Assignment – 3, QFT

TOPIC:- ALGEBRAIC STRUCTURES IN QFT

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Q.1) The Weyl representation of gamma matrices is given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$i = 1, 2, 3$

where σ^i are the pauli matrices and 1 denotes 2×2 Identity matrix.

Show that these matrices satisfy the Clifford Algebra.

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

Solution:→

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad i = 1, 2, 3$$

we know that pauli matrices satisfy

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} I_{2\times 2}$$

$$\Rightarrow \{\gamma^0, \gamma^0\} = (\gamma^0)^2 = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{4\times 4}$$

$$= 2\, I_{4\times 4} = 2\eta^{00} I$$

$$\Rightarrow \{\gamma^0, \gamma^i\} = \gamma^0\gamma^i + \gamma^i\gamma^0$$

$$= \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} + \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 = 2\eta^{0i} I$$

$$\Rightarrow \{\gamma^i, \gamma^j\} = \gamma^i\gamma^j + \gamma^j\gamma^i = -\begin{pmatrix} \sigma^i\sigma^j + \sigma^j\sigma^i & 0 \\ 0 & \sigma^i\sigma^j + \sigma^j\sigma^i \end{pmatrix}$$

$$= -\begin{pmatrix} \{\sigma^i, \sigma^j\} & 0 \\ 0 & \{\sigma^i, \sigma^j\} \end{pmatrix}$$

$$\{\gamma^i, \gamma^j\} = -2\delta^{ij} I = 2\eta^{ij} I$$

hence, $\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_{4\times4}}$

Hence the weyl Gamma matrices satisfy the clifford algebra

Q.2. Assume gamma matrices satisfy the clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

show that a) $Tr(\gamma^\mu) = 0$

b) $Tr(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$

c) $Tr(\gamma^\mu \gamma^\nu \gamma^\beta) = 0$

Solution :- As we have assumed $\gamma \rightarrow$ satisfies clifford algebra

b) $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} 1$

taking traces of both side $\rightarrow$

$$Tr(\gamma^\mu \gamma^\nu) + Tr(\gamma^\nu \gamma^\mu) = 2\eta^{\mu\nu} Tr(I)$$

$\Rightarrow$ $2\, Tr(\gamma^\mu \gamma^\nu) = 2\eta^{\mu\nu} Tr(I)$ $\quad [\because Tr(\gamma^\mu \gamma^\nu) = Tr(\gamma^\nu \gamma^\mu)]$

$\Rightarrow$ $Tr(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$ $\quad [\because Tr(I)_{4\times4} = 4]$

$\Rightarrow$ $\boxed{Tr(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}}$

a) $\gamma^\nu \gamma^\mu = -\gamma^\mu \gamma^\nu$ — from clifford Algebra

multiply by γ^ν on right

$$\gamma^\nu \gamma^\mu \gamma^\nu = -\gamma^\mu (\gamma^\nu)^2 \qquad [\because (\gamma^\nu)^2 = \eta^{\nu\nu} I]$$

$\Rightarrow$ $\gamma^\mu = -\eta^{\nu\nu} \gamma^\nu \gamma^\mu \gamma^\nu$

taking trace on both side,

$$Tr(\gamma^\mu) = -\eta^{\nu\nu}\, Tr(\gamma^\nu \gamma^\mu \gamma^\nu)$$

$$Tr(\gamma^\mu) = -Tr(\gamma^\mu)$$

[using cyclicity →
$Tr(\gamma^\nu\gamma^\mu\gamma^\nu) = Tr(\gamma^\mu(\gamma^\nu)^2)$
$= Tr(\gamma^\mu) \times \eta^{\nu\nu}$]

$$Tr(\gamma^\mu) = -Tr(\gamma^\mu)$$

$$\Rightarrow \boxed{Tr(\gamma^\mu) = 0}$$

e) from clifford algebra

$$\gamma^\mu\gamma^\nu = 2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu$$

multiplying by $\gamma\rho$ →

$$\gamma^\mu\gamma^\nu\gamma\rho = 2\eta^{\mu\nu}\gamma\rho - \gamma^\nu\gamma^\mu\gamma\rho$$

Taking Trace,

$$Tr(\gamma^\mu\gamma^\nu\gamma\rho) = 2\eta^{\mu\nu}\, Tr(\gamma\rho) - Tr(\gamma^\nu\gamma^\mu\gamma\rho)$$

from part (d) → $Tr(\gamma\rho) = 0$

hence, $Tr(\gamma^\mu\gamma^\nu\gamma\rho) = -Tr(\gamma^\nu\gamma^\mu\gamma\rho)$ — (1)

using cyclicity, $[\nu \leftrightarrow \mu]$ $[\nu \leftrightarrow \rho]$

$$Tr(\gamma^\mu\gamma^\nu\gamma\rho) = Tr(\gamma^\mu\gamma\rho\gamma^\nu) \quad \text{— (2)}$$

now $\gamma\rho\gamma^\nu = -\gamma^\nu\gamma\rho + 2\eta^{\rho\nu}$

so multiplying with $\gamma\mu$ and taking trace

$$Tr(\gamma^\mu\gamma\rho\gamma^\nu) = -Tr(\gamma^\mu\gamma^\nu\gamma\rho)$$

Substituting in eq (2), RHS →

$$Tr(\gamma^\mu\gamma^\nu\gamma\rho) = -Tr(\gamma^\mu\gamma^\nu\gamma\rho)$$

$$Tr(\gamma^\mu\gamma^\nu\gamma\rho) = 0$$

Q.3. i) Show that if $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ then

$$[\gamma^\kappa\gamma^\lambda, \gamma^\mu\gamma^\nu] = 2\eta^{\lambda\mu}\gamma^\kappa\gamma^\nu - 2\eta^{\kappa\mu}\gamma^\lambda\gamma^\nu + 2\eta^{\lambda\nu}\gamma^\mu\gamma^\kappa - 2\eta^{\kappa\nu}\gamma^\mu\gamma^\lambda$$

ii) Show further that $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] = \frac{1}{2}(\gamma^\mu\gamma^\nu - \eta^{\mu\nu})$

Use this to confirm that the matrices $S^{\mu\nu}$ form a representation of the Lie algebra of Lorentz group.

Solution:-

i) given $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ — (1)

$[\gamma^\kappa\gamma^\lambda, \gamma^\mu\gamma^\nu] = \gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda$ — (2)

reordering 1st product

$\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = \gamma^\kappa(\gamma^\lambda\gamma^\mu)\gamma^\nu$ — (3)

now $\{\gamma^\lambda, \gamma^\mu\} = 2\eta^{\lambda\mu}$

$\gamma^\lambda\gamma^\mu = 2\eta^{\lambda\mu} - \gamma^\mu\gamma^\lambda$

$\Rightarrow$

Substituting in (3), $\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = \gamma^\kappa(-\gamma^\mu\gamma^\lambda + 2\eta^{\lambda\mu})\gamma^\nu$

$= -\gamma^\kappa\gamma^\mu\gamma^\lambda\gamma^\nu + 2\eta^{\lambda\mu}\gamma^\kappa\gamma^\nu$ — (4)

reordering second product

$\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda = \gamma^\mu(\gamma^\nu\gamma^\kappa)\gamma^\lambda = \gamma^\mu\{-\gamma^\kappa\gamma^\nu + 2\eta^{\nu\kappa}\}\gamma^\lambda$ — (5)

$= -\gamma^\mu\gamma^\kappa\gamma^\nu\gamma^\lambda + 2\eta^{\nu\kappa}\gamma^\mu\gamma^\lambda$

now substracting (5) from (4) $\rightarrow$

$[\gamma^\kappa\gamma^\lambda, \gamma^\mu\gamma^\nu] = -\gamma^\kappa\gamma^\mu\gamma^\lambda\gamma^\nu + 2\eta^{\lambda\mu}\gamma^\kappa\gamma^\nu + \gamma^\mu\gamma^\kappa\gamma^\nu\gamma^\lambda - 2\eta^{\nu\kappa}\gamma^\mu\gamma^\lambda$ — (6)

now $\gamma^\mu\gamma^\kappa\gamma^\nu\gamma^\lambda - \gamma^\kappa\gamma^\mu\gamma^\lambda\gamma^\nu \Rightarrow$ can be reorderd

$$\gamma^\mu \gamma^\kappa \gamma^\nu \gamma^\lambda = \gamma^\mu \gamma^\kappa [-\gamma^\lambda \gamma^\nu + 2\eta^{\nu\lambda}]$$

$$= -\gamma^\mu \gamma^\kappa \gamma^\lambda \gamma^\nu + 2\eta^{\nu\lambda} \gamma^\mu \gamma^\kappa \quad \text{—} \ (7)$$

also $$\gamma^\kappa \gamma^\mu \gamma^\lambda \gamma^\nu = (-\gamma^\mu \gamma^\kappa + 2\eta^{\kappa\mu}) \gamma^\lambda \gamma^\nu$$

$$= -\gamma^\mu \gamma^\kappa \gamma^\lambda \gamma^\nu + 2\eta^{\kappa\mu} \gamma^\lambda \gamma^\nu \quad \text{—} \ (8)$$

So $$\gamma^\mu \gamma^\kappa \gamma^\nu \gamma^\lambda - \gamma^\kappa \gamma^\mu \gamma^\lambda \gamma^\nu$$

$$= [-\gamma^\mu \gamma^\kappa \gamma^\lambda \gamma^\nu + 2\eta^{\nu\lambda} \gamma^\mu \gamma^\kappa] + \gamma^\mu \gamma^\kappa \gamma^\lambda \gamma^\nu - 2\eta^{\kappa\mu} \gamma^\lambda \gamma^\nu \quad [\text{using } (7) \text{ and } (8)]$$

$$= 2\eta^{\nu\lambda} \gamma^\mu \gamma^\kappa - 2\eta^{\kappa\mu} \gamma^\lambda \gamma^\nu$$

now plugging back to Eq (6) $\rightarrow$

$$\boxed{[\gamma^\kappa \gamma^\lambda, \gamma^\mu \gamma^\nu] = 2\eta^{\lambda\mu} \gamma^\kappa \gamma^\nu - 2\eta^{\nu\kappa} \gamma^\mu \gamma^\lambda - 2\eta^{\kappa\mu} \gamma^\lambda \gamma^\nu + 2\eta^{\nu\lambda} \gamma^\mu \gamma^\kappa} \quad \text{—} \ (9)$$

ii b) By definition, $$[\gamma^\mu, \gamma^\nu] = \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \quad \text{—} \ (10)$$

from (1), $\gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} - \gamma^\mu \gamma^\nu$

putting in eq (10), $$[\gamma^\mu, \gamma^\nu] = 2(\gamma^\mu \gamma^\nu) - 2\eta^{\mu\nu}$$

$$= 2(\gamma^\mu \gamma^\nu - \eta^{\mu\nu})$$

therfore, $$\boxed{S^{\mu\nu} = \frac{1}{4} [\gamma^\mu, \gamma^\nu] = \frac{1}{2} (\gamma^\mu \gamma^\nu - \eta^{\mu\nu})}$$

Lorentz Lie Algebra :-

the lorentz lie algebra satisfies ,

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho} + \eta^{\mu\sigma} M^{\nu\rho} \quad (11)$$

using part (i) and definition of $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$

$$[S^{\kappa\lambda}, S^{\mu\nu}] = \frac{1}{16}[[\gamma^\kappa, \gamma^\lambda], [\gamma^\mu, \gamma^\nu]] \quad — \quad (12)$$

now $[\gamma^\kappa, \gamma^\lambda] = \gamma^\kappa\gamma^\lambda - \gamma^\lambda\gamma^\kappa$

$[\gamma^\mu, \gamma^\nu] = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$

$$[\gamma^\kappa\gamma^\lambda - \gamma^\lambda\gamma^\kappa, \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu]$$

$$= [\gamma^\kappa\gamma^\lambda, \gamma^\mu\gamma^\nu] - [\gamma^\kappa\gamma^\lambda, \gamma^\nu\gamma^\mu] - [\gamma^\lambda\gamma^\kappa, \gamma^\mu\gamma^\nu] + [\gamma^\lambda\gamma^\kappa, \gamma^\nu\gamma^\mu]$$

now using, result from i)

$$[\gamma^a\gamma^b, \gamma^c\gamma^d] = 2\eta^{bc}\gamma^a\gamma^d - 2\eta^{ac}\gamma^b\gamma^d + 2\eta^{bd}\gamma^c\gamma^a - 2\eta^{ad}\gamma^c\gamma^d$$

applying this to 1st term,

$$[\gamma^\kappa\gamma^\lambda, \gamma^\mu\gamma^\nu] = 2^{\lambda\mu}\gamma^\kappa\gamma^\nu - 2\eta^{\kappa\mu}\gamma^\lambda\gamma^\nu + 2\eta^{\lambda\nu}\gamma^\mu\gamma^\kappa - 2\eta^{\kappa\nu}\gamma^\mu\gamma^\lambda$$

similarly after applying it to all 4 terms →

$$[[\gamma^\kappa, \gamma^\lambda], [\gamma^\mu, \gamma^\nu]] = 8\,(\eta^{\lambda\mu}\gamma^\kappa\gamma^\nu - \eta^{\kappa\mu}\gamma^\lambda\gamma^\nu - \eta^{\lambda\nu}\gamma^\kappa\gamma^\mu + \eta^{\kappa\nu}\gamma^\lambda\gamma^\nu).$$

$$[S^{\kappa\lambda}, S^{\mu\nu}] = \frac{1}{16}[[\gamma^\kappa, \gamma^\lambda], [\gamma^\mu, \gamma^\nu]]$$

$$= \frac{8}{16}(\eta^{\lambda\mu}\gamma^\kappa\gamma^\nu - \eta^{\kappa\mu}\gamma^\lambda\gamma^\nu - \eta^{\lambda\nu}\gamma^\kappa\gamma^\mu + \eta^{\kappa\nu}\gamma^\lambda\gamma^\nu) \quad (13)$$

$$\gamma^a\gamma^b = 2S^{ab} + \eta^{ab}$$

and $S^{ab} = -S^{ba}$

$$\Rightarrow \eta^{\lambda\mu}\gamma^\kappa\gamma^\nu = \eta^{\lambda\mu}(2S^{\kappa\nu} + \eta^{\kappa\nu})$$

$$\Rightarrow -\eta^{\kappa\mu}\gamma^\lambda\gamma^\nu = -\eta^{\kappa\mu}(2S^{\lambda\nu} + \eta^{\lambda\nu})$$

$$\Rightarrow -\eta^{\lambda\nu}\gamma^\kappa\gamma^\mu = -\eta^{\lambda\nu}(2S^{\kappa\mu} + \eta^{\kappa\mu})$$

$$\Rightarrow \eta^{\kappa\nu}\gamma^\lambda\gamma^\mu = \eta^{\kappa\nu}(2S^{\lambda\mu} + \eta^{\lambda\mu})$$

Inserting all these 4 terms back

$$[S^{\kappa\lambda}, S^{\mu\nu}] = \frac{1}{2}\Big[2\eta^{\lambda\mu}S^{\kappa\nu} + \eta^{\lambda\mu}\eta^{\kappa\nu} - 2\eta^{\kappa\mu}S^{\lambda\nu} - \eta^{\kappa\mu}\eta^{\lambda\nu} - 2\eta^{\lambda\nu}S^{\kappa\mu} - \eta^{\lambda\nu}\eta^{\kappa\mu} + 2\eta^{\kappa\nu}S^{\lambda\mu} + \eta^{\kappa\nu}\eta^{\lambda\mu}\Big] \quad -(14)$$

metric terms $\rightarrow \frac{1}{2}(\eta^{\lambda\mu}\eta^{\kappa\nu} - \eta^{\kappa\mu}\eta^{\lambda\nu} - \eta^{\lambda\nu}\eta^{\kappa\mu} + \eta^{\kappa\nu}\eta^{\lambda\mu})$

$$\left.\begin{aligned} &\rightarrow \eta^{\lambda\nu}\eta^{\kappa\nu} + \eta^{\kappa\nu}\eta^{\lambda\mu} = 2\eta^{\lambda\mu}\eta^{\kappa\nu} \\ &\rightarrow -\eta^{\kappa\mu}\eta^{\lambda\nu} - \eta^{\lambda\nu}\eta^{\kappa\mu} = -2\eta^{\kappa\mu}\eta^{\lambda\nu} \end{aligned}\right\} \text{ canceles each other upon adding}$$

$$[S^{\kappa\lambda}, S^{\mu\nu}] = \frac{1}{2}\left[2\eta^{\lambda\mu}S^{\kappa\nu} - 2\eta^{\kappa\mu}S^{\lambda\nu} - 2\eta^{\lambda\nu}S^{\kappa\mu} + 2\eta^{\kappa\nu}S^{\lambda\mu}\right]$$

$$[S^{\kappa\lambda}, S^{\mu\nu}] = \eta^{\lambda\mu}S^{\kappa\nu} - \eta^{\kappa\mu}S^{\lambda\nu} - \eta^{\lambda\nu}S^{\kappa\mu} + \eta^{\kappa\nu}S^{\lambda\mu} \quad -(15)$$

And the Lorentz lie algebra condition,

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho} \quad -(11)$$

now comparing Eq (15) and Eq (11) $\rightarrow$ ~~they~~ both $M^{\mu\nu}$ and $S^{\mu\nu}$ follows the same condition

hence, $S^{\mu\nu}$ form a representation of lie algebra of Lorentz group.