FINM 34500/STAT 39000
Winter 2025
Problem Set 3 (due January 27)

Reading: Sections 3.1 – 3.2.

Exercise 1 *For each of these problems $\{A_t\}$ will be a simple process that only changes at time $1/2$, that is*

$$A_t = 1, \qquad 0 \le t < 1/2.$$

$$A_t = Y, \qquad 1/2 \le t \le 1,$$

where Y is a random variable measurable with respect to $\mathcal{F}_{1/2}$. Let

$$Z_t = \int_0^t A_s \, dB_s.$$

For each of these examples of Y, find $\mathbb{P}\{Z_1 \ge 0\}$.

1.

$$Y = \begin{cases} 0 & B_{1/2} \ge 0 \\ 1 & B_{1/2} < 0 \end{cases}$$

In this case, $Z_1 \ge 0$ if either $B_{1/2} > 0$ or $B_{1/2} \le 0$ and $B_1 \ge 0$. The first happens with probability $1/2$ and (going back to facts about Brownian motion we did earlier) we see that the second probability is the same as the probability that $B_{1/2} \le 0$ and $B_1 \ge 0$ which is $1/8$. The answer is $5/8$.

2.

$$Y = \begin{cases} 0 & B_{1/2} \ge 0 \\ -5 & B_{1/2} < 0 \end{cases}$$

This is more difficult because we have to determine

$$\mathbb{P}\{Z_1 \ge 0, B_{1/2} \le 0\} = \mathbb{P}\{B_{1/2} \le 0, -5[B_1 - B_{1/2}] > B_{1/2}\}.$$

As in one of the calculations we did for Brownian motion we can write

$$\mathbb{P}\{Z_1 \ge 0, B_{1/2} \le 0\} = \int_{-\infty}^{0} \mathbb{P}\{Z_1 \ge 0 \mid B_{1/2} = x\} \, \frac{1}{\sqrt{2\pi(1/2)}} \, e^{-x^2/2(1/2)} \, dx.$$

Given $B_{1/2} = x$, we can see that

$$\mathbb{P}\{Z_1 \ge 0 \mid B_{1/2} = x\} = \mathbb{P}\{-5[B_1 - B_{1/2}] \ge -x\} = \mathbb{P}\{B_1 - B_{1/2} \le x/5\}$$

$$= \mathbb{P}\{\frac{1}{\sqrt{2}} \, B_1 \le x/5\} = \Phi(\sqrt{2}x/5) = \int_{-\infty}^{\sqrt{2}x/5} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy.$$

If we integrate over x we get that

$$\mathbb{P}\{Z_1 \geq 0, B_{1/2} \leq 0\} = \int_{-\infty}^{0} \left[\int_{-\infty}^{-\sqrt{2}x/5} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy \right] \frac{1}{\sqrt{2\pi(1/2)}}\, e^{-x^2/2(1/2)}\, dx.$$

The final answer is this plus $1/2$.

3.

$$Y = \begin{cases} 1 & B_{1/2} \geq 0 \\ -1 & B_{1/2} < 0 \end{cases}$$

The distribution of $-[B_1 - B_{1/2}]$ given $\mathcal{F}_{1/2}$ is the same as the distribution of $B_1 - B_{1/2}$.

So "changing the bet from heads to tails" does not change the probabilities and the answer is $1/2$.

Exercise 2 *For the three cases in the last exercise, give*

$$\langle Z \rangle_t = \int_0^t A_s^2\, ds.$$

In each case, the answer should look like two functions of t — one for the event $B_{1/2} \geq 0$ and one on the event $B_{1/2} < 0$.

In the first case, if $B_{1/2} > 0$,

$$\int_0^t A_s^2\, ds = \begin{cases} t & t \leq 1/2 \\ 1/2 & t \geq 1/2 \end{cases},$$

and if $B_{1/2} \leq 0$, $\int_0^t A_s^2\, ds = t$.

In the second case, if $B_{1/2} > 0$,

$$\int_0^t A_s^2\, ds = \begin{cases} t & t \leq 1/2 \\ 1/2 & t \geq 1/2 \end{cases},$$

and if $B_{1/2} \leq 0$,

$$\int_0^t A_s^2\, ds = \begin{cases} t & t \leq 1/2 \\ \frac{1}{2} + 5^2 \cdot (t - \frac{1}{2}) & t \geq 1/2 \end{cases},$$

For the third case, $\int_0^t A_s^2\, ds = t$ in both cases.

Exercise 3 *For Case #2 in the last two exercises, verify directly the statement*

$$\mathrm{Var}[Z_1] = \int_0^1 \mathbb{E}[A_t^2]\, dt.$$

Since $\mathbb{E}[Z_1] = 0$***,*** $\mathrm{Var}[Z_1] = \mathbb{E}[Z_1^2]$***. We will calculate this expectation directly. Using "indicator function" notation we can write***

$$Z_1 = B_{1/2} + 1\{B_{1/2} \leq 0\}\,(-5)\,[B_1 - B_{1/2}].$$

$$Z_1^2 = B_{1/2}^2 - 10\,B_{1/2}\,1\{B_{1/2} \leq 0\}\,[B_1 - B_{1/2}] + 25\,1\{B_{1/2} \leq 0\}\,[B_1 - B_{1/2}]^2.$$

$$\mathbb{E}[B_{1/2}^2] = \frac{1}{2},$$

$$\begin{aligned}
\mathbb{E}\left[B_{1/2}\,1\{B_{1/2} \leq 0\}\,[B_1 - B_{1/2}]\right] &= \mathbb{E}\left[E(B_{1/2}\,1\{B_{1/2} \leq 0\}\,[B_1 - B_{1/2}] \mid \mathcal{F}_{1/2})\right] \\
&= \mathbb{E}\left[B_{1/2}\,1\{B_{1/2} \leq 0\}\,E([B_1 - B_{1/2}] \mid \mathcal{F}_{1/2})\right] \\
&= \mathbb{E}\left[B_{1/2}\,1\{B_{1/2} \leq 0\}\,\mathbb{E}(B_1 - B_{1/2})\right] = 0.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[1\{B_{1/2} \leq 0\}\,[B_1 - B_{1/2}]^2] &= \mathbb{E}[E(1\{B_{1/2} \leq 0\}\,[B_1 - B_{1/2}]^2 \mid \mathcal{F}_{1/2})] \\
&= \mathbb{E}[1\{B_{1/2} \leq 0\}\,E([B_1 - B_{1/2}]^2 \mid \mathcal{F}_{1/2})] \\
&= \mathbb{E}[1\{B_{1/2} \leq 0\}\,\mathbb{E}([B_1 - B_{1/2}]^2)] \\
&= \mathbb{E}[1\{B_{1/2} \leq 0\}\,\frac{1}{2}] \\
&= \frac{1}{2}\,\mathbb{P}\{B_{1/2} \leq 0\} \\
&= \frac{1}{4}.
\end{aligned}$$

Therefore,

$$\mathrm{Var}[Z_1] = \frac{1}{2} + \frac{25}{4}.$$

Using the formula for A_t ***we see that***

$$\mathbb{E}[A_t^2] = 1, \quad t < 1/2,$$

$$\mathbb{E}[A_t^2] = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 25, \quad t \geq 1/2.$$

and hence

$$\int_0^1 \mathbb{E}[A_t^2]\,dt = \frac{1}{2} + \frac{1}{2} \cdot \frac{25}{2}.$$