FINM 34500/STAT 39000
Problem Set 1 (due January 13)
Reading: Notes through Section 2.8. Chapter 1 as well as Chapter 2.1 and 2.2 should be review, they were covered in FINM 34000. The material in the small font including all of Section 2.5, is optional.

Exercise 1 *Let B_t be a standard Brownian motion. Find the following probabilities. If you cannot give the answer precisely give it up to at least three decimal places using a table of the normal distribution.*

1. $\mathbb{P}\{B_3 \leq 2\}$
2. $\mathbb{P}\{B_1 \leq 1, B_3 - B_1 \leq 1\}$
3. $\mathbb{P}(E)$ *where E is the event that the path stays below the line $y = 2$ up to time $t = 4$.*
4. $\mathbb{P}\{B_4 \geq 0 \mid B_8 \geq 0\}$.
5. $\mathbb{P}\{B_1(B_3 - B_1) \geq 0 \mid B_1 \leq 1, (B_3 - B_1)^2 \geq 2\}$.

Exercise 2 *Suppose B_t is a standard Brownian motion, $\lambda \in \mathbb{R}$, and let $\mathcal{F}_t$ be its corresponding filtration. Let*

$$M_t = e^{\lambda B_t - (\lambda^2/2)t}.$$

1. *Show that M_t is a martingale with respect to $\mathcal{F}_t$. In other words, show that if $s < t$, then*
$$E(M_t \mid \mathcal{F}_s) = M_s.$$
You may use the following fact. If $X \sim N(0,1)$, then the moment generating function of X is given by
$$m(t) = \mathbb{E}\left[e^{tX}\right] = e^{t^2/2}. \tag{1}$$
2. *Find $\mathbb{E}[M_7]$ and $\mathbb{E}[M_{43} - 2M_9]$.*

Exercise 3 *Textbook, Exercise 2.6 (You may wish to use* (1) *when doing part 4.)*

Exercise 4 *Prove Theorem 2.6.3 from the notes. In other words, show that if B_t is a standard Brownian motion, $a > 0$, and*

$$Y_t = a^{-1/2}\, B_{at},$$

then Y_t is a standard Brownian motion. (You need to show that Y_t satisfies the four properties that a standard Brownian motion satisfies.)

Exercise 5 *Write a program that will sample from a standard Brownian motion using step size $\Delta t = 1/500$.*

- *Make a graph of* $B_t, 0 \leq t \leq 3$ *from your simulation for one trial.*
- *Repeat the simulation* 1000 *times to give estimates for the following:*

$$\mathbb{P}\{\min_{0 \leq t \leq 3} B_t < -1/2\},$$

$$\mathbb{P}\{B_{1.5} > 0, B_3 < 0\}.$$

  Find the exact values of these probabilities and compare your simulations with the exact values. You may use any computer language and/or package that you choose.