ST102/ST109 Exercise 10

In this exercise you will practise various topics related to multivariate random variables. Question 1 covers discrete bivariate distributions. Questions 2, 3 and 4 concern linear combinations of independent random variables. Finally, Question 5 is a theoretical exercise.

Your answers to this problem set should be submitted as a pdf file upload to Moodle. It will be covered by your class teacher in your tenth class, which will take place in the week commencing Monday 5 December 2022.

1. A fair coin is tossed four times. Let X be the number of heads obtained on the first three tosses of the coin. Let Y be the number of heads on all four tosses of the coin.

   (a) Find the joint probability distribution of X and Y.

   (b) Find the mean and variance of X.

   (c) Find the conditional probability distribution of Y given that $X = 2$.

   (d) Find the mean of the conditional probability distribution of Y given that $X = 2$.

2. Cars pass a point on a busy road at an average rate of 150 per hour. Assume that the number of cars in an hour follows a Poisson distribution. Other motor vehicles (lorries, motorcycles etc.) pass the same point at the rate of 75 per hour. Assume a Poisson distribution for these vehicles too, and assume that the number of other vehicles is independent of the number of cars.

   (a) What is the probability that one car and one other motor vehicle pass in a two-minute period?

   (b) What is the probability that two motor vehicles of any type (cars, lorries, motorcycles etc.) pass in a two-minute period?

3. The random variable X has a discrete uniform distribution with values 1, 2 and 3, i.e. $P(X = i) = 1/3$ for $i = 1, 2, 3$. The random variable Y has a discrete uniform distribution with values $1, 2, 3$ and 4, i.e. $P(Y = i) = 1/4$ for $i = 1, 2, 3, 4$. X and Y are independent.

   (a) Derive the probability distribution of $X + Y$.

   (b) What are $\mathrm{E}(X + Y)$ and $\mathrm{Var}(X + Y)$?

4. At one stage in the manufacture of an engine a piston of circular cross-section has to fit into a similarly-shaped cylinder. The distributions of the diameters of pistons and cylinders are known to be normal with parameters as follows.

   – Piston diameters: mean 6.84 cm, standard deviation 0.03 cm.
   – Cylinder diameters: mean 6.94 cm, standard deviation 0.04 cm.

(a) If pairs of pistons and cylinders are selected at random for assembly, for what proportion will the piston not fit into the cylinder? In other words, what is the probability that the piston diameter exceeds the cylinder diameter?

(b) What is the probability that in 75 pairs (each consisting of a piston and cylinder) selected independently and at random, every piston will fit into the cylinder?

5.* Suppose that X and Y are random variables, and a, b, c and d are constants.

(a) Show that:

$$\mathrm{Cov}(aX+b, cY+d) = ac\,\mathrm{Cov}(X,Y).$$

(b) Derive $\mathrm{Corr}(aX+b, cY+d)$.

(c) Suppose that $Z = cX+d$, where c and d are constants. Using the result you obtained in (b), or in some other way, show that:

$$\mathrm{Corr}(X,Z) = 1 \quad \text{for } c > 0$$

and:

$$\mathrm{Corr}(X,Z) = -1 \quad \text{for } c < 0.$$