FINM 345X0/STAT 390X0
Exam 2 (May 25, 2022)

You have an hour and twenty minutes (80 minutes) to complete the exam. You can start your time after you finish reading this page. There are 8 problems/questions, many with multiple parts, worth varying amounts for a total of 100 points. For all multi-part problems all parts are worth the same amount. You may have one sheet of notes. No calculators, phones, etc., permitted.

For your reference, here is the standard Black-Scholes differential equation

$$\dot{f}(t,x) = r\, f(t,x) - r\, x\, f'(t,x) - \frac{\sigma^2}{2}\, x^2\, f''(t,x).$$

Also, if Y is a Poisson random variable with mean λ, the moment generating function is given by

$$\mathbb{E}\left[e^{sY}\right] = \exp\left\{\lambda\,(e^s - 1)\right\}.$$

Problem 1 (10)**.** For the following examples, state if $\mu_1 \ll \mu_2$. Give reasons.

1. μ_1 is the distribution of a normal random variable with mean 3 and variance 1 and μ_2 is an exponential random variable with rate 1.
2. μ_1 is the probability measure on continuous functions on $[0,1]$ given by standard Brownian motion and μ_2 is the probability measure given by Brownian motion starting at the origin with drift 4 and variance parameter 1.

Problem 2 (5)**.** Suppose that B_t^1, B_t^2 are independent Brownian motions and X_t, Y_t satisfy

$$dX_t = 3\, dt + dB_t^1 + B_t^1\, dB_t^2, \quad dY_t = 4\, B_t\, dt - 3\, dB_t^1 + dB_t^2.$$

1. Let $J_t = X_t\, Y_t$. Find dJ_t.

Problem 3 (15)**.** Suppose Y_t is a Poisson process with parameter $\lambda = 3$. Find the following. If the answer includes terms like e^{-4}, just leave the answer as e^{-4}.

1. $\mathbb{P}\{Y_3 = 2\}$
2. $E[e^{2(Y_1+Y_3)} \mid Y_1]$
3. $\mathbb{P}\{Y_1 = 2 \mid Y_2 = 3\}$.

Problem 4. (10) Suppose that we want to simulate a Brownian motion with drift $m = -3$ and variance parameter $\sigma^2 = 4$. We will choose a small time increment $\Delta t = .005$ and start with $X_0 = 0$.

1. Suppose we use the following method: for $k > 0$,

$$X_{k\Delta t} = X_{(k-1)\Delta t} + c_1\,(\Delta t)^{\beta} + c_2\,(\Delta t)^{r}\, N$$

where N is a standard normal random variable and c_1, c_2, β, r are constants. What are the correct values of c_1, c_2, β, r?

2. Suppose instead we use the following method

$$\mathbb{P}\{X_{k\Delta t} - X_{(k-1)\Delta t} = a \mid X_{(k-1)\Delta t}\} = p$$

$$\mathbb{P}\{X_{k\Delta t} - X_{(k-1)\Delta t} = -a \mid X_{(k-1)\Delta t}\} = 1 - p$$

where $a > 0$ and $0 < p < 1$. What are the correct values of a, p?

Problem 5 (10)**.** Suppose X_t is a compound Poisson process with $\lambda = 2$ and measure $\mu^{\#}$ given by an exponential random variable with parameter (rate) 1.

1. Find the function f such that the Lévy measure μ can be written as

$$d\mu(x) = f(x)\, dx.$$

2. What is the generator L of the process X_t?

Problem 6 (20)**.** Suppose that a stock price S_t follows the SDE

$$dS_t = S_t \,[(1.5)\, dt + 2\, dB_t]\,.$$

Suppose also there is a risk-free bond available growing at rate $r = .03$,

$$dR_t = r\, R_t\, dt.$$

1. Let $\tilde{S}_t = e^{-rt}\, S_t$ denote the discounted stock price. What is the SDE for $\tilde{S}_t$?

2. Let Q be the measure under which $\tilde{S}_t$ is a martingale. This is obtained using the Girsanov theorem by tilting by a martingale M_t. Write the SDE for M_t.

3. Suppose there is a payoff $F(S_4) = [S_4 - 34]_+$ at time 4. Let $V_t, 0 < t < 4$ denote the (undiscounted) value of this claim. Write an expression for V_t in terms of expectations or conditional expectations. Make sure you specify which measure the expectations are taken with respect to. You do not need to compute the expectations.

4. Let $v(t, x)$ be the value of this option at time t given that the stock price at time t equals x. What partial differential equation does $v(t, x)$ satisfy?

Problem 7 (15)**.** Suppose X_t satisfies the SDE

$$dX_t = X_t\, dt + e^{X_t}\, dB_t, \qquad X_0 = 1,$$

and let $T = \min\{t : |X_t| = 4\}$. We will consider $Y_t = X_{t\wedge T}$, the process stopped at time t. Here B_t is a standard Brownian motion with respect to probability measure $\mathbb{P}$.

1. Suppose we want to find a new probability measure Q that is mutually absolutely continuous with respect to $\mathbb{P}$ and such that Y_t is a martingale with respect to Q. We will do this by tilting by a martingale M_t. Write down the SDE that M_t should satisfy for $t < T$.

2. Your answer to the previous part should show immediately that $M_t, t < T$ is a local martingale. Explain why it is, in fact, a martingale.

3. Let $\phi(x), -4 < x < 4$, be the probability that $X_T = 4$ given $X_0 = x$. Find the second order ODE that ϕ satisfies. (You do not have to solve the equation.)

Problem 8 (15)**.** Suppose $X_1, X_2, \ldots$ are independent coin flips, where

$$\mathbb{P}\{X_j = 1\} = \frac{1}{2}, \qquad \mathbb{P}\{X_j = -1\} = \frac{1}{2},$$

and let $S_n = X_1 + \cdots + X_n$, where $S_0 = 0$. For each $\lambda > 0$, let

$$M_n = M_{n,\lambda} = \frac{e^{\lambda S_n}}{c_\lambda^n}$$

where c_λ is chosen so that M_n is a martingale with respect to $\{\mathcal{F}_n\}$ where $\mathcal{F}_n$ denotes the information in $X_1, \ldots, X_n$.

1. Find c_λ.
2. Let Q_λ be the probability measure obtained by tilting by $M_n = M_{n,\lambda}$, that is, if V is an F_n-measurable event

   $$Q_\lambda[V] = \mathbb{E}\left[1_V \, M_{n,\lambda}\right].$$

   For which value of λ is

   $$\mathbb{E}_{Q_\lambda}[S_n] = \frac{n}{2}?$$

3. Suppose the first 12 outcomes of the coin tosses give

   $$\omega = (1, 1, -1, 1, -1, -1, 1, 1, 1, 1, 1, -1).$$

   Using the λ from the previous part, give

   $$\frac{dQ_\lambda}{d\mathbb{P}}\,(\omega).$$