ST202/ST206 – Michaelmas Term

Problem set 8

Due: 12 noon, Wednesday MT Week 10

1. Let U, V be independent random variables with cumulant-generating functions $K_U(t)$ and $K_V(u)$. Define random variables X and Y by $X = aU + bV$ and $Y = cU + dV$.

   (a) Show that the joint CGF of X and Y is given by
   $$K_{X,Y}(t,u) = K_U(ta + uc) + K_V(tb + ud).$$

   (b) Now suppose that $U \sim \mathrm{N}(0,1)$ and $V \sim \mathrm{N}(0,1)$. Let
   $$X = U \quad \text{and} \quad Y = \rho U + \sqrt{1-\rho^2}V\,.$$
   Work out the joint cumulants of X and Y, $\kappa_{i,j}$, for all i and j.

2. Consider again the random variables X and Y with joint density
   $$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda x} & \text{if } 0 < y < x < \infty \\ 0 & \text{otherwise} \end{cases}$$
   Show that their joint moment-generating function is
   $$M_{X,Y}(t,u) = \frac{\lambda^2}{(\lambda - t)(\lambda - t - u)}\,.$$
   For which values of t and u is this well defined?

3. If X and Y are independent and identically distributed exponential random variables, find the joint density function of $U = X/Y$ and $V = X + Y$. Are U and V independent?

4.* Let $Y_1, \ldots, Y_n \sim \mathrm{N}(\mu, \sigma^2)$ be independent random variables and define the random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)^T$. Show that the joint density is
   $$f_{\mathbf{Y}}(\mathbf{y}) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{\sum_{i=1}^n (y_i - \mu)^2}{2\sigma^2}\right).$$
   Now let $\bar{y} = \frac{1}{n}\sum_{i=1}^n y_i$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^n (y_i - \bar{y})^2$. Prove that the joint density can also be written as
   $$f_{\mathbf{Y}}(\mathbf{y}) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{n(\bar{y} - \mu)^2 + (n-1)s^2}{2\sigma^2}\right).$$