FINM 34500/STAT 39000
Winter 2025
Problem Set 2 (due January 21)

Reading: Sections 2.8 — 2.10.

Exercise 1 *Let $f(t)$ be a continuous function for $0 \leq t \leq 1$ and let*

$$Q = \lim_{n\to\infty} \sum_{j=1}^{n} \left| f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)\right|^{5/4}.$$

What is Q

1. *If $f(t) = t^2$?*
   Since** $f'(t) = 2t$**,

$$\left| f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)\right| \sim 2(j/n)\,(1/n) \leq \frac{2}{n}$$

$$\sum_{j=1}^{n} \left| f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)\right|^{5/4} \leq n\,\left(\frac{2}{n}\right)^{5/4} \to 0.$$

2. *If $f(t) = B_t$ where B_t is a standard Brownian motion.* ***Since for** n **large,***

$$\left| f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)\right|^{5/4} \gg \left| f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)\right|^{2}$$

   this must be going to infinity. See the definition of quadratic variation.

Exercise 2 *Suppose B_t, W_t are independent standard Brownian motions.*

1. *Let $Y_t = 2\,B_t - W_t - t$. Show that Y_t is a (one-dimensional) Brownian motion starting at the origin. What are the drift and variance parameter?*
   One just checks that the conditions are satisfied. Note that if** $s < t$**, then

$$Y_t - Y_s = 2\,(B_t - B_s) - (W_t - W_s) - (t-s)$$

   is a normal random variable with

$$\mathbb{E}\,[Y_t - Y_s] = 2\,\mathbb{E}[B_t - B_s] - \mathbb{E}[W_t - W_s] - (t-s) = -(t-s),$$

$$\mathrm{Var}\,[Y_t - Y_s] = 2^2\,\mathrm{Var}[B_t - B_s] + \mathrm{Var}[W_t - W_s] = 5\,(t-s).$$

   Hence the drift is** -1 **and the variance parameter is** 5**.

2. *Let $Z_t = (Z_t^1, Z_t^2)$ denote the random vector where*

$$Z_t^1 = Y_t + t, \qquad Z_t^2 = -B_t + 3W_t,$$

and Y_t is as in the previous part. Explain why Z_t is a two-dimensional Brownian motion starting at the origin with zero drift. What is the covariance matrix Γ?
One just checks that the conditions are satisfied. Note that Z_t has a joint normal distribution with mean zero (centered) and

$$\mathrm{Var}[Z_t^1] = \mathbb{E}[(Z_t^1)^2] = 4\,\mathrm{Var}[B_t] + \mathrm{Var}[W_t] = 5t,$$

$$\mathrm{Var}[Z_t^2] = \mathbb{E}[(Z_t^2)^2] = (-1)^2\mathrm{Var}[B_t] + 9\,\mathrm{Var}[W_t] = 10t,$$

$$\mathbb{E}[Z_t^1\, Z_t^2] = \mathbb{E}[(2B_t - W_t)\,(-B_t + 3W_t)] = -2\mathbb{E}[B_t^2] - 3\mathbb{E}[W_t^2] = -5t.$$

The last calculation uses $\mathbb{E}[B_t W_t] = 0$. Similarly, if $s < t$, $Z_t - Z_s$ has a joint normal distribution with covariance matrix $(t-s)\,\Gamma$ where

$$\Gamma = \begin{bmatrix} 5 & -5 \\ -5 & 10 \end{bmatrix}.$$

3. *Find*

$$\langle Z^1 \rangle_t, \qquad \langle Z^1, Z^2 \rangle_t.$$

$$\langle Z^1 \rangle_t = \Gamma_{11}\, t = 5t, \qquad \langle Z^1, Z^2 \rangle_t = \Gamma_{12} t = -5t.$$

Exercise 3 *Let B_t, W_t be independent standard Brownian motions. Find*

$$\mathbb{P}\{B_t \geq 6W_t - 4 \text{ for all } 0 \leq t \leq 3\}.$$

Hint: You may wish to consider $6W_t - B_t$.
As in Exercise 2, if $Y_t = 6W_t - B_t$, then Y_t is a Brownian motion with drift 0 and variance parameter $6^2 + 1^2 = 37$. Using the reflection principle

$$\mathbb{P}\{B_t \geq 6W_t - 4 \textbf{ for all } 0 \leq t \leq 3\} = \mathbb{P}\{Y_t \leq 4 \textbf{ for all } 0 \leq t \leq 3\} = 1 - 2\,\mathbb{P}\{Y_3 \geq 4\}.$$

Since Y_t has variance parameter $\sigma^2 = 37$. we have $Y_3 \sim N(0, 3(37))$ and hence

$$\mathbb{P}\{Y_3 \geq 4\} = 1 - \Phi(4/\sqrt{3(37)}).$$

So the answer is

$$1 - 2\,[1 - \Phi(4/\sqrt{111})] = 2\,\Phi(4/\sqrt{111}) - 1 = 0.464964$$

Exercise 4 *Let B_t be a standard (one-dimensional) Brownian motion (not necessarily starting at the origin). For the following functions $\phi(t,x), 0 < t < 4$, state the PDE that it satisfies. If you use the L or L^* notation, you must say what L or L^* is in these cases.*

1. *$\phi(t,x)$ is the density of B_t (as a function of x) given that $B_0 = 0$.*
2. $\phi(t,x) = \mathbb{P}\{B_t < 4 \mid B_0 = x\}$
3. $\phi(t,x) = \mathbb{E}[B_t^3 \mid B_0 = x]$
4. $\phi(t,x) = E[B_4 - B_4^2 \mid B_t = x]$
5. *Repeat the examples above where B has drift 1 and variance parameter 4.*

I will write the answer in terms of L and L^*. For standard Brownian motion

$$Lf(x) = L^* f(x) = \frac{1}{2} f''(x),$$

For Brownian motion with drift 1 and variance parameter 4,

$$Lf(x) = f'(x) + \frac{4}{2} f''(x), \qquad L^* f(x) = -f'(x) + 2 f''(x).$$

1. *$\phi(t,x)$ is the density of B_t (as a function of x) given that $B_0 = 0$.*

   $$\dot{\phi}(t,x) = L^* \phi(t,x).$$

2. $\phi(t,x) = \mathbb{P}\{B_t < 4 \mid B_0 = x\}$
   This is the same as

   $$\phi(t,x) = \mathbb{E}[F(B_t) \mid B_0 = x]$$

   where

   $$F(x) = \begin{cases} 1 & x < 4 \\ 0 & x \geq 4. \end{cases}$$

   Therefore,

   $$\dot{\phi}(t,x) = L\phi(t,x).$$

3. $\phi(t,x) = \mathbb{E}[B_t^3 \mid B_0 = x\}$
   Same thing with $F(x) = x^3$.

   $$\dot{\phi}(t,x) = L\phi(t,x).$$

4. $\phi(t,x) = E[B_4 - B_4^2 \mid B_t = x]$
   This is the same as

   $$\phi(t,x) = \mathbb{E}[F(B_4) \mid B_t = x]$$

   where $F(x) = x - x^2$.

   $$\dot{\phi}(t,x) = -L\phi(t,x).$$

Exercise 5 *Suppose B_t, W_t are independent standard Brownian motions starting at the origin, and the random vector $Z_t = (Z_t^1, Z_t^2)$ is defined by*

$$Z_t^1 = B_t + W_t - t, \qquad Z_t^2 = 2\,B_t - 4\,W_t.$$

Note that Z_t is a two-dimensional Brownian motion.

1. *What are the drift and covariance matrix for Z?*
   The drift is $\vec{m} = (-1, 0)$. The covariance matrix is the same as if the drift term were not there.

$$\mathbb{E}\left[(B_t + W_t)^2\right] = 2t, \qquad \mathbb{E}\left[(2B_t - 4W_t)^2\right] = (2^2 + 4^2)\,t = 20t,$$

$$\mathbb{E}[(B_t + W_t)(2B_t - 4W_t)] = (2 - 4)t = -2t.$$

$$\Gamma = \begin{bmatrix} 2 & -2 \\ -2 & 20 \end{bmatrix}.$$

2. *What are the operators L, L^* associated to Z?*

$$\begin{aligned} Lf(\vec{x}) &= \vec{m} \cdot \nabla f(\vec{x}) + \frac{1}{2}\,\left[2\,\partial^2_{x_1x_1} f(\vec{x}) - 4\,\partial^2_{x_1x_2} f(\vec{x}) + 20\,\partial^2_{x_2x_2} f(\vec{x})\right] \\ &= \partial_{x_1} f(\vec{x}) + \frac{1}{2}\,\left[2\,\partial^2_{x_1x_1} f(\vec{x}) - 4\,\partial^2_{x_1x_2} f(\vec{x}) + 20\,\partial^2_{x_2x_2} f(\vec{x})\right] \\ L^* f(\vec{x}) &= -\vec{m} \cdot \nabla f(\vec{x}) + \frac{1}{2}\,\left[2\,\partial^2_{x_1x_1} f(\vec{x}) - 4\,\partial^2_{x_1x_2} f(\vec{x}) + 20\,\partial^2_{x_2x_2} f(\vec{x})\right] \\ &= -\partial_{x_1} f(\vec{x}) + \frac{1}{2}\,\left[2\,\partial^2_{x_1x_1} f(\vec{x}) - 4\,\partial^2_{x_1x_2} f(\vec{x}) + 20\,\partial^2_{x_2x_2} f(\vec{x})\right] \end{aligned}$$

3. *Let $\phi(t, x), t > 0, x \in \mathbb{R}^2$ be the density of Z_t at time t. Find the PDE satisfied by $\phi(t, x)$.*

$$\dot{\phi}(t, x) = L_x^*\,\phi(t, x)$$