Exercise 1

$\mu_j \ll \mu_k$] whether distribution of μ_j is absolutely continuous with respect to μ_k

This means μ_j doesnot have more 0 probability events than μ_k

X_1 : Normal (2, 7)] Continuous distribution in R

X_2 : Binomial(7, 0.3)] Discrete distribution with possible values 0 to 7

X_3 : Poisson(2)] Discrete distribution over non-negative integers

X_4 : e^{X_1}] X_4 is log normal since X_1 is normal

∴ Continuous distribution & in range $(0, \infty)$

X_5 : Uniform (0,1) : Continuous distribution on [0,1]

X_6 : $X_1 \times X_3 + X_2$: Mixed distribution

Note:- A continuous & discrete distribution are mutually singular i.e. they are not absolutely continuous

- 2 continuous distributions are absolutely continuous only if their ranges/supports overlap

$\mu_4 \ll \mu_1$ → If μ_1 assigns 0 prob to a set A, then μ_4 also assigns 0 prob to the set A

→ Absolutely continuous

because $A \subseteq (0, \infty)$

Let $A = [1, 2]$ → range

$\mu_1(A) > 0$

$\mu_4(A) > 0$

Let $A = 1$ [single point]

$\mu_1(A) = 0$

$\mu_4(A) = 0$

$\mu_1 \ll \mu_4$ → Not absolutely continuous

because $A \subseteq (-\infty, 0)$

$\mu_1(A) > 0$, $\mu_4(A) = 0$

[since out of range]

Similarily, $\mu_5 << \mu_1$ is absolutely continuous as support of μ_5 is within the support of μ_1

, $\mu_5 << \mu_4$ is absolutely continuous with the same reason above

, $\mu_2 << \mu_3$ $\rightarrow$ though discrete, the poisson process support contains the binomial process support. $\therefore$ absolutely continuous

$\mu_5 << \mu_6 \rightarrow$ Absolutely continuous

$$X_6 = \underbrace{X_1 X_3}_{\text{continuous component}} + \underbrace{X_2}_{\text{discrete component}}$$

$X_1 X_3 \rightarrow$ Normal distribution scaled by X_3
For each $X_3 = k \geq 1$, $X_1 k$ is Normal with mean $2k$ & variance $7k^2$

but still spans all real numbers

The continuous component of μ_6 dominates μ_5 on $[0,1]$

Similarily,

$\mu_4 << \mu_6 \rightarrow$ Absolutely continuous

Because the continuous component of μ_6 dominates μ_4 on $[0,\infty]$

$\mu_2 << \mu_6 \rightarrow$ Absolutely continuous
The discrete component of μ_6 has the same support as μ_2 at $[0, \dots 7]$

Therefore, the only absolutely continuous pairs are $\rightarrow$

- $\mu_4 << \mu_1$
- $\mu_5 << \mu_1$
- $\mu_5 << \mu_4$
- $\mu_2 << \mu_3$
- $\mu_4 << \mu_6$
- $\mu_2 << \mu_6$

other pairs are not absolutely continuous

Exercise 2

$W_n \rightarrow$ Winnings at time n

$M_n = 1 - W_n$

$M_0 = 1$, $T = \min(n : M_n = 0)$

Let F_n be the info in $M_0, \dots, M_n$

1) Why M_n is a non-negative martingale

For any n, if we have lost all games upto n

$W_n = 1 - 2^n$

$\therefore M_n = 1 - (1 - 2^n) = 2^n$

If we have won the n^{th} game

$W_n = 1$

$M_n = 1 - 1 = 0$

$\therefore M_n$ will always be positive

$$E[M_{n+1} / F_n] = E\left[(1 - W_{n+1}) / F_n\right]$$

$$= \left[1 - E(W_{n+1} / F_n)\right]$$

$$= \left[1 - \left(\tfrac{1}{2} \times 1 + \tfrac{1}{2} \times (1 - 2^{n+1})\right)\right]$$

$$= \left[1 - (1 - 2^n)\right] = 2^n = M_n$$

$\therefore M_n$ is a non negative martingale.

2) $Q_n(V) = E[M_n 1_V]$

To show $\Rightarrow$ $Q_m(V) = Q_n(V)$ where $m<n$

$$Q_n(V) = E[M_n 1_V] = E[E[M_n / F_m] 1_V] \quad \boxed{\text{Using Tower Property}}$$

$$= E[M_m 1_V] = Q_m(V)$$

3) $Q\{M_{n+1} = 2^{n+1} / M_n = 2^n\}$

$P(M_{n+1} = 2^{n+1} / M_n = 2^n) = 1/2$

Let $A = (M_{n+1} = 2^{n+1})$

$B = (M_n = 2^n)$

$$Q(A/B) = \frac{Q(A \cap B)}{Q(B)} = \frac{Q(M_{n+1} = 2^{n+1}, M_n = 2^n)}{Q(M_n = 2^n)}$$

$$= \frac{E^P[M_{n+1} 1_{(M_{n+1} = 2^{n+1}, M_n = 2^n)}]}{E^P[M_n 1_{(M_n = 2^n)}]}$$

$$= \frac{2^{n+1} P(M_{n+1} = 2^{n+1}, M_n = 2^n)}{2^n P(M_n = 2^n)}$$

$$= \frac{2^{n+1}}{2^n} P(M_{n+1} = 2^{n+1} / M_n = 2^n)$$

$$= 2 \times \frac{1}{2} = 1$$

4) Q probability that $T < \infty$

From part 3,

M_n always transitions to 2^{n+1} (never to 0)

Thus, $T = \infty$ with probability 1 under Q

$\therefore \; Q(T<\infty) = 0$

5) $E[M_{n+1}/F_n] = E[2M_n/F_n] = 2E[M_n/F_n]$
$= 2M_n$
$= 2^{n+1}$

$E[M_{n+1}/F_n] \neq M_n$

$\therefore$ M_n is not a martingale under Q

Exercise 3

B_t is a standard BM on (Ω, P)

Is there prob measure Q such that X_t $(0 \leq t \leq 1)$ is Standard BM

$B_0 = 0, \; X_0 = 0$

1) $dX_t = 2dt + dB_t$

drift $= 2$ & variance parameter $= 1$
(μ)

Novikov's Condition $\rightarrow$

$$E^P\left[\exp\left(\frac{1}{2}\int_0^1 \mu^2 dt\right)\right] = \exp(2) < \infty$$

$\therefore$ Q exists

We can apply Girsanov's theorem as volatility matches standard brownian motion $(\sigma = 1)$, we can remove the drift

The Radon-Nikodym derivative $\frac{dQ}{dP}$ is predefined as:

$$\frac{dQ}{dP} = \exp\left(-\mu B_t - \frac{1}{2}\mu^2 T\right)$$

where $\mu \to$ drift & $T \to$ Time horizon (variance parameter)

$$\frac{dQ}{dP} = \exp\left(-2B_t - \frac{1}{2} \times 2^2 \times 1\right) = \exp(-2B_t - 2)$$

The term $\exp(-2B_t - 2)$ reweights probabilities under the new measure Q. It tilts the original measure P to cancel the drift $2dt$ in X_t.

$$dX_t = 2dt + dB_t$$

By defining $\frac{dQ}{dP} = \exp(-2B_1 - 2)$, we define BM B_t^Q

$$dB_t^Q = dB_t + \mu dt = dB_t + 2dt \quad \text{[Girsanov Transformation]}$$

Substituting $dB_t = dB_t^Q - \mu dt$ into original SDE

$$dX_t = dB_t^Q$$

2) $dX_t = 2dt + 6dB_t$

drift $(\mu) = 2$, variance parameter $= 6$

Girsanov's Theorem only adjusts the drift, not the volatility.

$\therefore$ Q doesnot exist

3) $dX_t = 2B_t dt + dB_t$

drift $(\mu) = 2B_t$, variance parameter $= 1$

Novikov's Condition $\to$

$$E^P\left[\exp\left(\frac{1}{2}\int_0^1 (2B_t)^2 dt\right)\right] = E^P\left[\exp\left(2\int_0^1 B_t^2 dt\right)\right]$$

The integral has a heavy tailed behaviour because of the coeffecient 2 . ∴ it fails Novikov's Condition

∴ Q doesnot exist.

[Note: If coeffecient $< \frac{1}{2T}$, the expectation would be finite]

Exercise 4

B_t is Standard BM with $B_0 = 0$

$X_t = e^{-mB_t^2}$ where $m > 0$

1) $M_t = X_t \exp\left(\int_0^t g(B_s)\, ds\right)$ Then M_t is a local Martingale

Let $Y_t = \exp\left(\int_0^t g(B_s)\, ds\right)$

∴ $M_t = X_t Y_t$

$$dX_t = e^{-mB_t^2} \times -2mB_t\, dB_t + \frac{1}{2}\left(e^{-mB_t^2} \times -2m + (-2mB_t)^2 e^{-mB_t^2}\right) dt$$

$$= -2mX_t B_t\, dB_t + (-mX_t + 2m^2 B_t^2 X_t)\, dt$$

$$dY_t = \exp\left(\int_0^t g(B_s)\, ds\right) g(B_t)\, dt$$

$$= g(B_t) Y_t\, dt$$

[since Y_t is a time integral & not stochastic integral]

$$dM_t = Y_t\, dX_t + X_t\, dY_t + \underbrace{dX_t\, dY_t}_{0}$$

$$= Y_t\left[-2mX_t B_t\, dB_t - mX_t\, dt + 2m^2 B_t^2 X_t\, dt\right] + X_t\left[g(B_t) Y_t\, dt\right]$$

$$= M_t\left[-2mB_t\, dB_t + (2m^2 B_t^2 - m)\, dt\right] + M_t\left[g(B_t)\, dt\right]$$

$$dM_t = M_t\left[-2mB_t\,dB_t + (2m^2B_t^2 - m + g(B_t))\,dt\right]$$

For M_t to be a local martingale

$$2m^2B_t^2 - m + g(B_t) = 0$$

$$\therefore g(B_t) = m - 2m^2B_t^2$$

2) SDE for M_t

$$dM_t = -2mB_tM_t\,dB_t$$

3) $Q(V) = E[M_t 1_A]$

Find SDE satisfied by B_t wrt Q-Brownian motion

Girsanov's Theorem states that if M_t is a martingale under P, then under the new measure Q, the process:

$$\tilde{B}_t = B_t + \int_0^t \theta_s\,ds \quad \text{is a standard BM}$$

Since SDE of $M_t = dM_t = -2mB_tM_t\,dB_t$

$\theta_t = 2mB_t$ [Since $dM_t = -\theta_t M_t\,dB_t$ where θ_t is the drift adjustment]

$$\tilde{B}_t = B_t + \int_0^t 2mB_s\,ds$$

$$d\tilde{B}_t = dB_t + 2mB_t\,dt$$

$$\therefore dB_t = d\tilde{B}_t - 2mB_t\,dt$$

4) Why M_t is a martingale & not local martingale

To prove M_t is a martingale, we can use Novikov's condition

Novikov's condition states that if

$$E\left[\exp\left(\frac{1}{2}\int_0^t \theta_s^2\,ds\right)\right] < \infty$$

Then,

$$M_t = \exp\left(\int_0^t \theta_s\,dB_s - \frac{1}{2}\int_0^t \theta_s^2\,ds\right)$$

is a true martingale

$$M_t = X_t Y_t$$

$$= e^{-mB_t^2}\, e^{\int_0^t m - 2m^2 B_s^2\,ds}$$

$$= e^{-mB_t^2 + mt - 2m^2\int_0^t B_s^2\,ds}$$

Let $f(B_t) = -mB_t^2$

$$d(-mB_t^2) = -2mB_t\,dB_t - m\,dt$$

$$-mB_t^2 = -2m\int_0^t B_s\,dB_s - mt$$

$$-2m\int_0^t B_s\,dB_s = -mB_t^2 + mt$$

$$M_t = e^{-2m\int_0^t B_s\,dB_s - 2m^2\int_0^t B_s^2\,ds}$$

$$= e^{\int_0^t -2mB_s\,dB_s - \frac{1}{2}\int_0^t (2mB_s)^2\,ds}$$

This matches Novikov's martingale expression

If $E\left[\exp\left(\frac{1}{2}\int_0^t (2mB_s)^2\,ds\right)\right] < \infty$, then M_t is martingale

$$= E\left[\exp\left(2m^2\int_0^t B_s^2\,ds\right)\right]$$

For small m, the coeffecient $2m^2$ can be small enough to ensure light tails, making expectation finite.

Exercise 5

B_t is standard BM with $B_0 = 1$

$T = \min\{t : B_t = 0\}$

$r > 0$

$X_t = B_t^r$

1) $M_t = X_t \exp\left\{\int_0^t g(B_s)\,ds\right\}$

Let $Y_t = \exp\left\{\int_0^t g(B_s)\,ds\right\}$

$dY_t = g(B_t)Y_t\,dt$ [From Last question]

$dX_t = rB_t^{r-1}\,dB_t + \frac{1}{2}r(r-1)B_t^{r-2}\,dt$

$$dM_t = Y_t\,dX_t + X_t\,dY_t + \underbrace{dX_t\,dY_t}_{0}$$

$$= Y_t\left[rB_t^{r-1}\,dB_t + \frac{1}{2}r(r-1)B_t^{r-2}\,dt\right]$$

$$+ B_t^r\left[g(B_t)Y_t\,dt\right]$$

$$= Y_t\left[rB_t^{r-1}\,dB_t + \left(\frac{1}{2}r(r-1)B_t^{r-2} + B_t^r g(B_t)\right)dt\right]$$

For M_t to be local martingale

$$\frac{1}{2}r(r-1)B_t^{r-2} + B_t^r g(B_t) = 0$$

$$g(B_t) = \frac{-\frac{1}{2}r(r-1)B_t^{r-2}}{B_t^r} = -\frac{1}{2}r(r-1)B_t^{-2}$$

2) SDE of M_t

$$dM_t = Y_t\left[r B_t^{r-1} dB_t + \left(\tfrac{1}{2}r(r-1)B_t^{r-2} + B_t^{r}\left(-\tfrac{1}{2}r(r-1)B_t^{-2}\right)\right)dt\right]$$

$$= Y_t\left[r B_t^{r-1} dB_t\right] = M_t\, r\, B_t^{-1} dB_t$$

3) SDE satisfied by B_t wrt Q brownian motion

Girsanov's Theorem states that if M_t is a martingale under P, then under the new measure Q, the process:

$$\tilde{B}_t = B_t + \int_0^t \theta_s\, ds \qquad \text{is a standard BM}$$

$$dM_t = M_t\, r\, B_t^{-1} dB_t$$

$$dM_t = -\theta M_t dB_t \quad \Big] \therefore \theta = -r B_t^{-1}$$

$$\tilde{B}_t = B_t + \int_0^t -r B_s^{-1}\, ds$$

$$d\tilde{B}_t = dB_t - r B_t^{-1} dt$$

$$dB_t = d\tilde{B}_t + r B_t^{-1} dt$$

4) $Q(T < \infty) = 0$

$$dB_t = d\tilde{B}_t + r B_t^{-1} dt$$

This follows a Bessel process $dX_t = \frac{\delta - 1}{2} X_t^{-1} dt + dW_t$

$r = \frac{\delta - 1}{2} \quad \therefore \delta = 2r + 1$

If $\delta \geq 2$, the process never hits 0

If $\delta < 2$, the process can hit 0

$\delta \geq 2$, $r \geq \frac{2-1}{2} \geq \frac{1}{2}$

$\therefore$ if $r \geq \frac{1}{2}$, the process never hits 0

Hence $Q(T < \infty) = 0$