FINM 345X0/STAT 390X0
Midterm Exam (February 10, 2025)

You have an hour and twenty minutes (80 minutes) to complete the exam. There are 7 problems/questions, many with multiple parts, worth varying amounts for a total of 95 points. For all multi-part problems all parts are worth the same amount. This test is closed book. No calculators, phones, etc., permitted. You may have one two-sided, letter-sized sheet of notes that you have prepared.

- On any question you may leave your answer in terms of the normal distribution function
$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}\, dt.$$
- Also for your reference, if N is a $N(0,1)$ random variable N, then
$$\mathbb{E}[N^4] = 3,$$
and the moment generating function is given by
$$\mathbb{E}[e^{sN}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}\, e^{st}\, dt = e^{s^2/2}.$$

Problem 1 (10)**.** Give the definition of a (one-dimensional) Brownian motion B_t starting at the origin with drift m and variance parameter σ^2.

The solution should mention four things: $B_0 = 0$**; independent increments; if** $s < t$**, then** $B_t - B_s$ **is** $N(m(t-s), \sigma^2(t-s))$**; and continuous paths.**

Problem 2 (15)**.** Suppose B_t is a standard Brownian motion starting at the origin. Compute the following.

1. $\mathbb{P}\{B_3^2 \geq 1\}$

$$= 1 - \mathbb{P}\{-1 \leq B_3 \leq 1\} = 1 - \mathbb{P}\{-1/\sqrt{3} \leq B_1 \leq 1/\sqrt{3}\} = 1 - [\Phi(1/\sqrt{3}) - \Phi(-1/\sqrt{3})] = 2\,\Phi(-1/\sqrt{3})$$

2. $E[e^{4B_3} \mid B_1]$

$$= E[e^{4B_1}\, e^{4(B_3 - B_1)} \mid B_1] = e^{4B_1}\, \mathbb{E}[e^{4(B_3-B_1)}] = e^{4B_1}\, \mathbb{E}[e^{4\sqrt{2}N}] = e^{16}\, e^{4B_1}$$

The last step uses the formula for the moment generating function.

3. $\mathbb{P}\{B_t \leq 2$ **for all** $t \leq 3\}$

$$= 1 - \mathbb{P}\{B_t \geq 2 \textbf{ for some } t \leq 3\} = 1 - 2\mathbb{P}\{B_3 \geq 2\} = 1 - 2[1 - \Phi(2/\sqrt{3})].$$

Problem 3 (10)**.** Let B_t be a standard Brownian motion and let X_t be a geometric Brownian motion satisfying
$$dX_t = X_t\,[2\,dt + 3\,dB_t].$$
Find the following differentials, that is, write them as
$$[\text{something}]\,dt + [\text{something}]\,dB_t.$$
Note that
$$d\langle X\rangle_t = 6\,X_t^2\,dt$$

1. $d[t\,X_t^2]$
   Here one could use a product rule or one could write this as $f(t,X_t)$ where $f(t,x) = t\,x^2$. I will do it the latter way.
$$\dot f = x^2, \quad f' = 2tx, \quad f'' = 2t.$$
$$d[tX_t^2] = X_t^2\,dt + 2tX_t\,dX_t + t\,d\langle X\rangle_t = [X^2 + 4tX_t^2 + 9tX_t^2]\,dt + 6tX_t^2\,dB_t$$
$$= X_t^2\,[1+13t]\,dt + 6tX_t^2\,dB_t.$$

2. $d[X_tB_t + e^{-3t}]$
$$de^{-3t} = -3\,e^{-3t}\,dt$$
$$d[X_tB_t] = X_t\,dB_t + B_t\,dX_t + d\langle X,B\rangle_t = X_t\,dB_t + 2\,B_t\,X_t dt + 3\,B_t\,X_t\,dB_t + 3X_t\,dt$$
$$= [2B_tX_t + 3X_t]\,dt + [X_t + 3B_tX_t]\,dB_t$$
$$d[X_tB_t + e^{-3t}] = [2B_tX_t + 3X_t - 3e^{-3t}]\,dt + [X_t + 3B_tX_t]\,dB_t$$

3. $d\left[\exp\left\{\int_0^t X_s^2\,ds\right\}\right]$
   This is differentiable
$$= \exp\left\{\int_0^t X_s^2\,ds\right\}\,X_t^2\,dt.$$

Problem 4. (10) Let X_t, B_t be as in the previous problem. Find A_t, C_t such that
$$d\langle Y\rangle_t = A_t\,dt, \quad d\langle X,Y\rangle_t = C_t\,dt.$$
(Note that Y_t below come from previous problem.)

1. $Y_t = t\,X_t^2$.
$$A_t = (6tX_t^2)^2, \qquad C_t = (6tX_t^2)\,(3X_t)$$

2. $Y_t = X_t\,B_t + e^{-3t}$.
$$A_t = [X_t + 3B_tX_t]^2, \quad C_t = [X_t + 3B_tX_t]\,(3X_t).$$

Problem 5 (25). Suppose W_t is a (one-dimensional) Brownian motion with drift 3 and variance parameter 2. Let $\lambda \in \mathbb{R}$ and let $X_t = e^{\lambda W_t}$.

1. For which value of λ is X_t a martingale with respect to $\{\mathcal{F}_t\}$ where $\mathcal{F}_t$ is the information in $\{W_s : s \le t\}$?
   We need for $s < t$,
$$E[e^{\lambda W_t} \mid \mathcal{F}_s] = e^{\lambda\,W_s}.$$
$$E[e^{\lambda W_t} \mid \mathcal{F}_s] = E[e^{\lambda W_s}\,e^{\lambda(W_t - W_s)} \mid \mathcal{F}_s] = e^{\lambda W_s}\,\mathbb{E}[e^{\lambda(W_t-W_s)}].$$
   So we need $\mathbb{E}[e^{\lambda(W_t-W_s)}] = 1$. $W_t - W_s$ has the same distribution as $3\,(t-s) + \sqrt{2\,(t-s)}\,N$ where N has a standard normal distribution.
$$\mathbb{E}[e^{\lambda(W_t-W_s)}] = e^{3\lambda(t-s)}\,\mathbb{E}[e^{\lambda\sqrt{2\,(t-s)}\,N}] = e^{3\lambda(t-s)}e^{\lambda^2(t-s)}.$$
   The answer is $\lambda = -3$,

2. What is
$$\lim_{n\to\infty} \sum_{j=1}^{3n} \left[W\left(\frac{j}{n}\right) - W\left(\frac{j-1}{n}\right)\right]^2 \ ?$$
This is the quadratic variation. The limit is $2 \cdot 3 = 6$.

3. Suppose that B_t is a standard Brownian motion independent of W_t and
$$Z_t = 2\,B_t - W_t + 3t.$$
Then Z_t is a Brownian motion (you do not need to prove this). What are the drift and variance parameters for Z_t?
The drift is $-3 + 3 = 0$. The variance parameter is $4 \cdot 1 + 2 = 6$

4. Consider the two-dimensional Brownian motion $Y_t = (W_t, Z_t)$. What are the drift and covariance matrix for Y?
The drift is the vector $m = (3, 0)$. The covariance matrix is the same as if there were no drift terms.
$$\mathrm{Var}(W_t) = 2t, \quad \mathrm{Var}(Z_t) = (2^2 + 2)t, \quad \mathrm{Cov}(W_t, Z_t) = \mathrm{Cov}(W_t, -W_t) = -2t$$
$$\Gamma = \begin{bmatrix} 2 & -2 \\ -2 & 6 \end{bmatrix}.$$

Problem 6 (10)**.** Suppose W_t is as in Problem 5.

1. Let
$$u(t,x) = \mathbb{E}\left[W_t^3 \mid W_0 = x\right].$$
Then $u(t,x)$ satisfies a partial differential equation (PDE) for $t > 0$ with an initial condition $u(0,x)$. Give the PDE that $u(t,x)$ satisfies, that is:
$$\partial_t u(t,x) = ???????$$
$$= 3\,u'(t,x) + u''(t,x)$$

2. Suppose $\phi(t,x)$ is the density of the random variable W_t assuming that $W_0 = 0$. What PDE does $\phi(t,x)$ satisfy, that is:
$$\partial_t \phi(t,x) = ??????$$
$$= -3\,u'(t,x) + u''(t,x)$$

(If you use the L or L^* notation in your answers, you must state what L or L^* equals.)

Problem 7 (15)**.** Suppose B_t is a standard Brownian motion and consider the simple process A_t that changes values only at time $t = 1$. We set $A_t = 2$ for $t < 1$ and for $t > 1$,
$$A_t = \begin{cases} B_1^2 & \text{if } B_1 > 0 \\ 0 & \text{if } B_1 \leq 0 \end{cases}.$$
Let
$$Z_2 = \int_0^2 A_s\, dB_s$$
Find the following.

1. $\mathbb{E}[Z_2]$

$$= 0 \text{ (Martingale since } \mathbb{E}[Z_2^2] < \infty)$$

2. $\mathrm{Var}[Z_2]$

$$= \int_0^2 \mathbb{E}[A_s^2]\, ds.$$

For $s \leq 1$, $A_s^2 = 4$. For $1 \leq s \leq 2$ as in the previous part,

$$\mathbb{E}[A_s^2] = \mathbb{E}[B_1^2\, 1\{B_1 > 0\}] = \mathbb{E}[(B_1^2)^2]\, \mathbb{P}\{B_1 > 0\} = \frac{3}{2}.$$

Here we use the fact about $E[N^4]$ mentioned at the beginning of the exam.

$$= \int_0^1 4\, ds + \int_1^2 \frac{3}{2}\ ds = 4 + \frac{3}{2} = \frac{11}{2}.$$

3. Which of the following is true: $\mathbb{P}\{Z_2 > 0\} > 1/2$; $\mathbb{P}\{Z_2 > 0\} = 1/2$; $\mathbb{P}\{Z_2 < 0\} > 1/2$? You must justify your answer.
In order for $Z_2 > 0$ we need both $B_1 > 0$ and $Z_2 > 0$. The first happens with probability $1/2$ and given $B_1 > 0$, there is a positive probability that $Z_2 \leq 0$. Hence the total probability is less than $1/2$.

$$\mathbb{P}\{Z_2 < 0\} > 1/2.$$

No credit should be given to answer without justification.