Exercise 1

B_t is a standard BM starting at x with $0 < x < 3$

1) Prob that B_t will obtain a value of 3 before reaching 0?

Let τ be the first time B_t hits either 3 or 0

Since B_t has 0 drift, it is a martingale

According to optional sampling Theorem

$$E[B_\tau] = E[B_0] = x$$

B_τ can either be 3 or 0

$$E[B_\tau] = 3 \times P(B_\tau = 3) + 0 \times P(B_\tau = 0)$$

$$x = 3P(B_\tau = 3)$$

$$\therefore P(B_\tau = 3) = \frac{x}{3}$$

2) $X_t = x + \int_0^t B_s \, dB_s$

We need to check whether X_t is a martingale

For the stochastic integral $\int_0^t B_s \, dB_s$ to be a martingale, it needs to follow below 2 properties →

- Adaptedness [Depends only on information upto time s]

  B_s is adapted by definiton of Brownian motion

- Square Integrability $\left[E\left[\int_0^t B_s^2 \, ds\right] < \infty \text{ for all } t \right]$

$$E\left[\int_0^t B_s^2 \, ds\right] = \int_0^t E[B_s^2] \, ds = \int_0^t s \, ds = \frac{t^2}{2} < \infty$$

$\therefore$ it is square integrable

$\therefore \int_0^t B_s \, dB_s$ is a martingale

$$E[X_t / F_s] = x + E\left[\int_0^t B_u \, dB_u / F_s\right]$$

$$= x + \int_0^s B_u \, dB_u = X_s$$

$\therefore X_t$ is also a martingale

let τ be the first time X_t hits 0 or 3

$E[X_\tau] = E[X_0] = x$ [Using optimal Sampling Theorem]

let $p = P(X_\tau = 3)$

Then:

$$E[X_\tau] = 3p + 0(1-p) = 3p$$

$$x = 3p$$

$$\therefore p = \frac{x}{3}$$

Exercise 2

Bessel Process

$$dX_t = \frac{a}{X_t} dt + dB_t, \quad X_0 = 1$$

and let $T = T_{r,R} = \min\{t : X_t = r \text{ or } R\}$

[first time X_t hits r or R]

let $\varphi(r, R)$ be the prob that $X_T = R$

1) $a < 1/2$, Find $\lim_{r \to 0} \varphi(r, R)$

$$dX_t = \frac{a}{X_t} dt + dB_t$$

let $f(X_t)$ be a function of X_t

$$df(X_t) = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)(dX_t)^2$$

$$= f'(X_t)\left[\frac{a}{X_t}dt + dB_t\right] + \frac{1}{2} f''(X_t)\,dt$$

$$= \left[\frac{a}{X_t} f'(X_t) + \frac{1}{2} f''(X_t)\right]dt + f'(X_t)\,dB_t$$

let $s(x)$ be the scale function

$s(x)$ satisfies $\frac{a}{X_t} s'(X_t) + \frac{1}{2} s''(X_t) = 0$

Solving this ODE:

Let $g = s'(X_t)$

$$\therefore \frac{a}{x} g + \frac{1}{2} g' = 0$$

$$g' = -\frac{2a}{x} g \Rightarrow \frac{dg}{g} = \frac{-2a}{x} dx$$

Integrating both sides

$$\ln|g| = -2a \ln x + C$$

$$g = Cx^{-2a}$$

Integrating again

$$s(x) = C\left[\frac{x^{-2a+1}}{-2a+1}\right] + D$$

Choosing $C = -2a+1$ & $D = 0$ for simplicity

$$\therefore s(x) = x^{-2a+1}$$

$$\rho(r,R) = \frac{s(1) - s(r)}{s(R) - s(r)}$$

$$= \frac{1^{1-2a} - r^{1-2a}}{R^{1-2a} - r^{1-2a}} = \frac{1 - r^{1-2a}}{R^{1-2a} - r^{1-2a}}$$

$1-2a > 0$ as $a < 1/2$

$\therefore$ as $r \to 0$, $r^{1-2a} = 0$

$\therefore\ P(0,R) = \dfrac{1}{R^{1-2a}}$

2) $\lim_{R\to\infty} P(0,R) = 0$

$P(0,R) = \dfrac{1}{R^{1-2a}}$

As $R \to \infty$, $\dfrac{1}{R^{1-2a}} \to 0$

$\therefore\ P(0,R) = 0$ that is $P(X_T = R) = 0$

This means the prob that the Bessel process reaches 0 before reaching R $= 1 - P(0,R)$

$= 1$

3) $a = \dfrac{1}{2}$

$dX_t = \dfrac{1}{2X_t}dt + dB_t$

Now $s(x)$ satisfies $\dfrac{1}{x}s'(x) + s''(x) = 0$

let $g = s'(x)$

$\dfrac{1}{x}g + g' = 0$

$g' = -\dfrac{1}{x}g \Rightarrow \dfrac{dg}{g} = -\dfrac{1}{x}dx$

$\ln|g| = -\ln x + C \Rightarrow g = \dfrac{C}{x}$

$s(x) = C\ln x + D$

Choosing $C=1$ & $D=0$ $\therefore\ s(x) = \ln x$

$$P(r,R) = \frac{S(1)-S(r)}{S(R)-S(r)} = \frac{\ln 1 - \ln r}{\ln R - \ln r} = \frac{-\ln r}{\ln R - \ln r}$$

As $r \to 0$, $\ln r \to \infty$

$P(0,R) = 1$

$\therefore$ the probability of reaching R is 1

$\therefore$ the process never hits 0

4) $a = \frac{1}{2}$ and $r > 0$

$$P(r,R) = \frac{-\ln r}{\ln R - \ln r}$$

As $R \to \infty$:

$$P(r,\infty) = \lim_{R\to\infty} \frac{-\ln r}{\ln R - \ln r} = 0$$

Thus the process hits r before escaping to infinity implying that $X_t < r$ for some t

Exercise 3

B_t is standard brownian & $X_t = 2e^{B_t - t}$

1) $X_t = 2e^{B_t}e^{-t}$

$$dX_t = 2e^{-t}e^{B_t}dB_t + \frac{1}{2}\cdot 2e^{-t}e^{B_t}(dB_t)^2 - 2e^{B_t}e^{-t}dt$$

$$= X_t dB_t + e^{B_t - t}dt - X_t dt$$

$$= X_t dB_t + \left(e^{B_t - t} - X_t\right)dt$$

$$= X_t dB_t - \frac{1}{2}X_t dt$$

2) $\phi(t,x) = E\left[e^{-2(T-t)}(X_T-3)_+ / X_t = x\right]$, $0<t<T$

According to Feynman-Kac formula →

The PDE for $\phi(t,x)$ is :

$$\frac{d\phi}{dt} + m(x)\frac{d\phi}{dx} + \frac{1}{2}\sigma(x)^2\frac{d^2\phi}{dx^2} - r\phi = 0$$

with terminal condition $\phi(T,x) = (x-3)_+$

$m(x) = -\frac{1}{2}x$

$\sigma(x) = x$

r => discount rate = 2

$$\frac{d\phi}{dt} - \frac{1}{2}x\frac{d\phi}{dx} + \frac{1}{2}x^2\frac{d^2\phi}{dx^2} - 2\phi = 0$$

with $\phi(T,x) = (x-3)_+$

3) $\phi(t,x) = E\left[e^{-2(T-t)}X_T^2 e^{-X_T} / X_t = x\right]$, $0<t<T$

The PDE structure would remain the same with

$m(x) = 2$

$\sigma(x) = x$

$r = 2$

The only change is in the terminal condition (x^2e^{-x})

$$\frac{d\phi}{dt} - \frac{1}{2}x\frac{d\phi}{dx} + \frac{1}{2}x^2\frac{d^2\phi}{dx} - 2\phi = 0$$

with $\phi(t,x) = x^2e^{-x}$

Exercise 4

B_t is a standard brownian motion

$$Z_t = \int_0^t \frac{1}{(1-s)^\alpha} dB_s \quad , 0 \le t < 1$$

1) $\alpha = \frac{1}{4}$

$$E\left[\int_0^t \left(\frac{1}{(1-s)^{1/4}} \right)^2 ds \right]$$ should be less than ∞

for Z_t to be square integrable martingale

$$E\left[\int_0^t \frac{1}{(1-s)^{1/2}} ds \right]$$

Let $1-s = u$

$\frac{du}{ds} = -1 \qquad \therefore du = -ds$

Limits change $\rightarrow$ $1-0 \Rightarrow u$ lower

$1-t \Rightarrow u$ upper

$$\int_1^{1-t} -u^{-1/2} du = \int_{1-t}^1 u^{-1/2} du$$

$$= \left[2u^{1/2} \right]_{1-t}^1 = 2(1-(1-t)^{1/2})$$

When $t=0$, $E\left[\int_0^t \frac{1}{(1-s)^{1/2}} ds \right] = 0$

when $t=1$, $E\left[\int_0^t \frac{1}{(1-s)^{1/2}} ds \right] = 2$

$\therefore E[Z_t^2] \le 2$ Hence it is bounded

$\therefore$ it is square integrable martingale

$Var(Z_t) = E[Z_t^2] \qquad \therefore C = 2$

2) $\alpha = 1$

Now $Z_t = \int_0^t \frac{1}{1-s} dB_s$

$$E[Z_t^2] = \int_0^t \frac{1}{(1-s)^2} ds$$

let $u = 1-s$ $\therefore du = -ds$

lower bound $\to 1$

upper bound $\to 1-t$

$$E[Z_t^2] = \int_1^{1-t} -u^{-2} du = \int_{1-t}^{1} u^{-2} du = \left[-\frac{1}{u}\right]_{1-t}^{1}$$

$$= \left[-1 - \left(-\frac{1}{1-t}\right)\right] = \left(-1 + \frac{1}{1-t}\right)$$

$$= \frac{1}{1-t} - 1$$

As $t \to 1$, $E[Z_t^2] \to \infty$

$\therefore$ if $t < 1$,

Z_t is a continuous martingale with $Var(Z_t) \to \infty$

Let W_T be a standard brownian motion such that

$Z_t = W_T$ such that $T = Var(Z_t)$

As $t \to 1$, $Var(Z_t) \to \infty$

This means W_T evolves over infinite time horizon

With infinite time horizon, it is certain that it hits any level (1 in this case) with prob $= 1$.