STA201 Lecture-14

Joint Probability Distribution & Conditioning on Random Variables

14.1 – Joint Probability and Marginalisation

14.1.1 – Joint Probability and Marginalisation:

Joint Probability

If X and Y are two discrete random variables and $P_{X,Y}(x,\ y)$ is a function of X and Y, then $P_{X,Y}(x,\ y)$ is called the joint probability function if the following conditions are satisfied:

1. $P_{X,Y}(x,\ y) \geq 0$
2. $\sum_X \sum_Y P_{X,Y}(x,\ y) = 1$

A joint probability function is used to express the probability that X and Y simultaneously take the values x and y.

$$P_{X,Y}\ (x, y)\ =\ P_{X,Y}(X = x\ \cap\ Y = y)$$

Marginal Probability for Discrete Random Variables

If X and Y are two discrete random variables with joint probability function $P_{X,Y}(x,\ y)$, then

The marginal probability function of X is

$$P_X(x) = \sum_Y P_{X,Y}(x,\ y)$$

The marginal probability function of Y is

$$P_Y(y) = \sum_X P_{X,Y}(x,\ y)$$

Marginal Probability for Continuous Random Variables

If X and Y are two continuous random variables with joint probability density function $f_{X,Y}(x,\ y)$, then

The marginal probability function of X is

$$f_X(x) = \int f_{X,Y}(x,\ y)\ dy$$

The marginal probability function of Y is

$$f_Y(y) = \int f_{X,Y}(x,\ y)\ dx$$

Independence of Random Variable

Jointly distributed random variables, say X and Y, are said to be independent if and only if their joint probability function is the product of their marginal probability functions. i.e.

$$P_{X,Y}(x,\ y) = P_X(x) \cdot P_Y(y)$$

For all $x \in X$ and $y \in Y$

Consequently, a set of k random variables $\{X_1,\ X_2,\ \ldots,\ X_k\}$ is independent if and only if

$$P(X_1,\ X_2,\ \ldots,\ X_k) = P(X_1) \cdot P(X_2) \cdot \ldots \cdot P(X_k)$$

14.1.2 – Examples: Joint Probability & Marginalisation

Example 1:

The joint probability distribution of two random variables *X* and *Y* is as follows:

X \ Y	**0**	**1**	**2**	$P_X(x)$
0	0.1	0	0.2	**0.3**
1	0.2	0.1	0	**0.3**
2	0	0.2	0.2	**0.4**
$P_Y(y)$	**0.3**	**0.3**	**0.4**	**1**

a. Find the Marginal Probabilities of *X* and *Y*. (Done on table in red)

$X = x$	0	1	2
$P_X(x)$	0.3	0.3	0.4

$Y = y$	0	1	2
$P_Y(y)$	0.3	0.3	0.4

b. Compute the expected values of *X* and *Y*.
Sol:
$E(X) = \sum x \cdot P_X(x) = (0 \times 0.3) + (1 \times 0.3) + (2 \times 0.4) = 1.1$
$E(Y) = \sum y \cdot P_Y(y) = (0 \times 0.3) + (1 \times 0.3) + (2 \times 0.4) = 1.1$

c. Determine if *X* and *Y* are independent.
Sol:
X and Y are independent if and only if $P_{X,Y}(x,\ y) = P_X(x) \cdot P_Y(y)$
We can show that $P_{X,Y}(x,\ y) \neq P_X(x) \cdot P_Y(y)$ for some X and Y

Let $X = 0$ and $Y = 0$
$P_{X,Y}(0,0) = 0.1$
$P_X(0) \cdot P_Y(0) = 0.3 \times 0.3 = 0.09 \neq 0.1$
$\therefore P_{X,Y}(x,\ y) \neq P_X(x) \cdot P_Y(y)$ Therefore, X and Y are not independent

Example 2:

The joint probability distribution of Weather (*W*) and Temperature (*T*) is as follows

T / W	Hot	Cold	$P_W(w)$
Sunny	3/10	**1/5**	1/2
Rainy	**1/30**	2/15	**1/6**
Snowy	**0**	**1/3**	1/3
$P_T(t)$	1/3	**2/3**	**1**

a. Complete the probability distribution table.
Sol: Done in red

b. Are Weather and Temperature independent of each other?
Sol:
W and T are independent if and only if $P_{W,T}(w,\ t) = P_W(w) \cdot P_T(t)$
We can show that $P_{W,T}(w,\ t) \neq P_W(w) \cdot P_T(t)$ for some W and T

Let $W = Sunny$ and $T = Hot$
$P_{W,T}(Sunny, Hot) = \frac{3}{10}$
$P_W(Sunny) \cdot P_T(Hot) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \neq \frac{3}{10}$
$\therefore P_{W,T}(w,\ t) \neq P_W(w) \cdot P_T(t)$
Therefore, Weather and Temperature are not independent.

14.2 – Conditioning on Random Variables

14.2 – Extending Conditioning to Random Variables

Conditioning on Random Variables

If X and Y are two discrete random variables with joint probability function $P_{X,Y}(x,\ y)$ and marginal probability functions $P_X(x)$ and $P_Y(y)$, then

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,\ y)}{P_Y(y)}$$

$$P_{Y|X}(y|x) = \frac{P_{X,Y}(x,\ y)}{P_X(x)}$$

Example:

The joint probability distribution of Weather (*W)* and Temperature (*T)* is as follows

W \ T	Hot	Cold	$P_W(w)$
Sunny	3/10	1/5	1/2
Rainy	1/30	2/15	1/6
Snowy	0	1/3	1/3
$P_T(t)$	1/3	2/3	1

What is the probability that it will rain given it is cold?

Solution:

$$P_{W|T}\ (Rainy\ |Cold) = \frac{P_{W,T}(Rainy \cap Cold)}{P_T(Cold)} = \frac{2/15}{2/3} = \frac{2}{15} \times \frac{3}{2} = \frac{1}{5}$$

Practice Problems

Probability & Statistics for Engineering and the Sciences (Devore)

Joint Probability Distributions

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