# ST102/ST109 Outline solutions to Exercise 10

1. (a) The joint probability distribution is:

| $Y = y \setminus X = x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1/16 | 0 | 0 | 0 |
| 1 | 1/16 | 3/16 | 0 | 0 |
| 2 | 0 | 3/16 | 3/16 | 0 |
| 3 | 0 | 0 | 3/16 | 1/16 |
| 4 | 0 | 0 | 0 | 1/16 |

(b) The marginal distribution of $X$ is:

| $X = x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $p(x)$ | 1/8 | 3/8 | 3/8 | 1/8 |

Hence:

$$\mathrm{E}(X) = \sum_x x\,p(x) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{3}{2}$$

$$\mathrm{E}(X^2) = \sum_x x^2 p(x) = 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8} = 3$$

and:

$$\mathrm{Var}(X) = 3 - \frac{9}{4} = \frac{3}{4}.$$

(c) We have:

$$P(Y = 0 \,|\, X = 2) = \frac{p(2,0)}{p_X(2)} = \frac{0}{3/8} = 0$$

$$P(Y = 1 \,|\, X = 2) = \frac{p(2,1)}{p_X(2)} = \frac{0}{3/8} = 0$$

$$P(Y = 2 \,|\, X = 2) = \frac{p(2,2)}{p_X(2)} = \frac{3/16}{3/8} = \frac{1}{2}$$

$$P(Y = 3 \,|\, X = 2) = \frac{p(2,3)}{p_X(2)} = \frac{3/16}{3/8} = \frac{1}{2}$$

$$P(Y = 4 \,|\, X = 2) = \frac{p(2,4)}{p_X(2)} = \frac{0}{3/8} = 0.$$

Hence:

| $Y = y \,\vert\, X = 2$ | 2 | 3 |
|---|---|---|
| $p(y \,\vert\, X = 2)$ | 1/2 | 1/2 |

(d) We have:

$$\mathrm{E}(Y \,|\, X = 2) = 2 \times \frac{1}{2} + 3 \times \frac{1}{2} = \frac{5}{2}.$$

2. (a) Let $X$ denote the number of cars, and $Y$ denote the number of other motor vehicles in a two-minute period. We need the probability given by $P(X = 1, Y = 1)$, which is $P(X = 1)\,P(Y = 1)$ since $X$ and $Y$ are independent.

   A rate of 150 cars per hour is a rate of 5 per two minutes, so $X \sim \text{Poisson}(5)$. The probability of one car passing in two minutes is $P(X = 1) = \mathrm{e}^{-5}(5)^1/1! = 0.0337$. The rate for other vehicles over two minutes is 2.5, so $Y \sim \text{Poisson}(2.5)$ and so $P(Y = 1) = \mathrm{e}^{-2.5}(2.5)^1/1! = 0.2052$. Hence the probability for one vehicle of each type is $0.0337 \times 0.2055 = 0.0069$.

   (b) Here we require $P(Z = 2)$, where $Z = X + Y$. Since the sum of two independent Poisson variables is again Poisson (see Section 5.10.5), then $Z \sim \text{Poisson}(5 + 2.5) = \text{Poisson}(7.5)$. Therefore, the required probability is:

$$P(Z = 2) = \frac{\mathrm{e}^{-7.5}(7.5)^2}{2!} = 0.0156.$$

3. (a) The possible values of the sum are 2, 3, 4, 5, 6 and 7. Since $X$ and $Y$ are independent, the probabilities of the different sums are:

$$P(X + Y = 2) = P(X = 1, Y = 1) = P(X = 1)\,P(Y = 1) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$$

$$P(X + Y = 3) = P(X = 1)\,P(Y = 2) + P(X = 2)\,P(Y = 1) = \frac{2}{12} = \frac{1}{6}$$

$$\begin{aligned} P(X + Y = 4) &= P(X = 1)\,P(Y = 3) + P(X = 2)\,P(Y = 2) \\ &\quad + P(X = 3)\,P(Y = 1) = \frac{3}{12} = \frac{1}{4} \end{aligned}$$

$$\begin{aligned} P(X + Y = 5) &= P(X = 1)\,P(Y = 4) + P(X = 2)\,P(Y = 3) \\ &\quad + P(X = 3)\,P(Y = 2) = \frac{3}{12} = \frac{1}{4} \end{aligned}$$

$$P(X + Y = 6) = P(X = 2)\,P(Y = 4) + P(X = 3)\,P(Y = 3) = \frac{2}{12} = \frac{1}{6}$$

$$P(X + Y = 7) = P(X = 3)\,P(Y = 4) = \frac{1}{12}$$

   and 0 for all other real numbers.

   (b) You could find the expectation and variance directly from the distribution of $X + Y$ above. However, it is easier to use the expected value and variance of the discrete uniform distribution for both $X$ and $Y$, and then the results on the expectation and variance of sums of independent random variables to get:

$$\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y) = \frac{1 + 3}{2} + \frac{1 + 4}{2} = 4.5$$

   and:

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = \frac{3^2 - 1}{12} + \frac{4^2 - 1}{12} = \frac{23}{12} \approx 1.92.$$

4. (a) Let $P \sim N(6.84, (0.03)^2)$ for the pistons, and $C \sim N(6.94, (0.04)^2)$ for the cylinders. Let $D = C - P$ denote the difference of their diameters, so we have:

$$D \sim N(6.94 - 6.84, (0.04)^2 + (0.03)^2) = N(0.1, (0.05)^2).$$

The piston will not fit if $D < 0$, which occurs with probability:

$$P(D < 0) = P\left(\frac{D - 0.1}{0.05} < \frac{0 - 0.1}{0.05}\right) = P(Z < -2) = P(Z > 2) = 0.02275.$$

This also means that the probability that the piston will fit is $1 - 0.02275 = 0.97725$.

(b) The number of pistons, $X$, which will fit out of 75 will be a binomial random variable such that $X \sim \text{Bin}(75, 0.97725)$. Therefore, the required probability is:

$$P(X = 75) = \binom{75}{75}(0.97725)^{75}(0.02275)^0 = (0.97725)^{75} = 0.1780.$$

5.* (a) Note first that:

$$\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b \quad \text{and} \quad \mathrm{E}(cY + d) = c\,\mathrm{E}(Y) + d.$$

Therefore, the covariance is:

$$\begin{aligned}
\operatorname{Cov}(aX + b, cY + d) &= \mathrm{E}((aX + b)(cY + d)) - \mathrm{E}(aX + b)\,\mathrm{E}(cY + d) \\
&= \mathrm{E}(acXY + adX + bcY + bd) - (a\,\mathrm{E}(X) + b)(c\,\mathrm{E}(Y) + d] \\
&= ac\,\mathrm{E}(XY) + ad\,\mathrm{E}(X) + bc\,\mathrm{E}(Y) + bd \\
&\quad - ac\,\mathrm{E}(X)\,\mathrm{E}(Y) - ad\,\mathrm{E}(X) - bc\,\mathrm{E}(Y) - bd \\
&= ac\,\mathrm{E}(XY) - ac\,\mathrm{E}(X)\,\mathrm{E}(Y) \\
&= ac(\mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y)) \\
&= ac\operatorname{Cov}(X, Y)
\end{aligned}$$

as required.

(b) Note first that:

$$\operatorname{sd}(aX + b) = |a|\operatorname{sd}(X) \quad \text{and} \quad \operatorname{sd}(cY + d) = |c|\operatorname{sd}(Y).$$

Therefore, the correlation is:

$$\begin{aligned}
\operatorname{Corr}(aX + b, cY + d) &= \frac{\operatorname{Cov}(aX + b, cY + d)}{\operatorname{sd}(aX + b)\operatorname{sd}(cY + d)} \\
&= \frac{ac\operatorname{Cov}(X, Y)}{|ac|\operatorname{sd}(X)\operatorname{sd}(Y)} \\
&= \frac{ac}{|ac|}\operatorname{Corr}(X, Y).
\end{aligned}$$

(c) First, note that the correlation of a random variable with itself is 1, since:

$$\mathrm{Corr}(X, X) = \frac{\mathrm{Cov}(X, X)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(X)}} = \frac{\mathrm{Var}(X)}{\mathrm{Var}(X)} = 1.$$

In the result obtained in (b), select $a = 1$, $b = 0$ and $Y = X$. This gives:

$$\mathrm{Corr}(X, Z) = \mathrm{Corr}(X, cX + d) = \frac{c}{|c|} + \mathrm{Corr}(X, X) = \frac{c}{|c|}.$$

This gives the two cases mentioned in the question.

- For $c > 0$, then $\mathrm{Corr}(X, cX + d) = 1$.
- For $c < 0$, then $\mathrm{Corr}(X, cX + d) = -1$.