Synthesis of a PSD Compatible Acceleration Time-History

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Abstract

System survival to a random vibration environment is frequently defined in terms of the acceptable response to a prescribed Power Spectral Density (PSD). Commercial electro-dynamic shakers have control systems that will subject a test article to an input acceleration that is compatible with a required PSD. In some cases, it is desirable to have an acceleration time-history that is compatible with a prescribed PSD for pre-test and/or post-test random vibration analysis of the system. Study of the temporal response of a nonlinear system is one example where a spectrum compatible acceleration time-history could be beneficial.

An approach is presented herein to synthesize a base acceleration time-history that is compatible with a prescribed PSD. The synthesized acceleration is developed using the summation of sinusoids with frequencies that cover the frequency range of the PSD. Each sinusoid has a unique frequency, unique amplitude and random phase angle. The amplitude of each sinusoid is sized by equating the maximum power of the sinusoid with the area under the PSD curve corresponding to the frequency of the sinusoid. An envelope function $E(t)$ is used to control the overall shape and duration of the synthesized acceleration. The peak amplitude of the envelope is sized to match the total power of the prescribed PSD to that of the PSD that results from the synthesized base acceleration. This is done efficiently by Newton Raphson iteration. Three examples are presented to illustrate the process.

I. Introduction

Shock and vibration engineers are typically familiar with the process of converting acceleration in the time domain to spectral representations such as shock response spectra (SRS) and power spectral densities (PSD). However, occasionally the inverse

process is needed; namely synthesizing an acceleration time-history from a spectral representation. Manufacturers of electro-dynamic test machines create waveforms from prescribed spectral representations to drive a shaker head with an acceleration time-histories compatible with specified SRS or PSD. These algorithms are proprietary to the manufacturers of these systems. A number of papers have been published, largely by seismic engineers (Alexander, Gasparini, Levy, Lin, Preumont, Rizzo, Salginik, Scanlan) with methods to synthesize an acceleration time-history to be compatible with a prescribed SRS. However, the process of synthesizing an acceleration time-history that is compatible with a prescribed PSD is not as well published. Seismic engineers (Hou, Housner, Jennings, Kanai, Tajimi) have addressed the synthesis of wave-forms for seismic simulations based on PSD spectral representations, motivated by the random nature, relative long durations and the multi-frequency content of a single earthquake. However, there is a paucity of research for PSD spectra inverse transform to transient wave forms in the mechanical shock and vibration community. This may be due to the differences of the physical phenomena of mechanical shock and random vibrations. The shock response spectrum, as the name implies, is the spectral representation of a relatively short, high intensity transient shock event with a defined rise, peak and decay. The power spectral density on the other hand, at least for mechanical systems, is used largely to characterize random vibrations over relatively long durations.

The author, recently faced with a task to synthesize an acceleration wave form to meet a prescribed PSD, developed a simple procedure described herein. The procedure synthesizes an acceleration time-history, the PSD of which has the same total power as that of the prescribed PSD. The overall shape and duration of the synthesized wave form is controlled by an envelope function that can be tailored to be consistent with the temporal characteristics of the physical event being considered.

II. Approach

The objective of this study was to synthesize a base acceleration to match a prescribed Power Spectral Density (PSD). There are a number of ways that this could be achieved. The one examined herein is the summation of sinusoids of different frequencies with amplitudes A_n and random phase angles ϕ_n for each frequency n. The overall shape and duration of the synthesized wave form is controlled by an

envelope function $E(t)$. Judicious choices for the amplitudes A_n will tailor the synthesized base acceleration to yield a PSD that matches a target PSD within a certain error. The form of the synthesized base acceleration (Alexander) is given by,

$$a_b(t) = E(t) \sum_{n=1}^{N} A_n \sin(\omega_n t + \phi_n)$$

(1)

To determine the amplitude coefficients A_n, the total power of the PSD is matched to the power obtained from the summations of the sinusoids (Gasparini). The total power of a signal $a(t)$ of duration T is defined as,

$$Power = \frac{1}{T} \int_0^T a^2(t) dt$$

$$Power = \frac{A_{ave}^2 T}{T} = A_{ave}^2$$

For a sine wave of peak amplitude A_n, the average amplitude is given by $A_{ave} = sqrt(1/2)*A_n$ resulting in,

$$Power_n = (0.707 A_n)^2 = \frac{1}{2} A_n^2$$

The total power of the synthesized base acceleration can then be represented as the summation of the power contributions of each sinusoid from (1).

$$Power = \sum_{n=1}^{N} \left(\frac{A_n}{2} \right)^2$$

(2)

The total power of a PSD also corresponds to the area of the PSD. Figure 1 shows a graph of a notional PSD. For equal discrete frequency intervals Δf *the* area under the curve, and hence power, over all frequencies n is approximated as,

$$Power = \sum_{n=1}^{N} PSD_n \Delta f$$

(3)

For each frequency n, equating the power represented by the area ($PSD_n \Delta f$) with the power corresponding to the amplitude of the nth sinusoid of a synthesized base

acceleration (2) results in a relationship (4) between the amplitude A_n and the PSD_n for the nth frequency. This relationship is used to determine A_n at each frequency from the target PSD to yield an initial synthesized base acceleration.

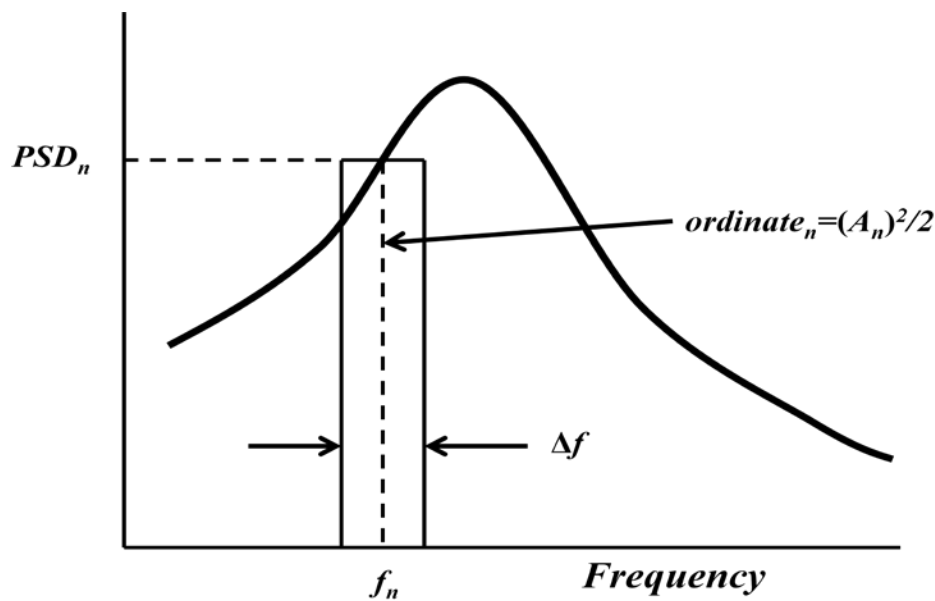$$\frac{A_n^2}{2} = PSD_n \Delta f$$

$$A_n = \sqrt{2 PSD_n \Delta f} \qquad\qquad (4)$$

Figure 1 – Relationship of Power Spectral Density Function to Sinusoid Amplitude

Synthesis Process

An initial synthesized acceleration time-history (1) can be readily determined from the A_n coefficients from (4) with the amplitude of the envelope function, A_e shown in Figure 2, set to unity. A PSD of the initial synthesized acceleration-time history can then be determined and compared with the target PSD. However, to get a match of the power calculated from (3) from the target PSD to that of the synthesized acceleration-time history, iteration is required to fine tune A_e. An efficient way to do this is to use Newton Raphson iteration. If it is assumed that the error in the power difference between the

target PSD and the PSD from the synthesized base acceleration can be characterized as a function of A_e, then a Newton Raphson iteration process can be defined by the relationships below. Figure 3 illustrates graphical how the iteration process works for improving successive choices of A_e for each iteration i.
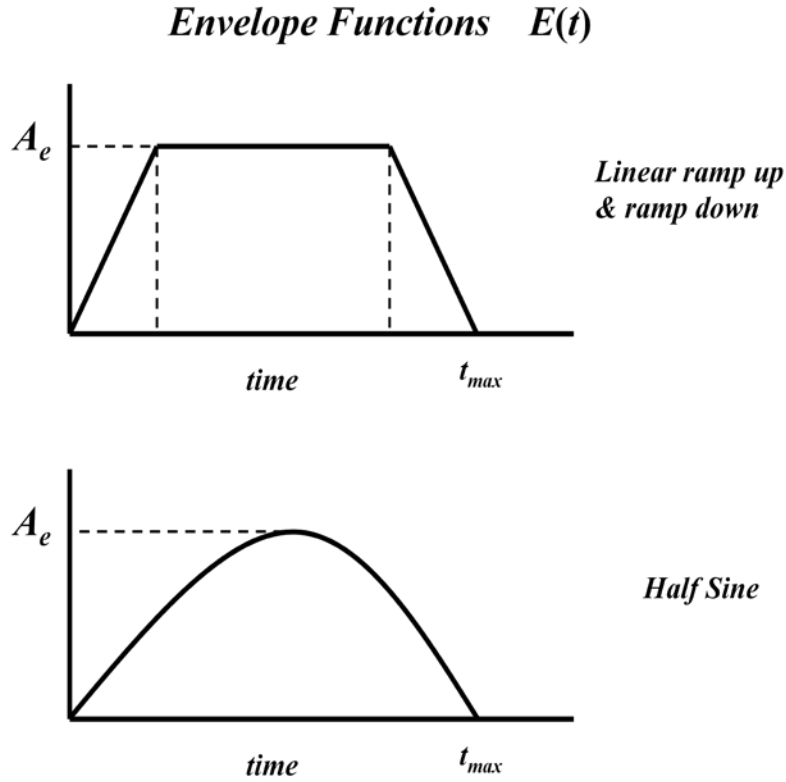
Envelope Functions $E(t)$

Figure 2 – Typical Envelope Functions $E(t)$

Newton-Raphson Iteration Procedure:

$$\% \ error \ power \ (PSD_{synthesized} : PSD_{t\,arg\,et}) \equiv f(A_e)$$

$$(A_e)_{i+1} = (A_e)_i - \frac{f\big[(A_e)_i\big]}{\dfrac{df}{dA_e}\big[(A_e)_i\big]}$$

where

$$\frac{df}{dA_e}\big[(A_e)_i\big] \cong \frac{(\% \ error)_i - (\% \ error)_{i-1}}{(A_e)_i - (A_e)_{i-1}}$$
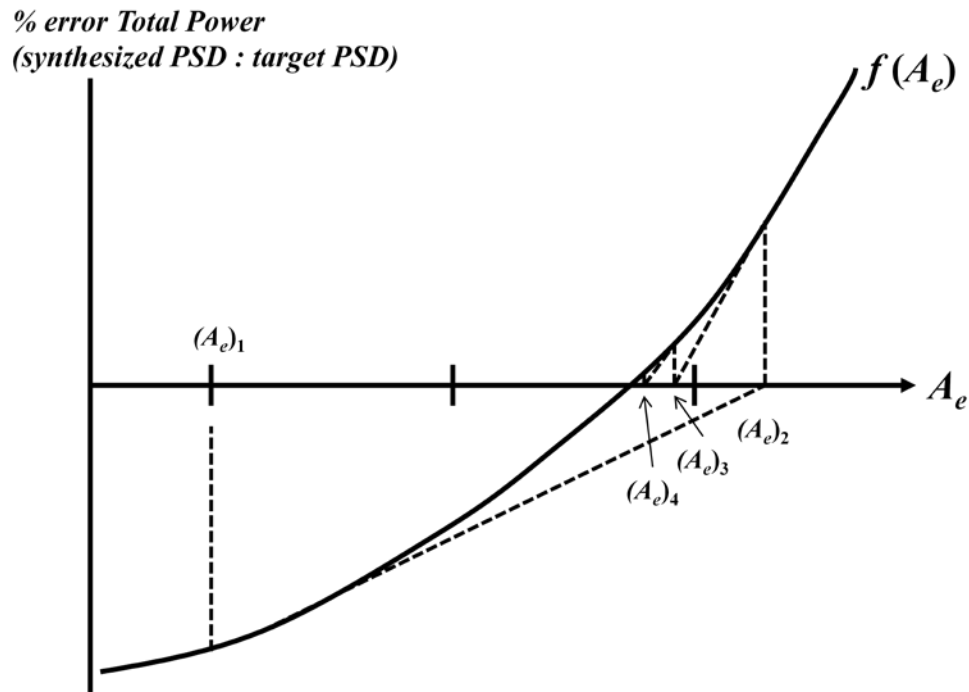
Figure 3: Newton Raphson Iteration to Determine A_e

III. Results

Example #1

A Matlab function was written to synthesize an initial base acceleration from (1) given a prescribed PSD using coefficients determined from (4). The process was checked using a known acceleration time-history data record from which a target PSD was calculated with Matlab's *PWELCH* function. Two envelope functions $E(t)$ shown on Figure 2, were developed for the synthesis process. Others can be readily developed depending on the specifics of the physical phenomena being simulated. As an initial guess, the $E(t)$ function is set with A_e=1.0. The acceleration-time history data used to test the Matlab function and the corresponding PSD are shown on Figures 4 and 5, respectively.
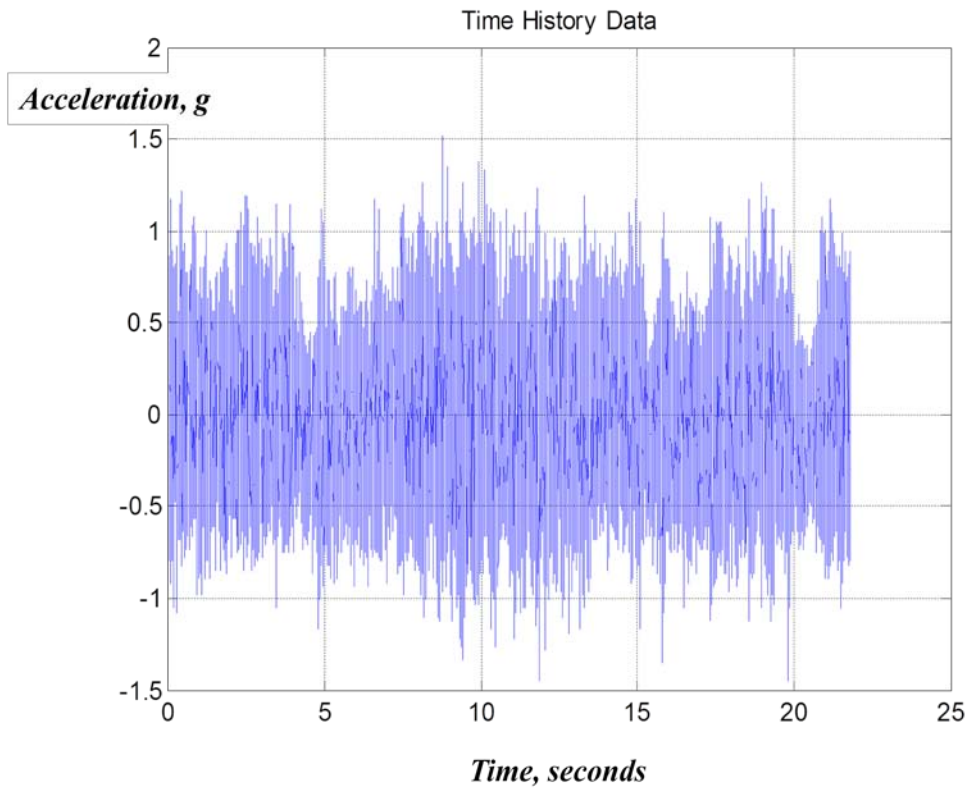
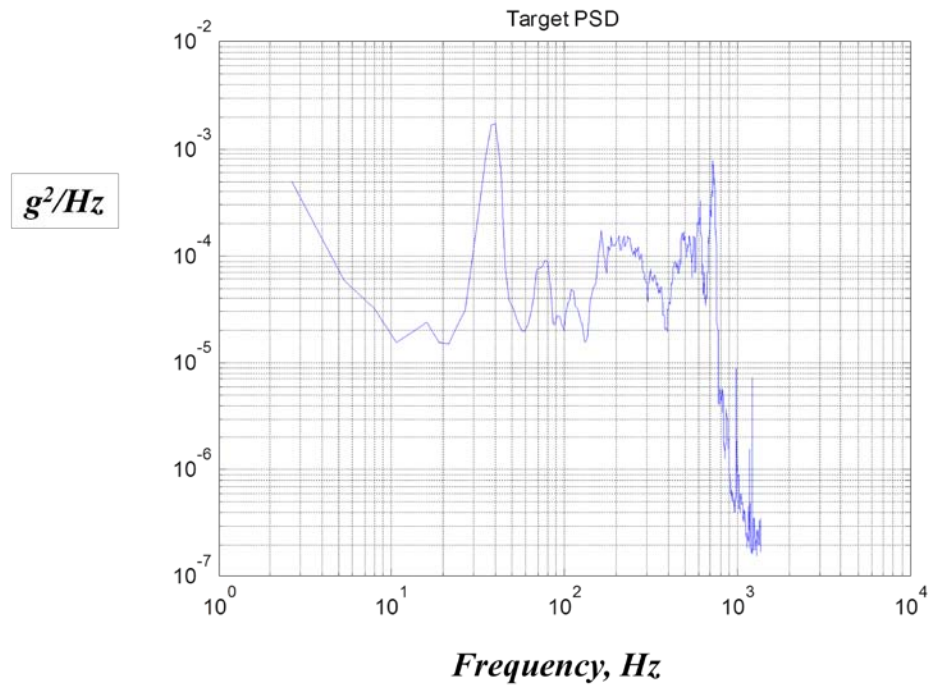## *Figure 4 - Test Data Acceleration Time-History*

Figure 5 – Target PSD Determined from Time-History Data

Using the relationship (4) and the target PSD of Figure 5, the amplitudes A_n for the sinusoids were determined for each frequency, plotted in Figure 6. The first calculation of the synthesized acceleration time-history was developed using these A_n amplitudes and the ramp-up, ramp-down envelope function in Figure 2 with A_e initially set to unity. The ramp-up, ramp-down durations were each set to 10% of the total duration of the data pulse. This resulted in a synthesized base acceleration with a PSD error of 13.3% (averaged over all frequencies) and a power error of -14.7% based on the total area under each PSD curve. These errors are prior to iterations to adjust A_e. After the iteration process characterized in Figure 3, the final value of A_e converged to 1.083 after five iterations. With this value, the power error was reduced to 1.5×10^{-5}%. The final synthesized base acceleration is plotted on Figure 7. The final PSD error was 6.7%. The PSD of the original time-history data and the PSD of the synthesized base acceleration are plotted on Figure 8.
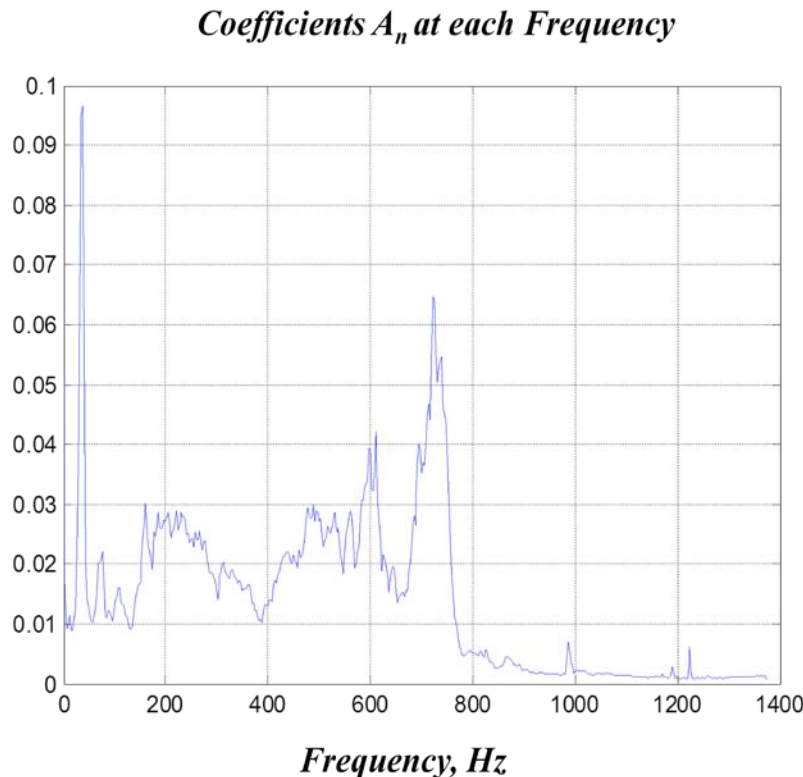

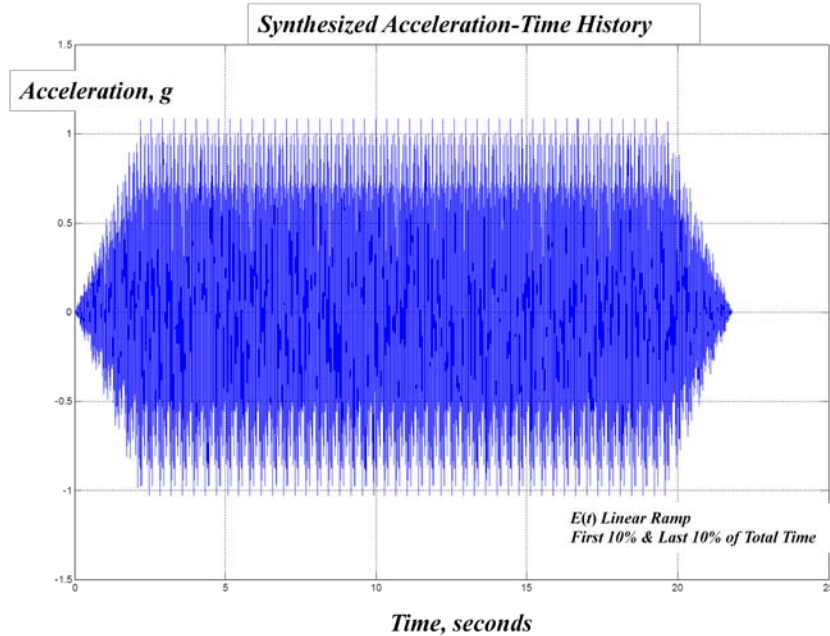**Figure 6 – Sinusoid Amplitudes A_n Determined from Target PSD**

Figure 7 – Final Synthesized Base Acceleration

PSD abs(% error) = 6.68%
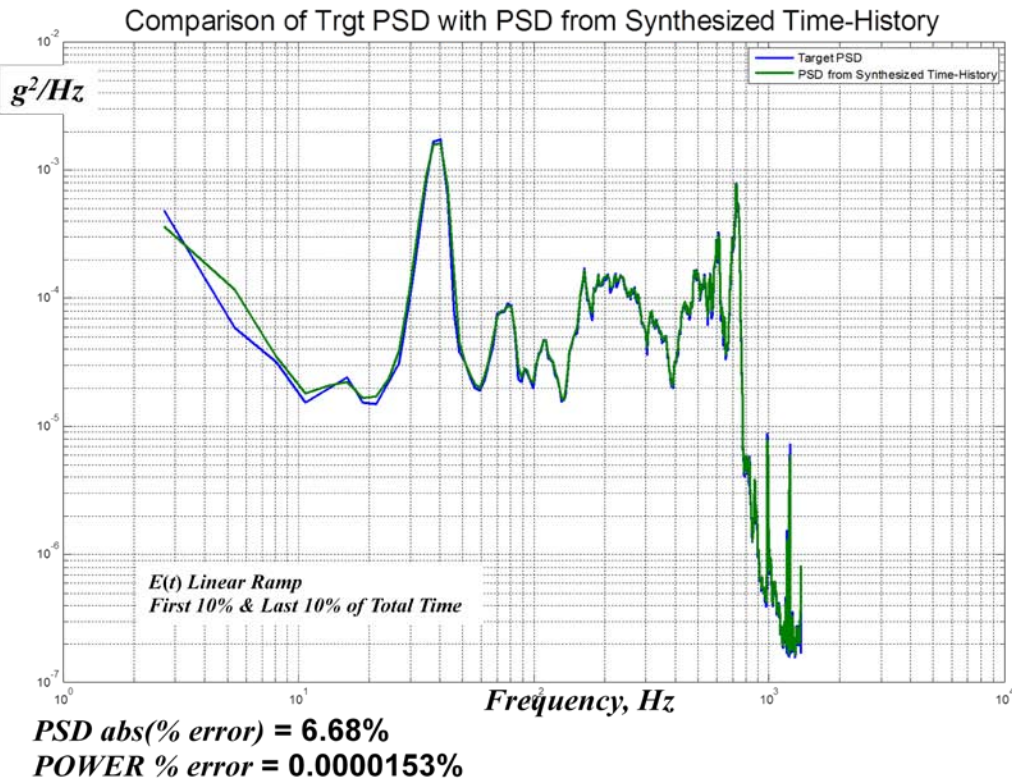**POWER % error = 0.0000153%**

Figure 8 –Comparison of Target PSD with PSD from the Final Synthesized Acceleration-Time History (ramp up/down)

Example #2

A second example was examined using the same test data as Example #1 (Figure 4) with the half sine envelope function of Figure 2. As before, the amplitude of the half sine envelope A_e was initially set to unity prior to iteration. Since the same time history data from Example #1 was used, the target PSD remained the same. Hence, the A_n coefficients from (4) are unchanged from those plotted in Figure 5. Prior to iteration to size A_e, the PSD error was 48.6% and a power error was -50.8%. After the iteration process shown in Figure 3, the final value of A_e converged to 1.4261 after five iterations. With this value, the power error was reduced to 2.7×10^{-4} %. The final synthesized base acceleration is plotted on Figure 9. The final PSD error was 6.3%. The PSD of the original time-history data and the PSD of the synthesized base acceleration are plotted on Figure 10.
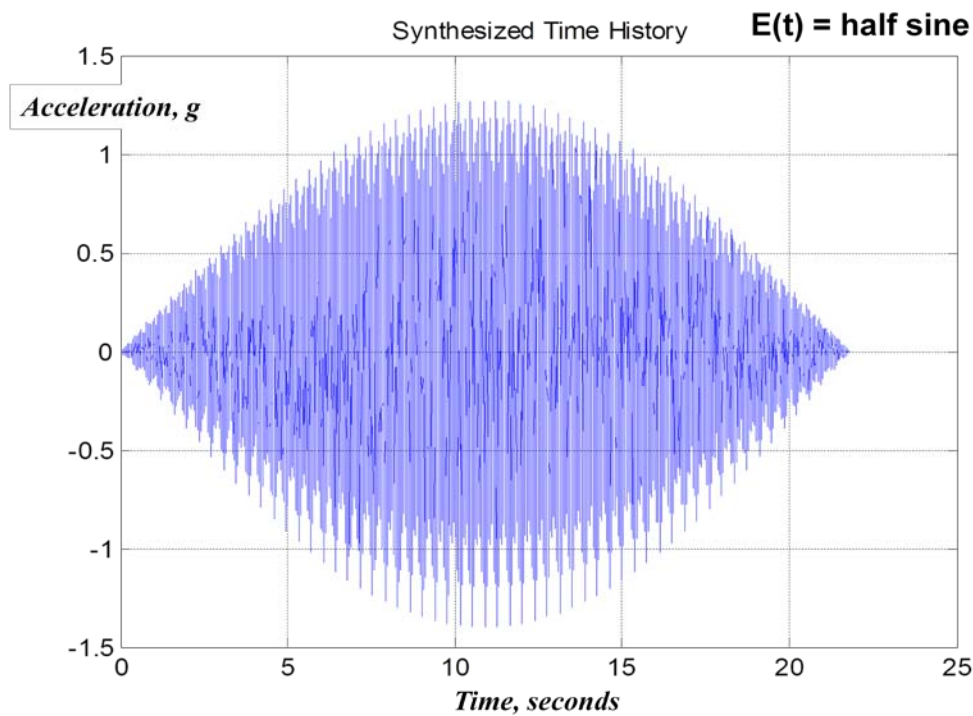

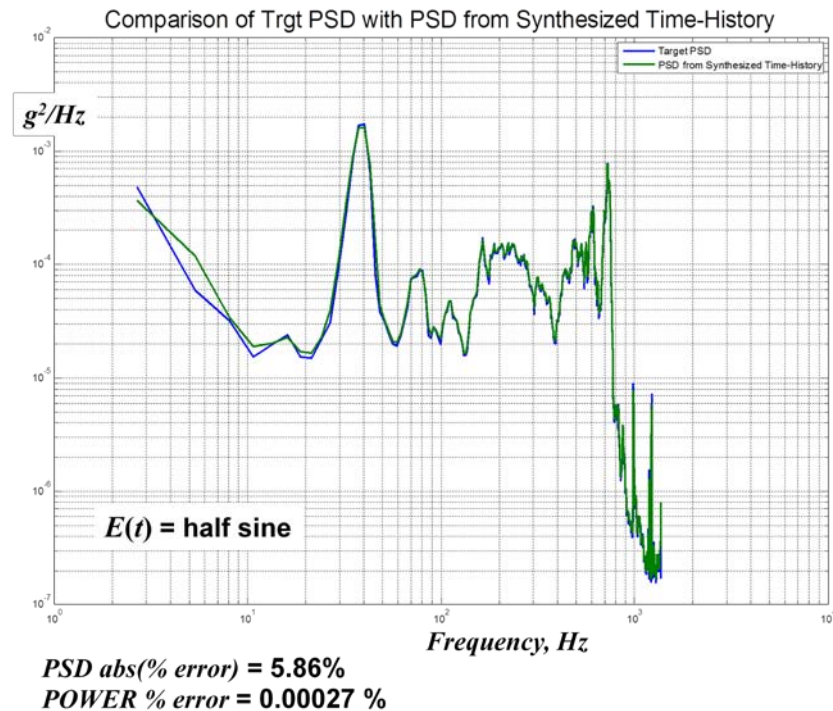*Figure 7 – Final Synthesized Base Acceleration*

Comparison of Trgt PSD with PSD from Synthesized Time-History

PSD abs(% error) = 5.86%
POWER % error = 0.00027 %

Figure 10 – Comparison of Target PSD with PSD from the Final Synthesized Acceleration-Time History (Half Sine)

Example #3

While for shock and vibration applications the PSD is typically a representation of an acceleration spectral density as a function of frequency, there are other applications where the PSD is used to represent a physical parameter in the frequency domain. An example of this is the characterization of the surface roughness of ground terrain for combat vehicle mobility testing. The US Army Aberdeen Test Center (ATC) publication TOP 1-1-010 (references list) includes PSD roughness plots of numerous ATC vehicle test courses (Belgian Block, Perryman, Munson, Churchville, and others). For this application, the PSD magnitude is characterized in units of $ft^2/(cycles/ft)$ as a function of Wave Number (frequency) with units of $cycles/ft$. The procedure presented herein is equally applicable to a spectral representation. Using these measured test course PSD's, a spectrum compatible terrain profile can be synthesized. In this case the resulting synthesized profile is the height of the surface (displacement) as a function of distance.

The Belgian Block course was used to demonstrate the utility of the synthesis process for this example. Figure 11 is a plot of the Belgian Block PSD which was extracted from a graphical plot in TOP 1-1-101. While the synthesis process works identically as the prior two examples, (1) is rewritten as (5) with symbols that traditionally correspond to distances rather than acceleration and time. In this case the ***height(x)*** corresponds to the height of the ground as a function of distance ***x.***

$$height(x) = E(x) \sum_{j=1}^{j_{max}} A_j \sin(\omega_j t + \phi_j)$$

(5)

The coefficients A_j of the sinusoids were determined from (4) using the target spectrum and are plotted in Figure 12. The synthesized terrain profile is plotted in Figure 13. For this example, the linear ramp-up, ram-down profile of Figure 2 was used where the duration of each ramp was 10% of the total duration. As a check on the procedure, Figure 14 compares a plot of a 160 foot segment of the synthesized terrain compared with Belgian Block course test data of the same distance. The peak amplitudes of the synthesized terrain profile exceed two inches and are less than three inches, which is also the case with the test data. The frequency content is also generally comparable. By visually counting the peaks, the test data has approximately 13 cycles in 160 feet and the synthesized terrain also has approximately 13 for the same distance. This corresponds to 0.081 cycles/foot which generally agrees with the PSD peak which occurs between 0.088 and 0.094 cycles/foot. Figure 15 is a plot of the Belgian Block target PSD with that of the synthesized terrain. Newton Raphson iteration sized the envelope peak amplitude, A_e, to match the power of the synthesized terrain PSD to that of the target PSD. The resulting value for A_e was 1.0335. The error of the synthesized PSD was 7.67% averaged over all frequencies when compared to the target PSD.
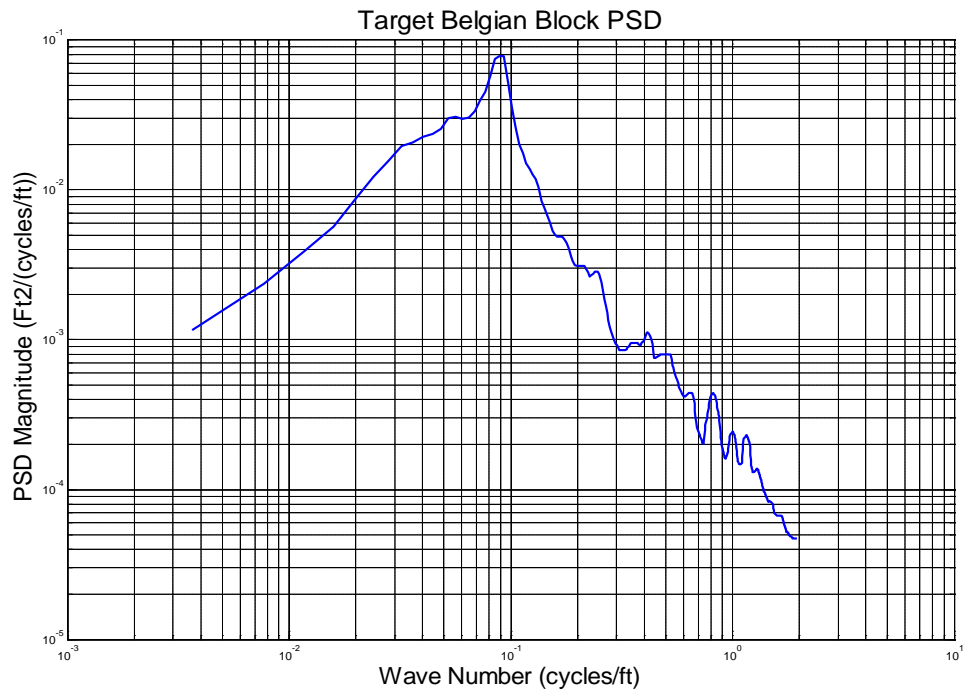
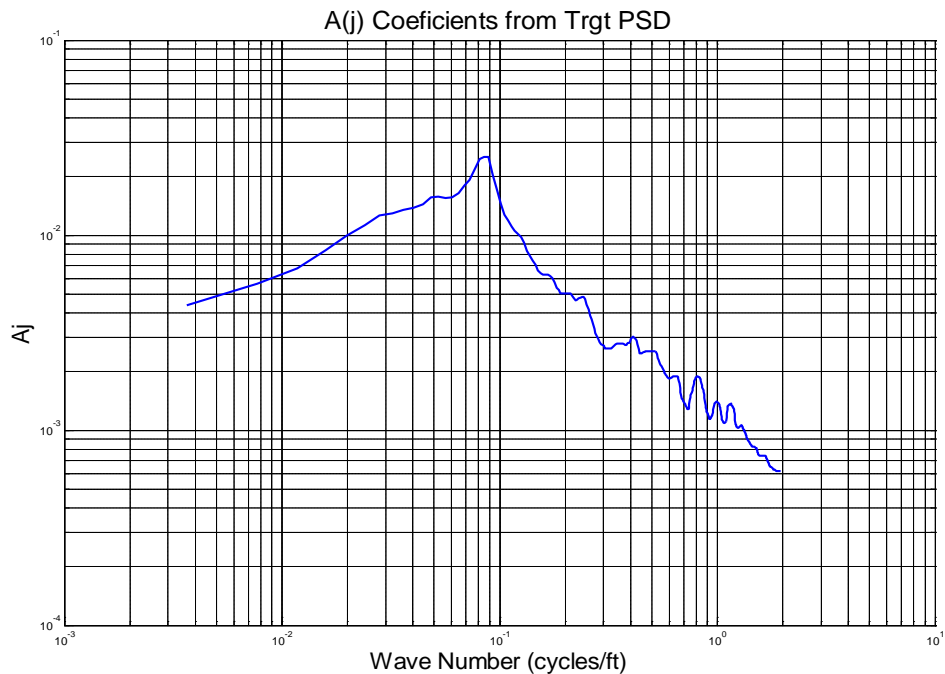## Figure 11 –Target Belgian Block PSD

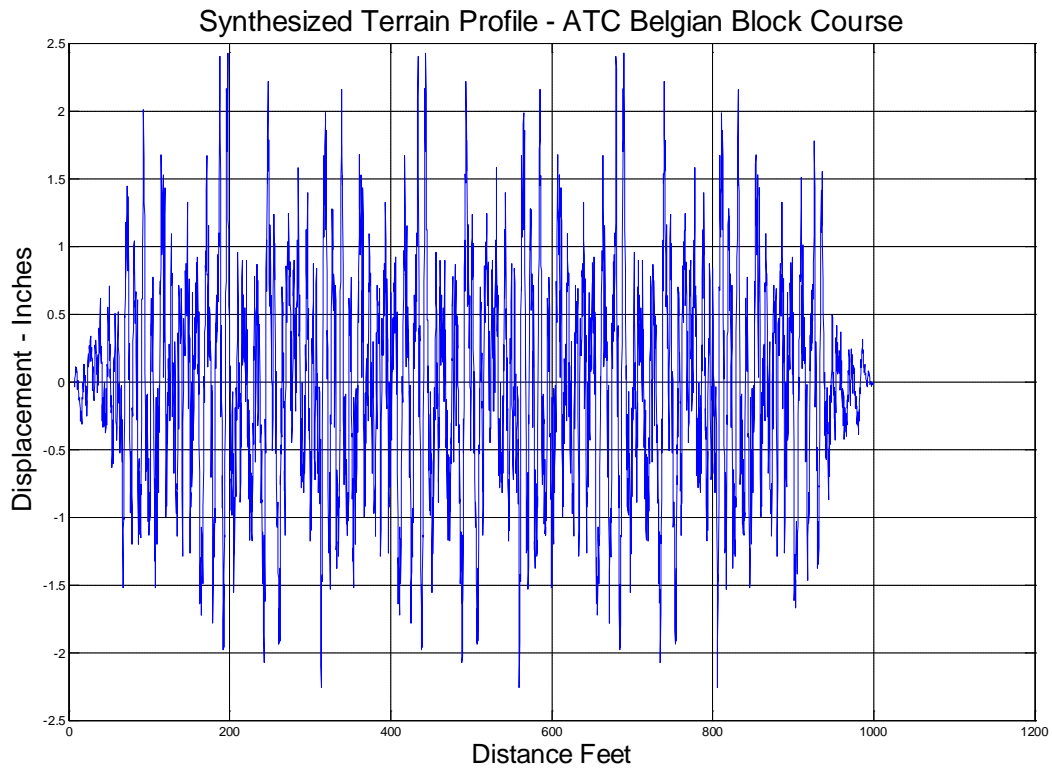## Figure 12 –Coefficients of the Sinusoids of Equation (5)

Figure 13 –Synthesized Terrain Profile Compatible with Belgian Block PSD
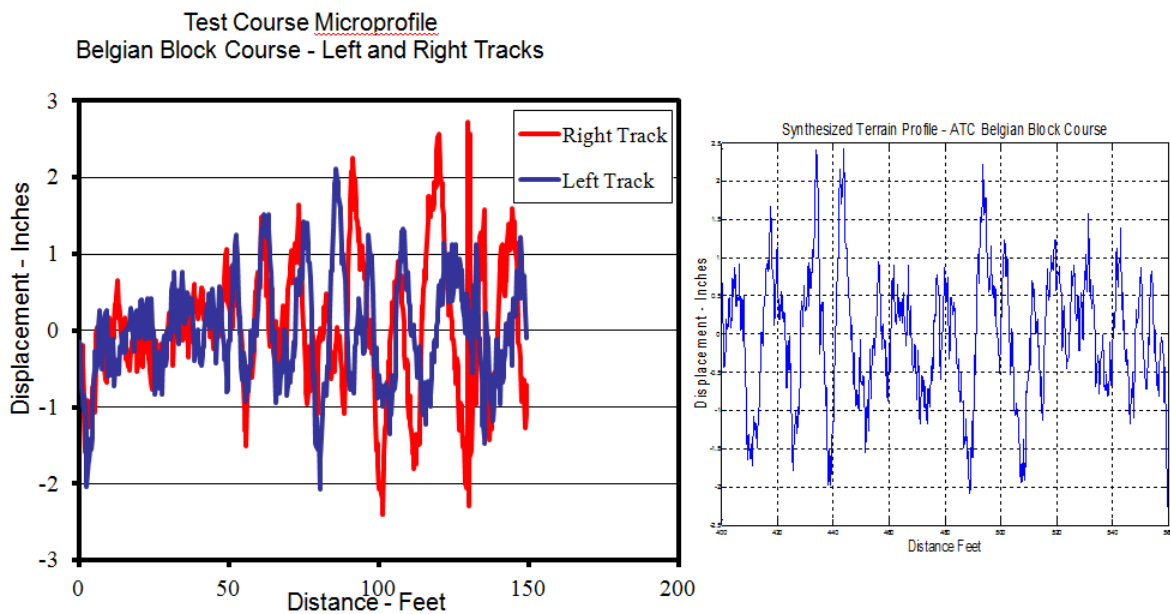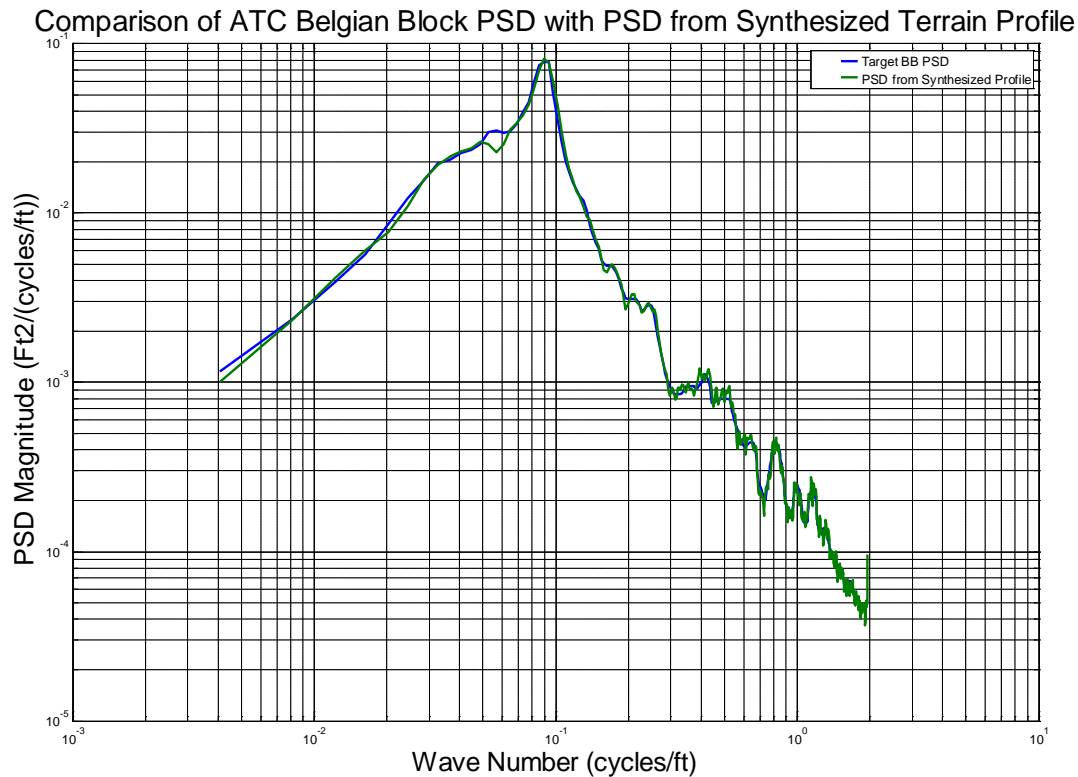
Figure 14 – Belgian Block Test Course & Synthesized Terrain Profiles

Comparison of ATC Belgian Block PSD with PSD from Synthesized Terrain Profile

Figure 15 – Belgian Block Test Course PSD & Synthesized Terrain PSD

IV. Conclusions

A simple procedure has been developed to synthesize a base acceleration time-history to be compatible with a prescribed PSD using a summation of sinusoids with random phase angles over frequencies that span the frequency range of the PSD. Each sinusoid amplitude is sized to match the power of the PSD corresponding to that frequency. An envelope function $E(t)$, used to control the overall shape and duration of the synthesized acceleration, is constructed to tailor the synthesized acceleration time-history to be compatible with the physical event. The peak amplitude of the envelope is determined by Newton Raphson iteration such that the total power of the PSD that results from the synthesized base acceleration matches that of the prescribed PSD.

Three examples were presented. The first two examples are based on the synthesis of an acceleration time-history to be compatible with a conventional PSD with units of g^2/Hz. A third example demonstrated the flexibility of the procedure for the synthesis of a ground terrain profile to be compatible with the US Army Aberdeen Test Center's Belgian Block test course PSD having units of $Ft^2/(cycle/Ft)$. Comparison of a segment of the Belgian Block test terrain profile to that of the synthesized terrain were in good agreement in terms of terrain height and frequency content.

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