FINM 34500/STAT 39000
Problem Set 1

Exercise 1 *Let B_t be a standard Brownian motion. Find the following probabilities. If you cannot give the answer precisely give it up to at least three decimal places using a table of the normal distribution.*

1. $\mathbb{P}\{B_3 \leq 2\}$
$$= \mathbb{P}\{\sqrt{3}\, B_1 \leq 2\} = \Phi(2/\sqrt{3}) = 0.876$$

2. $\mathbb{P}\{B_1 \leq 1, B_3 - B_1 \leq 1\}$
$$= \mathbb{P}\{B_1 \leq 1\}\, \mathbb{P}\{B_3 - B_1 \leq 1\} = \mathbb{P}\{B_1 \leq 1\}\, \mathbb{P}\{\sqrt{2}\, B_1 \leq 1\} = \Phi(1)\, \Phi(1/\sqrt{2}) = 0.$$

3. $\mathbb{P}(E)$ *where E is the event that the path stays below the line $y = 2$ up to time $t = 4$.* ***Use the reflection principle.***
$$\mathbb{P}(E) = \mathbb{P}\{\max_{0 \leq t \leq 4} B_t \leq 2\} = 1 - \mathbb{P}\{\max_{0 \leq t \leq 4} B_t \geq 2\} = 0.640.$$
$$\mathbb{P}\{\max_{0 \leq t \leq 4} B_t \geq 2\} = 2\mathbb{P}\{B_4 \geq 2\} = 2\, \mathbb{P}\{\sqrt{4}\, B_1 \geq 2\} = 2\ [1 - \Phi(1)]$$
$$\mathbb{P}(E) = 1 - 2\ [1 - \Phi(1)] =: 0.683$$

4. $\mathbb{P}\{B_4 \geq 0 \mid B_8 \geq 0\}$.
By the definition of conditional probability,
$$\mathbb{P}\{B_4 \geq 0 \mid B_8 \geq 0\} = \frac{\mathbb{P}\{B_4 \geq 0, B_8 \geq 0\}}{\mathbb{P}\{B_8 \geq 0\}} = \frac{\mathbb{P}\{B_4 \geq 0, B_8 \geq 0\}}{1/2} = 2\, \mathbb{P}\{B_4 \geq 0, B_8 \geq 0\}$$
Using the scaling rule and the result proved in class
$$\mathbb{P}\{B_4 \geq 0, B_8 \geq 0\} = \mathbb{P}\{B_1 \geq 0, B_2 \geq 0\} = \frac{3}{8}.$$
Hence,
$$\mathbb{P}\{B_4 \geq 0 \mid B_8 \geq 0\} = 2 \cdot \frac{3}{8} = \frac{3}{4}.$$

5. $\mathbb{P}\{B_1(B_3 - B_1) \geq 0 \mid B_1 \leq 1, (B_3 - B_1)^2 \geq 2\}$.
This just requires a little bit of thought. Note that $B_1(B_3 - B_1) \geq 0$ ***means that*** B_1 ***and*** $B_3 - B_1$ ***have the same sign. If we are given information up to time*** 1***, we know that*** $B_3 - B_1$ ***is independent of this information and has a distribution symmetric about the origin. Therefore, information about*** $(B_3 - B_1)^2$ ***is irrelevant (independent) of the event having to d****o with the sign of $B_3 - B_1$. The answer is $1/2$.*

Exercise 2 *Suppose B_t is a standard Brownian motion, $\lambda \in \mathbb{R}$, and let $\mathcal{F}_t$ be its corresponding filtration. Let*

$$M_t = e^{\lambda B_t - (\lambda^2/2)t}.$$

1. *Show that M_t is a martingale with respect to $\mathcal{F}_t$. In other words, show that if $s < t$, then*

$$E(M_t \mid \mathcal{F}_s) = M_s.$$

You may use the following fact. If $X \sim N(0,1)$, then the moment generating function of X is given by

$$m(t) = \mathbb{E}\left[e^{tX}\right] = e^{t^2/2}. \tag{1}$$

It is easy to see that $\mathbb{E}[M_t] < \infty$ and M_t is measurable with respect to $\mathcal{F}_t$. Suppose that $s < t$.

$$\begin{aligned}
E(M_t \mid \mathcal{F}_s) &= E(e^{\lambda(B_s + B_t - B_s) - (\lambda^2/2)t} \mid \mathcal{F}_s) \\
&= E(e^{\lambda B_s}\, e^{\lambda(B_t - B_s)}\, e^{-(\lambda^2/2)t} \mid \mathcal{F}_s) \\
&= e^{\lambda B_s}\, e^{-(\lambda^2/2)t} E(e^{\lambda(B_t - B_s)} \mid \mathcal{F}_s) \\
&= M_s\, e^{-(\lambda^2/2)(t-s)}\, \mathbb{E}[e^{\lambda(B_t - B_s)}] \\
&= M_s\, e^{-(\lambda^2/2)(t-s)}\, \mathbb{E}[e^{\lambda\sqrt{t-s}\, B_1}] \\
&= M_s
\end{aligned}$$

The last step uses (1).

2. *Find $\mathbb{E}[M_7]$ and $\mathbb{E}[M_{43} - 2M_9]$.*
Hopefully you got this very quickly. Since M_t is a martingale

$$\mathbb{E}[M_7] = \mathbb{E}[M_0] = 1, \qquad \mathbb{E}[M_{43} - 2M_9] = \mathbb{E}[M_0] - 2\,\mathbb{E}[M_0] = -1.$$

Exercise 3 *Textbook, Exercise 2.6 (You may wish to use (1) when doing part 4.)*

$$\begin{aligned}
E[B_t^2 \mid \mathcal{F}_s] &= E[(B_s + B_t - B_s)^2 \mid \mathcal{F}_s] \\
&= E[B_s^2 \mid \mathcal{F}_s] + 2\,\mathbb{E}[B_s\,(B_t - B_s) \mid \mathcal{F}_s] + E[(B_t - B_s)^2 \mid \mathcal{F}_s] \\
&= B_s^2 + \ 2\,B_s\,\mathbb{E}[B_t - B_s \mid \mathcal{F}_s] + \mathbb{E}[(B_t - B_s)^2] \\
&= B_s^2 + 0 + (t - s)
\end{aligned}$$

$$\begin{aligned}
E[B_t^3 \mid \mathcal{F}_s] &= E[(B_s + B_t - B_s)^3 \mid \mathcal{F}_s] \\
&= E[B_s^3 \mid \mathcal{F}_s] + 3\mathbb{E}[B_s^2(B_t - B_s) \mid \mathcal{F}_s] + 3\mathbb{E}[B_s(B_t - B_s)^2 \mid \mathcal{F}_s] + E[(B_t - B_s)^3 \mid \mathcal{F}_s] \\
&= B_s^3 + 3\,B_s^2\,\mathbb{E}[B_t - B_s] + 3\,B_s\,\mathbb{E}[(B_t - B_s)^2] + \mathbb{E}[(B_t - B_s)^3] \\
&= B_s^3 + 0 + 3\,B_s\,(t - s) + 0
\end{aligned}$$

If we do the same for the next one we get that $E[B_t^4 \mid \mathcal{F}_s]$ is the sum of four terms

$$B_s^4 + 4B_s^3\,\mathbb{E}[B_t - B_s] + 6\,B_s^2\,\mathbb{E}[(B_t - B_s)^2] + 4\,B_s\,\mathbb{E}[(B_t - B_s)^3] + \mathbb{E}[(B_t - B_s)^4].$$

Using the fact that if N is a standard normal, then $\mathbb{E}[N^4] = 3$, we get

$$B_s^4 + 0 + 6\,B_s^2\,(t - s) + 0 + 3\,(t - s)^2.$$

The last part is done very similarly to Exercise 2.

$$\mathbb{E}[e^{4B_t - 2} \mid \mathcal{F}_s] = e^{4B_s - 2}\,\mathbb{E}[e^{4(B_t - B_s)}] = e^{4B_s - 2}\,\mathbb{E}[e^{4\sqrt{t-s}\,B_t}] = e^{4B_s - 2 + 8(t-s)}.$$

Exercise 4 *Prove Theorem 2.6.3 from the notes. In other words, show that if B_t is a standard Brownian motion, $a > 0$, and*

$$Y_t = a^{-1/2}\,B_{at},$$

then Y_t is a standard Brownian motion. (You need to show that Y_t satisfies the four properties that a standard Brownian motion satisfies.)

One just quickly checks that it satisfies the conditions to be a Brownian motion. This should be straightforward. For example if $s < t$, $Y_t - Y_s = a^{-1/2}\,[B_{at} - B_{as}]$ with

$$\mathbb{E}[Y_s] = a^{-1/2}\,\mathbb{E}[B_{at} - B_{as}] = 0, \quad \mathrm{Var}[Y_s] = a^{-1}\,\mathrm{Var}[B_{at} - B_{as}] = a^{-1}\,[at - as] = t - s.$$

Exercise 5 *Write a program that will sample from a standard Brownian motion using step size $\Delta t = 1/500$.*

- *Make a graph of $B_t, 0 \le t \le 3$ from your simulation for one trial.*
- *Repeat the simulation 1000 times to give estimates for the following:*

$$\mathbb{P}\{\min_{0\le t\le 3} B_t < -1/2\},$$

$$\mathbb{P}\{B_{1.5} > 0, B_3 < 0\}.$$

Find the exact values of these probabilities and compare your simulations with the exact values. You may use any computer language and/or package that you choose.

The exact values are

$$\mathbb{P}\{\min_{0\le t\le 3} B_t < -1/2\} = \mathbb{P}\{\max_{0\le t\le 3} B_t > 1/2\} = 2\,\mathbb{P}\{B_3 > 1/2\} = 2\,\mathbb{P}\{\sqrt{3}\,B_1 > 1/2\}$$

$$= 2\,\left[1 - \Phi(1/2\sqrt{3})\right] = 0.773$$

$$\mathbb{P}\{B_{1.5} > 0, B_3 < 0\} = \mathbb{P}\{B_1 > 0, B_2 < 0\} = \mathbb{P}\{B_1 > 0\} - \mathbb{P}\{B_1 > 0, B_2 > 0\} = \frac{1}{8}.$$