ST102/ST109 Class 8 – Additional exercises

1. Suppose $X \sim N(-7, 9)$.
   (a) Find:
      i. $P(X > 0.8)$
      ii. $P(-10 < X < -4)$.
   (b) Find the value a such that $P(-7 - a < X < -7 + a) = 0.95$.
   (c) Find the value b such that $P(-7 - b < X < -7 + b) = 0.99$.
   (d) How far above the mean of the Z distribution must we go such that only 1% of the probability remains in the right-hand tail?
   (e) How far below the mean of the Z distribution must we go such that only 5% of the probability remains in the left-hand tail?

2. Let X be a non-negative random variable. We say that X has the *memoryless property* if:
   $$P(X > s + t \,|\, X > t) = P(X > s) \quad \text{for all } s, t \geq 0.$$
   Here the quantity $P(X > s + t \,|\, X > t)$ is a standard conditional probability, as defined in Chapter 2.
   (a) Show that the exponential distribution has the memoryless property. You can use the cumulative distribution function of the distribution, which has been given in the lectures.
   (b) Suppose that X denotes the number of hours that a light bulb gives light until it burns out. Let Y denote the hours that a wine glass is used, until it breaks.
   Explain briefly, in your own words, what the memoryless property means in these two examples. Suppose we wanted to make the assumption that these variables follow a distribution which has the memoryless property. Is this assumption more realistic in one of these examples than the other? If so, which one, and why?

3. In a survey of the population of employed workers in the UK, respondents are asked how many hours they usually work per week in their main job. The average of the responses is 38.74 hours, with a standard deviation of 15.72 hours. In answering the following question, suppose that working hours in the population follow a normal distribution with these values of the mean and standard deviation.
   (a) What percentage of people in the population work between 35 and 45 hours per week?
   (b) The middle 50% of people in the population have weekly working hours between which two figures?
   (c) What is the number of hours such that only 10% of people work that long or longer per week?

4.* Suppose you apply for a passport. If you do not receive further notice from the relevant government department (with probability $p > 1/2$), the waiting time T_1 until you receive your new passport is exponentially distributed with parameter λ, that is $T_1 \sim \text{Exp}(\lambda)$, with probability density function:

$$f_{T_1}(t) = \begin{cases} \lambda \mathrm{e}^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

However, there is a probability $q = 1 - p$ that there are problems in processing your application, and the waiting time T_2 until you receive your new passport will then be $T_2 \sim \text{Exp}(\lambda/4)$. All time units are in weeks.

(a) Let T be the overall waiting time until you receive the new passport. Find $P(T < 4)$. Express your answer in terms of p and q.

Hint: Use the total probability formula. The key to answering this part is to observe that T has different probability density functions with certain probabilities (processing problems or not). Conditioning on having processing problems or not first is then the natural thing to do.

(b) You have waited for 4 weeks and still have not received your passport and have received no notification from the government department. Show that the probability that there are actually processing problems with your application is:

$$\frac{q\mathrm{e}^{-\lambda}}{p\mathrm{e}^{-4\lambda} + q\mathrm{e}^{-\lambda}}.$$

Hint: Use Bayes' theorem.

(c) After how many weeks of waiting would you deduce that a processing problem is likelier than not (i.e. find the value of x for which the probability of processing problems given you have waited for x weeks or more is equal to $1/2$)?

(d) Find the mean and variance of T in terms of p and λ only. (You can use the mean and variance of an exponential random variable without proof.)

Hint 1: Calculate the expectation with the total probability formula. Again like (a), conditioning on having processing problems or not is the natural thing to do here.

Hint 2: For the variance of T, find $\mathrm{E}(T^2)$ first.