(1) $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

Eigenvalues: $(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow (4-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 2 & 4-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 4-\lambda \\ 2 & 2 \end{vmatrix} = 0$

$\Rightarrow (4-\lambda)\{(4-\lambda)(4-\lambda) - 2.2\} - 2\{2(4-\lambda) - 2.2\} + 2\{2.2 - 2(4-\lambda)\} = 0$

$\Rightarrow (4-\lambda)(16-4\lambda-4\lambda+\lambda^2-4) - 2(8-2\lambda-4) + 2(4-8+2\lambda) = 0$

$\Rightarrow (4-\lambda)(12-8\lambda+\lambda^2) - 2(4-2\lambda) + 2(-4+2\lambda) = 0$

$\Rightarrow 48 - 32\lambda + 4\lambda^2 - 12\lambda + 8\lambda^2 - \lambda^3 - 8 + 4\lambda - 8 + 4\lambda = 0$

$\Rightarrow -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$

$\Rightarrow -(\lambda^3 - 12\lambda^2 + 36\lambda - 32) = 0$

$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

~~$\Rightarrow \lambda^3$~~

After calculation, $\lambda_1 = 8$, $\lambda_2 = 2$,
$\lambda_3 = 2$

For, $\lambda_1 = 8$,

$$A - 8I = \begin{bmatrix} 4-8 & 2 & 2 \\ 2 & 4-8 & 2 \\ 2 & 2 & 4-8 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \begin{array}{l} \rightarrow R_1' = \frac{R_1}{2} \\ \rightarrow R_2' = \frac{R_2}{2} \\ \rightarrow R_3' = \frac{R_3}{2} \end{array}$$

$$= \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{array}{l} \rightarrow R_1' = R_1 + R_3 \\ \rightarrow R_2' = R_2 - R_3 \\ \end{array}$$

$$= \begin{bmatrix} -1 & 2 & -1 \\ 0 & -3 & 3 \\ 1 & 1 & -2 \end{bmatrix} \begin{array}{l} \rightarrow R_1' = R_1 \times (-1) \\ \rightarrow R_2' = R_2/3 \\ \end{array}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{array}{l} \\ \\ \rightarrow R_3' = R_1 - R_3 \end{array}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & -3 & 3 \end{bmatrix} \begin{array}{l} \\ \\ \rightarrow R_3' = R_3 - 3R_2 \end{array}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \rightarrow R_1' = R_1 - 2R_2 \\ \\ \end{array}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The reduced row echelon form,

$$x_1 - x_3 = 0$$
$$x_2 - x_3 = 0$$

$\therefore$ free variable, x_3

$$x_1 = x_3$$
$$x_2 = x_3$$
$$x_3 = x_3$$

$$X = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix}$$

Let, $x_3 = 1$, $X = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

<u>Now, $\lambda = 2$,</u>

$$A - 2I = \begin{bmatrix} 4-2 & 2 & 2 \\ 2 & 4-2 & 2 \\ 2 & 2 & 4-2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

After reduced row echelon form,

$$x_1 + x_2 + x_3 = 0$$

so, $x_1 = -x_2 - x_3$

$$x_2 = s$$

$$x_3 = t$$

$$\therefore x = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Now, constructing a inventable matrix P,

$$P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$\therefore$ Finding the inverse of matrix P,

we know, $P^{-1} = \dfrac{\operatorname{adj} P}{|P|}$.

$$P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix}.$$

$$\text{adj } P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1-(-1) & -1 \\ -1 & -1 & 1-(-1) \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$|P| = 1 + 1 + 1 = 3$$

$$\therefore P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

Finally, $$P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 4+2+2 & -4+2+0 & -4+0+2 \\ 2+4+2 & -2+4+0 & -2+0+2 \\ 2+2+4 & -2+2+0 & -2+0+4 \end{bmatrix}$$

$$\Rightarrow \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 8 & -2 & -2 \\ 8 & 2 & 0 \\ 8 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \frac{1}{3}\begin{bmatrix} (8+8+8) & (-2+2+0) & (-2+0+2) \\ (-8+16-8) & (2+4+0) & (2+0-2) \\ (-8-8+16) & (2-2+0) & (2+0+4) \end{bmatrix}$$

$$\Rightarrow \frac{1}{3}\begin{bmatrix} 24 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\therefore$ Which is a diagonal matrix.

② $u_1 = (1,1,1)$, $u_2 = (0,1,1)$, $u_3 = (0,0,1)$

$$v_1 = u_1 = (1,1,1)$$

Again:
$$v_2 = u_2 - \frac{\langle u_2 \cdot v_1 \rangle}{v_1 \cdot v_1} v_1$$

$$= (0,1,1) - \frac{(0+1+1)}{(1+1+1)} \cdot (1,1,1)$$

$$= (0,1,1) - \frac{2}{3}(1,1,1)$$

$$= \left(-2/3, 1/3, 1/3\right)$$

Again:
$$v_3 = u_3 - \frac{\langle u_3 \cdot v_1 \rangle}{v_1 \cdot v_1} v_1 - \frac{\langle u_3 \cdot v_2 \rangle}{v_2 \cdot v_2} \cdot v_2$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1/3}{2/3}\left(-2/3, 1/3, 1/3\right)$$

$$= \left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right) - \frac{1}{2}\left(-2/3, 1/3, 1/3\right)$$

$$= \left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right) - \left(-\frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$$

$$= \cancel{\left(\frac{7}{9}, -\frac{7}{18}, \frac{11}{18}\right)} \Rightarrow \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

Onthogonal basis $(v_1, v_2, v_3) = \left\{ \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}, \begin{vmatrix} -2/3 \\ 1/3 \\ 1/3 \end{vmatrix}, \begin{vmatrix} 0 \\ -1/2 \\ 1/2 \end{vmatrix} \right\}$

Now, normalizing to onthonormal basis,

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1)}{\sqrt{1^2+1^2+1^2}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{\left(-2/3, \frac{1}{3}, \frac{1}{3}\right)}{\sqrt{(-2/3)^2+(1/3)^2+(1/3)^2}} = \frac{(-2/3, 1/3, 1/3)}{\frac{\sqrt{6}}{3}}$$

$$= \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{0^2+(-\frac{1}{2})^2+(1/2)^2}}$$

$$\Rightarrow \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$\therefore$ Onthonormal basis $(q_1, q_2, q_3) = \left\{ \begin{vmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{vmatrix} + \begin{vmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{vmatrix} + \begin{vmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \sqrt{2}/2 \end{vmatrix} \right\}$

(Ans)

③ (i) $\sin(2x)$

If $f(x) = -f(-x)$ than $f(x)$ is odd.

$$f(\tfrac{\pi}{4}) = \sin 2\times\tfrac{\pi}{4} = 1$$

$$f(-\tfrac{\pi}{4}) = \sin 2\times -\tfrac{\pi}{4} = -1$$

$$\therefore f(\tfrac{\pi}{4}) = -f(-\tfrac{\pi}{4})$$

$\therefore \sin(2x)$ is an odd function.

(ii) $\sin x . \cos x$

$$f(\pi/6) = \sin \pi/6 . \cos \pi/6 = \frac{\sqrt{3}}{4}$$

$$f(-\tfrac{\pi}{6}) = \sin(-\pi/6) \cos(-\pi/6) = -\sqrt{3}/4$$

$$\therefore f(\pi/6) = -f(-\pi/6)$$

$\therefore \sin x . \cos x$ is an odd function.

(iii) $\tan x$

$$f(\pi/6) = \tan \pi/6 = 1/\sqrt{3}$$

$$f(-\pi/6) = \tan(-\pi/6) = -1/\sqrt{3}$$

$$\therefore f(\pi/6) = -f(-\pi/6)$$

$\therefore \tan x$ is an odd function