ST202/ST206 – Autumn Term

Problem set 4

Due: 12 noon, Wednesday AT Week 5

Please submit your answers to the questions marked †.

1.† Find the cumulative distribution functions corresponding to the following density functions:

(a) Cauchy: $f_X(x) = 1/[\pi(1+x^2)]$ for $x \in \mathbb{R}$
(b) Logistic: $f_X(x) = e^{-x}/(1+e^{-x})^2$ for $x \in \mathbb{R}$
(c) Pareto: $f_X(x) = (a-1)/(1+x)^a$ for $x > 0$, where $a > 1$
(d) Weibull: $f_X(x) = c\tau x^{\tau-1}e^{-cx^\tau}$ for $x > 0$, where $c, \tau > 0$

2.† Let $X \sim F_X$ be a continuous random variable. Work out the CDF and PDF of the following random variables:

(a) e^X (b) X^2 (c) $F_X(X)$ (d) $G^{-1}(F_X(X))$

where $G : \mathbb{R} \to [0,1]$ is a continuous and strictly increasing function

3. If X is a positive continuous random variable with density function $f_X(x)$ and mean μ, show that

$$g(y) = \begin{cases} yf_X(y)/\mu & y \geq 0 \\ 0 & y < 0 \end{cases}$$

is a valid density function, and hence show that

$$\mathbb{E}(X^3)\,\mathbb{E}(X) \geq \{\mathbb{E}(X^2)\}^2.$$

4.† Find the mean and the variance for the following distributions:

(a) Gamma: $f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}e^{-\lambda x}x^{\alpha-1}$ for $x > 0$, where $\alpha, \lambda > 0$
(b) Poisson: $f_X(x) = e^{-\lambda}\lambda^x/x!$ for $x = 0, 1, 2, \dots$, where $\lambda > 0$
(c) Pareto: $f_X(x) = (a-1)/(1+x)^a$ for $x > 0$, where $a > 1$

5.* Suppose that $X \sim F_X$ is a continuous random variable taking values between $-\infty$ and $+\infty$. Sometimes we want to *fold* the distribution of X about the value $x = a$, that is, we want the distribution of the random variable $Y = |X - a|$.

(a) Work out the density function of Y in terms of f_X.
[Hint: start with the CDF, $F_Y(y)$.]
(b) Apply the result to the case where $X \sim \mathrm{N}(\mu, \sigma^2)$ and $a = \mu$.