Assignment - Selected Solved Problems
Topic :- Topology and Differential Geometry
Solved by - Puja Mandal {Self-Study}

Q.1) Write Maxwell's action / Lagrangian in terms of differential forms.

Solution : The Lagrangian density $\mathcal{L}$ is an n-form on an n-dimensional spacetime (4-form in 4 dimensional spacetime) ensuring that it's integral over the manifold (the action) is a Scaler quantity which is essential for Lorentz invariance. The free field term in the Lagrangian density describes the dynamic of the electro-magnetic fields in the absence of sources, representing the energy stored within the field. The Lagrangian density for free field in differential form is given by

$$\mathcal{L}_{free} = \frac{-1}{2} F \wedge {}^* F$$

where $F = dA$, the two form Maxwell's field strength

$$\mathcal{L}_{free} = \frac{-1}{2} dA \wedge {}^* dA$$

the factor of $\frac{1}{2}$ is crucial to get correct field equations upon variation.

The action, $S[A] = \int_M \left(\frac{-1}{2} F \wedge {}^* F\right)$

this integration is to be performed ~~on~~ over the 4-D Space-time manifold M.

Q.2) Discuss the following terms briefly :-
i) Differential Forms ii) Exterior Derivative iii) Hodge Star Operator iv) Electromagnetic 2-form v) four current vi) Manifold vii) Orientable Manifold viii) Action Integral ix) Isomorphism x) Homeomorphism.

Solution →

i) Differential Forms - Mathematical objects that generalize scalar functions, vector fields and other quantities encountered in vector calculus.

ii) Exterior Derivative - $(d) \rightarrow$ An operator that maps a n-form to an $(n+1)$-form. It generalizes the gradient (for 0-form), curl (for 1-forms in $\mathbb{R}^3$) and divergence (for 2-forms in $\mathbb{R}^3$).

iii) Hodge Star Operator $(*)$: An operator that maps an n-form to an $(N-n)$-form in an N dimensional manifold. It requires a metric tensor for it's definition.

iv) Electromagnetic 2-form = A differential 2-form in spacetime whose components are elements of electromagnetic field tensor $F_{\mu\nu}$.

v) four current - A spacetime four vector field representing the charge and current densities.

vi) Manifold - A differentiable manifold is a topological space that locally resemble Euclidean space

vii) Orientable Manifold - A n-dimensional diffentiable manifold is orientable if there exist non-vanishing n-form, called a volume element on the manifold

viii) Action Integral – A Scalar Quantity that represent the total dynamics of a physical system.

ix) Isomorphism – Structure-preserving bijective map between two mathematical objects with a structure preserving inverse.

x) Homeomorphism – A homeomorphism is a bijective, continuous map with continuous inverse between topological Spaces

Q.3)

a) Show that $\{(a,\infty) \mid a \in \mathbb{R}\} \cup \{\phi, \mathbb{R}\}$ is a topology on $\mathbb{R}$

b) Define the collection $C = \{(a,\infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\}$

Show that the collection of finite intersection of sets in C forms a basis that generates the standard topology on $\mathbb{R}$.

Solution –

a) definition of Topology :- A collection $\mathcal{T}$ of subsets of $\mathbb{R}$ is a topology if

1. $\phi, \mathbb{R} \in \mathcal{T}$
2. Arbitary unions of sets in $\mathcal{T}$ are in $\mathcal{T}$
3. Finite intersections of sets in $\mathcal{T}$ are in $\mathcal{T}$.

by definition $\phi \in \mathcal{T}$, $\mathbb{R} \in \mathcal{T}$

hence condition 1 is satisfied

$\Rightarrow$ let $\{(a_i, \infty)\}_{i \in I} \subset \mathcal{T}$

then $\bigcup\limits_{i \in I} (a_i, \infty) = (\inf\limits_i a_i, \infty)$

$\rightarrow$ If $\inf_i a_i = -\infty$, union $= \mathbb{R}$

otherwise union $= (a, \infty)$ for $a \in \mathbb{R}$

thus, $\bigcup_{i \in I} (a_i, \infty) \in T$

$\therefore$ condition 2 is also satisfied

$\rightarrow$ $(a, \infty) \cap (b, \infty) = (\max\{a, b\}, \infty)$

this is again of the form (c, ∞)

condition 3 is satisfied.

Hence T is a topology on $\mathbb{R}$ and this is called ray topology.

b) $C = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\}$

we have to show that finite intersections of sets in C form a basis for the standard topology.

$\Rightarrow$ $(a, \infty) \cap (-\infty, b) = (a, b)$

$\Rightarrow$ $(a_1, \infty) \cap (a_2, \infty) = (\max\{a_1, a_2\}, \infty)$

$\Rightarrow$ $(-\infty, b_1) \cap (-\infty, b_2) = (-\infty, \min\{b_1, b_2\})$

So finite intersections given, (a, b) (a, ∞) $(-\infty, b)$.

now a collection $\mathcal{B}$ is a basis if

$\rightarrow$ for every $x \in \mathbb{R}$ there is $B \in \mathcal{B}$ with $x \in B$

$\rightarrow$ for any $x \in B_1 \cap B_2$ there exist $B_3 \subset B_1 \cap B_2$ with $x \in B_3$.

$\rightarrow$ but in topology, every open interval (a, b) appears and any open set in standard topology is a union of open interval.

Thus finite intersections of C generate exactly the standard topology.

Q. 4. Give an example of a non-metrizable topological space. Explain.

Sol:- An uncountable set with the co-countable topology

Let X be an uncountable set. Define a topology $\mathcal{T}$ on X by $\rightarrow$ $\mathcal{T} = \{\emptyset\} \cup \{U \subset X : X \setminus U \text{ is countable}\}$

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$ (since $X \setminus X = \emptyset$, which is countable)

2. Arbitary Unions :- if $U_i \in \mathcal{T}$, then $X \setminus \bigcup_i U_i = \bigcap_i (X \setminus U_i)$

3. Finite Intersections :- $X \setminus (U_1 \cap U_2) = (X \setminus U_1) \cup (X \setminus U_2)$

So $(X, \mathcal{T})$ is a topological Space

$\rightarrow$ In co-countable topology, every nonempty open set is uncountable, hence no-point can have countable local base therefore ~~every me~~ the space is not first countable hence it can not be metrizable, as every metric Space has to be first countable.

Q. 5. Introduce a topology on $\mathbb{N}$ by declaring that open sets are $\emptyset$, $\mathbb{N}$, and all sets that can be represented as unions of (infinite) arithmetic progressions. check that this is indeed a topological space. Prove that any finite set is closed. Is it true that any closed set is finite.

Solution :→ let τ be the collection

i) Empty set and whole space

→ $\emptyset \in \tau$

→ $\mathbb{N} \in \tau$

ii) Arbitory Unions

- A Union of unions of arithmatic progressions is still a union of arithmatic progressions. So arbitory union stays in τ

iii) Finite Intersection -
The intersection of two infinite arithmetic progression is either

→ Another infinite arithmatic progression

→ finite or empty

finite intersections of open sets are therefore unions of arithmatic progressions (possibly empty)

⇒ Hence τ is a topology on $\mathbb{N}$

▨ Let $F \subset \mathbb{N}$ be finite

then $\mathbb{N} \setminus F$ is cofinite, hence finite

for each $n \in F$, the set

→ $\mathbb{N} \setminus \{n\}$

Thus $\mathbb{N} \setminus F = \bigcap_{n \in F} (\mathbb{N} \setminus \{n\})$

Therefore, F is closed

hence any finite set is closed.

▨ Is every closed set finite ?

$C = \{1, 2, 3 \dots 100\} \cup \{200, 300, 400 \dots\}$

→ Its complement is a union of infinite arithmatic progresssions

→ Hence C is closed but not finite.

Q.3 Let $X = C([0,1], \mathbb{R})$

$\|f\| = \max_{x \in [0,1]} |f(x)|$

Define $g : [0,1] \longrightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} 0, & x \in [0, \frac{1}{2}), \\ 1, & x \in [\frac{1}{2}, 1] \end{cases}$$

Define a linear map $T : X \longrightarrow X$

$T(f) = (g)(f) = gf$

Show that T is continuous and find its norm.

Ans:- → $f \in C([0,1])$

→ g is bounded on $[0,1]$

the product gf is bounded and defined pointwise

Continuity of g is not required for boundedness in

sup norm.

Hence $T(f) \in X$, so T is well defined.

linearity of $T \Rightarrow$

for $f, h \in X$ and $\alpha \in \mathbb{R}$

$T(f+h) = g(f+h) = gf + gh = T(f) + T(h$

Thus T is linear

computing norm $\Rightarrow$

$\|T(f)\| = \|gf\| = \max_{x \in [0,1]} |g(x) f(x)|$

since $|g(x)| \leq 1$ for all x

$\|T(f)\| \leq |f(x)|$

$\|T(f)\| \leq \|f\|$

therefore $\|T(f)\| \leq 1 \cdot \|f\|$

So T is bounded and hence continuous.

Q. 6) i) Let S be the surface of the unit sphere

$$x^2+y^2+z^2=1$$

oriented with the outward normal. Compute

$$\iint_S z\,dx \wedge dy$$

ii) let S be same surface as mentioned, $\gamma = xy\,dz$

compute $\iint d\gamma$

Solution :- i) Consider the reflection $(x,y,z) \mapsto (x,y,-z)$

the unit sphere S is invariant under this map

as it contains 'z^2'

$\rightarrow dx \wedge dy$ also remains unchanged

Hence the 2 form $z\,dx \wedge dy$ changes sign :

$$z\,dx \wedge dy \longmapsto -z\,dx \wedge dy$$

therefore, the integrand is odd with respect to reflection across x-y plane, while the domain of integration is symmetric. The contributions from upper and lower hemispheres cancel,

$$\iint_S z\,dx \wedge dy = 0$$

ii) Since γ is a 1-form, $d\gamma$ is two form. the sphere S is a closed oriented surface, hence

$$\partial S = \phi$$

by Stoke's theorem, $\iint_S d\gamma = \int_{\partial S} \gamma$

Because $\partial S = \phi \Rightarrow \int_{\partial S} \gamma = 0$

therefore $\boxed{\iint_S d\gamma = 0}$

Q.7. Discuss Maxwell's equations using differential forms:

a) without Source

b) with Source

Solution :-

The electromagnetic fields are represented by 1-form $A(x) = A_\mu(x)\, dx^\mu$. The field strength is simply a 2-form

$$F(x) = dA(x) = \frac{1}{2} F_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu$$

where $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$

Since, $d^2 = 0$ we get $\rightarrow dF = d^2 A = 0$

$$\Rightarrow dF = F_{\mu\nu,\lambda}\, dx^\lambda \wedge dx^\mu \wedge dx^\nu = 0$$

$\Rightarrow$ which we get as, $F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0$

by total antisymmetry of the wedge product. This is called the Bianchi Identity.

Now we introduce the Minkowski metric of spacetime, the above question can be written as

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0$$

$$F_{0i,j} + F_{ij,0} + F_{j0,i} = 0$$

now, defining the magnetic and electric fields by

$$B^i = \frac{1}{2} \epsilon^{ijk} F_{jk}$$

$$E^i = -F_{0i}$$

Then the Bianchi Identity becomes

$$\boxed{\partial_i B^i = \vec{\nabla} \cdot \vec{B} = 0} \qquad \text{--- (1)}$$

$$\partial_i E_j - \partial_j E_i = -\epsilon_{ijk} \frac{\partial B^k}{\partial t}$$

$$\Rightarrow \boxed{\nabla \times \vec{E} = -\frac{\partial B}{\partial t}} \qquad \text{--- (2)}$$

thus Bianchi Identity $d^2A = dF = 0$ is equivalent to two of the Maxwell's equation in empty space ~~that is~~ without source.

To define rest of the Maxwell Equation we first define 4-dimensional anti-symmetric tensor on $\mathbb{R}^4$ as $\epsilon_{\mu\nu\lambda\rho}$, which takes values $+1, -1, 0$.

$$\epsilon_{\mu\nu\lambda\rho} = \begin{cases} +1 & \{\mu,\nu,\lambda,\rho\} = \text{even permutation of } \{0,1,2,3\} \\ -1 & \{\mu,\nu,\lambda,\rho\} = \text{odd permutation of } \{0,1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

We can think of this tensor as forming the components of the ~~4-vectors~~ 4-form:

$$a = \frac{1}{4!}\epsilon_{\mu\nu\lambda\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho \quad \text{— (3)}$$

Now, we can use it to define the dual of any arbitary form. For a 1-form $\boxed{a = a_\mu dx^\mu}$ — (4) we define it's dual 3-form $\boxed{{}^*a = \frac{1}{3!}\, {}^*a_{\mu\nu\lambda}\, dx^\mu \wedge dx^\nu \wedge dx^\lambda}$ — (5)

$$\Rightarrow \quad a^*_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda\rho}\, a^\rho \quad \text{— (6)}$$

Similarly the dual of a 2-form $b = \frac{1}{2} b_{\mu\nu} \cdot dx^\mu \wedge dx^\nu$ is the two form

$$({}^*b) = \frac{1}{2} \cdot {}^*b_{\mu\nu}\, dx^\mu \wedge dx^\nu \quad \text{— (7)}$$

and

$$ {}^*b_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\, b^{\lambda\rho} \quad \text{— (8)}$$

in general, the dual of an n-form is a $(4-n)$ form and taking the dual of any form twice gives the original form back.

$$\Rightarrow \quad \boxed{*(*a) = a} \quad \text{for any } n\text{-form } a, \quad \text{— (9)}$$

Considering, $^{*}F$ where F is the Maxwell field strength 2-form.

$$d(^{*}F) = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} \partial_{\alpha} F^{\lambda\rho} \, dx^{\alpha} \wedge dx^{\mu} \wedge dx^{\nu} \quad \text{— (10)}$$

Setting this to zero and taking dual of the ~~qu~~ eq →

$$d(^{*}F) = 0 = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} \partial_{\alpha} F^{\lambda\rho} dx^{\alpha} \wedge dx^{\mu} \wedge dx^{\nu} = 0$$

$$\Rightarrow \quad \epsilon^{\sigma\mu\nu\alpha} \epsilon_{\nu\nu\lambda\rho} \partial_{\sigma} F^{\lambda\rho} dx^{\alpha} = 0$$

⇒ after contracting the idices →

$$\boxed{\partial^{\mu} F_{\mu\nu} = 0} \quad \text{— (11)}$$

$$\Rightarrow \quad \boxed{\vec{\nabla} \cdot \vec{E} = 0} \qquad \boxed{\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t}}$$

these are rest of Max-well Eq in empty space with source.

thus, differential forms of the free space Maxwell's Eq →

$$\boxed{\begin{aligned} dF &= 0 \\ d(^{*}F) &= 0 \end{aligned}}$$