ASSIGNMENT-2, QFT

TOPIC :- QUANTIZATION OF FIELDS

Solved by - Puja Mondal { Self Study }

Q 1: Quantization of a classical String →

A classical string can be decomposed into an infinite set of normal modes, whose dynamics are described by the Hamiltonian

$$H = \sum_{n=1}^{\infty} \left(\frac{1}{2} p_n^2 + \frac{1}{2} \omega_n^2 q_n^2\right)$$

where q_n and p_n are the canonical coordinate and momentum associated with n th mode, and ω_n is corresponding frequency.

Upon quantization, the variables q_n and p_n are promoted to operators satisfying the canonical commutation relations

$$[q_n, q_m] = [p_n, p_m] = 0 \quad , \quad [q_n, p_m] = i\,\delta_{nm}$$

a) Introduce anhilation and creation operators a_n & a_n^+.

b) Show explicitely, $[a_n, a_m] = [a_n^+, a_m^+] = 0$

$$[a_n, a_m^+] = \delta_{nm}$$

c) Show that the Hamiltonian of the system can be written in the form $H = \sum_{n=1}^{\infty} \frac{1}{2}\omega_n (a_n a_n^+ + a_n^+ a_n)$.

d) Assuming existance of a ground state $|0\rangle$ such that $a_n|0\rangle = 0$ for all n, explain how, after removing the vacuum energy, the Hamiltonian can be written as $H = \sum_{n=1}^{\infty} \omega_n a_n^+ a_n$

e) Show that $[H, a_n^+] = \omega_n a_n^+$, determine the energy of the state.

Solution:- the classical Hamiltonian of the string

$$H = \sum_{n=1}^{\infty} \left(\frac{1}{2} p_n^2 + \frac{1}{2} \omega_n^2 q_n^2 \right)$$

also, $[q_n, q_m] = 0$, $[p_n, p_m] = 0$, $[q_n, p_m] = i\delta_{nm}$

a) Creation operator, $\boxed{a_n^+ = \sqrt{\frac{\omega_n}{2}}\, q_n - \frac{i}{\sqrt{2\omega_n}}\, p_n}$

$$\boxed{a_n = \sqrt{\frac{\omega_n}{2}}\, q_n + \frac{i}{\sqrt{2\omega_n}}\, p_n}$$

These definitions are chosen so that a_n^+ is the hermitian conjugate of a_n, since both q_n and p_n are hermitian operators.

b) $[a_n, a_m] = \left[\sqrt{\frac{\omega_n}{2}}\, q_n + \frac{i}{\sqrt{2\omega_n}}\, p_n , \sqrt{\frac{\omega_m}{2}}\, q_m + \frac{i}{\sqrt{2\omega_m}}\, p_m \right]$

use linearity of the commutator, expand →

$$[a_n, a_m] = \sqrt{\frac{\omega_n}{2}}\sqrt{\frac{\omega_m}{2}}\, [q_n, q_m] + \sqrt{\frac{\omega_n}{2}}\, \frac{i}{\sqrt{2\omega_m}}\, [q_n, p_m]$$
$$+ \frac{i}{\sqrt{2\omega_n}}\sqrt{\frac{\omega_m}{2}}\, [p_n, q_m] + \frac{i}{\sqrt{2\omega_n}}\, \frac{i}{\sqrt{2\omega_n}}\, [p_n, p_m]$$

we know, $[q_n, q_m] = 0$ $[p_n, p_m] = 0$ $[q_n, p_m] = i\delta_{nm}$

$[p_n, q_m] = -i\delta_{nm}$

$$[a_n, q_m] = -\frac{1}{2}\sqrt{\frac{\omega_n}{\omega_m}}\, \delta_{nm} + \frac{1}{2}\sqrt{\frac{\omega_m}{\omega_n}}\, \delta_{nm} = 0$$

- $[a_n^+, a_m^+] = \left[\sqrt{\frac{\omega_n}{2}}\, q_n - \frac{i}{\sqrt{2\omega_n}} p_n ,\ \sqrt{\frac{\omega_m}{2}}\, q_m - \frac{i}{\sqrt{2\omega_m}} p_m\right]$

$$[a_n^+, a_m^+] = \sqrt{\frac{\omega_n}{2}}\sqrt{\frac{\omega_n}{2}}\,[q_n, q_m] - i\sqrt{\frac{\omega_n}{2}}\sqrt{\frac{1}{2\omega_m}}\,[q_n, p_m]$$

$$-\frac{i}{\sqrt{2\omega_n}}\sqrt{\frac{\omega_m}{2}}\,[p_n, q_m] + \frac{i}{\sqrt{2\omega_n}}\frac{i}{\sqrt{2\omega_m}}\,[p_n, p_m]$$

$$= \frac{1}{2}\sqrt{\frac{\omega_n}{\omega_m}}\,(-i\,\delta_{nm} + i\,\delta_{nm}) = 0$$

- $[a_n, a_m^+] = \left[\sqrt{\frac{\omega_n}{2}}\, q_n + \frac{i}{\sqrt{2\omega_n}} p_n ,\ \sqrt{\frac{\omega_m}{2}}\, q_m - \frac{i}{\sqrt{2\omega_m}} p_m\right]$

$$= \sqrt{\frac{\omega_n}{2}}\sqrt{\frac{\omega_m}{2}}\,[q_n, q_m] - \sqrt{\frac{\omega_n}{2}}\,\frac{i}{\sqrt{2\omega_m}}\,[q_n, p_m]$$

$$+ \frac{i}{\sqrt{2\omega_n}}\sqrt{\frac{\omega_m}{2}}\,[p_n, q_m] - \frac{i}{\sqrt{2\omega_n}}\frac{i}{\sqrt{2\omega_m}}\,[p_n, p_m]$$

inserting results commutators $\rightarrow$

$$[a_n, a_m^+] = \sqrt{\frac{\omega_n}{2}}\frac{i}{\sqrt{2\omega_m}}\,(i\,\delta_{nm}) + \frac{i}{\sqrt{2\omega_n}}\sqrt{\frac{\omega_m}{2}}\,(-i\,\delta_{nm})$$

$$= \frac{1}{2}\left[\sqrt{\frac{\omega_n}{\omega_m}} + \sqrt{\frac{\omega_m}{\omega_n}}\right]\delta_{nm}$$

for $n = m$, $\delta_{nm} \Rightarrow 1$ also, $\omega_n = \omega_m$

$$\boxed{[a_n, a_m^+] = \delta_{nm}}$$

c) $q_n = \frac{1}{\sqrt{2\omega_n}} (a_n + a_n^+)$

$p_n = -i\sqrt{\frac{\omega_n}{2}} (a_n - a_n^+)$

kinetic term,

$$\frac{1}{2} p_n^2 = \frac{\omega_n}{4} (a_n - a_n^+)(a_n - a_n^+)$$

$$= \frac{\omega_n}{4} (a_n a_n - a_n a_n^+ - a_n^+ a_n + a_n^+ a_n^+)$$

potential term,

$$\frac{1}{2} \omega_n^2 q_n^2 = \frac{\omega_n}{4} (a_n + a_n^+)(a_n + a_n^+)$$

$$= \frac{\omega_n}{4} (a_n a_n + a_n a_n^+ + a_n^+ a_n + a_n^+ a_n^+)$$

$$H = \sum_{n=1}^{\infty} \frac{\omega_n}{2} (a_n a_n^+ + a_n^+ a_n)$$

d) Assuming a ground state $|0\rangle$ such that $a_n|0\rangle = 0$

using, $a_n a_n^+ = (a_n^+ a_n) + 1$

the hamitonian becomes

$$H = \sum_{n=1}^{\infty} \omega_n (a_n^+ a_n + \tfrac{1}{2})$$

the constant term is vacuum energy, Removing it gives

$$H = \sum_{n=1}^{\infty} \omega_n a_n^+ a_n$$

e) $[H, a_n^+] = \sum_{m=1}^{\infty} \omega_m [a_m^+ a_m, a_n^+]$

$$= \sum_m \omega_m \{a_m^+ [a_m, a_n^+] + [a_m^+, a_n^+] a_m\}$$

$[H, a_n^+] = \sum_m \omega_m a_m^+ \delta_{mn} = \omega_n a_n^+$ $[\because \delta_{mn} = 1$ for $m=n]$

Energy of the state,

$|l_1\, l_2 \cdots l_N\rangle = (a_1^{\dagger})^{l_1}(a_2^{\dagger})^{l_2} \cdots (a_N^{\dagger})^{l_N}|0\rangle$

using commutator $[H, a_n^{\dagger}] = \omega_n a_n^{\dagger}$

$H|l_1, l_2 \cdots l_N\rangle = \left(\sum_{n=1}^{N} l_n \omega_n\right) |l_1, l_2 \cdots l_N\rangle$

the energy of the state is, $E = \sum_{n=1}^{N} l_n \omega_n$

$$\boxed{E = \sum_{n=1}^{N} l_n E_n}$$

problem 2. The fourier decomposition of a real scaler field $\phi(\vec{x})$ and it's conjugate momentum $\pi(x)$ in the Schrödinger picture is given by

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\dagger}\, e^{-i\vec{p}\cdot\vec{x}}\right)$$

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} \left(-i\sqrt{\frac{E_{\vec{p}}}{2}}\right)\left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^{\dagger}\, e^{-i\vec{p}\cdot\vec{x}}\right)$$

where $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$

the canonical equal-time commutation relations are

$[\phi(x), \phi(y)] = 0$, $[\pi(x), \pi(y)] = 0$, $[\phi(\vec{x}), \pi(\vec{y})] = i\delta^3(\vec{x}-\vec{y})$

Show that these commutation relations imply

$[a_{\vec{p}}, a_{\vec{q}}] = 0$ $\quad [a_{p}^{\dagger}, a_{\vec{q}}^{\dagger}] = 0$

$[a_{\vec{p}}, a_{\vec{q}}^{\dagger}] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})$

given, $\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a^{+}_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}\right)$ — ①

$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \left(-i\sqrt{\frac{E_{\vec{p}}}{2}}\right) \cdot \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a^{+}_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}\right)$ — ②

— ③

$E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$

— ④

also, $[\phi(x), \phi(y)] = 0$

$[\pi(x), \pi(y)] = 0$ — ⑤

$[\phi(x), \pi(y)] = i\delta^3(\vec{x} - \vec{y})$ — ⑥

$$[\phi(x), \pi(y)] = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \left(\frac{-i}{2}\sqrt{\frac{E_{\vec{q}}}{E_{\vec{p}}}}\right)$$
$$\times \left([a_{\vec{p}}, a_{\vec{q}}] e^{i\vec{p}\cdot\vec{x}} e^{i\vec{q}\cdot\vec{x}} - [a_{\vec{p}}, a^{+}_{\vec{q}}] e^{i\vec{p}\cdot\vec{x}} e^{-i\vec{q}\cdot\vec{x}}\right.$$
$$\left. + [a^{+}_{\vec{p}}, a_{\vec{q}}] e^{-i\vec{p}\cdot\vec{x}} e^{i q\cdot y} - [a^{+}_{\vec{p}}, a^{+}_{\vec{q}}] e^{-i\vec{p}\cdot x} e^{-i\vec{q}\cdot\vec{y}}\right)$$ ↳ ⑦

$$[\phi(x), \pi(y)] = i \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}\cdot\vec{y}}$$ → from ⑥

$$[\phi(\vec{x}), \pi(\vec{y})] = i \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}\cdot\vec{y}}$$ — ⑧

now comparing Eq ⑦ and Eq ⑧

$$i\delta^3(\vec{x} - \vec{y}) = i\int \frac{d^3k}{(2\pi)^3} e^{+i\vec{k}\cdot\vec{x}} e^{-i\vec{k}\cdot\vec{y}}$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \left(c_1(\vec{p},\vec{q}) e^{ip\cdot x} e^{iq\cdot y}\right.$$
$$+ c_2(p,q) e^{ip\cdot x} e^{-iq\cdot y} + c_3(p,q) e^{-ip\cdot x} e^{iq\cdot y}$$
$$+ c_4(p,q) e^{-ip\cdot x} e^{-iq\cdot y}$$

Hence, $c_1(\bar{p},\bar{q}) = \frac{-i}{2}\sqrt{\frac{E_{\bar{q}}}{E_{\bar{p}}}}\,[a_{\bar{p}}, a_{\bar{q}}]$

$$c_2(p,q) = \frac{i}{2}\sqrt{\frac{E_{\bar{q}}}{E_{\bar{p}}}}\,[a_{\bar{p}}, a_q^{+}]$$

$$c_3(p,q) = \frac{-i}{2}\sqrt{\frac{E_{\bar{q}}}{E_{\bar{p}}}}\,[a_{\bar{p}}^{+}, a_{\bar{q}}]$$

$$c_4(p,q) = \frac{i}{2}\sqrt{\frac{E_{\bar{q}}}{E_{\bar{p}}}}\,[a_{\bar{p}}^{+}, a_{\bar{q}}^{+}]$$

the exponential functions

$e^{i\bar{p}\cdot\bar{x}}e^{i\bar{q}\cdot\bar{y}}$, $e^{i\bar{p}\cdot\bar{x}}e^{-i\bar{q}\cdot\bar{y}}$, $e^{-i\bar{p}\cdot\bar{x}}e^{i\bar{q}\cdot\bar{y}}$, $e^{-i\bar{p}\cdot\bar{x}}e^{-i\bar{q}\cdot\bar{y}}$

these all are linearly independent function of $(\bar{x},\bar{y})$

therefore, equality for all $\bar{x}$ and $\bar{y}$ implies

$c_1(\bar{p},\bar{q}) = 0 \quad \ldots \quad c_4(\bar{p},\bar{q}) = 0$

Also, the coefficient of $e^{i\bar{p}\cdot\bar{x}}e^{-i\bar{q}\cdot\bar{y}}$ must match the right hand side →

$$[a_{\bar{p}}, a_{\bar{q}}] = 0$$

$$[a_{\bar{p}}^{+}, a_{\bar{q}}^{+}] = 0$$

$$[a_{\bar{p}}, a_{\bar{q}}^{+}] = (2\pi)^3\,\delta^3(\bar{p}-\vec{q}).$$

Q. Show that in the Heisenberg picture the field operators satisfy,

$$\dot{\phi}(x) = i\,[H, \phi(x)] = \pi(x)$$

and $\dot{\pi}(x) = i\,[H, \pi(x)] = \nabla^2\phi(x) - m^2\phi(x)$

Hence show that the operator $\phi(x)$ satisfies the Klein-Gordon equation. $(\Box + m^2)\,\phi(x) = 0$, $\Box = \partial_\mu \partial^\mu$

Solution:- We consider a real scaler field in the Heisenberg picture with Hamiltonian

$$H = \int d^3x \left[\tfrac{1}{2}\pi^2(x) + \tfrac{1}{2}(\nabla\phi(x))^2 + \tfrac{1}{2}m^2\phi^2(x)\right]$$

the equal-time commutation relations are

$$[\phi(t,\bar{x}), \phi(t,\bar{y})] = 0 \quad , \quad [\pi(t,\bar{x}), \pi(t,\bar{y})] = 0$$

$$[\phi(t,\bar{x}), \pi(t,\bar{y})] = i\,\delta^{(3)}(\bar{x}-\bar{y})$$

→ In Heisenberg picture, the time evolution of any operator

$$\dot{O}(x) = i\,[H, O(x)]$$

$$\rightarrow [H, \phi(x)] = \int d^3y \left\{ \tfrac{1}{2}[\pi^2(y), \phi(x)] + \tfrac{1}{2}[(\nabla\phi(y))^2, \phi(x)] + \tfrac{1}{2}[m^2\phi^2(y), \phi(x)] \right\}$$

$$\rightarrow [\pi^2(y), \phi(x)] = \pi(y)\,[\pi(y), \phi(x)] + [\pi(y), \phi(x)]\,\pi(y)$$

$$= -i\,\delta^{(3)}(\bar{y}-\bar{x})\,\pi(y) - i\,\pi(y)\,\delta^3(\bar{y}-\bar{x})$$

$$= -2i\,\delta^{(3)}(\bar{y}-\bar{x})\,\pi(y)$$

$$[(\nabla\phi(y))^2, \phi(x)] = \nabla\phi(y)[\nabla\phi(y), \phi(x)] + [\nabla\phi(y), \phi(x)]\nabla\phi(y)$$

$$= 0$$

$$[\phi^2(y), \phi(x)] = \phi(y)[\phi(y), \phi(x)] + [\phi(y), \phi(x)]\phi(y)$$

$$= 0$$

therefore, $[H;\phi(x)] = \int d^3y \, \frac{1}{2}(-2i\delta^3(\bar{y}-\bar{x})\pi(y))$

$$= -i\pi(x)$$

thus, $\dot{\phi}(x) = i[H, \phi(x)] = \pi(x)$

<u>Calculation, $\dot{\pi}(x)$</u> : →

$$[H, \pi(x)] = \int d^3y \left\{ \frac{1}{2}[\pi^2(y), \pi(x)] + \frac{1}{2}[(\nabla\phi(y))^2, \pi(x)] + \frac{1}{2}[m^2\phi^2(y), \pi(x)] \right\}$$

→ $[\pi^2(y), \pi(x)] = 0$

→ $[(\nabla\phi(y))^2, \pi(x)] = \partial_i\phi(y)[\partial_i\phi(y), \pi(x) + [\partial_i\phi(y), \pi(x)]\partial_i\phi(y)$

$$= 2i\,\partial_i\phi(y)\,\partial_i\,\delta^{(3)}(\bar{y}-\bar{x})$$

→ Integrating by parts →

$$\int d^3y\, \partial_i\phi(y)\, \partial_i\delta^3(\bar{y}-x)$$

$$= -\int d^3y\,(\nabla^2\phi(y))\,\delta^3(\bar{y}-x) = -\nabla^2\phi(x)$$

thus, $\int d^3y\, \frac{1}{2}[(\nabla\phi(y))^2, \pi(x)] = -i\nabla^2\phi(x)$

also, $[\phi^2(y), \pi(x)]$

$= \phi(y)[\phi(y), \pi(x)] + [\phi(y), \pi(x)]\,\phi(y)$

$= 2i\,\delta^{(3)}(\vec{y}-\vec{x})\,\phi(y)$

$\int d^3y\, \frac{1}{2} m^2 [\phi^2(y), \pi(x)] = i m^2 \phi(x)$

combining all contributions →

$[H, \pi(x)] = -i\nabla^2\phi(x) + im^2\phi(x)$

therefore, $\dot{\pi}(x) = i[H, \pi(x)] = \nabla^2\phi(x) - m^2\phi(x)$

Klein-Gordon Equation →

$\dot{\phi}(x) = \pi(x)$

taking another derivative, $\ddot{\phi}(x) = \dot{\pi}(x)$

also $\dot{\pi}(x) = \nabla^2\phi(x) - m^2\phi(x)$

so, $\ddot{\phi}(x) = \nabla^2\phi(x) - m^2\phi(x)$

or, $(\partial_t^2 - \nabla^2 + m^2)\,\phi(x) = 0$

or, $(\partial_\mu\partial^\mu + m^2)\,\phi(x) = 0$

or, $\boxed{(\Box + m^2)\,\phi(x) = 0}$

Hence $\phi(x)$ satisfies Klein-Gordon Equation.