Problem Set 2

Exercise 1

$f(t)$ is a continuous function for $0 \le t \le 1$

$$Q = \lim_{n\to\infty} \sum_{j=1}^{n} \left| f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) \right|^{5/4}$$

1) $f(t) = t^2$

$$Q = \lim_{n\to\infty} \sum_{j=1}^{n} \left| \left(\frac{j}{n}\right)^2 - \left(\frac{j-1}{n}\right)^2 \right|^{5/4}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \left| \frac{j^2 - (j^2 - 2j + 1)}{n^2} \right|^{5/4}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \left| \frac{2j-1}{n^2} \right|^{5/4}$$

Since j can only take values >1, $2j-1$ will always be positive

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \left(\frac{2j-1}{n^2} \right)^{5/4}$$

$= 0$ as $n \to \infty$, $\frac{x}{n^2} \to 0$

2) $f(t) = B_t$ where $B_t \sim N(0, t)$

$$Q = \lim_{n\to\infty} \sum_{j=1}^{n} \left| B_{\frac{j}{n}} - B_{\frac{j-1}{n}} \right|^{5/4}$$

$$\left(B_{j/n} - B_{\frac{j-1}{n}}\right) \sim N(0, 1/n)$$

Also, $\left| B_{j/n} - B_{j-1/n} \right| \approx \sqrt{\frac{1}{n}} = \frac{1}{\sqrt{n}}$

$$Q = \lim_{n\to\infty} \sum_{j=1}^{n} \left(n^{-1/2}\right)^{5/4}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} n^{-5/8}$$

$$= \lim_{n\to\infty} n \cdot n^{-5/8} = \lim_{n\to\infty} n^{3/8}$$

$$= \infty$$

As $n \to \infty$, $n^{3/8} \to \infty$

Exercise 2

B_t, W_t are independent ~~standard~~ standard Brownian motions

1) $Y_t = 2B_t - W_t - t$

Y_t will be of the form $\sigma B_t + mt$

$$E[Y_t] = E[2B_t - W_t - t] = 2E[B_t] - E[W_t] - t = -t$$

$$Var[Y_t] = Var[2B_t - W_t - t]$$

$$= 4Var[B_t] + Var[W_t] + 0$$

$$= 4t + t$$

$$= 5t$$

$\therefore$ $Y_t = \sqrt{5} A_t - t$

where A_t is a different standard brownian motion

Drift parameter $= -1$

Diffusion parameter $= 5$

2) $Z_t = (Z_t^1, Z_t^2)$

$Z_t^1 = Y_t + t$, $Z_t^2 = -B_t + 3W_t$

$Z_t^1 = (\sqrt{5}A_t - t) + t = \sqrt{5}A_t$ where A_t is a standard BM

$E[Z_t^1] = E[\sqrt{5}A_t] = 0$

$\therefore Z_t^1$ has 0 drift

$Z_t^2 = -B_t + 3W_t$

$E[Z_t^2] = E[-B_t + 3W_t]$

$= -E[B_t] + 3E[W_t]$

$= 0$

$\therefore Z_t^2$ also has 0 drift

$\therefore Z_t = (Z_t^1, Z_t^2)$ has a 0 drift

Also $Z_0^1 = \sqrt{5}A_0 = 0$

$Z_0^2 = -B_0 + 3W_0 = 0$

$\therefore Z_0 = (Z_0^1, Z_0^2) = (0,0)$

Z_0 starts at origin

Covariance Matrix $= \begin{bmatrix} Var(Z^1) & Cov(Z^1, Z^2) \\ Cov(Z^1, Z^2) & Var(Z^2) \end{bmatrix}$

$Var(Z^1) = Var(\sqrt{5}A_t) = 5t$

$Var(Z^2) = Var(-B_t + 3W_t) = t + 9t = 10t$

$Cov(Z^1, Z^2) = Cov(\sqrt{5}A_t, -B_t + 3W_t)$

$= Cov(2B_t - W_t, -B_t + 3W_t)$

$= Cov(2B_t, -B_t) + Cov(-W_t, 3W_t)$
$+ Cov(2B_t, 3W_t) + Cov(-W_t, -B_t)$

$= E[-2B_t^2] - E(2B_t)E(-B_t) +$
$E[-3W_t^2] - E[-W_t]E[3W_t] + 0 + 0$

$$= -2E[B_t^2] - 0 - 3E[W_t^2] - 0$$

$$= -2t - 3t \quad = -5t$$

$$\Gamma = \begin{bmatrix} 5 & -5 \\ -5 & 10 \end{bmatrix}$$

3) $$\langle Z^1 \rangle_t = \lim_{n \to \infty} \sum_{j \leq nt} \left(Z_{\frac{j}{n}} - Z_{\frac{j-1}{n}} \right)^2$$

$$= \lim_{n \to \infty} 5 \sum_{j \leq nt} \left(A_{\frac{j}{n}} - A_{\frac{j-1}{n}} \right)^2$$

$$\left(A_{\frac{j}{n}} - A_{\frac{j-1}{n}} \right) \sim N\left(0, \sqrt{\tfrac{1}{n}}\right)$$

$$= \lim_{n \to \infty} 5 \sum_{j \leq nt} \left(n^{-1/2}\right)^2$$

$$= \lim_{n \to \infty} 5 \sum_{j \leq nt} \frac{1}{n}$$

$$= 5 \times \frac{tn}{n} = 5t \quad = Var(Z_1)$$

$$\langle Z^1, Z^2 \rangle = Cov(Z^1, Z^2)$$

$$= -5t$$

<u>Exercise 3</u>

$$P(B_t \geq 6W_t - 4 \text{ for all } 0 \leq t \leq 3)$$

$$= P(6W_t - B_t \leq 4 \text{ for all } 0 \leq t \leq 3)$$

According to Reflection Principle →

$$P(B_t \leq x \text{ for all } 0 \leq t \leq y) = 1 - P\left(B_t > x \text{ for all } 0 \leq t \leq y\right)$$

$$= 1 - 2P(B_y > x)$$

Therefore in this case,

$$= 1 - 2P(6W_3 - B_3 > 4)$$

Let $X_3 = 6W_3 - B_3$

$E[X_3] = 0$

$$\begin{aligned} Var[X_3] &= Var[6W_3 - B_3] \\ &= 36 Var[W_3] + Var[B_3] \\ &= 36 \times 3 + 3 = 111 \end{aligned}$$

$\therefore X_3 \sim N(0, 111)$

$$= 1 - 2P(X_3 > 4) \quad = 1 - 2P\left(Z > \frac{4}{\sqrt{111}}\right)$$

$$= 1 - 2\left[1 - P\left(Z < \frac{4}{\sqrt{111}}\right)\right]$$

$$= 1 - 2[0.3521]$$

$$= 0.2958$$

Exercise 4

B_t is a standard BM

1) $\phi(t,x)$ is the density of B_t (as a function of x) given that $B_0 = 0$

$$\frac{d}{dt}\phi(t,x) = \frac{\phi(t+\Delta t, x) - \phi(t,x)}{\Delta t}$$

$$\phi(t+\Delta t, x) = \frac{1}{2}\phi(t, x+\Delta x) + \frac{1}{2}\phi(t, x-\Delta x)$$

$$\frac{d}{dt}\phi(t,x) = \frac{1}{2}\frac{[\phi(t,x+\Delta x) - \phi(t,x)] + [\phi(t,x-\Delta x) - \phi(t,x)]}{\Delta x^2}$$

$$= \frac{1}{2}\frac{\frac{\phi(t,x+\Delta x) - \phi(t,x)}{\Delta x} - \frac{\phi(t,x) - \phi(t,x-\Delta x)}{\Delta x}}{\Delta x}$$

$$\frac{d}{dt}\phi(t,x) = \frac{1}{2}\frac{d^2}{dx^2}\phi(t,x)$$

$$\dot{\phi}(t,x) = \frac{1}{2}\phi''(t,x) = L_x\phi(t,x)$$

where $L f(x) = \frac{1}{2} f''(x)$

2) $\phi(t,x) = P(B_t < 4 \mid B_0 = x)$

If $x \geq 4$, $\phi(t,x) = 0$

PDE $\rightarrow$

$$\frac{d}{dt}\phi(t,x) = -\frac{1}{2}\frac{d^2}{dx^2}\phi(t,x)$$

$$\dot{\phi}(t,x) = L^*\phi(t,x)$$

where $L^* f(x) = \frac{1}{2} f''(x)$

3) $\phi(t,x) = E[B_t^3 \mid B_0 = x]$

Let $B_t^3 = F(y)$

let $P_t(x,y)$ be the density of B_t (using y as a variable

assuming $B_0 = x$

$$\phi(t,x) = \int_{-\infty}^{\infty} P_t(x,y)\, F(y)\, dy$$

$$\frac{d}{dt}\phi(t,x) = \frac{d}{dt}\int_{-\infty}^{\infty} P_t(x,y) F(y)\, dy$$

$$= \int_{-\infty}^{\infty} \dot{P}_t(x,y)\, F(y)\, dy$$

$$= \int_{-\infty}^{\infty} L_x P_t(x,y)\, F(y)\, dy$$

where $L_x f(x) = \frac{1}{2} P_t''(x,y)$ wrt x

$$= L_x \int_{-\infty}^{\infty} P_t(x,y) F(y)\, dy$$

$$\dot{\phi}(t,x) = L_x \phi(t,x)$$

4) $\phi(t,x) = E[B_4 - B_4^2 / B_t = x]$

Let $B_4 - B_4^2 = F(Y)$

Let $P_{T-t}(x,y)$ be the density of $F(Y)$ using y as a variable and $B_t = x$

$$\frac{\delta}{\delta t} \phi(t,x) = \frac{\delta}{\delta t} \int_{-\infty}^{\infty} P_{T-t}(x,y) F(Y)\, dy$$

$$= \int_{-\infty}^{\infty} -\dot{P}_{T-t}(x,y) F(y)\, dy$$

$$= \int_{-\infty}^{\infty} -L_x P_{T-t}(x,y) F(y)\, dy$$

where $L_x = \frac{1}{2} \frac{\delta^2}{\delta x^2} P_{T-t}(x,y)$

$$= -L_x \int_{-\infty}^{\infty} P_{T-t}(x,y) F(y)\, dy$$

$$\dot{\phi}(t,x) = -L_x \phi(t,x)$$

5) B has drift $= 1$, variance $= 4$

1) $\phi(t,x)$ is density of B_t given $B_0 = 0$

$$\dot{\phi}(t,x) = -m \frac{\delta}{\delta x} \phi(t,x) + \frac{\sigma^2}{2} \frac{\delta^2}{\delta x^2} \phi(t,x)$$

where $m = \text{drift} = 1$

$\sigma^2 = \text{variance} = 4$

$$\dot{\phi}(t,x) = -\frac{\partial}{\partial x}\phi(t,x) + 2\frac{\partial^2}{\partial x^2}\phi(t,x)$$

$$\dot{\phi}(t,x) = L^* \phi(t,x)$$

where $L^* f(x) = -m f'(x) + \frac{\sigma^2}{2} f''(x)$

$$= -f'(x) + 2f''(x)$$

2) $\phi(t,x) = P(B_t < 4 / B_0 = x)$

$$\dot{\phi}(t,x) = m\frac{\partial}{\partial x} P_t(x,y) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} P_t(x,y)$$

where $P_t(x,y)$ is the density of B_t using y as a variable given $B_0 = x$

$$\dot{\phi}(t,x) = 1\frac{\partial}{\partial x} P_t(x,y) + 2\frac{\partial^2}{\partial x^2} P_t(x,y)$$

$$= L_x P_t(x,y)$$

where $L_x f(x) = m f'(x) + \frac{\sigma^2}{2} f''(x)$

$$= f'(x) + 2f''(x)$$

3) $\phi(t,x) = E[B_t^3 / B_0 = x]$

$$\dot{\phi}(t,x) = m\frac{\partial}{\partial x}\phi(t,x) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\phi(t,x)$$

$$= \frac{\partial}{\partial x}\phi(t,x) + 2\frac{\partial^2}{\partial x^2}\phi(t,x)$$

$$\dot{\phi}(t,x) = L_x \phi(t,x)$$

where $L_x f(x) = m f'(x) + \frac{\sigma^2}{2} f''(x)$

$$= f'(x) + 2f''(x)$$

4) $\phi(t,x) = E[B_4 - B_4^2 / B_t = x]$

$$\dot{\phi}(t,x) = -\left[m \frac{d}{dx} \phi(t,x) + \frac{\sigma^2}{2} \frac{d^2}{dx^2} \phi(t,x)\right]$$

$$\dot{\phi}(t,x) = -\left[\frac{d}{dx} \phi(t,x) + 2 \frac{d^2}{dx^2} \phi(t,x)\right]$$

$$\dot{\phi}(t,x) = - L_x \phi(t,x)$$

where $L_x f(x) = m f'(x) + \frac{\sigma^2}{2} f''(x)$
$= f'(x) + 2f''(x)$

Exercise 5

B_t & W_t are independent standard BM starting at origin

$Z_t = (Z_t^1, Z_t^2)$

$Z_t^1 = B_t + W_t - t$, $Z_t^2 = 2B_t - 4W_t$

1) $E[Z_t^1] = E[B_t + W_t - t]$
$= -t$

$E[Z_t^2] = E[2B_t - 4W_t]$
$= 0$

$Var[Z_t^1] = Var[B_t + W_t - t]$
$= Var[B_t] + Var[W_t] + 0$
$= t + t$
$= 2t$

$Var[Z_t^2] = Var[2B_t - 4W_t]$
$= 4Var[B_t] + 16 Var[W_t]$
$= 4t + 16t$
$= 20t$

$$\begin{aligned}
Cov(Z_t^1, Z_t^2) &= Cov(B_t + W_t - t,\ 2B_t - 4W_t) \\
&= Cov(B_t, 2B_t) + Cov(W_t, -4W_t) \\
&= E[B_t 2B_t] - E[B_t]E[2B_t] + \\
&\quad E[-4W_t^2] - E[W_t]E[-W_t] \\
&= E[2B_t^2] + E[-4W_t^2] \\
&= 2t - 4t \quad = -2t
\end{aligned}$$

Covariance Matrix $\Rightarrow$ $\begin{bmatrix} 2 & -2 \\ -2 & 20 \end{bmatrix} t$

Drift $\Rightarrow$ $[-1, 0]t$

2) Z_t^1 has drift -1 & variance $2t$

Z_t^2 has 0 drift & variance $20t$

Cov $= -2t$

$$Lf(x) = \overline{m}\,\nabla f(x) + \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d}\Gamma_{jk}\frac{\partial^2 f(x)}{\partial x_j \partial x_k}$$

$$= m_1\frac{\partial f}{\partial x_1} + m_2\frac{\partial f}{\partial x_2} + \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\Gamma_{jk}\frac{\partial^2 f(x)}{\partial x_j \partial x_k}$$

$$= -1\frac{\partial f}{\partial x_1} + 0 + \frac{1}{2}\times 2\times\frac{\partial^2 f}{\partial x_1^2} - \frac{1}{2}\times 2\times\frac{\partial^2 f}{\partial x_1\partial x_2}$$
$$- \frac{1}{2}\times 2\times\frac{\partial^2 f}{\partial x_1\partial x_2} + \frac{1}{2}\times 20\times\frac{\partial^2 f}{\partial x_2^2}$$

$$= -\frac{\partial f}{\partial x_1} + \frac{\partial^2 f}{\partial x_1^2} - \frac{\partial^2 f}{\partial x_1\partial x_2} - \frac{\partial^2 f}{\partial x_1\partial x_2} + 10\frac{\partial^2 f}{\partial x_2^2}$$

$$= -\frac{\partial f}{\partial x_1} + \frac{\partial^2 f}{\partial x_1^2} - 2\frac{\partial^2 f}{\partial x_1\partial x_2} + 10\frac{\partial^2 f}{\partial x_2^2}$$

$$L^* f(x) = -\bar{m} \nabla f(x) + \frac{1}{2} \sum_{j=1}^{d} \sum_{k=1}^{d} \Gamma_{jk} \frac{\delta^2 f}{\delta x_j x_k}$$

$$= \frac{\delta f}{\delta x_1} + \frac{\delta^2 f}{\delta x_1^2} - \frac{2\delta^2 f}{\delta x_1 \delta x_2} + \frac{10\delta^2 f}{\delta x_2^2}$$

3) $\phi(t,x)$ is the density of Z_t at time t

PDE $\rightarrow$

$$\dot{\phi}(t,x) = \frac{\delta f}{\delta x_1} + \frac{\delta^2 f}{\delta x_1^2} - \frac{2\delta^2 f}{\delta x_1 \delta x_2} + \frac{10\delta^2 f}{\delta x_2^2}$$

$$\dot{\phi}(t,x) = L^* \phi(t,x)$$