Abstract

Search cost is one of the reasons why we do not observe perfect competition in real-life markets. To study the market with this condition, the paper presents a search model with heterogeneous search costs in which consumers look for the best match utility assuming the same equilibrium price. We derive formulas for the demand and first order condition on the equilibrium price. We compute prices for some specific cases and provide the proof and necessary condition on a utility function that prices in our model for two shops are higher than prices in a model with the same search costs for both shops.

Contents

1 Introduction

Different approaches were used to understand why firms put these very prices. Everyone has heard that one firm should put the monopolistic price, while if there are just two firms in a market, they put price equal to their costs — so-called Bertrand paradox (Joseph Bertrand's model, 1883). Economists have different explanations why in the real world prices are higher, and one of such explanation is that people have to pay to observe prices and match utilities at different shops.

One of the first models that introduced consumer search was Diamond [1] model in 1971 which gave a name to the "Diamond's paradox." He assumed that there are some firms in a market and a large number of consumers who search for the best option. In his model match utilities were constant for each of the pair consumer-shop, so a consumer looked for a better price paying a constant s for visiting each extra shop. However, in this formulation of a problem, it turned out that all firms should set the monopolistic price, while the best strategy for consumers is not to search at all. An informal explanation for this is the following: in the equilibrium (which are supposed to be symmetric in pure strategies for this result) all firms set a price p [1]. What happens if you come to the first shop and observe the price $p + s - \epsilon$? Despite the fact you have expected another price there is no sense in continuing searching.

In 1986 Wolinsky [4] presented a model based on Perloff-Salop [3] model. The Perloff-Salop model assumes that a consumer knows all prices in advance but has different preferences for different firms. Firms know a distribution of preferences and therefore try to maximize their surplus. In most cases, the distribution of preferences is supposed to be symmetric and independent over all shops.

Wolinsky wants to study Chamberlinian monopolistic competition [2]. In Perloff-Salop there was no locality - everyone knew all prices. There were some attempts to make their model truly monopolistic, by, for instance, assuming that a person may choose only among some of the firms (Hart, 1984-1985) - however, it seems unrealistic that a person would not buy the same product from a different firm with a lower price. Therefore, Wolinsky proposes another model based on Perloff-Salop model: he assumes that a consumer does not know the prices and has to pay to know a utility and a price of a firm. Even increasing number of firms to infinity does not move the price to competitive price - it has an asymptote depending on competitive price and a fixed search cost. In fact, in the equilibrium, a consumer is likely to visit only several firms and then leave the market, which means that despite large number of firms, each of them competes only with several others. Every next model we would discuss is built on a Wolinsky's model.

In 1999 Anderson and Renault [5] developed and examined the Wolinsky model. The main properties of Wolinsky model are that prices fall as search costs fall and decrease as the number of firm increases. Moreover, prices fall as the diversity in utilities or preferences rises. The explanation for the last fact is as follows: with greater diversity consumers search more, therefore involving more firms into competition with each other. This result may seem obvious, but in the same model without search costs, the equilibrium price rises with bigger diversity because of the distribution of the maximum of utility shifts

[1] To be more precise something between $p - s$ and p.

[2] "Which satisfies, more or less, the following four properties: (1) there are many firms producing differentiated commodities; (2) each firm is negligible in the sense that it can ignore its impact on, and hence reactions from, other firms; (3) each firm faces a downward-sloping demand curve and hence the equilibrium price exceeds marginal cost; (4) free entry results in zero-profit of operating firms." Wolinsky, 1986 [4]

to the right.

For instance, Anderson and Renault examination of the Wolinsky's model gives the explanation why in touristic streets a large number of shops has practically the same price as one shop with souvenirs: in fact, consumers have a little diversity in utilities and very high search costs compared to utilities. So tourist visits just several shops, observes the same high prices and utilities everywhere and being on vacation he has a low intensity to search further.

There were some other deviations from the original model, such as considering other equilibrium. Zhou [8] showed that while searching in the street in the same order, the equilibrium is ascending sequence of prices. The idea behind this fact is that if a person visits shops in the middle of the standard route, he has already visited the previous shops and match utilities were low for him. Therefore shops at the end of the route feel more monopolistic to this particular consumer. One more time, in a model with the same utilities, when consumers search only for prices, there is a descending sequence of prices as shown by Arbatskaya [7]. Some other authors introduced advertising.

Finally, Moraga-Gonzalezy, Sandorz and Wildenbeestx (2017) [10] study the model when there are different search costs - there is a fixed search cost for each consumer that comes from a distribution function. In brief, in real markets some consumers value their time more and therefore tend to search less or not to participate in the market at all, therefore now firms have greater participation in the market. They examine the influence of lowering search costs in the market. It seems that lower search costs (for instance, using the Internet) must imply lower prices due to the reasons discussed above, but some examples show that sometimes firms likely to raise prices - it happens because new people participate in market and that new population who did not search before may lead to lower overall elasticity of demand. They prove that for search cost densities with the decreasing likelihood ratio property [3], a decrease in search costs in the sense of first [4] or second [5] order stochastic dominance results in a higher equilibrium price.

In this paper, we introduce a model with heterogeneous search costs. For instance, in the case of ordinary shops in a city a consumer has different search costs for different shops and it has two effects: on the one hand, the lowest search costs are usually much smaller than in the same model by Moraga-Gonzalezy, Sandorz, and Wildenbeestx with the same cost distribution. Moreover, more people participate in the market as usually there at least several shops with low search costs. However, on the other hand, each shop feels more monopolistic due to the fact you have to pay more to visit each extra shop. In our model there is also a returning demand which means that you may visit 5 shops, decide that it is not worth searching anymore and return to the third shop (in a previous model there were infinite number of shops and the optimal strategy was to visit several shops and to buy at the last one). As opposed to Zhou, we are looking for a symmetric equilibrium as each consumer is searching in a specific order. We want to derive whether or not prices are lower when consumers have different search costs for each shop.

Section 2 presents a beautiful theorem by Weitzman, which explains how a person must search to get the highest profit. There is also a new elegant proof of the Weitzman's result. Section 3 introduces the model and optimal rule for consumers using Weitzman's

[3]The density $g(c;b)$ has the increasing likelihood ratio property iff for any $b' < b$, $g(c;b)g(d,b') \leq g(c,b')g(d,b)$ for any $c < d$ in the union of the supports of $g(c,b')$ and $g(c,b)$.

[4]F dominates G in the sense of first order if $F\,(x) \geq G(x)$ and for some x the inequality is strong.

[5]F dominates G in the sense of second order if $\mathbb{E}_{F(u(x))} \geq \mathbb{E}_{G(u(x))}$ for any concave and nondecreasing utility function $u(x)$ and for some x the inequality is strong.

rule. In section 4 we calculate the demand in a particular case with two shops, while section 5 extends this analysis to a general case. In section 6 there are some examples of equilibrium prices under uniform distribution of utility function. Section 7 provides a proof for the result that in a case with two shops the equilibrium price is higher. Results are summed up in section 8.

2 Solution to Pandora's problem

This section briefly provides the main result of Weitzman, 1979 [2] with an example from the same article.

Suppose there are two technologies you may develop, but at a time you can use only one of them. The surplus of the first one is 100 or 55 with equal probabilities, and the cost is 15. The surplus of the second one is 240 with probability 0.2 and 0 with probability 0.8, and the cost for investigation is 40.

Obviously, if a person has time for developing only one technology, he should choose the first one.

However, if a person has time to investigate both of them it occurs that the best strategy is first to investigate the second one and only next, in a case of fail, examine the first one.

Let us provide the general problem:

There are n different boxes, in each of them there is a reward x_i with the probability function $F_i(x_i)$. You may open i box for price c_i. You can collect no more than one reward. Your strategy may depend on rewards you have already gained.

Suppose you already have some value z_i. The net benefit from opening one more box is $-c_i + z_i \int\limits_{-\infty}^{z_i} \mathrm{d}F_i(x_i) + \int\limits_{z_i}^{+\infty} x_i \,\mathrm{d}F_i(x_i)$. A person is indifferent in searching under the following condition:

$$c_i = \int\limits_{z_i}^{+\infty} (x_i - z_i)\,\mathrm{d}F_i(x_i)$$

Let us call the solution of the above equation z_i the reservation price. Note, that is just the solution to $\mathbb{E}\ \max(x_i - z_i, 0) = c_i$. The good heuristics is to calculate reservation prices for all boxes and search in a descending order while it is profitable. More formal: let $v_i = \max(0, x_1, x_2, \ldots, x_i)$. Therefore, a person should search until current v_i is bigger than the biggest of left reservation prices.

Theorem 1 (Wolinsky)**.** *The above-formulated rule is the optimal strategy.*

The original proof is quite technical; however, there is a very short proof by Armstrong [9].

Proof.

Easy to see that the Wolinsky optimal rule is equal to choosing the box with greatest $w_i = \min(z_i, x_i)$ (w_i is equal to z_i until it is opened and then it equals to $\min(z_i, x_i)$ where x_i is a realization of random variable).

Let $\mathbb{A}_i$ be the indicator for selecting the box, and $\mathbb{I}_i$ be the indicator for inspecting the box. From the definition $\mathbb{A}_i \le \mathbb{I}_i$ and $\sum_i A_i \le 1$ because according to the rules a person could choose no more than 1 box.

Suppose there is a strategy of searching, than the expected surplus is:

$$\mathbb{E}\left(\sum_i \mathbb{A}_i x_i - \sum_i \mathbb{I}_i c_i\right) = \mathbb{E}\left(\sum_i \mathbb{A}_i x_i - \sum_i \mathbb{I}_i \cdot \max(x_i - z_i, 0)\right) \leq$$
$$\leq \mathbb{E}\left(\sum_i \mathbb{A}_i \left[x_i - \max(x_i - z_i, 0)\right]\right) = \mathbb{E}\left(\sum_i \mathbb{A}_i w_i\right) \leq \mathbb{E}\left(\max(w_i)\right)$$

We first have used the definition of z_i, then the fact that $\mathbb{A}_i \leq \mathbb{I}_i$, the the definition of w_i, the the fact that $\sum_i A_i \leq 1$.

However, in Pandora's rule we have two equalities: if $\mathbb{A}_i < \mathbb{I}_i$ then $x_i < z_i$ according to stopping rule, therefore we have $\mathbb{A}x - \mathbb{I} \cdot c = \mathbb{A}x - \mathbb{I} \cdot \max(x - z, 0) = \mathbb{A}x$; and for the last inequality it is Weitzman's rule that tells us to choose the biggest w_i. □

3 Model

We build our model Moraga-Gonzalezy, Sandorz and Wildenbeest's model which is built on Wolisky's (1986) model of consumer search for differentiated products; we assume that a person has unique search cost for each firm.

There are n firms and a large number of consumers with measure of one. Each consumer has tastes described by a utility function of the form

$$u_i(p_i) = \mu_i - p_i$$

if she buys product i at price p_i. Let μ_i be the realization of a random variable with distribution F and a continuously differentiable density f whose support is an interval $[\underline{\mu}, \overline{\mu}]$ of the extended real line. Realizations of random variable are independently distributed across consumers and products.

Consumers differ in their costs of search for each of the firm: c_{il} is search cost for consumer l for firm i. We will usually drop index l. Random realization c_i is drawn independently from a differentiable cumulative distribution function G with support $[\underline{c}, \overline{c}]$. Let g be the density of G.

Each consumer and each shop wants to maximize expected surplus. Firms compete in the market by simultaneously choosing their prices. We are looking as symmetric [6] Nash equilibrium (SNE), assuming it exists [7] and unique [8] — condition for it may be quite complicated [9], and for now we do not derive them.

3.1 Optimal stopping rule

Applying Pandora's rule and taking into account that all utility distribution function are equal we receive that all consumers search in order of increasing search costs.

Let $\hat{x}(c)$ is the solution of $h(x) = \int\limits_x^{\overline{\mu}} (\mu - x)\,\mathrm{d}F(\mu) = c$. $\hat{x}(c)$ is the reservation price for current search cost. It is well-known that $\hat{x}(c)$ is well-defined and uniquely defined for

[6] There exist many non-symmetric equilibria. For instance, it is enough for consumers to believe that a shop puts very high price and never visit it. In this case, that shop may put any price it wants.

[7] An SNE always exists in symmetric games, but it may be in mixed strategies. It seems reasonable to assume here that the SNE is a pure strategy here.

[8] Function of expected payoff for a firm is unimodal function in an equilibrium price, which is quite easy to believe, too.

[9] Usually it is enough to have log-concave $1 - F(x)$, but there is no general proof for all models.

$c \in \left[\underline{c}, \min(\overline{c}, \mathbb{E}(\mu)\right]$. Moreover, $\hat{x}\,(c)$ is a descreasing and convex function on its support:

$$\hat{x}'(c) = -\frac{1}{1 - F\left(\hat{x}\,(c)\right)} < 0$$
$$\hat{x}''(c) = \frac{f(\hat{x}\,(c)) \cdot (\hat{x}'(c))^2}{1 - F\left(\hat{x}\,(c)\right)} < 0$$

Sometimes we will need the inverse function of $\hat{x}\,(c)$ which we denote as $\tilde{x}(p)$). $\tilde{x}(p^*)$) is such a critical search cost that it is not worth searching anymore.

Let p^* be an SNE price. All consumers expect all shops to set the same equilibrium price.

If consumer has already visited k firms and the maximum available surplus so far $v_k := \max(0, u_1, \ldots, u_k)$. If $\mathbb{E}[\max(u_{k+1}, v_k)] - c_{(k+1)} \geq v_k$ then the consumer is going to search one more time; otherwise, she will leave the market with the greatest payoff she has found (maybe without buying at all).

Our plan is to consider a shop that deviates from an equilibrium and calculate his expected payoff. Then we maximize his payoff and find optimal price p when all other shops set price p^*. Obviously, in the equilibrium $p = p^*$ which gives us the desired equation.

4 2 shops

4.1 Demand and first order condition

Consider the case with 2 shops. Let the shops have prices $p \neq p^*$ and p^* respectively and c_1, c_2 be the search costs for a certain person.

At first, we calculate the demand for the first shop.

We will consider two cases - when a consumer may visit both shops according to his search costs and when may visit only one.

Let a consumer may visit only one firm. It means that $c_{(1)} < \tilde{x}(p^*) < c_{(2)}$. Obviously, it happens with probability $2G(\tilde{x}(p^*))[1 - G(\tilde{x}(p^*))]$. Moreover, here we have conditional density, so we have to divide it by $G(\tilde{x}(p^*))$ and we have to multiply by $\frac{1}{2}$ which means that a consumer will visit first shop. The expected demand is equal to :

$$2G(\tilde{x}(p^*))\left[1 - G(\tilde{x}(p^*))\right] \cdot \frac{1}{2} \cdot \int\limits_{c<\tilde{x}(p^*)} \frac{1 - F\,(p)}{G(\tilde{x}(p^*))}\,\mathrm{d}G(c) =$$
$$= G(\tilde{x}(p^*))\left[1 - G(\tilde{x}(p^*))\right]\left[1 - F\,(p)\,\right]$$

If both search costs are less than critical level $\tilde{x}(p^*)$, there is a competition among shops. On the one hand, we have to multiply everything above by $G^2(\tilde{x}(p^*))$, but conditional distribution will "kill" it later.

Let $c_1 < c_2$ so the person will firstly visit the first shop. We have 2 cases here.

First one - u_1 is very big so there is no profit in visiting the other shop. It happens with probability

$$\mathbb{P}\Big[u_1 > \hat{x}\,(c_2) - p^*\Big] = 1 - F(\hat{x}\,(c_2) - p^* + p)$$

The second one - it is worth searching one more time. However, it turns out that $u_1 > u_2$ so she returns to the first shop after visiting the second. It happens with probability

$$\mathbb{P}\Big[\mu_2 - p^* < \mu_1 - p < \hat{x}\left(c_2\right) - p^*\Big]$$

Fixing μ_1 at first and then integrating over it from p to $\hat{x}\left(c_2\right) - p^* + p$ we receive the following formula:

$$\int\limits_{p}^{\hat{x}(c_2)-p^*+p} F\Big(\mu - p + p^*\Big)\,\mathrm{d}F\left(\mu\right) = \int\limits_{p^*}^{\hat{x}(c_2)} F\left(\mu\right) f(\mu - p + p^*)\,\mathrm{d}\mu$$

Finally, we have

$$d_1(p,p^*) = \Big(1 - F(\hat{x}\left(c_2\right) - p^* + p)\Big) + \int\limits_{p^*}^{\hat{x}(c_2)} F\left(\mu\right) f(\mu - p + p^*) d\mu$$

Otherwise $c_2 < c_1$ so consumer visits the second shops first. She buys product at first shop if she continues searching and it turns out that $u_1 > u_2$. It happens with probability

$$\begin{aligned}
&Prob\Big(\mu_2 - p^* < \min(\hat{x}(c_1) - p^*, \mu_1 - p)\Big) = \\
&\qquad = Prob\Big(u_2 < \hat{x}(c_1) - p^* < u_1\Big) + Prob\Big(u_2 < u_1 < \hat{x}(c_1) - p^*\Big) = \\
&\qquad = F\left(\hat{x}\left(c_1\right)\right)\Big[1 - F\left(\hat{x}\left(c_1\right) + p - p^*\right)\Big] + \int\limits_{p_1}^{\hat{x}(c_1)-p^*+p} F\Big(\mu - p + p^*\Big)\,\mathrm{d}F\left(\mu\right) = \\
&\qquad = F\left(\hat{x}\left(c_1\right)\right)\Big[1 - F\left(\hat{x}\left(c_1\right) + p - p^*\right)\Big] + \int\limits_{p^*}^{\hat{x}(c_1)} F\left(\mu\right) f(\mu - p + p^*)\,\mathrm{d}\mu
\end{aligned}$$

We receive

$$d_2(p,p^*) = F\left(\hat{x}\left(c_1\right)\right)\Big[1 - F\left(\hat{x}\left(c_1\right) + p - p^*\right)\Big] + \int\limits_{p^*}^{\hat{x}(c_1)} F\left(\mu\right) f(\mu - p + p^*)\,\mathrm{d}\mu$$

Here we integrate over all people who participate in market and add the first part.

$$\begin{aligned}
d(p_1,p^*) = \frac{1}{2}\cdot 2 \cdot \iint\limits_{c_1<c_2<\tilde{x}(p^*)} &\Big(d_1(p,p^*) + d_2(p,p^*)\Big)\,\mathrm{d}G(c_1)\,\mathrm{d}G(c_2) + \\
&+ G(\tilde{x}(p^*))\big[1 - G(\tilde{x}(p^*))\big]\big[1 - F\left(p\right)\big]
\end{aligned}$$

The expected payoff is (assuming that cost of production is 0)

$$\pi(p,p^*) = p \cdot d(p,p^*)$$

Taking the FOC gives (we are using Leibniz integral rule and change $\hat{x}(c_{(2)})$ to $\hat{x}$ for convenience):

$$0 = -p \iint\limits_{c_1<c_2<\tilde{x}(p^*)} \left((1+F(\hat{x}))f(\hat{x}+p-p^*)+2\int\limits_{p^*}^{\hat{x}} F(\mu)\, f'(\mu-p+p^*)\,\mathrm{d}\mu\right)\mathrm{d}G(c_2)\,\mathrm{d}G(c_1)+$$
$$+\, d(p,p^*) - p\cdot G(\tilde{x}(p^*))\big[1-G(\tilde{x}(p^*))\big]f(p)$$

Using that p^* is a NE substituting $p = p^*$ must give us equality:

$$\iint\limits_{c_1<c_2<\tilde{x}(p^*)} \left(1-F^2(\hat{x})+2\int\limits_{p^*}^{\hat{x}} F(\mu)\,\mathrm{d}F(\mu)\right)\mathrm{d}G(c_1)\,\mathrm{d}G(c_2)-$$
$$-\,2p^* \iint\limits_{c_1<c_2<\tilde{x}(p^*)} \left(\frac{1}{2}(1+F(\hat{x}))f(\hat{x})+\int\limits_{p^*}^{\hat{x}} F(\mu)\, f'(\mu)d\mu\right)\mathrm{d}G(c_1)\,\mathrm{d}G(c_2)+$$
$$+\, G(\tilde{x}(p^*))\big[1-G(\tilde{x}(p^*))\big]\big[1-F(p^*)-p^*\cdot f(p^*)\big] =$$
$$= \iint\limits_{A(p^*)} \left(1-F^2(p^*)-p^*(1+F(\hat{x}))f(\hat{x})-2p^*\int\limits_{p^*}^{\hat{x}} F(\mu)\, f'(\mu)\,\mathrm{d}\mu\right)\mathrm{d}G(c_2)\,\mathrm{d}G(c_1)+$$
$$+\, G(\tilde{x}(p^*))\big[1-G(\tilde{x}(p^*))\big]\big[1-F(p^*)-p^*\cdot f(p^*)\big] = 0$$

4.2 Consumer Surplus

The original formula for surplus from appendix (the proof was omitted):

$$\Big(1+F(\hat{x}(c))\Big)\Big[\int\limits_{\hat{x}(c)}^{\overline{\mu}} (\mu-p^*)\,\mathrm{d}F(\mu) - c\Big] + \int\limits_{p^*}^{\hat{x}(c)} (\mu-p^*)\,\mathrm{d}F^2(\mu)$$

Proof.

Here we work under the assumption that $\hat{x}(c_{(2)}) > p^*$.

Counting $F(p^*) - F(0) = q_1$, $F(\hat{x}) - F(p^*) = q_2$ and $F(\overline{\mu}) - F(\hat{x}) = q_3$, we have the following:

The person spends c with probability q_3 and $2c$ otherwise. We receive $-\Big(1+F(\hat{x}(c))\Big)c$ as in original formula. (In our case with different search costs it would be c_1 with probability q_3 and $c_1 + c_2$ otherwise. That is the only change, not counting $\hat{x}(c_2)$ instead of $\hat{x}(c)$.)

The person buys the product unless she both times received utility less than p^*. So we have $-\Big(1-F^2(p^*)\Big)p^*$. Easy to check that it equals the same part from the original formula.

With probability q_3 the person stops after the first search. The surplus is $\int\limits_{\hat{x}}^{\overline{\mu}} \mu\,\mathrm{d}F(\mu)$.

It may also happen that at first search the person falls in case $q_1 + q_2 = F(\hat{x})$ and then in case q_3. The surplus is $F(\hat{x}(c)) \int\limits_{\hat{x}}^{\overline{\mu}} \mu\,\mathrm{d}F(\mu)$. So we have received the first part of the original formula.

The last possibility is when person fell in case q_1 or q_2 for both times. Then the surplus is integrate over maximum of two i.i.d., but starting from p^* which equals to $\int\limits_{p^*}^{\hat{x}} \mu \, \mathrm{d}F^2(\mu)$.

So the formula for two shops with different search costs is

$$S_2(p^*) = \Big(1 + F(\hat{x}(c_{(2)}))\Big)\Big[\int\limits_{\hat{x}(c_{(2)})}^{\overline{\mu}} \mu \, \mathrm{d}F\,(\mu)\Big] + \int\limits_{p^*}^{\hat{x}(c_{(2)})} \mu \, \mathrm{d}F^2(\mu) - $$
$$- p^*(1 - F^2(p^*)) - \Big(c_{(1)} + c_{(2)} \cdot F(\hat{x}(c_{(2)}))\Big)$$

Setting $c_1 = c_2 = c$ we receive the original formula. $\square$

To calculate common surplus we have consider another case. If person may visit only one shop, the formula is

$$S_1(p^*) = \int\limits_{p^*}^{\overline{\mu}} \mu \, \mathrm{d}F\,(\mu) - p^*(1 - F\,(p^*)) - c_{(1)}$$

So, common surplus is equal to

$$2\big[1 - G(\tilde{x}(p^*))\big] \cdot \int\limits_{0}^{\tilde{x}(p^*)} S_1(p^*, c) \, \mathrm{d}G(c) + 2 \iint\limits_{c_1 < c_2 < \tilde{x}(p^*)} S_2(p^*, c_1, c_2) \, \mathrm{d}G(c_1) \, \mathrm{d}G(c_2)$$

5 Demand for n shops

5.1 Low search costs

Consider the case with n shops. At first, we calculate the demand for the i shop. Our plan is to first calculate a demand for a particular consumer with a given sequence of search costs and then integrate over all consumers.

Let all the shops have equilibrium prices p^* while i shop has price $p \neq p^*$. Let a consumer has search costs equal to $c_{(1)}, \ldots, c_{(n)}$, such that i shop is k shop this consumer will visit.

We are going to use a variation of the following theorem from Zhou, 2011 [8].

Let there be n shops and consumers sample firms in an exogenously specified order. There is a constant search cost $s > 0$. Let $a = \hat{x}(s)$.

Assuming that consumers except an equilibrium where $p_1 < p_2 < \cdots < p_n < a$, the following theorem holds.

Theorem 2 (Zhou)**.** *Suppose consumers expect an increasing price sequence $p_1^e < p_2^e < \cdots < p_n^e < a$. Then the optimal stopping rule is characterized by a sequence of decreasing cutoff reservation surplus levels $z_1 > z_2 > \cdots > z_{n-1}$, where $z_k = a - p_{k+1}^e$. That is, if a consumer has already sampled $k \leq n-1$ firms, she will search on if the maximum available surplus so far $v_k = \max(0, u_1 - p_1, \ldots, u_k - p_k)$ is less than z_k; otherwise she will stop searching and buy the best product so far. If a consumer has sampled all products, she will either buy the best one with positive surplus, or leave the market without buying anything.*

In fact, the theorem uses only prices and reservation levels, so we will "hide" search costs in reservation levels. In our case and our notation the theorem turns into the following:

Theorem 3 (Reformulation)**.** *. Suppose consumer has an increasing search costs sequence $c_1 < c_2 < \cdots < c_n$ and expectation that all firms have the same equilibrium price p^*. Then the optimal stopping rule is characterized by a sequence of decreasing cutoff reservation surplus levels $\mathbb{E}(\mu) - p = z_0 > z_1 > z_2 > \cdots > z_{n-1} > z_n = 0$, where $z_k = \hat{x}\left(c_{k+1}\right) - p_{k+1}$ which equals [10] $z_k = \hat{x}\left(c_{k+1}\right) - p^*$ because all consumers excepts the equilibrium price. That is, if a consumer has already sampled $k \leq n-1$ firms, she will search on if the maximum available surplus so far $v_k = \max(0, \mu_1 - p_1, \ldots, \mu_k - p_k)$ is less than z_k; otherwise she will stop searching and buy the best product so far. If a consumer has sampled all products, she will either buy the best one with positive surplus, or leave the market without buying anything.*

As now we have converted our problem to the Zhou example, we may use the same formulas for demand:

Firm k's demand when it charges p is

$$d_k(p) = h_k[1 - F(z_{k-1} + p)] + r_k,$$

where

$$h_k = \prod_{j \leq k-1} F(z_{k-1} + p_j)$$

is the number of consumers who visit firm k, and

$$r_k = \sum_{i=k}^{n} \int\limits_{z_i}^{z_{i-1}} f(u+p) \prod_{j \leq i, j \neq k} F(u + p_j)\, \mathrm{d}u$$

is the number of consumers who return to our firm after visiting several firms after our firm (maybe 0).

(For this demand function to be valid for every k, we use $z_0 = \mathbb{E}\mu - p^*$, $z_n = 0$, and $\prod_{j \leq 0} = 1$.)

We can substitute out parameters to simplify the formula. We have:

1. $z_k = \hat{x}\left(c_{k+1}\right) - p^*$ and all $p_j = p^*$ for $j < k$, therefore

   $$h_k = \prod_{j \leq k-1} F(z_{k-1} + p_j)$$

   transforms into

   $$h_k = \prod_{j \leq k-1} F(\hat{x}\left(c_k\right)) = F^{k-1}(\hat{x}\left(c_k\right))$$

   which just means that at moment $k-1$ it is worth searching one more time $\rightarrow$ maximum utility is less than $\hat{x}\left(c_k\right) \rightarrow$ all previous utilities are less than $\hat{x}\left(c_k\right)$.

[10] In fact, it may happen that $\hat{x}\left(c_{k+1}\right) - p^* < 0$. In this case it will break the formulas, so we should write $z_k = \max(\hat{x}\left(c_{k+1}\right) - p^*, 0)$. However, it will make analysis very complicated because of non-differentiable function. For now, let us assume that distribution of search costs is such that $\hat{x}(\bar{c}) - p^*) > 0$. We will derive the formula without this assumption in the next section.

2. $[1 - F(z_{k-1} + p)]$ transforms into

$$1 - F(\hat{x}(c_k) - p^* + p)$$

It just shows the probability that product k is at least better than previous (not current!) reservation level: $P[\mu_k - p \geq z_{k-1}] = 1 - F(z_{k-1} + p) = 1 - F(\hat{x}(c_k) - p^* + p)$. Obviously, if a consumer visited k^{th} shop, it means that all previous utilities were less than this one.

3. Finally, we want to compute the return demand. In fact, it consists of two parts:

   (a) If a consumer at firm k finds that $\mu_k - p \in [z_k, z_{k-1})$, she will stop searching and will choose the best utility so far. The probability that she chooses k^{th} shop is:

$$Pr(\max(z_k, v_{k-1}) \leq \mu_k - p < z_{k-1}) = Pr(\max(z_k, v_{k-1}) + p \leq \mu_k < z_{k-1} + p) =$$
$$\int\limits_{z_k+p}^{z_{k-1}+p} \prod_{j \leq k-1} F(\mu_k - p + p^*)\, \mathrm{d}F(\mu_k) = \int\limits_{\hat{x}(c_{k+1})-p^*+p}^{\hat{x}(c_k)-p^*+p} \prod_{j \leq k-1} F(\mu_k - p + p^*)\, \mathrm{d}F(\mu_k) =$$
$$\int\limits_{\hat{x}(c_{k+1})}^{\hat{x}(c_k)} \prod_{j \leq k-1} F(\mu_k)\, \mathrm{d}F(\mu_k - p^* + p) = \int\limits_{\hat{x}(c_{k+1})}^{\hat{x}(c_k)} f(u - p^* + p) F(u)^{k-1}\, \mathrm{d}u$$

   It looks practically as $\frac{1}{k} \int\limits_{\hat{x}(c_{k+1})}^{\hat{x}(c_k)} \mathrm{d}F^k(u)$ which just means "the maximum of k utilities is between $\hat{x}(c_{k+1})$ and $\hat{x}(c_k)$ and our shop wins with probability $\frac{1}{k}$, but there are different prices, that is why the formula changes slightly.

   (b) Just a general case of 3.1 - a consumer visits first $k < l \leq n$ shops and it turns out that it does not worth searching anymore, and he returns to the best value. If in 3.1 $F(u)^{k-1}$ stayed for "$k - 1$ values are worse than our shop”’" , here we will have: $\int\limits_{\hat{x}(c_{l+1})}^{\hat{x}(c_l)} f(u - p^* + p) F(u)^{l-1}\, \mathrm{d}u$

   It may be shown by the same computations.

   Finally, we sum up over all l such that $k < l \leq n$ and add the border case $l = k$ (3.1) to receive:

$$r_k = \sum_{i=k}^{n} \int\limits_{\hat{x}(c_{i+1})}^{\hat{x}(c_i)} f(u - p^* + p_k) F(u)^{i-1}\, \mathrm{d}u$$

After all substitutions we receive:

$$d_k(p, p^*) = F^{k-1}(\hat{x}(c_k)) \Big[1 - F(\hat{x}(c_k) - p^* + p)\Big] + \sum_{i=k}^{n} \int\limits_{\hat{x}(c_{i+1})}^{\hat{x}(c_i)} f(u - p^* + p) F(u)^{i-1}\, \mathrm{d}u$$

where $\hat{x}(c_{n+1}) = p^*$ from the assumption that $z_n = 0$.

Substituting $k = 1$, 2, and using $\hat{x}(c_3) = p^*$ we have

$$d_1(p,p^*) = \Big[1 - F(\hat{x}(c_1) - p^* + p)\Big] + \int\limits_{\hat{x}(c_2)}^{\hat{x}(c_1)} f(u - p^* + p)\,\mathrm{d}u +$$

$$+ \int\limits_{\hat{x}(c_3)}^{\hat{x}(c_2)} f(u - p^* + p)F(u)\,\mathrm{d}u = 1 - F(\hat{x}(c_2) - p^* + p) + \int\limits_{p^*}^{\hat{x}(c_2)} f(u - p^* + p)F(u)\,\mathrm{d}u$$

$$d_2(p,p^*) = F(\hat{x}(c_2))\Big[1 - F(\hat{x}(c_2) - p^* + p)\Big] + \int\limits_{p^*}^{\hat{x}(c_2)} f(u - p^* + p)F(u)\,\mathrm{d}u$$

These formulas coincide with those that were derived in a previous part.

In fact, $d_k(p)$ also depends on search costs $c = (c_1, \dots, c_n)$, therefore, to find the whole demand for consumers for whom i shop has k^{th} position we have to integrate over them. For all this demand is similar so that we can integrate over all consumers.

$$D_k(p_i) = \frac{1}{n} \iint\limits_{[0,\bar{c}]^n} d_k(p,p^*)\,\mathrm{d}G(c_1)\,\mathrm{d}G(c_2)\dots\,\mathrm{d}G(c_n)$$

where $\frac{1}{n}$ appears because here we sum over all shops.

We have to sum over all possible positions to get final demand:

$$D(p,p^*) = \sum_{i=1}^{n} D_k(p_i) = \frac{1}{n} \iint\limits_{[0,\bar{c}]^n} \Big[\sum_{i=1}^{n} d_k(p,p^*)\Big]\,\mathrm{d}G(c_1)\,\mathrm{d}G(c_2)\dots\,\mathrm{d}G(c_n) =$$

$$= \frac{n!}{n} \iint\limits_{c_1 \le c_2 \le \dots \le c_n \le \tilde{x}(p^*)} \Big[\sum_{i=1}^{n} d_k(p,p^*)\Big]\,\mathrm{d}G(c_1)\,\mathrm{d}G(c_2)\dots\,\mathrm{d}G(c_n)$$

where

$$d_k(p,p^*) = F^{k-1}(\hat{x}\,(c_k))\Big[1 - F(\hat{x}\,(c_k) - p^* + p)\Big] + \sum_{i=k}^{n} \int\limits_{\hat{x}(c_{i+1})}^{\hat{x}(c_i)} f(u - p^* + p)F(u)^{i-1}\,\mathrm{d}u$$

It is worth noticing that we have not used anywhere that search costs are different. Therefore, the same formula is correct when all search costs are equal (some terms vanish):

$$d_k(p,p^*) = F^{k-1}(\hat{x})\Big[1 - F(\hat{x} - p^* + p)\Big] + \int\limits_{p^*}^{\hat{x}} f(u - p^* + p)F(u)^{n-1}\,\mathrm{d}u$$

$$\sum_{i=1}^{n} d_k(p,p^*) = \frac{1 - F^n(\hat{x})}{1 - F\,(\hat{x})}\Big[1 - F(\hat{x} - p^* + p)\Big] + n\int\limits_{p^*}^{\hat{x}} f(u - p^* + p)F(u)^{n-1}\,\mathrm{d}u$$

5.2 General case

In the previous section, we have assumed that there is a full participation in the market. Let us derive the formula for the general case of density.

We will use the notation $d_{k,n} = d_k$ when a consumer may visit exactly n shops according to his cost searches (we remind that in fact d_k depends not only on the price of a firm, equilibrium price but also on search costs in ascending order). Similarly, $D_n = D$ under the same conditions.

Note, that now $G(c)$ is such a distribution that the formula for D_k changes slightly (here we use that condition distribution $[g(c)|c < a] = \frac{g(c)}{G(a)} \cdot \mathbb{1}\{c < a\}$:

$$D_k(p, p^*) = \frac{k!}{G(\tilde{x}(p^*))^k} \frac{1}{k} \iint\limits_{c_1 \leq c_2 \leq \dots \leq c_n \leq \tilde{x}(p^*)} \Big[\sum_{i=1}^{k} d_{i,k}(p, p^*)\Big] \,\mathrm{d}G(c_1)\,\mathrm{d}G(c_2)\dots\,\mathrm{d}G(c_n)$$

Finally, a firm should expect that a part of consumers $p_k(a)$, where $\sum_{k=0}^{n} p_k = 1$ may visit only k firms. The firm will get from them $p_k(a)D_k(p, p^*)$ under condition that a firm may be visited. Due to the symmetry it happens with probability $\frac{k}{n}$.

$p_k(a)$ is a probability that exactly k realizations of random variables are less than p^*. Easy to see, it is equal to $\binom{n}{k} G^k(a)\big[1 - G(a)\big]^{n-k}$.

Finally, we receive:

$$\sum_{k=0}^{n} \Bigg[\binom{n}{k} G^k(\tilde{x}(p^*))\big[1 - G(\tilde{x}(p^*))\big]^{n-k} \frac{k}{n} \cdot \frac{k!}{G(\tilde{x}(p^*))^k} \frac{1}{k}$$
$$\iint\limits_{c_1 \leq c_2 \leq \dots \leq c_n \leq \tilde{x}(p^*)} \Big[\sum_{i=1}^{k} d_{i,k}(p, p^*)\Big] \,\mathrm{d}G(c_1)\,\mathrm{d}G(c_2)\dots\,\mathrm{d}G(c_n)\Bigg] =$$
$$= \frac{n!}{n} \sum_{k=0}^{n} \left[\frac{\big[[1 - G(\tilde{x}(p^*))\big]^{n-k}}{(n-k)!} \iint\limits_{c_1 \leq c_2 \leq \dots \leq c_n \leq \tilde{x}(p^*)} \left[\sum_{i=1}^{k} d_{i,k}(p, p^*)\right] \,\mathrm{d}G(c_1)\,\mathrm{d}G(c_2)\dots\,\mathrm{d}G(c_n) \right]$$

5.3 First order condition

At first we take FOC for full participation and then sum over all possible variants. Let $\pi(p, p^*) = p \cdot D(p, p^*)$.

FOC equation:

$$n \cdot \pi'(p) = n \cdot D(p, p^*) + np \cdot \frac{\partial D(p, p^*)}{\partial p} =$$
$$= n \cdot D(p, p^*) + p \cdot \iint\limits_{c_1 \leq c_2 \leq \dots \leq c_n \leq \tilde{x}(p^*)} \Big[\sum_{i=1}^{n} \frac{\partial d_k(p, p^*)}{\partial p}\Big] \,\mathrm{d}G(c_1)\,\mathrm{d}G(c_2)\dots\,\mathrm{d}G(c_n) = 0$$

As we are looking for SPNE the upper equation must be equal to zero at (p^*, p^*). Finally, we have:

$$\iint\limits_{c_1 \leq c_2 \leq \dots \leq c_n \leq \tilde{x}(p^*)} \Big[\sum_{i=1}^{n} d_k(p^*, p^*) + p^* \cdot \frac{\partial d_k(p, p^*)}{\partial p}\Big|_{(p^*, p^*)}\Big] \,\mathrm{d}G(c_1)\,\mathrm{d}G(c_2)\dots\,\mathrm{d}G(c_n) = 0$$

Some calculations:

$$d_k(p^*,p^*) = F^{k-1}(\hat{x}\,(c_k))\Big[1 - F(\hat{x}\,(c_k))\Big] + \sum_{i=k}^{n} \int\limits_{\hat{x}(c_{i+1})}^{\hat{x}(c_i)} f(u)F(u)^{i-1}\,\mathrm{d}u =$$

$$= F^{k-1}(\hat{x}\,(c_k))\Big[1 - F(\hat{x}\,(c_k))\Big] + \sum_{i=k}^{n} \frac{F(\hat{x}\,(c_i))^i - F(\hat{x}\,(c_{i+1}))^i}{i} \Rightarrow$$

$$\sum_{i=1}^{n} d_k(p^*,p^*) = \sum_{i=1}^{n} F^{i-1}(\hat{x}\,(c_i)) - \sum_{i=1}^{n} F^{i}(\hat{x}\,(c_i)) + \sum_{i=1}^{n} F^{i}(\hat{x}\,(c_i)) -$$

$$- \sum_{i=1}^{n} F^{i}(\hat{x}\,(c_{i+1})) = F^0(\hat{x}(c_1)) - F^n(\hat{x}(c_{n+1})) = 1 - F^n(p^*)$$

$$d_k'(p)\Big|_{p^*} = \Big[- F^{k-1}(\hat{x}\,(c_k)) f(\hat{x}\,(c_k) - p^* + p) + \sum_{i=k}^{n} \int\limits_{\hat{x}(c_{i+1})}^{\hat{x}(c_i)} f'(u - p^* + p)F(u)^{i-1}\,\mathrm{d}u\Big]\Big|_{p^*} =$$

$$= -F^{k-1}(\hat{x}\,(c_k)) f(\hat{x}\,(c_k)) + \sum_{i=k}^{n} \int\limits_{\hat{x}(c_{i+1})}^{\hat{x}(c_i)} f'(u)F(u)^{i-1}\,\mathrm{d}u$$

Let us denote $\pi_k'(p)$ as the derivative for case with k shops and $p_k = \frac{\left[[1-G(\tilde{x}(p^*))\right]^{n-k}}{(n-k)!}$ be the "probability" (it is a real probability multiplied by constant). Then final FOC equation is

$$\sum_{k=0}^{n} (p_k \cdot \pi_k'(p)) = 0$$

5.4 Consumer surplus for n shops

We want to derive a formula for the surplus of a consumer with search costs $c_1, \ldots, c_n$. Without loss of generality: $c_1 < c_2 < \cdots < c_k < \cdots < c_n$. Let us add a "fake" search cost c_{n+1} such that $\hat{x}(c_{n+1}) = p^*$ - it will mean that it is a final shop, and will not make any exceptions in our summation.

Let us at first compute a sum of search costs which a person will spend if she participates in a market. With probability $1 - F(\hat{x}\,(c_2))$ she stops at first shop, otherwise, she continues to search. It is easy to see that we receive the following formula (it may occur that $F(\hat{x}\,(c_k)) = 0$, it means that search costs are very high and the person will no visit next shops).

$$c_1 + F(\hat{x}\,(c_2))\big(c_2 + F(\hat{x}(c_3))\Big(c_3 + \ldots\Big)\big)$$

Now imagine, that person have came to k^{th} shop. It means that there were no utilities higher than $\hat{x}(c_{k-1})$. So, if now we get utility higher, we buy and receive $\int\limits_{\hat{x}(c_{k-1})}^{\overline{\mu}} (\mu - p^*)\,\mathrm{d}F\,(\mu)$. The consumer also stops here if at least one of previous utilities was greater than $\hat{x}\,(c_k)$.

The probability she visits k shop is $F(\hat{x}(c_{k_1}))^{k-1}$ - it just means that all previous utilities were smaller than $\hat{x}(c_{k-1})$ and they are all independent. We receive the following formula:

$$F(\hat{x}(c_{k-1}))^{k-1}\Big[\int\limits_{\hat{x}(c_{k-1})}^{\overline{\mu}} (\mu - p^*)\,\mathrm{d}F\,(\mu) + \frac{\int\limits_{\hat{x}(c_k)}^{\hat{x}(c_{k-1})} (\mu - p^*)\,\mathrm{d}F^k(\mu)}{F(\hat{x}(c_{k-1}))^{k-1}}\Big] =$$

$$= F(\hat{x}(c_{k-1}))^{k-1}\Big[\int\limits_{\hat{x}(c_{k-1})}^{\overline{\mu}} (\mu - p^*)\,\mathrm{d}F\,(\mu)\Big] + \int\limits_{\hat{x}(c_k)}^{\hat{x}(c_{k-1})} (\mu - p^*)\,\mathrm{d}F^k(\mu)$$

After summing over all shops and adding the final decision among all shops, we receive

$$- p^*(1 - F^n(p^*)) - \Big[c_1 + F(\hat{x}\,(c_2))\big(c_2 + F(\hat{x}(c_3))\Big(c_3 + \dots\Big)\big)\Big] +$$
$$+ \sum_{k=1}^{n} \Big[F(\hat{x}(c_{k-1}))^{k-1} \int\limits_{\hat{x}(c_{k-1})}^{\overline{\mu}} \mu f(\mu)\,\mathrm{d}\mu + k \int\limits_{\hat{x}(c_k)}^{\hat{x}(c_{k-1})} \mu \cdot f(\mu) F^{k-1}(\mu)\,\mathrm{d}\mu\Big] +$$
$$+ n \int\limits_{p^*}^{\hat{x}(c_n)} \mu \cdot f(\mu) F^{n-1}(\mu)\,\mathrm{d}\mu = S_n(p^*, c_1, c_2, \dots, c_n)$$

Here we were looking at a consumer who may visit all shops: $c_{(n)} < \tilde{x}(p^*)$. The formula is just the same for all others consumers except the fact they think there are just k shops in the market. Finally, to find common surplus we do as in the previous section:

$$\frac{n!}{n} \sum_{k=0}^{n} \left[\frac{\big[[1 - G(\tilde{x}(p^*))\big]^{n-k}}{(n-k)!} \iint\limits_{c_1 \le c_2 \le \dots \le c_k < \tilde{x}(p^*)} [S_k(p^*, c_1, c_2, \dots, c_k)]\ \mathrm{d}G(c_1) \dots \mathrm{d}G(c_k) \right]$$

In case when a consumer has equal search costs the second term in summation vanishes (everywhere, except the last term, where the person chooses the best utility) and the formula becomes

$$\sum_{k=1}^{n} \Big[F\,(\hat{x}\,(c))^{k-1} \int\limits_{\hat{x}(c)}^{\overline{\mu}} (\mu - p^*)\,\mathrm{d}F\,(\mu) - c \cdot F\,(\hat{x}\,(c))^{k-1}\Big] + \int\limits_{p^*}^{\hat{x}(c)} (\mu - p^*)\,\mathrm{d}F^n(\mu) =$$

$$= \frac{1 - F\,(\hat{x}\,(c))^n}{1 - F\,(\hat{x}\,(c))} \Big[\int\limits_{\hat{x}(c)}^{\overline{\mu}} (\mu - p^*)\,\mathrm{d}F\,(\mu) - c\Big] + \int\limits_{p^*}^{\hat{x}(c)} (\mu - p^*)\,\mathrm{d}F^n(\mu)$$

It coincides with the formula derived for $n = 1$ or $n = 2$.

6 Formulas for uniform distribution

In this section, we calculate the demand in a specific case when match utilities are uniform. We will need them for numerical calculation. Assuming $f \sim U[0, 1]$ and full participation in the market, we have the following:

$$\int_{x}^{\overline{\mu}} (\mu - x) \, \mathrm{d}F(\mu) = \int_{x}^{1} (\mu - x) \, \mathrm{d}\mu = \frac{(1-x)^2}{2} = c \Rightarrow$$

$$\hat{x}(c) = 1 - \sqrt{2c}, \;\; \tilde{x}(p^*) = \frac{(1-p)^2}{2}$$

$$\left.\frac{\partial d_k(p, p^*)}{\partial p}\right|_{(p^*, p^*)} = -\hat{x}^{k-1}(c_k)$$

$$\Big[\sum_{i=1}^{n} d_k(p^*, p^*) + p^* \cdot \left.\frac{\partial d_k(p, p^*)}{\partial p}\right|_{(p^*, p^*)}\Big] = 1 - (p^*)^n - p^*\Big(1 + \hat{x}(c_2) + \dots + \hat{x}^{n-1}(c_n)\Big)$$

Equilibrium price:

$$\iint\limits_{c_1 \le c_2 \le \dots \le c_n} \Big[1 - (p^*)^n - p^*\Big(1 + \hat{x}(c_2) + \dots + \hat{x}^{n-1}(c_n)\Big)\Big] \, \mathrm{d}G(c_1) \, \mathrm{d}G(c_2) \dots \, \mathrm{d}G(c_n) = 0 \Rightarrow$$

$$\frac{1 - (p^*)^n}{p} = \iint\limits_{c_1 \le c_2 \le \dots \le c_n} \left[1 + \hat{x}(c_2) + \dots + \hat{x}^{n-1}(c_n)\right] \, \mathrm{d}G(c_1) \, \mathrm{d}G(c_2) \dots \, \mathrm{d}G(c_n)$$

Expected Payoff:

$$\pi = \frac{n!}{n} p^* \cdot \iint\limits_{c_1 \le c_2 \le \dots \le c_n} \Big[1 - (p^*)^n\Big] \, \mathrm{d}G(c_1) \, \mathrm{d}G(c_2) \dots \, \mathrm{d}G(c_n) = \frac{p^*(1 - (p^*)^n}{n}$$

Consumer Surplus:

$$\sum_{k=1}^{n} \Big[\hat{x}(c_k)^{k-1} \int_{\hat{x}(c_k)}^{\overline{\mu}} \mu \, \mathrm{d}\mu + k \int_{\hat{x}(c_{k+1})}^{\hat{x}(c_k)} \mu^k \, \mathrm{d}\mu\Big] =$$

$$= \sum_{k=1}^{n} \Big[\frac{\hat{x}(c_k)^{k-1} - \hat{x}(c_k)^{k+1}}{2} + \frac{k\Big(\hat{x}(c_k)^{k+1} - \hat{x}(c_{k+1})^{k+1}\Big)}{k+1}\Big]$$

Finally, we have:

$$s_k(p^*) = -p^*(1 - (p^*)^n) - \Big[c_1 + \hat{x}(c_2)\left(c_2 + \hat{x}(c_3)\Big(c_3 + \dots\Big)\right)\Big] +$$

$$+ \sum_{k=1}^{n} \Big[\frac{\hat{x}(c_k)^{k-1} - \hat{x}(c_k)^{k+1}}{2} + \frac{k\Big(\hat{x}(c_k)^{k+1} - \hat{x}(c_{k+1})^{k+1}\Big)}{k+1}\Big]$$

For search costs, we use the Kumaraswamy distribution [11]. We will use Kumaraswamy distribution with $a = 1$; in brief β accounts for range, while b shifts mass to the left ($b < 1$)

[11]

$$g(x) = \frac{ab}{\beta}\left(\frac{x}{\beta}\right)^{a-1}\left[1 - \left(\frac{x}{\beta}\right)^{a}\right]^{b} \quad x \in [0, \beta], \quad a, b > 0$$

$$G(x) = 1 - \left[1 - \left(\frac{x}{\beta}\right)^{a}\right]^{b-1}$$

or to the right. It looks like Beta distribution, but has closed density and distribution functions and its likelihood ratio is increasing for $b > 0$, decreasing for $0 < b < 1$ and constant if $b = 1$.

At first we calculate different characteristics for two firms in case when there is a full participation:

Table 1: Duopoly model (uniform-Kumaraswamy with $a = 1$)

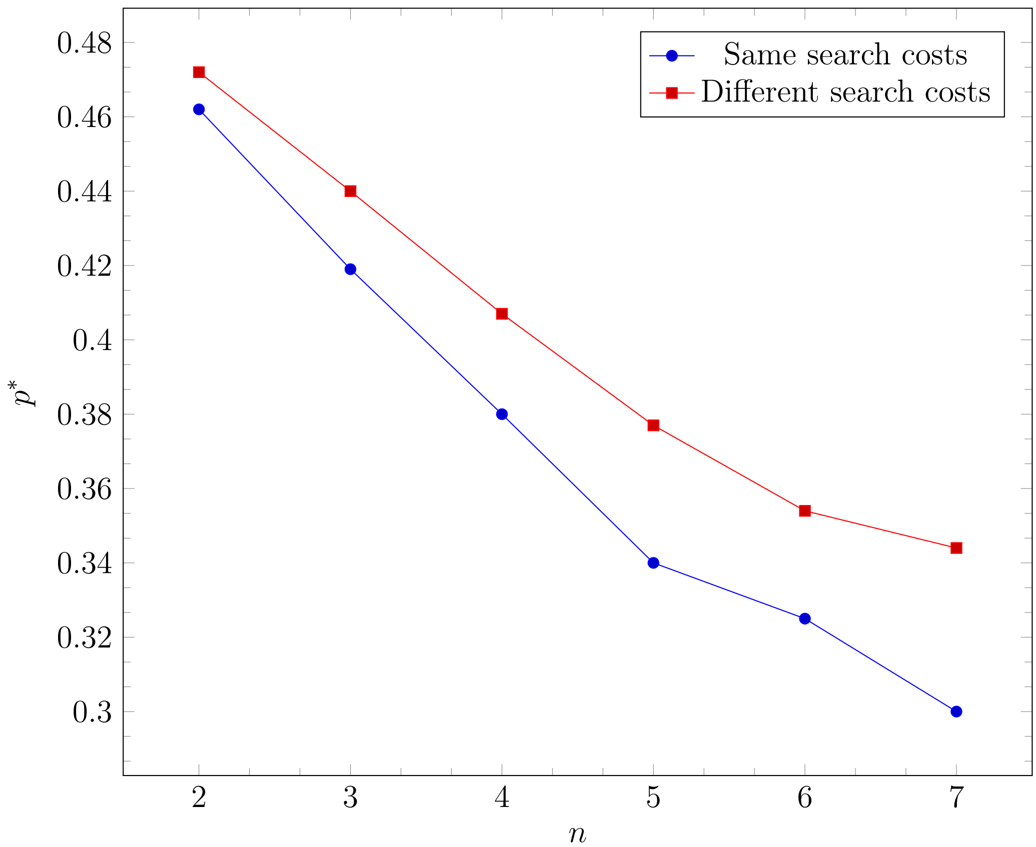
	$b = 3/2$			$b = 1$			$b = 1/2$		
	$\beta = 0.05$	$\beta = 0.08$	$\beta = 0.11$	$\beta = 0.05$	$\beta = 0.08$	$\beta = 0.11$	$\beta = 0.05$	$\beta = 0.08$	$\beta = 0.11$
p^*	0.443 0.450	0.451 0.460	0.458 0.469	0.447 0.454	0.456 0.465	0.464 0.476	0.453 0.459	0.464 0.473	0.474 0.485
π	0.1786 0.1810	0.1828 0.1811	0.1810 0.1813	0.1808 0.1833	0.1814 0.1824	0.1826 0.1846	0.1770 0.1801	0.1900 0.1797	0.1813 0.1799
CS	0.215 0.211	0.188 0.184	0.162 0.158	0.203 0.199	0.170 0.166	0.139 0.136	0.184 0.180	0.142 0.138	0.102 0.099

Notes: CS is consumer surplus. The first line corresponds to the model with equal search costs, the second — to different.

As we see, although we expected that firms would lower prices, in fact, they are higher at every density.

Let us plot equilibrium prices when $G(c) \sim U[0, 0.1]$. Surprisingly, it turns out that even for seven shops we receive higher prices in our model:

Equilibrium price depending on a number of firms



7 Comparison of prices for two shops

Let $n = 2$. We want to prove that equilibrium price with different search costs is higher than the same in the ordinary case. Why should this be true? In brief, we know that under full participation lower search costs imply lower equilibrium price. When a person comes to a shop, the shop understands (we have derived the same formulas in a case of 2 shops) that a person has to pay more to visit the second shop. Moreover, we know that $\mathbb{E}(c_{(2)}) > \mathbb{E}(c_{(2)})$, so we can look at it as at increasing search costs. The above "proof" must work for any distribution function F. However, we have proved it only for concave F.

Proof.

We will use following notation: $l(p) = \hat{x}^{-1}(p)$ and $h(p^*, c) = 1 - F^2(p^*) - p^* \cdot \Big(f(\hat{x}\,(c)) + F\,(\hat{x}\,(c))\, f(\hat{x}\,(c)) - 2 \int\limits_{p^*}^{\hat{x}(c)} f'(u)F(u)\,\mathrm{d}u \Big)$.

Suppose that $\int\limits_0^{\tilde{x}(p^*)} h(p,c)\,\mathrm{d}G(c)$ is decreasing[12] in p and that $h(p,c)$ is increasing in c. Let p^* will be the solution. Then, under the same assumptions, it is sufficient to show that: $\iint\limits_{0 \le c_1 \le c_2 \le \tilde{x}(p^*)} h(a', c)\,\mathrm{d}G(c_1)\,\mathrm{d}G(c_2) > 0$.

$$\iint\limits_{0 \le c_1 \le c_2 \le \tilde{x}(p^*)} h(p^*, c_2)\,\mathrm{d}G(c_1)\,\mathrm{d}G(c_2) = \int\limits_0^{\tilde{x}(p^*)} \Big(\int\limits_0^{c_2} \mathrm{d}G(c_1) \Big) h(p^*, c_2)\,\mathrm{d}G(c_2) =$$

$$= \int\limits_0^{\tilde{x}(p^*)} G(c_2) h(p^*, c_2)\,\mathrm{d}G(c_2) \ge 0$$

The inequality holds due to the fact that we multiple the increasing (first negative, then positive) $h(p^*, c)$ on increasing $G(c) \ge 0$.

We have $h'(p^*, c) = \hat{x}'(c_2) \cdot (-f' - f^2 - Ff' + 2Ff')$. Taking into account that $\hat{x}'(c_2) < 0$, we should show that $-f' - f^2 + Ff' < 0 \Leftarrow (F - 1)f' \le 0 \Leftarrow f' \ge 0$. Finally, it is sufficient to have concave $F\,(x)$.

To prove the theorem in general case it is enough to notice that payoff is the sum of two payoffs - when person visit only one shop or two shops. The solution to the first case is a monopoly price which is higher (it is easy to show) than p^*. Using that there are two unimodal functions, such that $f_1'(x_1) = 0$ and $f_2'(x_2) = 0$, it is easy to show that each x such that $f_1'(x) + f_2'(x) = 0$ satisfies $x_1 < x < x_2$. It is exactly what we needed. $\square$

8 Conclusion

Most models assume that there is a fixed search cost c for a consumer or even for all consumers. As shown in [10] rejection of the last assumptions give us the unexpected effect that prices may go up when search costs go down. This effect was observed by Hortacsu and Syverson's, 2004 [6] study of the mutual fund industry - it turned out that prices (percents) went up despite the fact that the Internet should lower the search costs.

[12] One of the assumption in our model was unimodular payoff in p^*

We think that our model may simulate many real situations such as shops somehow distributed in the city. We were expecting that prices will be smaller compared to Moraga-Gonzalezy and others' model because a consumer in an equilibrium usually visits only several shops and the expected value lowers (it is an ordered statistics), however simulation for a small number of shops did not give us expected results. As we have noted in the introduction there is another effect of giving different search costs to a consumer — a shop knows that a consumer has to pay more to go to the next shop which can be seen from demand formulas as well, and therefore may set a higher price.

Most theorems in such models are proved for an infinite number of firms which makes the analysis simpler due to the fact there is no return demand. Main results of our work are calculating demand and optimal price for the model and proving inequality about prices for two shops. We are going to calculate prices for a larger number of shops and depending on results prove one of the following: prices are always higher in our model or from a large number of shops prices are lower. There are some other technical theorems such as the necessary condition for existence and uniqueness of equilibrium or proving that price falls as a number firms raise which I will try to derive and prove.

9 Current Updates

9.1 Another formula for $\hat{x}(c)$

Theorem 4.

$$\hat{x}(c) := \left(\int_{x}^{\overline{\mu}} (1 - F(\mu))\, \mathrm{d}\mu = c \right)$$

Proof.

We know that

$$\hat{x}(c) := \left(\int_{x}^{\overline{\mu}} (\mu - x)\, \mathrm{d}F(\mu) = c \right)$$

On the other hand, we know that for non-negative random variable we have the following:

$$\int_{0}^{+\infty} x\, \mathrm{d}F(\mu) = \mathbb{E}[X] = \mathbb{E}\left[\int_{0}^{x} 1\, \mathrm{d}t \right] = \mathbb{E}\left[\int_{0}^{+\infty} 1_{\{t<X\}}\, \mathrm{d}t \right] =$$

$$= \int_{0}^{+\infty} \mathbb{P}\Big[X > t\Big]\, \mathrm{d}t = \int_{0}^{+\infty} (1 - F_X(t))\, \mathrm{d}t$$

After applying this fact to random variable $\mu - x$ and changing variable we receive what was stated:

$$\hat{x}(c) := \left(\int_{x}^{\overline{\mu}} (1 - F(\mu))\, \mathrm{d}\mu = c \right)$$

□

9.2 Formulas for (almost) conditional expected value if $\mu > x$

They are multiplied by probability actually.

$$h(x) = \int_{x}^{\overline{\mu}} (\mu - x)\, \mathrm{d}F(\mu) = \int_{x}^{\overline{\mu}} (1 - F(\mu))\, \mathrm{d}\mu$$

$$h'(x) = F(x) - 1$$

$$h''(x) = f(x) \geq 0$$

9.3 Rewrite our FOC equation to 1 dimensional integrals

I thought it might be helpful but new formulas are even more threatening. Actually, it might be easier to code them... (And now I understand each multiplier in the second part)

Assuming full participation we have the following formula for RHS of FOC now:

$$\iint\limits_{c_1\le c_2\le\cdots\le c_n} \sum_{k=1}^{n}\Big(F^{k-1}(\hat{x}\,(c_k))f(\hat{x}\,(c_k))- \\ -\sum_{i=k}^{n}\int\limits_{\hat{x}(c_{i+1})}^{\hat{x}(c_i)} f'(u)F^{i-1}\,(u)\;\mathrm{d}u\Big)\,\mathrm{d}G(c_1)\,\mathrm{d}G(c_2)\ldots\,\mathrm{d}G(c_n)$$

Theorem 5. *It equals*[13]

$$\sum_{k=1}^{n}\Bigg(\int\limits_{\underline{c}}^{\overline{c}} F^{k-1}(\hat{x}\,(c))f(\hat{x}\,(c))g_{k,n}(c)+ \\ +\,(f(h(c)))'F^{k-2}(h(c))\cdot\big(kF(h(c))-(k-1)\big)\cdot G_{k,n}(c)\,\mathrm{d}c\Bigg) \quad (1)$$

where

$$g_{k,n}(x)=\frac{n!}{(k-1)!(n-k)!}\cdot G^{k-1}(x)\cdot(1-G(x))^{n-k}\cdot g(x)$$

and

$$G_{k,n}(x)=\int\limits_{0}^{x} g_{k,n}(t)\,\mathrm{d}t$$

Proof.

Rewriting first part of RHS

Let's look at the $F^{k-1}(\hat{x}\,(c_k))f(\hat{x}\,(c_k))$ term.

As it has only $c_{(k)}$ inside we may rewrite this part of integral as

$$\sum_{k=1}^{n}\Big(\int F^{k-1}(\hat{x}\,(c))f(\hat{x}\,(c))g_{k,n}(c)\,\mathrm{d}c\Big)$$

Rewriting second part of RHS

[13] It might be further simplified by using $f(h(c))' = f'(h(c)\cdot(F(c)-1)$

Let us omit the outer integral for now.

$$\sum_{k=1}^{n}\Big(\sum_{i=k}^{n}\int\limits_{\hat{x}(c_{i+1})}^{\hat{x}(c_i)} f'(u)F^{i-1}(u)\ \mathrm{d}u\Big) = \sum_{k=1}^{n}\Big(k\cdot\int\limits_{\hat{x}(c_{k+1})}^{\hat{x}(c_k)} f'(u)F^{k-1}(u)\ \mathrm{d}u\Big) =$$

$$= \sum_{k=1}^{n}\int\limits_{0}^{\hat{x}(c_k)} kf'(u)F(u)^{k-1}\,\mathrm{d}u - \sum_{k=1}^{n}\int\limits_{0}^{\hat{x}(c_{k+1})} kf'(u)F(u)^{k-1}\,\mathrm{d}u =$$

$$= \sum_{k=1}^{n}\int\limits_{0}^{\hat{x}(c_k)} kf'(u)F(u)^{k-1}\,\mathrm{d}u - \sum_{k=2}^{n+1}\int\limits_{0}^{\hat{x}(c_k)} (k-1)f'(u)F(u)^{k-2}\,\mathrm{d}u = \big[\hat{x}\left(c_{n+1}\right) = 0\big] =$$

$$= \sum_{k=1}^{n}\int\limits_{0}^{\hat{x}(c_k)} kf'(u)F(u)^{k-1}\,\mathrm{d}u - \sum_{k=1}^{n}\int\limits_{0}^{\hat{x}(c_k)} (k-1)f'(u)F(u)^{k-2}\,\mathrm{d}u =$$

$$= \sum_{k=1}^{n}\int\limits_{0}^{\hat{x}(c_k)} \Big(kf'(u)F(u)^{k-1} - (k-1)f'(u)F(u)^{k-2}\Big)\,\mathrm{d}u =$$

$$= \sum_{k=1}^{n}\int\limits_{0}^{\hat{x}(c_k)} \Big(f'(u)F(u)^{k-2}\cdot\big(kF(u)-(k-1)\big)\Big)\,\mathrm{d}u = \big[h(c) = \hat{x}^{-1}(c)\big] =$$

$$= \sum_{k=1}^{n}\int\limits_{\mathbb{E}\mu}^{c_k} \Big(f'(h(u))F^{k-2}(h(u))\cdot\big(kF(h(u))-(k-1)\big)\cdot h'(u)\Big)\,\mathrm{d}u =$$

$$= -\sum_{k=1}^{n}\int\limits_{c_k}^{\mathbb{E}\mu} \Big(f'(h(u))F^{k-2}(h(u))\cdot\big(kF(h(u))-(k-1)\big)\cdot h'(u)\Big)\,\mathrm{d}u$$

We will use the following fact:

Lemma 1. *Let μ be a random variable on $[a,b]$ with density f and cumulative density function F. Then:*

$$\mathbb{E}\left[\int\limits_{\mu}^{b} g(x)\,\mathrm{d}x\right] = \int\limits_{a}^{b} g(x)F\left(x\right)\ \mathrm{d}x$$

Proof.

$$\mathbb{E}\left[\int\limits_{\mu}^{b} g(x)\,\mathrm{d}x\right] = \int\limits_{a}^{b}\int\limits_{\mu}^{b} g(x)f(\mu)\,\mathrm{d}x\,\mathrm{d}\mu = \int\limits_{a}^{b}\int\limits_{a}^{\mu} g(x)f(\mu)\,\mathrm{d}\mu\,\mathrm{d}x = \int\limits_{a}^{b} g(x)F\left(x\right)\ \mathrm{d}x \quad \square$$

Now we use the same trick we used for first part (it depends on only one order statistics so we may change distribution):

$$\iint\limits_{c_1 \le c_2 \le \dots \le c_n} \left(-\sum_{k=1}^{n} \int_{c_k}^{\mathbb{E}\mu} \Big(\dots \Big) \,\mathrm{d}u \right) \mathrm{d}G(c_1)\,\mathrm{d}G(c_2)\dots\mathrm{d}G(c_n) =$$

$$= -\sum_{k=1}^{n} \int_{\underline{c}}^{\overline{c}} \Big(f'(h(u))F^{k-2}(h(u)) \cdot \big(kF(h(u)) - (k-1)\big) \cdot h'(u) \cdot G_{k,n}(u) \Big) \,\mathrm{d}u$$

Where we used that $\overline{c} \le \mathbb{E}\mu$. Finally, $h'(u) = F(c) - 1$ $\square$

9.4 Old computational results updates by rewriting the integrals and moving to Python

9.4.1 New plots for uniform distribution of match values

Reminder: $\hat{x}(c) = 1 - \sqrt{2c}$

Equilibrium price for infinite number of shops in case with same search costs:

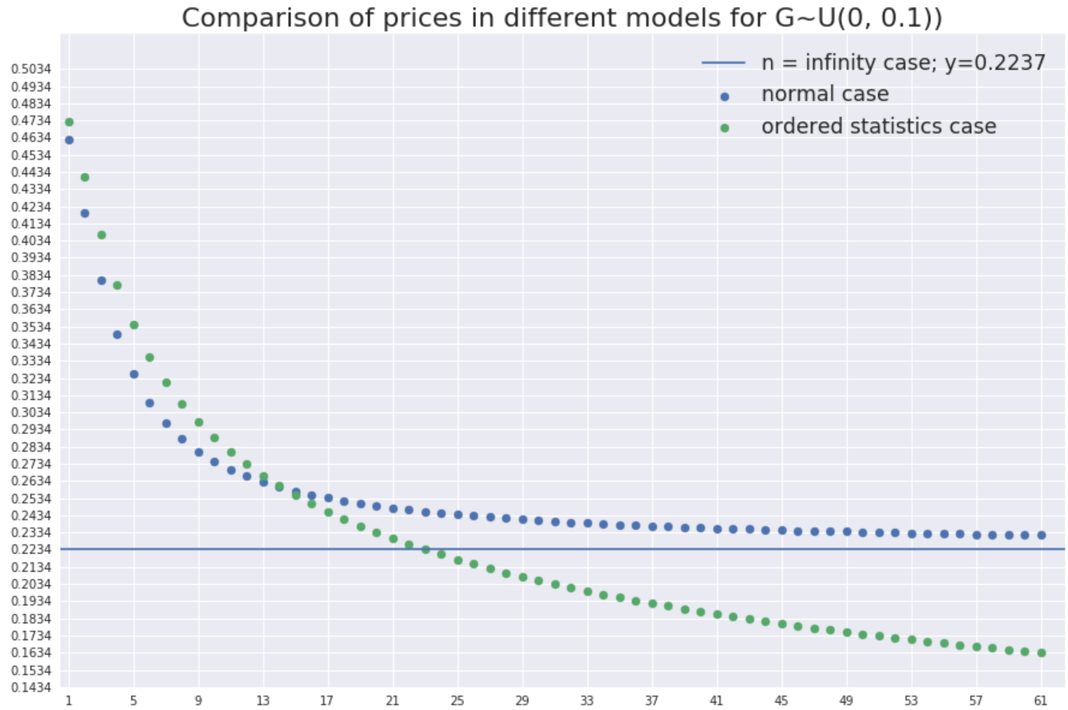
$$\frac{1 - p*^n}{p*} = \int \frac{1}{1 - \hat{x}(c)} \,\mathrm{d}G(c)$$

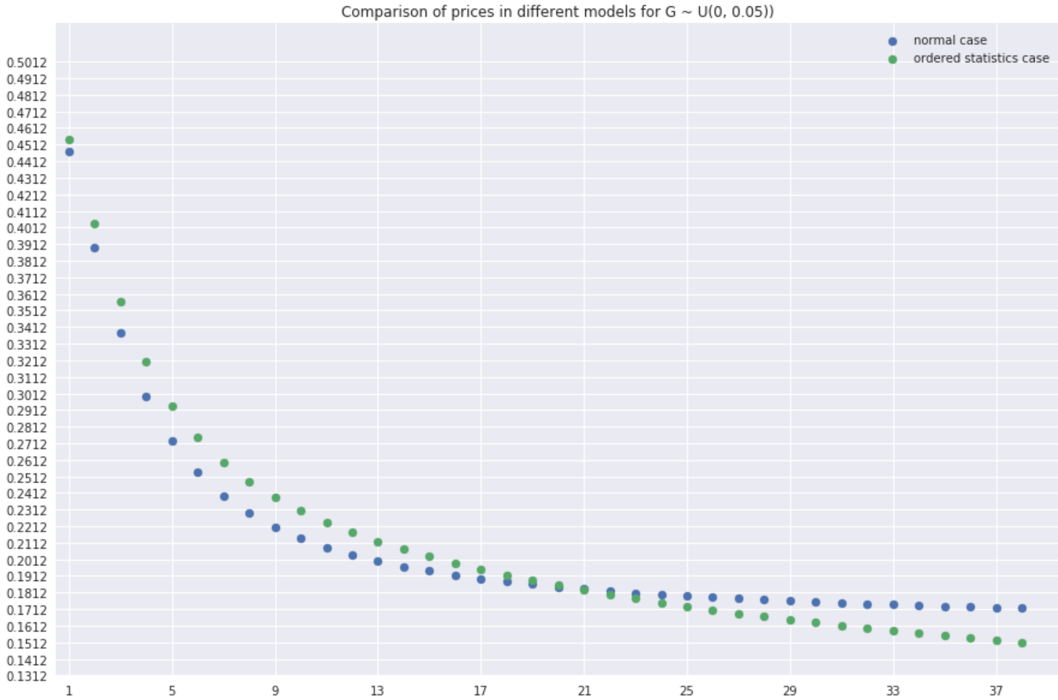
which turns into

$$\frac{1}{p*} = \int \frac{1}{\sqrt{2x}} \,\mathrm{d}G(x)$$

We will assume that $G(x) \sim U[0, 0.1]$. Then price tends to $\frac{1}{10\sqrt{0.2}} = 0.2236...$

As we have obtained stability in computing n-dimensional integrals by making them 1-dimensional (changing the density though) I recalculated the optimal prices:





The points coincide with previous results so they are quite trustworthy. Though now numerical methods stop working at $n = 40$ to $n = 60$.

Question: I think I may prove the following lemma

Lemma 2. *There exists such n that optimal price for number of shops greater than n in our model is lower than the same in the original model.*

Should I do this? I think I have an idea how to do it for uniform F (F is a distribution for match values) and, most probably, for all concave match values (I mean $F'' = f' < 0$ from which additional positive term will appear in right side of equation; at the same time the part we have now will be uglier.)

9.5 Rewriting our problem as discrete choice model

Demand is not log-concave => no result for uniqueness and existence

In "Consumer Search and Price Competition" [12] almost the same problem is rewritten as discrete choice game. It is stated that by defining the following random variable (they have much better variable names, so maybe it worth moving to them...)

$$W_i = v_i + min(\mu_i, \hat{x}(c_i))$$

where v_i in their model is a prior knowledge about utility (final match value is $v_i + \mu_i$, $v_i \sim V(x)$ and a sample of v_i is drawn in advance; we do not have this part in our model, so $v_i \equiv 0$).

Let $H_i(w_i) \sim W_i$. Actually, it is a mixed distribution:

$$H(w, c) = \begin{cases} F(w) & \text{if } w < \hat{x}(c) \\ 1 & \text{if } w \geq \hat{x}(c) \end{cases}$$

and

$$h(w, c) = \begin{cases} f(w) & \text{if } w < \hat{x}(c) \\ 1 - F(\hat{x}(c)) & \text{if } w = \hat{x}(c) \end{cases}$$

Then, according to [12]

$$d_i = \int_{p_i}^{+\infty} \left(\prod_{j \neq i} H_j(w_i - p_i + p_j) \right) \mathrm{d}H_i(w_i)$$

is a demand[14] for our problem for fixed (!) vector $(c_1, c_2, \ldots, c_n)$ as in fact in our model w_i depends on $c_i \Rightarrow H_i$ depends on c_i as well. Thus we have to integrate over all possible permutations to receive total demand:

$$D_i = \iint\limits_{c_1 \leq c_2 \leq \cdots \leq c_n \leq \tilde{x}(p^*)} d_i(c_1, c_2, \ldots, c_n) \, \mathrm{d}G(c_1) \, \mathrm{d}G(c_2) \ldots \mathrm{d}G(c_n)$$

It is stated that if demand function is log-concave than there is a unique solution. According to Quint [13] if for each i both $H_i(w_i)$ and $1 - H_i(w_i)$ are both log-concave, then d_i is log-concave (and by Prekopa theorem we have that D_i is log-concave, too).

Unfortunately, as stated above H_i is not continuous, thus it cannot be log-concave.

[14] I have checked that it is the same formula as we have received. Reviewing the cases of w_i we receive different number of multipliers H_j (as some of them equals to 1). And H_i itself is mixed distribution so the atom probability separates in separate term. I may show it here but it just was a kind of sanity check.

10 Summing up

Definitions:
Let $g_{k,n}$ be the k-order statistics for an initial distribution G. Then:

1. $g_{k,n}(x) = n\binom{n-1}{k-1} \cdot (1 - G(x))^{(n-1)-(k-1)} \cdot G^{k-1}(x) \cdot g(x)$
2. $\mathbb{E}[g_{k,n}(x)] = \frac{k}{n+1}$ for uniform distribution
3. $\mathbb{E}[x^\alpha] = \frac{\Gamma(n+1)\Gamma(k+\alpha)}{\Gamma(n+1+\alpha)\Gamma(k)}$
   1) It equals $\frac{k}{n+1} \cdot \frac{k+1}{n+2} \dots \frac{k+\alpha-1}{n+\alpha}$ for $\alpha \in \mathbb{Z}$
   2) and something ugly ($\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$) for $\alpha \in \mathbb{Z} + \frac{1}{2}$
4. No formulas for non-uniform distributions (like the desired $G^{-1}\left(\frac{k}{n+1}\right)$)

Assumptions:

1. $F \sim U[0,\ 1]$
2. $\hat{x}\,(c) = 1 - \sqrt{2c}$ for uniform F
3. Full participation

Then right hand sides are:

$$D_n = \sum_{k=1}^{n} \int \hat{x}^k(c) g_{k,n}\, \mathrm{d}c \qquad \text{Different search costs}$$

$$S_n = \sum_{k=1}^{n} \int \hat{x}^k(c) g\, \mathrm{d}c \qquad \text{Same search costs}$$

Some formulas:

$$D_{n+1} - D_n = \sum_{k=1}^{n} \int \hat{x}^k(c)(g_{k,n+1} - g_{k,n})\, \mathrm{d}c + \int \hat{x}^{n+1}(c) g_{n+1,n+1}\, \mathrm{d}c$$

$$S_{n+1} - S_n = \int \hat{x}\,(c)^{n+1}\, g\, \mathrm{d}c$$

$$D_n - S_n = \sum_{k=1}^{n} \int \hat{x}^k(c) \left(\frac{g_{k,n}}{g} - 1\right) g\, \mathrm{d}c$$

Obviously, there are compact formulas for uniform g as we have explicit formulas for moments and binomial theorem! (Though I do not think it will help as there are some "half" moments which look differently.)

11 Properties of Order Statistics

Definition 1. *F dominates G in the sense of first order if $F\,(x) \geq G(x)$ and for some x the inequality is strong.*

CDF is:

$$F_{k,n}(x) = \sum_{i=k}^{n} \binom{n}{i} F^i(x)(1 - F(x))^{n-i}$$

PDF is:

$$f_{k,n} = n\binom{n-1}{k-1} \cdot (1 - F(x))^{(n-1)-(k-1)} \cdot F^{k-1}(x) \cdot f(x)$$

It looks (and actually is) as a binomial distribution with $p = F(x)$ and then we can rewrite the sum above as (using substitution $k' = k - 1$ and representation in terms of the regularized incomplete beta function)

$$1 - F_{Binom}(k-1; n; p) = 1 - (n - k')\binom{n}{k'} \int_0^{1-p} t^{n-k'-1}(1-t)^{k'}\,\mathrm{d}t$$

1. Both $f_{k,n}(x;t)$ and $F_{k,n}(x;t)$ are continuous in t if $f(x;t) = f_t(x)$ and $F(x;t) = F_t(x)$ are continuous.

   Proof.

   Directly follows from explicit formulas for PDF and CDF. $\square$

2. $F_{k,n} > F_{k+1,n}$.

   Proof.

   Obvious – one more positive term is a sum. $\square$

3. $F_{k,n+1} > F_{k,n}$.

   Proof.

   Suppose there are $n+1$ samples of a random variable, so the distribution of a vector is the product of $n+1$ independent variables. If we take one of these variables (does not matter, which, e.g. the first one), we will receive the product of n random variables. We induce the distribution this way, thus, proving inequality for one point in the space will prove the same inequality in general.

   Now let us sort the vector of samples. As we took the first sample, now it is uniformly distributed across $n+1$ order statistics. If we took away sample with index greater than k, than $x_{k,n+1} = x_{k,n}$. Otherwise, with $\frac{k}{n+1}$ probability $x_{k+1,n+1} = x_{k,n}$ for this very sample. Integrating over all possible samples we receive that

$$\begin{aligned}F_{k,n+1}(x) = \mathbb{P}\Big[x_{k,n+1} < x\Big] &> \\ > \frac{n+1-k}{n+1} \cdot \mathbb{P}\Big[x_{k,n+1} < x\Big] &+ \frac{k}{n+1} \cdot \mathbb{P}\Big[x_{k+1,n+1} < x\Big] = \\ &= \mathbb{P}\Big[x_{k,n} < x\Big] = F_{k,n}(x) \quad \square\end{aligned}$$

4. Let $F < G$ in terms first order stochastic dominance. Then $F_{k,n} < G_{k,n}$ for any k.

Proof.

Let $F(t) = p_1 < p_2 = G(t)$. Using regularized incomplete beta function form:

$$
\begin{aligned}
F_{k+1,n}(t) &< G_{k+1,n}(t) \Leftrightarrow \\
1-(n-k)\binom{n}{k}\int_0^{1-p_1} t^{n-k-1}(1-t)^k \,\mathrm{d}t &< 1-(n-k)\binom{n}{k}\int_0^{1-p_2} t^{n-k-1}(1-t)^k \,\mathrm{d}t \Leftrightarrow \\
(n-k)\binom{n}{k}\int_0^{1-p_1} t^{n-k-1}(1-t)^k \,\mathrm{d}t &> (n-k)\binom{n}{k}\int_0^{1-p_2} t^{n-k-1}(1-t)^k \,\mathrm{d}t \Leftrightarrow \\
\int_0^{1-p_1} t^{n-k-1}(1-t)^k \,\mathrm{d}t &> \int_0^{1-p_2} t^{n-k-1}(1-t)^k \,\mathrm{d}t \Leftrightarrow \\
\int_{1-p_2}^{1-p_1} t^{n-k-1}(1-t)^k \,\mathrm{d}t &> 0
\end{aligned}
$$

□

12 Solution is Continuous in G

Assumption: Distribution of utility is bounded and continuous.

Then the distribution of search costs is clearly bounded, too[15]. Without loss of generality, let g be defined on $[0, 1]$.

Let $g(x, t) = g_t(x)$ be a continuous function in t. Then

$$G_t(x) = \int_0^x g_t(y)\,\mathrm{d}y$$

is also continuous. As stated in 11 both density and distribution of order statistics are continuous, too.

From formula 1 we receive that right hand side is continuous in t. Obviously, the solution to

$$\frac{1 - F^n(p)}{p} = \text{RHS}$$

is continuous, too, as left hand side is continuous under our assumptions, too.

[15]Otherwise we may look at interval $[0, \bar{c}]$ where $\bar{c}$ is such a search cost that it is not profitable to participate in the market.

13 Existence and uniqueness

Assumptions: Full participation.

Formula for demand: (see 5.1 for derivation)

$$D(p,p^*) = \frac{n!}{n} \iint\limits_{c_1 \le c_2 \le \dots \le c_n} \Big[\sum_{i=1}^{n} d_k(p,p^*)\Big] \, \mathrm{d}G(c_1)\, \mathrm{d}G(c_2) \dots \mathrm{d}G(c_n)$$

where:

$$\begin{aligned}
d_k(p,p^*) &= F^{k-1}(\hat{x}\,(c_k))\Big[1 - F(\hat{x}\,(c_k) - p^* + p)\Big] &&+ \sum_{i=k}^{n} \int\limits_{\hat{x}(c_{i+1})}^{\hat{x}(c_i)} f(u - p^* + p) F(u)^{i-1}\, \mathrm{d}u \\
\frac{\partial d_k}{\partial p} &= -F^{k-1}(\hat{x}\,(c_k)) f(\hat{x}\,(c_k) - p^* + p) &&+ \sum_{i=k}^{n} \int\limits_{\hat{x}(c_{i+1})}^{\hat{x}(c_i)} f'(u - p^* + p) F(u)^{i-1}\, \mathrm{d}u \\
\frac{\partial^2 d_k}{\partial p^2} &= -F^{k-1}(\hat{x}\,(c_k)) f'(\hat{x}\,(c_k) - p^* + p) &&+ \sum_{i=k}^{n} \int\limits_{\hat{x}(c_{i+1})}^{\hat{x}(c_i)} f''(u - p^* + p) F(u)^{i-1}\, \mathrm{d}u
\end{aligned}$$

Different derivatives of profit function:

$$\begin{aligned}
\pi &= p \cdot D(p,p^*) \\
\frac{\partial^2 \pi}{\partial p^2} &= p \cdot \frac{\partial^2 D}{\partial p^2} + 2\frac{\partial D}{\partial p} \\
\frac{\partial^2 \log \pi}{\partial p^2} &= -\frac{1}{p^2} + \frac{\frac{\partial^2 D}{\partial p^2} D - \left(\frac{\partial D}{\partial p}\right)^2}{D^2}
\end{aligned}$$

I do not see any options apart from requiring $\frac{\partial^2 D}{\partial p^2} \le 0$ and, even more, requiring $\frac{\partial^2 d_k}{\partial p^2} \le 0$ for each k.

As c_k and c_{k+1} may take any values in its domain (well, it is possible to rewrite it, actually, see 1, there are no insights), we should require $f'' \le 0$.

Talking about the sum of first parts of second derivative, we want the following:

$$\sum_{k=1}^{n} \bigg(\int \Big(-F^{k-1}(\hat{x}\,(c)) f'(\hat{x}\,(c) - p^* + p) \Big) g_{k,n}(c)\, \mathrm{d}c \bigg) \le 0$$

Again, in my opinion, the best we can do here is to require $f' \ge 0$.

Final conditions:

$$f'' \le 0$$
$$f' \ge 0$$

14 Lower Mean Of Search Costs Does Not Imply Lower Price

This one is quite obvious – let us take two different distribution of search costs with the same expected value. It is almost obvious that prices (or Right Hand Sides described in 13) will be different as it depends not only on expected value (we have sum of moments in uniform case, for instance).

On the other hand, if we take a distribution where price is higher and will change it a little (say making $g_{\text{new}}(x) = g(cx)$ for $c = 1 - \epsilon$), price will change continuously in c. Then we may found such c that price will still be higher though lower mean.

15 Dominance in Distribution Function Implies Lower Price (Not Proved)

Theorem 6. *Let $G_1(x) \geq G_2(x)$, $G_1(0) = G_2(0) = 0$, $G_1(\overline{x}) = G_2(\overline{x}) = 1$ and F be a differentiable decreasing function. Than*

$$\int\limits_0^{\overline{x}} F(x)\,\mathrm{d}G_1(x) > \int\limits_0^{\overline{x}} F(x)\,\mathrm{d}G_2(x)$$

Proof.

$$\int\limits_0^{\overline{x}} F(x)\,\mathrm{d}G_1(x) > \int\limits_0^{\overline{x}} F(x)\,\mathrm{d}G_2(x) \Leftrightarrow$$

$$FG_1\Big|_0^{\overline{x}} - \int\limits_0^{\overline{x}} G_1(x)F'(x)\,\mathrm{d}x > FG_2\Big|_0^{\overline{x}} - \int\limits_0^{\overline{x}} G_2(x)F'(x)\,\mathrm{d}x \Leftrightarrow$$

$$\int\limits_0^{\overline{x}} (G_2(x) - G_1(x))F'(x)\,\mathrm{d}x > 0 \Leftrightarrow$$

$$\int\limits_0^{\overline{x}} -F'(x)(G_1(x) - G_2(x))\,\mathrm{d}x > 0$$

□

Theorem 7. *Let $G_1(x) \geq G_2(x)$, $\int\limits_0^{\overline{x}} F(x)G_1(x)\,\mathrm{d}x > \int\limits_0^{\overline{x}} F(x)G_2(x)\,\mathrm{d}x$. Than*

$$\int\limits_0^{\overline{x}} F(x)(G_1(x) - G_2(x))\,\mathrm{d}x > 0$$

15.1 Analyzing RHS

Assumptions: $f' \geq 0$.

RHS:

$$\sum_{k=1}^{n}\left(\int_0^{\overline{c}}\left(F^{k-1}(\hat{x}\,(c))f(\hat{x}\,(c))g_{k,n}(c)\right)-\right.$$

$$\left.-\,f(h(c))'F^{k-2}(h(c))\cdot\left(\big(kF(h(c))-(k-1)\big)\cdot\big(1-G_{k,n}(c)\big)\right)\mathrm{d}c\right)$$

15.1.1 Analyzing $F^{k-1}(\hat{x}\,(c))f(\hat{x}\,(c))$

Using the assumption: $x\nearrow\implies\begin{cases}F(x)\nearrow\\ f(x)\nearrow\end{cases}$

$$c\nearrow\implies\hat{x}\,(c)\searrow\implies\begin{cases}F(\hat{x}\,(c))\searrow\\ f(\hat{x}\,(c))\searrow\end{cases}\implies F^{k-1}(\hat{x}\,(c))f(\hat{x}\,(c))\searrow$$

So $F^{k-1}(\hat{x}\,(c))f(\hat{x}\,(c))$ is decreasing in c.

Using points 3 and 4 from 11 we receive that price decreases if number of shops increases or if search costs fall in terms of first order stochastic dominance.

15.1.2 Analyzing Second Term In a Sum

As we are interested only in part with G or g, we receive the following

$$(f(h(c)))'F^{k-2}(h(c))\cdot\big(kF(h(c))-(k-1)\big)\cdot G_{k,n}(c)$$

Important: $kF(h(c))-(k-1)$ might be both positive and negative: $h(u)$ falls from $\mathbb{E}(x)$ to 0. As F is concave function under our assumptions, $F(\mathbb{E}(x))\geq\frac{1}{2}$ while $F(0)=0$. Therefore, even for $k=2$ it has the same sign if and only if f is uniform, which is not interesting.

(**Not Important and Not Finished**

$f(h(c))''=(f'(h)h')'=(f'(h))'h'+f'(h)h''=f''(h)h'^2+f'(h)h''$p

$$c\nearrow\implies h(c)\searrow\implies\begin{cases}f'(h(c))?????\\ \text{Fill}\\ \text{Fill}\end{cases}\implies\text{Fill}\searrow$$

)

16 Larger Number of Shops Implies Lower Price (Not Proved)

Well, I wanted to use point 3 from 11, but we need the same property as in the section above.

17 May You Please Check This and 1 Formula

We want to prove something about

$$F^{k-1}(\hat{x}\,(c))f(\hat{x}\,(c))g_{k,n}(c) \;+\; (f(h(c)))'F^{k-2}(h(c)) \;\cdot\; \big(kF(h(c)) \;-\; (k \;-\; 1)\big) \;\cdot\; G_{k,n}(c)$$

depending on g, but it depends both on g and G.

Let's take second part in consideration ($H(c)$ is a huge function inside):

$$\int\limits_{\underline{c}}^{\overline{c}} H(x)G(x)\,\mathrm{d}x = \int\limits_{\underline{c}}^{\overline{c}} G(x)\,\mathrm{d}\left(\int_0^x H(t)\,\mathrm{d}t\right) = \int\limits_{\underline{c}}^{\overline{c}} H(t)\,\mathrm{d}t - \int\limits_{\underline{c}}^{\overline{c}} \left(\int\limits_0^x H(t)\,\mathrm{d}t\right) g(x)\,\mathrm{d}x$$

Therefore, the integrand (assuming we are not interested in terms free of G and g) will turn into:

$$\left(F^{k-1}(\hat{x}\,(t))f(\hat{x}\,(t)) - \int\limits_0^t \big(f(h(c))'F^{k-2}(h(c)) \cdot (kF(h(c)) - (k-1))\big)\,\mathrm{d}c\right) g_{k,n}(t)$$

As it turned out above, we are interested in a decreasing function; a derivative diminish the integral:

$$\begin{aligned}(k-1)F^{k-2}(\hat{x}\,(t))f^2(\hat{x}\,(t)) \cdot \hat{x}\,(t)' + F^{k-1}(\hat{x}\,(t))f'(\hat{x}\,(t)) \cdot \hat{x}\,(t)' - \\ - kF^{k-1}(h(t))f'(h(t) \cdot h'(t) + (k-1)(F^{k-2}(h(t))f'(h(t)) \cdot h'(t)\end{aligned}$$

I am not sure whether I have done everything correctly. Moreover, we want this to be negative and I failed.

References

[1] Peter A. Diamond. A Model of Price Adjustment // *Journal of Economic Theory 3, 156-168, 1971*

[2] Martin L. Weitzman. Optimal Search for the Best Alternative. // *Econometrica, Vol. 47, No. 3, 1979*

[3] Jeffrey M. Perloff, Steven C. Salop. Equilibrium with Product Differentiation. // *Review of Economic Studies LII, 107-120, 1985*

[4] Asher Wolinsky. True Monopolistic Competition as a Result of Imperfect Information. // *The Quarterly Journal of Economics, vol. 101, issue 3, 493-511, 1986*

[5] Simon P. Anderson, Regis Renault. Pricing, product diversity, and search costs: a Bertrand-Chamberlin-Diamond Model. // *RAND Journal of Economics Vol. 30, No. 4, 719–735, 1999*

[6] Ali Hortacsu, Chad Syverson. Product Differentiation, Search Costs, and Competition in the Mutual Fund Industry: A Case Study of the S&P 500 Index Funds. // *Quarterly Journal of Economics 119 : 403-456, 2004*

[7] Arbatskaya, M.. Ordered search. // *RAND Journal of Economics, 38(1), 119-126, 2007*

[8] Jidong Zhou. Ordered search in differentiated markets. // *International Journal of Industrial Organization, Volume 29, Issue 2, Pages 253-262, 2011*

[9] Mark Armstrong. Ordered Consumer Search. // *https://mpra.ub.uni-muenchen.de/72194/, 2016*

[10] Jose Luis Moraga-Gonzalezy, Zsolt Sandorz, Matthijs R. Wildenbeestx. Prices and Heterogeneous Search Costs. // *Volume 48, Issue 1, 125–146, Spring 2017*

[11] Jose Luis Moraga-Gonzalezy, Zsolt Sandorz, Matthijs R. Wildenbeestx. Supplementary Appendix to Prices and Heterogeneous Search Costs.

[12] Michael Choi, Anovia Yifan Dai, Kyungmin Kim, Consumer Search and Price Competition. // *Available at SSRN: https://ssrn.com/abstract=2865162 or http://dx.doi.org/10.2139/ssrn.2865162, November 6, 2016*

[13] Quint, Daniel. Imperfect competition with complements and substitutes. // *Journal of Economic Theory, 2014, 152, 266–290.*