FINM 34500/STAT 39000
Winter 2025
Problem Set 2 (due January 21)

Reading: Sections 2.8 — 2.10.

Exercise 1 *Let $f(t)$ be a continuous function for $0 \leq t \leq 1$ and let*

$$Q = \lim_{n\to\infty} \sum_{j=1}^{n} \left| f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) \right|^{5/4}.$$

What is Q

1. *If $f(t) = t^2$?*
2. *If $f(t) = B_t$ where B_t is a standard Brownian motion.*

Exercise 2 *Suppose B_t, W_t are independent standard Brownian motions.*

1. *Let $Y_t = 2\,B_t - W_t - t$. Show that Y_t is a (one-dimensional) Brownian motion starting at the origin. What are the drift and variance parameter?*
2. *Let $Z_t = (Z_t^1, Z_t^2)$ denote the random vector where*

$$Z_t^1 = Y_t + t, \qquad Z_t^2 = -B_t + 3W_t,$$

and Y_t is as in the previous part. Explain why Z_t is a two-dimensional Brownian motion starting at the origin with zero drift. What is the covariance matrix Γ?

3. *Find*

$$\langle Z^1 \rangle_t, \qquad \langle Z^1, Z^2 \rangle_t.$$

Exercise 3 *Let B_t, W_t be independent standard Brownian motions. Find*

$$\mathbb{P}\{B_t \geq 6W_t - 4 \text{ for all } 0 \leq t \leq 3\}.$$

Hint: You may wish to consider $6W_t - B_t$.

Exercise 4 *Let B_t be a standard (one-dimensional) Brownian motion (not necessarily starting at the origin). For the following functions $\phi(t,x), 0 < t < 3$, state the PDE that it satisfies. If you use the L or L^* notation, you must say what L or L^* is in these cases.*

1. *$\phi(t,x)$ is the density of B_t (as a function of x) given that $B_0 = 0$.*
2. *$\phi(t,x) = \mathbb{P}\{B_t < 4 \mid B_0 = x\}$*
3. *$\phi(t,x) = \mathbb{E}[B_t^3 \mid B_0 = x]$*

4. $\phi(t,x) = E[B_4 - B_4^2 \mid B_t = x]$

5. *Repeat the examples above where B has drift 1 and variance parameter 4.*

Exercise 5 *Suppose B_t, W_t are independent standard Brwonian motions starting at the origin, and the random vector $Z_t = (Z_t^1, Z_t^2)$ is defined by*

$$Z_t^1 = B_t + W_t - t, \qquad Z_t^2 = 2\,B_t - 4\,W_t.$$

Note that Z_t is a two-dimensional Brownian motion.

1. *What are the drift and covariance matrix for Z?*

2. *What are the operators L, L^* associated to Z?*

3. *Let $\phi(t,x), t > 0, x \in \mathbb{R}^2$ be the density of Z_t at time t. Find the PDE satisfied by $\phi(t,x)$.*