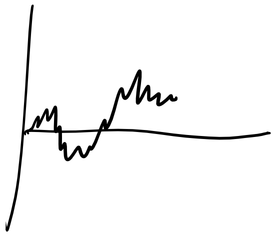
$B_t \to$ sBM

$F_t \to$ info in $B_s, s \le t$



$m \in \mathbb{R}$ {Real #}

$M_t = e^{mB_t} e^{-\frac{m^2}{2}t}$] Martingale

$$d e^{mB_t} = m e^{mB_t} dB_t + \frac{m^2}{2} e^{mB_t} dt$$

$$d e^{-\frac{m^2 t}{2}} = -\frac{m^2}{2} e^{-\frac{m^2}{2}t} dt$$

$$d\left[e^{mB_t} e^{-m^2 t/2}\right] = e^{mB_t} d e^{-\frac{m^2}{2}t} + e^{-\frac{m^2 t}{2}} d e^{mB_t} - d\langle e^{mB_t}, e^{-m^2 t/2}\rangle$$

$$= m e^{mB_t} e^{-\frac{m^2}{2}t} dB_t$$

$$dM_t = m M_t dB_t, \quad M_0 = 1$$

<u>Define</u> a measure Q_t by: if A is an event in F_t ($A \in F_t$)

then $Q_t[A] = E[1_A M_t]$] Prob Measure

$E[M_t] = E[M_0] = 1$

"$dQ_t = M_t dP$"

$M_t = \frac{dQ_t}{dP}$ [Radon - Nikodym]

If $s<t$ and $A \in F_s$ then $A \in F_t$, and

$Q_t(A) = E[1_A M_t]$

Using Conditional Expectation

$$E[E[1_A M_t / F_s]] = E[1_A E[M_t / F_s]]$$

$$= E[1_A M_s]$$ since M_t is Martingale

$$= Q_s(A)$$

We can define the prob measure Q by: if $A \in F_t$, then

$$Q[A] = E[1_A M_t]$$

$$M_1 = e^{mB_1} e^{-m^2/2}$$

Theorem: In the new measure Q, B_t is a brownian motion with drift m and variance parameter 1.

Under the new measure Q:

1. $B_0 = 0$
2. If $s<t$, then $B_t - B_s$ is independent of F_s
3. If $s<t$, then $B_t - B_s \sim N(m(t-s), t-s)$
4. Paths are continuous

2&3

The conditional distribution of $B_{t+s} - B_s$ given F_s is $N(mt, t)$

Recall $\rightarrow$ If $Y \sim N(mt, t)$

$$E[e^{\lambda Y}] = e^{\lambda m t} e^{\lambda^2 t/2} \quad] \text{ mgf}$$

$$E_Q\left[e^{\lambda(B_{t+s}-B_s)} / F_s\right] = e^{\lambda m t} e^{\lambda^2 t/2}$$

measure Q now and not P

Note:- If X is F_t-measurable

$$E_Q[X] = E_P[X M_t]$$

If V is F_s measurable

Then $E_Q\left[1_V e^{\lambda(B_{t+s}-B_s)}\right] = E_Q\left[e^{\lambda m t} e^{\lambda^2 t/2} 1_V\right]$

This means

removing Q but including Martingale

$$E\left[1_V e^{\lambda(B_{t+s}-B_s)} M_{t+s}\right] = e^{\lambda m t} e^{\lambda^2 t/2} E[M_{t+s} 1_V]$$

since martingale

$$= e^{\lambda m t} e^{\lambda^2 t/2} E[M_s 1_V]$$

Geometric BM

$$dX_t = X_t[m\,dt + \sigma\,dB_t]$$

We want to change the measure

$$M_t = e^{\lambda B_t - \frac{\lambda^2}{2} t}, \quad dM_t = \lambda M_t\,dB_t$$

$$M_0 = 1$$

In the new measure Q,

B_t is not SBM in Q but SBM in P

B_t is SBM with drift λ & variance parameter $= 1$

that is $B_t = \lambda t + W_t$

SBM in Q but not SBM in P

where W_t is a standard BM with respect to Q

$$dB_t = \lambda dt + dW_t \quad , \lambda = -\frac{m}{\sigma}$$

$$dX_t = \left[m dt + \sigma(\lambda dt + dW_t)\right] X_t$$
$$= \left[(m + \sigma\lambda) dt + \sigma dW_t\right] X_t$$

Non Negative martingale M_t satisfying

$$dM_t = A_t M_t dB_t \quad , \quad M_0 = 1$$

$$\boxed{\begin{aligned} M_t &= e^{mt - \frac{m^2}{2} t} \\ dM_t &= m M_t dB_t \end{aligned}}$$

$$M_t = \exp\left(\int_0^t A_s dB_s - \frac{1}{2}\int_0^t A_s^2 ds\right)$$

$$= e^{Y_t} \qquad \text{where } Y_t = \int_0^t A_s dB_s - \frac{1}{2}\int_0^t A_s^2 ds$$

$$E[M_t] = E[M_0] = 1$$

Let Q_t be the prob measure on F_t given by

$$Q_t[A] = E[1_A M_t]$$

$$E_{Q_t}[X] = E[X M_t] \quad \text{if } X \text{ is } F_t \text{ measurable}$$

If $s < t$, and A is F_s-measurable

Then $Q_s(A) = Q_t(A)$

Q measure if A is F_t-measurable

then $Q(A) = E[1_A M_t]$

Girsanov Theorem

Let $W_t = B_t - \int_0^t A_s \, ds$

then W_t is a standard BM wrt Q.

$$dW_t = dB_t - A_t dt$$

$$dB_t = A_t dt + dW_t$$

Example:

let $B_t \to$ SBM where $B_0 = 1$

$$T = \min\{t : B_t = 0\}$$

$M_t = B_{t \wedge T}$ [Now M_t is bigger than or equal to 0]

so non-negative martingale

$dM_t = A_t M_t dB_t$ $\quad A_t = \frac{1}{B_t}$

$dB_t = \frac{1}{B_t} dt + dW_t$ $\longrightarrow$ similar to Bessel equation where $a = 1$

where W_t is a standard BM wrt Q

$P(T < \infty) = 1$, $Q(T < \infty) = 0$