FINM 34500/STAT 39000 **Exam 2** (March 4, 2024)

You have two hours (120 minutes) to complete the exam. There are 6 problems/questions, many with multiple parts, worth varying amounts for a total of 100 points. For all multi-part problems all parts are worth the same amount. You may have one sheet of notes. No calculators, phones, etc., permitted.

For your reference, here is the standard Black-Scholes differential equation

$$\dot{f}(t,x) = r\,f(t,x) - r\,x\,f'(t,x) - \frac{\sigma^2}{2}\,x^2\,f''(t,x).$$

Problem 1 (10)**.** Consider the stochastic integral

$$Z_t = \int_0^t A_s\,dB_s,$$

where $A_t, 0 \le t < 1$ is as below with $A_t = 0$ for $t \ge 1$. Let $T = \min\{t : Z_t = 2\}$ and let $M_t = Z_{t\wedge T}$. In each case say whether or not $Z_t, 0 \le t \le 1$ is a martingale. Give reason.

1. $A_t = (1-t)^{-1/3}$
2. $A_t = (1-t)^{-2}$.

Problem 2 (15)**.** Suppose that X_t satisfies

$$dX_t = X_t\,dt + dB_t, \quad X_0 = 1,$$

where B_t is a standard Brownian motion. Let $T = \min\{t : X_t = 0 \text{ or } 3\}$. Suppose F is a function with $F(0) = 0$ that is C^2 for $0 < x < 3$ and such that $F(X_{t\wedge T})$ is a nonnegative martingale.

1. Find a second order ODE that $F(x), 0 < x < 3$ satisfies.
2. Find one such F with $F(x) > 0$ for $x > 0$ (you can leave your answer in terms of a definite integral).
3. Find the probability that $X_T = 3$. You can give your answer in terms of F.

Problem 3 (20)**.** Suppose that a stock price S_t follows the geometric Brownian motion

$$dS_t = S_t\,[3\,dt + 3\,dB_t]\,.$$

Suppose also there is a risk-free bond available growing at rate $r = .04$,

$$dR_t = r\,R_t\,dt.$$

1. Let $\tilde{S}_t = e^{-rt}\,S_t$ denote the discounted stock price. What is the SDE for $\tilde{S}_t$ (in terms of B_t)?
2. Let Q be the measure under which $\tilde{S}_t$ is a martingale. This is obtained using the Girsanov theorem by tilting by a martingale M_t. Write the SDE for M_t.
3. Write the SDE for $\tilde{S}_t$ in terms of a Brownian motion with respect to the measure Q.
4. Suppose there is a payoff $F(S_3) = [S_3 - 4]_+$ at time 3. Let $v(t,x)$ be the value of this option at time t given that the stock price at time t equals x. What partial differential equation does $v(t,x)$ satisfy?

Problem 4 (20)**.** Suppose X_t satisfies the SDE

$$dX_t = X_t\,dt + 4\,dB_t, \qquad X_0 = 0, \tag{1}$$

and let $T = \min\{t : X_t = -2 \text{ or } X_t = 3\}$. We will consider $Y_t = X_{t\wedge T}$, the process stopped at time T. Here B_t is a standard Brownian motion with respect to probability measure $\mathbb{P}$.

1. What is the (infinitesimal) generator L associated to the process satisfying (**??**)?

2. Suppose we want to find a new probability measure Q that is mutually absolutely continuous with respect to $\mathbb{P}$ and such that Y_t is a martingale with respect to Q. We will do this by tilting by a martingale M_t. Write down the SDE that M_t should satisfy for $t < T$. (It may be useful to note that $|X_t| \leq 3$ for $t < T$.)

3. Your answer to the previous part should show that $M_t, t < T$ is a local martingale. In fact it is a martingale (you do not need to show this and may use this fact). Give the SDE that X_t satisfies for $t < T$ with respect to a standard Brownian motion W_t with respect to the new measure Q.

4. Let $u(x), -2 < x < 3$, be the Q-probability that $X_T = 3$ given $X_0 = x$. Find $u(x)$.

Problem 5 (20)**.** Suppose $X_1, X_2, \ldots$ are independent, identically distributed, random variables with

$$\mathbb{P}\{X_j = 2\} = q, \qquad \mathbb{P}\{X_j = \frac{1}{2}\} = 1 - q.$$

Let

$$Y_n = X_1\, X_2\, \cdots\, X_n,$$

with $Y_0 = 1$.

1. For what value of q will Y_n be a martingale with respect to $\{\mathcal{F}_n\}$ where $\mathcal{F}_n$ is the information in $X_1, \ldots, X_n$? For the remainder of the problem use this value of q.

2. Find $\mathbb{P}\{Y_3 = 2\}$.

3. Let T be the first time n that $Y_n = 64$ or $Y_n = 1/8$. Find $\mathbb{P}\{Y_T = 64\}$.

4. Let Q denote the probability measure obtained by tilting by Y, that is, if V is an $\mathcal{F}_n$-measurable event, then $Q(V) = \mathbb{E}\left[1_V\, Y_n\right]$. Find $Q\{Y_3 = 2\}$.

Problem 6 (15)**.** For the following measures μ_1, μ_2, state if $\mu_1 \ll \mu_2$. Give reasons.

1. μ_1 is the distribution of a uniform random variable on $\{1, 2, 3, 4, 5, 6\}$ and μ_2 is the distribution of a normal random variable with mean 3 and variance 1.

2. μ_1 is the probability measure on continuous functions on $[0, 1]$ given by standard Brownian motion and μ_2 is the probability measure given by Brownian motion starting at the origin with drift 1 and variance parameter 4.

3. μ_1 is the distribution of a uniform random variable on $\{1, 2, 3, 4, 5, 6\}$ and μ_2 is the distribution of

$$X_1 + X_2 + \cdots + X_{20}$$

where $X_1, X_2, \ldots, X_{20}$ are independent random variables each with

$$\mathbb{P}\{X_j = 1\} = \mathbb{P}\{X_j = -1\} = \frac{1}{2}.$$