ST102/ST109 Class 9 – Solutions to Additional exercises

1. (a) The joint probability distribution is:

$Y = y \setminus X = x$	0	1	2	3
0	1/16	0	0	0
1	1/16	3/16	0	0
2	0	3/16	3/16	0
3	0	0	3/16	1/16
4	0	0	0	1/16

(b) The marginal distribution of X is:

$X = x$	0	1	2	3
$p(x)$	1/8	3/8	3/8	1/8

Hence:

$$\mathrm{E}(X) = \sum_{x=0}^{3} x\, p(x) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{3}{2}$$

also:

$$\mathrm{E}(X^2) = \sum_{x=0}^{3} x^2 p(x) = 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8} = 3$$

and:

$$\mathrm{Var}(X) = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}.$$

(c) We have:

$$P(Y = 0 \,|\, X = 2) = \frac{p(2,0)}{p_X(2)} = \frac{0}{3/8} = 0$$

$$P(Y = 1 \,|\, X = 2) = \frac{p(2,1)}{p_X(2)} = \frac{0}{3/8} = 0$$

$$P(Y = 2 \,|\, X = 2) = \frac{p(2,2)}{p_X(2)} = \frac{3/16}{3/8} = \frac{1}{2}$$

$$P(Y = 3 \,|\, X = 2) = \frac{p(2,3)}{p_X(2)} = \frac{3/16}{3/8} = \frac{1}{2}$$

$$P(Y = 4 \,|\, X = 2) = \frac{p(2,4)}{p_X(2)} = \frac{0}{3/8} = 0.$$

Hence:

$Y = y \,\vert\, X = 2$	2	3
$p(y \,\vert\, X = 2)$	1/2	1/2

(d) We have:

$$\mathrm{E}(Y \,|\, X = 2) = 2 \times \frac{1}{2} + 3 \times \frac{1}{2} = \frac{5}{2}.$$

2. (a) $W = X + Y$, hence:

$$\mathrm{E}(W) = \mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y) = \frac{1}{3} \times (1+2+3) + \frac{1}{3} \times (1+2+3) = 4.$$

(b) We have:

$$\mathrm{Var}(W) = \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = 2 \times \left(\frac{1}{3} \times (1+4+9) - 2^2 \right) = \frac{4}{3}.$$

(c) The distribution of W is:

$W = w$	2	3	4	5	6
$p(w)$	1/9	2/9	3/9	2/9	1/9

Hence the moment generating function is:

$$M_W(t) = \mathrm{E}(\mathrm{e}^{tW}) = \sum_{w=2}^{6} \mathrm{e}^{tw} p(w) = \frac{1}{9} \left(\mathrm{e}^{2t} + 2\mathrm{e}^{3t} + 3\mathrm{e}^{4t} + 2\mathrm{e}^{5t} + \mathrm{e}^{6t} \right).$$