FINM 34500/STAT 39000
Problem Set 6 (due February 17)
Reading: 4.1 — 4.3

Exercise 1 *Suppose B_t is a standard Brownian motion starting at x with $0 < x < 3$.*

1. *What is the probability that the Brownian motion will obtain a value of 3 before reaching 0?*

2. *Do the same question for X_t defined by*

$$X_t = x + \int_0^t B_s \, dB_s.$$

Exercise 2 *Consider the Bessel process*

$$dX_t = \frac{a}{X_t} \, dt + dB_t, \qquad X_0 = 1 > 0.$$

and let

$$T = T_{r,R} = \min\{t : X_t = r \text{ or } R\}.$$

Let $\rho(r, R)$ be the probability that $X_T = R$.

1. *Show that if $a < 1/2$, find*

$$\rho(0, R) = \lim_{r \downarrow 0} \rho(r, R).$$

2. *Show that if $a < 1/2$,*

$$\lim_{R \to \infty} \rho(0, R) = 0.$$

   Conclude that the Bessel process reaches zero with probability one.

3. *Show that if $a = 1/2$, then $\rho(0, R) = 1$. Conclude that with probability one $X_t > 0$ for all t.*

4. *Show that if $a = 1/2$ and $r > 0$,*

$$\rho(r, \infty) := \lim_{R \to \infty} \rho(r, R) = 0.$$

   Conclude that with probability one for all $r > 0$, $X_t < r$ for some t.

Exercise 3 *Suppose B_t is a standard Brownian motion and let $X_t = 2e^{B_t - t}$.*

1. *Show that X_t is a (time homogeneous) diffusion by writing down the appropriate SDE. In other words, find $m(\cdot), \sigma(\cdot)$ such that*

$$dX_t = m(X_t) \, dt + \sigma(X_t) \, dB_t.$$

2. *Let*

$$\phi(t,x) = \mathbf{E}\left[e^{-2(T-t)}\,(X_T-3)_+ \mid X_t = x\right], \quad 0<t<T.$$

Use the Feynman-Kac theorem to find a second-order PDE that ϕ satisfies. Recall that $(x-3)_+ = \max\{x-3,0\}$.

3. *Repeat the last part with*

$$\phi(t,x) = \mathbf{E}\left[e^{-2(T-t)}\,X_T^2\,e^{-X_T} \mid X_t = x\right], \quad 0<t<T.$$

Exercise 4 *Let B_t be a standard Brownian motion and let*

$$Z_t = \int_0^t \frac{1}{(1-s)^\alpha}\,dB_s, \qquad 0 \le t < 1.$$

1. *Show that if $\alpha = 1/4$, then $Z_t, 0 \le t < 1$ is a square integrable martingale and find a $C<\infty$ such that for all $t<1$, $\mathrm{Var}(Z_t) \le C$.*

2. *Show that if $\alpha = 1$, then with probability one there will exist $t<1$ with $Z_t = 1$.*