Exercise 1

$$df(t,B_t) = \dot{f}(t,B_t) + f'(t,B_t) + \frac{1}{2}f''(t,B_t)$$
$$= \left[\dot{f}(t,B_t) + \frac{1}{2}f''(t,B_t)\right]dt + f'(t,B_t)\,dB_t$$

since x follows a standard BM

1) $f(t,x) = xe^{-x}$

$$f'(t,x) = -xe^{-x} + e^{-x} = e^{-x}(1-x)$$
$$f''(t,x) = (xe^{-x} - e^{-x}) - e^{-x} = xe^{-x} - 2e^{-x}$$
$$= e^{-x}(x-2)$$
$$d(f(t,x)) = \frac{1}{2}e^{-x}(x-2)\,dt + e^{-x}(1-x)\,dB_t$$

2) $f(t,x) = xte^{-tx}$

$$\dot{f}(t,x) = -x^2te^{-tx} + xe^{-tx} = (1-tx)xe^{-tx}$$
$$f'(t,x) = -t^2xe^{-tx} + te^{-tx} = (1-tx)te^{-tx}$$
$$f''(t,x) = \left(t^3xe^{-tx} - t^2e^{-tx}\right) - t^2e^{-tx}$$
$$= t^2e^{-tx}(tx-2)$$
$$d(f(t,x)) = \left[(1-tx)xe^{-tx} + \frac{1}{2}t^2e^{-tx}(tx-2)\right]dt$$
$$+ (1-tx)te^{-tx}\,dB_t$$
$$= \left((1-tx)x + \frac{1}{2}t^2(tx-2)\right]e^{-tx}dt +$$
$$(1-tx)te^{-tx}\,dB_t$$

3) $f(t,x) = t + [\sin x]^3$

$$\dot{f}(t,x) = 1$$
$$f'(t,x) = 3(\sin x)^2\cos x$$

$$f''(t,x) = 6\sin x \cos x . \cos x - 3(\sin x)^2 . \sin x$$

$$= 6\sin x(\cos x)^2 - 3(\sin x)^3$$

$$df(t,x) = \left(1 + \frac{1}{2}\left(6\sin x(\cos x)^2 - 3(\sin x)^3\right)\right) dt +$$

$$3(\sin x)^2 \cos x \, dB_t$$

4) $dX_t = X_t[2dt + 2dB_t]$

$$df(t,X_t) = \left[\dot{f}(t,X_t) + \mu f'(t,X_t) + \frac{1}{2}\sigma^2 f''(t,X_t)\right]dt$$

$$+ \sigma f'(t,X_t)\,dB_t$$

Since X_t follows a stochastic differential equation

$$\underbrace{\mu = 2X_t}_{\text{drift}} , \quad \underbrace{\sigma = 2X_t}_{\text{diffusion}}$$

1. $f(t,x) = xe^{-x}$

$$df(t,X_t) = \left[2X_t e^{-X_t}(1-X_t) + 2X_t^2 e^{-X_t}(X_t-2)\right]dt$$

$$+ 2X_t e^{-X_t}(1-X_t)\,dB_t$$

$$= \left[(1-X_t) + X_t(X_t-2)\right]2X_t e^{-X_t} dt +$$

$$(1-X_t)2X_t e^{-X_t} dB_t$$

2. $f(t,x) = xte^{-tx}$

$$df(t,X_t) = \left[(1-tX_t)X_t e^{-tX_t} + 2X_t(1-tX_t)te^{-tX_t} +\right.$$

$$\left.2X_t^2 t^2 e^{-tX_t}(tX_t-2)\right] dt +$$

$$2X_t(1-tX_t)te^{-tX_t} dB_t$$

$$= \left[(1-tX_t) + 2(1-tX_t)t + 2X_t t^2(tX_t-2)\right]X_t e^{-tX_t} dt$$

$$+ (1-tX_t)2X_t te^{-tX_t} dB_t$$

3. $f(t,x) = t + (\sin x)^3$

$$df(t,X_t) = \Big(1 + 2X_t \cdot 3(\sin X_t)^2 \cdot \cos X_t + 2X_t^2\left[6\sin X_t(\cos X_t)^2 - 3(\sin X_t)^3\right]\Big)dt + 2X_t \cdot 3(\sin X_t)^2 \cdot \cos X_t \, dB_t$$

$$= \Big(1 + 6X_t(\sin X_t)^2\cos X_t + 2X_t^2\left[6\sin X_t(\cos X_t)^2 - 3(\sin X_t)^3\right]\Big)dt + 6X_t(\sin X_t)^2 \cdot \cos X_t \, dB_t$$

Exercise 2

To Prove ->

$M_t = Z_t^2 - \langle Z\rangle_t$

$Z_t = \int_0^t A_s \, dB_s$ where A_s is a bounded, adapted process with continuous paths

$\langle Z\rangle_t = \int_0^t A_s^2 \, ds$] Quadratic Variation of Z_t

Since A_s is bounded.

Let $|A_s| \leq C$

$E[Z_t^2] = E\left[\int_0^t A_s^2 \, ds\right] \leq C^2 . t < \infty$

This means Z_t has finite second moments, making it square integrable.

$E[M_t/F_s] = E[Z_t^2 - \langle Z\rangle_t / F_s] = E[Z_t^2/F_s] - E[\langle Z\rangle_t / F_s]$

Using decomposition property of square integrable martingales

$E[Z_t^2/F_s] = Z_s^2 + E[\langle Z\rangle_t - \langle Z\rangle_s / F_s]$

$E[\langle Z\rangle_t / F_s] = \langle Z\rangle_s + E[\langle Z\rangle_t - \langle Z\rangle_s / F_s]$

$E[M_t/F_s] = Z_s^2 - \langle Z\rangle_s = M_s$

Since both Z_t & $\langle Z\rangle_t$ are continuous, M_t is a continuous martingale

Exercise 3

$$dX_t = X_t[2dt + dB_t]$$

1) $$d(\log X_t) = \frac{1}{X_t}dX_t - \frac{1}{2}\times\frac{1}{X_t^2}(dX_t)^2$$

$$= \frac{1}{X_t}[X_t(2dt + dB_t)] - \frac{1}{2X_t^2}[X_t(2dt + dB_t)]^2$$

$$= 2dt + dB_t - \frac{1}{2X_t^2}X_t^2 dt$$

$$= 2dt + dB_t - \frac{1}{2}dt \quad = \frac{3}{2}dt + dB_t$$

$$\log X_t = \log X_0 + \frac{3}{2}\int_0^t ds + \int_0^t dB_s$$

$$= \log X_0 + \frac{3}{2}t + \int_0^t dB_s$$

$$\int_0^t dB_s \sim N(0, t)$$

$$\log X_0 + \frac{3}{2}t + \int_0^t dB_s \sim N(\log X_0 + \frac{3}{2}t, t)$$

$$\log X_t \sim N(\log X_0 + \frac{3}{2}t, t)$$

$$X_t \sim \log\text{Normal}(\log X_0 + \frac{3}{2}t, t)$$

or

$$X_t = X_0 e^{3/2 t + \int_0^t dB_s} = X_0 e^{3/2 t + B_t}$$

2) $X_0 = 1$

$$P(X_1 > 3) = P(e^{3/2 + B_1} > 3)$$

$$= P(3/2 + B_1 > \ln(3))$$

$$= P(B_1 > \ln(3) - 3/2)$$

$$= P(Z > \ln(3) - 3/2)$$

$$= P(Z < 0.4014)$$

$$\approx 0.6554$$

Using normal distribution

$$P(X_1 > 3) = P(\ln(X_1) > \ln(3))$$

$$= P\left(Z > \frac{\ln(3) - 3/2}{1}\right)$$

$$= P(Z < 0.4014)$$

$$\approx 0.6554$$

3) $X_0 = 1/2$

$$P(X_2 < 3) = P\left(\frac{1}{2} e^{3/2 \times 2 + B_2} < 3\right)$$

$$= P\left(\frac{1}{2} e^{3+B_2} < 3\right) = P\left(e^{3+B_2} < 6\right)$$

$$= P\left(3 + B_2 < \ln(6)\right) = P\left(B_2 < \ln(6) - 3\right)$$

$$= P\left(Z < \frac{\ln(6)-3}{\sqrt{2}}\right) = P(Z < -0.85435)$$

$$\approx 0.1965$$

4) $Y_t = \log X_t$

$$d(\log X_t) = \frac{3}{2} dt + dB_t$$

$$dY_t = \frac{3}{2} dt + dB_t$$

$$Y_t = Y_0 + \frac{3}{2} dt + dB_t$$

or

$$Y_t = Y_0 + \frac{3}{2} \int_0^t ds + \int_0^t dB_s$$

$$Y_t \sim N\left(Y_0 + \frac{3}{2}t, B_t\right)$$

Exercise 4

$$dX_t = X_t [2dt + 2dB_t]$$

$$dY_t = Y_t [3dt - dB_t]$$

$$X_0 = Y_0 = 1$$

1) $Z_t = X_t Y_t$

$$dZ_t = dX_t Y_t + dY_t X_t + dX_t dY_t$$

$$= Y_t X_t [2dt + 2dB_t] + X_t Y_t [3dt - dB_t]$$

$$+ X_t [2dt + 2dB_t] Y_t [3dt - dB_t]$$

$$= X_t Y_t\left[2dt + 2dB_t + 3dt - dB_t\right] - 2X_tY_tdt$$

$$= Z_t\left[5dt + dB_t\right] - 2Z_tdt$$

$$dZ_t = 3Z_tdt + Z_tdB_t \qquad = Z_t\left[3dt + dB_t\right]$$

2) $Z_t = \frac{X_t}{Y_t}$

$$dZ_t = dX_t\cdot\frac{1}{Y_t} + d\left(\frac{1}{Y_t}\right)X_t + dX_t\, d\left(\frac{1}{Y_t}\right) +$$
$$+ \frac{1}{2}\frac{d^2Z_t}{dX_t^2}(dX_t)^2 + \frac{1}{2}\frac{d^2Z_t}{dY_t^2}(dY_t)^2$$

$$d\left(\frac{1}{Y_t}\right) = -\frac{1}{Y_t^2}dY_t = \frac{-1}{Y_t^2}Y_t\left[3dt - dB_t\right] = -\frac{1}{Y_t}\left[3dt - dB_t\right]$$

$$\frac{d^2Z_t}{dX_t^2} = \frac{d}{dX_t}\left(\frac{1}{Y_t}\right) = 0$$

$$\frac{d^2Z_t}{dY_t^2} = \frac{d}{dY_t}\left(-\frac{X_t}{Y_t^2}\right) = 2\frac{X_t}{Y_t^3}$$

$$dZ_t = \frac{X_t\left[2dt + 2dB_t\right]}{Y_t} - \frac{1}{Y_t}\left[3dt - dB_t\right]X_t$$
$$- X_t\left[2dt + 2dB_t\right]\cdot\frac{1}{Y_t}\left[3dt - dB_t\right] + \frac{X_t}{Y_t^3}\left[Y_t(3dt - dB_t)\right]^2$$

$$= \frac{X_t}{Y_t}\left[2dt + 2dB_t - 3dt + dB_t\right] + \frac{X_t}{Y_t}2dt + \frac{X_t}{Y_t}dt$$

$$= Z_t\left[3dB_t - dt + 2dt + dt\right]$$

$$dZ_t = Z_t\left[2dt + 3dB_t\right]$$

Exercise 5

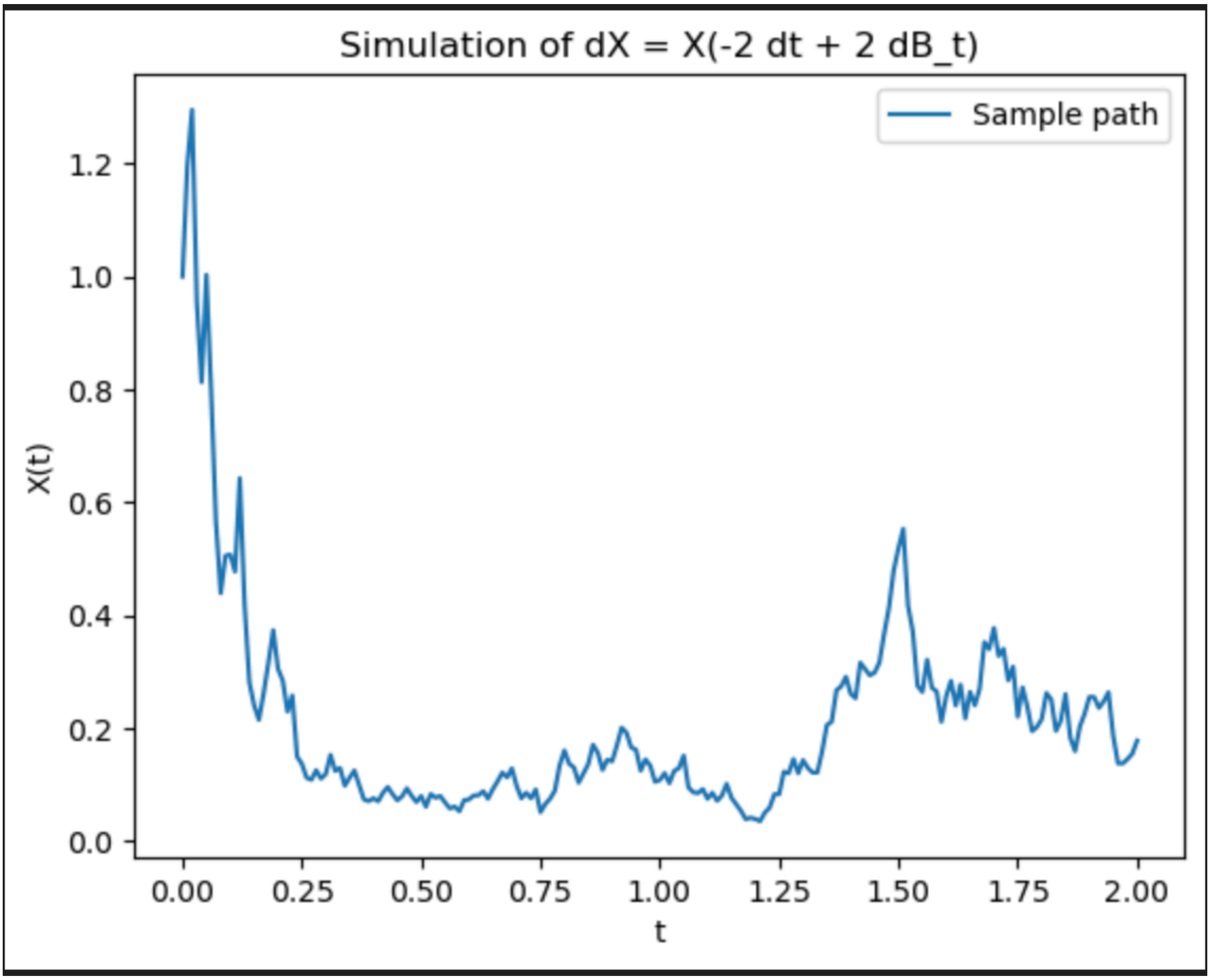
$$dX_t = X_t\left[-2dt + 2dB_t\right]$$

$$X_0 = 1, \ \Delta t = 0.01, \ 0 \leq t \leq 2$$

```python
def simulation(num_steps=200, dt=0.01, x0=1.0):
    X = np.zeros(num_steps + 1)
    X[0] = x0
    for i in range(num_steps):
        dW = np.random.normal(loc=0.0, scale=np.sqrt(dt))
        X[i+1] = X[i] + X[i]*(-2.0*dt) + X[i]*(2.0*dW)
    return X


T = 2.0
dt = 0.01
num_steps = int(T / dt)

# 1) Plot one sample path
single_path = simulation(num_steps, dt, x0=1.0)
time_grid = np.linspace(0, T, num_steps + 1)
plt.plot(time_grid, single_path, label="Sample path")
plt.xlabel("t")
plt.ylabel("X(t)")
plt.title("Simulation of dX = X(-2 dt + 2 dB_t)")
plt.legend()
plt.show()
```



```python
# 2) Estimate probability that X(2) >= 3 using 1000000 simulations
n_sims = 1000000
final_vals = np.zeros(n_sims)
for i in range(n_sims):
    path = simulation(num_steps, dt, x0=1.0)
    final_vals[i] = path[-1]

prob_est = np.mean(final_vals >= 3.0)
print(f"Estimated P(X_2 >= 3) from {n_sims} sims = {prob_est:.6f}")
```

✓ 2m 40.4s

```
Estimated P(X_2 >= 3) from 1000000 sims = 0.000631
```

3) $P(X_2 \geq 3)$

$$d\log X_t = \frac{1}{X_t} dX_t - \frac{1}{2X_t^2}(dX_t)^2$$

$$= \frac{1}{X_t}\left[X_t\left[-2dt + 2dB_t\right]\right] - \frac{1}{2X_t^2}\left[X_t\left[-2dt + 2dB_t\right]\right]^2$$

$$= -2dt + 2dB_t - \frac{1}{2X_t^2} 4X_t^2 dt$$

$$= -2dt + 2dB_t - 2dt = -4dt + 2dB_t$$

$$\log X_t = \log X_0 - 4\int_0^t ds + 2\int_0^t dB_s$$

$$= \log X_0 - 4t + 2\int_0^t dB_s$$

$$2\int_0^t dB_s \sim N(0, 4t)$$

$$\log X_0 - 4t + 2\int_0^t dB_s \sim N(\log X_0 - 4t, 4t)$$

$$\therefore \log X_t = \log X_0 - 4t + 2B_t$$

$$\log X_2 = \log 1 - 4\times 2 + 2B_2$$

$$= 0 - 8 + 2B_2$$

$$X_2 = e^{2B_2 - 8}$$

$$P(X_2 \geq 3) = P(e^{2B_2 - 8} \geq 3)$$
$$= P(2B_2 - 8 \geq \ln(3))$$
$$= P\left(B_2 \geq \frac{\ln(3)+8}{2}\right)$$
$$= P\left(Z \geq \frac{4.55}{\sqrt{2}}\right)$$
$$= 1 - P(Z < 3.217)$$
$$= 0.00064853$$

This answer is really close to the answer in the 1000000 simulation case