ST102 Exercise 12

In this exercise you will practise standardisation and transformations of random variables. Question 1 requires sample size determination, via the central limit theorem, to ensure the sampling error (the difference between the sample mean and the population mean) is within a particular tolerance with a minimum probability. Questions 2 to 5 concern the definitions of (some of) the normal, χ^2, t and F distributions, with Questions 3 and 5 involving practice of working with Murdoch and Barnes' *Statistical Tables* (you may use the nearest tabulated value).

Your answers to this problem set should be submitted as a pdf file upload to Moodle. It will be covered by your class teacher in your twelfth class, which will take place in the week commencing Monday 30 January 2023.

1. Suppose that a random sample of size n is to be taken from a non-normal distribution for which $\mu = 6$ and $\sigma = 3$. Use the central limit theorem to determine, approximately, the smallest value of n for which:

$$P(|\bar{X}_n - \mu| < 0.3) \geq 0.90$$

   where $\bar{X}_n$ denotes the sample mean, which depends on n.

2. Let X_i, for $i = 1, 2, 3, 4$, be independent random variables such that $X_i \sim N(i, i^2)$. For each of the following situations, use the X_is to construct a statistic with the indicated distribution. Note there could be more than one possible answer for each.

   (a) χ^2_2.

   (b) t_3.

   (c) $F_{1,3}$.

3. Suppose that $X_i \sim N(0, 9)$, for $i = 1, 2, 3, 4$. Assume all these random variables are mutually independent. Derive the value of k in each of the following.

   (a) $P(X_1 + 5X_2 > 8) = k$.

   (b) $P(X_1^2 + X_2^2 + X_3^2 + X_4^2 < k) = 0.95$.

   (c) $P(X_1 < k\sqrt{X_2^2 + X_3^2}) = 0.025$.

4.* The random variables X_1, X_2 and X_3 are independent and identically distributed as $X_i \sim N(0, 4)$, for $i = 1, 2, 3$. Express the distributions of the following random variables as *functions* of χ^2-distributed or t-distributed random variables:

   (a) $X_1^2/4$

   (b) $X_1^2 + X_2^2 + X_3^2$

   (c) $X_1/\sqrt{X_2^2 + X_3^2}$.

Note that in (b) and (c) you can take as true the following result – if random variables are independent, then transformations of them are also independent. For example, if X_1, X_2 and X_3 are independent, then aX_i and bX_j are independent for any $i \neq j$, and for any constants a and b. Similarly, $\sqrt{X_2^2 + X_3^2}$ is independent of X_1.

5.* For the random variables X_1, X_2 and X_3 defined as in Question 4, calculate the following probabilities:

(a) $P(X_1^2/4 < 1.25)$

(b) $P(X_1^2 + X_2^2 + X_3^2 < 7)$

(c) $P(X_1/\sqrt{X_2^2 + X_3^2} > 2)$.

Note you should use Tables 7 and 8 of Murdoch and Barnes' *Statistical Tables*. You will not be able to determine precise values for the probabilities. You can, however, conclude that they must be between specific values.