Scalar Field Dynamics in Global AdS_3 and the BTZ Black Hole

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Abstract

This independent study develops a set of tools for analyzing a real scalar field in global AdS_3 and in the non-rotating BTZ black hole background. We begin with a review of canonical quantization of a free scalar field in flat spacetime, emphasizing the construction of Fock space, conserved charges, and the role of symmetries. [1] We then formulate the Klein–Gordon equation in curved spacetime and apply it to global AdS_3, deriving the radial equation, studying its asymptotic behavior, and determining the normal mode spectrum under appropriate boundary conditions. [2] Finally, we investigate scalar field dynamics in the BTZ black hole geometry, focusing on the structure of mode solutions and their behavior near the horizon and at the asymptotic AdS boundary, and we use these solutions to discuss quasinormal modes. [3, 4] The results provide a concrete example of quantum field theory in curved spacetime and a useful stepping stone toward studying aspects of the AdS/CFT correspondence in three dimensions. [5]

1 Motivation

Quantum field theory in curved spacetime plays a central role in modern approaches to black hole physics, cosmology, and the interface between gravity and quantum theory. [1] Phenomena such as Hawking radiation and particle production in expanding universes demonstrate that quantum fields can exhibit qualitatively new behavior in nontrivial gravitational backgrounds, but explicit calculations are often technically challenging.

Anti-de Sitter (AdS) spacetime provides a particularly valuable laboratory for studying quantum fields in curved geometry. [1, 2] Its high degree of symmetry allows many computations to be carried out explicitly, while its timelike conformal boundary leads to interesting boundary conditions and a discrete spectrum of normal modes. In three dimensions, global AdS_3 is especially tractable, yet already exhibits many of the conceptual features that appear in higher-dimensional AdS spaces. [2]

The discovery of the AdS/CFT correspondence has further enhanced the importance of AdS spacetimes. [5] In this duality, quantum gravity (or string theory) in an asymptotically AdS spacetime is conjectured to be equivalent to a conformal field theory living on the boundary. AdS_3/CFT_2 is the best-understood example and has provided deep insights into black hole microstates, entanglement, and holography. From this perspective, understanding the dynamics of simple bulk fields, such as a free scalar, is an essential first step toward exploring more sophisticated holographic phenomena. [5]

The BTZ black hole is a solution of three-dimensional gravity with a negative cosmological constant and can be viewed as a quotient of AdS_3. [1] It provides a remarkably simple model of a black hole that nevertheless possesses many familiar features of higher-dimensional black holes, including an event horizon, thermodynamic properties, and a well-defined notion of Hawking temperature. Studying scalar fields in the BTZ geometry therefore offers a clean setting in which to investigate quantum field theory in the presence of a black hole in AdS and to analyze quasinormal modes as characteristic damped oscillations of the geometry. [3,4]

The goal of this project is to develop a coherent set of techniques for analyzing a real scalar field in global AdS_3 and in the non-rotating BTZ black hole background. By starting from the canonical quantization of a free scalar field in flat spacetime and then systematically extending the analysis to curved geometries, we aim to build intuition for how geometry, boundary conditions, and symmetry structures affect field dynamics. [1,2] The resulting tools and explicit solutions are intended as a foundation for future studies of quantum fields in curved spacetime and for applications to the AdS/CFT correspondence in three dimensions. [5]

2 Scalar Field in Curved Spacetime

2.1 Klein–Gordon equation in curved spacetime

Let $\phi(x)$ be a real scalar field on a spacetime with metric $g_{\mu\nu}(x)$. A natural curved-space generalization of the flat Lagrangian is

$$\mathcal{L} = \frac{1}{2}\, g^{\mu\nu}(x)\, \nabla_\mu\phi\, \nabla_\nu\phi - \frac{1}{2}m^2\phi^2, \tag{1}$$

where ∇_μ is the Levi–Civita covariant derivative associated with $g_{\mu\nu}$. [1,2] The corresponding action is

$$S = \int d^{d+1}x\, \sqrt{-g}\, \left[\frac{1}{2}\, g^{\mu\nu}\nabla_\mu\phi\, \nabla_\nu\phi - \frac{1}{2}m^2\phi^2\right], \tag{2}$$

with $g = \det(g_{\mu\nu})$. Varying with respect to ϕ yields

$$\delta S = \int d^{d+1}x\, \sqrt{-g}\, \left[g^{\mu\nu}\nabla_\mu\phi\, \nabla_\nu(\delta\phi) - m^2\phi\, \delta\phi\right]. \tag{3}$$

Integrating by parts and discarding boundary terms, one arrives at the curved-space Klein–Gordon equation

$$\left(\Box_g - m^2\right)\phi = 0, \qquad \Box_g\phi \equiv \frac{1}{\sqrt{-g}}\, \partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\right). \tag{4}$$

This is the basic equation we will study on global AdS_3 and on the BTZ black hole background. [2,3]

2.2 Plane waves and the failure of flat-space intuition

In flat space, solutions of the Klein–Gordon equation can be written as plane waves $e^{-ik_\mu x^\mu}$ with a simple dispersion relation. [1] In a generic curved spacetime, such global plane-wave solutions do not exist; instead, the geometry typically forces a more complicated mode structure with discrete or continuous spectra depending on the global properties of $g_{\mu\nu}$ and on boundary conditions. [2,3] AdS_3 is a key example: the presence of a timelike conformal boundary at spatial infinity leads to a discrete set of normal modes for ϕ once suitable boundary conditions are imposed.

3 Scalar Field on Global AdS_3

3.1 Geometry of global AdS_3

Three-dimensional anti-de Sitter space (AdS_3) can be defined as the maximally symmetric solution of the vacuum Einstein equations with negative cosmological constant $\Lambda = -1/L^2$. [1,2] In global coordinates (t, ρ, φ), the metric takes the form

$$ds^2 = L^2 \left(-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\varphi^2\right), \qquad \rho \in [0, \infty), \quad \varphi \sim \varphi + 2\pi. \tag{5}$$

The inverse metric and determinant are

$$g^{tt} = -\frac{1}{L^2 \cosh^2 \rho}, \quad g^{\rho\rho} = \frac{1}{L^2}, \quad g^{\varphi\varphi} = \frac{1}{L^2 \sinh^2 \rho}, \tag{6}$$

$$\sqrt{-g} = L^3 \cosh \rho \sinh \rho. \tag{7}$$

The conformal boundary of AdS_3 lies at $\rho \to \infty$ and has the topology of $S^1 \times \mathbb{R}$, which plays a central role in AdS/CFT constructions. [2,5]

3.2 Covariant d'Alembertian on AdS_3

For a scalar field, the covariant d'Alembertian on AdS_3 is

$$\Box_g \phi = \frac{1}{\sqrt{-g}} \partial_\mu \Big(\sqrt{-g} \, g^{\mu\nu} \partial_\nu \phi \Big). \tag{8}$$

Using the explicit metric components above, one finds

$$\Box_g \phi = -\frac{1}{L^2 \cosh^2 \rho} \, \partial_t^2 \phi + \frac{1}{L^2 \sinh \rho \cosh \rho} \, \partial_\rho \big(\sinh \rho \cosh \rho \, \partial_\rho \phi \big) + \frac{1}{L^2 \sinh^2 \rho} \, \partial_\varphi^2 \phi. \tag{9}$$

Substituting into the Klein–Gordon equation $(\Box_g - m^2)\phi = 0$ gives

$$-\frac{1}{\cosh^2 \rho} \, \partial_t^2 \phi + \frac{1}{\sinh \rho \cosh \rho} \, \partial_\rho \big(\sinh \rho \cosh \rho \, \partial_\rho \phi \big) + \frac{1}{\sinh^2 \rho} \, \partial_\varphi^2 \phi - m^2 L^2 \, \phi = 0. \tag{10}$$

This is the master equation for a scalar field on global AdS_3. [2]

3.3 Separation of variables and radial equation

The AdS_3 metric is invariant under time translations and rotations in φ. It is therefore natural to seek separable solutions of the form

$$\phi(t, \rho, \varphi) = e^{-i\omega t} \, e^{i\ell\varphi} \, R(\rho), \qquad \ell \in \mathbb{Z}. \tag{11}$$

Substituting this ansatz into the Klein–Gordon equation yields a radial ordinary differential equation for $R(\rho)$,

$$\frac{1}{\sinh \rho \cosh \rho} \frac{d}{d\rho} \left(\sinh \rho \cosh \rho \, \frac{dR}{d\rho} \right) - \left(\frac{\omega^2}{\cosh^2 \rho} - \frac{\ell^2}{\sinh^2 \rho} + m^2 L^2 \right) R(\rho) = 0. \tag{12}$$

This radial equation, together with appropriate boundary conditions at $\rho = 0$ and $\rho \to \infty$, encodes the scalar field dynamics in global AdS_3. [2]

3.4 Asymptotic analysis and boundary conditions

To understand the allowed mode solutions, it is useful to analyze the behavior of $R(\rho)$ near the origin and near the boundary. [2, 5]

3.4.1 Near the origin $\rho \to 0$

For small ρ, the hyperbolic functions behave as

$$\sinh\rho \sim \rho, \qquad \cosh\rho \sim 1. \tag{13}$$

In this limit, the radial equation reduces to

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\,\frac{dR}{d\rho}\right) - \frac{\ell^2}{\rho^2}\,R(\rho) \simeq 0. \tag{14}$$

Looking for power-law solutions $R(\rho) \sim \rho^{\alpha}$ gives

$$\alpha(\alpha-1) + \alpha - \ell^2 = 0 \quad\Rightarrow\quad \alpha^2 = \ell^2, \tag{15}$$

so the two independent behaviors are

$$R(\rho) \sim \rho^{|\ell|}, \qquad R(\rho) \sim \rho^{-|\ell|}. \tag{16}$$

Regularity at the origin requires the scalar field to remain finite there, which excludes the divergent solution and enforces

$$R(\rho) \sim \rho^{|\ell|} \quad \text{as} \quad \rho \to 0. \tag{17}$$

3.4.2 Near the AdS boundary $\rho \to \infty$

For large ρ, one has

$$\sinh\rho \sim \cosh\rho \sim \tfrac{1}{2}e^{\rho}. \tag{18}$$

In this limit, the radial equation simplifies to

$$\frac{d^2R}{d\rho^2} + 2\,\frac{dR}{d\rho} - m^2L^2\,R(\rho) \simeq 0. \tag{19}$$

Looking for exponential behavior $R(\rho) \sim e^{-\Delta\rho}$ leads to

$$\Delta(\Delta-2) = m^2L^2, \tag{20}$$

with solutions

$$\Delta_{\pm} = 1 \pm \sqrt{1+m^2L^2}. \tag{21}$$

Thus, near the AdS boundary the radial function behaves as

$$R(\rho) \sim A\,e^{-\Delta_-\rho} + B\,e^{-\Delta_+\rho}. \tag{22}$$

For m^2 above the Breitenlohner–Freedman bound in AdS$_3$,

$$m^2L^2 \geq -1, \tag{23}$$

both solutions are stable, and for $-1 < m^2L^2 < 0$ both falloffs can be normalizable, allowing for different choices of boundary conditions. [2, 5] In the standard quantization used here, one typically imposes that the slower-decaying mode vanish ($A = 0$), so that the field decays as $e^{-\Delta_+\rho}$ at the boundary. [5]

3.5 Normal mode spectrum (qualitative)

The combination of regularity at the origin and admissible falloff at infinity leads to a discrete set of normal modes with frequencies $\omega_{n,\ell}$, where n is a non-negative radial quantum number. [2,5] In AdS_3 these frequencies can be written in terms of the conformal dimension Δ_+ associated with the scalar mass,

$$\omega_{n,\ell} = \pm\left(2n + |\ell| + \Delta_+\right), \tag{24}$$

so that the spectrum is evenly spaced in n and depends on the angular momentum ℓ. [5] These normal modes reflect the fact that global AdS acts effectively like a confining box: waves reflect off the timelike boundary and do not escape to infinity.

4 Scalar Field on the BTZ Black Hole

4.1 Non-rotating BTZ geometry

The non-rotating BTZ black hole is a solution of three-dimensional gravity with negative cosmological constant $\Lambda = -1/L^2$ and line element [1]

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2, \qquad f(r) = -M + \frac{r^2}{L^2}, \tag{25}$$

with $M > 0$ the mass parameter and $\varphi \sim \varphi + 2\pi$. The event horizon is located at

$$r_+ = L\sqrt{M}, \tag{26}$$

and the spacetime is asymptotically AdS_3 as $r \to \infty$. [1,2]

4.2 Klein–Gordon equation and radial ODE

In this background the inverse metric and determinant are

$$g^{tt} = -\frac{1}{f(r)}, \quad g^{rr} = f(r), \quad g^{\varphi\varphi} = \frac{1}{r^2}, \quad \sqrt{-g} = r. \tag{27}$$

The d'Alembertian acting on ϕ is

$$\Box_g\phi = -\frac{1}{f(r)}\,\partial_t^2\phi + \frac{1}{r}\partial_r\big(rf(r)\,\partial_r\phi\big) + \frac{1}{r^2}\,\partial_\varphi^2\phi. \tag{28}$$

The Klein–Gordon equation $(\Box_g - m^2)\phi = 0$ thus reads

$$-\frac{1}{f(r)}\,\partial_t^2\phi + \frac{1}{r}\partial_r\big(rf(r)\,\partial_r\phi\big) + \frac{1}{r^2}\,\partial_\varphi^2\phi - m^2\phi = 0. \tag{29}$$

We again separate variables as

$$\phi(t, r, \varphi) = e^{-i\omega t}\, e^{i\ell\varphi}\, R(r), \qquad \ell \in \mathbb{Z}, \tag{30}$$

which leads to the radial equation

$$\frac{1}{r}\frac{d}{dr}\left(rf(r)\,\frac{dR}{dr}\right) + \left(\frac{\omega^2}{f(r)} - \frac{\ell^2}{r^2} - m^2\right) R(r) = 0. \tag{31}$$

This is the starting point for the analysis of scalar perturbations of the BTZ black hole. [3,4]

4.3 Near-horizon and asymptotic behavior

Near the event horizon $r \to r_+$, one has $f(r) \simeq f'(r_+)(r - r_+)$ with $f'(r_+) = 2r_+/L^2$. Introducing the tortoise coordinate r_* defined by $dr_* = dr/f(r)$, the radial equation for modes behaves near the horizon like

$$\frac{d^2R}{dr_*^2} + \omega^2 R \simeq 0, \tag{32}$$

with solutions $R \sim e^{\pm i\omega r_*}$. The sign choice corresponds to ingoing or outgoing waves at the horizon; for quasinormal modes one imposes purely ingoing behavior. [3, 4]

As $r \to \infty$, the BTZ metric approaches that of AdS_3, and the scalar field again exhibits two possible falloffs characterized by $\Delta_\pm$ as in the global AdS_3 case. [2, 5] For standard AdS boundary conditions, one requires that the field vanish sufficiently fast so that the energy flux through the boundary is zero. This selects the faster-decaying falloff and plays an analogous role to the $e^{-\Delta_+\rho}$ condition in global AdS_3.

4.4 Quasinormal modes (qualitative)

Quasinormal modes (QNMs) are defined as solutions of the radial equation that satisfy

- purely ingoing behavior at the horizon,
- normalizable (or Dirichlet-like) behavior at the AdS boundary.

These conditions turn the radial equation into an eigenvalue problem for complex frequencies $\omega_{n,\ell}$. [3, 4] In the BTZ case, this problem can be solved exactly, and the resulting QNM spectrum can be written in terms of left- and right-moving temperatures in the dual CFT. [4, 5] The imaginary part of $\omega_{n,\ell}$ encodes the damping rate of the mode, reflecting the decay of perturbations due to absorption at the horizon and reflection at the AdS boundary.

Although a full derivation lies beyond the scope of this short report, the structure of the BTZ scalar QNMs provides a clean example of how black hole perturbation theory and AdS boundary conditions combine to produce a discrete set of damped oscillations, which play an important role in holographic interpretations and in the study of thermalization in the dual CFT. [4]

5 Conclusion

We have derived and analyzed the Klein–Gordon equation for a massive scalar field in global AdS_3 and in the non-rotating BTZ black hole background. In global AdS_3, regularity at the origin and appropriate falloff at the timelike boundary lead to a discrete normal mode spectrum whose frequencies are determined by the scalar mass and angular momentum quantum number. [2, 5] In the BTZ geometry, the presence of an event horizon changes the boundary-value problem: imposing ingoing behavior at the horizon and suitable decay at infinity selects a discrete set of quasinormal modes with complex frequencies, characterizing the relaxation of perturbations. [3, 4]

These examples provide a compact but instructive illustration of scalar field dynamics in AdS and black hole spacetimes, and they form a useful foundation for future work on holography, correlation functions, and more general field content in $\mathrm{AdS}_3/\mathrm{CFT}_2$. [5]

References

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