Exercise 1

$A_t = 1, \quad 0 \le t \le 1/2$

$A_t = Y, \quad 1/2 \le t \le 1$

Y is a random variable measurable with respect to $F_{1/2}$

Let $Z_t = \int_0^t A_s \, dB_s$

1)

$$Y = \begin{cases} 0 & B_{1/2} \geq 0 \\ 1 & B_{1/2} < 0 \end{cases}$$

$$Z_1 = \int_0^{1/2} 1 \, dB_s + \int_{1/2}^{1} Y \, dB_s = (B_{1/2} - B_0) + Y(B_1 - B_{1/2})$$

$$= B_{1/2} + Y(B_1 - B_{1/2})$$

Let $B_{1/2} = X$ & $B_1 - B_{1/2} = W$

$X \sim N(0, 1/2)$ & $W \sim N(0, 1/2)$

but X & Z are independent

$$Z_1 = X + YW$$

$$P(Z_1 \geq 0) = P(X + YW \geq 0)$$

$$= P(X + YW \geq 0 / X \geq 0) \times P(X \geq 0) + P(X + YW \geq 0 / X < 0) \times P(X < 0)$$

① $P(X + YW \geq 0 / X \geq 0) \, P(X \geq 0)$

$$= P(X \geq 0 / X \geq 0) \times P(X \geq 0) = 1 \times 1/2 = 1/2$$

since $Y = 0$

② $P(X+YW \geq 0 / X<0)\ P(X<0)$

$$= P(X+W \geq 0/X<0) \times \frac{1}{2}$$

$$= \frac{1}{2} P\left(B_{1/2} + (B_1 - B_{1/2}) \geq 0 / x<0\right)$$

$$= \frac{1}{2} P(B_1 \geq 0 / B_{1/2} < 0) = \frac{1}{2} \frac{P(B_1 \geq 0, B_{1/2} < 0)}{P(B_{1/2} < 0)}$$

B_1 & $B_{1/2}$ are dependent

$Cov(B_1, B_{1/2}) = 1/2$

$$\text{Correlation} = \frac{1/2}{\sqrt{1 \times 1/2}} = \frac{1}{\sqrt{2}}$$

$$P(B_1 > 0, B_{1/2} > 0) = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{4} + \frac{1}{2\pi} \times \frac{\pi}{4} = \frac{3}{8}$$

$$P(B_1 > 0) = \frac{1}{2}$$

$$P(B_1 > 0, B_{1/2} < 0) = \frac{1}{2} - \frac{3}{8} = \frac{1}{8}$$

$$\therefore \frac{1}{2} P(B_1 \geq 0 / B_{1/2} < 0) = \frac{1}{2}\left(\frac{1/8}{1/2}\right) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$

$$\therefore P(Z_1 \geq 0) = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$$

2) $$Y = \begin{cases} 0 & B_{1/2} \geq 0 \\ -5 & B_{1/2} < 0 \end{cases}$$

$$Z_1 = \int_0^1 A_s\, ds = \int_0^{1/2} dB_s + \int_{1/2}^1 Y\, dB_s$$

$$= \underbrace{\left(B_{1/2} - B_0\right)}_{X} + Y \underbrace{\left[B_1 - B_{1/2}\right]}_{W}$$

$$P(Z_1 \geq 0) = P(X + YW \geq 0)$$

$$= P(X + YW \geq 0 / X \geq 0)\, P(X \geq 0) +$$

$$P(X + YW \geq 0 / X < 0)\, P(X < 0)$$

$$P(X + YW \geq 0 / X \geq 0)\, P(X \geq 0) = \frac{1}{2}$$ [From previous question]

$$P(X + YW \geq 0 / X < 0)\, P(X < 0) = \frac{1}{2} P\left(B_{1/2} - 5(B_1 - B_{1/2}) \geq 0 / B_{1/2} < 0\right)$$

$$= \frac{1}{2} P\left(-5B_1 + 6B_{1/2} \geq 0 / B_{1/2} < 0\right)$$

$$= \frac{1}{2} \frac{P\left(6B_{1/2} - 5B_1 \geq 0,\, B_{1/2} < 0\right)}{1/2}$$

$$= \int_{-\infty}^{0} P(6x - 5B_1 \geq 0)\, dP(B_{1/2} = x)$$

$$= \int_{-\infty}^{0} P\left(B_1 \leq \frac{6x}{5}\right) \frac{1}{\sqrt{\pi}} e^{-x^2} dx$$

Subtracting x from both sides

$$= \int_{-\infty}^{0} P\left(B_1 - B_{1/2} \leq \frac{x}{5}\right) \frac{1}{\sqrt{\pi}} e^{-x^2} dx$$

$$= \int_{-\infty}^{0} \int_{-\infty}^{x/5} \frac{1}{\pi} e^{-(x^2+y^2)}\, dx\, dy$$

Let $\phi = \tan^{-1}(1/5)$

$$= \int_{\theta = \phi - r}^{-r/2} \int_{r=0}^{\infty} \frac{r e^{-r^2}}{\pi}\, dr\, d\theta$$

$$= \frac{\left(\frac{\pi}{2} - \phi\right)}{2\pi} \int_{t=0}^{\infty} e^{-t}\, dt$$

$r^2 = t$

$2r\,dr = dt$

$$= \frac{1}{4} - \frac{\phi}{2\pi}$$

$$\therefore P(Z_1 \geq 0) = \frac{1}{2} + \frac{1}{4} - \frac{\phi}{2\pi} = \frac{3}{4} - \frac{\phi}{2\pi}$$

where $\phi = \tan^{-1}(1/5)$

3) $Y = \begin{cases} 1 & B_{1/2} \geq 0 \\ -1 & B_{1/2} < 0 \end{cases}$

$$Z_1 = \int_0^1 A_s\, ds = \int_0^{1/2} dB_s + \int_{1/2}^{1} Y\, dB_s$$

$$= \underbrace{(B_{1/2} - B_0)}_{X} + Y \underbrace{\left[B_1 - B_{1/2}\right]}_{W}$$

$$P(Z_1 \geq 0) = P(X + YW \geq 0)$$

$$= P(X + YW \geq 0 / X \geq 0)\, P(X \geq 0) +$$
$$P(X + YW \geq 0 / X < 0)\, P(X < 0)$$

$$P(X + YW \geq 0 / X \geq 0)\, P(X \geq 0)$$

$$= P(X + W \geq 0 / X \geq 0) \times \frac{1}{2}$$

$$= P(B_1 \geq 0 / B_{1/2} \geq 0) \times \frac{1}{2}$$

$$= \frac{P(B_1 \geq 0,\ B_{1/2} \geq 0)}{1/2} \times \frac{1}{2}$$

$$= P(B_1 \geq 0,\ B_{1/2} \geq 0)$$

$$\mathrm{Cov}(B_1, B_{1/2}) = 1/2$$

$$\mathrm{Corr}(B_1, B_{1/2}) = \frac{1/2}{\sqrt{1 \times 1/2}} = \frac{1}{\sqrt{2}}$$

$$P(B_1 \geq 0, B_{1/2} \geq 0) = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{4} + \frac{1}{2\pi} \times \frac{\pi}{4} = \frac{3}{8}$$

$$P(X + YW \geq 0 / X < 0) \times P(X<0) = P(X - W \geq 0 / X < 0)\, P(X<0)$$

$$= P(2B_{1/2} - B_1 \geq 0 / B_{1/2} < 0)\, P(B_{1/2} < 0)$$

$$= P(2B_{1/2} - B_1 \geq 0,\ B_{1/2} < 0)$$

$$= \int_{-\infty}^{0} P(2x - B_1 \geq 0)\, dP(B_{1/2} = x)$$

$$= \int_{-\infty}^{0} P(B_1 \leq 2x) \frac{1}{\sqrt{\pi}} e^{-x^2} dx$$

$$= \int_{-\infty}^{0} P(B_1 - B_{1/2} \leq x) \frac{1}{\sqrt{\pi}} e^{-x^2} dx$$

$$= \int_{-\infty}^{0} \int_{-\infty}^{x} \frac{1}{\pi} e^{-(x^2+y^2)} dx\, dy$$

$$= \int_{\theta = -3\pi/4}^{-\pi/2} \int_{r=0}^{\infty} \frac{1}{\pi} e^{-r^2} r\, dr\, d\theta$$

$$= \frac{\pi}{4} \frac{1}{2\pi} \int_{0}^{\infty} e^{-t} dt \qquad \begin{array}{c} r^2 = t \\ 2r\,dr = dt \end{array}$$

$$= 1/8$$

$$\therefore P(Z_1 \geq 0) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$$

Exercise 2

$$\langle Z \rangle_t = \int_0^t A_s^2 \, ds$$

If $0 \le t < 1/2$

$$\langle Z \rangle_t = \int_0^t 1 \, ds = t$$

If $1/2 \le t < 1$

$$\langle Z \rangle_t = \int_0^{1/2} 1 \, ds + \int_{1/2}^t Y^2 \, ds = \frac{1}{2} + Y^2 [t - 1/2]$$

$$\therefore \langle Z \rangle_t = \frac{1}{2} + Y^2 [t - 1/2]$$

Case 1

For $B_{1/2} \ge 0$

$\langle Z \rangle_t = t$ if $0 < t < 1/2$ or $\langle Z \rangle_t = \frac{1}{2}$ if $1/2 < t < 1$

For $B_{1/2} < 0$

$\langle Z \rangle_t = t$

Case 2

For $B_{1/2} \ge 0$

$\langle Z \rangle_t = t$ if $0 < t < 1/2$ and $\langle Z \rangle_t = \frac{1}{2}$ if $1/2 < t < 1$

For $B_{1/2} < 0$

$\langle Z \rangle_t = t$ if $0 < t < 1/2$ and $\langle Z \rangle_t = \frac{1}{2} + 25 \left(t - \frac{1}{2}\right)$ if $1/2 < t < 1$

$= 25t - 12$

Case 3

For $B_{1/2} \geq 0$

$$\langle Z \rangle_t = t$$

For $B_{1/2} < 0$

$$\langle Z \rangle_t = t$$

for $0 < t < 1$

Exercise 3

$$Y = \begin{cases} 0 & B_{1/2} \geq 0 \\ -5 & B_{1/2} < 0 \end{cases}$$

$$Z_1 = \int_0^1 A_s \, dB_s = \int_0^{1/2} dB_s + \int_{1/2}^1 Y \, dB_s$$

$$= B_{1/2} + Y\left[B_1 - B_{1/2}\right]$$

$$E[Z_1] = E\left[B_{1/2} + Y\left[B_1 - B_{1/2}\right]\right]$$

$$= E(B_{1/2}) + E(Y)E(B_1 - B_{1/2})$$

$$= 0$$

LHS

$$\therefore \mathrm{Var}(Z_1) = E(Z_1^2)$$

$$E(Z_1^2) = E\left[\left[B_{1/2} + Y\left[B_1 - B_{1/2}\right]\right]^2\right]$$

$$= E\left[B_{1/2}^2 + 2YB_{1/2}(B_1 - B_{1/2}) + Y^2\left[B_1 - B_{1/2}\right]^2\right]$$

$$= E(B_{1/2}^2) + 2E(Y)E(B_{1/2})E(B_1 - B_{1/2}) + E(Y^2)E\left[(B_1 - B_{1/2})^2\right]$$

$$= \frac{1}{2} + 0 + E(Y^2)\frac{1}{2}$$

$$E(Y^2) = 0 \times P(B_{1/2} \geq 0) + 25 \times P(B_{1/2} < 0)$$

$$= 25 \times \frac{1}{2} = 12.5$$

$$\therefore E(Z_1^2) = \frac{1}{2} + \frac{12.5}{2} = 6.75$$

<u>RHS</u>

$$\int_0^1 E[A_t^2]\,dt$$

$$= \int_0^{1/2} E[A_t^2]\,dt + \int_{1/2}^1 E[A_t^2]\,dt$$

$$= \int_0^{1/2} dt + \int_{1/2}^1 E[Y^2]\,dt$$

$$= \frac{1}{2} + E(Y^2)\int_{1/2}^1 dt$$

$$= \frac{1}{2} + 12.5 \times \frac{1}{2} = 6.75$$

Hence LHS = RHS