Problem set 1

Exercise 1

B_t is a standard Brownian Motion

$B_0 = 0$

$B_t \sim N(0, t)$

1) $P(B_3 \leq 2)$

$B_3 \sim N(0, 3)$

$$P(B_3 \leq 2) = P\left(Z \leq \frac{2-0}{\sqrt{3}}\right) = P(Z \leq 1.155)$$
$$\approx 0.876$$

2) $P(B_1 \leq 1, B_3 - B_1 \leq 1)$

$B_1 \sim N(0, 1)$, $B_3 - B_1 \sim N(0, 2)$

B_1 & $(B_3 - B_1)$ are both independent

$$\therefore P(B_1 \leq 1, B_3 - B_1 \leq 1) = P(B_1 \leq 1)\, P(B_3 - B_1 \leq 1)$$
$$= P\left(Z \leq \frac{1-0}{\sqrt{1}}\right) P\left(Z \leq \frac{1-0}{\sqrt{2}}\right)$$
$$= P(Z \leq 1)\, P\left(Z \leq \frac{1}{\sqrt{2}}\right)$$
$$= 0.84134 \times 0.76025$$
$$= 0.6396$$

3) $P(E)$ where E is the event that the path stays below the line $y = 2$ up to time $t = 4$

$P(E) = P(B_t < 2 \text{ for all } t \in [0, 4])$

Let M_4 denote maximum of B_t up to time $t = 4$

$\therefore P(E) = P(M_4 < 2)$

According to Reflection Principle:

If B_t reaches point a at time t_1, then the distribution of B_t is same as distribution of $2a - B_t$ beyond t_1

$P(M_t < a) = P(B_t < a) - P(\text{Brownian motion hits } a \text{ before } t)$

Let t_a be the time Brownian motion hits a before t

$P(t_a \leq t) = P(B_t \geq 2a)$

This is because after hitting a, the path of Brownian motion can be reflected about a without changing its distribution.

$$P(M_t < a) = P(B_t < a) - P(B_t \geq 2a)$$
$$= P(B_t < a) - (1 - P(B_t < 2a))$$
$$= 2P(B_t < a) - 1$$

$$\therefore P(M_4 < 2) = 2P(B_4 < 2) - 1$$
$$= 2 \times P\left(Z < \frac{2-0}{\sqrt{4}}\right) - 1$$
$$= 2P(Z < 1) - 1$$
$$= 2 \times 0.84134 - 1$$
$$= 0.68268$$

or Approach 2

$$P(M_t < a) = 1 - P(M_t > a)$$
$$= 1 - 2P(B_t > a)$$
$$\therefore P(M_4 < 2) = 1 - 2P(B_4 > 2)$$
$$= 1 - 2[1 - P(B_4 < 2)]$$
$$= 1 - 2[0.15876]$$
$$= 0.68268$$

4) $P(B_4 \geq 0 \mid B_8 \geq 0)$

$B_4 \sim N(0, 4)$ These are not independent

$B_8 \sim N(0, 8)$

The joint distribution of B_4 & B_8 is multivariate normal

$$Cov(B_4, B_8) = E[B_4 B_8] - E[B_4]E[B_8]$$
$$= E[B_4 B_8]$$
$$= E[B_4(B_4 + (B_8 - B_4))]$$
$$= E[B_4^2] + E[B_4(B_8 - B_4)]$$
$$= Var(B_4) + 0$$
$$= 4$$

$$\rho = \frac{Cov(B_4, B_8)}{\sqrt{Var(B_4)}\sqrt{Var(B_8)}}$$

$$= \frac{4}{\sqrt{4}\sqrt{8}} = \frac{4}{2\times 2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$P(B_4 \geq 0 \mid B_8 \geq 0) = \frac{P(B_4 \geq 0, B_8 \geq 0)}{P(B_8 \geq 0)}$$

$$= \frac{P(B_4 \geq 0, B_8 \geq 0)}{(1/2)}$$

$$= 2P(B_4 \geq 0, B_8 \geq 0)$$

$B_4 \geq 0$ is same as $\frac{B_4}{\sqrt{4}} \geq 0$

$B_8 \geq 0$ is same as $\frac{B_8}{\sqrt{8}} \geq 0$

Let $X = \frac{B_4}{\sqrt{4}}$, $Y = \frac{B_8}{\sqrt{8}}$

$\therefore$ X & Y follow standard normal distribution with correlation $\frac{1}{\sqrt{2}}$

For two standard normal variables X & Y with correlation ρ, the probability that both are nonnegative is known to be:

$$\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(\rho)$$

$$P(X \geq 0, Y \geq 0) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4} + \frac{1}{2\pi} \times \frac{\pi}{4}$$

$$= \frac{1}{4} + \frac{1}{8}$$

$$= \frac{3}{8}$$

$\therefore \quad P(B_4 \geq 0 / B_8 \geq 0) = \dfrac{0.375}{0.5} = 0.75$

5) $P(B_1(B_3 - B_1) \geq 0 \mid B_1 \leq 1, (B_3 - B_1)^2 \geq 2)$

Let $B_1 = x$

Let $B_3 - B_1 = y$

$x \sim N(0,1)$

$y \sim N(0,2)$

$$P(XY \geq 0 / X \leq 1, Y^2 \geq 2) = \frac{P(XY \geq 0, X \leq 1, Y^2 \geq 2)}{P(X \leq 1, Y^2 \geq 2)}$$

Note: X & Y are independent

- For $XY \geq 0$,

  X & Y should have same signs or atleast one of them should be 0

- $X \leq 1$
- $Y^2 \geq 2$ means $|Y| \geq \sqrt{2}$

We have 2 cases in order to meet all these conditions →

Case 1

X in $[0,1]$ & $Y \geq \sqrt{2}$

Case 2

$X \leq 0$ & $Y \leq -\sqrt{2}$

Case 1 Calculation →

$P(X \leq 1) - P(X \leq 0) = 0.84134 - 0.5 = 0.34134$

$P(Y \geq \sqrt{2}) = P\left(Z \geq \dfrac{\sqrt{2} - 0}{\sqrt{2}}\right) = 1 - P(Z \leq 1) = 1 - 0.84134$

$= 0.15866$

$\therefore P(XY \geq 0, X \text{ in } [0,1], Y \geq \sqrt{2}) = 0.34134 \times 0.15866$

$= 0.05416$

Case 2 Calculation $\rightarrow$

$$P(X \le 0) = 0.5$$

$$P(Y \le -\sqrt{2}) = P\left(Z \le \frac{-\sqrt{2}-0}{\sqrt{2}}\right) = P(Z \le -1)$$

$$= 1 - P(Z \le 1)$$

$$= 1 - 0.84134 = 0.15866$$

$$\therefore P(XY \ge 0, X \le 0, Y \le -\sqrt{2}) = 0.5 \times 0.15866$$
$$= 0.07933$$

$$P(XY \ge 0, X \le 1, Y^2 \ge 2) = 0.05416 + 0.07933$$
$$= 0.13349 \quad] \text{ Numerator}$$

$$P(X \le 1, Y^2 \ge 2) \quad] \text{ Denominator}$$

$$P(X \le 1) = 0.84134$$

$$P(|Y| \ge \sqrt{2}) = P(Y \ge \sqrt{2}) \text{ or } P(Y \le -\sqrt{2})$$

$$= P(Z \ge 1) \text{ or } P(Z \le -1)$$

$$= 0.15866 + 0.15866$$

$$= 0.31732$$

$$\therefore P(X \le 1, Y^2 \ge 2) = 0.84134 \times 0.31732$$
$$= 0.26697$$

$$\therefore P\left(B_1(B_3 - B_1) \ge 0 \mid B_1 \le 1, (B_3 - B_1)^2 \ge 2\right)$$

$$= \frac{0.13349}{0.26697} \approx 0.5$$

Exercise 2

$$M_t = e^{\lambda B_t - (\lambda^2/2)t}$$

1) $E[M_t / F_s]$

$$= E\left[e^{\lambda B_t - (\lambda^2/2)t} / F_s\right]$$

$$= E\left[e^{\lambda (B_s + (B_t - B_s)) - (\lambda^2/2)t} / F_s\right]$$

$$= e^{\lambda B_s - \left(\frac{\lambda^2}{2}\right)t} E\left[e^{\lambda (B_t - B_s)} / F_s\right]$$

$B_t - B_s \sim N(0, t-s)$

The moment generating function of a normal random variable is

$E[e^{tx}] = e^{t^2/2 \, Var(x)}$ where $X \sim N(0, Var)$

$$E\left[e^{\lambda (B_t - B_s)}\right] = e^{\lambda^2/2 \, (t-s)}$$

$$= e^{\lambda B_s - \left(\frac{\lambda^2}{2}\right)t} e^{\frac{\lambda^2}{2}(t-s)}$$

$$= e^{\lambda B_s - \left(\frac{\lambda^2}{2}\right)t + \frac{\lambda^2}{2}t - \frac{\lambda^2}{2}s}$$

$$= e^{\lambda B_s - \frac{\lambda^2}{2}s} = M_s$$

2) $E[M_7] = E\left[e^{\lambda B_7 - 7(\lambda^2/2)}\right]$

$$= E\left[e^{\lambda B_7}\right] e^{-7/2 \, \lambda^2}$$

$$= e^{\frac{\lambda^2}{2} \times 7} e^{-7/2 \, \lambda^2}$$

$$= 1$$

Also, since M_t is a martingale

$E[M_7] = M_0 = 1$

$$E[M_{43} - 2M_9] = E[M_{43}] - 2E[M_9]$$

$$= 1 - 2$$

$$= -1$$

Exercise 3

$B_t \to$ Standard Brownian Motion

$s<t$

1) $E[B_t^2/F_s]$

$= E[(B_s + (B_t - B_s))^2/F_s]$

$= E[B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2/F_s]$

$= E[B_s^2/F_s] + 2E[B_s(B_t - B_s)/F_s] + E[(B_t - B_s)^2/F_s]$

B_s & $B_t - B_s$ are independent

$B_s \sim N(0, s)$

$B_t - B_s \sim N(0, t-s)$ $\therefore E[(B_t - B_s)/F_s] = 0$

$Var(B_t - B_s) = E[(B_t - B_s)^2] - E[(B_t - B_s)]^2$

$t-s = E[(B_t - B_s)^2]$

$\therefore E[B_t^2/F_s] = B_s^2 + t - s$

2) $E[B_t^3/F_s]$

$= E[[B_s + (B_t - B_s)]^3/F_s]$

$= E[B_s^3 + (B_t - B_s)^3 + 3B_s^2(B_t - B_s) + 3B_s(B_t - B_s)^2/F_s]$

$= B_s^3 + E[(B_t - B_s)^3/F_s] + 3E(B_s^2/F_s)E(B_t - B_s/F_s)$
$+ 3E(B_s/F_s)E[(B_t - B_s)^2/F_s]$

$E[(B_t - B_s)^3/F_s] = 0$ because $B_t - B_s \sim N(0, t)$ which is a symmetric distribution & odd moments would be 0

$= B_s^3 + 0 + 0 + 3B_s(t-s)$

$= B_s^3 + 3B_s(t-s)$

3) $E[B_t^4 / F_s]$

$= E[(B_s + (B_t - B_s))^4 / F_s]$

According to binomial theorem

$(x+y)^n = \sum_{k=0}^{n} {}^nC_k x^{n-k} y^k$

$= E[(B_s^4 + 4B_s^3(B_t - B_s) + 6B_s^2(B_t - B_s)^2 + 4B_s(B_t - B_s)^3 + (B_t - B_s)^4) / F_s]$

$= E[B_s^4 / F_s] + 4E[B_s^3(B_t - B_s) / F_s] + 6E[B_s^2(B_t - B_s)^2 / F_s] +$

$4[B_s(B_t - B_s)^3 / F_s] + E[(B_t - B_s)^4 / F_s]$

$= B_s^4 + 0 + 6B_s^2(t-s) + 0 + E[(B_t - B_s)^4 / F_s]$

$B_t - B_s \sim N(0, t-s)$

If $X \sim N(0, \sigma^2)$

$E[X^k] = \begin{cases} (k-1)(k-3)(k-5) \cdots (1)(\sigma^2)^{k/2} + \cdots, & k \text{ is even} \\ 0, & k \text{ is odd} \end{cases}$

$\therefore E[(B_t - B_s)^4 / F_s] = 3 \times 1 \times (t-s)^{4/2}$

$= 3(t-s)^2$

$= B_s^4 + 6B_s^2(t-s) + 3(t-s)^2$

4) $E[e^{4B_t - 2} / F_s]$

$= E[e^{4(B_s + (B_t - B_s)) - 2} / F_s]$

$= E[e^{4B_s - 2} / F_s] \, E[e^{4(B_t - B_s)} / F_s]$

$4(B_t - B_s) \sim N(0, 16(t-s))$

The moment generating function of a normal random variable is

$E[e^{tx}] = e^{t^2/2 \, Var(x)}$ where $X \sim N(0, Var)$

$E[e^{4(B_t - B_s)}] = e^{8(t-s)}$

$= e^{4B_s - 2} e^{8(t-s)} = e^{4B_s + 8(t-s) - 2}$

Exercise 4

B_t is a standard brownian motion & $a>0$

$$Y_t = \frac{1}{\sqrt{a}} B_{at}$$

Property 1

$$Y_0 = \frac{1}{\sqrt{a}} B_{a\times 0} = \frac{1}{\sqrt{a}} B_0 = B_0 = 0$$

Property 2

Independent Increments

$$Y_t - Y_s = \frac{1}{\sqrt{a}} B_{at} - \frac{1}{\sqrt{a}} B_{as}$$

$$= \frac{1}{\sqrt{a}} \left(B_{at} - B_{as} \right)$$

$B_{at} - B_{as}$ are independent of the past values of B_t

Thus Y_t has independent increments

Property 3

Stationary Increments

$$Y_t - Y_s = \frac{1}{\sqrt{a}} B_{at} - \frac{1}{\sqrt{a}} B_{as}$$

$$= \frac{1}{\sqrt{a}} \left[B_{at} - B_{as} \right]$$

$$= \frac{1}{\sqrt{a}} \left[B_{a(t-s)} \right]$$

[Since B_t is standard brownian & follows the property of independent increments]

$$= Y_{t-s}$$

$\therefore$ Y_t also has stationary increments

Property 4

Continuous Paths

Since B_t has a continuous path, B_{at} will also have a continuous path

Y_t is scaled version of B_{at}

$\therefore$ Y_t is also continuous.

Exercise 5

1) $P\left(\min\limits_{0\le t\le 3} B_t < -\frac{1}{2}\right)$

$$= 2P\left(B_3 < -\tfrac{1}{2}\right)$$ [using reflection principle]

$$= 2P\left(Z < \frac{-0.5-0}{\sqrt{3}}\right)$$

$= 2P(Z < -0.2887)$

$= 2 \times 0.386415$

$= 0.77283$

2) $P(B_{1.5} > 0, B_3 < 0)$

$B_{1.5}$ & B_3 are not independent

$Cov(B_{1.5}, B_3) = E[B_{1.5} B_3] = \min(1.5, 3)$

$= 1.5$

$$\text{Correlation} = \frac{Cov(B_{1.5}, B_3)}{\sqrt{Var(B_{1.5}) Var(B_3)}}$$

$$= \frac{1.5}{\sqrt{1.5 \times 3}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$B_{1.5} \sim N(0, 1.5)$, $B_3 \sim N(0, 3)$

Let $X = \frac{B_{1.5}}{\sqrt{1.5}}$ & $Y = \frac{B_3}{\sqrt{3}}$

Now X & Y follow standard normal dist

$$P(X > 0, Y > 0) = \frac{1}{4} + \frac{1}{2\pi} \times \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{4} + \frac{1}{2\pi} \times \frac{\pi}{4} = \frac{1}{4} + \frac{1}{8}$$

$= 0.375$

$P(X > 0) = 0.5$

$\therefore P(X > 0, Y < 0) = 0.5 - 0.375$

$= 0.125$

Intuition: We know $P(B_{1.5} > 0) = 0.5$
since the correlation of $B_{1.5}$ & B_3 is high (≈ 0.7),
the event $(B_{1.5} > 0$ & $B_3 < 0)$ is less likely than
if they were uncorrelated.

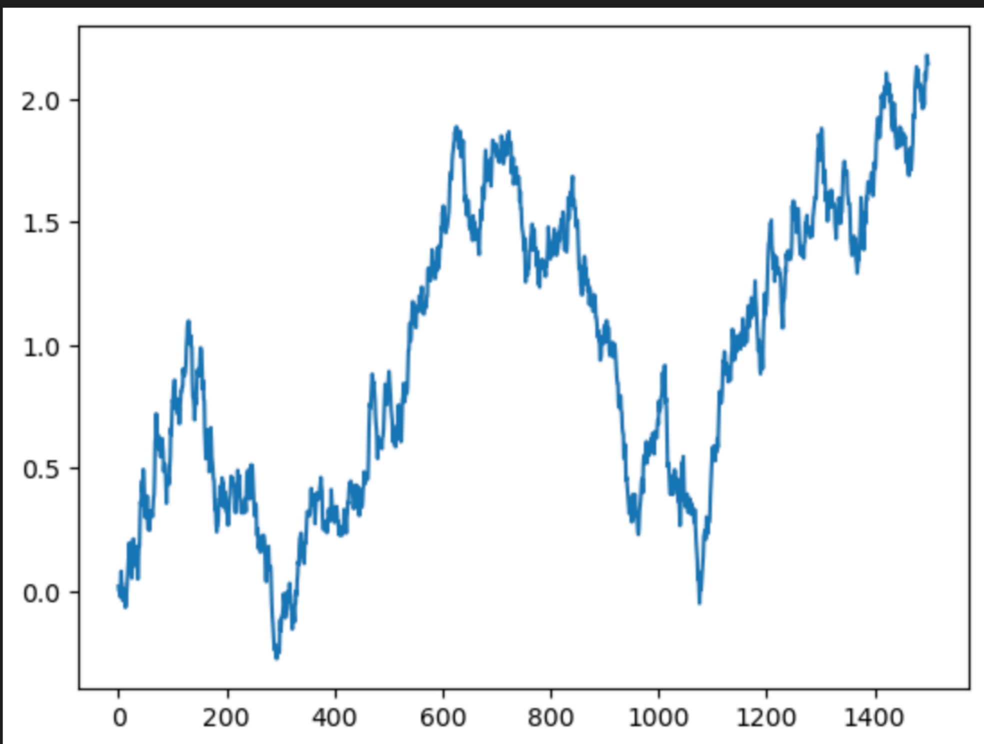
```python
delta_t = 1/500
max_t = 3

Bt = 0
t = delta_t
Bt_list = []

while t <= max_t:
    Bt = Bt + np.sqrt(delta_t) * norm.rvs()
    Bt_list.append(Bt)
    t += delta_t

plt.plot(Bt_list)
```

```
[<matplotlib.lines.Line2D at 0x166b3ead0>]
```



P(min Bt < -0.5) for t = [0,3]

```
counter = 0
for i in range(1000):
    Bt = 0
    t = delta_t
    Bt_list = []
    while t <= max_t:
        Bt = Bt + np.sqrt(delta_t) * norm.rvs()
        Bt_list.append(Bt)
        t += delta_t
    if (min(Bt_list) < -0.5) == True:
        counter += 1

counter/1000
```

```
0.76
```

P(B1.5 > 0, B3 < 0)

```
counter = 0
for i in range(1000):
    Bt = 0
    t = delta_t
    Bt_list = []
    while t <= max_t:
        Bt = Bt + np.sqrt(delta_t) * norm.rvs()
        Bt_list.append(Bt)
        t += delta_t
    if ((Bt_list[750] > 0) and (Bt_list[-1] < 0)) == True:
        counter += 1

counter/1000
```

```
0.131
```