Lecture Note

Inspiring Excellence

Math and Natural Science Department

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Chapter 1

Circular motion

Until now, we have learned motion along straight line, projectile motions, etc. In this chapter, we shall investigate a special class of motion, motion in a plane about a central point, a motion we shall refer to as central motion, the most outstanding case of which is circular motion.
let me give you some motivation!
You love to enjoy moonlit night and you know that the moon is orbiting our earth.Now you want to calculate when we will have next Moonlit night. In order to find this, you need to know about circular motion. As, the motion of the moon around the earth is nearly circular. The motions of the planets around the sun are also nearly circular. Our sun moves in nearly a circular orbit about the center of galaxy, 50,000 light years from a massive black hole at the center of the galaxy. After the understanding of gravitation, Newton solved the two-body under a gravitational central force and discovered that the orbit can be circular, elliptical, parabolic or hyperbolic.
Now, let's talk about our everyday life. There are many instances of central motion about a point; a bicycle rider on a circular track, a ball spun around by a string, and the rotation of a spinning wheel are just a few examples of circular motion.

We shall begin by describing the kinematics of circular motion, the position, velocity, and acceleration, as a special case of two-dimensional motion. Some key point's about circular motion are:

- Unlike linear motion where the velocity, acceleration are always directed towards along the line of motion, In circular motion the velocity always directs toward the tangent of the circle. That means the direction of velocity is always changing, when something is moving along a circular path.
- In case of circular motion, we shall see a non-zero component of the acceleration directed radially inward.It is referred as the centripetal acceleration. Even if the object is moving with a constant speed in a circle, we will have the centripetal acceleration which is non-zero.
- If our object is increasing its speed or slowing down, there is also a non-zero tangential acceleration in the direction of motion.

1.1 Velocity and Angular Velocity

Before defining velocity and angular velocity, Let's begin with choosing the polar Coordinates.

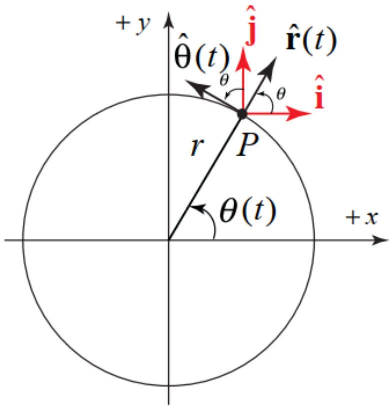

Figure 1.1: A circular orbit with unit vectors

At time t , the particle is located at the point P with coordinates $(r, \theta(t))$ and position vector given by

$$\vec{r}(t) = r\hat{r}(t) \tag{1.1}$$

'At point p, consider the two sets of unit vector $(\hat{r}, \hat{\theta})$ and $(\hat{\imath}, \hat{\jmath})$ as shown in figure (1.1). Using the rule of vector decomposition, we can write the unit vectors $(\hat{r}, \hat{\theta})$ in terms of $(\hat{\imath}, \hat{\jmath})$.

$$\hat{r}(t) = \cos\theta(t)\hat{\imath} + \sin\theta(t)\hat{\jmath} \tag{1.2}$$

$$\hat{\theta}(t) = -\sin\theta(t)\hat{\imath} + \cos\theta(t)\hat{\jmath} \tag{1.3}$$

Now we can define the velocity as,

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}(t)) = r\frac{d\hat{r}}{dt} \tag{1.4}$$

Let's first begin with $\frac{d\hat{r}}{dt}$,

$$\begin{aligned}
\frac{d\hat{r}}{dt} &= \frac{d}{dt}(\cos\theta(t)\hat{\imath} + \sin\theta(t)\hat{\jmath}) \\
&= -\sin\theta(t)\frac{d\theta(t)}{dt}\hat{\imath} + \cos\theta(t)\frac{d\theta(t)}{dt}\hat{\jmath} \\
&= \frac{d\theta(t)}{dt}(-\sin\theta(t)\hat{\imath} + \cos\theta(t)\hat{\jmath}) \\
&= \frac{d\theta(t)}{dt}\hat{\theta(t)}
\end{aligned}$$

We have used,

$$\begin{aligned}
\frac{d}{dt}(\sin\theta(t)) &= -\cos\theta(t)\frac{d\theta(t)}{dt} \\
\frac{d}{dt}(\cos\theta(t)) &= \sin\theta(t)\frac{d\theta(t)}{dt}
\end{aligned}$$

Now, we can write the velocity vector as,

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}(t)) = r\frac{d\hat{r}}{dt} = r\frac{d\theta}{dt}\hat{\theta}(t) = v_\theta\, \hat{\theta}(t) \tag{1.5}$$

Where the θ component of velocity is given as,

$$v_\theta = r\frac{d\theta}{dt},$$

This is referred as the tangential component of the velocity.Denote the magnitude of the velocity by $v = |\vec{v}|$, The angular speed is the magnitude of the rate of change of angle with respect to time, which we denote by the Greek letter ω,

$$\omega = |\frac{d\theta}{dt}|$$

1.2 Circular Motion: Tangential and Radial Acceleration

In case of circular motion, The acceleration has two components, the tangential component a_θ, and the radial component, a_r. We can write the acceleration vector as

$$\vec{a}(t) = a_\theta\, \hat{\theta}(t) + a_r\, \hat{r}(t) \tag{1.6}$$

Keep in mind that as the object moves in a circle, the unit vectors $\hat{r}(t)$ and $\hat{\theta}(t)$ change direction and hence are not constant in time. Suppose that the tangential velocity $v_\theta = rd\theta/dt$ is changing in magnitude due to the presence of some tangential force; we shall now consider that $d\theta/dt$ is changing in time, (the magnitude of the velocity is changing in time). Recall that in polar coordinates the velocity vector can be written as,

$$\vec{v}(t) = r\frac{d\theta}{dt}\hat{\theta}(t)$$

Now using the product rule, we can find the acceleration as,

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = r\frac{d^2\theta(t)}{dt^2}\hat{\theta(t)}(t) + r\frac{d\theta(t)}{dt}\frac{d\hat{\theta}(t)}{dt} \tag{1.7}$$

Recall that $\hat{\theta(t)} = -\sin\theta(t)\hat{\imath} + \cos\theta(t)\hat{\jmath}$. So we can write,

$$\begin{aligned}
\vec{a}(t) &= r\frac{d^2\theta(t)}{dt^2}\hat{\theta}(t) + r\frac{d\theta(t)}{dt}\frac{d}{dt}(-\sin\theta(t)\hat{\imath} + \cos\theta(t)\hat{\jmath}) \\
&= r\frac{d^2\theta(t)}{dt^2}\hat{\theta}(t) + r\frac{d\theta(t)}{dt}(-\cos\theta(t)\frac{d\theta}{dt}\hat{\imath} - \sin\theta(t)\frac{d\theta}{dt}\hat{\jmath}) \\
&= r\frac{d^2\theta(t)}{dt^2}\hat{\theta}(t) - r(\frac{d\theta(t)}{dt})^2\hat{r}(t)
\end{aligned}$$

So, now the tangential component of the acceleration is,

$$a_\theta = r\frac{d^\theta(t)}{dt^2} \tag{1.8}$$

and the radial component of the acceleration is given by

$$a_r = -r(\frac{d\theta(t)}{dt})^2 = -r\omega^2 < 0 \tag{1.9}$$

Because $a_r < 0$, that radial vector component $\vec{a}_r(t) = -r\omega^2\hat{r}(t)$ is always directed towards the center of the circular orbit.

1.3 Uniform Circular Motion

If the object is constrained to move in a circle and the total tangential force acting on the object is zero, $F_\theta^{total} = 0$, then (Newton's Second Law), the tangential acceleration is zero,

$$a_\theta = 0$$

This means that the magnitude of the velocity (the speed) remains constant. This motion is known as uniform circular motion. The acceleration is then given by only the acceleration radial component vector.

$$\vec{a}_r(t) = -r\omega^2(t)\hat{r}(t) \quad \text{Uniform circular motion} \tag{1.10}$$

This is called centripetal acceleration. Because the speed $v = r\omega$ is constant, the amount of time that the object takes to complete one circular orbit of radius r is also constant. This time interval, T , is called the period. In one period the object travels a distance $s = vT$ equal to the circumference, $s = 2\pi r$; thus

$$s = 2\pi r = vT$$

So, time period is given as,

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{r\omega} = \frac{2\pi}{\omega} \tag{1.11}$$

The frequency f is defined to be the reciprocal of the period,

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{1.12}$$

he SI unit of frequency is the inverse second, which is defined as the hertz, $[s^{-1}]$ or $[Hz]$.

Let's go back to centripetal acceleration. we can write it as,

$$|a_r| = r\omega^2 = \frac{v^2}{r}$$

Recall that the magnitude of the angular velocity is related to the frequency by $\omega = 2\pi f$. we have a another alternate expression for the magnitude of the centripetal acceleration in terms of the radius and frequency,

$$|a_r| = 4\pi^2 r f^2 \tag{1.13}$$

You can have another expression of the centripetal acceleration by putting $f = \frac{1}{T}$. Often we decide which expression to use based on information that describes the orbit. A convenient measure might be the orbit's radius. We may also independently know the period, or the frequency, or the angular velocity, or the speed. If we know one, we can calculate the other three but it is important to understand the meaning of each quantity.