ST202/ST206 – Autumn Term

Solutions to problem set 4

1. We start with $F_X(x) = \int_{-\infty}^{x} f_X(y)dy$ and plug in each density function.

   (a)

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} \frac{1}{\pi(1+y^2)} dy \\
&= \left[\frac{1}{\pi} \arctan y \right]_{-\infty}^{x} \\
&= \frac{1}{\pi} \arctan x - \frac{1}{\pi} \frac{-\pi}{2} \\
&= \frac{1}{\pi} \arctan x + \frac{1}{2} \,.
\end{aligned}$$

   (b)

$$\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} \frac{e^{-y}}{(1+e^{-y})^2} dy \\
&= \left[\frac{1}{1+e^{-y}} \right]_{-\infty}^{x} \\
&= \frac{1}{1+e^{-x}} \,.
\end{aligned}$$

   (c) Notice that $F_X(x) = 0$ for $x < 0$. For $x \geq 0$ we have

$$\begin{aligned}
F_X(x) &= \int_{0}^{x} \frac{a-1}{(1+y)^a} dy \\
&= \left[-\frac{1}{(1+y)^{a-1}} \right]_{0}^{x} \\
&= 1 - \frac{1}{(1+x)^{a-1}} \,.
\end{aligned}$$

   (d) Notice that $F_X(x) = 0$ for $x < 0$. For $x \geq 0$ we have

$$\begin{aligned}
F_X(x) &= \int_{0}^{x} c\tau y^{\tau-1} e^{-cy^\tau} dy \\
&= \left[-e^{-cy^\tau} \right]_{0}^{x} \\
&= 1 - e^{-cx^\tau} \,.
\end{aligned}$$

2. (a) Let $Y = e^X$, so $F_Y(y) = 0$ for $y < 0$. If $y \geq 0$ we have

$$F_Y(y) = P(e^X \leq y) = P(X \leq \log y) = F_X(\log y)\,.$$

Applying the chain rule, the PDF is

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(\log y) = \frac{1}{y}f_X(\log y)$$

(b) Let $Y = X^2$, so $F_Y(y) = 0$ for $y < 0$. If $y \geq 0$ we have

$$\begin{aligned} F_Y(y) &= P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y})\,. \end{aligned}$$

Applying the chain rule, the PDF is

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) = \frac{d}{dy}(F_X(\sqrt{y}) - F_X(-\sqrt{y})) \\ &= \frac{1}{2\sqrt{y}}f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}}f_X(-\sqrt{y})\,. \end{aligned}$$

(c) Let $Y = F_X(X)$. The function F_X can only take values in $[0, 1]$, so for $0 \leq y \leq 1$ we have

$$F_Y(y) = P(F_X(X) \leq y) = P(X \leq F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y\,.$$

The full CDF is

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y & 0 \leq y \leq 1 \\ 1 & y > 1 \end{cases}$$

The PDF is, thus,

$$f_Y(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $Y \sim \text{Unif}[0, 1]$, the continuous uniform distribution on $[0, 1]$.

(d) Let $Y = G^{-1}(F_X(X))$. We have

$$\begin{aligned} F_Y(y) &= P(G^{-1}(F_X(X)) \leq y) = P(F_X(X) \leq G(y)) \\ &= P(X \leq F_X^{-1}(G(y))) = F_X(F_X^{-1}(G(y))) = G(y)\,, \end{aligned}$$

i.e. the CDF is G and the corresponding PDF is its derivative. We can write this result as $G^{-1}(U) \sim G$, where $U \sim \text{Unif}[0, 1]$. This suggests a method for generating values from a particular distribution. If G is the CDF of the desired distribution, we can just apply the inverse of G to the output of a random number generator (i.e. a method for generating observations from $\text{Unif}[0, 1]$).

3. If X is positive then $\mu = \mathbb{E}(X) > 0$, so $yf_X(y)/\mu \geq 0$ for all $y \geq 0$. We just need to show that $g(y)$ integrates to 1 over the whole real line,

$$\int_{-\infty}^{\infty} g(y)dy = \int_0^{\infty} \frac{yf_X(y)}{\mu}dy = \frac{1}{\mu}\underbrace{\int_0^{\infty} yf_X(y)dy}_{=\mu} = 1\,.$$

To prove the inequality, let Y be a random variable with PDF $g(y)$ and notice that

$$\mathbb{E}(Y^k) = \int_0^{\infty} y^k\,\frac{yf_X(y)}{\mu}dy = \frac{1}{\mu}\int_0^{\infty} y^{k+1}f_X(y)dy = \frac{\mathbb{E}(X^{k+1})}{\mu}\,,$$

or, equivalently, $\mathbb{E}(X^{k+1}) = \mu\,\mathbb{E}(Y^k)$ for all k. This implies that

$$\mathbb{E}(X^3)\,\mathbb{E}(X) = \mu\,\mathbb{E}(Y^2)\mu\,\mathbb{E}(Y^0) = \mu^2\,\mathbb{E}(Y^2)$$

and

$$\left\{\mathbb{E}(X^2)\right\}^2 = \{\mu\,\mathbb{E}(Y)\}^2 = \mu^2\,\mathbb{E}(Y)^2\,.$$

The result we want to prove is, thus, equivalent to

$$\mu^2\,\mathbb{E}(Y^2) \geq \mu^2\,\mathbb{E}(Y)^2 \quad\Longleftrightarrow\quad \mathbb{E}(Y^2) \geq \mathbb{E}(Y)^2\,,$$

which holds.

4. (a) We can save a bit of time by working out the s^{th} moment and using this result to find the mean and variance. We have

$$\begin{aligned}
\mathbb{E}(X^s) &= \int_{-\infty}^{\infty} x^s f_X(x)dx = \int_0^{\infty} x^s \frac{\lambda^\alpha}{\Gamma(\alpha)}e^{-\lambda x}x^{\alpha-1}dx \\
&= \int_0^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)}e^{-\lambda x}x^{s+\alpha-1}dx \\
&= \frac{\Gamma(s+\alpha)}{\Gamma(\alpha)}\lambda^{-s}\int_0^{\infty} \frac{\lambda^{s+\alpha}}{\Gamma(s+\alpha)}e^{-\lambda x}x^{s+\alpha-1}dx \\
&= \frac{\Gamma(s+\alpha)}{\Gamma(\alpha)}\lambda^{-s}
\end{aligned}$$

since the integrand is the PDF of a Gamma$(s+\alpha,\lambda)$. Now recall the property $\Gamma(y) = (y-1)\Gamma(y-1)$. This gives

$$\begin{aligned}
\mathbb{E}(X^s) &= \frac{(s+\alpha-1)\Gamma(s+\alpha-1)}{\Gamma(\alpha)}\lambda^{-s} = \ldots \\
&= \frac{(s+\alpha-1)(s+\alpha-2)\ldots\alpha\Gamma(\alpha)}{\Gamma(\alpha)}\lambda^{-s} \\
&= (s+\alpha-1)(s+\alpha-2)\ldots\alpha\lambda^{-s}\,.
\end{aligned}$$

We deduce that

$$\mathbb{E}(X) = \frac{\alpha}{\lambda} \quad \text{and} \quad \mathbb{E}(X) = (\alpha+1)\alpha/\lambda^2\,,$$

so $\mathrm{Var}(X) = \alpha/\lambda^2$.

(b) Finding $\mathbb{E}(X)$ by direct calculation is straightforward, as the x term in the sum cancels out with the final term in $x!$. Working out $\mathbb{E}(X^2)$ is fiddlier; instead, it is easier to evaluate $\mathbb{E}[X(X-1)]$ and use this to compute the variance. In general, if $\mathbb{E}(X)^{(r)}$ is the r^{th} **factorial moment**, defined as

$$\mathbb{E}(X)^{(r)} = \mathbb{E}[X(X-1)\dots(X-r+1)]\,,$$

we can write

$$\begin{aligned}
\mathbb{E}(X)^{(r)} &= \sum_{x=0}^{\infty} x^{(r)} \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=r}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-r)!} && (x! = x^{(r)}(x-r)!) \\
&= \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^{y+r}}{y!} && (y = x-r) \\
&= \lambda^r \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^{y}}{y!} = \lambda^r\,,
\end{aligned}$$

because the final summand is a Poisson PMF. This gives

$$\begin{aligned}
&\mathbb{E}(X)^{(1)} = \mathbb{E}(X) = \lambda\,, \\
&\mathbb{E}(X)^{(2)} = \mathbb{E}[X(X-1)] = \lambda^2\,, \text{ and} \\
&\mathrm{Var}(X) = \mathbb{E}[X(X-1)] - \mathbb{E}(X)\,\mathbb{E}(X-1) = \lambda^2 - \lambda(\lambda-1) = \lambda\,.
\end{aligned}$$

(c) Direct calculation of $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$ is a bit awkward, as there is no term in the density that cancels out with the x or x^2 term. It is far easier to consider $X+1$, noting that $\mathbb{E}(X+1) = \mathbb{E}(X)+1$ and $\mathrm{Var}(X+1) = \mathrm{Var}(X)$. We have

$$\begin{aligned}
\mathbb{E}[(X+1)^r] &= \int_0^\infty (x+1)^r \frac{a-1}{(1+x)^a} = \int_0^\infty \frac{a-1}{(1+x)^{a-r}} \\
&= \frac{a-1}{a-r-1} \int_0^\infty \frac{a-r-1}{(1+x)^{a-r}} = \frac{a-1}{a-r-1},
\end{aligned}$$

because the final integrand is the density of a Pareto$(a-r)$. This requires $a-r>1$ or else the integral diverges. We conclude that

$$\mathbb{E}(X+1) = \frac{a-1}{a-2} \;\Rightarrow\; \mathbb{E}(X) = \frac{1}{a-2}$$

as long as $a > 2$, and

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{Var}(X+1) = \mathbb{E}[(X+1)^2] - \mathbb{E}(X+1)^2 \\ &= \frac{a-1}{a-3} - \left(\frac{a-1}{a-2}\right)^2 = \frac{a-1}{(a-2)^2(a-3)}\end{aligned}$$

provided $a > 3$.

5. (a) Notice that $F_Y(y) = 0$ for $y < 0$. For $y \geq 0$ we have

$$\begin{aligned}F_Y(y) &= P(|X-a| \leq y) = P(-y \leq X - a \leq y) \\ &= P(a - y \leq X \leq a + y) = F_X(a+y) - F_X(a-y)\,.\end{aligned}$$

The PDF is then

$$\begin{aligned}f_Y(y) &= \frac{d}{dy} F_Y(y) = \frac{d}{dy}(F_X(a+y) - F_X(a-y)) \\ &= f_X(a+y) + f_X(a-y)\,.\end{aligned}$$

(b) If $X \sim \mathrm{N}(\mu, \sigma^2)$ and $a = \mu$, notice that

$$f_X(\mu + y) = f_X(\mu - y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\mu+y-\mu)^2/2\sigma^2}$$

so the density function of Y is

$$f_Y(y) = \sqrt{\frac{2}{\pi\sigma^2}} e^{-y^2/2\sigma^2}$$

This is known as the **half-normal** distribution.