Exercise 1

$dX_t = 3X_t dt - 2\sqrt{X_t}\, dB_t \quad , \quad X_0 = 1$

Find A_t & C_t such that $d\langle Y \rangle_t = A_t dt \quad , \quad d\langle Y, X \rangle_t = C_t dt$

1) $Y_t = B_t^3 + 2t^2$

$$dY_t = 3B_t^2 dB_t + \frac{1}{2} 6 B_t (dB_t)^2 + 4t\, dt$$

$$= 3B_t^2 dB_t + 3B_t dt + 4t\, dt$$

$$= 3B_t^2 dB_t + (3B_t + 4t) dt$$

Quadratic Variation A_t:

$$A_t = (3B_t^2)^2 = 9B_t^4$$

Covariation C_t:

Diffusion term of $Y_t \to 3B_t^2$

Diffusion term of $X_t \to 2\sqrt{X_t}$

$$C_t = 3B_t^2 (-2\sqrt{X_t}) = -6B_t^2 \sqrt{X_t}$$

2) $Y_t = X_t + \int_0^t X_s^3\, ds$

$$dY_t = dX_t + X_t^3 dt$$

$$= 3X_t dt - 2\sqrt{X_t}\, dB_t + X_t^3 dt$$

$$= (3 + X_t^2) X_t dt - 2\sqrt{X_t}\, dB_t$$

Quadratic Variation $\rightarrow$ (A_t)

$$A_t = (-2\sqrt{X_t})^2 = 4X_t$$

Covariation (C_t)

$$(-2\sqrt{X_t})(-2\sqrt{X_t}) = 4X_t$$

3) $Y_t = X_t^3 + \exp\left(2\int_0^t X_s^2\,ds\right)$

$$dY_t = 3X_t^2 dX_t + \frac{1}{2} 6X_t (dX_t)^2 + e^{2\int_0^t X_s^2 ds} \times 2X_t^2 dt$$

$$= 3X_t^2\left[3X_t dt - 2\sqrt{X_t}\,dB_t\right] + \frac{1}{2} 6X_t\left[4X_t dt\right]$$

$$+ \exp\left(2\int_0^t X_s^2 ds\right) 2X_t^2 dt$$

$$= 9X_t^3 dt - 6X_t^{5/2} dB_t + 12X_t^2 dt + \exp\left(2\int_0^t X_s^2 ds\right) 2X_t^2 dt$$

$$= \left[9X_t - 12 + 2\exp\left(2\int_0^t X_s^2 ds\right)\right] X_t^2 dt - 6X_t^{5/2} dB_t$$

Quadratic Variation:

$$A_t = \left[-6X_t^{5/2}\right]^2 = 36X_t^5$$

Covariation

$$C_t = \left(-6X_t^{5/2}\right)\left(-2\sqrt{X_t}\right) = 12X_t^3$$

Exercise 2

$$B_t = (B_t^1, B_t^2)$$

$$dX_t = X_t\left[m_1(B_t)dt + \sigma_1(B_t)dB_t^1 + \rho_1(B_t)\,dB_t^2\right]$$

$$dY_t = Y_t\left[m_2(B_t)dt + \sigma_2(B_t)\,dB_t^1 + \rho_2(B_t)\,dB_t^2\right]$$

1) $Z_t = X_t^3$

$$d(X_t^3) = 3X_t^2 dX_t + 3X_t(dX_t)^2$$

$(dX_t)^2$ is computed only using the diffusion part of dX_t

Quadratic Variation term $= \left[\left(\sigma_1(B_t)dB_t^1\right)^2 + \left(\rho_1(B_t)^2 dB_t^2\right)^2\right] X_t^2$

$$= X_t^2\left[\sigma_1^2(B_t) + \rho_1^2(B_t)\right] dt$$

$$\therefore d(X_t^3) = 3X_t^2\left[X_t(m_1(B_t)dt + \sigma_1(B_t)dB_t^1 + \rho_1(B_t)dB_t^2)\right]$$

$$+ 3X_t\left[X_t^2(\sigma_1^2(B_t) + \rho_1^2(B_t))dt\right]$$

$$= 3X_t^3\left[m_1(B_t) + \sigma_1^2(B_t) + \rho_1^2(B_t)\right]dt$$
$$+ \ 3X_t^3\left[\sigma_1(B_t)\,dB_t^1 + \rho_1(B_t)\,dB_t^2\right]$$

2) $Z_t = X_t Y_t$

$$dZ_t = d(X_t Y_t) = Y_t\,dX_t + X_t\,dY_t + dX_t\,dY_t$$

$$dX_t\,dY_t = X_t\left[m_1(B_t)dt + \sigma_1(B_t)dB_t^1 + \rho_1(B_t)\,dB_t^2\right] \times$$
$$Y_t\left[m_2(B_t)dt + \sigma_2(B_t)\,dB_t^1 + \rho_2(B_t)\,dB_t^2\right]$$
$$= X_t Y_t\,\sigma_1(B_t)\,\sigma_2(B_t)(dB_t^1)^2$$
$$+ X_t Y_t\,\rho_1(B_t)\,\rho_2(B_t)\,(dB_t^2)^2$$
$$= X_t Y_t\left[\sigma_1(B_t)\sigma_2(B_t) + \rho_1(B_t)\rho_2(B_t)\right]dt$$

$$dZ_t = Y_t\left[X_t\left(m_1(B_t)dt + \sigma_1(B_t)\,dB_t^1 + \rho_1(B_t)\,dB_t^2\right)\right]$$
$$+ X_t\left[Y_t\left(m_2(B_t)dt + \sigma_2(B_t)\,dB_t^1 + \rho_2(B_t)\,dB_t^2\right)\right]$$
$$+ X_t Y_t\left[\sigma_1(B_t)\sigma_2(B_t) + \rho_1(B_t)\rho_2(B_t)\right]dt$$
$$= X_t Y_t\left[\left(m_1(B_t) + m_2(B_t) + \sigma_1(B_t)\sigma_2(B_t) + \rho_1(B_t)\rho_2(B_t)\right)\right]dt$$
$$+ X_t Y_t\left[\sigma_1(B_t) + \sigma_2(B_t)\right]dB_t^1$$
$$+ X_t Y_t\left[\rho_1(B_t) + \rho_2(B_t)\right]dB_t^2$$

Exercise 3

$$X_t = B_t^1 + B_t^2 \quad , \quad Y_t = 2B_t^1 - B_t^2$$

1) $Z_t = e^{X_t + Y_t} = e^{X_t}e^{Y_t}$

$$dZ_t = e^{Y_t}e^{X_t}\,dX_t + e^{X_t}e^{Y_t}\,dY_t + e^{X_t}e^{Y_t}\,dX_t\,dY_t$$
$$+ \frac{1}{2}e^{Y_t}e^{X_t}(dX_t)^2 + \frac{1}{2}e^{Y_t X_t}(dY_t)^2$$
$$= Z_t\left[dX_t + dY_t + dX_t\,dY_t + \frac{1}{2}(dX_t)^2 + \frac{1}{2}(dY_t)^2\right]$$

$dX_t = dB_t^1 + dB_t^2$, $dY_t = 2dB_t^1 - dB_t^2$

$(dX_t)^2 = 2dt$, $(dY_t)^2 = 4dt + dt = 5dt$

$$= Z_t \left[dB_t^1 + dB_t^2 + 2dB_t^1 - dB_t^2 + (dB_t^1 + dB_t^2)(2dB_t^1 - dB_t^2) + \frac{1}{2} 2dt + \frac{1}{2} 5dt \right]$$

$$= Z_t \left[3dB_t^1 + (2dt - dt) + dt + \frac{5}{2} dt \right]$$

$$= Z_t \left[3dB_t^1 + \frac{9}{2} dt \right]$$

Standard Form Representation $\Rightarrow \frac{9}{2} Z_t dt + 3Z_t dB_t^1 + 0\, dB_t^2$

2) $Z_t = \int_0^t e^{X_s + Y_s} ds$

$dZ_t = e^{X_t + Y_t} dt$

$\therefore dZ_t = e^{X_t + Y_t} dt + 0\, dB_t^1 + 0\, dB_t^2$

3) $Z_t = \exp(tX_t)$

$$dZ_t = X_t \exp(tX_t) dt + t\exp(tX_t)\, dX_t + \frac{1}{2} t^2 \exp(tX_t)(dX_t)^2$$

$$= e^{tX_t}(B_t^1 + B_t^2) dt + te^{tX_t}[dB_t^1 + dB_t^2] + \frac{1}{2} t^2 e^{tX_t} \times 2dt$$

$$= e^{tX_t}\left[B_t^1 + B_t^2 + t^2\right] dt + te^{tX_t} dB_t^1 + te^{tX_t} dB_t^2$$

$$= e^{tX_t}\left[X_t + t^2\right] dt + te^{tX_t} dB_t^1 + te^{tX_t} dB_t^2$$

Exercise 4

We need to find a function f such that $f(0) = 0$ & f is a martingale

1) $dX_t = dt + dB_t$

Let $M_t = f(X_t)$

$$dM_t = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)(dX_t)^2$$

$$= f'(X_t)[dt + dB_t] + \frac{1}{2} f''(X_t)[dt]$$

$$= \left[f'(X_t) + \frac{1}{2} f''(X_t)\right] dt + f'(X_t)\, dB_t$$

$f'(X_t) + \frac{1}{2} f''(X_t) = 0$ for M_t to be a martingale

[ODE]

Let $g = f'(X_t)$

$\therefore$ ODE $\rightarrow$ $g + \frac{1}{2} g' = 0$

$$g' = -2g \Rightarrow \frac{dg}{dx} = -2g \Rightarrow \frac{dg}{g} = -2\,dx$$

Integrating on both sides

$$\ln|g| = -2x + C$$

$$g = e^{-2x + C} = Ce^{-2x}$$

[as C is a constant]

Integrating g to get $f(x)$

$$f(x) = \int g = \int Ce^{-2x}\, dx = C\left[\frac{-e^{-2x}}{2}\right] + D$$

$$= -\frac{C}{2} e^{-2x} + D$$

Since $f(0) = 0$

$$0 = -\frac{C}{2} + D \Rightarrow D = \frac{C}{2}$$

$\therefore$ $C = 2$ so that $f(0) = 0$

$$\therefore f(x) = -\frac{C}{2} e^{-2x} + D = -e^{-2x} + 1$$

2) $dX_t = X_t\, dt + dB_t$

Let $f(X_t) = M_t$

$$dM_t = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)(dX_t)^2$$

$$= f'(X_t)\left[X_t\, dt + dB_t\right] + \frac{1}{2}\left[f''(X_t)\, dt\right]$$

$$= \left[f'(X_t) X_t + \frac{1}{2} f''(X_t)\right] dt + f'(X_t)\, dB_t$$

For $f(X_t)$ to be a Martingale,

$$f'(X_t)X_t + \frac{1}{2}f''(X_t) = 0$$

let $g = f'(X_t)$

$$gx + \frac{1}{2}g' = 0$$

$$g' = -2gx \Rightarrow \frac{dg}{g} = -2x\,dx$$

$$\ln|g| = -2\frac{x^2}{2} + c = -x^2 + c$$

$$g = Ce^{-x^2}$$

Integrating to get value of $f(X_t)$

$$f(X_t) = \int_0^s Ce^{-x^2}dx + D$$

Since $f(0) = 0$

$$f(0) = 0 + D \qquad \therefore D = 0$$

$$\& \; C = 1$$

$$\therefore \quad f(x) = \int_0^x e^{-s^2} ds$$

3) $dX_t = \frac{1}{X_t}dt + dB_t$

$f : (0, \infty) \to R$ with $f(1) = 0$

Let $f(X_t)$ be M_t

$$dM_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2$$

$$= f'(X_t)\left(\frac{1}{X_t}dt + dB_t\right) + \frac{1}{2}f''(X_t)dt$$

$$= \left[\frac{1}{X_t}f'(X_t) + \frac{1}{2}f''(X_t)\right]dt + f'(X_t)dB_t$$

For M_t to be a martingale,

$$\frac{1}{X_t} f'(X_t) + \frac{1}{2} f''(X_t) = 0$$

Let $g = f'(X_t)$

$$\frac{g}{X} + \frac{1}{2} g' = 0 \Rightarrow g' = -2\frac{g}{x} \Rightarrow \frac{dg}{dx} = -2\frac{g}{x}$$

$$\frac{dg}{g} = \frac{-2}{x} dx$$

Integrating on both sides

$$\log|g| = -2\log x + c$$

$$g = e^{-2\log x} e^{c}$$

[Using property $e^{k\log x} = x^{k}$]

$$g = Cx^{-2} = \frac{C}{x^2}$$

Integrating to get in terms of f

$$\int g = \int \frac{C}{x^2} dx = C\left[-\frac{1}{x}\right] + D = -\frac{C}{x} + D$$

Since $f(1) = 0$

$$-C + D = 0 \quad \therefore C = D$$

If $C = 1$

$$f(X_t) = \frac{-1}{X_t} + 1$$