FINM 345X0/STAT 390X0
Exam (February 5, 2024)

You have an hour and a half (90 minutes) to complete the exam. There are 7 problems/questions, many with multiple parts, worth varying amounts for a total of 100 points. For all multi-part problems all parts are worth the same amount. This test is closed book. You may have one standard sheet of paper (two-sided) with self-created notes. No calculators, phones, etc., permitted.

Answers should go into the blue book. You will be given some white paper that you can use as scratch paper while working problems out, but the answers must be put in the blue books.

- On any question you may leave your answer in terms of the normal distribution function

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt.$$

- Also for your reference, the moment generating function of a $N(0,1)$ random variable N is

$$\mathbb{E}[e^{sN}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} e^{st} \, dt = e^{s^2/2}.$$

Problem 1 (10)**.** Give the definition of a standard Brownian motion B_t starting at the origin.

Problem 2 (20)**.** Suppose B_t is a standard Brownian motion starting at the origin and let $\mathcal{F}_t$ denote the information in $\{B_s : s \leq t\}$. Find the following.

1. $\mathbb{P}\{B_3^2 \leq 4\}$
2. $\mathbb{P}\{B_1 \leq 2, B_3 > B_1 + 2\}$
3. $E[e^{4B_3} \mid \mathcal{F}_1]$
4. $$\lim_{n \to \infty} \sum_{j=1}^{5n} \left[B\left(\frac{j}{n}\right) - B\left(\frac{j-1}{n}\right) \right]^2 .$$

Problem 3 (20)**.** Suppose that B_t is a standard Brownian motion and X_t, Y_t satisfy the following.

$$dX_t = 2\, X_t \, dB_t, \qquad dY_t = 3 \, dt + Y_t \, dB_t.$$

Find the following differentials. The answers should be in the form

$$[\text{ something }] \, dt + [\text{ something }] \, dB_t$$

1. dB_t^5

2. $d[t^2 e^{4X_t}]$

3. $d[1/Y_t]$

4. $d[X_t\,Y_t]$.

Problem 4 (15)**.** Suppose B_t is a Brownian motion with drift -2 and variance parameter $\sigma^2 = 4$.

1. Give the corresponding operators, that is, give $Lf(x), L^* f(x)$ corresponding to this Brownian motion.

2. Let $\phi(t,x) = \mathbb{E}[B_t^4 \mid B_0 = x]$. What partial differential equation does ϕ satisfy for $0 < t < 1$? (You can write the answer in terms of L or L^*.)

3. Let $\phi(t,x) = \mathbb{E}[B_1^4 \mid B_t = x]$. What partial differential equation does ϕ satisfy for $0 < t < 1$? (You can write the answer in terms of L or L^*.)

Problem 5 (15)**.** Suppose $X_1, X_2, \ldots$ are independent standard normal random variables and let $\mathcal{F}_n$ denote the information in $X_1, \ldots, X_n$. Let $S_n = X_1 + \cdots + X_n$ with $S_0 = 0$. Consider the discrete stochastic integral

$$Z_n = \sum_{j=1}^{n} S_{j-1}\, X_j$$

with $Z_0 = 0$. We showed in class, and you may use, that Z_n is a martingale with respect to $\{\mathcal{F}_n\}$.

1. Find $\mathbb{E}[(Z_n - Z_{n-1})^2]$.

2. Find $\mathbb{E}[Z_{10}^2]$.

3. Find $\mathbb{E}[Z_{20}\, Z_{10}]$.

Problem 6 (10)**.** Let B_t be a standard Brownian motion. Suppose $A_t = 1$ for $0 \leq t < 1$ and for $t \geq 1$,

$$\begin{aligned} A_t &= 0 \qquad \text{if } B_1 \geq 0 \\ A_t &= -B_1 \quad \text{if } B_1 \leq 0. \end{aligned}$$

Let

$$Z_t = \int_0^t A_s\, dB_s.$$

1. Find $\mathbb{E}[A_t^2]$.

2. Find $\mathbb{P}\{Z_2 > 0\}$.

Problem 7 (10)**.** Let B_t be a standard Brownian motion starting at the origin. Let

$$T = \min\{t : B_t = 1\}.$$

1. Use the reflection principle to find the distribution function for T, that is, the function $F(t) = \mathbb{P}\{T \leq t\}$.

2. Find the density of T. (Recall that the density of a random variable is the derivative of the distribution function.)