Two-Dimensional Conformal Field Theory: A Self-Study with the Free Boson

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Abstract

These notes are a self-contained introduction to quantum field theory basics and to the simplest two-dimensional conformal field theory: the free massless boson. Starting from internal and spacetime symmetries and Noether's theorem, I review the path integral formulation in quantum mechanics and in quantum field theory, including generating functionals and the free scalar propagator. I then move to two dimensions, introduce light-cone and complex coordinates, and formulate the free boson on the cylinder and the plane. The final part develops the stress tensor, operator product expansions, the Virasoro algebra, two-point functions of primary fields, and a first conformal Ward identity. The goal of this document is purely pedagogical: to organise and understand standard material from modern QFT and CFT courses, rather than to present new results.

Motivation

Learning is, in my view, the central part of doing physics. I fully understand the importance of formal coursework and supervised projects, and this document does not claim to replace them. At the same time, I did not want to stop exploring subjects that fascinate me just because I did not yet have formal courses or an official thesis project in those topics.

We are fortunate to live in a time when high-quality lecture notes and textbooks from leading universities are freely available. This makes it possible, with enough effort and care, to study advanced subjects from written resources such as books and lecture notes, even outside a formal program. These notes are my attempt to do exactly that: to learn quantum field theory and two-dimensional conformal field theory step by step, by carefully following standard references and working through the derivations myself.

My research interests lie in understanding the theoretical structures that govern our universe. I have already followed courses on quantum mechanics, general relativity, and cosmology, and I see quantum field theory and conformal field theory as essential pieces that complete my basic theoretical prerequisites. Writing these notes has been a way to build that foundation as solidly as I can at this stage.

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I would like to express my deep gratitude to the universities and individual lecturers who have made their lecture notes publicly available. Without this open access to high-quality material, it would not have been possible for me to study quantum field theory and conformal field theory at this level on my own.

In particular, I would like to thank ETH Zürich and Professor Niklas Beisert. His lecture notes on Quantum Field Theory I and II have been central to my understanding of QFT. I am especially grateful that he kindly changed the licence and granted me access to notes that were not originally public. These notes have shaped much of my approach to the subject and form the backbone of my understanding of both QFT1 and QFT2.

I also wish to thank Professor David Tong from the University of Cambridge. His lecture notes on Quantum Field Theory and String Theory are exceptionally clear and pedagogical. They have been a constant reference and inspiration while I was working through many of the topics covered in this document.

Beyond these, I am thankful to all the authors of lecture notes and textbooks whose work I have used and cited throughout these notes: the authors of various ETH courses, and the many lecturers who have taken the time to write and share detailed material on QFT, CFT, and string theory. Their generosity in making advanced topics accessible to students all over the world is what made this project possible.

Any errors, gaps, or misunderstandings in these notes are entirely my own and not the responsibility of any of the sources or teachers I have relied on.

Contents

1 Quantum Field Theory Basics

1.1 Symmetries

1.1.1 Internal Symmetries

We begin by discussing internal symmetries, namely transformations that act on the fields without referring to spacetime coordinates. Such symmetries play a central role in quantum field theory, especially in interacting theories.

Consider a complex scalar field $\phi(x)$ described by the Lagrangian density

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \tag{1}$$

Treating ϕ and ϕ^* as independent fields, the Euler–Lagrange equations yield

$$(\Box + m^2)\phi = 0, \qquad (\Box + m^2)\phi^* = 0, \tag{2}$$

where $\Box \equiv \partial_\mu \partial^\mu$ is the d'Alembertian operator.

We now consider the following global transformation of the fields:

$$\phi(x) \to \phi'(x) = e^{i\alpha}\phi(x), \qquad \phi^*(x) \to \phi'^*(x) = e^{-i\alpha}\phi^*(x), \tag{3}$$

where α is a real constant. Since this transformation does not depend on spacetime coordinates and acts only in field space, it represents a global internal symmetry.

Applying the Klein–Gordon operator to the transformed field, we obtain

$$(\Box + m^2)\phi' = (\Box + m^2)(e^{i\alpha}\phi) = e^{i\alpha}(\Box + m^2)\phi. \tag{4}$$

Because ϕ satisfies the equation of motion, it follows immediately that

$$(\Box + m^2)\phi' = 0. \tag{5}$$

Hence, the transformation maps solutions of the equations of motion to other solutions.

We now verify that the Lagrangian is invariant under this transformation. The derivatives of the fields transform as

$$\partial_\mu \phi' = e^{i\alpha}\partial_\mu \phi, \qquad \partial_\mu \phi'^* = e^{-i\alpha}\partial_\mu \phi^*. \tag{6}$$

The kinetic term therefore transforms as

$$\partial_\mu \phi'^* \partial^\mu \phi' = e^{-i\alpha} e^{i\alpha} \partial_\mu \phi^* \partial^\mu \phi = \partial_\mu \phi^* \partial^\mu \phi, \tag{7}$$

while the mass term transforms according to

$$\phi'^* \phi' = e^{-i\alpha} e^{i\alpha} \phi^* \phi = \phi^* \phi. \tag{8}$$

Consequently, the Lagrangian density is invariant:

$$\mathcal{L}(\phi', \partial_\mu \phi') = \mathcal{L}(\phi, \partial_\mu \phi). \tag{9}$$

Since the action is defined as

$$S[\phi] = \int d^4x\, \mathcal{L}, \tag{10}$$

the invariance of the Lagrangian implies

$$S[\phi'] = S[\phi]. \tag{11}$$

We therefore conclude that the theory possesses a global U(1) internal symmetry[2].

1.1.2 Noether's Theorem and Conserved Currents

Noether's theorem establishes a fundamental connection between continuous symmetries and conservation laws. Specifically, every continuous global symmetry of the action gives rise to a conserved current and, consequently, a conserved charge for solutions of the equations of motion.

To derive this result, consider a field configuration $\phi(x)$ that satisfies the equations of motion. The variation of the Lagrangian density under an arbitrary infinitesimal variation $\delta\phi$ is given by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu\delta\phi. \tag{12}$$

Using the equations of motion,

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right),$$

the variation of the Lagrangian evaluated on-shell can be rewritten as a total derivative:

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi\right). \tag{13}$$

Now suppose that $\delta\phi$ arises from an infinitesimal continuous symmetry transformation parameterized by α. Since the action is invariant under this symmetry, its variation must vanish,

$$\delta S = 0.$$

This implies that the corresponding change in the Lagrangian density can differ from zero only by a total derivative:

$$\delta\mathcal{L} = \alpha\,\partial_\mu K^\mu, \tag{14}$$

where K^μ is some function of the fields and their derivatives.

Comparing the two expressions for $\delta\mathcal{L}$, we identify the Noether current associated with the symmetry:

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\delta\alpha} - K^\mu. \tag{15}$$

Taking the divergence of this current and using the equations of motion, we find

$$\partial_\mu J^\mu = 0, \tag{16}$$

showing that the current is conserved for all classical solutions of the theory.

A conserved current naturally leads to a conserved charge, defined as

$$Q(t) = \int d^d x \, J^0(t, \vec{x}). \tag{17}$$

Assuming that the current vanishes sufficiently rapidly at spatial infinity, the time derivative of the charge is

$$\dot{Q} = \int d^d x \, \partial_0 J^0 = -\int d^d x \, \partial_i J^i = 0. \tag{18}$$

Hence, the charge Q is conserved in time. This completes the derivation of Noether's theorem, demonstrating that continuous global symmetries of the action give rise to conserved currents and charges. An important consequence of Noether's theorem is that the conserved charge associated with a continuous symmetry acts as the generator of the corresponding infinitesimal transformation. This relationship is most transparently expressed in the Hamiltonian formulation of the theory.

Let Q denote the conserved charge associated with a continuous symmetry parameterized by α, and let F be any functional of the canonical fields and their conjugate momenta. The infinitesimal variation of F under the symmetry transformation is generated through the Poisson bracket with Q:

$$\{Q, F\} = -\frac{\delta F}{\delta \alpha}. \tag{19}$$

This relation follows directly from the defining properties of Poisson brackets and reflects the role of Q as the generator of symmetry transformations in phase space.

In particular, when F is chosen to be a field variable, this expression reproduces the infinitesimal symmetry transformation of the field. Thus, conserved charges not only encode conservation laws but also determine how fields transform under the associated continuous symmetries[2].

1.1.3 Symmetry Generators

Continuous spacetime symmetries of the action give rise to conserved currents and charges. These conserved quantities act as generators of the corresponding symmetry transformations.

Energy–Momentum Tensor. Under spacetime translations $x^\mu \to x^\mu + a^\mu$, Noether's theorem yields the conserved energy–momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}, \tag{20}$$

which satisfies the conservation law

$$\partial_\mu T^{\mu\nu} = 0. \tag{21}$$

Energy and Momentum. The associated conserved four-momentum is

$$P^\nu = \int d^d x\, T^{0\nu}. \tag{22}$$

In particular,

$$H = P^0 = \int d^d x\, T^{00}, \qquad P^i = \int d^d x\, T^{0i}. \tag{23}$$

Lorentz Transformations. Infinitesimal Lorentz transformations are given by

$$x^\mu \to x^\mu + \omega^\mu_{\ \nu} x^\nu, \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{24}$$

The corresponding conserved current defines the angular momentum tensor

$$J^{\mu\nu\rho} = x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}, \tag{25}$$

with conservation law

$$\partial_\rho J^{\mu\nu\rho} = 0. \tag{26}$$

The conserved Lorentz generators are

$$M^{\mu\nu} = \int d^d x \left(x^\mu T^{\nu 0} - x^\nu T^{\mu 0} \right). \tag{27}$$

Here, M^{ij} generate spatial rotations, while M^{0i} generate Lorentz boosts.

Generators in Quantum Theory. Upon quantization, conserved charges become operators that generate symmetry transformations:

$$\delta F = i[Q, F], \qquad U(\alpha) = e^{-i\alpha Q}. \tag{28}$$

For spacetime translations,

$$U(a) = e^{-ia_\mu P^\mu}. \tag{29}$$

The operators P^μ and $M^{\mu\nu}$ satisfy the Poincaré algebra, which encodes the full spacetime symmetry structure of relativistic quantum field theory[2].

1.1.4 Poincaré Symmetry

In the previous sections, we identified the generators of spacetime translations and Lorentz transformations, namely the momentum operators P^μ and the Lorentz generators $M^{\mu\nu}$, for a relativistic scalar field. We now briefly summarize the algebraic structures underlying these symmetries[2].

Basic Definitions. A *group* G is a set equipped with an associative composition law $G \times G \to G$, together with an identity element and inverses.

An *algebra* A is a vector space endowed with a bilinear product $A \otimes A \to A$.

A *Lie group* is a group that also has the structure of a smooth manifold. Continuous symmetries in physics are described by Lie groups.

The associated *Lie algebra* $\mathfrak{g}$ consists of the generators of infinitesimal transformations and is equipped with an antisymmetric bracket $[\cdot, \cdot]$ satisfying the Jacobi identity

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0. \tag{30}$$

The Lie algebra can be identified with the tangent space to the Lie group at the identity.

In quantum field theory, symmetry generators are represented by operators acting on a Hilbert space. Their algebra is defined through commutators,

$$[a, b] = ab - ba,$$

which automatically satisfy the Jacobi identity, thereby realizing a Lie algebra structure.

A *representation* of a group or algebra is a map $R : X \to \mathrm{End}(V)$ from the abstract symmetry generators to linear operators acting on a vector space V, such that the algebraic relations are preserved. In particular, for a Lie algebra,

$$[a, b] = c \quad \Rightarrow \quad [R(a), R(b)] = R(c).$$

In physics, it is common to use the same notation for abstract generators and their operator representations, and to identify Lie brackets with operator commutators when working in the quantum theory.

Representations and Representation Spaces. A representation specifies not only the operator realization of symmetry generators but also the vector space on which these operators act. In quantum field theory, this space is typically a Hilbert space of states or a space of fields.

Unitary Groups. An important class of Lie groups in quantum field theory is given by the unitary groups $\mathrm{U}(N)$, consisting of unitary $N \times N$ matrices. These groups frequently appear as global and gauge symmetries. The matrices themselves define the fundamental representation. The special unitary subgroup $\mathrm{SU}(N)$ is defined by the additional condition of unit determinant. The corresponding Lie algebra $\mathfrak{u}(N)$ is spanned by Hermitian matrices, while the algebra $\mathfrak{su}(N)$ consists of traceless Hermitian matrices.

Orthogonal Groups. Another important class of Lie groups is formed by the orthogonal groups $\mathrm{SO}(N)$, which describe rotations in N-dimensional real space. More generally, the groups $\mathrm{SO}(N, M)$ represent rotations in a space with signature (N, M). In particular, $\mathrm{SO}(d, 1)$ is the Lorentz group of $(d + 1)$-dimensional spacetime. The associated Lie algebras $\mathfrak{so}(N)$ and $\mathfrak{so}(N, M)$ are generated by antisymmetric matrices with respect to the corresponding metric.

Double Cover and Spin Groups. The orthogonal groups $\mathrm{SO}(N, M)$ admit a double cover known as $\mathrm{Spin}(N, M)$. In this covering group, a 2π rotation

corresponds to a non-trivial central element. As a result, vector representations are single-valued, while spinor representations acquire a minus sign under a 2π rotation. Although the global topology differs, both groups share the same local Lie algebra $\mathfrak{so}(N, M)$.

Poincaré Algebra. The generators of spacetime translations P^μ and Lorentz transformations $M^{\mu\nu}$ satisfy the commutation relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\big(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho}\big), \tag{31}$$

$$[M^{\mu\nu}, P^\rho] = i\big(\eta^{\nu\rho}P^\mu - \eta^{\mu\rho}P^\nu\big), \tag{32}$$

$$[P^\mu, P^\nu] = 0. \tag{33}$$

The operators $M^{\mu\nu}$ generate the Lorentz algebra $\mathfrak{so}(d, 1)$, while the spatial components M^{ij} generate the rotation algebra $\mathfrak{so}(d)$.

Poincaré Group. The Poincaré group is obtained by exponentiating the algebra,

$$U(\omega, a) = \exp\left(\frac{i}{2}\,\omega_{\mu\nu}M^{\mu\nu} + ia_\mu P^\mu\right). \tag{34}$$

This expression represents the component of the Poincaré group connected to the identity. It consists of spacetime translations together with the proper orthochronous Lorentz transformations, whose double cover is $\mathrm{Spin}^+(d, 1)$[2].

1.1.5 Poincaré Representations

In the previous sections we constructed the Poincaré generators P^μ and $M^{\mu\nu}$ and studied their action on fields. We now reverse the perspective: instead of starting from fields, we begin with the abstract Poincaré algebra and use its unitary irreducible representations to define relativistic particles.

The physical states of a relativistic quantum theory furnish unitary irreducible representations (UIRs) of the Poincaré group. This classification, originally developed by Wigner, determines the possible types of elementary particles in relativistic quantum field theory [**?**, **?**].

Casimir Operators. The classification is based on the Casimir operators of the Poincaré algebra. The first invariant is

$$P^2 = P_\mu P^\mu\,, \tag{35}$$

whose eigenvalue defines the mass,

$$P^2 p = m^2 p\,. \tag{36}$$

The second invariant is constructed from the Pauli–Lubanski vector,

$$W^\mu = \frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}M_{\nu\rho}P_\sigma\,, \tag{37}$$

which satisfies

$$W_\mu W^\mu = -\,m^2 s(s+1) \tag{38}$$

for massive representations. The eigenvalue s corresponds to the intrinsic spin of the particle.

Massive Representations. For $m^2 > 0$, one may go to the rest frame $p^\mu = (m, \mathbf{0})$. The little group leaving this momentum invariant is SO(3) (or its double cover SU(2)). Therefore, massive particles are classified by mass m and spin s, corresponding to finite-dimensional representations of SU(2).

Massless Representations. For $m = 0$, there is no rest frame. Choosing a reference momentum $p^\mu = (E, 0, 0, E)$, the little group becomes ISO(2). Physical massless particles are characterized by helicity λ, corresponding to one-dimensional representations of the rotation subgroup SO(2).

Thus, unitary irreducible representations of the Poincaré group are labelled by

$$(m, s) \quad \text{for massive particles,} \qquad (m = 0, \lambda) \quad \text{for massless particles.}$$

These representations provide the foundation for constructing quantum fields with definite mass and spin in relativistic quantum field theory[2].

1.2 Free Spinor Field

In addition to scalar fields, relativistic quantum field theory admits massive representations with intrinsic spin. The simplest non-trivial case corresponds to spin $j = \frac{1}{2}$, described by the Dirac field. As a field with half-integer spin, it obeys Fermi–Dirac statistics[2].

1.2.1 Dirac Equation and Clifford Algebra

Dirac Equation. To construct a relativistic first-order wave equation consistent with special relativity, Dirac proposed

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \tag{39}$$

This equation can be viewed as a factorization of the Klein–Gordon operator. Acting with $(i\gamma^\nu \partial_\nu + m)$ gives

$$(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi = 0. \tag{40}$$

For consistency with the Klein–Gordon equation, the gamma matrices must satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}. \tag{41}$$

Under this condition, every solution of the Dirac equation also satisfies

$$(\Box + m^2)\psi = 0,$$

ensuring that the field describes particles of mass m.

Clifford Algebra. The Clifford algebra cannot be realized by ordinary numbers; it requires matrix representations. In four-dimensional Minkowski spacetime, the gamma matrices are 4×4 matrices, and their explicit form depends on the chosen representation. The structure of the Clifford algebra is determined by the spacetime dimension and signature.

1.2.2 Clifford Algebra Representations

The simplest non-trivial realization of a Clifford algebra arises in three-dimensional Euclidean space. In this case, the algebra is represented by the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{42}$$

These matrices satisfy the multiplication rule

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k, \tag{43}$$

which implies the three-dimensional Clifford algebra

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}. \tag{44}$$

In relativistic quantum field theory, we are primarily interested in $(3+1)$-dimensional spacetime. In this case, the smallest non-trivial representation of the Clifford algebra is four-dimensional. The corresponding elements of the representation space are called Dirac spinors (or four-spinors)[2, 17].

There exist several equivalent matrix representations of the four-dimensional Clifford algebra, including the Dirac, Weyl, and Majorana representations. In this work, we adopt the Weyl representation, in which the gamma matrices take the block form

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}. \tag{45}$$

One may verify that these matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{46}$$

Although explicit representations are useful for concrete calculations, many theoretical results can be derived directly from the abstract properties of the Clifford algebra, independent of any particular matrix realization.

1.2.3 Solutions of the Dirac Equation

The free Dirac equation is a linear and homogeneous differential equation and can therefore be solved conveniently by Fourier transformation. In momentum space, the equation reduces to an algebraic condition on the spinor wave functions.

The explicit construction of the solutions involves projecting onto the mass shell and decomposing the spinor space into positive- and negative-energy components. This procedure leads to the standard spinor solutions of the Dirac equation, commonly denoted by $u(p)$ and $v(p)$, which form a complete basis for on-shell momenta.

Since this derivation is standard and well documented in the literature, we do not reproduce it here. Detailed constructions of the Dirac spinor solutions can be found in any textbook on relativistic quantum field [2, 17].

1.3 Interference and the Path Integral Picture

The path integral interpretation of quantum mechanics can be understood through interference. When a particle propagates from a source to a detector, it can follow many different paths. Each possible path contributes a complex amplitude, and the observed intensity arises from the coherent sum of these [2, 18, 22, 9].

For a discrete set of paths labelled by k, the amplitude at a point x is given by

$$A(x) = \sum_k e^{i\phi_k(x)}, \tag{47}$$

where the phase $\phi_k(x)$ depends on the length of the corresponding path. The measured intensity is

$$I(x) = |A(x)|^2. \tag{48}$$

Interference occurs because different paths generally have different phases.

If the number of allowed paths becomes continuous, for example due to a finite-width slit, the sum over paths is replaced by an integral,

$$A(x) = \int dy\, e^{i\phi(y,x)}. \tag{49}$$

This again leads to non-trivial interference patterns.

As more obstacles are introduced, the number of possible trajectories increases, including curved paths. However, paths whose phases vary rapidly tend to cancel each other through destructive interference. Only trajectories close to those of stationary phase contribute coherently. This explains why classical paths emerge in the appropriate limit.

These considerations motivate the path integral formulation of quantum mechanics, in which transition amplitudes are obtained by summing over all possible paths, each weighted by a phase determined by the classical action.

1.3.1 Path Integral for the Transition Amplitude

We derive the path integral representation of the quantum mechanical transition amplitude starting from the time evolution operator in Hilbert space[2]. The transition amplitude between position eigenstates is defined as

$$A_{fi} = \langle q_f, t_f \mid q_i, t_i \rangle = \langle q_f | \hat{U}(t_f, t_i) | q_i \rangle, \tag{50}$$

where the time evolution operator is given by

$$\hat{U}(t_f, t_i) = \exp\left[-\frac{i}{\hbar}(t_f - t_i)\hat{H}\right]. \tag{51}$$

Let $T = t_f - t_i$ and divide the time interval into N equal segments of length $\epsilon = T/N$. The evolution operator can then be written as

$$\hat{U}(T) = \left(e^{-\frac{i}{\hbar}\epsilon\hat{H}}\right)^N, \tag{52}$$

so that the transition amplitude takes the form

$$A_{fi} = \langle q_f | \left(e^{-\frac{i}{\hbar}\epsilon\hat{H}}\right)^N |q_i\rangle. \tag{53}$$

To evaluate this expression, we insert $N-1$ resolutions of the identity in the position basis,

$$\mathbb{1} = \int dq_k \, |q_k\rangle\langle q_k|,$$

which yields

$$A_{fi} = \int \prod_{k=1}^{N-1} dq_k \prod_{j=0}^{N-1} \langle q_{j+1}|e^{-\frac{i}{\hbar}\epsilon\hat{H}}|q_j\rangle, \tag{54}$$

where $q_0 = q_i$ and $q_N = q_f$.

We now assume a Hamiltonian of the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}).$$

For sufficiently small ϵ, the exponential of the Hamiltonian may be approximated using the Trotter formula,

$$e^{-\frac{i}{\hbar}\epsilon\hat{H}} = e^{-\frac{i}{\hbar}\epsilon\frac{\hat{p}^2}{2m}} e^{-\frac{i}{\hbar}\epsilon V(\hat{q})} + \mathcal{O}(\epsilon^2). \tag{55}$$

The potential term acts diagonally in the position basis, while the kinetic term is treated by inserting a resolution of the identity in momentum space,

$$\mathbb{1} = \int \frac{dp}{2\pi\hbar} |p\rangle\langle p|.$$

Using the relation

$$\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}pq},$$

one finds

$$\langle q_{j+1}|e^{-\frac{i}{\hbar}\epsilon\hat{H}}|q_j\rangle = \int \frac{dp_j}{2\pi\hbar} \exp\left[\frac{i}{\hbar}\left(p_j(q_{j+1} - q_j) - \epsilon\frac{p_j^2}{2m} - \epsilon V(q_j)\right)\right]. \tag{56}$$

The momentum integral is Gaussian and can be evaluated explicitly. This results in

$$\langle q_{j+1}|e^{-\frac{i}{\hbar}\epsilon\hat{H}}|q_j\rangle = \sqrt{\frac{m}{2\pi i\hbar\epsilon}}\exp\left[\frac{i}{\hbar}\epsilon\left(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\epsilon}\right)^2 - V(q_j)\right)\right]. \tag{57}$$

Substituting this expression back into the transition amplitude and collecting all factors, we obtain

$$A_{fi} = \int\prod_{k=1}^{N-1} dq_k \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{N/2}\exp\left[\frac{i}{\hbar}\sum_{j=0}^{N-1}\epsilon\left(\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\epsilon}\right)^2 - V(q_j)\right)\right]. \tag{58}$$

In the limit $N\to\infty$ and $\epsilon\to 0$, the discrete sum in the exponent approaches a time integral. Defining the Lagrangian

$$L(q,\dot{q}) = \frac{m}{2}\dot{q}^2 - V(q),$$

the exponent becomes the classical action

$$S[q] = \int_{t_i}^{t_f} dt\, L(q,\dot{q}).$$

The transition amplitude can therefore be written in the path integral form

$$\boxed{\langle q_f, t_f \mid q_i, t_i\rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f}\mathcal{D}q(t)\,\exp\left[\frac{i}{\hbar}S[q]\right].} \tag{59}$$

1.3.2 Path Integral in Configuration Space

The path integral derived above is naturally expressed in phase space and is therefore closely tied to the Hamiltonian formulation, where time plays a distinguished role. For many physical systems, however, it is convenient to rewrite the path integral purely in configuration space and recover the standard Lagrangian form[2, 22].

This is possible provided the Hamiltonian is at most quadratic in the canonical momentum. We consider a general Hamiltonian of the form

$$H(q,p) = \frac{p^2}{2M(q)} + p\,K(q) + V(q), \tag{60}$$

which includes most systems of physical interest. For example, a particle of mass m moving in a potential $V(q)$ corresponds to $M(q)=m$ and $K(q)=0$.

Since the dependence on the momentum is at most quadratic, all momentum integrals appearing in the phase-space path integral are Gaussian and can

be evaluated explicitly. Integrating out the momenta yields a path integral expressed entirely in terms of the configuration variable $q(t)$,

$$A_{f,i} = \int \mathcal{D}q \, \exp\left[\frac{i}{\hbar} S[q]\right], \tag{61}$$

where the phase is given by the classical action in Lagrangian form,

$$S[q] = \int_{t_i}^{t_f} dt \, \left[\frac{1}{2} M(q)\big(\dot{q} - K(q)\big)^2 - V(q)\right] = \int_{t_i}^{t_f} dt \, L(q, \dot{q}). \tag{62}$$

This action is classically equivalent to the Hamiltonian action used previously, and both formulations lead to the same quantum transition amplitude in this case. The resulting path integral depends only on the classical Lagrangian and does not require explicit reference to the Hamiltonian formulation.

The measure associated with the configuration-space path integral is more involved than in phase space and may depend on $M(q)$. For a standard particle with constant mass, this dependence reduces to an overall normalization factor. In general, however, different discretizations of the action correspond to different operator orderings in the quantum theory.

The configuration-space path integral thus provides a compact and practical formulation of quantum mechanics, directly in terms of the classical action, and serves as the natural starting point for generalization to quantum field theory.

1.3.3 Path Integral in Phase Space

An alternative formulation of the transition amplitude is obtained by expressing the path integral directly in phase space. In this representation, the transition amplitude is written as an integral over all intermediate positions and momenta, while the initial and final positions are held fixed. The corresponding measure naturally combines integrations over both variables and reflects the elementary cell structure of phase space[2].

Taking the limit of infinitely many time slices, the transition amplitude assumes the compact form

$$A_{f,i} = \int \mathcal{D}q \, \mathcal{D}p \, \exp\left[\frac{i}{\hbar} S[q, p]\right], \tag{63}$$

where the phase is given by the phase-space action

$$S[q, p] = \int_{t_i}^{t_f} dt \, \big(p\dot{q} - H(q, p)\big). \tag{64}$$

The path integral thus sums over all trajectories $(q(t), p(t))$ in phase space.

The equations of motion associated with this action follow from extremizing $S[q, p]$ and reproduce Hamilton's equations. Paths for which the action varies rapidly acquire rapidly oscillating phases and cancel upon integration, while trajectories close to those satisfying the classical equations of motion contribute

coherently. In this sense, the classical limit arises naturally through a stationary-phase argument.

In the phase-space path integral, the boundary conditions fix the initial and final positions, whereas the momenta are unconstrained. More general boundary conditions can be imposed if required. Although the functional integration measures generally contain non-trivial normalization factors, these details are usually absorbed into an overall normalization and do not affect physical results.

While the phase-space formulation is conceptually useful, especially for connecting with classical mechanics, it is often more practical to integrate out the momenta and work directly with the configuration-space path integral. This latter form will be the preferred starting point for quantum field theory.

1.3.4 Free Particle

As a simple illustration of the path integral formalism, we consider a free non-relativistic particle of mass m. The classical action is given by

$$S[q] = \int_{t_0}^{t_n} dt\, \frac{m}{2}\dot{q}^2. \tag{65}$$

Discretising the time interval into n segments, the action takes the form

$$S_{n,0}[q] = \frac{m}{2}\sum_{k=1}^{n} \frac{(q_k - q_{k-1})^2}{t_k - t_{k-1}}. \tag{66}$$

The transition amplitude is therefore written as

$$A_{n,0} = \int \prod_{k=1}^{n-1} dq_k \prod_{k=1}^{n} \sqrt{\frac{m}{2\pi i\hbar(t_k - t_{k-1})}}\, \exp\left[\frac{i}{\hbar} S_{n,0}[q]\right]. \tag{67}$$

Since the action is quadratic in the coordinates, all integrals over the intermediate positions are Gaussian and can be evaluated exactly. Performing these integrations sequentially eliminates all intermediate variables without introducing any corrections. As a result, the final expression depends only on the initial and final positions.

Taking the continuum limit, one obtains the exact transition amplitude for the free particle,

$$A_{f,i} = \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}}\, \exp\left[\frac{im(q_f - q_i)^2}{2\hbar(t_f - t_i)}\right]. \tag{68}$$

This expression agrees with the result obtained from canonical quantization and provides a non-trivial check of the path integral formalism. Owing to the quadratic nature of the action, the discretisation becomes exact already at finite time slicing, and no subtleties arise in the continuum limit.

1.4 Path Integral for Fields

We now generalise the path integral formulation from quantum mechanics to quantum field theory. In this framework, quantum dynamics is encoded in functional integrals over field configurations, and correlation functions arise naturally as expectation values of operator insertions[2, 22, 9].

1.4.1 Time-Ordered Correlators

The path integral representation of quantum mechanical transition amplitudes extends straightforwardly to fields. For a set of field operators $\mathcal{O}_1, \ldots, \mathcal{O}_n$, the transition amplitude between initial and final field configurations can be written as

$$\langle \Psi_f, t_f \mid T(\mathcal{O}_1 \cdots \mathcal{O}_n) \mid \Psi_i, t_i \rangle = \int \mathcal{D}\Psi \; \mathcal{O}_1[\Psi] \cdots \mathcal{O}_n[\Psi] \; \exp(iS_{f,i}[\Psi]) \,, \quad (69)$$

where Ψ denotes a field configuration interpolating between the boundary values $\Psi_i = \Psi(t_i)$ and $\Psi_f = \Psi(t_f)$. The action appearing in the exponent is given by

$$S_{f,i}[\Psi] = \int_{t_i}^{t_f} dt \, L[\Psi(t), \dot{\Psi}(t)] = \int d^D x \, \mathcal{L}(\Psi(x), \partial_\mu \Psi(x)). \quad (70)$$

Although this expression is close to being relativistically covariant, it still depends on the choice of initial and final time slices. In practice, one is typically interested not in transition amplitudes between specific field configurations, but in vacuum expectation values of time-ordered operators,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \langle 0 \mid T(\mathcal{O}_1 \cdots \mathcal{O}_n) \mid 0 \rangle. \quad (71)$$

To isolate the vacuum contribution, one exploits the fact that a generic state has non-vanishing overlap with the ground state. By evolving the system for a long time and slightly tilting the time axis into the complex plane, higher-energy contributions are exponentially suppressed, leaving only the vacuum. Taking this limit, the time-ordered correlator can be written as

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \frac{\int \mathcal{D}\Psi \; \mathcal{O}_1[\Psi] \cdots \mathcal{O}_n[\Psi] \; \exp(iS[\Psi])}{\int \mathcal{D}\Psi \; \exp(iS[\Psi])}. \quad (72)$$

Here the functional integration runs over all field configurations defined on spacetime, and the action is given by

$$S[\Psi] = \int d^D x \, \mathcal{L}(\Psi(x), \partial_\mu \Psi(x)). \quad (73)$$

The denominator ensures proper normalization of the vacuum expectation value and removes any overall constant factors in the path integral measure.

With the standard prescription for tilting the time contour, the path integral automatically reproduces Feynman propagators and time ordering. In this form, the path integral provides a fully relativistic expression for time-ordered correlation functions and constitutes the primary tool for computing observables in quantum field theory.

1.4.2 Sources and Generating Functional

Correlation functions of quantum fields can be expressed directly in the path integral formalism. For example, the two-point function of a scalar field may be written as

$$\langle \Psi(x)\Psi(y)\rangle = \frac{\int \mathcal{D}\Psi\ \Psi(x)\Psi(y)\, e^{iS[\Psi]}}{\int \mathcal{D}\Psi\ e^{iS[\Psi]}}. \tag{74}$$

In interacting theories, as well as in free theories with operator insertions, such expressions quickly become cumbersome to evaluate directly.

A convenient and systematic way to generate correlation functions is to introduce external sources. One defines the generating functional

$$Z[J] = \int \mathcal{D}\Psi\ \exp(iS[\Psi] + iS_{\rm src}[\Psi, J])\,, \tag{75}$$

where the source term is given by

$$S_{\rm src}[\Psi, J] = \int d^D x\, \Psi(x) J(x). \tag{76}$$

The generating functional differs from the original path integral only by the presence of this linear coupling to an external field $J(x)$.

The key property of the source term is that functional differentiation with respect to $J(x)$ produces insertions of the field $\Psi(x)$. Indeed, one finds

$$\frac{\delta S_{\rm src}[\Psi, J]}{\delta J(x)} = \Psi(x), \tag{77}$$

and therefore

$$\frac{\delta}{\delta J(x)} \exp(iS_{\rm src}[\Psi, J]) = i\,\Psi(x)\ \exp(iS_{\rm src}[\Psi, J])\,. \tag{78}$$

Since the source does not appear in the original action, functional derivatives of the generating functional insert powers of the field into the path integral,

$$\Psi(x)\ \longleftrightarrow\ \frac{1}{i}\frac{\delta}{\delta J(x)}. \tag{79}$$

As a consequence, correlation functions can be obtained by functional differentiation of $Z[J]$ followed by setting the source to zero. For example, the time-ordered two-point function is given by

$$\langle \Psi(x)\Psi(y)\rangle = \frac{1}{Z[0]}\ \frac{1}{i^2}\frac{\delta^2 Z[J]}{\delta J(x)\,\delta J(y)}\bigg|_{J=0}. \tag{80}$$

Higher-point correlation functions are generated analogously by taking higher functional derivatives with respect to the source.

The generating functional thus encodes all time-ordered correlation functions of the theory in a compact form. It provides the natural starting point for perturbative expansions, symmetry constraints, and, ultimately, the derivation of Ward identities and conformal properties[2].

1.5 Free Scalar Field

The generating functional introduced previously can be evaluated explicitly for a free scalar field. This provides an exact and instructive example, and forms the basis for perturbative treatments of interacting theories[2].

We consider a real scalar field $\phi(x)$ with action supplemented by a source term,

$$S[\phi] + S_{\rm src}[\phi, j] = \int d^D x \left[\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + j\phi \right]. \tag{81}$$

By integrating by parts and neglecting boundary terms, the action can be rewritten such that all derivatives act on a single field,

$$S[\phi] + S_{\rm src}[\phi, j] = \int d^D x \left[\frac{1}{2} \phi(-\partial^2 - m^2)\phi + j\phi \right]. \tag{82}$$

The path integral is therefore Gaussian, with kernel given by the differential operator

$$\mathcal{O} = -\partial^2 - m^2.$$

The inverse of this operator defines the Green function $G_F(x-y)$, which satisfies

$$(-\partial^2 - m^2)\, G_F(x-y) = \delta^{(D)}(x-y). \tag{83}$$

The choice of boundary conditions corresponding to the standard $i\epsilon$ prescription selects the Feynman propagator.

To evaluate the Gaussian integral, we perform a shift of the integration variable,

$$\phi(x) = \tilde{\phi}(x) + \int d^D y \, G_F(x-y)\, j(y). \tag{84}$$

Substituting this expression into the action, all terms linear in $\tilde{\phi}$ cancel by construction, and the action separates into a $\tilde{\phi}$-dependent part and a purely source-dependent contribution,

$$S[\phi] + S_{\rm src}[\phi, j] = \int d^D x \, \frac{1}{2} \tilde{\phi}(x)(-\partial^2 - m^2)\tilde{\phi}(x) + W[j], \tag{85}$$

where

$$W[j] = \frac{1}{2} \int d^D x \, d^D y \; j(x)\, G_F(x-y)\, j(y). \tag{86}$$

Since the functional measure is invariant under shifts of the integration variable, the path integral over $\tilde{\phi}$ factorises and contributes only an overall normalization. The generating functional therefore takes the form

$$Z[j] = Z[0] \, \exp(iW[j]) \, . \tag{87}$$

In momentum space, the quadratic form simplifies further. Writing $j(x) = \int \frac{d^D p}{(2\pi)^D} e^{ipx} j(p)$, one finds

$$W[j] = \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \frac{j(p) j(-p)}{p^2 + m^2 - i\epsilon}. \tag{88}$$

Formally, the normalization factor is given by

$$Z[0] \propto \left[\det(-\partial^2 - m^2 + i\epsilon)\right]^{-1/2}. \tag{89}$$

Correlation functions are obtained by functional differentiation with respect to the source. In particular, the two-point function is given by

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{Z[0]}\,\frac{1}{i^2}\,\frac{\delta^2 Z[j]}{\delta j(x)\,\delta j(y)}\bigg|_{j=0}. \tag{90}$$

Using the explicit form of $Z[j]$, this yields

$$\langle \phi(x)\phi(y)\rangle = -\,i\,G_F(x-y). \tag{91}$$

This result demonstrates that the Feynman propagator arises naturally as the two-point function of the free scalar field. Higher-order correlators follow analogously and reduce to sums over products of propagators, in accordance with Wick's theorem.

1.5.1 Generating Functional and Propagator

In this subsection we evaluate the generating functional for a free real scalar field and show how the Feynman propagator arises as the two-point function. This provides a complete and exactly solvable example of the general functional methods introduced above [10, 20, 8, 4].

Free scalar with source. We consider a real scalar field $\phi(x)$ in D-dimensional Minkowski spacetime, with action

$$S[\phi] + S_{\rm src}[\phi, j] = \int d^D x \left[\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + j(x)\,\phi(x)\right]. \tag{92}$$

Here $j(x)$ is a classical external source that couples linearly to the field. It is convenient to integrate by parts (and drop surface terms) so that all derivatives act on a single field:

$$S[\phi] + S_{\rm src}[\phi, j] = \int d^D x \left[\frac{1}{2}\,\phi(x)\,(-\partial^2 - m^2)\,\phi(x) + j(x)\,\phi(x)\right]. \tag{93}$$

We define the differential operator

$$\mathcal{O} = -\partial^2 - m^2\,. \tag{94}$$

Definition of the generating functional. The generating functional of time-ordered correlation functions in the presence of the source j is

$$Z[j] = \int \mathcal{D}\phi\ \exp(iS[\phi] + iS_{\rm src}[\phi, j])\,. \tag{95}$$

Using the quadratic form above, this becomes

$$Z[j] = \int \mathcal{D}\phi \, \exp\left\{ \frac{i}{2} \int d^D x \, \phi(x) \, \mathcal{O}\phi(x) + i \int d^D x \, j(x) \, \phi(x) \right\}. \tag{96}$$

This is a Gaussian functional integral over ϕ, analogous to a multidimensional Gaussian integral with kernel $\mathcal{O}$ [8, 20].

Green function and completion of the square. The inverse of $\mathcal{O}$ defines the Green function (Feynman propagator) $G_F(x-y)$ by

$$\mathcal{O}_x \, G_F(x-y) = (-\partial_x^2 - m^2) \, G_F(x-y) = \delta^{(D)}(x-y). \tag{97}$$

To evaluate $Z[j]$, we complete the square by shifting the integration variable,

$$\phi(x) = \tilde{\phi}(x) + \int d^D y \, G_F(x-y) \, j(y). \tag{98}$$

Substituting this into the action, all terms linear in $\tilde{\phi}$ cancel by construction, and the action separates into a $\tilde{\phi}$-dependent part and a purely source-dependent contribution,

$$S[\phi] + S_{\rm src}[\phi, j] = \int d^D x \, \frac{1}{2} \, \tilde{\phi}(x) \, \mathcal{O}\tilde{\phi}(x) + W[j], \tag{99}$$

where

$$W[j] = \frac{1}{2} \int d^D x \, d^D y \; j(x) \, G_F(x-y) \, j(y). \tag{100}$$

Since the functional measure is invariant under shifts, the path integral over $\tilde{\phi}$ factorizes and contributes only an overall normalization constant $Z[0]$ (the vacuum functional at zero source). Thus the generating functional takes the compact form

$$Z[j] = Z[0] \, \exp(iW[j]) \, . \tag{101}$$

In momentum space, writing

$$j(x) = \int \frac{d^D p}{(2\pi)^D} \, e^{ip\cdot x} j(p) \, , \tag{102}$$

the quadratic form simplifies to

$$W[j] = \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \, \frac{j(p) \, j(-p)}{p^2 - m^2 + i\epsilon} \, , \tag{103}$$

where the $i\epsilon$ prescription implements the correct boundary conditions for the Feynman propagator [10, 8, 20].

Formally, the normalization factor is

$$Z[0] \propto \left[\det(-\partial^2 - m^2 + i\epsilon) \right]^{-1/2} , \tag{104}$$

but this overall constant drops out of normalized correlation functions and will not concern us further.

Two-point function and propagator. Correlation functions are obtained by functional differentiation with respect to the source $j(x)$. In particular, the time-ordered two-point function is

$$\langle T\,\phi(x)\phi(y)\rangle = \frac{1}{Z[0]}\,\frac{1}{i^2}\,\frac{\delta^2 Z[j]}{\delta j(x)\,\delta j(y)}\bigg|_{j=0}. \tag{105}$$

Using the explicit form $Z[j] = Z[0]\exp(iW[j])$, we find

$$\frac{\delta Z[j]}{\delta j(x)} = iZ[0]\,\frac{\delta W[j]}{\delta j(x)}\,e^{iW[j]}, \tag{106}$$

and

$$\frac{\delta^2 Z[j]}{\delta j(x)\,\delta j(y)} = iZ[0]\frac{\delta^2 W[j]}{\delta j(x)\,\delta j(y)}\,e^{iW[j]} - Z[0]\frac{\delta W[j]}{\delta j(x)}\,\frac{\delta W[j]}{\delta j(y)}\,e^{iW[j]}. \tag{107}$$

Evaluated at $j = 0$, the second term vanishes because $\delta W/\delta j|_{j=0} = 0$, and we obtain

$$\frac{\delta^2 Z[j]}{\delta j(x)\,\delta j(y)}\bigg|_{j=0} = iZ[0]\;\frac{\delta^2 W[j]}{\delta j(x)\,\delta j(y)}\bigg|_{j=0}. \tag{108}$$

Differentiating $W[j]$ explicitly gives

$$\frac{\delta^2 W[j]}{\delta j(x)\,\delta j(y)}\bigg|_{j=0} = G_F(x-y). \tag{109}$$

Therefore

$$\langle T\,\phi(x)\phi(y)\rangle = \frac{1}{Z[0]}\,\frac{1}{i^2}\;\frac{\delta^2 Z[j]}{\delta j(x)\,\delta j(y)}\bigg|_{j=0} = -\,i\,G_F(x-y). \tag{110}$$

This shows that the Feynman propagator $G_F(x-y)$ is precisely the two-point function of the free scalar field,

$$\boxed{\langle 0|\,T\big(\phi(x)\phi(y)\big)\,|0\rangle = -\,i\,G_F(x-y)\,.} \tag{111}$$

Higher-point functions follow similarly and reduce to sums over products of propagators, in accordance with Wick's theorem for Gaussian theories [10, 20, 4].

1.6 Euclidean path integral

We start from the Minkowski path integral for a real scalar field ϕ in d-dimensional spacetime with metric signature $(+,-,\ldots,-)$.[1] The Minkowski action is

$$S[\phi] \;=\; \int d^d x\,\mathcal{L}(\phi,\partial\phi) \tag{112}$$

[1] See e.g. Peskin and Schroeder, *An Introduction to Quantum Field Theory* [10]. See also the ETH notes on the Euclidean path integral formalism [1].

with Lagrangian for a free scalar

$$\mathcal{L}_M \;=\; \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi \;-\; \frac{1}{2}m^2\phi^2\,. \tag{113}$$

In the path–integral formulation, the vacuum–to–vacuum amplitude in the presence of a source $J(x)$ is

$$Z_M[J] \;=\; \int \mathcal{D}\phi\,\exp\Big\{i\int d^dx\,\big(\mathcal{L}_M + J\phi\big)\Big\}\,. \tag{114}$$

To define the Euclidean path integral, we perform a Wick rotation of time,

$$t \;=\; -\,i\tau\,, \qquad dt \;=\; -\,i\,d\tau\,, \tag{115}$$

and analytically continue the fields to imaginary time τ.[2] The Minkowski line element

$$ds^2 \;=\; dt^2 - d\vec{x}^{\,2} \tag{116}$$

becomes Euclidean

$$ds_E^2 \;=\; d\tau^2 + d\vec{x}^{\,2}\,, \tag{117}$$

and the kinetic term changes sign so that the Euclidean Lagrangian is

$$\mathcal{L}_E \;=\; \frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\,. \tag{118}$$

[3] Correspondingly, the action becomes

$$S_E[\phi] \;=\; \int d\tau\,d^{d-1}x\,\mathcal{L}_E(\phi,\partial\phi)\,, \tag{119}$$

and the weight in the functional integral turns from oscillatory to exponentially damped,

$$Z_E[J] \;=\; \int \mathcal{D}\phi\,\exp\Big\{-\int d\tau\,d^{d-1}x\,\big(\mathcal{L}_E + J\phi\big)\Big\} \;=\; \int \mathcal{D}\phi\,e^{-S_E[\phi]+\int J\phi}\,. \tag{120}$$

The Euclidean generating functional $Z_E[J]$ defines Euclidean correlation functions (Schwinger functions) via functional derivatives:

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle_E \;=\; \frac{1}{Z_E[0]}\,\frac{\delta^n Z_E[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}\,. \tag{121}$$

For the free scalar field, the Euclidean propagator in momentum space is

$$G_E(p) \;=\; \frac{1}{p_E^2 + m^2}\,, \qquad p_E^2 = p_0^2 + \vec{p}^{\,2}\,, \tag{122}$$

[2] For a detailed discussion of Wick rotation and Euclidean actions, see Peskin and Schroeder [10] and the ETH Euclidean notes [1].

[3] See e.g. Wipf, *Euclidean Path Integral*, Chapter 6 of his QFT lecture notes [21].

which is the Wick–rotated version of the Feynman propagator $1/(p^2 - m^2 + i\epsilon)$.[4]

This Euclidean path integral formulation is the basis for defining QFT (and later CFT) on Euclidean manifolds such as the plane, the cylinder, or the torus, where correlation functions and partition functions are computed as Euclidean functional integrals.[5]

Glossary

- **Minkowski vs Euclidean**: Minkowski spacetime is the usual relativistic spacetime with time and space having different signs in the metric ($ds^2 = dt^2 - d\vec{x}^2$). Euclidean spacetime is obtained by taking time to be imaginary, so time and space appear symmetrically ($ds_E^2 = d\tau^2 + d\vec{x}^2$) [10].
- **Wick rotation**: The change of variables $t \to -i\tau$ that turns oscillatory factors e^{iS} into decaying factors e^{-S_E}. This makes the path integral mathematically similar to a statistical mechanics partition function [10, 1, 21].
- **Path integral / functional integral** $\int \mathcal{D}\phi$: An integral over all possible field configurations $\phi(x)$, not just over a finite number of variables. It is the field-theory analogue of summing over all paths in quantum mechanics [10, 21].
- **Generating functional** $Z[J]$: A functional of a source $J(x)$ whose derivatives with respect to J generate all correlation functions. It plays the role of a "master object" from which n-point functions are obtained [1].
- **Schwinger functions**: Euclidean correlation functions $\langle \phi(x_1) \dots \phi(x_n) \rangle_E$. After analytic continuation back to real time, they give ordinary time-ordered correlators [21].
- **Propagator**: The two-point function of the field, which describes how disturbances of the field propagate from one point to another. In momentum space, for a free scalar, it is $1/(p_E^2 + m^2)$ in Euclidean signature [10, 21].

2 Two-Dimensional Conformal Field Theory and the Free Boson

2.1 Light-cone coordinates in two dimensions

Consider two-dimensional Minkowski spacetime with coordinates (t, x) and metric

$$ds^2 \;=\; dt^2 - dx^2 \,. \tag{123}$$

[4] Compare with the free propagator formulas in Peskin and Schroeder [10] and in Wipf's notes [21].

[5] See the ETH Euclidean path integral notes for a concise summary [1].

This setup and the use of light-cone coordinates in 1+1 dimensions are standard in quantum field theory and conformal field theory; see for example [10, 11].

We introduce light-cone coordinates

$$x^+ \;=\; \frac{1}{\sqrt{2}}\,(t+x)\,, \qquad x^- \;=\; \frac{1}{\sqrt{2}}\,(t-x)\,. \tag{124}$$

Inverting these relations gives

$$t \;=\; \frac{1}{\sqrt{2}}\,(x^+ + x^-)\,, \qquad x \;=\; \frac{1}{\sqrt{2}}\,(x^+ - x^-)\,, \tag{125}$$

so that

$$dt \;=\; \frac{1}{\sqrt{2}}\,(dx^+ + dx^-)\,, \qquad dx \;=\; \frac{1}{\sqrt{2}}\,(dx^+ - dx^-)\,. \tag{126}$$

Substituting into the line element, we find

$$\begin{aligned} ds^2 &= dt^2 - dx^2 \\ &= \frac{1}{2}\big[(dx^+ + dx^-)^2 - (dx^+ - dx^-)^2\big] \\ &= 2\,dx^+ dx^-\,. \end{aligned} \tag{127}$$

Thus curves of constant x^+ or constant x^- satisfy $ds^2 = 0$ and correspond to lightlike (null) directions [10].

The derivatives with respect to the new coordinates are

$$\partial_+ \;=\; \frac{\partial}{\partial x^+} \;=\; \frac{1}{\sqrt{2}}\Big(\partial_t + \partial_x\Big)\,, \qquad \partial_- \;=\; \frac{\partial}{\partial x^-} \;=\; \frac{1}{\sqrt{2}}\Big(\partial_t - \partial_x\Big)\,, \tag{128}$$

and the two-dimensional d'Alembertian

$$\Box \;=\; \partial_t^2 - \partial_x^2 \tag{129}$$

can be written in light-cone form as

$$\Box \;=\; 4\,\partial_+\partial_-\,. \tag{130}$$

This expression is standard in discussions of two-dimensional field theories [10].

For a massless scalar field $\phi(t,x)$, the Klein–Gordon equation

$$(\partial_t^2 - \partial_x^2)\,\phi(t,x) \;=\; 0 \tag{131}$$

becomes

$$4\,\partial_+\partial_-\,\phi(x^+,x^-) \;=\; 0\,, \tag{132}$$

whose general solution is

$$\phi(x^+,x^-) \;=\; \phi_L(x^+) + \phi_R(x^-)\,. \tag{133}$$

Here ϕ_L depends only on x^+ and describes a right-moving mode, while ϕ_R depends only on x^- and describes a left-moving mode. This left/right-moving factorization in two dimensions is the starting point for the holomorphic and antiholomorphic sectors of two-dimensional conformal field theory [11].

Glossary

- **Light-cone coordinates**: The combinations $x^{\pm} = (t \pm x)/\sqrt{2}$ that align with the directions along which light rays propagate in $1+1$ dimensions; they simplify the metric and the wave equation [10].
- **Null (lightlike) directions**: Directions along which $ds^2 = 0$; in two dimensions, curves with constant x^+ or constant x^- are null [10].
- **Left-/right-moving modes**: Solutions depending only on x^+ (right-moving) or only on x^- (left-moving); they represent waves travelling at the speed of light in opposite directions and form the chiral sectors in two-dimensional CFT [11].

2.2 Mode expansion in light-cone coordinates

In this subsection we study a very simple system: a free massless scalar field $\phi(t, x)$ in $1+1$ dimensions. We will see how any solution can be written as a sum of simple waves (modes) when space is a circle.

2.2.1 The theory on a circle (cylinder picture).

We choose space to be a circle of length L:

$$x \sim x + L\,. \tag{134}$$

This means the point with coordinate x is physically the same as the point with coordinate $x + L$. If we draw time vertically and the circle horizontally, the spacetime looks like a cylinder.

Because of this identification, the field must be periodic in x:

$$\phi(t, x + L) = \phi(t, x)\,. \tag{135}$$

This is the key condition that will force a Fourier (mode) expansion [12, 5].

2.2.2 Equation of motion in light-cone coordinates

We introduce light-cone coordinates

$$x^+ = \frac{t + x}{\sqrt{2}}\,, \qquad x^- = \frac{t - x}{\sqrt{2}}\,. \tag{136}$$

These are just new coordinates built from t and x. In these coordinates, the massless Klein–Gordon equation

$$(\partial_t^2 - \partial_x^2)\,\phi(t, x) = 0$$

becomes

$$4\,\partial_+\partial_-\,\phi(x^+, x^-) = 0\,. \tag{137}$$

The most general solution of this equation is

$$\phi(x^+, x^-) = \phi_L(x^+) + \phi_R(x^-)\,. \tag{138}$$

Here $\phi_L(x^+)$ is a wave that depends only on x^+ and moves to the right, and $\phi_R(x^-)$ is a wave that depends only on x^- and moves to the left [10, 11].

2.2.3 Periodicity condition in light-cone variables

The condition $\phi(t, x + L) = \phi(t, x)$ must hold for all t and x. When we shift $x \to x + L$ at fixed t, the light-cone coordinates change as

$$x^+ \to x^+ + \frac{L}{\sqrt{2}}\,, \qquad x^- \to x^- - \frac{L}{\sqrt{2}}\,. \tag{139}$$

Therefore,

$$\phi(t, x + L) = \phi\Big(x^+ + \tfrac{L}{\sqrt{2}},\, x^- - \tfrac{L}{\sqrt{2}}\Big) \tag{140}$$

$$= \phi_L\Big(x^+ + \tfrac{L}{\sqrt{2}}\Big) + \phi_R\Big(x^- - \tfrac{L}{\sqrt{2}}\Big)\,. \tag{141}$$

The periodicity condition now reads

$$\phi_L\Big(x^+ + \tfrac{L}{\sqrt{2}}\Big) + \phi_R\Big(x^- - \tfrac{L}{\sqrt{2}}\Big) = \phi_L(x^+) + \phi_R(x^-)\,. \tag{142}$$

This must hold for all x^+ and x^-. A simple way to make this true is to require that ϕ_L and ϕ_R are each periodic (up to possible linear terms) with period $L/\sqrt{2}$ [12].

2.2.4 Fourier expansion of left- and right-moving modes

Any (sufficiently nice) periodic function on a circle can be written as a Fourier series. So we expand the left-moving part as

$$\phi_L(x^+) = q_L + \frac{p_L}{L}\,x^+ + i\sum_{n\neq 0}\frac{1}{n}\,\alpha_n\,\exp\Big(-\frac{2\pi i n}{L}\,x^+\Big)\,. \tag{143}$$

Similarly, we expand the right-moving part as

$$\phi_R(x^-) = q_R + \frac{p_R}{L}\,x^- + i\sum_{n\neq 0}\frac{1}{n}\,\tilde{\alpha}_n\,\exp\Big(-\frac{2\pi i n}{L}\,x^-\Big)\,. \tag{144}$$

Interpretation of the terms:

- q_L and q_R are constant pieces: they give the average value of the field in the left- and right-moving sectors. - $\frac{p_L}{L}x^+$ and $\frac{p_R}{L}x^-$ are linear pieces: they describe a uniform slope or drift of the field along the cylinder. - The sum over $n \neq 0$ contains oscillating exponentials. Each integer n labels a wave that fits an integer number of times around the circle. The coefficients α_n and $\tilde{\alpha}_n$ tell you how big each wave is [5, 12].

2.2.5 The full field on the cylinder

Putting everything together, the field on the cylinder is

$$\phi(t,x) = \phi_L\Big(\tfrac{t+x}{\sqrt{2}}\Big) + \phi_R\Big(\tfrac{t-x}{\sqrt{2}}\Big)\ . \tag{145}$$

You can picture this as a superposition of right-moving and left-moving waves travelling around a circular string. Later, when we quantize the theory, $q_{L,R}$, $p_{L,R}$, α_n and $\tilde{\alpha}_n$ will become operators, and the left/right split will match the holomorphic/antiholomorphic sectors of the free boson conformal field theory [11, 5].

Glossary

- **Cylinder**: The spacetime obtained by taking space to be a circle of length L and time to be an infinite line. Visually: time is vertical, space is a loop; so the picture is a cylinder [12].
- **Light-cone coordinates** $x^{\pm}$: New coordinates defined by $x^{\pm} = (t \pm x)/\sqrt{2}$. In these coordinates, massless waves move simply along lines of constant x^+ (right-moving) or constant x^- (left-moving) [10, 11].
- **Left-moving mode** ϕ_L: The part of the field that depends only on x^+. It keeps its shape and moves to the right at the speed of light.
- **Right-moving mode** ϕ_R: The part of the field that depends only on x^-. It keeps its shape and moves to the left at the speed of light.
- **Periodic boundary condition**: The requirement $\phi(t, x + L) = \phi(t, x)$. It expresses that space is a circle rather than an infinite line [12].
- **Zero modes**: The constant term $q_{L,R}$ and linear term $p_{L,R}x^{\pm}/L$ in the expansion. They describe the overall average value and total momentum of the field, not its oscillations [5, 12].
- **Oscillator modes**: The coefficients α_n and $\tilde{\alpha}_n$ with $n \neq 0$. They describe the actual vibrations of the field, like the different harmonics of a vibrating guitar string on a circle [5].
- **Fourier / mode expansion**: Writing a periodic function as a sum of waves with integer labels n. Each label n is a mode with wavelength $L/|n|$. A very similar construction is used for the string coordinate $X(\tau, \sigma)$ in introductory string theory texts such as Zwiebach [23] and in ETH string theory notes [16].
- **Left-moving and right-moving modes**: Parts of the field that move only in one direction along the circle. In the language of 2d CFT, they become holomorphic and antiholomorphic fields; see Runkel's notes [12] or Ribault's lectures [11] for a systematic treatment.

- **Free boson as an example**: The mode expansion and left/right decomposition are explained with many pictures in video lectures on the free boson, for example the series by Steiner [13].

2.3 Euclidean complex coordinates and chiral fields

In this subsection we take the free massless scalar field to Euclidean two-dimensional spacetime. We introduce complex coordinates and see that the field naturally splits into a "left-moving" (holomorphic) and a "right-moving" (antiholomorphic) part, which is the standard language of two-dimensional conformal field theory [19, 12, 14].

2.3.1 From Minkowski to Euclidean signature

We start from $1+1$-dimensional Minkowski spacetime with coordinates (t, x) and metric

$$ds^2 = dt^2 - dx^2 \,. \tag{146}$$

For a massless scalar field $\phi(t, x)$, the equation of motion is

$$(\partial_t^2 - \partial_x^2)\, \phi(t, x) = 0 \,. \tag{147}$$

To go to Euclidean signature, we perform a Wick rotation

$$t = -i\tau \,, \tag{148}$$

where τ is Euclidean time. The line element becomes

$$ds_E^2 = d\tau^2 + dx^2 \,. \tag{149}$$

This is the standard flat Euclidean metric on the plane $\mathbb{R}^2$ [19, 12].

2.3.2 Real Euclidean coordinates and the Laplacian

We now rename the Euclidean coordinates as

$$\sigma^1 = x \,, \qquad \sigma^2 = \tau \,, \tag{150}$$

so that the metric reads

$$ds_E^2 = (d\sigma^1)^2 + (d\sigma^2)^2 \,. \tag{151}$$

The massless equation becomes the Laplace equation

$$(\partial_1^2 + \partial_2^2)\, \phi(\sigma^1, \sigma^2) = 0 \,, \tag{152}$$

where

$$\partial_1 = \frac{\partial}{\partial \sigma^1} \,, \qquad \partial_2 = \frac{\partial}{\partial \sigma^2} \,. \tag{153}$$

This is just the ordinary two-dimensional Laplacian in flat space [12, 14].

2.3.3 Introducing complex coordinates z and $\bar{z}$

Instead of using (σ^1, σ^2), it is very convenient in two dimensions to combine them into a complex coordinate

$$z = \sigma^1 + i\sigma^2 \,, \qquad \bar{z} = \sigma^1 - i\sigma^2 \,. \tag{154}$$

Here: - z plays the role of a point in the complex plane. - $\bar{z}$ is its complex conjugate.

In terms of these, we can also recover

$$\sigma^1 = \frac{z + \bar{z}}{2} \,, \qquad \sigma^2 = \frac{z - \bar{z}}{2i} \,. \tag{155}$$

The metric can be written as

$$ds_E^2 = d\sigma_1^2 + d\sigma_2^2 = dz\, d\bar{z} \,, \tag{156}$$

up to a conventional factor that one can absorb into the definition of the complex coordinates [19, 14].

2.3.4 Complex derivatives and the Laplacian

We now define the complex derivatives

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\Big(\partial_1 - i\,\partial_2\Big) \,, \qquad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\partial_1 + i\,\partial_2\Big) \,. \tag{157}$$

From these definitions one checks (by a short algebra) that

$$\partial_1^2 + \partial_2^2 = 4\,\partial\bar{\partial} \,. \tag{158}$$

Therefore the massless equation

$$(\partial_1^2 + \partial_2^2)\,\phi(\sigma^1, \sigma^2) = 0 \tag{159}$$

becomes

$$\partial\bar{\partial}\,\phi(z, \bar{z}) = 0 \,. \tag{160}$$

So in complex coordinates, the Laplace equation factorizes into a product of two first-order operators ∂ and $\bar{\partial}$ [19, 12].

2.3.5 General solution: holomorphic and antiholomorphic parts

The equation

$$\partial\bar{\partial}\,\phi(z, \bar{z}) = 0 \tag{161}$$

means that if we first differentiate with respect to $\bar{z}$, then with respect to z, the result is zero. This implies that

$$\bar{\partial}\,\phi(z, \bar{z}) \tag{162}$$

is a function whose derivative with respect to z vanishes, so $\bar{\partial}\phi$ depends only on $\bar{z}$. Similarly, one can argue that

$$\partial\,\phi(z,\bar{z}) \tag{163}$$

depends only on z. A standard consequence (which can be verified explicitly) is that the general solution can be written as a sum of a holomorphic function and an antiholomorphic function:

$$\phi(z,\bar{z}) = \phi_L(z) + \phi_R(\bar{z})\,. \tag{164}$$

Here: - $\phi_L(z)$ depends only on z and is called a *holomorphic* or *left-moving* piece. - $\phi_R(\bar{z})$ depends only on $\bar{z}$ and is called an *antiholomorphic* or *right-moving* piece [12, 11].

This is the Euclidean version of the left- and right-movers we found in Minkowski light-cone coordinates. There we had $\phi(t,x) = \phi_L(x^+) + \phi_R(x^-)$; here we have $\phi(z,\bar{z}) = \phi_L(z) + \phi_R(\bar{z})$.

2.3.6 Relation to the cylinder and the previous mode expansion

Earlier, we put the theory on a cylinder with spatial coordinate x periodic and time non-compact. There is a standard conformal map between the Euclidean plane (with coordinate z) and the Euclidean cylinder (with coordinates (τ, x)). For example, one can use

$$z = e^{\frac{2\pi}{L}(\tau+ix)}\,, \tag{165}$$

which maps the infinite cylinder (in τ) to the punctured complex plane (in z) [19, 12]. Under this map, the holomorphic field $\phi_L(z)$ on the plane corresponds to the left-moving field on the cylinder, and similarly for the antiholomorphic field $\phi_R(\bar{z})$.

This is why in conformal field theory we often work on the complex plane with fields that depend on z or on $\bar{z}$, and then translate to the cylinder when we need a Hamiltonian picture or a mode expansion.

Glossary

- **Euclidean coordinates** (σ^1, σ^2): Real coordinates on the Euclidean plane obtained after Wick rotating time. They are just another name for (x, τ) [12].
- **Complex coordinates** $z, \bar{z}$: The combinations $z = \sigma^1 + i\sigma^2$, $\bar{z} = \sigma^1 - i\sigma^2$ that turn the Euclidean plane into the complex plane. Many conformal maps are easiest to describe in these variables [19, 14].
- **Complex derivatives** ∂**,** $\bar{\partial}$: The operators

  $$\partial = \frac{1}{2}(\partial_1 - i\partial_2)\,, \qquad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)\,,$$

  which satisfy $\partial_1^2 + \partial_2^2 = 4\,\partial\bar{\partial}$ and are natural in complex analysis [14].

- **Holomorphic function**: A function $f(z)$ that depends only on z and not on $\bar{z}$, i.e. it satisfies $\bar{\partial} f = 0$. In CFT, fields depending only on z are called chiral or left-moving [12, 11].

- **Antiholomorphic function**: A function $g(\bar{z})$ that depends only on $\bar{z}$ and satisfies $\partial g = 0$. In CFT, these are antichiral or right-moving fields [12].

- **Chiral / left-moving field** $\phi_L(z)$: The part of the field that depends only on z. On the cylinder, it corresponds to waves moving in one direction along the circle; on the plane, it transforms nicely under $z \to f(z)$ [19, 11].

- **Antichiral / right-moving field** $\phi_R(\bar{z})$: The part of the field that depends only on $\bar{z}$. On the cylinder, it corresponds to waves moving in the opposite direction; on the plane, it transforms under $\bar{z} \to \bar{f}(\bar{z})$.

- **Plane–cylinder map**: A conformal map such as $z = e^{\frac{2\pi}{L}(\tau + ix)}$ that relates the complex plane to the Euclidean cylinder. It sends radial directions on the plane to time on the cylinder and circles around the origin to spatial circles [19, 12].

2.4 Stress tensor of the free boson

In this subsection we study the energy–momentum (stress) tensor for the free massless scalar field in two dimensions. We work in Euclidean signature, in complex coordinates $z, \bar{z}$, and we see how the stress tensor naturally splits into a holomorphic part $T(z)$ and an antiholomorphic part $\bar{T}(\bar{z})$ [19, **?**, 14, 5].

2.4.1 Free boson action in complex coordinates.

For a free massless scalar field ϕ in two Euclidean dimensions, a convenient form of the action is

$$S[\phi] = \frac{1}{4\pi} \int d^2\sigma \, (\partial_1 \phi)(\partial_1 \phi) + \frac{1}{4\pi} \int d^2\sigma \, (\partial_2 \phi)(\partial_2 \phi) \,, \tag{166}$$

where σ^1, σ^2 are Cartesian coordinates on the plane and $d^2\sigma = d\sigma^1 d\sigma^2$. This is the usual kinetic term for a free scalar field [**?**, 5].

We now introduce complex coordinates

$$z = \sigma^1 + i\sigma^2 \,, \qquad \bar{z} = \sigma^1 - i\sigma^2 \,, \tag{167}$$

and the corresponding derivatives

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - i\partial_2) \,, \qquad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2) \,. \tag{168}$$

One can check that

$$(\partial_1 \phi)^2 + (\partial_2 \phi)^2 = 4 \, (\partial \phi)(\bar{\partial} \phi) \,, \tag{169}$$

and that $d^2\sigma = \frac{i}{2}\, dz \wedge d\bar{z}$, so up to an overall normalization which is conventional, the action can be written as

$$S[\phi] = \frac{1}{2\pi} \int d^2z\, \partial\phi(z,\bar{z})\, \bar{\partial}\phi(z,\bar{z})\,. \tag{170}$$

Here d^2z is a shorthand for $dz\, d\bar{z}$; the normalizations are chosen so that later the central charge of the free boson comes out as $c = 1$ [19, 5].

2.4.2 Idea of the stress tensor.

In general, the (Euclidean) stress tensor $T_{\mu\nu}$ is defined as the response of the action to a change of the metric:

$$\delta S = \frac{1}{4\pi} \int d^2\sigma\, \sqrt{g}\, T_{\mu\nu}\, \delta g^{\mu\nu}\,, \tag{171}$$

where $g_{\mu\nu}$ is the metric on the manifold. Intuitively, $T_{\mu\nu}$ measures how the theory reacts if we gently stretch or compress spacetime; in flat space it contains the energy density, momentum density, and so on [19, **?**, 7].

In two dimensions and for the free boson, this general definition simplifies a lot. We can use standard formulas (derived once in general textbooks) to write the stress tensor directly in terms of derivatives of ϕ, and then translate to complex coordinates [19, 5].

2.4.3 Stress tensor in real coordinates.

In flat Euclidean space with coordinates σ^1, σ^2 and metric δ_{ij}, a classical expression for the stress tensor of the free scalar is

$$T_{ij} = \frac{1}{2\pi}\Big(\partial_i\phi\, \partial_j\phi - \frac{1}{2}\, \delta_{ij}\, \delta^{k\ell}\partial_k\phi\, \partial_\ell\phi\Big)\,, \tag{172}$$

where indices $i, j, k, \ell \in \{1, 2\}$ and we sum over repeated indices. This form is symmetric in i, j and (classically) traceless, $\delta^{ij}T_{ij} = 0$, which is a signal of conformal invariance in two dimensions [14, 5].

2.4.4 Complex components $T(z)$ and $\bar{T}(\bar{z})$.

We now rewrite the stress tensor in complex coordinates. We define the complex components by contracting with the appropriate vectors:

$$T_{zz} = T_{ij}\, \frac{\partial\sigma^i}{\partial z}\, \frac{\partial\sigma^j}{\partial z}\,, \qquad T_{\bar{z}\bar{z}} = T_{ij}\, \frac{\partial\sigma^i}{\partial \bar{z}}\, \frac{\partial\sigma^j}{\partial \bar{z}}\,, \tag{173}$$

and similarly for the mixed component $T_{z\bar{z}}$. Using

$$\partial_1 = \partial + \bar{\partial}\,, \qquad \partial_2 = -i(\partial - \bar{\partial})\,, \tag{174}$$

one can do a short computation (substituting into T_{ij} and simplifying) to show that, classically,

$$T_{z\bar{z}} = 0\,, \tag{175}$$

and

$$T_{zz} = -\,(\partial\phi)(\partial\phi)\,, \qquad T_{\bar{z}\bar{z}} = -\,(\bar{\partial}\phi)(\bar{\partial}\phi)\,. \tag{176}$$

This is a standard result quoted in many CFT texts: for the free boson, the holomorphic and antiholomorphic components of the stress tensor are quadratic in the derivatives of the field [19, 5, 12].

By convention, we define

$$T(z) \equiv T_{zz}(z) = -\,(\partial\phi(z,\bar{z}))^2\,, \qquad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}) = -\,(\bar{\partial}\phi(z,\bar{z}))^2\,. \tag{177}$$

Classically, this is the stress tensor; quantum mechanically we will have to normal order the products, but for now we stay at the classical level.

2.4.5 Holomorphic and antiholomorphic dependence.

We already know that the general solution of the free equation is

$$\phi(z,\bar{z}) = \phi_L(z) + \phi_R(\bar{z})\,, \tag{178}$$

with

$$\bar{\partial}\phi_L(z) = 0\,, \qquad \partial\phi_R(\bar{z}) = 0\,. \tag{179}$$

Therefore

$$\partial\phi(z,\bar{z}) = \partial\phi_L(z) + \partial\phi_R(\bar{z}) = \partial\phi_L(z)\,, \tag{180}$$

since $\partial\phi_R(\bar{z}) = 0$. Similarly,

$$\bar{\partial}\phi(z,\bar{z}) = \bar{\partial}\phi_L(z) + \bar{\partial}\phi_R(\bar{z}) = \bar{\partial}\phi_R(\bar{z})\,, \tag{181}$$

since $\bar{\partial}\phi_L(z) = 0$.

Plugging this into the stress tensor, we get

$$T(z) = -\,(\partial\phi(z,\bar{z}))^2 = -\,(\partial\phi_L(z))^2\,, \tag{182}$$

and

$$\bar{T}(\bar{z}) = -\,(\bar{\partial}\phi(z,\bar{z}))^2 = -\,(\bar{\partial}\phi_R(\bar{z}))^2\,. \tag{183}$$

We see explicitly that $T(z)$ depends only on z (through $\phi_L(z)$), and $\bar{T}(\bar{z})$ depends only on $\bar{z}$ (through $\phi_R(\bar{z})$). This matches the general statement that in a 2d CFT the stress tensor has holomorphic and antiholomorphic components [19, 14, 12].

2.4.6 Quantum normal ordering (brief preview).

In the quantum theory, products such as $(\partial\phi_L(z))^2$ are singular at coincident points and must be defined carefully. The standard prescription is to use *normal ordering*, denoted by colons $:\cdots:$, to subtract the singular part. The quantum stress tensor is then written as

$$T(z) = -\,:(\partial\phi_L(z))^2:\,, \qquad \bar{T}(\bar{z}) = -\,:(\bar{\partial}\phi_R(\bar{z}))^2:\ . \tag{184}$$

With this definition, the operator product expansion of $T(z)$ with itself reproduces the Virasoro algebra with central charge $c = 1$ for the free boson [19, 5, 7]. We will return to these quantum aspects later; for now, it is enough to remember the classical form and the idea that T generates conformal transformations.

Glossary

- **Stress tensor / energy–momentum tensor** $T_{\mu\nu}$: An object that measures how the theory responds to changes in the metric. In flat space it encodes energy density, momentum flow, and stresses. In CFT it also generates conformal transformations [19, 12].
- **Components** $T(z)$ **and** $\bar{T}(\bar{z})$: In complex coordinates, the nonzero components of the stress tensor of a 2d CFT are T_{zz} and $T_{\bar{z}\bar{z}}$. We write them as $T(z)$ and $\bar{T}(\bar{z})$ and call them the holomorphic and antiholomorphic stress tensors [14, 5].
- **Free boson action in complex form**: The action

  $$S = \frac{1}{2\pi}\int d^2z\,\partial\phi\,\bar{\partial}\phi$$

  is the complex-coordinate version of the usual kinetic term for a free scalar in 2D. It is chosen so that the theory is conformally invariant and has central charge $c = 1$ [19, 12].
- **Chiral stress tensor**: The holomorphic stress tensor $T(z)$ depends only on z and generates holomorphic conformal transformations $z \to f(z)$; the antiholomorphic stress tensor $\bar{T}(\bar{z})$ generates $\bar{z} \to \bar{f}(\bar{z})$ [19, 7].
- **Normal ordering** $:\cdots:$ A prescription in quantum field theory to reorder creation and annihilation operators (or subtract short-distance singularities) so that composite operators like $(\partial\phi)^2$ are finite. For the free boson CFT, the quantum stress tensor is defined using normal ordering, $T(z) = -:(\partial\phi_L)^2:$ [5, 7].
- **Central charge** c: A number that appears in the algebra of the modes of the stress tensor (the Virasoro algebra). It measures the number of degrees of freedom in the CFT; for a single free boson, $c = 1$ [19, 12].

2.5 Operator product expansions in the free boson

In two-dimensional CFT, we often care about what happens when two operators are inserted very close to each other. The *operator product expansion* (OPE) is a systematic way to replace such a product of nearby operators by a sum of other local operators with known coefficient functions [11, 19].

2.5.1 Idea of an OPE.

Consider two local operators $\mathcal{O}_1(z_1)$ and $\mathcal{O}_2(z_2)$. When z_1 is close to z_2, their product can be expanded as

$$\mathcal{O}_1(z_1)\,\mathcal{O}_2(z_2) \;\sim\; \sum_k C_k(z_1 - z_2)\,\mathcal{O}_k(z_2)\,, \tag{185}$$

where the symbol $\sim$ means that this expansion is valid inside correlation functions in the limit $z_1 \to z_2$, and the sum runs over some set of local operators $\mathcal{O}_k$ [11]. The coefficient functions $C_k(z_1 - z_2)$ are (possibly singular) ordinary functions, typically powers of $(z_1 - z_2)$ or $1/(z_1 - z_2)^n$.

Very roughly, this is similar in spirit to a Taylor expansion of an ordinary function, but now we are expanding a *product of operators* instead of a single function . The most singular terms (with the highest negative powers of $z_1 - z_2$) dominate when the two operator insertions approach each other, while the less singular or regular terms are subleading [11].

2.5.2 A simple free-boson example.

Let us specialize to the free massless scalar (boson) in two dimensions and focus on the holomorphic part $\phi_L(z)$. We define the holomorphic current

$$J(z) \equiv \partial\phi_L(z)\,. \tag{186}$$

In this theory, the two-point function of J takes the form

$$\langle J(z)J(w)\rangle = \frac{1}{(z-w)^2}\,, \tag{187}$$

up to overall normalization conventions [**?**, 19]. This suggests the following OPE for $J(z)$ with itself:

$$J(z)\,J(w) \;\sim\; \frac{1}{(z-w)^2} + \text{less singular terms as } z \to w\,. \tag{188}$$

We usually write the omitted contributions as "..." or "regular terms". The key point is that the leading behaviour when $z \to w$ is captured by the simple singular piece $1/(z-w)^2$ [11].

2.5.3 OPEs inside correlation functions.

Equation (185) is not just a formal operator identity; it is a rule for simplifying correlation functions. For example, consider a three-point function with $J(z_1)J(z_2)$ and some other operator $\mathcal{O}(z_3)$:

$$\big\langle J(z_1)J(z_2)\mathcal{O}(z_3)\big\rangle. \tag{189}$$

If z_1 is close to z_2, we can replace the product $J(z_1)J(z_2)$ by its OPE (188), schematically

$$J(z_1)J(z_2) \sim \frac{1}{(z_1-z_2)^2} + \cdots, \tag{190}$$

and obtain

$$\big\langle J(z_1)J(z_2)\mathcal{O}(z_3)\big\rangle \;\sim\; \frac{1}{(z_1-z_2)^2}\big\langle \mathcal{O}(z_3)\big\rangle + \cdots, \tag{191}$$

where the dots denote less singular contributions coming from subleading terms in the OPE [11, 19]. In this way, OPEs allow us to systematically reduce N-point functions to $(N-1)$-point functions by fusing two operators together.

2.5.4 Stress tensor and its OPEs (preview).

For conformal field theories, the most important OPEs involve the holomorphic stress tensor $T(z)$. In any 2d CFT, there is a universal OPE of $T(z)$ with a primary operator $\mathcal{O}(w)$ of holomorphic weight h,

$$T(z)\,\mathcal{O}(w) \sim \frac{h}{(z-w)^2}\,\mathcal{O}(w) + \frac{1}{z-w}\,\partial\mathcal{O}(w) + \text{regular terms}, \tag{192}$$

and an OPE of $T(z)$ with itself of the form

$$T(z)\,T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular terms}, \tag{193}$$

where c is the central charge [11, 19]. In the free boson CFT we will later see explicitly that $c = 1$ [19].

These particular OPEs are extremely powerful. They encode how conformal symmetry acts on local operators and will be the main input for deriving the Virasoro algebra from the modes of $T(z)$ in the next subsection [11].

Glossary

- **Local operator**: An operator $\mathcal{O}(z,\bar{z})$ inserted at a specific point on the worldsheet; in the operator formalism it corresponds to a state in the Hilbert space [11].
- **Operator product expansion (OPE)**: An expansion of a product of nearby local operators, of the form

$$\mathcal{O}_1(z_1)\mathcal{O}_2(z_2) \sim \sum_k C_k(z_1-z_2)\,\mathcal{O}_k(z_2),$$

valid when $z_1 \to z_2$ inside correlation functions [11, **?**].

- **Singular term**: A term in the OPE whose coefficient behaves like a negative power of $(z_1 - z_2)$, such as $1/(z_1 - z_2)^2$, and therefore diverges as the two insertion points coincide .
- **Regular term**: A term in the OPE whose coefficient stays finite as $z_1 \to z_2$; these are less important very close to coincidence compared to the singular terms [11].
- **Free-boson current** $J(z)$: The holomorphic derivative field $J(z) = \partial\phi_L(z)$ in the free boson theory; its OPE with itself has the leading behaviour
$$J(z)J(w) \sim \frac{1}{(z-w)^2} + \cdots ,$$
which is the simplest explicit OPE example in this theory [19].

2.6 Virasoro algebra of the free boson

In this subsection we explain what the Virasoro generators L_n are, how they are defined from the stress tensor $T(z)$, how their commutators follow from the TT OPE, and what the central charge c means. We then state the Virasoro algebra and comment on its physical meaning in the free boson CFT [19, 11, 3, 7].

2.6.1 Modes of the stress tensor: definition.

On the complex plane, the holomorphic stress tensor $T(z)$ is expanded in a Laurent series,
$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}} . \tag{194}$$
The coefficients L_n are extracted by contour integrals:
$$L_n = \frac{1}{2\pi i} \oint_{|z|=\epsilon} dz\, z^{n+1} T(z) , \tag{195}$$
where the contour is a small circle around the origin [19, 3]. Mathematically, these L_n are just the Fourier/Laurent modes of $T(z)$; physically, they are the operators that generate infinitesimal conformal transformations acting on states and fields [11, 3].

2.6.2 Input: the TT OPE.

From the OPE subsection, we take as input the universal operator product expansion of the holomorphic stress tensor with itself,
$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular terms} , \tag{196}$$

where c is the central charge of the CFT [19, 11, 7]. For the free boson, one can compute $T(z)$ explicitly in terms of the field and show that this OPE holds with $c = 1$ [19, 3].

2.6.3 Strategy: commutators from a double contour integral.

We now want to compute the commutator $[L_m, L_n]$ using the definitions (195) and the OPE (196). Start from

$$L_m = \frac{1}{2\pi i} \oint dz\, z^{m+1} T(z), \qquad L_n = \frac{1}{2\pi i} \oint dw\, w^{n+1} T(w). \tag{197}$$

Then

$$[L_m, L_n] = \frac{1}{(2\pi i)^2} \left(\oint dz\, z^{m+1} \oint dw\, w^{n+1} T(z)T(w) - \oint dw\, w^{n+1} \oint dz\, z^{m+1} T(w)T(z) \right). \tag{198}$$

The difference between the two orderings of the contours can be written more compactly by using a small circle around w for the z–integral [19, **?**]:

$$[L_m, L_n] = \frac{1}{2\pi i} \oint dw\, w^{n+1} \left(\frac{1}{2\pi i} \oint_{|z-w|=\epsilon} dz\, z^{m+1} T(z)T(w) \right). \tag{199}$$

The inner integral only sees the singular part of the OPE as $z \to w$, so we can replace $T(z)T(w)$ by its OPE (196) inside that small contour [19, **?**].

2.6.4 Inner contour: picking residues from the OPE.

Write $z = w + \zeta$, so $dz = d\zeta$ and $z^{m+1} = (w+\zeta)^{m+1}$. Expand $(w+\zeta)^{m+1}$ as a power series in ζ:

$$(w+\zeta)^{m+1} = \sum_{k=0}^{\infty} \binom{m+1}{k} w^{m+1-k}\, \zeta^k. \tag{200}$$

We now insert the OPE (196) into (199) and integrate term by term. The basic residue rule is that

$$\oint_{|\zeta|=\epsilon} d\zeta\, \zeta^{\ell} = \begin{cases} 2\pi i & \text{if } \ell = -1, \\ 0 & \text{if } \ell \neq -1. \end{cases} \tag{201}$$

Thus only those powers of ζ that combine with the denominators in the OPE to give ζ^{-1} contribute [19, 3].

Central term (from the $(z-w)^{-4}$ piece).

From the first term in (196) we have

$$\frac{c}{2} \oint d\zeta\, \frac{(w+\zeta)^{m+1}}{\zeta^4}. \tag{202}$$

We need the ζ^3 term in $(w+\zeta)^{m+1}$ so that $\zeta^3/\zeta^4 = \zeta^{-1}$. This term is

$$\binom{m+1}{3} w^{m-2}\zeta^3. \tag{203}$$

Therefore the residue is

$$\frac{c}{2}\binom{m+1}{3} w^{m-2}\cdot 2\pi i. \tag{204}$$

Inserting this into the outer integral gives

$$\frac{1}{2\pi i}\oint dw\, w^{n+1}\left(\frac{c}{2}\cdot 2\pi i\binom{m+1}{3}w^{m-2}\right) = \frac{c}{2}\binom{m+1}{3}\frac{1}{2\pi i}\oint dw\, w^{m+n-1}. \tag{205}$$

The remaining contour integral is $2\pi i\,\delta_{m+n,0}$, so we obtain

$$\frac{c}{2}\binom{m+1}{3}\delta_{m+n,0} = \frac{c}{12}(m^3-m)\,\delta_{m+n,0}. \tag{206}$$

This is the central contribution to the commutator [19, 3, **?**].

Non-central term (from the $(z-w)^{-2}$ and $(z-w)^{-1}$ pieces).

From the second term in (196),

$$2\oint d\zeta\,\frac{(w+\zeta)^{m+1}}{\zeta^2}\,T(w), \tag{207}$$

we need the ζ^1 term in $(w+\zeta)^{m+1}$, which is $(m+1)w^m\zeta$. This gives

$$2(m+1)w^m T(w)\cdot 2\pi i. \tag{208}$$

From the third term in (196),

$$\oint d\zeta\,\frac{(w+\zeta)^{m+1}}{\zeta}\,\partial T(w), \tag{209}$$

we need the ζ^0 term in $(w+\zeta)^{m+1}$, which is w^{m+1}, giving

$$w^{m+1}\partial T(w)\cdot 2\pi i. \tag{210}$$

Putting these into the outer integral, we find

$$\frac{1}{2\pi i}\oint dw\, w^{n+1}\left[2(m+1)w^m T(w) + w^{m+1}\partial T(w)\right]. \tag{211}$$

For the ∂T term, integrate by parts,

$$\oint dw\, w^{m+1+n}\partial T(w) = -(m+n+2)\oint dw\, w^{m+n+1}T(w), \tag{212}$$

so the total coefficient becomes

$$\big[2(m+1)-(m+n+2)\big]\frac{1}{2\pi i}\oint dw\, w^{m+n+1}T(w) = (m-n)\frac{1}{2\pi i}\oint dw\, w^{m+n+1}T(w). \tag{213}$$

Using the definition (194), the last integral is exactly L_{m+n}, so this part of the commutator gives

$$(m-n)L_{m+n}. \tag{214}$$

2.6.5 Result: the Virasoro algebra.

Combining the central and non-central contributions, we arrive at

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m+n,0}\,. \tag{215}$$

This is the Virasoro algebra [19, 11, 3, 7]. If the central term were absent (i.e. $c = 0$), this would reduce to the Witt algebra, which is the algebra of infinitesimal conformal vector fields on the circle.

For a single free boson, one can start from $T(z) = -:(\partial\phi_L(z))^2:$ and the basic OPE $\partial\phi_L(z)\partial\phi_L(w) \sim 1/(z-w)^2$, and then compute the TT OPE explicitly to find $c = 1$ [19, 3]. Thus the free boson CFT realizes the Virasoro algebra with central charge $c = 1$.

2.6.6 Physical interpretation (radial quantization).

In radial quantization, circles $|z| = \text{const}$ play the role of equal-time slices, and the radial coordinate plays the role of time [**?**]. In this picture, L_0 acts as a Hamiltonian: its eigenvalues are the holomorphic scaling dimensions of states. The operators L_{-n} with $n > 0$ act like creation operators, generating descendants from a highest-weight state, while L_n with $n > 0$ act as annihilation operators [11, 3].

A highest-weight state $|\Delta\rangle$ satisfies $L_0|\Delta\rangle = \Delta|\Delta\rangle$ and $L_n|\Delta\rangle = 0$ for all $n > 0$, and the infinite tower of states obtained by acting with L_{-n} forms a Verma module over the Virasoro algebra [3]. In the free boson CFT, states created by vertex operators $V_\alpha(z,\bar{z}) =: e^{i\alpha\phi(z,\bar{z})}:$ are concrete examples of highest-weight states whose descendants are generated by the L_{-n} [19, 11].

2.7 Two-point functions on the plane

In any quantum field theory, correlation functions encode the observable information of the theory. In a two-dimensional conformal field theory, the simplest non-trivial correlators are the *two-point functions* of local operators, which already feel the effect of conformal symmetry in a strong way [19, 11, 6, 12, 15].

2.7.1 Definition and physical meaning.

Let $\mathcal{O}_1(z_1,\bar{z}_1)$ and $\mathcal{O}_2(z_2,\bar{z}_2)$ be local primary fields. Their two-point function on the complex plane is

$$G_2(z_1,z_2;\bar{z}_1,\bar{z}_2) \;\equiv\; \langle \mathcal{O}_1(z_1,\bar{z}_1)\,\mathcal{O}_2(z_2,\bar{z}_2)\rangle\,. \tag{216}$$

In the path integral language, this is the vacuum expectation value of the product of operators,

$$G_2(z_1,z_2;\bar{z}_1,\bar{z}_2) = \frac{1}{Z}\int \mathcal{D}\Phi\; \mathcal{O}_1(z_1,\bar{z}_1)\,\mathcal{O}_2(z_2,\bar{z}_2)\,e^{-S[\Phi]}, \tag{217}$$

where $S[\Phi]$ is the Euclidean action and Z is the partition function without insertions. Physically, G_2 measures the correlation between the two operator insertions; in a scale-invariant theory such as a CFT, this correlation will decay as a power of the distance between the points rather than exponentially [19, 11].

2.7.2 translation invariance.

We now derive the general form of G_2 using symmetry. First, the theory is invariant under translations

$$z \to z + a, \qquad \bar{z} \to \bar{z} + \bar{a}, \tag{218}$$

for constant complex a. This implies that the two-point function can only depend on the differences of coordinates, not on their absolute positions [19, 6]:

$$G_2(z_1, z_2; \bar{z}_1, \bar{z}_2) = G_2(z_{12}, \bar{z}_{12}), \qquad z_{12} \equiv z_1 - z_2,\ \bar{z}_{12} \equiv \bar{z}_1 - \bar{z}_2. \tag{219}$$

Intuitively: if we shift both insertion points by the same amount, the correlator should not change.

2.7.3 rotation invariance (Euclidean).

On the Euclidean plane, the theory is invariant under global rotations

$$z \to e^{i\theta} z, \qquad \bar{z} \to e^{-i\theta} \bar{z}, \tag{220}$$

where θ is a real angle. This implies that G_2 can only depend on the distance $|z_{12}|$ between the two points, up to possible phase factors associated with spin; for scalar primaries, one finds

$$G_2(z_{12}, \bar{z}_{12}) = f(|z_{12}|), \tag{221}$$

for some function f [15, 6]. In two dimensions we usually keep the holomorphic and antiholomorphic dependence explicit and write f as a product of a function of z_{12} and a function of $\bar{z}_{12}$.

2.7.4 scale invariance and power-law behaviour.

In a conformal field theory, we also have invariance under global dilations

$$z \to \lambda z, \qquad \bar{z} \to \lambda \bar{z}, \qquad \lambda > 0. \tag{222}$$

Primary fields transform under such a scaling as

$$\mathcal{O}_i(z, \bar{z}) \ \to\ \lambda^{-h_i} \lambda^{-\bar{h}_i}\, \mathcal{O}_i(\lambda z, \lambda \bar{z}), \tag{223}$$

where $(h_i, \bar{h}_i)$ are the holomorphic and antiholomorphic conformal weights of $\mathcal{O}_i$ [19, **?**].

Under a global scaling, the two-point function transforms as

$$G_2(z_1, z_2; \bar{z}_1, \bar{z}_2) = \langle \mathcal{O}_1(z_1, \bar{z}_1)\, \mathcal{O}_2(z_2, \bar{z}_2) \rangle \\ \to \lambda^{-h_1-\bar{h}_1} \lambda^{-h_2-\bar{h}_2} \langle \mathcal{O}_1(\lambda z_1, \lambda \bar{z}_1)\, \mathcal{O}_2(\lambda z_2, \lambda \bar{z}_2) \rangle \,. \tag{224}$$

But by scale invariance of the vacuum we must also have

$$G_2(z_1, z_2; \bar{z}_1, \bar{z}_2) = \langle \mathcal{O}_1(\lambda z_1, \lambda \bar{z}_1)\, \mathcal{O}_2(\lambda z_2, \lambda \bar{z}_2) \rangle \,. \tag{225}$$

Combining these two relations, and using translation invariance to express everything in terms of z_{12} and $\bar{z}_{12}$, we get the constraint

$$G_2(z_{12}, \bar{z}_{12}) = \lambda^{-h_1-\bar{h}_1-h_2-\bar{h}_2}\, G_2(\lambda z_{12}, \lambda \bar{z}_{12}). \tag{226}$$

The only functions that satisfy this scaling property are power laws. Thus we must have

$$G_2(z_{12}, \bar{z}_{12}) = \frac{C_{12}}{z_{12}^{\alpha}\, \bar{z}_{12}^{\bar{\alpha}}}, \tag{227}$$

for some exponents $\alpha, \bar{\alpha}$ and constant C_{12}, with the additional requirement that

$$\alpha + \bar{\alpha} = h_1 + \bar{h}_1 + h_2 + \bar{h}_2. \tag{228}$$

More refined arguments using the full conformal group (including inversions) show that for primaries one can choose

$$\alpha = h_1 + h_2, \qquad \bar{\alpha} = \bar{h}_1 + \bar{h}_2 \tag{229}$$

up to normalizations [11, 6, 15].

2.7.5 matching weights and final form.

We now use the fact that the two-point function must transform covariantly under the global conformal group generated by L_{-1}, L_0, L_1. Solving the corresponding global Ward identities shows that the two-point function can be non-zero only if the conformal weights of the two operators match: $h_1 = h_2$ and $\bar{h}_1 = \bar{h}_2$ [11, 12, 15].

Putting everything together, the final, standard result for the two-point function of primary fields on the plane is

$$\boxed{\langle \mathcal{O}_1(z_1, \bar{z}_1)\, \mathcal{O}_2(z_2, \bar{z}_2) \rangle = \frac{C_{12}\, \delta_{h_1,h_2}\, \delta_{\bar{h}_1,\bar{h}_2}}{z_{12}^{2h_1}\, \bar{z}_{12}^{2\bar{h}_1}}} \,, \tag{230}$$

where C_{12} is a constant that depends on the normalization of the operators. A common convention is to choose a basis of primaries such that $C_{12} = \delta_{12}$ when the operators are identical, but the precise normalization is a matter of choice [19, 11, 6].

Glossary

- **Two-point function**: Correlator of two local operators,

$$\langle \mathcal{O}_1(z_1, \bar{z}_1)\, \mathcal{O}_2(z_2, \bar{z}_2) \rangle,$$

  measuring how the insertion of $\mathcal{O}_1$ is correlated with the insertion of $\mathcal{O}_2$.

- **Primary field**: A local operator with definite conformal weights $(h, \bar{h})$ whose transformation under conformal maps is simple, and whose $T\mathcal{O}$ OPE has the form used earlier in the chapter.

- **Translation invariance**: Symmetry under $z \to z + a$; it implies that correlators depend only on coordinate differences such as $z_{12} = z_1 - z_2$.

- **Scale invariance**: Symmetry under $z \to \lambda z$; it implies that correlators behave as power laws in distances, with exponents fixed by the scaling dimensions.

- **Global conformal invariance**: Symmetry under the full $SL(2, \mathbb{C})$ Möbius group generated by L_{-1}, L_0, L_1; it further constrains 2- and 3-point functions and forces matching of the conformal weights in non-zero two-point functions.

2.8 A first conformal Ward identity

In this subsection we write one basic conformal Ward identity and explain it in simple terms. The idea is that the stress tensor $T(z)$ inside a correlator tells us how that correlator changes when we make an infinitesimal conformal transformation [19, 11, 6].

2.8.1 Setup and notation.

We consider N primary fields

$$\mathcal{O}_i(z_i, \bar{z}_i), \qquad i = 1, \ldots, N, \tag{231}$$

with holomorphic weights h_i. We write their N-point function as

$$G_N(z_1, \ldots, z_N; \bar{z}_1, \ldots, \bar{z}_N) \equiv \langle \mathcal{O}_1(z_1, \bar{z}_1) \cdots \mathcal{O}_N(z_N, \bar{z}_N) \rangle\,. \tag{232}$$

From our OPE subsection we know that the stress tensor $T(z)$ has the OPE with a primary $\mathcal{O}_i$:

$$T(z)\, \mathcal{O}_i(w, \bar{w}) \sim \frac{h_i}{(z-w)^2}\, \mathcal{O}_i(w, \bar{w}) + \frac{1}{z-w}\, \partial_w \mathcal{O}_i(w, \bar{w}) + \text{regular terms}, \tag{233}$$

valid when $z \to w$. Here ∂_w means the derivative with respect to w [19, 11].

2.8.2 Meaning of the $T\mathcal{O}$ OPE again.

The two singular terms in (233) have very concrete meanings:

- The $\dfrac{h_i}{(z-w)^2}\,\mathcal{O}_i$ term knows about the conformal weight h_i, i.e. how $\mathcal{O}_i$ scales under local dilations around w.
- The $\dfrac{1}{z-w}\,\partial_w\mathcal{O}_i$ term knows about shifts of the position w, i.e. how the operator moves when we change its insertion point.

So this OPE is really a compact way to encode the effect of a small conformal transformation on the operator $\mathcal{O}_i$ at w [19, 11].

2.8.3 Ward identity: $T(z)$ inside a correlator.

We now look at a correlator with one insertion of the stress tensor:

$$\left\langle T(z)\prod_{i=1}^{N}\mathcal{O}_i(z_i,\bar{z}_i)\right\rangle. \tag{234}$$

As z moves on the plane, it can come close to any of the points z_i. Near $z=z_i$, the product $T(z)\mathcal{O}_i(z_i,\bar{z}_i)$ behaves as in the OPE (233), and the other operators $\mathcal{O}_j(z_j,\bar{z}_j)$ with $j\neq i$ just come along for the ride.

Using this, one finds that the singular behaviour of the correlator as $z\to z_i$ is

$$\left\langle T(z)\prod_{i=1}^{N}\mathcal{O}_i(z_i,\bar{z}_i)\right\rangle \sim \frac{h_i}{(z-z_i)^2}\left\langle\prod_{j=1}^{N}\mathcal{O}_j(z_j,\bar{z}_j)\right\rangle+\frac{1}{z-z_i}\partial_{z_i}\left\langle\prod_{j=1}^{N}\mathcal{O}_j(z_j,\bar{z}_j)\right\rangle \tag{235}$$

for each i, where ∂_{z_i} means derivative with respect to the coordinate z_i in the correlator [19, **?**, 6].

Adding the contributions from all insertions, we obtain the *holomorphic conformal Ward identity*:

$$\boxed{\left\langle T(z)\prod_{i=1}^{N}\mathcal{O}_i(z_i,\bar{z}_i)\right\rangle=\sum_{i=1}^{N}\left(\frac{h_i}{(z-z_i)^2}+\frac{1}{z-z_i}\,\partial_{z_i}\right)\left\langle\prod_{j=1}^{N}\mathcal{O}_j(z_j,\bar{z}_j)\right\rangle.} \tag{236}$$

This formula says: inserting $T(z)$ is equivalent to letting T act on each operator with its OPE and summing the effects. The $1/(z-z_i)^2$ and $1/(z-z_i)$ poles tell us exactly how the correlator responds to a small conformal transformation around z_i [19, 11].

Explanation of each term in (236)

- $T(z)$: The holomorphic stress tensor; it generates local conformal transformations in the holomorphic sector of the theory.

- $\mathcal{O}_i(z_i, \bar{z}_i)$: Primary fields with holomorphic weights h_i; they transform in a simple, known way under conformal maps.
- h_i: The holomorphic conformal weight of $\mathcal{O}_i$; it measures how the operator scales when we zoom in or out around z_i.
- ∂_{z_i}: The derivative with respect to z_i acting on the whole N-point function. It appears because conformal transformations move the insertion point z_i.
- $(z - z_i)^{-2}$: The coefficient of this term in the OPE is tied to h_i; it reflects the local scaling of $\mathcal{O}_i$.
- $(z - z_i)^{-1}$: The coefficient of this term is tied to ∂_{z_i}; it reflects the shift of the insertion point.

These meanings are standard in 2d CFT discussions of Ward identities and can be found, in slightly different words, in many references [19, 11, 6, 12].

Check for $N = 2$ using the primary two-point function We now check (236) in a simple case with $N = 2$. Take two identical primary fields $\mathcal{O}(z, \bar{z})$ with weights $(h, \bar{h})$, and use the standard two-point function on the plane:

$$\langle \mathcal{O}(z_1, \bar{z}_1)\, \mathcal{O}(z_2, \bar{z}_2) \rangle = \frac{C}{(z_{12})^{2h} (\bar{z}_{12})^{2\bar{h}}}, \qquad z_{12} = z_1 - z_2, \tag{237}$$

with C a constant [**?**, 6, **?**].

The Ward identity (236) for $N = 2$ says

$$\begin{aligned} \langle T(z)\, \mathcal{O}(z_1, \bar{z}_1)\, \mathcal{O}(z_2, \bar{z}_2) \rangle = &\left(\frac{h}{(z - z_1)^2} + \frac{1}{z - z_1}\, \partial_{z_1} \right) \langle \mathcal{O}(z_1, \bar{z}_1)\, \mathcal{O}(z_2, \bar{z}_2) \rangle \\ &+ \left(\frac{h}{(z - z_2)^2} + \frac{1}{z - z_2}\, \partial_{z_2} \right) \langle \mathcal{O}(z_1, \bar{z}_1)\, \mathcal{O}(z_2, \bar{z}_2) \rangle . \end{aligned} \tag{238}$$

We already know the explicit form of the two-point function, so we can compute the derivatives:

$$\partial_{z_1} \langle \mathcal{O}(z_1, \bar{z}_1)\, \mathcal{O}(z_2, \bar{z}_2) \rangle = \partial_{z_1} \left(\frac{C}{(z_1 - z_2)^{2h} (\bar{z}_{12})^{2\bar{h}}} \right) = -\frac{2h\, C}{(z_{12})^{2h+1} (\bar{z}_{12})^{2\bar{h}}}, \tag{239}$$

$$\partial_{z_2} \langle \mathcal{O}(z_1, \bar{z}_1)\, \mathcal{O}(z_2, \bar{z}_2) \rangle = \frac{2h\, C}{(z_{12})^{2h+1} (\bar{z}_{12})^{2\bar{h}}}, \tag{240}$$

since $\partial_{z_2} z_{12} = -1$ [6, **?**]. Plugging these expressions into (238) gives a concrete expression for the three-point function $\langle T\mathcal{O}\mathcal{O} \rangle$ in terms of known functions of z, z_1, z_2. This explicit expression matches what one finds by directly computing the OPE of $T(z)$ with two primaries, and is the simplest non-trivial check of the Ward identity (236) [19, 11, 6].

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