satisfied if:

$$0.1\sqrt{n} \geq 1.645 \quad \Rightarrow \quad n \geq 270.6025.$$

Hence the smallest possible value of n is 271.

2. The following are possible, but not exhaustive, solutions.

   (a) We could have:

$$\sum_{i=1}^{2} \left(\frac{X_i - i}{i}\right)^2 \sim \chi^2_2.$$

   (b) We could have:

$$\frac{X_1 - 1}{\sqrt{\sum\limits_{i=2}^{4} \left(\frac{X_i - i}{i}\right)^2 \Big/ 3}} \sim t_3.$$

   (c) We could have:

$$\frac{(X_1 - 1)^2}{\sum\limits_{i=2}^{4} \left(\frac{X_i - i}{i}\right)^2 \Big/ 3} \sim F_{1,3}.$$

3. (a) Since $X_1 + 5X_2 \sim N(0, 234)$, then:

$$P(X_1 + 5X_2 > 8) = P\left(\frac{X_1 + 5X_2}{\sqrt{234}} > \frac{8}{\sqrt{234}}\right) = P(Z > 0.52) = 0.3015$$

   where $Z \sim N(0, 1)$.

(b) $X_i^2/9 \sim \chi_1^2$ for $i = 1, 2, 3, 4$, hence $(X_1^2 + X_2^2 + X_3^2 + X_4^2)/9 \sim \chi_4^2$, so:

$$P(X_1^2 + X_2^2 + X_3^2 + X_4^2 < k) = P\left(X < \frac{k}{9}\right) = 0.95$$

where $X \sim \chi_4^2$. Hence $k/9 = 9.488$, so $k = 85.392$.

(c) $X_1/\sqrt{9} \sim N(0,1)$ and $(X_2^2 + X_3^2)/9 \sim \chi_2^2$, hence:

$$\frac{X_1/\sqrt{9}}{\sqrt{\frac{(X_2^2+X_3^2)/9}{2}}} = \frac{\sqrt{2}X_1}{\sqrt{X_2^2 + X_3^2}} \sim t_2.$$

Therefore:

$$P(T < \sqrt{2}k) = 0.025$$

where $T \sim t_2$. Hence $\sqrt{2}k = -4.303$, so $k = -3.043$.

4.* (a) Since $X_1 \sim N(0, 4) = N(0, 2^2)$, $Z_1 = (X_1 - 0)/2 = X_1/2 \sim N(0, 1)$. Therefore, $X_1^2/4 = (X_1/2)^2 = Z_1^2$. This is distributed as χ_1^2, by the definition of the χ^2 distribution.

(b) As in (a), define $Z_i = X_i/2 \sim N(0, 1)$ for $i = 1, 2, 3$, so that $X_i = 2Z_i$. Since X_1, X_2 and X_3 are independent, then so are the Z_is. Therefore, by definition of the χ^2 distribution:

$$X_1^2 + X_2^2 + X_3^2 = 4(Z_1^2 + Z_2^2 + Z_3^2) = 4W$$

where $W \sim \chi_3^2$.

(c) Here $X_1 = 2Z_1$, where $Z_1 \sim N(0, 1)$. Using the same argument as in (b), we have $X_2^2 + X_3^2 = 4V$ where $V \sim \chi_2^2$. Using the result given in the note to the question, Z_1 and V are independent. Therefore, by the definition of the t distribution:

$$\frac{X_1}{\sqrt{X_2^2 + X_3^2}} = \frac{2Z_1}{\sqrt{4V}} = \frac{2Z_1}{\sqrt{8V/2}} = \frac{1}{\sqrt{2}}\frac{Z_1}{\sqrt{V/2}} = \frac{T}{\sqrt{2}}$$

where $T \sim t_2$.

5.* Using the distributions determined in Question 4 we have the following.

(a) We need $P(Y < 1.25)$ where $Y \sim \chi_1^2$. From Table 8 we see that for $Y \sim \chi_1^2$, $P(Y > 1.074) = 0.30$ and $P(Y > 1.323) = 0.25$, so $P(Y < 1.074) = 0.70$ and $P(Y < 1.323) = 0.75$. Since 1.25 is between 1.074 and 1.323, $P(Y < 1.25)$ must be between 0.70 and 0.75. The precise value of the probability is in fact 0.7364.

(b) Here $P(X_1^2 + X_2^2 + X_3^2 < 7) = P(4W < 7) = P(W < 1.75)$, where $W \sim \chi_3^2$. Based on Table 8, this is somewhere between 0.30 and 0.50. The precise value of the probability is in fact 0.3741.

(c) This probability is:

$$P\left(\frac{X_1}{\sqrt{X_2^2 + X_3^2}} > 2\right) = P\left(\frac{T}{\sqrt{2}} > 2\right) = P(T > 2.82)$$

where $T \sim t_2$. Table 7 shows that this is between 0.10 and 0.05, probably rather close to 0.05 since the table tells us that $P(T > 2.920) = 0.05$. The precise value of the probability is in fact 0.0531.