ST202/ST206 – Autumn Term

Solutions to problem set 3

1. We can work out the CDF by summing the geometric series directly,

$$\begin{aligned} F_X(x) &= \sum_{u \le x} f_X(u) = \sum_{u=1}^{x} (1-p)^{u-1} p \\ &= p\,\frac{1-(1-p)^x}{1-(1-p)} = 1-(1-p)^x \quad \text{for} \quad x = 1, 2, \ldots \end{aligned}$$

The survival function is, thus,

$$S_X(x) = P(X > x) = 1 - P(X \le x) = (1-p)^x \quad \text{for} \quad x = 1, 2, \ldots$$

The survival function evaluated at x is the probability that X takes a value *greater than* x, i.e. we have not observed a success after x trials. This is the same as the probability that the first x trials were all failures, which gives the answer directly.

2. Let Y denote the claims paid. The CDF of Y is

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(X \le y | X > a) \\ &= \frac{P(X \le y \cap X > a)}{P(X > a)} = \frac{P(a < X \le y)}{P(X > a)} \\ &= \begin{cases} 0 & y \le a \\ \frac{F_X(y) - F_X(a)}{1 - F_X(a)} & y > a \end{cases} \end{aligned}$$

Similarly, if Z are the claims not paid, the CDF is

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(X \le z | X \le a) \\ &= \frac{P(X \le z \cap X \le a)}{P(X \le a)} = \begin{cases} \frac{P(X \le z)}{P(X \le a)} & z \le a \\ 1 & z > a \end{cases} \\ &= \begin{cases} \frac{F_X(z)}{F_X(a)} & z \le a \\ 1 & z > a \end{cases} \end{aligned}$$

Notice that both $F_Y(y)$ and $F_Z(z)$ are linear functions of the original CDF.

3. Clearly, $f_Y(y) \geq 0$ for all $y \in \mathbb{R}$, so we just need to verify that $f_Y(y)$ sums to 1. We have

$$
\begin{aligned}
\sum_y f_Y(y) &= \sum_{y=r}^{\infty} \binom{y-1}{r-1} p^r (1-p)^{y-r} \\
&= p^r \sum_{j=0}^{\infty} \binom{j+r-1}{r-1} (1-p)^j \qquad \text{(setting } j = y-r\text{)} \\
&= p^r [1-(1-p)]^{-r} \qquad \text{(neg. bin. expansion)} \\
&= p^r p^{-r} = 1
\end{aligned}
$$

4. (a) For $t = 1$ we have

$$\Gamma(1) = \int_0^{\infty} e^{-u} du = -[e^{-u}]_0^{\infty} = -(0-1) = 1\,.$$

Integrating by parts, we have

$$
\begin{aligned}
\Gamma(t) &= \int_0^{\infty} u^{t-1} e^{-u} du = -\int_0^{\infty} u^{t-1} \left(\frac{d}{du} e^{-u}\right) du \\
&= -[u^{t-1}e^{-u}]_0^{\infty} + \int_0^{\infty} \left(\frac{d}{du} u^{t-1}\right) e^{-u} du \\
&= -[\underbrace{0^{t-1}e^{-0}}_{=0} - \underbrace{\lim_{u\to\infty} u^{t-1}e^{-u}}_{=0}] + (t-1)\int_0^{\infty} u^{t-2}e^{-u} du \\
&= (t-1)\Gamma(t-1)\,.
\end{aligned}
$$

The limit in the penultimate line is zero because the exponential goes to zero much faster than the polynomial goes to infinity.

(b) If $k \in \mathbb{Z}^+$, applying the two results from part (a) gives

$$
\begin{aligned}
\Gamma(k) &= (k-1)\Gamma(k-1) = (k-1)(k-2)\Gamma(k-2) \\
&= \ldots = (k-1)(k-2)\ldots 1\ \Gamma(1) = (k-1)!
\end{aligned}
$$

We can think of the Gamma function as a way of generalising the factorial function to non-integers.

If $X \sim \text{Polya}(r, p)$ and r is a positive integer, the PMF is

$$
\begin{aligned}
f_X(x) &= \frac{\Gamma(r+x)}{x!\Gamma(r)} p^r (1-p)^x = \frac{(r+x-1)!}{x!(r-1)!} p^r (1-p)^x \\
&= \binom{x+r-1}{r-1} p^r (1-p)^x \quad \text{for} \quad x = 0, 1, \ldots
\end{aligned}
$$

This is the same as the formulation of the negative binomial where X is the number of failures before we observe the r^{th} success. In other words, the negative binomial is the special case of the Polya where r is a positive integer.