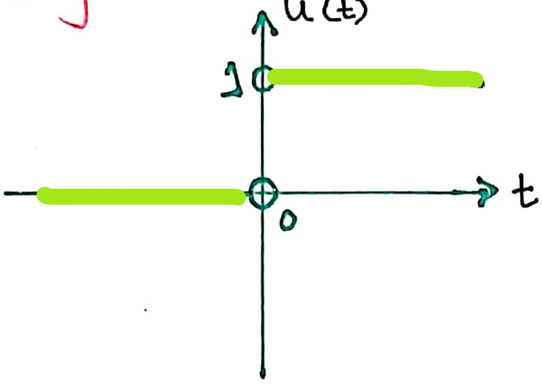
Laplace Transformation (Part B)

→ UNIT STEP FUNCTION or HEAVISIDE FUNCTION

Say we have the following signal:

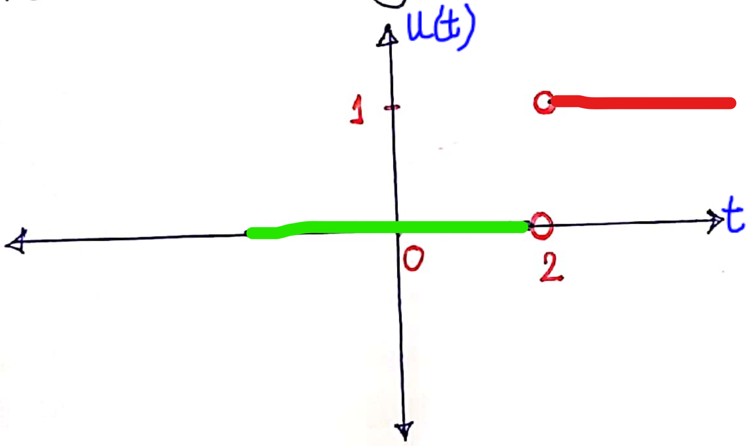


$$u(t) = \begin{cases} 0 & , t < 0 \\ 1 & , t > 0 \end{cases}$$

This is unit step function

$u(t) \Rightarrow u(t-0)$

Write the following graph in terms of function:



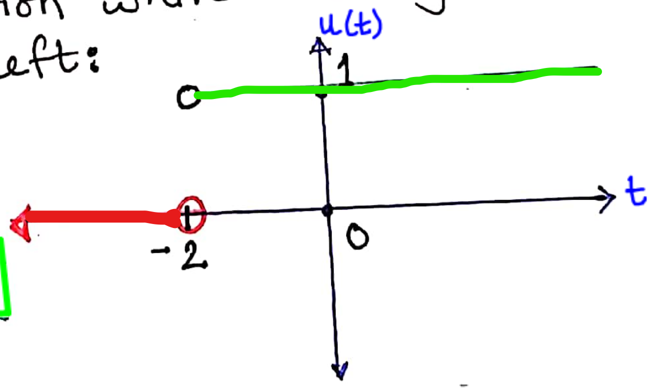
We are shifting towards right side

$$u(t-2) = \begin{cases} 0 & , \boxed{t < 2} \\ 1 & , \boxed{t > 2} \end{cases}$$

$$\begin{cases} t < 2 \Rightarrow t-2 < 0 \\ t > 2 \Rightarrow t-2 > 0 \end{cases}$$

Graph the following function while the signal is shifting two units towards left:

$$u(t+2) = \begin{cases} 0 , & \boxed{t < -2} \\ 1 , & \boxed{t > -2} \end{cases}$$



$$\begin{cases} t < -2 \Rightarrow t+2 < 0 \\ t > -2 \Rightarrow t+2 > 0 \end{cases}$$

Consider the following step function:

"a" is an arbitrary +ve number

General structure of Unit Step Function:

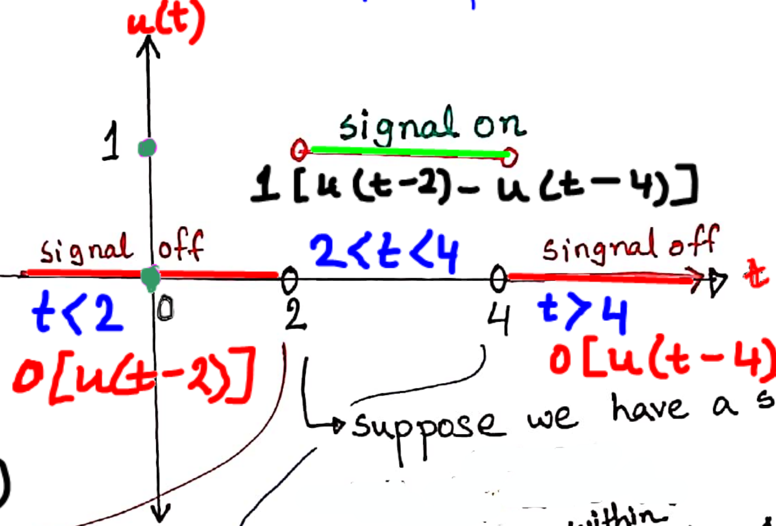
$$u(t-a) = \begin{cases} 0, & \underbrace{t<a}_{} \text{ -ve input} \\ 1 & \underbrace{t>a}_{} \text{ +ve input} \end{cases} \qquad a \geq 0$$

$t-a<0 \Rightarrow t<a$

$t-a>0 \Rightarrow t>a$

$f(t) = 0[u(t-2)] + 1[u(t-2)-u(t-4)] + 0[u(t-4)]$

$\therefore f(t) = u(t-2) - u(t-4)$



Suppose we have a signal while t within 2 to 4 signal is on & when t > 4, the signal is off.

Lets express this in terms of step function.

$$\boxed{f(t) = u(t-2) - u(t-4)}$$

↔ The signal of the above graph can be presented mathematically in this way.

let t = 1 → -ve input, -ve input: $u(t-2) - u(t-4)$

$t<2 \Rightarrow f(t) = 0 - 0 = 0$

graphically output should be "0"

let t = 3 → +ve input, -ve input

$2<t<4 \Rightarrow f(t) = u(t-2) - u(t-4)$ $\begin{cases} t>2 \;\&\; t<4 \Rightarrow u(t-2) \text{ is +ve} \\ t>2 \;\&\; t<4 \Rightarrow u(t-4) \text{ is -ve} \end{cases}$

graphically output should be "1" $= 1 - 0 = 1$

$\begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$

$t>4 \Rightarrow f(t) = u(t-2) - u(t-4)$ (+ve input, +ve input)

let t = 5

graphically output should be "0"

$= 1 - 1 = 0$

it has to be "–" so the final answer is "0"

t>4 ⇒ u(t-2) is +ve output = 1

t>4 ⇒ u(t-4) is +ve ∴ output = 1

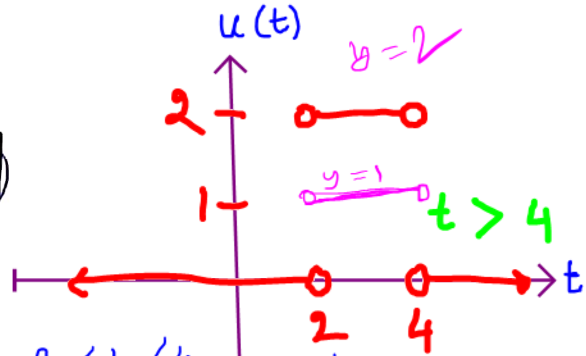
If we change the function

$f(t) = 2[u(t-2) - u(t-4)]$

+ve input +ve input

t>4, let t=5

$= 2(1-1) = 0$

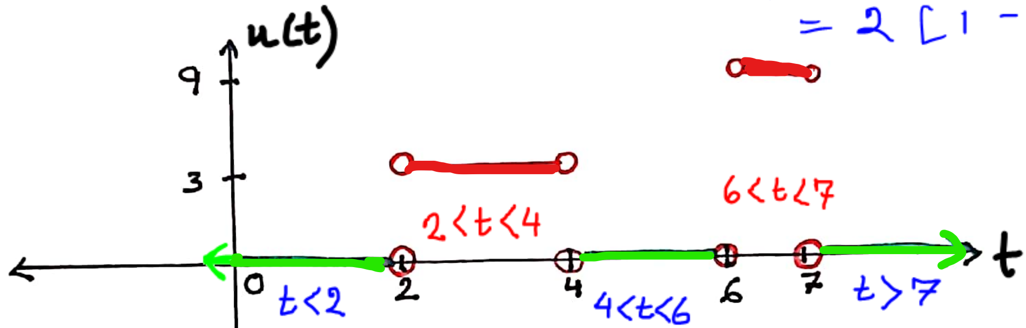


For: $2<t<4$ Let t=3

$f(t) = 2[u(t-2) - u(t-4)]$

$= 2[1-0] = 2$ etc.

Consider another signal:



$f(t) = 0[u(t-2)] + 3[u(t-2) - u(t-4)] + 0[u(t-4) - u(t-6)] + 9[u(t-6) - u(t-7)] + 0[u(t-7)]$

$\therefore f(t) = 3[u(t-2) - u(t-4)] + 9[u(t-6) - u(t-7)]$

Represent the above function:

Step Function

$$f(t) = \begin{cases} 3, & 2<t<4 \\ 9, & 6<t<7 \\ 0, & o/w \end{cases}$$

$t<2$ *, $t>7$

$4<t<6$ **

Not given $\therefore$ assuming over those intervals the signal is off.

Express this function in one line: (Unit step function)

$f(t) = 3[u(t-2) - u(t-4)] + 9[u(t-6) - u(t-7)]$

* $t<2 \Rightarrow f(t) = 3[\underbrace{u(t-2)}_{\substack{-ve \\ \text{"0"}}} - \underbrace{u(t-4)}_{\substack{-ve \\ \text{"0"}}}] + 9[\underbrace{u(t-6)}_{\substack{-ve \\ \text{"0"}}} - \underbrace{u(t-7)}_{\substack{-ve \\ \text{"0"}}}] = 0$

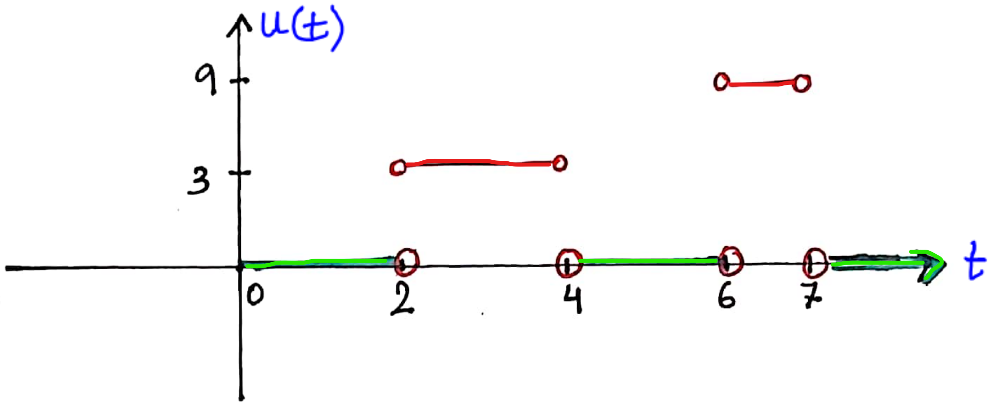
* let "t = 1"

$2<t<4 \Rightarrow f(t) = 3[\underbrace{u(t-2)}_{\substack{+v \\ \text{"1"}}} - \underbrace{u(t-4)}_{\substack{-ve \\ \text{"0"}}}] + 9[\underbrace{u(t-6)}_{\substack{-ve \\ \text{"0"}}} - \underbrace{u(t-7)}_{\substack{-ve \\ \text{"0"}}}] = 3$

let "t=3"

** $4<t<6 \Rightarrow f(t) = 3[\underbrace{u(t-2)}_{\substack{+ve \\ \text{"1"}}} - \underbrace{u(t-4)}_{\substack{+ve \\ \text{"1"}}}] + 9[\underbrace{u(t-6)}_{\substack{-ve \\ \text{"0"}}} - \underbrace{u(t-7)}_{\substack{-ve \\ \text{"0"}}}] = 0$

let "t=5"



$$6<t<7 \Rightarrow f(t) = 3[u(\underbrace{t-2}_{\substack{+ve \\ "1"}}) - u(\underbrace{t-4}_{\substack{+ve \\ "1"}})] + 9[u(\underbrace{t-6}_{\substack{+ve \\ "1"}}) - u(\underbrace{t-7}_{\substack{-ve \\ "0"}})] = 9$$

let "$t = 6.5$"

Unit step function helps us to write the **step** function in one line.

Example: **Reading**

$$f(t) = \begin{cases} t & ; \ 1<t<3 \\ \sin t & ; \ 6<t<7 \\ e^{2t} & ; \ t>7 \end{cases}$$

$\rightarrow \because t<1 \rightarrow$ output $= 0$

$3<t<6 \rightarrow$ " $= 0$

$\because$ this intervals are missing in the information

Unit step function will be as follows:

$$f(t) = t[u(t-1) - u(t-3)] + \sin t[u(t-6) - u(t-7)] + e^{2t}[u(t-7)]$$

$$t<1 \Rightarrow f(t) = t[u(\underbrace{t-1}_{\substack{-ve \\ "0"}}) - u(\underbrace{t-3}_{\substack{-ve \\ "0"}})] + \sin t[u(\underbrace{t-6}_{\substack{-ve \\ "0"}}) - u(\underbrace{t-7}_{\substack{-ve \\ "0"}})] + e^{2t}(u(\underbrace{t-7}_{\substack{-ve \\ "0"}}))$$

$$= 0$$

$$1<t<3 \Rightarrow f(t) = t[u(\underbrace{t-1}_{\substack{+ve \\ "1"}}) - u(\underbrace{t-3}_{\substack{-ve \\ "0"}})] + \sin t[u(\underbrace{t-6}_{\substack{-ve \\ "0"}}) - u(\underbrace{t-7}_{\substack{-ve \\ "0"}})] + e^{2t}[u(\underbrace{t-7}_{\substack{-ve \\ "0"}})]$$

$$= t$$

$$6<t<7 \Rightarrow f(t) = t\,[u(\underbrace{t-1}_{\substack{+ve \\ \text{"}1\text{"}}}) - u(\underbrace{t-3}_{\substack{+ve \\ \text{"}1\text{"}}})] + \sin t\,[u(\underbrace{t-6}_{\substack{+ve \\ \text{"}1\text{"}}}) - u(\underbrace{t-7}_{\substack{-ve \\ \text{"}0\text{"}}})] + e^{2t}\,[u(\underbrace{t-7}_{\substack{-ve \\ \text{"}0\text{"}}})]$$

$$= t\,[1-1] + \sin t\,[1-0] + e^{2t}\,[0]$$

$$= \sin t$$

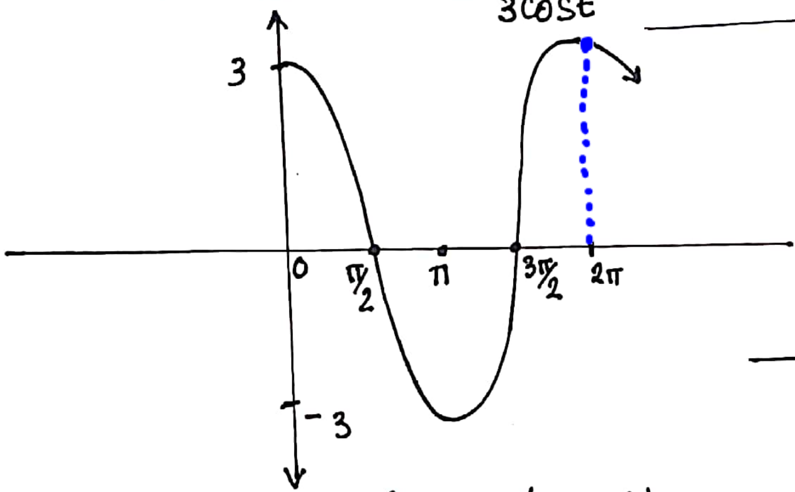
$$t>7 \Rightarrow f(t) = t\,[u(\underbrace{t-1}_{+ve}) - u(\underbrace{t-3}_{+ve})] + \sin t\,[u(\underbrace{t-6}_{+ve}) - u(\underbrace{t-7}_{+ve})] + e^{2t}[u(\underbrace{t-7}_{+ve})]$$

$$= t\,[1-1] + \sin t\,[1-1] + e^{2t}(1)$$

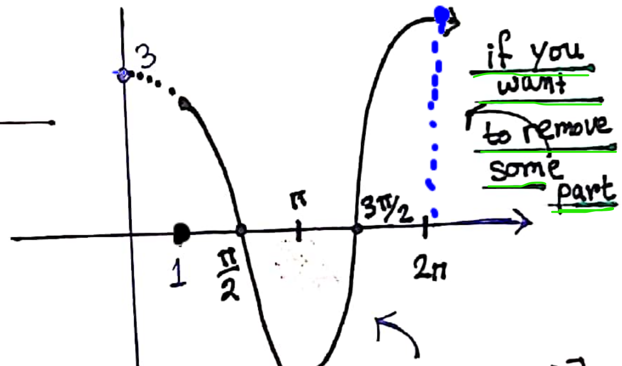
$$= e^{2t}$$

Omit these Graphs ⟶

TRANSLATIONS



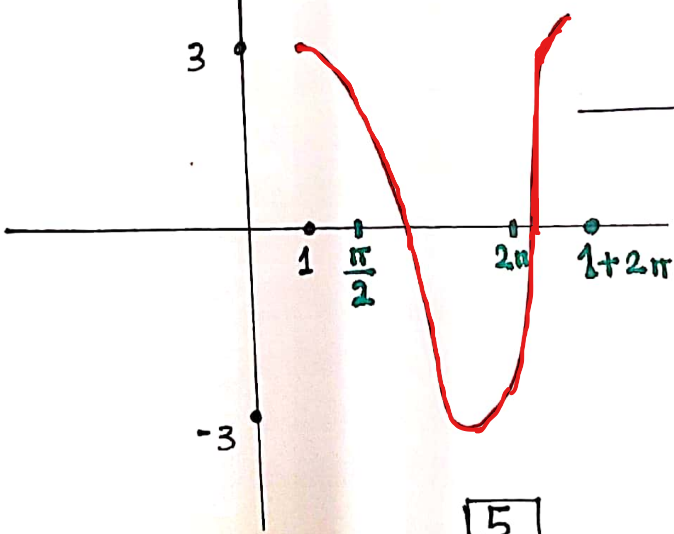
→ $f(t) = 3\cos t$



-3 $f(t) = 3\cos t\,[u(t-1)]$

$= f(t)\,u(t-1)$

Shift it forward by 1 unit



→ $f(t) = 3\cos(t-1)\,[u(t-1)]$

$= f(t-1)\,u(t-1)$

Translation

Example ①

Piecewise function

$$f(t)=\begin{cases}0 & ; \quad 0<t<1\\ t-1 & ; \quad 1<t<2\\ t+1 & ; \quad t>2\end{cases}$$

Find the laplace transformation of the above function

Unit step function

$$f(t)=0[u(t-0)-u(t-1)]+(t-1)[u(t-1)-u(t-2)]$$
$$+(t+1)[u(t-2)]$$
$$=tu(t-1)-\cancel{tu(t-2)}-u(t-1)+u(t-2)$$
$$+\cancel{tu(t-2)}+u(t-2)$$
$$=\underbrace{tu(t-1)}_{\text{i } f(t)=t}-\underbrace{u(t-1)}_{\text{ii } f(t)=1}+\underbrace{2u(t-2)}_{\text{iii } f(t)=1}$$

2nd Translation Theorem

Formula (i)

$$\mathcal{L}(u(t-a)f(t))=e^{-sa}\mathcal{L}\{f(t+a)\}$$

$$\Rightarrow \mathcal{L}\{f(t)\}=e^{-s(1)}\mathcal{L}\{t+1\}-e^{-s(1)}\mathcal{L}\{1\}+2e^{-s(2)}\mathcal{L}\{1\}$$

(i) $f(t)=t$

$f(t+a)=f(t+1)=t+1$ $\because a=1$

(ii) $f(t)=1$

$f(t+a)=f(t+1)=1$ $\because a=1$

(iii) $f(t)=1$

$f(t+a)=f(t+2)=1$ $\because a=2$

$$\mathcal{L}\{f(t)\}=e^{-s}\mathcal{L}\{t+1\}-e^{-s}\mathcal{L}\{1\}+2e^{-2s}\mathcal{L}\{1\}$$
$$=e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)-e^{-s}\cdot\frac{1}{s}+2e^{-2s}\cdot\frac{1}{s}$$
$$=e^{-s}\frac{1}{s^2}+e^{-s}\cdot\frac{1}{s}-e^{-s}\cdot\frac{1}{s}+2e^{-2s}\cdot\frac{1}{s}$$
$$=\frac{1}{s^2}e^{-s}+\frac{2}{s}e^{-2s}$$

Example ② Find the Laplace Transformation of

$$f(t)=\begin{cases} 2 \ ; & 0<t<3 \\ t^2 \ ; & 3<t<5 \\ t+1 \ ; & t>5 \end{cases}$$

$\mathcal{L}\{u(t-a)f(t)\}$
$=e^{-sa}\mathcal{L}\{f(t+a)\}$

$$f(t)=2\,[u(t-0)-u(t-3)]+t^2[u(t-3)-u(t-5)]$$
$$+(t+1)[u(t-5)]$$
$$=2u(t-0)-2u(t-3)+t^2u(t-3)-t^2u(t-5)$$
$$+t(u(t-5))+1u(t-5)$$

$$=\underbrace{2u(t-0)}_{\substack{a=0\\ f(t)=1}}^{\text{i}}-\underbrace{2u(t-3)}_{f(t)=1,\ a=3}^{\text{ii}}+\underbrace{t^2u(t-3)}_{f(t)=t^2,\ a=3}^{\text{iii}}-\underbrace{t^2u(t-5)}_{\substack{a=5\\ f(t)=t^2,}}^{\text{iv}}$$
$$+\underbrace{t\,u(t-5)}_{\substack{a=5\\ f(t)=t}}^{\text{v}}+\underbrace{u(t-5)}_{\substack{a=5\\ f(t)=1}}^{\text{vi}}$$

$\mathcal{L}(u(t-a)f(t))$
$=e^{-sa}\mathcal{L}\{f(t+a)\}$

i $f(t)=1$
$f(t+a)=f(t+0)=1$

ii $f(t)=1$
$f(t+a)=f(t+3)=1$

iii $f(t)=t^2$
$f(t+a)=f(t+3)=(t+3)^2$

iv $f(t)=t^2$
$f(t+a)=f(t+5)=(t+5)^2$

vi $f(t)=1$
$f(t+a)=f(t+5)=1$

v $f(t)=t$
$f(t+a)=f(t+5)=t+5$

Take Laplace on both sides:

$$\therefore \mathcal{L}\{f(t)\}=2e^{-s(0)}\mathcal{L}\{1\}-2e^{-s(3)}\mathcal{L}\{1\}+e^{-s(3)}\mathcal{L}\{(t+3)^2\}$$
$$+e^{-s(5)}\mathcal{L}\{(t+5)^2\}+e^{-s(5)}\mathcal{L}\{t+5\}+e^{-s(5)}\mathcal{L}\{1\}$$
$$=2\cdot\frac{1}{s}-2e^{-3s}\frac{1}{s}+e^{-3s}\mathcal{L}\{t^2+6t+9\}$$
$$+e^{-5s}\mathcal{L}\{t^2+10t+25\}+e^{-5s}\left(\frac{1}{s^2}+\frac{5}{s}\right)+e^{-5s}\left(\frac{1}{s}\right)$$

$$= \frac{2}{s} - \frac{2}{s}e^{-3s} + e^{-3s}\left(\frac{2!}{s^{2+1}} + 6\cdot\frac{1}{s^2} + \frac{9}{s}\right)$$

$$+ e^{-5s}\left(\frac{2!}{s^{2+1}} + 10\cdot\frac{1}{s^2} + 25\frac{1}{s}\right) + e^{-5s}\left(\frac{1}{s^2} + \frac{5}{s}\right)$$

$$+ \frac{e^{-5s}}{s}$$

$$= \frac{2}{s} - \frac{2e^{-3s}}{s} + \frac{2e^{-3s}}{s^3} + \frac{6e^{-3s}}{s^2} + \frac{9e^{-3s}}{s}$$

$$+ \frac{2e^{-5s}}{s^3} + \frac{10}{s^2} + \frac{25}{s} + \frac{e^{-5s}}{s^2} + \frac{5e^{-5s}}{s} + \frac{e^{-5s}}{s}$$

$$= \frac{27}{s} + \frac{7e^{-3s}}{s} + \frac{6e^{-5s}}{s} + \frac{2e^{-3s}}{s^3} + \frac{2e^{-5s}}{s^3}$$

$$+ \frac{6e^{-3s}}{s^2} + \frac{e^{-5s}}{s^2} + \frac{10}{s^2}.$$