Examiners' commentary 2019

ST102 Elementary Statistical Theory (LT0 examination)

General remarks

Learning outcomes

By the end of this module you should:

- be able to summarise the ideas of randomness and variability, and the way in which these link to probability theory to allow the systematic and logical collection of statistical techniques of great practical importance in many applied areas
- be competent users of standard statistical operators and be familiar with a variety of well-known distributions and their moments
- understand the fundamentals of statistical inference and be able to apply these principles to choose an appropriate model and test in a number of different settings
- recognise that statistical techniques are based on assumptions and in the analysis of real problems the plausibility of these assumptions must be thoroughly checked.

Examination structure

After an initial ten minutes of reading time, you have one hour to complete this paper, which has two compulsory questions. The questions are given equal weight, and carry 50 marks each.

What are the Examiners looking for?

The Examiners are looking for you to demonstrate command of the course material. Although the final solution is 'desirable', the Examiners are more interested in how you approach each solution, as such, most marks are awarded for the 'method steps'. They want to be sure that you:

- have covered the syllabus
- know the various definitions and concepts covered throughout the year and can apply them as appropriate to examination questions
- understand and answer the questions set.

You are not expected to write long essays where explanations or descriptions are required, and note-form answers are acceptable. However, clear and accurate language, both mathematical and written, is expected and marked.

Key steps to improvement

The most important thing you can do is answer the question set! This may sound very simple, but these are some of the things that candidates did not do. Remember:

- **Always show your working.** The bulk of the marks are awarded for your approach, rather than the final answer.
- **Write legibly!**
- **Keep solutions to the same question in one place.** Avoid scattering your solutions randomly throughout the answer booklet – the Examiners will not appreciate having to spend a lot of time searching for different elements of your solutions.
- **Where appropriate, underline your final answer.**
- **Do not waste time calculating things which are not required by the Examiners!**

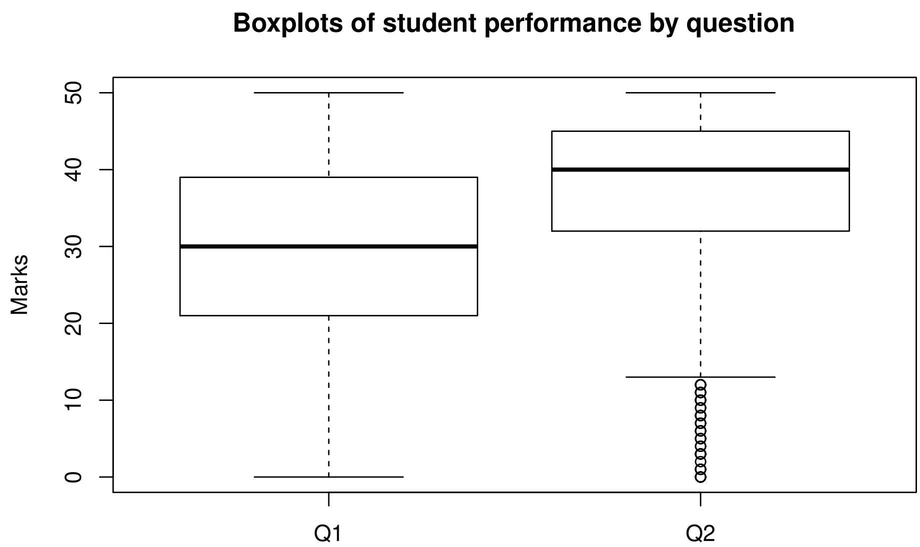
Using the commentary

We hope that you find the commentary useful. For each question and subquestion, it gives:

- the answers, or keys to the answers, which the Examiners were looking for
- common mistakes, as identified by the Examiners.

Student performance by question

Question #	Number of attempts	Mean mark	Standard deviation
1	672	29.49	11.77
2	669	36.46	11.72



Dr James Abdey, ST102 Lecturer, February 2019

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Specific comments on questions

Question 1

(a) **Suppose that $A \subset B$. Show that A and $B \cap A^c$ form a partition of B. You may use any properties of set operators, as defined in the course, and clearly show where these are applied.**

(10 marks)

(b) **There are three cards: one is black on both sides, one is red on both sides, and one is black on one side and red on the other side. We choose a card at random and we see one side (also chosen at random). If the side we see is black, what is the probability that the other side is also black?**

(10 marks)

(c) **Suppose X is a continuous random variable with the following probability density function:**

$$f(x) = \begin{cases} \dfrac{k}{\lambda}\left(\dfrac{x}{\lambda}\right)^{k-1} \mathrm{e}^{-(x/\lambda)^k} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $k > 0$ and $\lambda > 0$ are parameters of this distribution.

i. **Derive the cumulative distribution of X.**

(15 marks)

ii. **Calculate $P(2 < X \leq 6)$ when $k = 1$ and $\lambda = 2$.**

(4 marks)

iii. **State the probability density function which results when $k = 1$ and $\lambda = 1$, and identify which specific distribution this is.**

(4 marks)

iv. **When $k = 1$ and $\lambda = 1$, *derive* the moment generating function of X.**
Hint: You may use any known properties of the common probability distributions seen in the course without proof.

(7 marks)

Feedback on this question

Part (a) is about the property of sets, and not on probability. Solutions answering in terms of probability were penalised as they often mixed set notation with probability. Separately, some candidates did not fully address the question as they did not show that the events are mutually exclusive. In part (b) some candidates attempted to solve this problem using intuition, which tended to result in the wrong answer, and hence no marks. In part (c) the most common error was not recognising that the random variable is continuous, as well as mistakes in adding probabilities in parts ii. and iv. Disappointingly, numerous candidates had the wrong limits for finding the cumulative distribution function in i. often integrating from 0 to infinity, or performing an indefinite integral, while in part iv. integrating from 0 to x. Finally, in part iv. careful answers noted the condition on t.

Full solutions are as follows.

(a) Using associativity, A and $B \cap A^c$ are mutually exclusive since:

$$A \cap (B \cap A^c) = (A \cap A^c) \cap B = \emptyset \cap B = \emptyset.$$

Using the distributive laws, A and $B \cap A^c$ are collectively exhaustive of B since:

$$A \cup (B \cap A^c) = (A \cup B) \cap (A \cup A^c) = B \cap S = B.$$

Hence A and $B \cap A^c$ are mutually exclusive and collectively exhaustive of B, and so they form a partition of B.

(b) Let S_b and S_r denote, respectively, the side we see is black or red. Let C_{bb}, C_{rr} and C_{br} denote, respectively, the card we have is two-sided black, two-sided red and the one with different colours. The required probability is:

$$\begin{aligned}
P(C_{bb} \mid S_b) = \frac{P(C_{bb} \cap S_b)}{P(S_b)} &= \frac{P(C_{bb})\, P(S_b \mid C_{bb})}{P(S_b \cap C_{bb}) + P(S_b \cap C_{rr}) + P(S_b \cap C_{br})} \\
&= \frac{(1/3) \times 1}{(1/3)(1 + 0 + 1/2)} \\
&= \frac{2}{3}.
\end{aligned}$$

Alternatively, see each side as a separate object, we then have in total 6 objects: three in black, and three in red. Furthermore, two in black are 'linked', two in red are 'linked', and the other two (one in black and one in red) are 'linked'. Now we have randomly taken one out, which happens to be black. Of course, the probability that this black was from the 'linked' pair is 2 out of 3.

(c) i. For $x \geq 0$, we have:

$$F(x) = \int_0^x \frac{k}{\lambda} \left(\frac{t}{\lambda}\right)^{k-1} \mathrm{e}^{-(t/\lambda)^k}\, \mathrm{d}t = \Big[- \mathrm{e}^{-(t/\lambda)^k} \Big]_0^x = 1 - \mathrm{e}^{-(x/\lambda)^k}.$$

For $x < 0$ it is obvious that $F(x) = 0$, so we write the result in full as:

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - \mathrm{e}^{-(x/\lambda)^k} & \text{for } x \geq 0. \end{cases}$$

ii. For $k = 1$ and $\lambda = 2$, we have:

$$P(2 < X \leq 6) = F(6) - F(2) = (1 - \mathrm{e}^{-3}) - (1 - \mathrm{e}^{-1}) = \mathrm{e}^{-1} - \mathrm{e}^{-3} = 0.3181.$$

iii. When $k = 1$ and $\lambda = 1$, the probability density function reduces to:

$$f(x) = \begin{cases} \mathrm{e}^{-x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

hence $X \sim \mathrm{Exp}(1)$.

iv. When $k = 1$ and $\lambda = 1$, we have:

$$\begin{aligned}
M_X(t) = \mathrm{E}(\mathrm{e}^{tX}) = \int_0^\infty \mathrm{e}^{tx} \mathrm{e}^{-x}\, \mathrm{d}x &= \int_0^\infty \mathrm{e}^{-(1-t)x}\, \mathrm{d}x \\
&= \frac{1}{1-t} \int_0^\infty (1-t)\mathrm{e}^{-(1-t)x}\, \mathrm{d}x \\
&= \frac{1}{1-t}
\end{aligned}$$

for $t < 1$, using the fact that the integral is that of an exponential probability density function over its sample space, hence is equal to 1.

Question 2

(a) **Suppose we have a biased coin which comes up heads with probability π. An experiment is carried out so that X is the number of independent flips of the coin required until r heads show up, where $r \geq 1$ is known. Explain why the probability function of X is:**

$$p(x) = \begin{cases} \binom{x-1}{r-1}\pi^r(1-\pi)^{x-r} & \text{for } x = r, r+1, r+2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

(10 marks)

(b) **Consider two random variables X and Y. They both take the values 0, 1 and 2. The joint probabilities for each pair are given by the following table, where $\theta \in \mathbb{R}$ is a parameter.**

	$X = 0$	$X = 1$	$X = 2$
$Y = 0$	$1 - \dfrac{8\theta}{10}$	0	$\dfrac{\theta}{10}$
$Y = 1$	$\dfrac{\theta}{10}$	$\dfrac{\theta}{10}$	$\dfrac{2\theta}{10}$
$Y = 2$	$\dfrac{\theta}{10}$	$\dfrac{\theta}{10}$	$\dfrac{\theta}{10}$

i. **What is the range of values the parameter θ can take?**

(10 marks)

ii. **Calculate $P(X = 2 \mid X + Y = 3)$ and state the possible values of θ for which this probability is defined. (Be precise in your use of limits in any inequalities.)**

(10 marks)

iii. **Calculate $\text{Cov}(X, Y)$. Your answer should be expressed as a function of θ.**

(20 marks)

Feedback on this question

In part (a), full marks were awarded for explaining in detail all parts of the derivation of the pdf. For instance, omitting to explain the range of values of x resulted in a one-mark penalty. In (b) part i. candidates had to show that *all* probabilities of the joint distribution satisfied the required property. In part ii. often the range of θ values was incorrectly calculated, and the $\theta \neq 0$ condition was ignored. Finally, part iii. was a straightforward problem, although with occasional calculation errors.

Full solutions are as follows.

(a) To wait for r heads to show up, suppose x flips are required. The last flip must be a head, with $r - 1$ heads randomly appearing in the first $x - 1$ flips. In each particular combination of heads and tails, there must be r heads by definition of the experiment, as well as $x - r$ tails (so adding together, x flips in total), with probability due to independence of:

$$\pi^r(1-\pi)^{x-r}.$$

There are $\binom{x-1}{r-1}$ combinations of outcomes with this probability. Hence we have:

$$p(x) = \begin{cases} \binom{x-1}{r-1}\pi^r(1-\pi)^{x-r} & \text{for } x = r, r+1, r+2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

(b) i. We need $\theta \geq 0$ for all cases except $(X = 0, Y = 0)$ and $(X = 1, Y = 0)$. We also need $1 - 8\theta/10 \geq 0$ for the case $(X = 0, Y = 0)$, so the range is:

$$0 \leq \theta \leq \frac{10}{8}.$$

ii. We have:

$$P(X = 2 \mid X + Y = 3) = \frac{P(X = 2, Y = 1)}{P(X = 2, Y = 1) + P(X = 1, Y = 2)} = \frac{2\theta/10}{2\theta/10 + \theta/10} = \frac{2}{3}$$

except when $\theta = 0$, when it is not defined. Hence this probability is defined for:

$$0 < \theta \leq \frac{10}{8}.$$

(Note: Although θ is unknown, we see that the answer simplifies which shows that this probability is independent of θ, provided $\theta \neq 0$.)

iii. We have:

$$\mathrm{E}(X) = 0 \times \left(1 - \frac{8\theta}{10} + \frac{\theta}{10} + \frac{\theta}{10}\right) + 1 \times \left(0 + \frac{\theta}{10} + \frac{\theta}{10}\right) + 2 \times \left(\frac{\theta}{10} + \frac{2\theta}{10} + \frac{\theta}{10}\right) = \theta$$

also:

$$\mathrm{E}(Y) = 0 \times \left(1 - \frac{8\theta}{10} + 0 + \frac{\theta}{10}\right) + 1 \times \left(\frac{\theta}{10} + \frac{\theta}{10} + \frac{2\theta}{10}\right) + 2 \times \left(\frac{\theta}{10} + \frac{\theta}{10} + \frac{\theta}{10}\right) = \theta$$

and:

$$\mathrm{E}(XY) = 0 + 1 \times 1 \times \frac{\theta}{10} + 1 \times 2 \times \left(\frac{2\theta}{10} + \frac{\theta}{10}\right) + 2 \times 2 \times \frac{\theta}{10} = \frac{11\theta}{10}.$$

Therefore:

$$\mathrm{Cov}(X, Y) = \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y) = \frac{11\theta}{10} - \theta^2.$$