ST202/ST206 – Michaelmas Term

Problem set 6

Due: 12 noon, Wednesday MT Week 8

1. Let X be a positive, continuous random variable.
   (a) Find an expression for the density of $1/X$ in terms of f_X.
   (b) Now suppose that $X \sim \text{Exp}(\lambda)$. What is the density of $1/X$?

2. If X is a positive random variable, recall that its survival function is $S_X(x) = P(X > x) = 1 - F_X(x)$. We define the **hazard function** to be $\lambda_X(x) = f_X(x)/S_X(x-)$, where $S_X(x-) = \lim_{h \downarrow 0} S_X(x - h)$. Find the hazard function for the following distributions:
   (a) Exponential: $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$, where $\lambda > 0$.
   (b) Geometric: $f_X(x) = p(1-p)^{x-1}$ for $x = 1, 2, \ldots$, where $0 < p < 1$.
   (c) Pareto: $f_X(x) = (a-1)/(1+x)^a$ for $x > 0$, where $a > 1$.
   (d) Weibull: $f_X(x) = c\tau x^{\tau-1} e^{-cx^\tau}$ for $x > 0$, where $c, \tau > 0$.

3. A monkey is typing at random on a typewriter and hits each of the 26 letter keys with equal probability. We assume the monkey lives forever and keeps on typing keys indefinitely at constant speed, each stroke independent of all other keys typed. We would like to know whether the monkey can reproduce works of art such as the complete works of Shakespeare. Denote by k the number of characters if you string together these works of fiction (I estimate k to be a few million, but admittedly I haven't checked this). For simplicity, we shall ignore details such as spaces, capital letters, and any punctuation.
   (a) Explain why the probability that the monkey will immediately write the complete works of Shakespeare is given by 26^{-k}.
   (b) What is the probability that the monkey will eventually produce infinitely many copies of the complete works of Shakespeare?
   (c) Does your answer depend on the assumption that the monkey hits each key with equal probability?
   (d) Is the independence assumption important?

4.* (a) Consider a sequence of random variables $X_1, X_2, \ldots$ with cumulant-generating functions $K_{X_1}, K_{X_2}, \ldots$, and a random variable X with cumulant-generating function $K_X(t)$. Suppose, in addition, that all these cumulant-generating functions are well-defined for $|t| < c$. If $K_{X_n}(t) \to K_X(t)$ as $n \to \infty$ for all $|t| < c$, what can we conclude?

(b) Now suppose that $Y_n \sim \text{Pois}(n)$ and we define

$$X_n = (Y_n - n)/\sqrt{n}\,.$$

Show that $K_{X_n}(t) \to t^2/2$ as $n \to \infty$. What does this tell us about the distribution of Y_n for large values of n?