FINM 34500/STAT 39000 Final Exam, Winter 2023

You have an hour and twenty minutes (80 minutes) to complete the exam. There are 8 problems/questions, each with multiple parts, for a total of 100 points. For all multi-part problems each part is worth the same amount, 5 points. You may use the one (double sided) sheet that you prepared for the test, but you may not use any other books, notes, or electronic devices.

Problem 1 (10)**.** Suppose $X_t, t \geq 0$ is a process with $X_0 = 0$.

1. Define what it means for $\{X_t\}$ to be a Brownian motion.
2. Define what it means for $\{X_t\}$ to be a Lévy process.

Problem 2 (15)**.** Suppose a process X_t satisfies

$$dX_t = -4\, X_t\, dt + dB_t, \qquad X_0 = 1,$$

where B_t is a standard Brownian motion. Let T be the first time t that $X_t = 3$ or $X_t = 0$. Suppose F is a C^2 function with $F(0) = 0$, $F(x) > 0$ for $x > 0$, and such that $F(X_{t\wedge T})$ is a martingale.

1. Find the second order ordinary differential equation that F satisfies.
2. Find one such function F. (You may leave your answer in terms of a definite integral.)
3. Find the probability that $X_T = 3$. You can write the answer in terms of the function F.

Problem 3 (10)**.** Let B_t be a standard Brownian motion and let

$$Z_t = \int_0^t \frac{1}{(1-s)^\alpha}\, dB_s, \qquad 0 \leq t < 1.$$

1. For which values of $\alpha > 0$ is it the case that $Z_t, 0 \leq t < 1$ is a square integrable martingale?
2. For which values of $\alpha > 0$ is it true that there exists $C = C_\alpha < \infty$ such that $\mathrm{Var}(Z_t) \leq C$ for all $t < 1$?

Problem 4 (10)**.** Suppose B_t is a standard Brownian motion with respect to probability P and X_t satisfies the following. State whether or not there is an equivalent (mutually absolutely continuous) measure Q such that with respect to Q, X_t is a standard Brownian motion, Give reasons.

1. $dX_t = dt + 2\, dB_t$.
2. $dX_t = 2\, dt + dB_t$

Problem 5 (10)**.** Suppose X_t is a compound Poisson process with $\lambda = 4$, and measure $\mu^\#$ is given by a uniform continuous random variable on $[0, 2]$. In other words, the process jumps when a Poisson process of rate 4 jumps and when it jumps it chooses a jump size uniformly on $[0, 2]$.

1. Find the function f so that the Lévy measure μ can be written as $d\mu = f(x)\, dx$.
2. Find $\mathrm{Var}[X_t]$

Problem 6 (20)**.** Let B_t be a standard Brownian motion with $B_0 = 1$. Let $T = \min\{t : B_t = 0\}$, and let $X_t = B_t^3$.

1. Find a function g such that
$$M_t := X_t \exp\left\{\int_0^t g(B_s)\, ds\right\}$$
is a local martingale for $t < T$. (Do not worry about what happens after time T.)

2. What SDE does M_t satisfy?

3. Let Q be the probability measure obtained by tilting by M_t. Find the SDE for B_t in terms of a Q-Brownian motion.

4. Is the local martingale in part 1 actually a martingale? Give reasons.

Problem 7 (15)**.** Suppose that a stock price S_t satisfies the SDE
$$dS_t = S_t\ [2\, dt + 3\, dB_t]$$
and its value at time 0 is $S_0 > 0$. Suppose that the interest rate is r, that is, one can obtain a bond worth $R_t = e^{rt}\, R_0$.

1. Give the SDE satisfied by the discounted stock price $\tilde{S}_t = e^{-rt}\, S_t$ (with respect to B_t).

2. Suppose we wish to find a probability measure Q and a standard Brownian motion with respect to Q such that the discounted stock price is a martingale with respect to Q. This will require tilting by a martingale M_t. Give the SDE that M_t satisfies (with respect to B_t).

3. Give the SDE that $\tilde{S}_t$ satisfies with respect to W_t, a standard Brownian motion with respect to Q.

Problem 8 (10)**.** Suppose X, Y are independent random variables; X is a standard normal and Y has a Poisson distribution with parameter (mean) 3. Let $Z = XY$. Let μ_X, μ_Y and μ_Z be the probability distributions for X, Y, and Z respectively. For the following statements say whether it is true or false and give reasons.

1. $\mu_X \perp \mu_Y$.

2. $\mu_Z \ll \mu_X$.