FINM 34500/STAT 39000
Problem Set 5 (due February 10 with extension to February 12)
Reading: Notes through Chapter 3 (remember: indented parts are optional reading)

EXAM in-class on Monday, February 10. Includes material in lectures through February 3 and Problem Sets 1-5. Even though you have until Wednesday, February 12 to hand in Problem 5, the material will be included on the Monday exam. The exam is closed book with no electronic equipment, but you may bring one standard sheet (two-sided) page of notes that you have created yourself (no copied material).

Exercise 1 *Suppose B_t is a standard Brownian motion and X_t satisfies*

$$dX_t = 3X_t\,dt - 2\sqrt{X_t}\,dB_t, \quad X_0 = 1.$$

For each of the following find A_t, C_t such that

$$d\langle Y\rangle_t = A_t\,dt, \quad d\langle Y, X\rangle_t = C_t\,dt.$$

1. $Y_t = B_t^3 + 2t^2$.
2. $Y_t = X_t + \int_0^t X_s^3\,ds$.
3. $$Y_t = X_t^3 + \exp\left\{2\int_0^t X_s^2\,ds\right\}.$$

Exercise 2 *Suppose that $B_t = (B_t^1, B_t^2)$ is a standard two-dimensional Brownian motion and X_t, Y_t satisfy*

$$dX_t = X_t\,\left[m_1(B_t)\,dt + \sigma_1(B_t)\,dB_t^1 + \rho_1(B_t)\,dB_t^2\right].$$

$$dY_t = Y_t\,\left[m_2(B_t)\,dt + \sigma_2(B_t)\,dB_t^1 + \rho_2(B_t)\,dB_t^2\right].$$

Here $m_1, m_2, \sigma_1, \sigma_2, \rho_1, \rho_2$ are functions from $\mathbb{R}^2$ to $\mathbb{R}$. Write out in detail the SDE satisfied by the following. The answers should be in the form

$$dZ_t = [\textit{something}]\,dt + [\textit{something}]\,dB_t^1 + [\textit{something}]\,dB_t^2.$$

1. $Z_t = X_t^3$
2. $Z_t = X_t Y_t$

Exercise 3 *Suppose B_t^1, B_t^2 are independent standard (one-dimensional) Brownian motions and let*

$$X_t = B_t^1 + B_t^2, \qquad Y_t = 2\,B_t^1 - B_t^2.$$

Find the following differentials. The answers should be in the form

$$dZ_t = [\textit{something}]\,dt + [\textit{something}]\,dB_t^1 + [\textit{something}]\,dB_t^2.$$

1. $Z_t = e^{X_t+Y_t}$.

2. $Z_t = \int_0^t e^{X_s+Y_s}\, ds$

3. $Z_t = \exp\{tX_t\}$.

Exercise 4 *In the following X_t satisfies a particular stochastic differential equation with $X_0 = 1$. In each case, find a function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ such that if $M_t = f(X_t)$, then M_t is a martingale.*

1. $dX_t = dt + dB_t$

2. $dX_t = X_t\, dt + dB_t$

3. $dX_t = \frac{1}{X_t}\, dt + dB_t$. *On this one let $f : (0, \infty) \to \mathbb{R}$ with $f(1) = 0$. (We will find out later that this process will never reach the origin so do not worry about what happens when $X_t = 0$.)*

(Hint: use Itô's formula to see what differential equation f has to satisfy. This equation should be in terms of f' and f'' and hence can be considered a first-order differential equation for the function $g := f'$. You can write f as an antiderivative of g.)