Exercise 1

- $\mu_1 \sim Pois(1)$
- $\mu_2 \sim Uniform[0,1]$
- $\mu_3 = \mu_1 + \mu_2$
- μ_4 is the counting measure on the integers
  $\mu_4(V) =$ the number of integers in V
- $\mu_5(V) =$ the length of V $\rightarrow$ infinite measure

1) μ_1 & μ_2

$\mu_1 \perp \mu_2$ since μ_1 is discrete & μ_2 is continuous

2) μ_1 & μ_3

$\mu_1 << \mu_3$ as μ_3 includes μ_1

Not $\mu_3 << \mu_1$ as μ_3 includes μ_2 which has continuous support

3) μ_1 & μ_4

$\mu_1 << \mu_4$ as μ_4 includes μ_1

Not $\mu_4 << \mu_1$ because both are discrete & μ_4 contains negative numbers but μ_1 doesnot.

4) μ_1 & μ_5

$\mu_1 \perp \mu_5$

Let $A =$ integers

Then $\mu_1(A^c) = 0$ & $\mu_5(A) = 0$

5) μ_2 & μ_3

$\mu_2 << \mu_3$

Not $\mu_3 << \mu_2$

because μ_3 includes μ_2 but not vice versa.

6) μ_2 & μ_4

$\mu_2 \perp \mu_4$ as μ_2 is continuous & μ_4 is discrete.

7) μ_2 & μ_5

$\mu_2 << \mu_5$

Not $\mu_5 << \mu_2$

as μ_5 includes region outside $[0,1]$

8) μ_3 & μ_4 $\to$ no relation

μ_3 includes discrete & continuous parts

μ_4 is discrete on all integers

There is no subset A which satisfies mutual singularity or absolute continuity.

9) μ_3 & μ_5 $\to$ no relation

μ_3 includes discrete & continuous parts

μ_5 is purely continuous

There is no subset A which satisfies mutual singularity or absolute continuity

10) μ_4 & μ_5

$\mu_4 \perp \mu_5$ as μ_4 is discrete & μ_5 is continuous

Exercise 2

$$dS_t = S_t[3dt + dB_t]$$

B_t is a Standard BM wrt P

$r = 0.05$

Q measure [mutually absolutely continuous to P] under which discounted stock price $\tilde{S}_t$ is a Martingale.

1) To make the discounted stock price $\tilde{S}_t = e^{-0.05t} S_t$ a martingale under Q, we need to adjust the drift term.

Using Girsanov's theorem,

$=$ market price of risk (θ) $= \dfrac{\text{Original drift} - \text{risk free rate}}{\text{Volatility}}$

$$\theta = \frac{3-0.05}{1} = 2.95$$

$\therefore$ transformed BM $\rightarrow dW_t = dB_t + \theta dt$

$$= dB_t + 2.95 dt$$

$\therefore$ SDE under $Q \rightarrow$

$$dS_t = S_t[3dt + dW_t - 2.95dt] = S_t[0.05dt + dW_t]$$

$$d\tilde{S}_t = d(e^{-0.05t} S_t)$$

$$= d(e^{-0.05t} S_t)$$

$$= -0.05e^{-0.05t} S_t dt + e^{-0.05t} dS_t + 0$$

$$= e^{-0.05t}[-0.05 S_t dt + S_t(0.05dt + dW_t)]$$

$$= e^{-0.05t}(S_t dW_t) = \tilde{S}_t dW_t$$

2) A contingent claim is a financial derivative whose payoff:

- Depends on the value of the underlying asset at a future time T
- is measurable & integrable

1. Dependence

$V = S_2^2$ depends on the stock price S_2 at time $T=2$

2. Measurability

The payoff $V = S_2^2$ depends only on S_2, which is fully observable at $T=2$

Since S_2 is F_2 measurable, $V = S_2^2$ is also F_2-measurable.

3. Integrability:

$$dS_t = 0.05 S_t dt + S_t dW_t$$

let $f(S_t) = \ln S_t$

$$df(S_t) = \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2$$

$$= \frac{1}{S_t}\left[0.05 S_t dt + S_t dW_t\right] - \frac{1}{2S_t^2}\left[S_t^2 dt\right]$$

$$d(\ln S_t) = \left(0.05 - \frac{1}{2}\right)dt + dW_t$$

$$\ln S_t - \ln S_0 = -0.45t + W_t \Rightarrow \ln\left(\frac{S_t}{S_0}\right) = -0.45t + W_t$$

$$S_t = S_0 \exp(-0.45t + W_t)$$

For $t=2$

$$S_2 = S_0 \exp(-0.9 + W_2)$$

$$E_Q[S_2^2] = S_0^2 \exp(-1.8) \; E_Q[\exp(2W_2)]$$

$$E[e^{tX}] = e^{t^2/2 \, Var(X)} \quad] \text{ Mgf}$$

$$E_Q[S_2^2] = S_0^2 \exp(-1.8)\exp(4)$$

$$= S_0^2 \exp(2.2) \quad < \infty$$

$\therefore$ it is integrable.

3) V_t for $0 \le t \le 2$ under Q

$$S_2 = S_t \exp(-0.45t + W_t) \quad \rightarrow \text{from } 2$$

$$S_2^2 = S_t^2 \exp\left(-0.45(2-t) \times 2 + 2 \times (W_2 - W_t)\right)$$

$$= S_t^2 \exp\left(-1.8 + 0.9t + 2(W_2 - W_t)\right)$$

$$\mathbb{E}[S_2^2] = S_t^2 \exp(0.9t - 1.8)\, E\left[\exp(2(W_2 - W_t))\right]$$

$$W_2 - W_t \sim N(0, 2-t)$$

$$E\left[\exp(2(W_2 - W_t))\right] = \exp\left(2(2-t)\right) = \exp(4-2t)$$

$$\mathbb{E}[S_2^2] = S_t^2 \exp(2.2 - 1.1t) = S_t^2 \exp\left(1.1(2-t)\right)$$

$$\tilde{V}_t = \exp\left(-(2-t) \times 0.05\right) S_t^2 \exp\left(1.1(2-t)\right)$$

$$= S_t^2 \exp\left(1.05(2-t)\right)$$

4) $$d\tilde{V}_t = d\left(S_t^2 e^{1.05(2-t)}\right)$$

$$= 2S_t e^{1.05(2-t)} dS_t - 1.05 e^{1.05(2-t)} S_t^2 dt + \frac{1}{2} \times 2 e^{1.05(2-t)} (dS_t)^2$$

$$= \frac{2\tilde{V}_t}{S_t} dS_t - 1.05 \tilde{V}_t dt + \frac{\tilde{V}_t}{S_t^2} S_t^2 dt$$

$$= \frac{2\tilde{V}_t}{S_t}\left[0.05 S_t dt + S_t dW_t\right] - 1.05 \tilde{V}_t dt + \tilde{V}_t dt$$

$$= 0.1 \tilde{V}_t dt + 2\tilde{V}_t dW_t - 1.05 \tilde{V}_t dt + \tilde{V}_t dt$$

$$= 0.05 \tilde{V}_t dt + 2\tilde{V}_t dW_t$$

5) Portfolio (a_t, b_t) that hedges the claim

$$\tilde{P}_t = a_t \tilde{S}_t + b_t \tilde{B}_t$$

$$\tilde{B}_t = e^{-0.05t} B_t = 1$$

$$\therefore \tilde{P}_t = a_t \tilde{S}_t + b_t$$

$$d\tilde{P}_t = a_t d\tilde{S}_t$$

$$d\tilde{P}_t = d\tilde{V}_t$$

$$a_t d\tilde{S}_t = 0.05\tilde{V}_t\,dt + 2\tilde{V}_t dW_t$$

$$a_t[\tilde{S}_t dW_t] = 0.05\tilde{V}_t\,dt + 2\tilde{V}_t\,dW_t$$

$$a_t\tilde{S}_t = 2\tilde{V}_t$$

$$a_t[S_t\exp(-0.05t)] = 2[S_t^2\exp(1.05(2-t))]$$

$$a_t = 2S_t e^{1.05(2-t)+0.05t}$$

$$= 2S_t e^{2.1-t}$$

$$\tilde{P}_t = \tilde{V}_t = 2S_t e^{2.1-t}[\tilde{S}_t] + b_t$$

$$\therefore b_t = S_t^2 e^{1.05(2-t)} - 2S_t e^{2.1-t}[\tilde{S}_t]$$

$$= S_t^2 e^{1.05(2-t)} - 2S_t e^{2.1-t}[S_t e^{-0.05t}]$$

$$= \tilde{V}_t - 2S_t^2 e^{2.1-1.05t}$$

$$= \tilde{V}_t - 2\tilde{V}_t = -\tilde{V}_t$$

$$\text{Portfolio} = [2S_t e^{2.1-t},\ -S_t^2 e^{1.05(2-t)}]$$

6) $V_t = e^{0.05t}\tilde{V}_t = e^{0.05t}S_t^2 e^{1.05(2-t)}$

$$V_t = S_t^2 e^{2.1-t}$$

Exercise 3

$V = S_2^3$

1) $d\tilde{S}_t = \tilde{S}_t dW_t$ [From exercise 2]

2) Yes the claim at time $T=2$ of $V = S_2^3$ is a contingent claim as $\rightarrow$

- S_2^3 is dependent on the underlying S_2 at time 2
- Since S_2 is F_2 measurable, $V = S_2^3$ is also F_2-measurable.
- $S_2 = S_0 \exp(-0.9 + W_2)$ → from exercise 2

$E_Q[S_2^3] = S_0^3 \exp(-2.7)\ E_Q[\exp(3W_2)]$

$E[e^{tX}] = e^{t^2/2 \, Var(X)}$] Mgf

$E_Q[S_2^3] = S_0^3 \exp(-2.7) \exp(9)$

$= S_0^3 \exp(6.3) < \infty$

∴ it is integrable.

3) $\tilde{V}_t$ for $0 \leq t \leq 2$

$\tilde{V}_t = V_t \exp(-0.05t)$

$S_2 = S_t \exp(-0.45t + W_t)$ → from 2

$S_2^3 = S_t^3 \exp(-0.45(2-t) \times 3 + 3 \times (W_2 - W_t))$

$= S_t^3 \exp(-2.7 + 1.35t + 3(W_2 - W_t))$

$E[S_2^3] = S_t^3 \exp(1.35t - 2.7)\ E[\exp(3(W_2 - W_t))]$

$W_2 - W_t \sim N(0, 2-t)$

$E[\exp(3(W_2 - W_t))] = \exp(\frac{9}{2}(2-t)) = \exp(4.5(2-t))$

$E[S_2^3] = S_t^3 \exp(3.15(2-t))$

$\tilde{V}_t = \exp(-(2-t) \times 0.05)\ S_t^3 \exp(3.15(2-t))$

$= S_t^3 \exp(3.1(2-t))$

4) SDE for $\tilde{V}_t$

$$d\tilde{V}_t = 3S_t^2 \exp(3.1(2-t))\, dS_t - 3.1 S_t^3 \exp(3.1(2-t))\,dt + \frac{1}{2} 6S_t \exp(3.1(2-t))(dS_t)^2$$

$$= 3\frac{\tilde{V}_t}{S_t} dS_t - 3.1\tilde{V}_t dt + 3\frac{\tilde{V}_t}{S_t^2} S_t^2 dt$$

$$= 3\frac{\tilde{V}_t}{S_t}\left[S_t(3dt + dB_t)\right] - 3.1\tilde{V}_t dt + 3\tilde{V}_t dt$$

$$= 3\tilde{V}_t(3dt + dB_t) - 0.1\tilde{V}_t dt$$

$$= 8.9\tilde{V}_t dt + 3\tilde{V}_t dB_t$$

$$dW_t = dB_t + 2.95\,dt$$

$$d\tilde{V}_t = 8.9\tilde{V}_t dt + 3\tilde{V}_t\left[dW_t - 2.95\,dt\right]$$

$$= 0.05\tilde{V}_t dt + 3\tilde{V}_t dW_t$$

5) Portfolio (a_t, b_t)

$$\tilde{P}_t = a_t\tilde{S}_t + b_t\tilde{B}_t$$

$$\tilde{B}_t = e^{0.05t} B_t = 1$$

$$\tilde{P}_t = a_t\tilde{S}_t + b_t$$

$$d\tilde{P}_t = a_t d\tilde{S}_t \quad \text{[self financing]}$$

$$d\tilde{P}_t = d\tilde{V}_t = 0.05\tilde{V}_t dt + 3\tilde{V}_t dW_t$$

$$a_t\left[\tilde{S}_t dW_t\right] = 0.05\tilde{V}_t dt + 3\tilde{V}_t dW_t$$

$$a_t e^{-0.05t} S_t = 3\tilde{V}_t = 3S_t^3 \exp\left(3.1(2-t)\right)$$

$$\therefore a_t = 3S_t^2 \exp(3.1(2-t) + 0.05t)$$

$$\tilde{P}_t = \tilde{V}_t = a_t \tilde{S}_t + b_t$$

$$S_t^3 \exp(3.1(2-t)) = 3S_t^2 \exp(3.1(2-t) + 0.05t) \tilde{S}_t + b_t$$

$$= 3S_t^3 \exp(3.1(2-t) + 0.05t) \exp(-0.05t) + b_t$$

$$\tilde{V}_t = 3\tilde{V}_t + b_t$$

$$\therefore b_t = -2\tilde{V}_t$$

$$\text{Portfolio} = \left[3S_t^2 \exp(3.1(2-t) + 0.05t), -2S_t^3 \exp(3.1(2-t))\right]$$

6) $$V_t = e^{0.05(2-t)} \tilde{V}_t$$

$$= e^{0.05(2-t)} S_t^3 e^{3.1(2-t)}$$

$$= e^{3.15(2-t)} S_t^3$$

<u>Exercise 4</u>

$$V = \int_0^2 s S_s \, ds$$

1) $d\tilde{S}_t = \tilde{S}_t dW_t$ [From exercise 2]

2) Yes the claim at time $T=2$ of $V = \int_0^2 s S_s \, ds$ a contingent claim as →

- $\int_0^2 s S_s \, ds$ is dependent on the underlying
- $\int_0^2 s S_s \, ds$ is F_2 measurable
- $E\left[\int_0^2 s S_s ds\right] = \int_0^2 s E(S_s) ds$

$S_s = S_0 \exp(-0.45s + W_s)$ [From Question 2]

$E[S_s] = S_0 \exp(-0.45s) E[\exp(W_s)]$

$$= S_0 e^{-0.45s + \frac{1}{2}s} = S_0 e^{0.05s}$$

$$\therefore E\left[\int_0^2 s S_s\right] = \int_0^2 s S_0 e^{0.05s} ds < \infty$$

$\therefore$ it is integrable

3) $$\tilde{V}_t = \exp(-0.05 \times (2-t))\, E\left[\int_0^2 s S_s\, ds / F_t\right]$$

$$E\left[\int_0^2 s S_s\, ds\right] = \int_0^t s S_s\, ds + \int_t^2 s E^{Q}[S_s/F_t]\, ds$$

$$E^{Q}[S_s/F_t] = S_t e^{0.05(s-t)} \rightarrow \text{From } 2$$

$$\int_t^2 s S_t e^{0.05(s-t)} ds = S_t \int_t^2 s e^{0.05(s-t)} ds$$

let $s-t = u$

$s = t+u$

$ds = du$

$$\therefore S_t \int_0^{2-t} (u+t) e^{0.05u} du = S_t\left[t\int_0^{2-t} e^{0.05u} du + \int_0^{2-t} u e^{0.05u} du\right]$$

$$t\int_0^{2-t} e^{0.05u} du = t\left[\frac{e^{0.05(2-t)}}{0.05} - \frac{1}{0.05}\right] = 20t\left(e^{0.05(2-t)} - 1\right)$$

$$\int_0^{2-t} u e^{0.05u} du = u\int_0^{2-t} e^{0.05u} du - \int_0^{2-t}\left(\frac{d}{du} u \int_0^{2-t} e^{0.05u} du\right) du$$

$$= \left[u \cdot \frac{e^{0.05u}}{0.05}\right]_0^{2-t} - \int_0^{2-t} \frac{e^{0.05u}}{0.05} du$$

$$= \frac{(2-t)e^{0.05(2-t)}}{0.05} - 400\left[e^{0.05(2-t)} - 1\right]$$

$$= 20(2-t)e^{0.05(2-t)} - 400 e^{0.05(2-t)} + 400$$

Combining results,

$$S_t\left[20t\left(e^{0.05(2-t)}-1\right)+20(2-t)e^{0.05(2-t)}-400e^{0.05(2-t)}+400\right]$$

$$=\left[-20t-360e^{0.05(2-t)}+400\right]S_t$$

$$E\left[\int_0^2 sS_s\,ds\right]=\int_0^t sS_s\,ds+\left[-20t-360e^{0.05(2-t)}+400\right]S_t$$

$$\tilde{V}_t=e^{-0.05(2-t)}\left[\int_0^t sS_s\,ds+\left[-20t-360e^{0.05(2-t)}+400\right]S_t\right]$$

4) SDE for $\tilde{V}_t$

$$\tilde{V}_t=e^{-0.05(2-t)}\int_0^t sS_s\,ds+20\left[-t-18e^{0.05(2-t)}+20\right]e^{-0.05(2-t)}S_t$$

$$=\underbrace{e^{-0.05(2-t)}\int_0^t sS_s\,ds}_{A}+\underbrace{20\left[-te^{-0.05(2-t)}+20e^{-0.05(2-t)}-18\right]S_t}_{B}$$

$$dA=0.05e^{-0.05(2-t)}\int_0^t sS_s\,ds\,dt+e^{-0.05(2-t)}tS_t\,dt$$

$$B=F(t)S_t$$

$$dB=F'(t)S_t\,dt+dS_tF(t)$$

$$=F'(t)S_t\,dt+\left[0.05S_t\,dt+S_t\,dW_t\right]F(t)$$

$$=\left[F'(t)+0.05F(t)\right]S_t\,dt+F(t)S_t\,dW_t$$

$$d\tilde{V}_t=dA+dB$$

$$=\left(0.05e^{-0.05(2-t)}\int_0^t sS_s\,ds+F'(t)S_t+0.05F(t)S_t+te^{-0.05(2-t)}S_t\right)dt$$

$$+F(t)S_t\,dW_t$$

$$= \left[0.05 \left(e^{-0.05(2-t)} \int_0^t s S_s ds + F(t) S_t \right) + F'(t) S_t + t e^{-0.05(2-t)} S_t \right] dt$$

$$+ F(t) S_t dW_t$$

$$d\tilde{V}_t = \left[0.05 \tilde{V}_t + F'(t) S_t + t e^{-0.05(2-t)} S_t \right] dt + F(t) S_t dW_t$$

where $F(t) = 20\left[-t e^{-0.05(2-t)} + 20 e^{-0.05(2-t)} - 18 \right] S_t$

5) Portfolio (a_t, b_t)

$\tilde{P}_t = a_t \tilde{S}_t + b_t \tilde{B}_t$

$$\tilde{B}_t = e^{-0.05t} B_t = 1$$

$$\therefore \tilde{P}_t = a_t \tilde{S}_t + b_t$$

$$d\tilde{P}_t = a_t d\tilde{S}_t$$

$$d\tilde{V}_t = d\tilde{P}_t = a_t d\tilde{S}_t$$

$$= a_t \left[\tilde{S}_t dW_t \right] = a_t \left[e^{-0.05t} S_t dW_t \right]$$

$$\therefore F(t) S_t dW_t = a_t \left[e^{-0.05t} S_t dW_t \right]$$

$$\therefore a_t = \frac{F(t)}{e^{-0.05t}}$$

$$\tilde{V}_t = \tilde{P}_t = \frac{F(t)}{e^{-0.05t}} \tilde{S}_t + b_t$$

$$e^{-0.05(2-t)} \left[\int_0^t s S_s ds + \left[-20t - 360 e^{0.05(2-t)} + 400 \right] S_t \right]$$

$$= F(t) S_t + b_t$$

$$\therefore b_t = e^{-0.05(2-t)} \int_0^t s S_s ds$$

6) V_t

$$V_t = \tilde{V}_t e^{0.05(2-t)}$$

$$= \int_0^t sS_s\,ds + [-20t - 360e^{0.05(2-t)} + 400]S_t$$

Exercise 5

1) $a \neq 0$ & y are constants, $N \to$ standard normal dist

Show: Density of $Z = \exp(aN + y)$ is $g(z) = \frac{1}{aZ}\phi\left(\frac{-y + \log z}{a}\right)$

Let $X = aN + y$

$X \sim N(y, a^2)$

$\therefore Z \sim \log Normal(y, a^2)$ as $Z = \exp(X)$

PDF of lognormal distribution $\Rightarrow$

$$g(z) = \frac{1}{z\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln z - \mu)^2}{2\sigma^2}\right)$$

$\mu = y$ & $\sigma = a$

$$\therefore g(z) = \frac{1}{za\sqrt{2\pi}}\exp\left(-\frac{(\ln z - y)^2}{2a^2}\right)$$

PDF of normal dist $\rightarrow$

$$\phi(n) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{n^2}{2}\right)$$

If $n = \frac{\ln z - y}{a}$

$$\phi\left(\frac{\ln z - y}{a}\right) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(\ln z - y)^2}{2a^2}\right)$$

Multiplying $\frac{1}{za}$ throughout

$$\frac{1}{za}\phi\left(\frac{\ln z - y}{a}\right) = \frac{1}{za\sqrt{2\pi}}\exp\left(\frac{-(\ln z - y)^2}{2a^2}\right) = g(z)$$

$$\therefore \quad g(z) = \frac{1}{za}\phi\left(\frac{-y+\ln z}{a}\right)$$ [Symmetry of Normal dist]

2) To show :

$$\int_x^\infty (z-x)g(z)\,dz = e^{y+(a^2/2)}\Phi\left(\frac{y-\log x + a^2}{a}\right) - x\Phi\left(\frac{y-\log x}{a}\right)$$

$$\int_x^\infty (z-x)g(z)\,dz = \underbrace{\int_x^\infty z\,g(z)\,dz}_{A} - \underbrace{x\int_x^\infty g(z)\,dz}_{B}$$

Ⓐ $\rightarrow$
$$\int_x^\infty z\,g(z)\,dz = \int_x^\infty z\,\frac{1}{az}\phi\left(\frac{-y+\ln z}{a}\right)dz$$
$$= \frac{1}{a}\int_x^\infty \phi\left(\frac{-y+\ln z}{a}\right)dz$$

Let $u = \dfrac{-y+\ln z}{a}$

$\therefore z = e^{au+y} \qquad \therefore dz = a e^{au+y}\,du$

$$= \frac{1}{a}\int_{\frac{-y+\ln x}{a}}^\infty \phi(u)\,a e^{au+y}\,du$$

$$= e^y\int_{\frac{-y+\ln x}{a}}^\infty \phi(u)e^{au}\,du$$

$$\phi(u)e^{au} = \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}+au} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(u^2-2au)}$$

$\therefore$ Completing the square

$$= e^{y} e^{\frac{a^2}{2}} \int_{\frac{-y+\ln x}{a}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u^2 - 2au + a^2)} du$$

$$= e^{y} e^{a^2/2} \int_{\frac{-y+\ln x}{a}}^{\infty} \phi(u-a) \, du$$

$$= e^{y} e^{a^2/2} \int_{\frac{-y+\ln x}{a} - a}^{\infty} \phi(v) \, dv$$

let $v = u - a$
$dv = du$

$$= e^{y} e^{a^2/2} \left[1 - \Phi\left(\frac{-y + \ln x - a^2}{a} \right) \right]$$

$$= e^{y + a^2/2} \, \Phi\left(\frac{y + a^2 - \ln x}{a} \right)$$

(B) $\rightarrow$ $x \int_{x}^{\infty} g(z) \, dz = x \, P(z \geq x)$

$z \sim \ln N(y, a^2)$

$\ln z \sim N(y, a^2)$

$P(z \geq x) = P(\ln z \geq \ln x) = P\left(N > \frac{\ln x - y}{a} \right)$

$= \Phi\left(\frac{y - \ln x}{a} \right)$

$$\therefore \quad x \int_{x}^{\infty} g(z) \, dz = x \, \Phi\left(\frac{y - \ln x}{a} \right)$$

Combining A & B

$$e^{y + a^2/2} \, \Phi\left(\frac{y + a^2 - \ln x}{a} \right) - x \, \Phi\left(\frac{y - \ln x}{a} \right)$$

Hence Proved

3) $\tilde{V}_t = \tilde{S}_t \Phi(d_1) - \tilde{K}\Phi(d_2)$

$$\tilde{S}_T = \tilde{S}_t \exp\left(\sigma(B_T - B_t) - \frac{\sigma^2(T-t)}{2}\right) = \exp\left(\sigma(B_T - B_t) + \ln(S_t) - \frac{\sigma^2(T-t)}{2}\right)$$

Call option Payoff : $(\tilde{S}_T - \tilde{K})_+$

From Part 1 : $Z = \exp(aN + y)$

$Z = \tilde{S}_T$

$aN = \sigma(B_T - B_t) = \sigma\sqrt{T-t}\, N(0,1)$

$\therefore a = \sigma\sqrt{T-t}$

$y = \ln(S_t) - \frac{\sigma^2(T-t)}{2} = \ln(S_t) - \frac{a^2}{2}$

$\therefore x = \tilde{K} = e^{-rT}K$

From Part 2,

$$\int_x^\infty (z-x)g(z)\,dz = e^{y+(a^2/2)}\Phi\left(\frac{y - \log x + a^2}{a}\right) - x\Phi\left(\frac{y - \log x}{a}\right)$$

$$= e^{\ln\tilde{S}_t - \frac{a^2}{2} + \frac{a^2}{2}}\Phi\left(\frac{\ln(\tilde{S}_t) - \frac{a^2}{2} - \ln\tilde{K} + a^2}{a}\right) - \tilde{K}\Phi\left(\frac{\ln(\tilde{S}_t) - \frac{a^2}{2} - \ln\tilde{K}}{a}\right)$$

$$= \tilde{S}_t\Phi\left(\frac{\ln(\tilde{S}_t/\tilde{K}) + a^2/2}{a}\right) - \tilde{K}\Phi\left(\frac{\ln(\tilde{S}_t/\tilde{K}) - a^2/2}{a}\right)$$

Let $d_1 = \dfrac{\ln(\tilde{S}_t/\tilde{K}) + a^2/2}{a} = \dfrac{\ln(S_t e^{-rt}/Ke^{-rT}) + \frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}}$

$$= \frac{\ln(S_t/K) + r(T-t) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

Similarily,

$$d_2 = \frac{\ln(\tilde{S}_t/\tilde{K}) - a^2/2}{a} = \frac{\ln(S_t/K) + r(T-t) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}$$

$$\therefore \tilde{V}_t = \tilde{S}_t \Phi(d_1) - \tilde{K}\Phi(d_2)$$

$$V_t = e^{r(T-t)}\left[\tilde{S}_t \Phi(d_1) - \tilde{K}\Phi(d_2)\right]$$

$$= S_t \Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

This is the black scholes formula.