ST202/ST206 – Autumn Term
Solutions to problem set 1

1. This is just a question of checking that the requirements of the definition are met. For (a) this comes directly from the properties of counting. For (b), property ii. holds since no element can be in the empty set, $\mathbf{1}_{\varnothing}(\omega) = 0$ for any given ω. Property iii. holds since

$$\mathrm{P}(\cup_{i=1}^{n} A_i) = \begin{cases} 1 & \text{if } \omega \in A_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

   It is clear that $\sum_{i=1}^{\infty} \mathrm{P}(A_i) = \sum_{i=1}^{\infty} \mathbf{1}_{A_i}(\omega)$ can be defined in precisely the same manner. Finally, $\mathrm{P}(\Omega) = 1$ because $\omega \in \Omega$.

2. A and B partition Ψ into four subsets that are irreducible in terms of A and B. These are $A \cap B$, $A \cap B^c$, $A^c \cap B$, and $A^c \cap B^c$ (a Venn diagram really helps to visualise this). Any σ-algebra containing A and B will also contain all possible unions of these elements. Thus, the size of the smallest σ-algebra containing A and B is $2^4 = 16$. Try writing down all of the elements of this σ-algebra in terms of A and B.

   With N sets, $\{A_1, A_2, \dots, A_N\}$, we need a more efficient way of counting the subsets in the partition. We have $\binom{N}{0} = 1$ subset that contains none of the sets (i.e. $A_1^c \cap A_2^c \cap \dots A_N^c$), $\binom{N}{1} = N$ subsets that contain exactly one set, and so on. The number of subsets is, thus,

$$\binom{N}{0} + \binom{N}{1} + \dots + \binom{N}{N},$$

   which is equal to 2^N (can you see why?). Applying the same argument as before, the size of the smallest σ-algebra is 2^{2^N}.

   If $A \subset B$, our partition only has three subsets: A, $A^c \cap B$, and B^c. The smallest σ-algebra will thus have $2^3 = 8$ elements. If A and B are disjoint, there are also three subsets in the partition (can you see what they are?), so the answer is the same.

   Generalising the disjoint case to N sets, there are only $N + 1$ sets in the partition, and the size of the smallest σ-algebra is 2^{N+1}.

3. If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ then $A_1^c \subseteq A_2^c \subseteq A_3^c \subseteq \dots$. Using the result we proved for increasing sets, we have

$$\lim_{n\to\infty} \mathrm{P}(A_n^c) = \mathrm{P}\left(\bigcup_{i=1}^{\infty} A_i^c\right) = \mathrm{P}\left(\left(\bigcap_{i=1}^{\infty} A_i\right)^c\right) = 1 - \mathrm{P}\left(\bigcap_{i=1}^{\infty} A_i\right).$$

The result follows by noting that $\mathrm{P}(A_n^c) = 1 - \mathrm{P}(A_n)$.

4. There are a number of equivalent ways to tackle this problem. The most reliable is to think in terms of a sample space of equally likely outcomes. We will use A to denote the event that the fraud goes undetected.

   (a) There are 10 ways of choosing the first claim to investigate and 9 ways of choosing the second. Similarly in order for the fraud to go undetected we must choose one of the 7 real claims first and one of the 6 remaining real claims second. Thus,

   $$\begin{aligned} |\Omega| &= 10 \times 9 = 90 \\ |A| &= 7 \times 6 = 42 \\ P(A) &= \frac{42}{90} = \frac{7}{15}. \end{aligned}$$

   (b)

   $$\begin{aligned} |\Omega| &= 5 \times 5 = 25 \\ |A| &= 4 \times 3 = 12 \\ P(A) &= \frac{12}{25}. \end{aligned}$$

   (c)

   $$\begin{aligned} |\Omega| &= 25 \\ |A| &= 5 \times 2 = 10 \\ P(A) &= \frac{10}{25} = \frac{2}{5}. \end{aligned}$$

   The optimal strategy is the one with the highest probability that the fraud goes undetected, that is, (b).