ST202/ST206 – Michaelmas Term

Solutions to problem set 8

1. (a) The key is to express the generating function in terms of the random variables U and V, which are independent. We have

$$\begin{aligned} M_{X,Y}(t,u) &= \mathbb{E}[e^{tX+uY}] = \mathbb{E}[e^{t(aU+bV)+u(cU+dV)}] \\ &= \mathbb{E}[e^{(ta+uc)U+(tb+ud)V}] = \mathbb{E}[e^{(ta+uc)U}]\,\mathbb{E}[e^{(tb+ud)V}] \\ &= M_U(ta+uc)M_V(tb+ud)\,, \end{aligned}$$

and it follows that

$$K_{X,Y}(t,u) = \log M_{X,Y}(t,u) = K_U(ta+uc) + K_V(tb+ud)\,.$$

(b) This corresponds to the first part with $a = 1$, $b = 0$, $c = \rho$, and $d = \sqrt{1-\rho^2}$. We have already shown that the CGF of a standard normal is $M_U(t) = t^2/2$, so we have

$$\begin{aligned} K_{X,Y}(t,u) &= K_U(t+u\rho) + K_V\left(u\sqrt{1-\rho^2}\right) \\ &= \frac{1}{2}(t+u\rho)^2 + \frac{1}{2}\left(u\sqrt{1-\rho^2}\right)^2 \\ &= \frac{1}{2}\left[t^2 + u^2\rho^2 + 2tu\rho + u^2(1-\rho^2)\right] \\ &= tu\rho + \frac{t^2}{2} + \frac{u^2}{2}\,. \end{aligned}$$

We deduce that $\kappa_{1,1} = \rho$, $\kappa_{2,0} = 1$, $\kappa_{0,2} = 1$, and all the other joint cumulants are zero.

2. Working directly from the definition of the joint moment-generating

function,

$$
\begin{aligned}
M_{X,Y}(t,u) = \mathbb{E}(e^{tX+uY}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{tx+uy} f_{X,Y}(x,y)dxdy \\
&= \int_0^{\infty}\int_y^{\infty} e^{tx}e^{uy}\lambda^2 e^{-\lambda x}dxdy \\
&= \int_0^{\infty} \lambda^2 e^{uy}\int_y^{\infty} e^{-(\lambda-t)x}dxdy \\
&= \int_0^{\infty} \lambda^2 e^{uy}\left[-\frac{1}{\lambda-t}e^{-(\lambda-t)x}\right]_{x=y}^{x\to\infty} dy \\
&= \int_0^{\infty} \frac{\lambda^2}{\lambda-t}e^{-(\lambda-t-u)y}dy \\
&= \left[-\frac{\lambda^2}{(\lambda-t)(\lambda-t-u)}e^{-(\lambda-t-u)y}\right]_0^{\infty} \\
&= \frac{\lambda^2}{(\lambda-t)(\lambda-t-u)}\,.
\end{aligned}
$$

For the inner integral to converge, we assumed that $\lambda - t > 0$. Similarly, the outer integral converges when $\lambda - t - u > 0$.

3. The joint density of X and Y is

$$
f_{X,Y}(x,y) = f_X(x)f_Y(y) = \lambda e^{-\lambda x}\lambda e^{-\lambda y} = \lambda^2 e^{-\lambda(x+y)}\,.
$$

Now let $(X,Y) = \mathbf{h}(U,V)$ denote the inverse transformation, that is,

$$
\begin{aligned}
X &= h_1(U,V) = \frac{UV}{1+U} \\
Y &= h_2(U,V) = \frac{V}{1+U}
\end{aligned}
$$

The Jacobian of this transformation is

$$
\begin{aligned}
J_{\mathbf{h}}(u,v) &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v/(1+u)^2 & u/(1+u) \\ -v/(1+u)^2 & 1/(1+u) \end{vmatrix} \\
&= \frac{v}{(1+u)^3} + \frac{uv}{(1+u)^3} = \frac{v(1+u)}{(1+u)^3} = \frac{v}{(1+u)^2}\,,
\end{aligned}
$$

which is always non-negative. Putting everything together, we have

$$
\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}\left(\frac{uv}{1+u}, \frac{v}{1+u}\right)|J_{\mathbf{h}}(u,v)| \\
&= \lambda^2 e^{-\lambda v}\frac{v}{(1+u)^2} \qquad \text{for } u, v > 0.
\end{aligned}
$$

Notice that we can express the joint as the product of the marginal densities $f_U(u) = 1/(1+u)^2$ and $f_V(v) = \lambda^2 v e^{-\lambda v}$, that is, $U \sim$ Pareto(2) and $V \sim \text{Gamma}(2,\lambda)$. We conclude that U and V are independent.

4. The joint density is

$$
\begin{aligned}
f_{\mathbf{Y}}(\mathbf{y}) &= f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) = f_{Y_1}(y_1)\ldots f_{Y_n}(y_n) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_i-\mu)^2}{2\sigma^2}\right) \\
&= (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{\sum_{i=1}^{n}(y_i-\mu)^2}{2\sigma^2}\right)
\end{aligned}
$$

To prove the last part, we need to show that

$$
\sum_{i=1}^{n}(y_i-\mu)^2 = n(\bar{y}-\mu)^2 + (n-1)s^2 = n(\bar{y}-\mu)^2 + \sum_{i=1}^{n}(y_i-\bar{y})^2.
$$

We have

$$
\sum_{i=1}^{n}(y_i-\mu)^2 = \sum_{i=1}^{n}(y_i^2 - 2\mu y_i + \mu^2) = \sum_{i=1}^{n} y_i^2 - 2\mu n\bar{y} + n\mu^2\,,
$$

and also

$$
\begin{aligned}
\sum_{i=1}^{n}(y_i-\bar{y})^2 &= \sum_{i=1}^{n}(y_i^2 - 2\bar{y}y_i + \bar{y}^2) = \sum_{i=1}^{n} y_i^2 - 2n\bar{y}^2 + n\bar{y}^2 \\
&= \sum_{i=1}^{n} y_i^2 - n\bar{y}^2\,.
\end{aligned}
$$

Subtracting the second equation from the first yields

$$
\sum_{i=1}^{n}(y_i-\mu)^2 - \sum_{i=1}^{n}(y_i-\bar{y})^2 = n\bar{y}^2 - 2\mu n\bar{y} + n\mu^2 = n(\bar{y}-\mu)^2\,.
$$