GROUP THEORY ASSIGNMENT

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FROM – COURSE OUTSIDE FORMAL COURSEWORK

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Q.1, a) Show that lorentz transformations satisfy the condition $\boxed{\Lambda^T g \Lambda = g}$

Also prove that they form a group.

b) Given an infinitesimal Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}$$

Show that the infinitesimal parameters $\omega_{\mu\nu}$ are antisymmetric.

c) Prove the following relation

$$\epsilon_{\alpha\beta\gamma\delta} A^{\alpha}_{\mu} A^{\beta}_{\nu} A^{\gamma}_{\lambda} A^{\delta}_{\sigma} = \epsilon_{\mu\nu\lambda\sigma} \det A$$

where A^{α}_{μ} are the matrix elements of the matrix A.

Solution :—

a) $g = \text{diag}(1,-1,-1,-1)$ be the Minkowski metric.

In case of lorentz transformation $x^T g x$ remains invariant. For $x' = \Lambda x$

$$(x')^T g (x') = x^T g x \qquad \forall x$$

Substitute, $x' = \Lambda x$

$$(\Lambda x)^T g (\Lambda x) = x^T g x$$

$$\Rightarrow x^T \Lambda^T g \Lambda x = x^T g x$$

$$\Rightarrow \boxed{\Lambda^T g \Lambda = g}$$

let us define the Lorentz group as

$$\mathcal{L} = \{\Lambda \in GL(4,\mathbb{R}) \mid \Lambda^T g \Lambda = g\}$$

$$g = \text{diag}\{1,-1,-1,-1\}$$

axiom 1:- closure

If Λ_1 and $\Lambda_2 \in L$ then $(\Lambda_1\Lambda_2) \in L$

proof → $\Lambda_1^T g \Lambda_1 = g$ — (1)

$\Lambda_2^T g \Lambda_2 = g$ — (2)

$$(\Lambda_1\Lambda_2)^T g (\Lambda_1\Lambda_2) = \Lambda_2^T \Lambda_1^T g \Lambda_1 \Lambda_2 = \Lambda_2^T g \Lambda_2 = g$$

Hence, $\Lambda_1\Lambda_2 \in L$

axiom 2 : - Associativity

For all $\Lambda_1, \Lambda_2, \Lambda_3 \in L$

$\Rightarrow$ $(\Lambda_1\Lambda_2)\Lambda_3 = \Lambda_1(\Lambda_2\Lambda_3)$

Proof :→ from matrix multiplication, $(\Lambda_1\Lambda_2)\Lambda_3 = \Lambda_1(\Lambda_2\Lambda_3)$

axiom 3 :- Identity element

There exist an identity element $I \in L$ Such that

$I\Lambda = \Lambda I = \Lambda \quad \forall \Lambda \in L$

Proof → $I^T g I = g$

also $I\Lambda = \Lambda I = \Lambda$

Thus $I \in L$

axiom 4 : Inverse Element → for every $\Lambda \in L$ there exist $\Lambda^{-1} \in L$

such that $\Lambda \Lambda^{-1} = \Lambda^{-1}\Lambda = I$

Proof →

$$\Lambda^T g \Lambda = g$$

multiply g^{-1} on ~~right~~ left → $g^{-1}\Lambda^T g \Lambda = g^{-1}g = I$

multiply Λ^{-1} on right → $g^{-1}\Lambda^T g \Lambda(\Lambda^{-1}) = g^{-1}g(\Lambda^{-1}) = I\Lambda^{-1}$

$$\boxed{g^{-1}\Lambda^T g = \Lambda^{-1}}$$

also $(\Lambda^{-1})^T g \Lambda^{-1} = (\Lambda^{-1})^T g g^{-1}\Lambda^T g$

$$= (\Lambda^{-1})^T I \Lambda^T g$$

$$= (\Lambda^{-1})^T \Lambda^T g$$

$$= I g$$

$$= g$$

Hence $\Lambda^{-1} \in L$

As Λ under L satifies all 4 axioms, hence they form a group.

b) Let $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$ → given $|\omega^{\mu}{}_{\nu}| << 1$

according to lorentz condition, $\Lambda^T g \Lambda = g$ — ①

writing Λ with Infinitesimal form

$$\Lambda = I + \omega \quad , \quad \Lambda^T = I + \omega^T$$

from Eq ① → $(I+\omega^T) g (I+\omega) = g$

$$(g + \omega^T g)(I+\omega) = g$$

$$g + g\omega + \omega^T g + O(\omega^2) = g$$

we discard $O(\omega^2)$ as ω is infinitesimal

$$g + \omega^T g + g\omega = g$$

$$\Rightarrow \quad \omega^T g + g\omega = 0$$

$\Rightarrow$ Index notation,

$$(\omega^T)_{\mu\nu}\, g^{\nu\rho} + g_{\mu\nu}\, \omega^{\nu\rho} = 0$$

But, $(\omega^T)_{\mu\nu} = \omega_{\nu\mu}$ [we know]

So, $(\omega^T)_{\mu\nu}\, g^{\nu\rho} = \omega_{\nu\mu}\, g^{\nu\rho}$

$$\Rightarrow \quad \omega_{\nu\mu}\, g^{\nu\rho} + g_{\mu\nu}\, \omega^{\nu\rho} = 0$$

$\Rightarrow$ As, $\omega_{\mu\nu} \equiv g_{\mu\rho}\, \omega^{\rho}{}_{\nu}$

or $\omega^{\nu\rho} = g^{\nu\sigma}\, \omega^{\rho}{}_{\sigma}$

$$\Rightarrow \quad \omega_{\nu\mu}\, g^{\nu\rho} + g_{\mu\nu}\, g^{\nu\sigma}\, \omega^{\rho}{}_{\sigma} = 0$$

$$\Rightarrow \quad \omega_{\nu\mu}\, g^{\nu\rho} + \delta^{\sigma}_{\mu}\, \omega^{\rho}{}_{\sigma} = 0$$

$$\Rightarrow \quad \omega_{\nu\mu}\, g^{\nu\rho} + \omega^{\rho}{}_{\mu} = 0$$

$$\Rightarrow \quad \omega_{\nu\mu}\, g^{\nu\rho} + g^{\rho\sigma}\, \omega_{\sigma\mu} = 0$$

$$\Rightarrow \quad \omega_{\nu\mu}\, g^{\nu\rho} + g^{\rho\sigma}\, \omega_{\sigma\mu} = 0$$

$\sigma \rightarrow \mu$ [change of indices] then $\mu \rightarrow \nu$

$$\Rightarrow \quad \omega_{\nu\mu}\, g^{\nu\rho} + g^{\nu\rho}\, \omega_{\mu\nu} = 0$$

$$\Rightarrow \quad \omega_{\nu\mu}\, g^{\nu\rho} + g^{\nu\rho}\, \omega_{\mu\nu} = 0$$

$$\Rightarrow \quad [\omega_{\nu\mu} + \omega_{\mu\nu}]\, g^{\nu\rho} = 0$$

$$\Rightarrow \quad \boxed{\omega_{\nu\mu} = -\omega_{\mu\nu}}$$

c)

the L.H.S

$$\epsilon_{\alpha\beta\gamma\delta} A^{\alpha}_{\mu} A^{\beta}_{\mu} A^{\gamma}_{\lambda} A^{\delta}_{\sigma} = T_{\mu\nu\lambda\sigma}$$

The tensor $T_{\mu\nu\lambda\sigma}$ is totally anti symmetric in $\mu, \nu, \lambda, \sigma$

so it must be proportional to $\epsilon_{\mu\nu\lambda\sigma}$

$$T_{\mu\nu\lambda\sigma} = C \, \epsilon_{\mu\nu\lambda\sigma}$$

multiplying with $\epsilon^{\mu\nu\lambda\sigma} \rightarrow$

$$\epsilon^{\mu\nu\lambda\sigma} T_{\mu\nu\lambda\sigma} = C \, \epsilon^{\mu\nu\lambda\sigma} \epsilon_{\mu\nu\lambda\sigma} = 4! \, C$$

from the definition of determinant of any matrix $\rightarrow$

$$\det A = \sum \operatorname{sign}(\pi) \, A^{\pi(0)}{}_{0} \, A^{\pi(1)}{}_{1} \, A^{\pi(2)}{}_{2} \, A^{\pi(3)}{}_{3}$$

$$\text{so, } \epsilon^{\alpha\beta\gamma\delta} \det A = \epsilon^{\mu\nu\lambda\sigma} A^{\alpha}{}_{\mu} A^{\beta}{}_{\nu} A^{\gamma}{}_{\lambda} A^{\delta}{}_{\sigma}$$

$$4! \det A = \epsilon^{\mu\nu\lambda\sigma} T_{\mu\nu\lambda\sigma}$$

thus $C = \det A$

therefore $\boxed{\epsilon_{\alpha\beta\gamma\delta} A^{\alpha}{}_{\mu} A^{\beta}{}_{\nu} A^{\gamma}{}_{\lambda} A^{\delta}{}_{\sigma} = \epsilon_{\mu\nu\lambda\sigma} \det A}$

:)

Q: 2

a) The Poincaré transformation (Λ, a) is defined by

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} + a^{\mu}$$

Determine the multiplication rule, i.e. the product $(\Lambda_1, a_1)(\Lambda_2, a_2)$, as well as the units and inverse element in the group.

b) Verify the multiplication rule

$$U^{-1}(\Lambda, 0)\, U(1, \epsilon)\, U(\Lambda, 0) = U(1, \Lambda^{-1}, \epsilon)$$

in the Poincaré group. In addition, show that from the previous relation it follows that

$$U^{-1}(\Lambda, 0)\, P_{\mu}\, U(\Lambda, 0) = (\Lambda^{-1})^{\nu}{}_{\mu}\, P_{\nu}$$

Calculate commutator $[M_{\mu\nu}, P_{\rho}]$

c) Show that $U^{-1}(\Lambda, 0)\, U(\Lambda', 0)\, U(\Lambda, 0) = U(\Lambda^{-1}\Lambda'\Lambda, 0)$

and find the commutator $[M_{\mu\nu}, M_{\rho\sigma}]$

d) Finally show that the generators of translations commute between themselves. i.e

$$[P_{\mu}, P_{\nu}] = 0$$

Solutions :→

a) Let (Λ_1, a_1) and (Λ_2, a_2) be two Poincaré transformations, then →

$$(x')^{\mu} = \Lambda_2{}^{\mu}{}_{\nu}\, x^{\nu} + a_2{}^{\mu} \quad \text{—} \quad (1)$$

$$(x'')^{\mu} = \Lambda_1{}^{\mu}{}_{\nu}\, x'^{\nu} + a_1{}^{\mu} \quad \text{—} \quad (2)$$

Substituting ,

$$(x'')^\mu = \Lambda_1{}^\mu{}_\nu \left(\Lambda_2{}^\nu{}_\rho x^\rho + a_2^\nu \right) + a_1^\mu$$

$$= (\Lambda_1 \Lambda_2)^\mu{}_\rho \, x^\rho + (\Lambda_1 a_2)^\mu + a_1^\mu$$

$$(x'')^\mu = (\Lambda_1 \Lambda_2)^\mu{}_\rho \, x^\rho + (\Lambda_1 a_2 + a_1)^\mu$$

Hence $\boxed{(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1 \Lambda_2, \Lambda_1 a_2 + a_1)}$ → multiplication rule

<u>Unit element</u> :→ Let consider the unit element is $(I, 0)$
and the inverse element of (Λ, a) is (Λ^{-1}, b)
then these two should satisfy

$$(\Lambda, a)(\Lambda^{-1}, b) = (I, 0)$$

which gives $\quad b = -\Lambda^{-1} a$

thus $\quad (\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1} a)$

So the inverse element is $(\Lambda^{-1}, -\Lambda^{-1} a)$
the identity elemet is $(I, 0)$

b) Verification of given multiplication rule →

A pure translation:- $\quad U(1, \epsilon) = e^{-i\epsilon^\mu P_\mu}$

⇒ writing L.H.S of given eq that is to be verified

$$U^{-1}(\Lambda, 0)\, U(1, \epsilon)\, U(\Lambda, 0)$$

using group law $\quad U^{-1}(\Lambda, 0) = U(\Lambda^{-1}, 0)$

$$U^{-1}(\Lambda, 0)\, U(1, \epsilon)\, U(\Lambda, 0) = U(\Lambda^{-1}, 0)\, U(1, \epsilon)\, U(\Lambda, 0)$$

we have

$$U(\Lambda^{-1},0)\,U(1,\epsilon) = U(\Lambda^{-1},\Lambda^{-1}\epsilon+0) = U(\Lambda^{-1},\Lambda^{-1}\epsilon)$$

$$U(\Lambda^{-1},0)\,U(1,\epsilon)\,U(\Lambda,0) = U(\Lambda^{-1},\Lambda^{-1}\epsilon)\,U(\Lambda,0)$$

$$= U(1,\Lambda^{-1}\epsilon)$$

$$\Rightarrow \quad U(\Lambda^{-1},0)\,U(1,\epsilon)\,U(\Lambda,0) = U(1,\Lambda^{-1}\epsilon) \quad —\ ①$$

now, a translation by ϵ^{μ} is generated by P_μ

$$U(1,\epsilon) = e^{-i\epsilon^{\mu}P_{\mu}}$$

substituting in eq ①

L.H.S $\rightarrow$ $U^{-1}(\Lambda,0)\, e^{-i\epsilon^{\mu}P_{\mu}}\, U(\Lambda,0)$

$$\Rightarrow U^{-1}e^{A}U = e^{U^{-1}AU}$$

Hence, $U^{-1}(\Lambda,0)\, e^{-i\epsilon^{\mu}P_{\mu}}\, U(\Lambda,0) = e^{-i\epsilon^{\mu}(U^{-1}(\Lambda,0)\,P_{\mu}\,U(\Lambda,0))}$

R.H.S $\rightarrow$ $U(1,\Lambda^{-1}\epsilon) = e^{-i(\Lambda^{-1}\epsilon)^{\mu}P_{\mu}}$

also $(\Lambda^{-1}\epsilon)^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu}\,\epsilon^{\nu}$

R.H.S $= e^{-i\epsilon^{\nu}(\Lambda^{-1})^{\mu}{}_{\nu}P_{\mu}}$

So L.H.S = R.H.S

$$\Rightarrow e^{-i\epsilon^{\mu}\{U^{-1}(\Lambda,0)\,P_{\mu}\,U(\Lambda,0)\}} = e^{-i\epsilon^{\nu}(\Lambda^{-1})^{\mu}{}_{\nu}P_{\mu}}$$

$$\Rightarrow \quad U^{-1}(\Lambda,0)\,P_{\mu}\,U(\Lambda,0) = (\Lambda^{-1})^{\nu}{}_{\mu}\,P_{\nu}$$

now, unitary representation of infinitesimal Lorentz transformation

$$U(\Lambda,0) = 1 - \frac{i}{2}\,\omega^{\alpha\beta}M_{\alpha\beta}$$

$$U^{-1}(\Lambda,0) = 1 + \frac{i}{2}\,\omega^{\alpha\beta}\cdot M_{\alpha\beta}$$

$$U^{-1} P_\mu U = \left(1 + \frac{i}{2}\omega^{\alpha\beta} M_{\alpha\beta}\right) P_\mu \left(1 - \frac{i}{2}\omega^{\gamma\delta} M_{\gamma\delta}\right)$$

Expanding it to first order in $\omega \rightarrow$

$$U^{-1} P_\mu U = P_\mu + \frac{i}{2}\omega^{\alpha\beta}[M_{\alpha\beta}, P_\mu]$$

Expanding the RHS : $\rightarrow$

$$(\Lambda^{-1})^\nu{}_\mu = \delta^\nu{}_\mu - \omega^\nu{}_\mu$$

we get, $(\Lambda^{-1})^\nu{}_\mu P_\nu = P_\mu - \omega^\nu{}_\mu P_\nu$

Equating both sides $\rightarrow$

$$P_\mu + \frac{i}{2}\omega^{\alpha\beta}[M_{\alpha\beta}, P_\mu] = P_\mu - \omega^\nu{}_\mu P_\nu$$

$$\Rightarrow \quad \frac{i}{2}\omega^{\alpha\beta}[M_{\alpha\beta}, P_\mu] = -\omega^\nu{}_\mu P_\nu$$

$$\Rightarrow \quad \frac{i}{2}\omega^{\alpha\beta}[M_{\alpha\beta}, P_\mu] = \eta_{\mu\rho}\omega^{\nu\rho} P_\nu = \eta_{\mu\beta}\,\omega^{\alpha\beta} P_\alpha = \frac{1}{2}\omega^{\alpha\beta}(\eta_{\mu\beta} P_\alpha - \eta_{\mu\alpha} P_\beta)$$

Comparing Coefficient of $\omega^{\alpha\beta}$ on both sides

$$\boxed{[M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu)}$$

e), we have to prove " $U^{-1}(\Lambda, 0)\, U(\Lambda', 0)\, U(\Lambda, 0) = U(\Lambda^{-1}\Lambda'\Lambda, 0)$ "

from group's law $\rightarrow$ $U^{-1}(\Lambda, 0) = U(\Lambda^{-1}, 0)$

$$U^{-1}(\Lambda, 0)\, U(\Lambda', 0) = U(\Lambda^{-1}, 0)\, U(\Lambda', 0)$$
$$= U(\Lambda^{-1}\Lambda', 0)$$

$$U^{-1}(\Lambda, 0)\, U(\Lambda', 0)\, U(\Lambda, 0) = U(\Lambda^{-1}\Lambda', 0)\, U(\Lambda, 0)$$
$$= U(\Lambda^{-1}\Lambda'\Lambda, \Lambda^{-1}\Lambda' \times 0 + 0)$$

$$\boxed{U^{-1}(\Lambda, 0)\, U(\Lambda', 0)\, U(\Lambda, 0) = U(\Lambda^{-1}\Lambda'\Lambda, 0)}$$

for a small Lorentz transformation :

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \text{ with } \omega_{\mu\nu} = -\omega_{\nu\mu}$$

$$U(\Lambda,0) = 1 - \frac{i}{2}\omega^{\alpha\beta} M_{\alpha\beta}$$

$$U(\Lambda',0) = 1 - \frac{i}{2}\omega'^{\rho\sigma} M_{\rho\sigma}$$

$$U^{-1}(\Lambda,0) = 1 + \frac{i}{2}\omega^{\mu\nu} M_{\mu\nu}$$

$$U^{-1}(\Lambda,0)\,U(\Lambda',0)\,U(\Lambda,0) = (1+A)(1+B)(1+C)$$

where $A = \frac{i}{2}\omega^{\mu\nu} M_{\mu\nu}$

$B = -\frac{i}{2}\omega'^{\rho\sigma} M_{\rho\sigma}$

$C = -\frac{i}{2}\omega^{\alpha\beta} M_{\alpha\beta}$

$$U^{-1}(\Lambda,0)\,U(\Lambda',0)\,U(\Lambda,0) = (1+A)(1+B)(1+C) = 1 + B + [A,B]$$

$\because$ Since ω and ω' are small we drop terms like ω^2, ω'^2 and $\omega\omega'$ except it produces commutator like $[A,B]$

$$U^{-1}(\Lambda,0)\,U(\Lambda',0)\,U(\Lambda,0) = 1 - \frac{i}{2}\omega'^{\rho\sigma} M_{\rho\sigma} + \frac{i}{2}\omega^{\mu\nu}\left[M_{\mu\nu}, -\frac{i}{2}\omega'^{\rho\sigma} M_{\rho\sigma}\right]$$

$$= 1 - \frac{i}{2}\omega'^{\rho\sigma} M_{\rho\sigma} + \frac{1}{4}\omega^{\mu\nu}\omega'^{\rho\sigma}[M_{\mu\nu}, M_{\rho\sigma}]$$

now, the RHS $\rightarrow$ $U(\Lambda^{-1}\Lambda'\Lambda, 0) \rightarrow$ It is still a Lorentz transformation, so it looks like $\sim 1 - \frac{i}{2}\tilde{\omega}^{\rho\sigma} M_{\rho\sigma}$

where $\tilde{\omega}^{\rho\sigma}$ is transformed parameter.

$$\tilde{\omega}^{\rho\sigma} = \omega'^{\rho\sigma} + \omega^\rho{}_\lambda \omega'^{\lambda\sigma} + \omega^\sigma{}_\lambda \omega'^{\rho\lambda}$$

$$\Rightarrow \quad R.H.S = 1 - \frac{i}{2}\omega'^{\rho\sigma} M_{\rho\sigma} - \frac{i}{2}\left(\omega^\rho{}_\lambda \omega'^{\lambda\sigma} + \omega^\sigma{}_\lambda \omega'^{\rho\lambda}\right) \times M_{\rho\sigma}$$

comparing L.H.S with R.H.S →

$$\frac{1}{4}\omega^{\mu\nu}\omega'^{\rho\sigma}\left[M_{\mu\nu}, M_{\rho\sigma}\right] = -\frac{i}{2}\left(\omega^{\rho}{}_{\lambda}\,\omega'^{\lambda\sigma} + \omega^{\sigma}{}_{\lambda}\,\omega'^{\rho\lambda}\right) M_{\rho\sigma}$$

$$\Rightarrow \left[M_{\mu\nu}, M_{\rho\sigma}\right] = i\left[\eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}\right]$$

d) we have to show $[P_\mu, P_\nu] = 0$

1st translation, $T(a)$: $x^\mu \longrightarrow x'^\mu = x^\mu + a^\mu$

applying two translations successively →

$$x^\mu \xrightarrow{T(a_2)} x^\mu + a_2^\mu$$

$$x^\mu + a_2^\mu \xrightarrow{T(a_1)} (x^\mu + a_2^\mu) + a_1^\mu$$

$$x^\mu \xrightarrow{T(a_2)\circ T(a_1)} x^\mu + (a_1^\mu + a_2^\mu)$$

$$\Rightarrow T(a_1)\circ T(a_2) = T(a_1 + a_2)$$

now representing translation by operator →

$$U(1,a) \cong T(a)$$

$$\Rightarrow \boxed{U(1,a_1)\,U(1,a_2) = U(1, a_1 + a_2)} \quad \text{—— (1)}$$

$$U(1,a) = e^{-ia^\mu P_\mu}$$

$$\Rightarrow U(1,a_1)\,U(1,a_2) = e^{-ia_1^\mu P_\mu}\, e^{-ia_2^\nu P_\nu} \neq \cancel{e^{-i(a_1^\mu)(a_2^\mu)P_\mu}}$$

$$\Rightarrow U(1, a_1 + a_2) = e^{-i(a_1^\mu + a_2^\mu)P_\mu}$$

Hence, from Eq (1) →

$$e^{-ia_1^\mu P_\mu}\, e^{-ia_2^\nu P_\nu} = e^{-i(a_1^\mu + a_2^\mu)P_\mu}$$

$\Rightarrow \quad e^{-ia_1^{\mu}P_\mu} e^{-ia_2^{\nu}P_\nu} = e^{-ia_1^{\mu}P_\mu} e^{-ia_2^{\mu}P_\mu}$

$\Rightarrow \quad e^{-ia_1^{\mu}P_\mu} e^{-ia_2^{\nu}P_\nu} = e^{-ia_2^{\nu}P_\nu} e^{-ia_1^{\mu}P_\mu}$

$\Rightarrow \quad e^A e^B = e^B e^A \quad , A = -ia_1^{\mu}P_\mu$ and $B = ia_2^{\nu}P_\nu$

$\Rightarrow \quad [A, B] = 0 = [-ia_1^{\mu}P_\mu , -ia_2^{\nu}P_\nu]$

$\Rightarrow \quad i^2 a_1^{\mu} a_2^{\nu} [P_\mu , P_\nu] = 0$

$\Rightarrow \quad [P_\mu , P_\nu] = 0 \quad \forall \; a_1 , a_2$

$\Rightarrow \quad \boxed{[P_\mu , P_\nu] = 0}$

Q.3) The Pauli - Lubanski vector is defined by

$$W_\mu = \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} M^{\nu\lambda} p^\sigma$$

a) Show that $W_\mu p^\mu = 0$ and $[W_\mu, P_\nu] = 0$

b) Show that $W^2 = -\frac{1}{2} M_{\mu\nu} M^{\mu\nu} p^2 + M_{\mu\nu} M^{\mu\rho} p^\nu p_\rho$

Solution :→

a) given that $W_\mu = \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} M^{\nu\lambda} p^\sigma$

$$W_\mu p^\mu = \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} M^{\nu\lambda} p^\sigma p^\mu$$

since , $[p^\sigma, p^\mu] = 0 \rightarrow p^\sigma p^\mu = p^\mu p^\sigma$

hence ~~antis.~~ $p^\sigma p^\mu$ is symmetric under $\sigma \leftrightarrow \mu$

while $\epsilon_{\mu\nu\lambda\sigma}$ is antisymmetric under the same exchange.

$$\epsilon_{\mu\nu\lambda\sigma} p^\sigma p^\mu = 0$$

$$\boxed{W_\mu p^\mu = 0}$$

$$[W_\mu, P_\nu] = \frac{1}{2} \epsilon_{\mu\alpha\beta\gamma} [M^{\alpha\beta} p^\gamma, P_\nu]$$

$$[M^{\alpha\beta} p^\gamma, P_\nu] = M^{\alpha\beta} [p^\gamma, P_\nu] + [M^{\alpha\beta}, P_\nu] p^\gamma$$

Since $[p^\gamma, P_\nu] = 0$

$$[M^{\alpha\beta} p^\gamma, P_\nu] = [M^{\alpha\beta}, P_\nu] p^\gamma$$

$$[W_\mu, P_\nu] = \frac{i}{2} \epsilon_{\mu\alpha\beta\gamma} (\eta_{\beta\nu} p^\alpha - \eta_{\alpha\nu} p^\beta) p^\gamma$$

interchanging $\alpha \leftrightarrow \beta$

$$[W_\mu, P_\nu] = \frac{i}{2} \epsilon_{\mu\alpha\beta\gamma} (\eta_{\beta\nu} P^\alpha - \eta_{\beta\nu} P^\alpha) P^\gamma$$

$$[W_\mu, P_\nu] = 0$$

$$W^2 = W_\mu W^\mu = \frac{1}{4} \epsilon_{\mu\nu\lambda\sigma} \epsilon^{\mu\alpha\beta\gamma} M^{\nu\lambda} P^\sigma M_{\alpha\beta} P_\gamma$$

$$\epsilon_{\mu\nu\lambda\sigma} \epsilon^{\mu\alpha\beta\gamma} = -(\delta^\alpha_\nu \delta^\beta_\lambda \delta^\gamma_\sigma + \delta^\beta_\nu \delta^\gamma_\lambda \delta^\alpha_\sigma + \delta^\gamma_\nu \delta^\alpha_\lambda \delta^\beta_\sigma - \delta^\alpha_\nu \delta^\gamma_\lambda \delta^\beta_\alpha - \delta^\beta_\nu \delta^\alpha_\lambda \delta^\gamma_\sigma - \delta^\gamma_\nu \delta^\beta_\lambda \delta^\alpha_\sigma)$$

$$W^2 = -\frac{1}{4} \Big[(\delta^\alpha_\nu \delta^\beta_\lambda \delta^\gamma_\sigma) + (\delta^\beta_\nu \delta^\gamma_\lambda \delta^\alpha_\sigma) + (\delta^\gamma_\nu \delta^\alpha_\lambda \delta^\beta_\sigma) - (\delta^\alpha_\nu \delta^\gamma_\lambda \delta^\beta_\sigma) - (\delta^\beta_\nu \delta^\alpha_\lambda \delta^\gamma_\sigma) - (\delta^\gamma_\nu \partial^\beta_\lambda \delta^\alpha_\sigma) \Big] \times M^{\nu\lambda} P^\sigma M_{\alpha\beta} P_\gamma$$

Expanding term 1 →

$$-\frac{1}{4} (\delta^\alpha_\nu \delta^\beta_\lambda \delta^\gamma_\sigma) M^{\nu\lambda} P^\sigma M_{\alpha\beta} P_\gamma$$

$$= -\frac{1}{4} M^{\alpha\beta} P^\gamma M_{\alpha\beta} P_\gamma$$

$$= -\frac{1}{4} M_{\alpha\beta} M^{\alpha\beta} P_\gamma P^\gamma$$

similarly treating all other terms we get →

$$W^2 = -\frac{1}{4} M_{\mu\nu} M^{\mu\nu} P^2 - \frac{1}{4} M_{\mu\nu} M^{\mu\nu} P^2 - \frac{1}{4} M_{\mu\nu} M^{\nu\rho} P^\mu P_\rho - \frac{1}{4} M_{\mu\nu} M^{\mu\rho} P^\nu P_\rho + \frac{1}{4} M_{\mu\nu} M^{\mu\rho} P^\nu P_\rho + \frac{1}{4} M_{\mu\nu} M^{\nu\rho} P^\mu P_\rho$$

$$\boxed{W^2 = -\frac{1}{2} M_{\mu\nu} M^{\mu\nu} P^2 + M_{\mu\nu} M^{\mu\rho} P^\nu P_\rho}$$

Q.4) Verify the following relations :→

a) $[M_{\mu\nu}, W_\sigma] = i(g_{\nu\sigma} W_\mu - g_{\mu\sigma} W_\nu)$

b) $[W_\mu, W_\nu] = -i\epsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma$

Solution :→

a) $W_\sigma = \frac{1}{2}\epsilon_{\sigma\alpha\beta\gamma} M^{\alpha\beta} P^\gamma$ — (1)

we also know , $[P_\mu, P_\nu] = 0$ — (2)

$[M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu)$ — (3)

$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\mu\sigma} M_{\nu\rho})$ — (4)

so $[M_{\mu\nu}, W_\sigma] = \frac{1}{2}\epsilon_{\sigma\alpha\beta\gamma}[M_{\mu\nu}, M^{\alpha\beta} P^\gamma]$ — (4)

$[M_{\mu\nu}, M^{\alpha\beta} P^\gamma] = [M_{\mu\nu}, M^{\alpha\beta}] P^\gamma + M^{\alpha\beta}[M_{\mu\nu}, P^\gamma]$ — (5)

$[M_{\mu\nu}, M^{\alpha\beta}] = i[\eta_\nu^\alpha M_\mu^\beta - \eta_\mu^\alpha M_\nu^\beta - \eta_\nu^\beta M_\mu^\alpha + \eta_\mu^\beta M_\nu^\alpha]$

$[M_{\mu\nu}, P^\gamma] = i[\eta_\nu^\gamma P_\mu - \eta_\mu^\gamma P_\nu]$

substituting in Eq (5) →

$$[M_{\mu\nu}, W_\sigma] = \frac{i}{2}\epsilon_{\sigma\alpha\beta\gamma}\Big[(\eta_\nu^\alpha M_\mu^\beta - \eta_\mu^\alpha M_\nu^\beta - \eta_\nu^\beta M_\mu^\alpha + \eta_\mu^\beta M_\nu^\alpha) P^\gamma$$
$$+ M^{\alpha\beta}(\eta_\nu^\gamma P_\mu - \eta_\mu^\gamma P_\nu)\Big]$$

mix tensor :- $\eta_\nu^\alpha = \delta_\nu^\alpha$

$\Rightarrow \epsilon_{\sigma\alpha\beta\gamma}\eta_\nu^\alpha M_\mu^\beta P^\gamma = \epsilon_{\sigma\nu\beta\gamma} M_\mu^\beta P^\gamma$

we can contract all 6 terms in similar way

$T_1 = \epsilon_{\sigma\alpha\mu\gamma} M_\nu{}^\alpha p^\gamma$, $T_2 = \epsilon_{\sigma\nu\beta\gamma} M_\mu{}^\beta p^\gamma$

$T_3 = -\epsilon_{\sigma\mu\beta\gamma} M_\nu^\beta p^\gamma$, $T_4 = -\epsilon_{\sigma\alpha\nu\gamma} M_\mu^\alpha p^\gamma$

$T_5 = \epsilon_{\sigma\alpha\beta\nu} M^{\alpha\beta} P_\mu$, $T_6 = -\epsilon_{\sigma\alpha\beta\mu} M^{\alpha\beta} P_\nu$

$$T_1 + T_3 = \epsilon_{\sigma\alpha\mu\gamma} M_\nu{}^\alpha p^\gamma - \epsilon_{\sigma\mu\beta\gamma} M_\nu{}^\beta p^\gamma$$

$$= \epsilon_{\sigma\beta\mu\gamma} M_\nu{}^\beta p^\gamma - \epsilon_{\sigma\beta\mu\gamma} M_\nu{}^\beta p^\gamma$$

$$= 0$$

also $T_2 + T_4 = 0$

$$[M_{\mu\nu}, W_\sigma] = \frac{i}{2}(\epsilon_{\sigma\alpha\beta\nu} M^{\alpha\beta} P_\mu - \epsilon_{\sigma\alpha\beta\mu} M^{\alpha\beta} P_\nu)$$

$$= \frac{i}{2}[\,2\eta_{\nu\sigma} W_\mu - 2\eta_{\mu\sigma} W_\nu]$$

$$[M_{\mu\nu}, W_\sigma] = i\,[\eta_{\nu\sigma} W_\mu - \eta_{\mu\sigma} W_\nu]$$

b) $[W_\mu, W_\nu] = \frac{1}{4}\epsilon_{\mu\alpha\beta\gamma}\epsilon_{\nu\rho\sigma\delta}[M^{\alpha\beta}p^\gamma, M^{\rho\sigma}p^\delta]$

we know, $[AB, CD] = A[B,C]D + AC[B,D] + [B,A]CD$
$+ C[A,D]B$

$$[W_\mu, W_\nu] = \frac{1}{4}\epsilon_{\mu\alpha\beta\gamma}\epsilon_{\nu\rho\sigma\delta}\big(M^{\alpha\beta}[p^\gamma, M^{\rho\sigma}]p^\delta + [M^{\alpha\beta}, M^{\rho\sigma}]p^\gamma p^\delta$$
$$+ M^{\rho\sigma}[M^{\alpha\beta}, p^\delta]p^\gamma + M^{\rho\sigma}M^{\alpha}[p^\gamma, p^\delta]\big)$$

$$= \frac{1}{4}\epsilon_{\mu\alpha\beta\gamma}\epsilon_{\nu\rho\sigma\delta}\big(M^{\alpha\beta}[p^\gamma, M^{\rho\sigma}]p^\delta + [M^{\alpha\beta}, M^{\rho\sigma}]p^\gamma p^\delta$$
$$+ M^{\rho\sigma}[M^{\alpha\beta}, p^\delta]p^\gamma\big) \qquad [\because [p^\gamma, p^\delta] = 0]$$

$$[P^\gamma, M^{\rho\sigma}] = -i(\eta^{\sigma\gamma} P^\rho - \eta^{\rho\gamma} P^\sigma)$$

$$[M^{\alpha\beta}, P^\delta] = i(\eta^{\beta\delta} P^\alpha - \eta^{\alpha\delta} P^\beta)$$

Substituting,

$$[W_\mu, W_\nu] = -\frac{i}{4}\, \epsilon_{\mu\alpha\beta\gamma}\, \epsilon_{\nu\rho\sigma\delta} (\eta^{\sigma\gamma} M^{\alpha\beta} P^\rho P^\delta - \eta^{\rho\gamma} M^{\alpha\beta} P^\sigma P^\delta)$$

$$+\frac{i}{4}\, \epsilon_{\mu\alpha\beta\gamma}\, \epsilon_{\nu\rho\sigma\delta} [M^{\alpha\beta}, M^{\rho\sigma}] P^\gamma P^\delta$$

$$+\frac{i}{4}\, \epsilon_{\mu\alpha\beta\gamma}\, \epsilon_{\nu\rho\sigma\delta} M^{\rho\sigma} (\eta^{\beta\delta} P^\alpha - \eta^{\alpha\delta} P^\beta) P^\gamma$$

$$\Rightarrow P^\gamma P^\delta = P^\delta P^\gamma$$

So $\epsilon_{\mu\alpha\beta\gamma}\, \epsilon_{\nu\rho\sigma\delta}\, \eta^{\sigma\gamma} M^{\alpha\beta} P^\rho P^\delta = \epsilon_{\mu\alpha\beta\sigma}\, \epsilon_{\nu\rho\sigma\delta} M^{\alpha\beta} P^\rho P^\delta$

$$T_1 = -\frac{i}{4}\, \epsilon_{\mu\alpha\beta\sigma}\, \epsilon_{\nu\rho\sigma\delta} M^{\alpha\beta} P^\rho P^\delta$$

$$T_2 = +\frac{i}{4}\, \epsilon_{\mu\alpha\beta\rho}\, \epsilon_{\nu\rho\sigma\delta} M^{\alpha\beta} P^\sigma P^\delta \qquad \{[\rho \longleftrightarrow \sigma]$$

$$T_1 + T_2 = -\frac{i}{2}\, \epsilon_{\mu\nu\rho\sigma} \left(\frac{1}{2} \epsilon^{\rho\alpha\beta\gamma} M_{\alpha\beta} P_\gamma\right) P^\sigma$$

now Term 3 has $[M^{\alpha\beta}, M^{\rho\sigma}]$

$$[M^{\alpha\beta}, M^{\rho\sigma}] = i(\eta^{\alpha\sigma} M^{\beta\rho} + \eta^{\beta\rho} M^{\alpha\sigma} - \eta^{\alpha\rho} M^{\beta\sigma} - \eta^{\beta\sigma} M^{\alpha\rho})$$

$$T_3 = \frac{i}{4} \left(\qquad\qquad \right) \times \epsilon_{\mu\alpha\beta\gamma}\, \epsilon_{\nu\rho\sigma\delta} P^\gamma P^\delta$$

Each term in T_3 has structure $\epsilon_{\mu\alpha\beta\gamma}\, \eta^{\alpha\sigma} = \epsilon_{\mu\sigma\beta\gamma}$

four such contractions, the sum will be zero

$$T_3 = 0$$

the Last term

$$T_4 = -\frac{i}{2} \epsilon_{\mu\alpha\beta\gamma} \epsilon_{\nu\rho\gamma\delta} \{ \eta^{\beta\delta} P^{\alpha} P^{\gamma} M^{\rho\sigma} - \eta^{\alpha\delta} M^{\rho\sigma} P^{\beta} P^{\gamma} \}$$

$$= \frac{-i}{4} \epsilon_{\mu\nu\rho\sigma} \left(\frac{1}{2} \epsilon^{\rho\alpha\beta\gamma} M_{\alpha\beta} P_{\gamma} \right) P^{\sigma}$$

Sum of all contributions →

$$[W_\mu, W_\nu] = -i \epsilon_{\mu\nu\rho\sigma} \left(\frac{1}{2} \epsilon^{\rho\alpha\beta\gamma} M_{\alpha\beta} P_{\gamma} \right) P^{\sigma}$$

$$\boxed{[W_\mu, W_\nu] = -i \epsilon_{\mu\nu\rho\sigma} W^{\rho} P^{\sigma}}$$