ST102/ST109 Outline solutions to Exercise 3

1. The sample space consists of all (unordered) subsets of 6 out of the 15 light bulbs in the box. There are $\binom{15}{6}$ such subsets. The number of subsets which contain the two defective bulbs is the number of subsets of size 4 out of the other 13 bulbs, $\binom{13}{4}$, so the probability we want is:
$$\frac{\binom{13}{4}}{\binom{15}{6}} = \frac{6 \times 5}{15 \times 14} = 0.1429.$$

2. Let A = numbers differ by more than 2, hence A^c = numbers differ by 1 or by 2. Therefore, using classical probability:
$$P(A) = 1 - P(A^c) = 1 - \frac{19}{\binom{20}{2}} - \frac{18}{\binom{20}{2}} = \frac{153}{190} = 0.8053.$$

3.* The total number of ways of choosing the n seats which the n people will occupy is $\binom{2n}{n}$. One might suspect that there are only two ways of choosing these seats so that no two adjacent seats are occupied, namely:
$$\text{X0X0}\ldots\text{X0} \quad \text{and} \quad \text{0X0X}\ldots\text{0X}.$$
However, $n-1$ more ways can be found, namely:
$$\text{X00X0X}\ldots\text{0X}, \quad \text{X0X00X0X}\ldots\text{0X}, \quad \text{etc.}$$
Therefore, the total number of ways of choosing the seats so that no two adjacent seats are occupied is $n+1$. The probability is:
$$\frac{n+1}{\binom{2n}{n}}.$$

4. (a) In Question 5 of Exercise 2 we noted that $A = (A \cap B) \cup (A \cap B^c)$ is a partition, and hence $P(A) = P(A \cap B) + P(A \cap B^c)$. It follows from this that:
$$\begin{aligned} P(A \cap B^c) &= P(A) - P(A \cap B) \\ &= P(A) - P(A)\,P(B) \qquad \text{(since } A \perp\!\!\!\perp B\text{)} \\ &= P(A)(1 - P(B)) \\ &= P(A)\,P(B^c). \end{aligned}$$

   (b) Here we first use one of De Morgan's laws (see page 29) such that:
$$\begin{aligned} P(A^c \cap B^c) &= P((A \cup B)^c) \\ &= 1 - P(A \cup B) \\ &= 1 - (P(A) + P(B) - P(A \cap B)) \\ &= 1 - P(A) - P(B) + P(A)\,P(B) \qquad \text{(since } A \perp\!\!\!\perp B\text{)} \\ &= (1 - P(A))(1 - P(B)) \\ &= P(A^c)\,P(B^c). \end{aligned}$$

5.* The player wins immediately if the total score is 7 or 11, which occurs with probability 2/9 (see Example 2.12 which shows the 6 ways for the total score to be 7, and the 2 ways for the total score to be 11, hence the probability is $8/36 = 2/9$). Similarly, the player loses immediately with probability $(1+2+1)/36 = 1/9$.

Therefore, the game continues with probability $1 - 2/9 - 1/9 = 2/3$. Clearly, the subsequent probabilities depend on the value of the 'point', so let the probability of rolling the point in a single roll be π (following the hint). Hence the probability of rolling the point (and so winning with the point) is given by the infinite series:

$$\pi \left(\sum_{i=1}^{\infty} \left(1 - \pi - \frac{1}{6} \right)^{i-1} \right) \pi = \frac{\pi^2}{\pi + 1/6}$$

where:

– the first factor, π, is the probability of obtaining the point

– the middle factor (which note is the sum to infinity of a geometric series) is the probability of not obtaining the point nor a 7 (noting the probability of a total score of 7 is 1/6) a sequence of times

– the final factor, π, is the probability of obtaining the point again and hence winning the game.

We have that the total probability of winning is the original 2/9 plus the sum of $\pi^2/(\pi + 1/6)$ over all possible point values. The probabilities of each possible point are:

– $\pi = 1/12$ for a point value of 4, hence $\pi^2/(\pi + 1/6) = 0.02\dot{7}$

– $\pi = 1/9$ for a point value of 5, hence $\pi^2/(\pi + 1/6) = 0.0\dot{4}$

– $\pi = 5/36$ for a point value of 6, hence $\pi^2/(\pi + 1/6) = 0.06\dot{3}\dot{1}$

– $\pi = 5/36$ for a point value of 8, hence $\pi^2/(\pi + 1/6) = 0.06\dot{3}\dot{1}$

– $\pi = 1/9$ for a point value of 9, hence $\pi^2/(\pi + 1/6) = 0.0\dot{4}$

– $\pi = 1/12$ for a point value of 10, hence $\pi^2/(\pi + 1/6) = 0.02\dot{7}$.

Therefore, the probability that the player wins is:

$$\frac{2}{9} + 2 \times 0.02\dot{7} + 2 \times 0.0\dot{4} + 2 \times 0.06\dot{3}\dot{1} = 0.4\dot{9}\dot{2}.$$

This can also be calculated in R by defining π to be a vector of probabilities for each point and then summing the terms. For example:

```
> pi <- c(1/12,1/9,5/36,5/36,1/9,1/12)
> 2/9 + sum(pi^2/(pi+1/6))
[1] 0.4929293
```