Department of Mathematics and Natural Sciences

Semester: Summer 2023

Final Examination

Course Title: Complex Variables and Laplace Transformations

Course ID: MAT-215

Total Marks: 35 Date: September 3, 2023

Time: 1 hour 30 minutes

Answer any five questions $[5 \times 7 = 35]$

1. [7]

Let $f(z)$ be analytic inside and on a simple closed curve C and let a be any point inside C, Then $f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-a)} dz$ where C is traversed in the positive (counterclockwise) sense. Also, the nth derivative of $f(z)$ at $z = a$ is given by $f^n(a) = \frac{n!}{2\pi i} \oint_c \frac{f(z)}{(z-a)^{n+1}} dz$, $n = 1,2,3 \ldots$. Evaluate $\oint \frac{\cos^6 z}{(z-\frac{\pi}{6})^3} dz$ around the circle C with $|z| = 1$.

2.

Let $F(z)$ be analytic inside and on a simple closed curve C except for a pole of order m at $z = a$ inside C, then $\frac{1}{2\pi i} \oint F(z)dz = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \{(z-a)^n F(z)\}$. Now Evaluate $\oint_c \frac{e^{zt}}{(z^2+1)^2} dz$ when $t > 0$ and C is the circle $|z| = 3$.

3. Evaluate $\int_c |z|^2 dz$ around the square with vertices at $(0,0), (1,0), (1,1), (0,1)$. [7]

4. [4+3]

   a. Find the Laplace Transformation of $f(t) = 3 \sin 2t \cos 2t$ by using definition.

   b. Find the Laplace Transform of $f(t) = te^{-3t} \sin 3t$.

5. [4+3]

a. Find the Inverse Laplace Transformation of $\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$.

b. Show that $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$.

6. Consider the following: [7]

i. The unit step function $u(t-a)$ is defined to be $u(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$.

ii. A general pricewise-defined function of the type $f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$ is the same as $f(t) = g(t) - g(t)u(t-a) + h(t)u(t-a)$.

iii. If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$, then $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$. If $f(t) = \mathcal{L}^{-1}\{F(s)\}, a > 0$, then $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$.

iv. Alternative form of second translation theorem is $\mathcal{L}\{g(t)u(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$.

Now use these information, and solve $y' + y = f(t), y(0) = 5$,

where $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\sin t, & t \ge \pi \end{cases}$.

7. [7]

Solve the ordinary differential equation by using Laplace transform and check the solution:

$y''(t) - 3y'(t) + 2y(t) = 4t + 12e^{-t}, y(0) = 6, y'(0) = -1.$