Assignment :→ 1

Topic :→ Classical Field Theory

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From - QFT Course {Lecture Series by Prof. David Tong, Problem Sheet}

Q.1 A string of length a, mass per unit length σ and under tension T is fixed at each end. The lagrangian governing the evolution of the transverse displacement $y(x,t)$ is —

$$L = \int_0^a dx \left[\frac{\sigma}{2} \left(\frac{\partial y}{\partial t}\right)^2 - \frac{T}{2} \left(\frac{\partial y}{\partial x}\right)^2 \right]$$

where x identifies position along the string from one end point. By expressing the displacement as a sine series fourier expansion

$$y(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) q_n(t)$$

i) Show that the lagrangian becomes

$$L = \sum_{n=1}^{\infty} \left(\frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} \left(\frac{n\pi}{a}\right)^2 q_n^2 \right)$$

ii) Derive the equation of motion.

iii) Hence show that the string is equivalent to ~~an~~ set of decoupled harmonic oscillators with frequency $\omega_n = \sqrt{\frac{T}{\sigma}} \left(\frac{n\pi}{a}\right)$

Ans :→

i) given $y(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) q_n(t)$

$$\frac{\partial y}{\partial t} = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \dot{q}_n(t)$$

$$\frac{\partial y}{\partial x} = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \left(\frac{n\pi}{a}\right) \cos\left(\frac{n\pi}{a}x\right) q_n(t)$$

$$L = \int_0^{\infty} dx \left\{ \frac{\sigma}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 \right\}$$

$$L = \int_0^{\infty} dx \left\{ \frac{\sigma}{2}\left(\frac{\partial y}{\partial t}\right)^2 \right\} - \int_0^{\infty} dx \left\{ \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 \right\}$$

$$L = \frac{\sigma}{2} \times \frac{2}{a} \int_0^{a} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \dot{q}_n \dot{q}_m \, dx$$

$$- \frac{T}{2} \times \frac{2}{a} \int_0^{a} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} dx \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi x}{a}\right) \left(\frac{n\pi}{a}\right)\left(\frac{m\pi}{a}\right) q_n q_m$$

$$\text{OR, } L = \frac{\sigma}{a} \sum_m \sum_n \left\{ \int_0^{a} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{a}x\right) dx \right\} \dot{q}_n \dot{q}_m$$

$$- \frac{T}{a} \sum_m \sum_n \left\{ \int_0^{a} \cos\left(\frac{n\pi}{a}x\right) \cos\left(\frac{m\pi}{a}x\right) dx \right\} q_n q_m \left(\frac{n\pi}{a}\right)\left(\frac{m\pi}{a}\right)$$

$$\text{OR, } L = \sum_n \sum_m \frac{\sigma}{a} \dot{q}_n \dot{q}_m \frac{a}{2}\delta_{mn} - \sum_n \sum_m \frac{T}{a} \times q_n q_m \left(\frac{n\pi}{a}\right)\left(\frac{m\pi}{a}\right) \frac{a}{2}\delta_{mn}$$

$$\text{OR, } L = \sum_n \frac{\sigma}{2} (\dot{q}_n)^2 - \frac{T}{2} \left(\frac{n\pi}{a}\right)^2 q_n^2$$

ii) Equation of motion :→

from Euler- lagrang equation →

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \frac{\partial L}{\partial q_n} = 0$$

OR $$\sum_{n=1}^{\infty} \sigma \ddot{q}_n + T\left(\frac{n\pi}{a}\right)^2 q_n^2 = 0$$

E.O.M → $$\sum_{n=1}^{\infty} \ddot{q}_n + \frac{T}{\sigma}\left(\frac{n\pi}{a}\right)^2 q_n = 0$$

OR, $$\sum_{n=1}^{\infty} \ddot{q}_n = -\frac{T}{\sigma}\sum_{n=1}^{\infty}\left(\frac{n\pi}{a}\right)^2 q_n$$

this follows the E.O.M for ~~harmonic~~ n- decoupled harmo
-nic oscillators,

$$\boxed{\omega_n = \sqrt{\frac{T}{\sigma}}\left(\frac{n\pi}{a}\right)}$$

Q.2. Show directly that if $\phi(x)$ satisfies the Klein-Gordon Equation, then $\phi(\Lambda^{-1}x)$ also satisfy this equation for any Lorentz transformation Λ.

Ans:→ Klein-Gordon Equation for ϕ :→

$$(\Box + m^2)\phi = 0$$

$$\Box = \partial_\mu \partial^\mu = \partial^\mu \partial_\mu$$

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \longrightarrow (1)$$

$$\phi(\Lambda^{-1}x) = \phi(y)$$ Inverse Lorentz transformation

where $y = \Lambda^{-1}x$ → {Lorentz transformation}.

So substituting $(\Lambda^{-1}x)$ in Klein-Gorden Equation's L.H.S →

$$\Rightarrow \partial_\mu \partial^\mu (\phi(y)) + m^2 \phi(y) = ?$$

$$\Rightarrow \partial_\mu \partial^\mu (\phi(y)) = \partial^\mu \partial_\mu \{\phi(y)\}$$

$$\partial_\mu(\phi(y)) = \frac{\partial}{\partial y^\nu}\{\phi(y)\} \frac{\partial (y)^\nu}{\partial x^\mu}$$ [Chain Rule]

$$\partial_\mu\{\phi(y)\} = \frac{\partial}{\partial x^\mu}\{y\}\, \partial_\nu \phi(y)$$

$$= \frac{\partial}{\partial x^\mu}\{\Lambda^{-1}x\}\, \partial_\nu \phi(y)$$

$$= \frac{\partial}{\partial x^\mu}\{(\Lambda^{-1})^\nu{}_\rho x^\rho\}\, \partial_\nu \phi(y)$$

$$= (\Lambda^{-1})^\nu{}_\rho \frac{\partial x^\rho}{\partial x^\mu}\, \partial_\nu \phi(y)$$

$$= (\Lambda^{-1})^\nu{}_\rho\, \delta^\rho_\mu\, \partial_\nu \phi(y)$$

$$\boxed{\partial_\mu\{\phi(y)\} = (\Lambda^{-1})^\nu{}_\mu\, \partial_\nu \phi(y)} \longrightarrow (2)$$

Second derivative →

$$\partial^{\mu} = \eta^{\mu\rho}\,\partial_{\rho}$$

$$\Rightarrow \quad \partial^{\mu}[\partial_{\mu}\phi(y)] = \eta^{\mu\rho}\,\partial_{\rho}[\partial_{\mu}\phi(y)]$$

$$= \eta^{\mu\rho}\,\partial_{\rho}\{(\Lambda^{-1})^{\nu}_{\mu}\,\partial_{\nu}\phi(y)\}$$

$$= \eta^{\mu\rho}(\Lambda^{-1})^{\nu}_{\mu}\,\partial_{\rho}[\partial_{\nu}\phi(y)]$$

$$= \eta^{\mu\rho}(\Lambda^{-1})^{\nu}_{\mu}\,\partial_{\nu}[\partial_{\rho}\phi(y)]$$

$$= \eta^{\mu\rho}(\Lambda^{-1})^{\nu}_{\mu}\,\partial_{\nu}[(\Lambda^{-1})^{\sigma}_{\rho}\,\partial_{\sigma}\phi(y)] \quad \text{from Eq } 2$$

$$= \eta^{\mu\rho}(\Lambda^{-1})^{\nu}_{\mu}(\Lambda^{-1})^{\sigma}_{\rho}\,\partial_{\nu}\partial_{\sigma}\{\phi(y)\}$$

$$= \eta^{\mu\rho}(\Lambda^{-1})^{\sigma}_{\rho}(\Lambda^{-1})^{\nu}_{\mu}\,\partial_{\nu}\partial_{\sigma}\{\phi(y)\}$$

$$= \eta^{\sigma\nu}\,\partial_{\nu}\partial_{\sigma}\{\phi(y)\}$$

$$= \cancel{\partial^{\sigma}\partial^{\nu}}\,(\eta^{\sigma\nu}\partial_{\nu})(\partial_{\sigma})\{\phi(y)\}$$

$$= \partial^{\sigma}\partial_{\sigma}\{\phi(y)\}$$

$$= \Box\,\phi(y)$$

Hence :→

$$(\Box + m^2)(\Lambda^{-1}x) = (\Box + m^2)(y) \quad \text{— } 3$$

as x, y are dummy variables we can write

$$\boxed{(\Box + m^2)\phi(\Lambda^{-1}x) = (\Box + m^2)\phi(y) = 0}$$

hence if $\phi(x)$ satisfies Klein-Gordon Equation then $\phi(\Lambda^{-1}x)$ will also satisfy Klein-Gordon Equation.

Q.3) The motion of a complex field $\psi(x)$ is governed by lagrangian, $L = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2}(\psi^* \psi)^2$

i) Write down the Euler-lagrange field Equation

ii) Verify that the lagrangian is invariant under the infinitesimal transformation,

$$\boxed{\delta\psi = i\alpha\psi} \quad , \quad \boxed{\delta\psi^* = -i\alpha\psi}$$

iii) Derive Noether Current associated with this transformation and verify explicitely that it is conserved using the field equations satisfied by ψ.

Ans:→ Given lagrangian

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2(\psi^*\psi) - \frac{\lambda}{2}(\psi^*\psi)^2 \quad — (1)$$

i) $$\frac{\partial L}{\partial \psi^*} = -m^2\psi - 2 \times \frac{\lambda}{2}(\psi^*\psi)\psi$$

$$\frac{\partial L}{\partial \psi^*} = -m^2\psi - \lambda(\psi^*\psi)\psi \quad — (2)$$

$$\frac{\partial L}{\partial(\partial_\mu \psi^*)} = \partial^\mu \psi \quad — (3)$$

now plugging (2) and (3) in ~~Eq (1)~~ Euler - lagrong Eq →

$$\frac{\partial L}{\partial \psi^*} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \psi^*)}\right) = 0$$

$$\Rightarrow \quad -m^2\psi - \lambda(\psi^*\psi)\psi - \partial_\mu(\partial^\mu\psi) = 0$$

$$\partial_\mu\partial^\mu(\psi) + m^2\psi + \lambda(\psi^*\psi).\psi = 0 \quad — (4)$$

$$\Rightarrow \quad \boxed{\Box\psi + m^2\psi + \lambda(\psi^*\psi).\psi = 0}$$

$\Rightarrow$

similarly we can show i by taking derivatives with respect to ψ and $\partial_\mu \psi$ →

$$\boxed{\Box \psi^* + m^2 \psi^* + \lambda (\psi^* \psi)\psi^* = 0} \quad -(5)$$

ii) Infinitesimal transformation :-

$$\delta\psi = i\alpha\psi \quad , \quad \delta\psi^* = -i\alpha\psi^*$$

In the lagrangian there are two parts → kinetic term and mass-interaction term

→ kinetic part ⇒ $\partial_\mu \psi^* \partial^\mu \psi$

→ change in kinetic part due to infinitesimal transformation in ψ and ψ^* ⟶

$$\delta(\partial_\mu \psi^* \partial^\mu \psi) = \{\delta(\partial_\mu \psi^*)\partial^\mu \psi\} + \{\partial_\mu \psi^* (\delta(\partial^\mu \psi))\}$$

$$\delta(T) = \partial_\mu(\delta\psi^*)(\partial^\mu \psi) + (\partial_\mu \psi^*)\partial^\mu(\delta\psi)$$

$$\delta(T) = \{\partial_\mu(-i\alpha\psi^*)\}(\partial^\mu \psi) + (\partial_\mu \psi^*)\{\partial^\mu(i\alpha\psi)\}$$

$$\delta(T) = -i\alpha(\partial_\mu \psi^*)(\partial^\mu \psi) + i\alpha(\partial_\mu \psi^*)(\partial_\mu \psi)$$

$$\delta(T) = 0$$

now, change in the mass-interaction term due to infinitesimal transformation ⟶

let say $\quad -m(\psi^*\psi) - \frac{\lambda}{2}(\psi^*\psi)^2 = V$

$$\delta(V) = \delta\left[-m(\psi^*\psi) - \frac{\lambda}{2}(\psi^*\psi)^2\right]$$

$$= -m\,\delta(\psi^*\psi) - \frac{\lambda}{2}\,\delta\{(\psi^*\psi)^2\}$$

$$\Rightarrow \delta(\psi^*\psi) = \psi^*\,\delta(\psi) + \{\delta(\psi^*)\}\psi$$

$$\Rightarrow \quad \delta(\psi^*\psi) = \psi^*\{i\alpha\psi\} + \{-i\alpha\psi^*\}\psi$$
$$= (i\alpha\psi^*\psi) - (i\alpha\psi^*\psi)$$
$$= 0$$

$$\Rightarrow \quad \boxed{\delta(V) = 0}$$

$$\Rightarrow \quad L = T + V$$

$$\boxed{\delta L = \delta T + \delta V = 0}$$

[note :→ I have taken the minus sign into consideration for V, $V = -m(\psi^*\psi) - \frac{\lambda}{2}(\psi^*\psi)^2$]

So the lagrangian is invariant under the global U(1) symmetry.

iii) Derivation of Noether Current using Gauge ~~...~~ :→

$$L = \partial_\mu\psi^* \, \partial^\mu\psi - m^2(\psi^*\psi) - \frac{\lambda}{2}(\psi^*\psi)^2$$

as we have seen earlier in (ii), the lagrangian is invariant under U(1) transformation.

$\psi \longrightarrow e^{i\alpha}\psi$ | $\psi^* \longrightarrow e^{-i\alpha}\psi^*$

$\Rightarrow \delta\psi = i\alpha\psi$ | $\delta\psi^* = -i\alpha\psi^*$.

→ we consider the parameter $\alpha \rightarrow \alpha(x)$

→ $\partial_\mu(\delta\psi) = i\{\partial_\mu(\alpha)\}\psi + i\alpha\,\partial_\mu\psi$

$\partial_\mu(\delta\psi^*) = -i\{\partial_\mu(\alpha)\}\psi^* - i\alpha\,\partial_\mu\psi^*$

let $T = \partial_\mu\psi^* \, \partial^\mu\psi$

$$\delta(T) = \delta[\partial_\mu\psi^* \, \partial^\mu\psi]$$
$$\delta(T) = \partial_\mu\psi^* \, \delta(\partial^\mu\psi) + \delta(\partial_\mu\psi^*) \, \partial^\mu\psi$$

⑨

$$\delta(T) = [-i(\partial_\mu \alpha)\psi^* - i\alpha\,\partial_\mu \psi^*]\,\partial^\mu \psi$$
$$+ \partial_\mu \psi^* [i(\partial^\mu \alpha)\psi + i\alpha(\partial^\mu \psi)]$$

$\Rightarrow$

$$\delta(T) = -i(\partial_\mu \alpha)\psi^* \partial^\mu \psi + i(\partial^\mu \alpha)\psi\,\partial_\mu \psi^*$$
$$-i\alpha(\partial_\mu \psi^*)(\partial^\mu \psi) + i\alpha(\partial_\mu \psi^*(\partial^\mu \psi)$$

$$\delta(T) = -i(\partial_\mu \alpha)\psi^* \partial^\mu \psi + i(\partial^\mu \alpha)\psi(\partial_\mu \psi^*)$$
$$= i[(\partial^\mu \alpha)\psi(\partial_\mu \psi^*) - (\partial_\mu \alpha)\psi^*(\partial^\mu \psi)]$$
$$= i[(\eta^{\mu\nu}\partial_\nu \alpha)\psi(\partial_\mu \psi^*) - (\partial_\mu \alpha)\psi^*(\partial^\mu \psi)]$$
$$= i[\psi(\partial_\nu \alpha)(\eta^{\mu\nu}\partial_\mu \psi^*) - (\partial_\mu \alpha)\psi^*(\partial^\mu \psi)]$$
$$= i[\psi(\partial_\nu \alpha)(\partial^\nu \psi^*) - (\partial_\mu \alpha)\psi^*(\partial^\mu \psi)]$$
$$= i[\psi(\partial_\mu \alpha)(\partial^\mu \psi^*) - (\partial_\mu \alpha)\psi^*(\partial^\mu \psi)]$$

$$\delta(T) = i[(\partial_\mu \alpha)\{\psi\,\partial^\mu \psi^* - \psi^* \partial^\mu \psi\}]$$

$$V = -m(\psi^*\psi) - \frac{\lambda}{2}(\psi^*\psi)^2$$

$\Rightarrow V \propto (\psi^*\psi)$

$\Rightarrow \quad \delta V = 0 \quad [\because \delta(\psi^*\psi) = 0]$

So, $\delta L = \delta T + \delta V = \delta T = i[(\partial_\mu \alpha)\{\psi\,\partial^\mu \psi^* - \psi^* \partial^\mu \psi\}] \rightarrow$ ⑥

So for ~~δL to be 0~~ $\boxed{\delta \mathcal{L} = (\partial_\mu \alpha)\,j^\mu} \rightarrow$ ⑦

$\Rightarrow$ Comparing Eq ⑥ and Eq ⑦

$$i(\partial_\mu \alpha)\{\psi\,\partial^\mu \psi^* - \psi^* \partial^\mu \psi\} = \{\partial_\mu(\alpha)\}\,j^\mu$$

$$\Rightarrow \boxed{j^\mu = i\{\psi\,\partial^\mu \psi^* - \psi^* \partial^\mu \psi\}}$$

Noether Current, $j^\mu = i\,[\psi\,\partial^\mu\psi^* - \psi^*\partial^\mu\psi]$

⇒ verifying that derived Noether Current is conserved

from Eq (4) and (5) →

$\Box\psi + m^2\psi + \lambda(\psi^*\psi)\psi = 0$ — (4)

$\Box\psi^* + m^2\psi^* + \lambda(\psi^*\psi)\psi^* = 0$ — (5)

~~so~~

$\Box\psi = -m^2\psi - \lambda(\psi^*\psi)\psi$ → (8)

$\Box\psi^* = -m^2\psi^* - \lambda(\psi^*\psi)\psi^*$ → (9)

$\partial_\mu j^\mu = i\,[\partial_\mu\{\psi\,\partial^\mu\psi^* - \psi^*\partial^\mu\psi\}]$

$= i\,[(\partial_\mu\psi)\partial^\mu\psi^* + \psi\,\partial_\mu\partial^\mu\psi^* - (\partial_\mu\psi^*)(\partial^\mu\psi) - \psi^*\partial_\mu\partial^\mu\psi]$

$= i\,[\psi\,\partial_\mu\partial^\mu\psi^* - \psi^*\partial_\mu\partial^\mu\psi]$

$= i\,[\psi\,\Box\psi^* - \psi^*\Box\psi]$

now from equations (8) and (9) →

$\partial_\mu j^\mu = i\,[\psi\{-m^2\psi^* - \lambda(\psi^*\psi)\psi^*\} - \psi^*\{-m^2\psi - \lambda(\psi^*\psi)\psi\}]$

$\partial_\mu j^\mu = i\,[-m^2\psi\psi^* + m^2\psi\psi^* - \lambda(\psi^*\psi)(\psi\psi^*) + \lambda(\psi^*\psi)(\psi\psi^*)]$

$$\boxed{\partial_\mu j^\mu = 0}$$

hence the Noether Current is Conserved.

Q.4) Verify that the lagrangian density

$$L = \frac{1}{2}\, \partial_\mu \phi_a\, \partial^\mu \phi_a - \frac{1}{2} m^2 \phi_a \phi_a$$

for a triplet of real field ϕ_a $(a = 1,2,3)$ is invariant under the infinitesimal $SO(3)$ rotation by θ

$$\phi_a \longrightarrow \phi_a + \theta\, \varepsilon_{abc}\, n_b \phi_c$$

where n_a is a unit vector, Compute the Noether current j^μ. Deduce that the three quantities

$$Q_a = \int d^3x\, \varepsilon_{abc}\, \dot{\phi}_b \phi_c$$

are all conserved and verify this directly using the field equations satisfied by ϕ_a.

Ans :→ $\delta \phi_a = \theta\, \varepsilon_{abc}\, n_b \phi_c$ → given

→ again Lagrangian has two terms

$$L = T - V$$

$$T = \frac{1}{2}\, \partial_\mu \phi_a\, \partial^\mu \phi_a$$

$$V = \frac{1}{2} m^2 \phi_a \phi_a$$

→ variation in kinetic term ⇒

$$\frac{1}{2}\, \delta(\partial_\mu \phi_a \partial^\mu \phi_a) = \frac{1}{2}\left[\delta(\partial_\mu \phi_a)(\partial^\mu \phi_a) + (\partial_\mu \phi_a)\, \delta(\partial^\mu \phi_a)\right]$$

$$\frac{1}{2}\left[\partial_\mu(\delta\phi_a)(\partial^\mu \phi_a) + (\partial_\mu \phi_a)\, \partial^\mu(\delta \phi_a)\right]$$

$$\delta T = \frac{1}{2}\left[\partial_\mu(\delta\phi_a)(\partial^\mu\phi_a) + (\partial_\mu \phi_a)\, \eta^{\mu\nu} \partial_\nu (\delta\phi_a)\right]$$

$$\delta T = \frac{1}{2}\left[\partial_\mu (\delta\phi_a)\, \eta^{\nu\mu} (\partial_\nu \phi_a) + (\partial_\mu \phi_a)\, \partial_\nu(\delta \phi_a)\right]$$

$$\delta T = \frac{2}{2}\, \eta^{\mu\nu} \left[\partial_\mu (\phi_a)\, \partial_\nu (\delta \phi_a)\right]$$

→ symmetry approach

let X be any quantity that on interchange of indices gives $-X$

hence, $X = -X$ [anti symmetric]

$$\boxed{X = 0}$$

hence being anti symmetric makes the whole term zero.

So $\delta T = \eta^{\mu\nu}\{\partial_\mu(\phi_a)\,\partial_\nu(\delta\phi_a)\}$

now $\partial_\nu(\delta\phi_a) = \partial_\nu(\theta\,\varepsilon_{abc}\,n_b\,\phi_c)$

$= \theta\,\varepsilon_{abc}\,n_b\,\partial_\nu(\phi_c)$

So, $\delta(T) = \eta^{\mu\nu}\,\theta\,\varepsilon_{abc}\,n_b\,\partial_\nu\phi_c\,\partial_\mu\phi_a$

$\Rightarrow a \longleftrightarrow c$, $\varepsilon_{abc} \neq \varepsilon_{cba}$ [Anti-symmetric]

$\partial_\nu\phi_c\,\partial_\mu\phi_a = \partial_\nu\phi_a\,\partial_\mu\phi_c$ [symmetric]

δT = Antisymmetric $= 0$

$V = \frac{1}{2}m^2\phi_a\phi_a$

$\delta(\phi_a\phi_a) = 2\phi_a\,\delta\phi_a = 2\theta\,\varepsilon_{abc}\,n_b\,\phi_a\phi_c$

$a \longleftrightarrow c$

$\phi_a\phi_c = \phi_c\phi_a$ [symmetric]

but $\varepsilon_{abc} \neq \varepsilon_{cba}$ [anti-sym]

$$\boxed{\delta V = 0}$$

$$\Rightarrow \boxed{\delta L = \delta T - \delta V = 0}$$

The lagrangian is invariant under $SO(3)$ rotations.

ii) Using Gauge trick to find Noether current $\rightarrow$

let consider $\theta \rightarrow \theta(x)$

$$\delta\phi_a = \theta(x)\, \varepsilon_{abc}\, n_b \phi_c$$

$$\delta \mathcal{L} = \delta(T) - \delta(V)$$

$$\delta(V) = -\delta\left(\tfrac{1}{2} m^2 \phi_a \phi_a\right) = -\tfrac{1}{2} m^2\, 2\theta(x)\, \varepsilon_{abc}\, n_b \phi_c(x) \phi_a$$

again this term is anti-symmetric

$$\delta V = 0$$

$\delta T \rightarrow$

$$\delta(\partial_\mu \phi_a \partial^\mu \phi_a) = \delta(\partial_\mu \phi_a)\{\partial^\mu \phi_a\} + \partial_\mu \phi_a \{\delta(\partial^\mu \phi_a)\} \longrightarrow (1)$$

$\rightarrow$ $\delta(\partial_\mu \phi_a) = \partial_\mu(\delta\phi_a)$

$\rightarrow$ $\delta(\partial_\mu \phi_a) = \partial_\mu [\theta(x)\, \varepsilon_{abc}\, n_b \phi_c(x)]$

$$= \partial_\mu(\theta(x))\, \varepsilon_{abc}\, n_b \phi_c(x) + \theta(x)\, \varepsilon_{abc}\, n_b \partial_\mu \phi_c(x)$$

the 1st term in Eq (1), RHS $\rightarrow$

$$\partial(\partial_\mu \phi_a)\, \partial^\mu \phi_a = [\partial_\mu(\theta(x))\, \varepsilon_{abc}\, n_b \phi_c(x)\, \partial^\mu \phi_a + \theta(x)\, \varepsilon_{abc}\, n_b \partial_\mu \phi_c(x)\, \partial^\mu \phi_a]$$

the second term $\rightarrow$ $\partial_\mu \phi_a (\delta \partial^\mu \phi_a)$

$$= \partial_\mu \phi_a [(\partial^\mu \theta)\, \varepsilon_{abc}\, n_b \phi_c + \theta\, \varepsilon_{abc}\, n_b \partial^\mu \phi_c]$$

plugging into Eq (1) we get $\rightarrow$

$$= \theta \cdot \varepsilon_{abc}\, n_b [(\partial_\mu \phi_c)(\partial^\mu \phi_a) + (\partial_\mu \phi_a)(\partial^\mu \phi_c)]$$
$$+ (\partial_\mu \theta)(\varepsilon_{abc})(n_b)[\phi_c \partial^\mu \phi_a + \phi_c \partial^\mu \phi_a]$$

the 1st term $\longrightarrow$ $\theta\, \varepsilon_{abc}\, n_b [\partial_\mu \phi_c (\partial^\mu \phi_a) + (\partial_\mu \phi_a)(\partial^\mu \phi_c)]$ is proportional to θ $\rightarrow$ it is symmetric but ε_{abc} makes it anti symmetric so it vanishes.

the 2nd term, $\quad \varepsilon_{abc} =$ Anti Symmetric

$$\rightarrow \quad \partial_a \partial^\mu \phi_c \neq \partial_c \partial^\mu \phi_a \quad \rightarrow \text{anti symmetric}$$

So $\quad (\partial_\mu \theta)\, \varepsilon_{abc}\, n_b [\phi_c \partial^\mu \phi_a + \phi_a \partial^\mu \phi_c] \rightarrow$ ~~anti~~ symmetric

So it does not vanishes.

$$\delta \mathcal{L} = \frac{2}{2}(\partial_\mu \theta)\, \varepsilon_{abc}\, n_b \phi_c \partial^\mu \phi_a$$

$$\delta \mathcal{L} = (\partial_\mu \theta)\, \varepsilon_{abc}\, n_b \phi_c (\partial^\mu \phi_a)$$

also $\quad \delta \mathcal{L} = \partial_\mu (\theta)\, j^\mu$

so $\quad j^\mu = \varepsilon_{abc}\, n_b \phi_c (\partial^\mu \phi_a)$

$$j^\mu(\vec{n}) = n_b j^\mu_b = \varepsilon_{abc}\, n_b \phi_c (\partial^\mu \phi_a)$$

$$\Rightarrow \quad j^\mu_b = \varepsilon_{abc} \phi_c (\partial^\mu \phi_a)$$

$$\Rightarrow \quad j^0_b = \varepsilon_{abc} \phi_c \dot{\phi}_a$$

$$j^0_a = \varepsilon_{abc} \phi_c \dot{\phi}_b$$

So, conserved charges are -

$$Q_a = \int d^3x\, j^0_a = \int d^3x\, \varepsilon_{abc} \phi_c \dot{\phi}_b$$

iii) Veryfing the conservation using field equation

$$\Box \phi_a + m^2 \phi_a = 0$$

$$\partial_\mu j^\mu_a = \partial_\mu (\varepsilon_{abc} \phi_c \partial^\mu \phi_b)$$

$$\partial_\mu j^\mu_a = \partial_\mu(\varepsilon_{abc}\phi_c\,\partial^\mu\phi_b) = \varepsilon_{abc}\left[(\partial_\mu\phi_c)(\partial^\mu\phi_b) + \phi_c\,\partial_\mu\partial^\mu(\phi_b)\right]$$

$$= \varepsilon_{abc}\left[(\partial_\mu\phi_c)(\partial^\mu\phi_b) + \phi_c\,\Box\phi_b\right]$$

$\varepsilon_{abc} \rightarrow$ anti symmetric

$\varepsilon_{abc}(\partial_\mu\phi_c)(\partial^\mu\phi_b) \mp \varepsilon_{abc}(\partial_\mu\phi_b)(\partial^\mu\phi_c)$ $\rightarrow$ anti symmetric

hence $\quad \partial_\mu j^\mu_a = \varepsilon_{abc}\,\phi_c\,\Box\phi_b$

$$= -m^2\phi_b \times \varepsilon_{abc}\phi_c \qquad [\because \Box\phi_b = -m^2\phi_b]$$

$$\partial_\mu j^\mu_a = -m^2\varepsilon_{abc}\,\phi_b\phi_c$$

as $\phi_b\phi_c \rightarrow$ symmetric , $\varepsilon_{abc} \rightarrow$ anti-symmetric

$$\partial_\mu j^\mu_a = \text{Anti symmetric} = 0$$

$$\boxed{\partial_\mu j^\mu_a = 0}$$

Q.5> Maxwell's Lagrangian for the electromagnetic field is $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the 4-Vector potential.

i) Show that L is invariant under gauge transformation

ii) Use Noether's theorem and Spacetime translational invariance of the action, to construct the energy momentum tensor $T^{\mu\nu}$ for the electro-magnetic field.

iii) Show that the resulting object is neither symmetric nor gauge invariant.

iv) Consider a new tensor given by
$$\Theta^{\mu\nu} = T^{\mu\nu} - (F^{\rho\mu})\partial_\rho A^\nu$$
, show that this also defines four conserved currents.

v) Show that it is symmetric, gauge invariant and traceless.

Ans :→

i) Gauge transformation $A_\mu \longrightarrow A_\mu + \partial_\mu \xi(x)$

where $\xi(x)$ is a scaler with arbitary dependence on x.

$F_{\mu\nu} \longrightarrow$ transform to $F_{\mu\nu}'$ (under Gauge transformation)

$$F_{\mu\nu}' = \partial_\mu (A_\nu + \partial_\nu \xi(x)) - \partial_\nu (A_\mu + \partial_\mu \xi(x))$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu \xi(x) - \partial_\mu \partial_\nu \xi(x)$$
$$F_{\mu\nu}' = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$$

Similarly $(F^{\mu\nu})' = F^{\mu\nu}$

Hence $\mathcal{L}' = -\frac{1}{4} (F_{\mu\nu})' (F^{\mu\nu})' = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \mathcal{L}$

∴ L remains invariant under the gauge transformation.

ii) Space time translation →

$$(x')^{\mu} = x^{\mu} + \xi^{\mu} \quad , \quad \xi^{\mu} = \text{constant vector}$$

$$A_\rho(x) = A_\rho'(x')$$

$$A_\rho'(x) = A_\rho(x - \varepsilon)$$

$$A_\rho'(x) = A_\rho(x) - \varepsilon^{\nu}\partial_\nu A_\rho(x) \quad [\text{upto first order}]$$

So, $\delta A_\rho(x) = A_\rho'(x) - A_\rho(x) = -\varepsilon^{\nu}\partial_\nu A_\rho$

So, $\delta S = \int d^4x \, \delta \mathcal{L}$

$$\delta L = \frac{\partial L}{\partial A_\rho}\delta A_\rho + \frac{\partial L}{\partial(\partial_\mu A_\rho)}\delta(\partial_\mu A_\rho)$$

from Euler-lagrange Eq → $\dfrac{\partial L}{\partial A_\rho} = \partial_\mu\left(\dfrac{\partial L}{\partial(\partial_\mu A_\rho)}\right)$

$$\delta L = \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu A_\rho)}\right)\delta A_\rho + \frac{\partial L}{\partial(\partial_\mu A_\rho)}\delta(\partial_\mu A_\rho)$$

now, $\delta(\partial_\mu A_\rho) = \partial_\mu(\delta A_\rho) = -\varepsilon^{\nu}\partial_\mu\partial_\nu A_\rho$

$$\delta L = \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu A_\rho)}\right)\delta A_\rho + \frac{\partial L}{\partial(\partial_\mu A_\rho)}(-\varepsilon^{\nu})\partial_\mu\partial_\nu A_\rho$$

$$= \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu A_\rho)}\right)(-\varepsilon)^{\nu}\partial_\nu A_\rho + \frac{\partial L}{\partial(\partial_\mu A_\rho)}\partial_\mu(-\varepsilon^{\nu}\partial_\nu A_\rho)$$

$$\delta L = \partial_\mu\left\{(-\varepsilon^{\nu})\cdot\left(\frac{\partial L}{\partial(\partial_\mu A_\rho)}\right)\partial_\nu A_\rho\right\}$$

$$\delta S = \int d^4x \, \delta L$$

$$\delta S = -\varepsilon^{\nu} \int d^4x \; \partial_\mu \left\{ \frac{\partial L}{\partial(\partial_\mu A_\rho)} \partial_\nu A_\rho \right\}$$

this change in action is due to the change in field which is due to Space time translation

but the Space-time translation itself generate a change in action $\rightarrow$

$$\delta S = -\varepsilon^{\nu} \int d^4x \; \partial_\mu \left\{ \frac{\partial L}{\partial(\partial_\mu A_\rho)} \partial_\nu A_\rho \right\}$$

$$\underbrace{- \varepsilon^{\nu} \int d^4x \; \partial_\mu \left(\delta^\mu_\nu L \right)}$$

$\hookrightarrow$ Extra term due to change in space time itself.

$$\delta S = \partial_\mu j^\mu = 0$$

$$j^\mu_\rho = \frac{\partial L}{\partial(\partial_\mu A_\nu)} \partial_\rho A_\nu - \delta^\mu_\rho L =: T^\mu_\rho$$

$$T^\mu_\rho = \frac{\partial L}{\partial(\partial_\mu A_\nu)} \partial_\rho A_\nu - \delta^\mu_\rho L$$

from E.O.M, $\frac{\partial L}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}$, and $\cancel{\delta^\mu_\rho L = \eta^{\nu\rho}}\, \delta.$

So, $\cancel{T^\mu_\rho = -F^{\mu\rho} \partial^\nu A_\rho + \frac{\eta^{\mu\nu}}{4} F_{\alpha\beta} F^{\alpha\beta}}$

$$T^{\mu\nu} = \eta^{\nu\rho} T^\mu_\rho$$

$$T^{\mu\nu} = \eta^{\nu\rho} T^{\mu}_{\rho}$$

$$\Rightarrow \quad T^{\mu\nu} = \eta^{\nu\rho} \frac{\partial L}{\partial(\partial_\mu A_\sigma)} \partial_\rho A_\sigma - \eta^{\nu\rho} \delta^{\mu}_{\rho} L$$

$$\Rightarrow T^{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu A_\sigma)} \eta^{\nu\rho}_{\rho} \partial_\rho A_\sigma - \eta^{\nu\mu} L.$$

$$\Rightarrow T^{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu A_\sigma)} \partial^\nu A_\sigma - \eta^{\nu\mu} L$$

$$\Rightarrow \boxed{T^{\mu\nu} = -F^{\mu\sigma} \partial^\nu A_\sigma + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}}$$

$T^{\mu\nu}$ is the energy momentum tensor for electromagnetic field.

ii) Gauge Invariance

there are two terms in $T^{\mu\nu}$,

$\rightarrow F_{\nu\mu} \rightarrow$ do not change under gauge transformation

$\Rightarrow \partial^\nu A_\rho \rightarrow \partial^\nu A_\rho + \partial^\nu \partial_\rho \xi$

$$\Rightarrow \delta T^{\mu\nu} = -F^{\mu\beta} \partial^\nu \partial_\beta \xi \neq 0$$

So $T^{\mu\nu}$ is not gauge invariant.

Symmetry : $\rightarrow \mu \leftrightarrow \nu$

$$T^{\nu\mu} = -F^{\nu\rho} \partial^\mu A_\rho + \frac{1}{4} \eta^{\nu\mu} F_{\alpha\beta} F^{\alpha\beta}$$

$$T^{\nu\mu} - T^{\mu\nu} = -F^{\nu\rho} \partial^\mu A_\rho + F^{\mu\rho} \partial^\nu A_\rho - \frac{1}{4} \eta^{\nu\mu} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

$$T^{\nu\mu} - T^{\mu\nu} = F^{\mu\rho} \partial^\nu A_\rho - F^{\nu\rho} \partial^\mu A_\rho \neq 0$$

Hence $T^{\mu\nu}$ is not symmetric.

iii) $\Theta^{\mu\nu} = T^{\mu\nu} - F^{\rho\mu}\partial_\rho A^\nu$

taking derivative $\rightarrow$

$$\partial_\mu[\Theta^{\mu\nu}] = \partial_\mu T^{\mu\nu} - \partial_\mu\{F^{\rho\mu}\partial_\rho A^\nu\} \quad \text{— ①}$$

$$\partial_\mu[\Theta^{\mu\nu}] = 0 - \partial_\mu F^{\rho\mu}(\partial_\rho A^\nu) - F^{\rho\mu}(\partial_\mu\partial_\rho A^\nu) \quad \text{— ②}$$

now, from Euler-lagrang Eq of motion $\rightarrow$

$$\boxed{\partial_\mu F^{\rho\mu} = 0}$$

the Second term, $F^{\rho\mu}(\partial_\mu\partial_\rho A^\nu)$

$\rho \longleftrightarrow \mu \Rightarrow F^{\rho\mu} = -F^{\mu\rho}$ [Anti-Symmetric]

$\partial_\mu\partial_\rho A^\nu = \partial_\rho\partial_\mu A^\nu$ [Symmetric]

So both the terms of Eq ② vanishes

$$\boxed{\partial_\mu[\Theta^{\mu\nu}] = 0} \quad \text{— ③}$$

So, $\Theta^{\mu\nu}$ also gives four conserved currents.

$$\Theta^{\mu\nu} = T^{\mu\nu} - F^{\rho\mu}\partial_\rho A^\nu$$
$$= -F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^{\rho\mu}\partial_\rho A^\nu$$
$$= -F^{\mu\rho}\partial^\nu A_\rho - F^{\rho\mu}\partial_\rho A^\nu + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$
$$= -F^{\mu\rho}\partial^\nu A_\rho + F^{\mu\rho}\partial_\rho A^\nu + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$
$$= -F^{\mu\rho}(\partial^\nu A_\rho - \partial_\rho A^\nu) + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

$$\Theta^{\mu\nu} = -F^{\mu\rho}(F^\nu{}_\rho) + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

$$\boxed{\Theta^{\mu\nu} = -F^{\mu\rho}F^\nu_\rho + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}}$$

iv) Gauge Invariant, Symmetric and Traceless $\rightarrow$

$$A_\mu \rightarrow A_\mu + \partial_\mu \xi$$

we have shown $F_{\mu\nu} \rightarrow F_{\mu\nu}$ under gauge transformation

$$F^{\mu\rho} \longrightarrow F^{\mu\rho} \quad ''$$

now $F^\nu{}_\rho = \eta_{\sigma\rho} F^{\nu\sigma}$

So, $\eta_{\sigma\rho}$ does not change under gauge transformation

$\Rightarrow$ $F^\nu{}_\rho \rightarrow F^\nu{}_\rho$ under gauge transformation

So, $\Theta^{\mu\nu} = -F^{\mu\rho} F^\nu{}_\rho + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$

- Every term in the above equation is gauge invariant

$\Theta^{\mu\nu}$ is invariant under gauge transformation.

Symmetry :$\rightarrow$

$$\Theta^{\mu\nu} = -F^{\mu\rho} F^\nu{}_\rho + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

the 2nd term is Symmetric.

$$F^{\mu\rho} F^\nu{}_\rho = F^{\mu\rho} \eta_{\rho\sigma} F^{\nu\sigma} = (X)^{\mu\nu}$$

$$(X)^{\mu\nu} = (F^{\mu\rho} \eta_{\rho\sigma}) F^{\nu\sigma} = (F^\mu{}_\sigma)(F^{\nu\sigma})$$

$$(X)^{\mu\nu} = (F \cdot F^T)^{\mu\nu}$$

we know $F_{\mu\nu} = -F_{\nu\mu}$

$$F^T = -F$$

$\Rightarrow$

$$F \cdot F^T = -F^2$$

$$(\chi^{\mu\nu}) = -(F^2)^{\mu\nu} = -(F^2)^{\nu\mu} = (\chi)^{\nu\mu}$$

$$F^{\mu\rho} F^{\nu}_{\ \rho} = F^{\nu\rho} F^{\mu}_{\ \rho}$$

So both the terms in $\Theta^{\mu\nu}$ are Symmetric

hence $\Theta^{\mu\nu}$ is Symmetric.

Traceless $:\longrightarrow$ Trace $= \Theta^{\mu}_{\ \mu}$

$$\Theta^{\mu}_{\ \mu} = \eta_{\mu\nu}\, \Theta^{\mu\nu}$$

$$= -\eta_{\mu\nu}\, F^{\mu\rho} F^{\nu}_{\ \rho} + \frac{1}{4}\eta_{\mu\nu}\,\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

$$= -\eta_{\mu\nu} F^{\mu\rho} F^{\nu}_{\ \rho} + \frac{1}{4} \times 4\, F_{\alpha\beta} F^{\alpha\beta}$$

$$= -F_{\nu}^{\ \rho} F^{\nu}_{\ \rho} + F_{\alpha\beta} F^{\alpha\beta}$$

$$= -(\eta^{\sigma\rho} F_{\nu\sigma})(\eta_{\rho\lambda} F^{\nu\lambda}) + F_{\alpha\beta} F^{\alpha\beta}$$

$$= -\eta^{\sigma\rho}\eta_{\rho\lambda} F_{\nu\sigma} F^{\nu\lambda} + F_{\alpha\beta} F^{\alpha\beta}$$

$$= -F_{\nu\lambda} F^{\nu\lambda} + F_{\alpha\beta} F^{\alpha\beta} \qquad \left[\because \eta^{\sigma\rho}\eta_{\rho\lambda} = \delta^{\sigma}_{\lambda}\right]$$

$$\Rightarrow \delta^{\sigma}_{\lambda} = 1 \text{ for } \sigma = \lambda$$

$$\Theta^{\mu}_{\ \mu} = 0 \Rightarrow [\text{Trace} = 0]$$

Hence $\Theta^{\mu\nu}$ is traceless.