ST102/ST109 Exercise 6

In this exercise you will practise moment generating functions (mgfs) and properties of random variables. Question 1 involves the derivation of a moment generating function and then using it to find the mean and variance of a distribution. Question 2 requires the mean and variance of one random variable to be expressed in terms of another. Question 3 requires the derivation of a probability distribution from an mgf. Question 4 relates to a continuous random variable. Finally, Question 5 concerns a continuous random variable whose probability density function (pdf) takes different functional forms depending on the values of x.

Your answers to this problem set should be submitted as a pdf file upload to Moodle, *as directed by your class teacher*. It will be covered by your class teacher in your sixth class, which will take place in the week commencing Monday 11 November 2024.

1. The random variable X has a discrete distribution such that we have $P(X = 1) = \pi$, $P(X = 0) = 1 - \pi$, and $P(X = x) = 0$ for all other real x.

   (a) Derive the moment generating function, $M_X(t)$, of X.

   (b) Use $M_X(t)$ to calculate the expected value and variance of the distribution.

2.* Let X be a random variable with mean μ and variance σ^2, and let $M_X(t)$ denote the mgf of X for $-\infty < t < \infty$. Let c be a given positive constant, and let Y be a random variable for which the mgf is:

$$M_Y(t) = \mathrm{e}^{c(M_X(t)-1)}$$

   for $-\infty < t < \infty$. Find expressions for the mean and the variance of Y in terms of the mean and the variance of X.

3. Suppose that X is a discrete random variable for which the mgf is:

$$M_X(t) = \frac{2}{11}\mathrm{e}^{t} + \frac{4}{11}\mathrm{e}^{2t} + \frac{4}{11}\mathrm{e}^{4t} + \frac{1}{11}\mathrm{e}^{8t}$$

   for $-\infty < t < \infty$. Write down the probability distribution of X.

4. A random variable, X, has the following probability density function:

$$f(x) = \begin{cases} \alpha x + \beta x^2 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   where α and β are constants. You are also given that $\mathrm{E}(X) = 25/36$.

   (a) Determine the values of α and β.

   (b) Does $\mathrm{Var}(1/X)$ exist? Explain your answer.

   (c) Calculate $P((2X - 1)^2 < 1/4 \,|\, X < 1/2)$.

5.* A random variable, X, has the following pdf:

$$f(x) = \begin{cases} 2x/3 & \text{for } 0 \leq x < 1 \\ (3-x)/3 & \text{for } 1 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Sketch the graph of $f(x)$.

(b) Derive the cumulative distribution function, $F(x)$, of X.

(c) Find the mean and the standard deviation of X.