

Beta Gamma Function

Beta function is also known as the Euler's integral defined by:

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx;$$

$$m > 0, n > 0.$$

It is a useful distribution to evaluate

- Laplace transformation
- Probability density function in statistics
- Quantum field theory in physics
- MATLAB programing

Gamma function (Γ) is an extension of the factorial function, with its argument shifted down by 1, to real and complex numbers.

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx = (n-1)!$$

Gamma Beta function can be related to each other: $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$

Trigonometric Function: $\int_0^{\pi/2} \sin^p x \cos^q x dx = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})}.$

OR: $\int_0^{\pi/2} 2\sin^{2x-1}(t)\cos^{2y-1}(t)dt = \beta(x, y)$

2. (ix) $\int_0^\infty \frac{1}{1+x^4} dx$

Trigonometric Substitution:

$$x^2 = \tan \theta; \therefore x^4 = \tan^2 \theta; 1 + x^4 = 1 + \tan^2 \theta = \sec^2 \theta.$$

$$x = \sqrt{\tan \theta}$$

$$dx = \frac{1}{2\sqrt{\tan \theta}} \sec^2 \theta d\theta$$

$$\text{Limits: } x = 0 \Rightarrow \theta = 0$$

$$x = \infty \Rightarrow \theta = \frac{\pi}{2}.$$

$$\int_0^\infty \frac{1}{1+x^4} dx$$

$$= \int_0^{\pi/2} \frac{1}{1+\tan^2 \theta} \cdot \frac{1}{2\sqrt{\tan \theta}} \sec^2 \theta d\theta$$

$$= \int_0^{\pi/2} \frac{\sec^2 \theta}{\sec^2 \theta} \cdot \frac{1}{2\sqrt{\tan \theta}} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \sqrt{\cot \theta} d\theta = \frac{1}{2} \int_0^{\pi/2} \sqrt{\frac{\cos \theta}{\sin \theta}} d\theta = \frac{1}{2} \int_0^{\pi/2} \sin^{-1/2} \theta \cos^{1/2} \theta d\theta$$

Alternative Method

$$= \frac{1}{4} \int_0^{\pi/2} 2 \sin^{\color{red}{-1/2}} \theta \cos^{\color{blue}{1/2}} \theta d\theta$$

$$= \frac{1}{4} \beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

\uparrow x \uparrow y

$$2y-1 = \frac{1}{2}$$

$$2y = \frac{1}{2} + 1 = \frac{3}{2}$$

$$y = \frac{3}{4}$$

$$2x-1 = -\frac{1}{2}$$

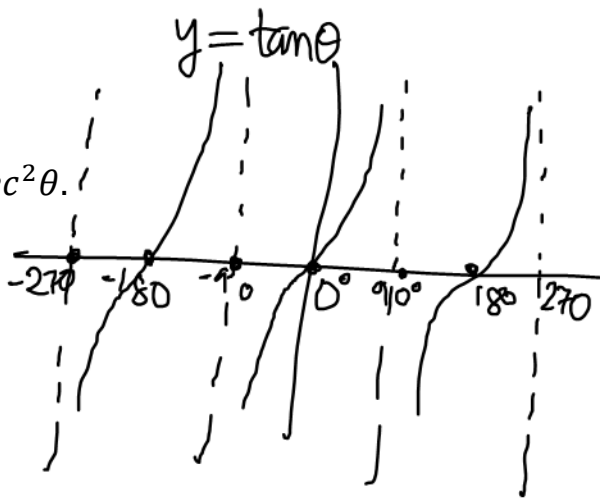
$$2x = -\frac{1}{2} + 1$$

$$2x = \frac{1}{2}$$

$$x = \frac{1}{4}$$

$$\int_0^{\pi/2} 2 \sin^{2x-1}(t) \cos^{2y-1}(t) dt = \beta(x, y)$$

$$1 + \tan^2 \theta = \sec^2 \theta$$



$$\star \tan 0^\circ = \frac{\sin 0^\circ}{\cos 0^\circ} = \frac{0}{1} = 0$$

$$\star \tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0} = \infty$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right) \Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)}$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}{\Gamma(1)}$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4} + \frac{3}{4}\right)}$$

$$= \frac{1}{4} \beta\left(\frac{1}{4}, \frac{3}{4}\right).$$

$$\frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = \beta(m, n)$$

$$\int_0^{\pi/2} \sin^p x \cos^q x dx = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$3(i) \int_0^{\pi} \sin^5 \theta \cos^4 \theta d\theta$$

$$= 2 \int_0^{\pi/2} \sin^5 \theta \cos^4 \theta d\theta$$

$$= 2 \frac{\Gamma\left(\frac{5+1}{2}\right) \Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{5+4+2}{2}\right)}$$

$$= 2 \frac{\Gamma(3) \Gamma\left(\frac{5}{2}\right)}{2\Gamma\left(\frac{11}{2}\right)}$$

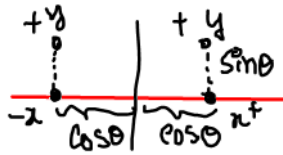
$$= \frac{\Gamma(3) \Gamma\left(\frac{5}{2}\right)}{\Gamma\left(3+\frac{5}{2}\right)} = \beta\left(3, \frac{5}{2}\right).$$

$$\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx \text{ when } f(2a-x) = -f(x)$$

$$\therefore \int_0^{\pi} \sin^5 \theta \cos^4 \theta d\theta = 2 \int_0^{\pi/2} \sin^5 \theta \cos^4 \theta d\theta \text{ if } f(\pi-\theta) = -f(\theta).$$

$$f(\theta) = \sin^5 \theta \cos^4 \theta$$

$$l.s. = f(\pi-\theta) = \sin^5(\pi-\theta) \cos^4(\pi-\theta) = (\sin^5 \theta)(-\cos^4 \theta)$$



$$= -\sin^5 \theta \cos^4 \theta = -f(\theta)$$

$$\therefore \int_0^{\pi} \sin^5 \theta \cos^4 \theta d\theta = 2 \int_0^{\pi/2} \sin^5 \theta \cos^4 \theta d\theta. \quad 3(ii) \text{ is similar to } 3(i)$$

$$\text{Alternative Method: } \int_0^{\pi/2} 2 \sin^{2x-1}(t) \cos^{2y-1}(t) dt = \beta(x, y)$$

$$3(i) \int_0^{\pi} \sin^5 \theta \cos^4 \theta d\theta$$

$$= 2 \int_0^{\pi/2} \sin^5 \theta \cos^4 \theta d\theta$$

$$= \int_0^{\pi/2} 2 \sin^5 \theta \cos^4 \theta d\theta$$

$$= \int_0^{\pi/2} 2 \sin^{2x-1}(\theta) \cos^{2y-1}(\theta) d\theta, \text{ where } 2x-1=5, 2y-1=4 \text{ hence } x=3, y=\frac{5}{2}$$

$$= \beta(x, y) = \beta\left(3, \frac{5}{2}\right)$$