PART A

MAT 215

TOPIC 4

ANTIDERIVATIVE

$\int f(z)\,dz = F(z)$

If $F'(z) = f(z)$ then $\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$

Example: $\int_0^{2+i} z^3\,dz = \left[\frac{z^4}{4}\right]_0^{2+i}$

$= \frac{1}{4}\left[(2+i)^4 - 0^4\right]$

$(x+y)^4$

$= x^4 + 6x^3y + 10x^2y^2 + 6xy^3 + y^4$

$= \frac{1}{4}(16 + 32i - 24 - 8i + 1)$ Considered $i^2 = -1$

$= \frac{1}{4}(24i - 7)$

Let a function f of a complex variable is continuous in a domain D. Then there exists a function F such that $F'(z) = f(z)$, $\forall z \in D$ is called an antiderivative of f. i.e. $\int f(z)\,dz = F(z)$.

Complex Valued function on Real Domain:

$\{ f : D \to \mathbb{C}$ while $D \subset \mathbb{R}$

↳domain ↳complex variable

$f(t) = u(t) + iv(t)$, $t \in [a, b]$

Definite Integral

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Example 1. $f(t) = t^2 + 1 + it^3$; $0 \leq t \leq 1$

$$\int_0^1 (t^2+1)\,dt + i\int_0^1 t^3\,dt$$

$$= \left[\frac{t^3}{3} + t\right]_0^1 + i\left[\frac{t^4}{4}\right]_0^1$$

$$= \frac{1}{3} + 1 + i\frac{1}{4} = \frac{4}{3} + \frac{1}{4}i$$

2. $f(t) = e^{i2t}$; $0 \leq t \leq \pi/6$

$$\int_0^{\pi/6} e^{i2t}\,dt = \int_0^{\pi/6} \cos 2t\,dt + i\int_0^{\pi/6} \sin 2t\,dt$$

e^{i2t}
$= \cos 2t + i\sin 2t$

$$= \left[\frac{\sin 2t}{2}\right]_0^{\pi/6} + i\left[-\frac{\cos 2t}{2}\right]_0^{\pi/6}$$

$$= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{6}\right) - \sin 2(0)\right] + i\left(-\frac{1}{2}\right)\left[\cos 2\left(\frac{\pi}{6}\right) - \cos 2(0)\right]$$

$$= \frac{1}{2}\sin\frac{\pi}{3} - \frac{i}{2}\cos\frac{\pi}{3} + \frac{i}{2}\cos(0)$$

$$= \frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{i}{2}\cdot\frac{1}{2} + \frac{i}{2}\cdot 1$$

$$= \frac{\sqrt{3}}{4} - i\frac{1}{4} + i\frac{1}{2} = \frac{\sqrt{3}}{4} + i\frac{1}{4}.$$

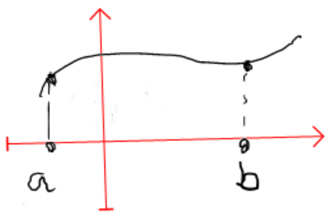
Properties:

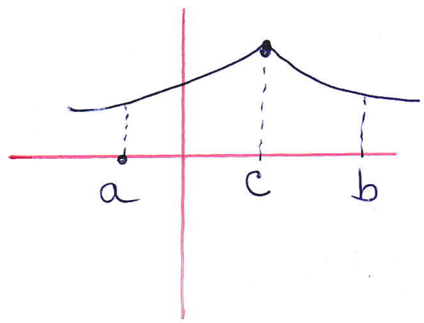
Let $f(t) = u(t) + iv(t)$, $g(t) = p(t) + iq(t)$

$t \in [a, b]$;

(i) $$\int_a^b [f(t) \pm g(t)]\,dt = \int_a^b f(t)\,dt \pm \int_a^b g(t)\,dt$$

(ii) $$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt, \quad c \in [a, b]$$





{ when there is no direct path between a and b.

(iii) $$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$$

(iv) $$\int_a^b f(t)g(t)\,dt = \int_a^b (u+iv)(p+iq)\,dt$$
$$= \int_a^b (up + ivp + iuq + i^2 vq)\,dt$$
$$= \int_a^b (up - vq)\,dt + i\int_a^b (vp + uq)\,dt$$

(v) $$\mathrm{Re}\int_a^b f(t)\,dt = \int_a^b \mathrm{Re}(f(t))\,dt$$

(vi) $$\int_a^b z_0 f(t)\,dt = \int_a^b (x_0 + iy_0)(u + iv)\,dt$$

complex constant $z_0 = x_0 + iy_0$

$$= \int_a^b (x_0 u + i y_0 u + i x_0 v + i^2 y_0 v)\,dt$$
$$= \int_a^b [u(x_0 + iy_0) + iv(x_0 + iy_0)]\,dt$$
$$= \int_a^b (x_0 + iy_0)(u + iv)\,dt = (x_0 + iy_0)\left[\int_a^b u\,dt + i\int_a^b v\,dt\right]$$

(vii) Absolute Value of integral:

$$f(t) = u + iv$$
$$|f(t)| = |u+iv|$$
$$= \sqrt{u^2+v^2}$$

$$\left| \int_a^b f(t)\,dt \right| \leqslant \int_a^b |f(t)|\,dt$$

Improper Integrals

while there is no direct path or there exist asymptote

$f: \underset{\text{domain}}{D} \longrightarrow \underset{}{\overset{\text{Complex Variable}}{C}}$, $D \subset R$

$f(t) = u(t) + iv(t)$, $t \in \mathbb{R}$

$\int f(t)dt = \int u(t)dt + i\int v(t)dt$

Provided that the right side integral exist

All properties of real integrals hold true

Improper to Definite integral

Let $U'(t) = u(t)$, $V'(t) = v(t)$

$f(t) = u(t) + iv(t)$

Then $\int u(t)\,dt = U(t)$, $\int v(t) = V(t)$

$\therefore \int f(t)\,dt = U(t) + iV(t)$

Then $\int_a^b f(t)\,dt = \left[U(t) + iV(t)\right]_a^b$
$= U(b) - U(a) + i(V(b) - V(a))$
$= U(b) + iV(b) - (U(a) + iV(a))$
$= F(b) - F(a)$

$\int f(t)dt = F(t)$

Paths: arcs & curves

An arc C in complex plane

$z(x,y), \quad x = x(t), \quad y = y(t), \quad a \leq t \leq b$

- t is a free parameter
- t represents characteristic of an equation.

Eqn of arc: $z = z(t) = x(t) + i\,y(t), \quad a \leq t \leq b$

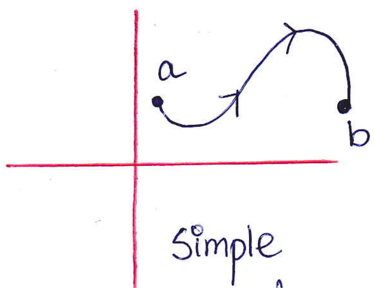
ex

$x^2 + y^2 = r^2$

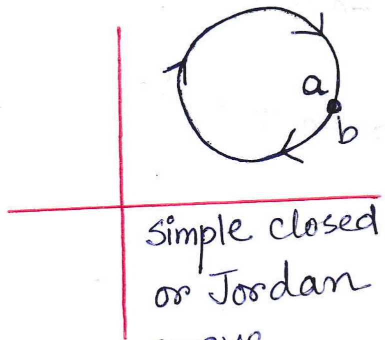
$r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$\cos^2\theta + \sin^2\theta = 1$

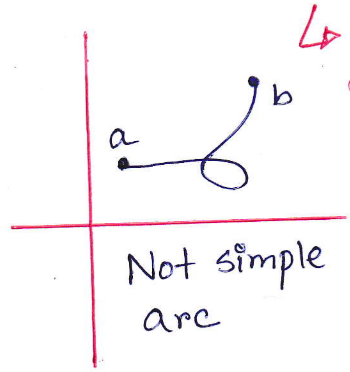
↳ eqn of circle.



Simple or Jordan Arc



Simple closed or Jordan curve "oriented"



Not simple arc

Examples

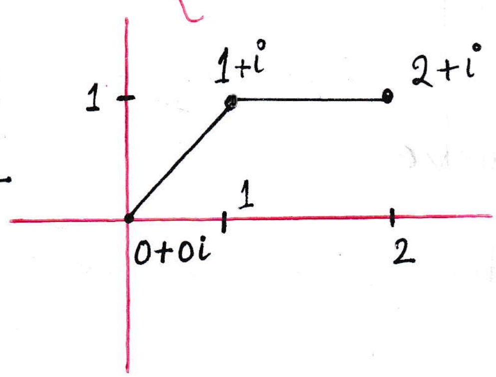
1. $f(t) = \begin{cases} t + it, & 0 \leq t \leq 1 \\ t + i, & 1 \leq t \leq 2 \end{cases}$

$t = 0 \Rightarrow 0 + 0i$

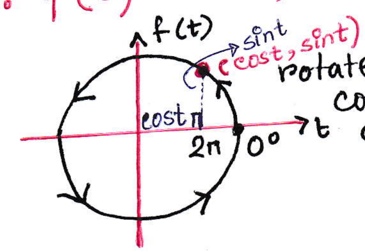
$t = 1 \Rightarrow 1 + i$

$t = 1 \Rightarrow 1 + i$

$t = 2 \Rightarrow 2 + i$



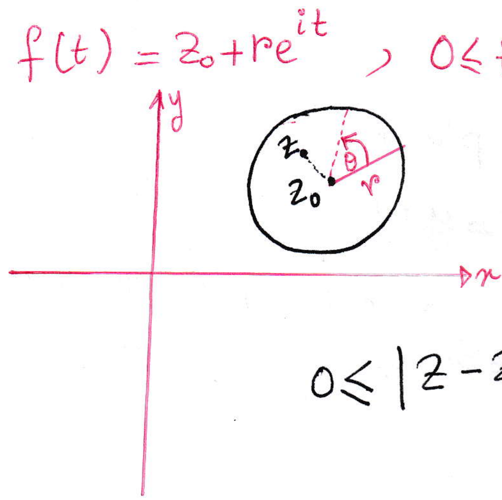
2. $f(t) = e^{it}, \quad 0 \leq t \leq 2\pi$



rotate counter clockwise

$e^{it} = \cos t + i\sin t$

3. $f(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$



$0 \leq |z - z_0| \leq R$ by def^n of neighbourhood

$\Rightarrow z - z_0 \leq re^{i\theta}$

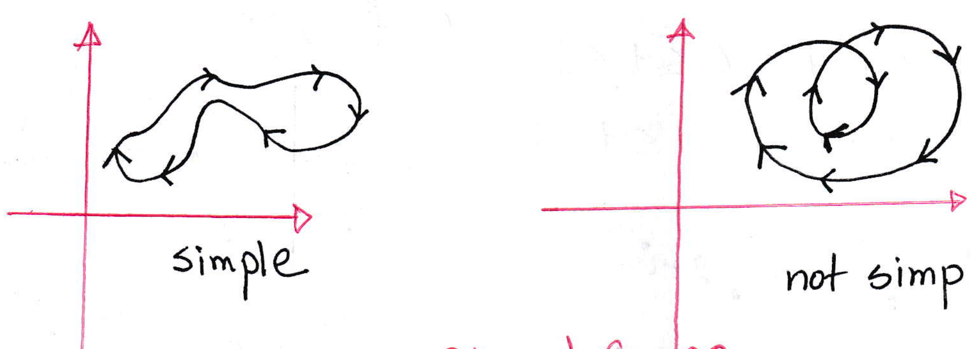
$\Rightarrow z \leq z_0 + re^{i\theta}$

$z_0 \rightarrow$ Center of circle

Curves

Consider parametric eqn:

$z = z(t) = x(t) + iy(t)$; $t_1 \leq t \leq t_2$



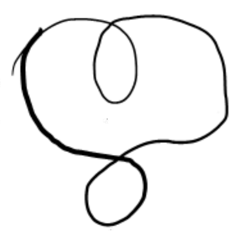
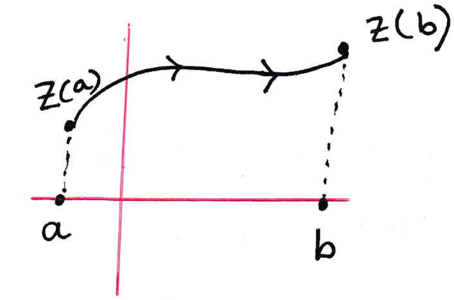
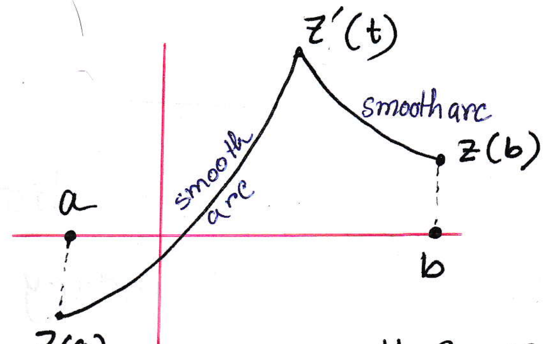


Figure: Closed Curve

A closed curve which does not intersect itself anywhere is called a simple closed curve.



smooth curve



Not smooth Curve

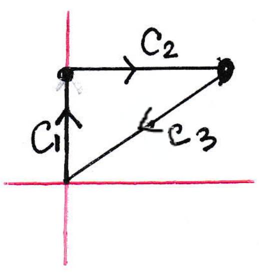
$z(t)$ or $(x(t), y(t))$ has continuous derivatives in $a \leq t \leq b$.

It is a piecewise or sectionally smooth curve.

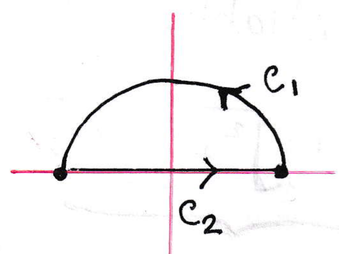
Smooth arc:
→ $z(t)$ is smooth if $z'(t)$ is continuous and $z'(t) \neq 0$, $\forall(t) \in [a, b]$

Contour or Path

When C is constructed by joining finitely many smooth curves end to end is called a contour.



$C = C_1 + C_2 + C_3$



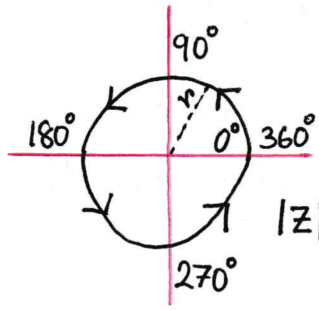
$C = C_1 + C_2$

$C = C_1 + C_2 + C_3 + \cdots + C_n$

C could be a combination of n many smooth curves

Contour Integral

$\oint_c f(z)dz \rightarrow$ integration of $f(z)$ around the boundary c in a '+ve' sence. (counter clockwise)



$|z|=r$

$\rightarrow$ the integral around c is called contour integral.

Length of Arc

Let

$Z = z(t) = x(t)+iy(t)$; $a \leq t \leq b$

↳ arc on curve

Suppose $x'(t)$ and $y'(t)$ exist and are continuous throughout the interval $a \leq t \leq b$

$\therefore z'(t) = x'(t) + i\,y'(t)$; $a \leq t \leq b$

$z(t)$ is differentiable over $[a,b]$

$|x'(t)+i\,y'(t)|$

$$|z'(t)| = \underbrace{\sqrt{[x'(t)]^2+[y'(t)]^2}}_{\text{It is integrable on } [a,b]}$$

$\because z = x+iy$

$\Rightarrow |z| = \sqrt{x^2+y^2}$

$\therefore$ Length of arc $L = \int_a^b |z'(t)|\,dt$

$$= \int_a^b \sqrt{[x'(t)]^2+[y'(t)]^2}\;dt$$

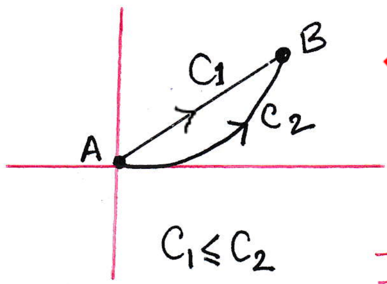
Recall MAT120

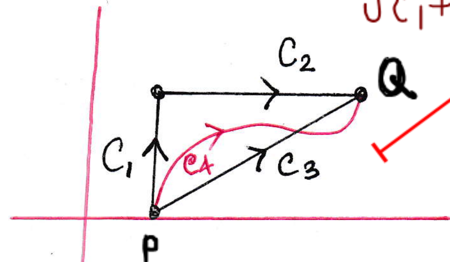
Length of Parametric curve

$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\;dt$

LINE Integral $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$

$$\int_{C_1+C_2} f(z)\,dz = \int_{C_3} f(z)\,dz = \int_{C_4} f(z)\,dz$$





$C_1 \leq C_2$

Independent PATH

$C_1 + C_2 \geqslant C_3$

C_1 & C_2 are independent path

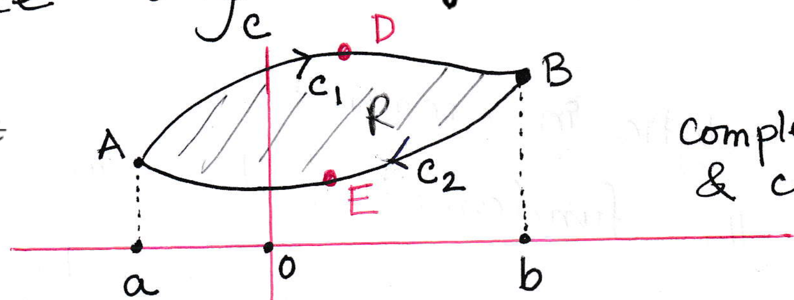
$C_1 + C_2$, C_3, C_4 are all independent of path

Cauchy's Theorem or Cauchy Goursat Theorem

If $f(z)$ is analytic everywhere within a simply connected region then $\oint_C f(z)\,dz = 0$ for every simple curve 'C' lying in the region R,

since $\because \oint_C \ldots dz = \int [C_1 + C_2 + C_3 + \cdots + C_n]$

$\int_C (C_1 + C_2)\,dz$



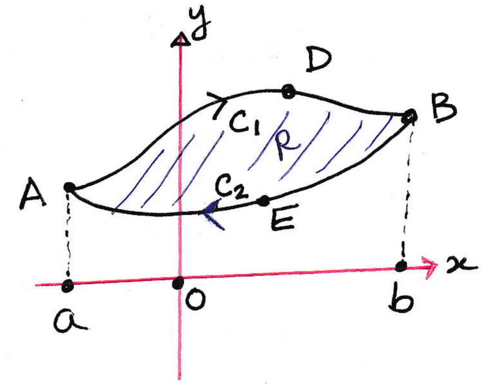
completing full turn & coming back to original point.

Independence of Path: (Theorem)

If $f(z)$ is analytic in a simply connected region R, then the line integral $\int_a^b f(z)\,dz$ is independent of the path in R joining any two points "a" and "b" in R.

Proof

By Cauchy's Theorem:



$$\int_{ADBEA} f(z)dz = 0$$

$$\Rightarrow \int_{ADB} f(z)dz + \int_{BEA} f(z)dz = 0$$

$$\Rightarrow \int_{ADB} f(z)dz = -\int_{BEA} f(z)dz = \int_{AEB} f(z)dz$$

$C_1 \rightarrow$ heading to +ve direction

$$\Rightarrow \int_{C_1} f(z)dz = \int_{C_2} f(z)dz = \int_a^b f(z)dz$$

$C_2 \rightarrow$ heading to -ve direction

Hence the integral is independent of path.

Let $f(z)$ be analytic in a region 'R' and on its boundary 'c'. Then $\oint_c f(z)dz = 0$. This fundamental theorem often called Cauchy-Goursat Theorem.

Integrals (Recall)

$\omega(t) = u(t) + i v(t)$ t is a real variable
$\omega(t)$ is a complex valued function

$\Rightarrow \omega'(t) = u'(t) + i v'(t)$

$\frac{d}{dt}[z_0 \omega(t)] = z_0 w'(t)$ $z_0 = x_0 + i y_0$

$\frac{d}{dt}[e^{z_0 t}] = z_0 e^{z_0 t}$

$\int_a^b \omega(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$

$\int_a^b \omega(t)dt = \int_a^c \omega(t)dt + \int_c^b \omega(t)dt$ for $c \in [a,b]$

Examples:

1. Evaluate $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$

$\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$

$= \int_1^2 \left[\frac{1}{t^2} - 2 \cdot i \cdot \frac{1}{t} + i^2\right] dt$

$= \int_1^2 \left(\frac{1}{t^2} - 1\right) dt - 2i \int_1^2 \frac{1}{t} dt$

$= \left[-\frac{1}{t} - t\right]_1^2 - i\left[2 \ln t\right]_1^2$

$= \left[\left(-\frac{1}{2} - 2\right) - (-1-1)\right] - i\left[2 \ln 2 - 2 \ln 1\right]$

$= -\frac{1}{2} - 2 + 2 - i\left(2 \ln\left(\frac{2}{1}\right)\right] = -\frac{1}{2} - i\, 2 \ln 2$

2. $\int_0^\infty e^{-zt}\,dt \qquad (\mathrm{Re}\, z > 0)$

$$= \lim_{l \to \infty} \int_0^l e^{-zt}\,dt \qquad l \in \mathbb{C}$$

$$= \lim_{l \to \infty} \left[\frac{e^{-zt}}{-z} \right]_0^l$$

$$= \lim_{l \to \infty} \left[-\frac{e^{-zl}}{z} + \frac{e^0}{z} \right]$$

$$= -\frac{e^{-\infty}}{z} + \frac{1}{z}$$

$$= -\frac{1}{ze^{\infty}} + \frac{1}{z}$$

$$= -\frac{1}{\infty} + \frac{1}{z}$$

$$= 0 + \frac{1}{z}$$

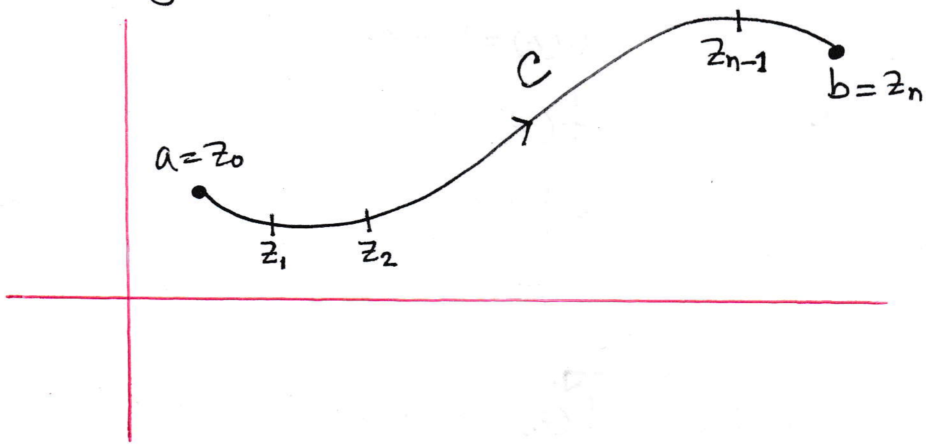
$$= \frac{1}{z}$$

Complex line Integral

The definite integral of $f(z)$ from a to b along curve c is:

$$\int_a^b f(z)\,dz \quad \text{or} \quad \oint_c f(z)\,dz$$

It is also known as path integral, contour integral, curve integral.



$$\int_a^b f(z)\,dz = \oint_c f(z)\,dz = \int_{t_1}^{t_2} f(z(t))\, z'(t)\,dt$$

$f(z) = f(z(t))$ $\qquad z = z(t)$

$dz = z'(t)\,dt$ / $dz = z'(t)dt$

$$= \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\; dt$$

(considering parametric eqn $z = z(t) = x(t) + iy(t)$

$t_1 \leq t \leq t_2$

Theorem ⟶ PAGE 8

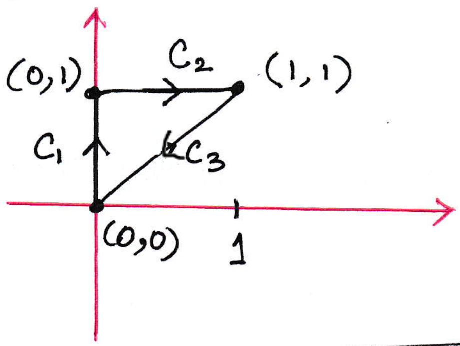
$f(z)$ is continuous on C. let $z(t)$ be any parametric function of c for $a \leq t \leq b$, then

$$\int_c f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$ ⟶ (It is known as chain rule)

$$= \int_a^b |z'(t)|\,dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\; dt = \text{arc length}$$

Examples:

(1) Evaluate $\int_c f(z)dz$ over the close triangular contour with vertices $(0,0)$, $(0,1)$, $(1,1)$ where $f(z) = y - x - 3ix^2$



$C_1 + C_2 \geq C_3$ (triangular inequality)

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \left(-\int_{C_3} f(z)dz\right)$$

C_1: $x = 0 \Rightarrow z = iy$ $(z = x + iy)$

$dz = i\,dy$

$f(z) = y - x - 3ix^2 = y - 0 - 3i(0)^2 = y$

$f(z) = y$

$$\int_{C_1} f(z)dz = \int_0^1 y\,i\,dy = i\left[\frac{y^2}{2}\right]_0^1 = \frac{i}{2}$$

C_2: $y = 1 \Rightarrow z = x + iy = x + i$

$dz = dx$

$f(z) = y - x - 3ix^2 = 1 - x - 3ix^2$

$$\int_{C_2} f(z)\,dz = \int_0^1 (1 - x - 3ix^2)\,dx = \left[x - \frac{x^2}{2} - 3i\frac{x^3}{3}\right]_0^1 = 1 - \frac{1}{2} - i = \frac{1}{2} - i$$

C_3: $y = x \Rightarrow z = x + iy = x + ix$

$z = (1+i)x$

$dz = (1+i)dx$

$f(z) = y - x - 3ix^2 = x - x - 3ix^2 = -3ix^2$

$f(z) = -3ix^2$

$$\int_{C_3} f(z)\,dz = \int_0^1 (-3ix^2)(1+i)\,dx = \int_0^1 \left(-3ix^2 - 3i^2x^2\right)dx$$

$$= \int_0^1 (-3ix^2+3x^2)dx$$

$$= \int_0^1 3x^2(1-i)dx$$

$$= 3(1-i)\left[\frac{x^3}{3}\right]_0^1$$

$$=(1-i)(1-0)$$

$$= 1-i$$

$$\therefore \int_c f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz - \int_{C_3} f(z)\,dz$$

$$= \frac{i}{2} + \frac{1}{2} - i - (1-i)$$

$$= \frac{i}{2} - \frac{1}{2} = \frac{1}{2}(-1+i)$$

(2) Evaluate $\int_c \bar{z}\,dz$ along the curve c given by

(a) from $z=0$ to $z=4+2i$ along the $z=t^2+it$

(b) the line from $z=0$ to $z=2i$ and then the line from $z=2i$ to $z=4+2i$

a) when $z=0$; $\rightarrow 0 = 0+0i = t^2+it$ by equating factor of like terms: $t=0$

when $z=4+2i$; $4+2i = t^2+it$ by equating factor of like terms: $t=2$

$\therefore 0 \le t \le 2$ or $t \in [0,2]$

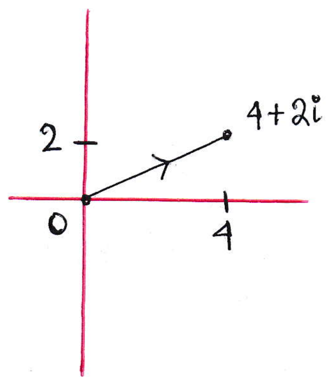
$$2t\,dt + i\,dt = (2t+i)\,dt$$

$$z = t^2 + it \Longrightarrow dz = (2t+i)\,dt \longrightarrow z'(t)$$

$$\bar{z} = t^2 - it = f(z(t)) \quad \because f(z) = \bar{z} \ (\text{given})$$

$$\int_c f(z)\,dz = \int_c \bar{z}\,dz = \int_c f(z(t))\, z'(t)\,dt$$

$$= \int_0^2 (t^2 - it)(2t+i)\,dt = \int_0^2 \left(2t^3 + it^2 - 2it^2 - i^2 t\right) dt$$

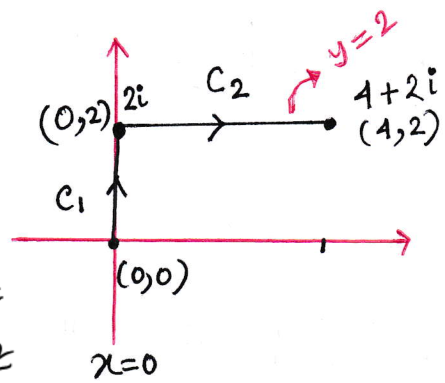


$$= \int_0^2 \left(2t^3 - it^2 + t\right) dt$$

$$= \left[\frac{2t^4}{4} - \frac{it^3}{3} + \frac{t^2}{2}\right]_0^2$$

$$= \frac{16}{2} - i\frac{8}{3} + \frac{4}{2} - 0 = 10 - \frac{8}{3}i$$

b)



$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz$$

$$\int f(z)\,dz = \int \bar{z}\,dz$$

$C_1 : x = 0, \Rightarrow z = x + iy \quad \because \bar{z} = -iy = f(z)$

$\Rightarrow z = iy$

$dz = i\,dy$

$0 \le y \le 2$

$$\int_{c_1} \bar{z}\,dz = \int_{c_1} f(z)\,dz = \int_0^2 (-iy)\,i\,dy = \int_0^2 -i^2 y\,dy = \left[\frac{y^2}{2}\right]_0^2 = 2$$

$C_2 : y = 2, \quad z = x + iy$

$z = x + 2i \qquad f(z) = \bar{z} = x - 2i$

$0 \le x \le 4 \qquad dz = dx$

$$\int_{c_2} \bar{z}\,dz = \int_{c_2} f(z)\,dz = \int_0^4 (x - 2i)\,dx$$

$$= \left[\frac{x^2}{2} - 2ix\right]_0^4$$

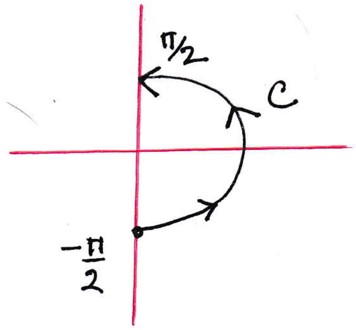
$$= 8 - 8i$$

$$\int_c f(z)\,dz = \int_c \bar{z}\,dz = \int_{c_1} dz + \int_{c_2} dz = 2 + 8 - 8i = 10 - 8i$$

③ Evaluate $\int_c \bar{z}\, dz$ where c is a right handed half of circle $|z| = 2$ or $z = 2e^{i\theta}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

The contour is given by

$|z| = 2$ $\quad\because |z| = R$



$z = 2e^{i\theta}$ $\quad \Rightarrow z = re^{i\theta}$

$\bar{z} = 2e^{-i\theta}$ $\quad \int_c f(z)\,dz$

$$\left\{\begin{array}{l} z = 2e^{i\theta} \\ \Rightarrow dz = 2i\,e^{i\theta}\,d\theta \end{array}\right.$$

$$= \int_c \bar{z}\, dz$$

$$= \int_{-\pi/2}^{\pi/2} (2e^{-i\theta})(2ie^{i\theta}\,d\theta)$$

$$= 4i\,[\theta]_{-\pi/2}^{\pi/2} = 4i\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 4\pi i$$

(Exercise Sheet #4)

A **counter** clockwise oriented contour is called a positively oriented contour.

① Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$ along

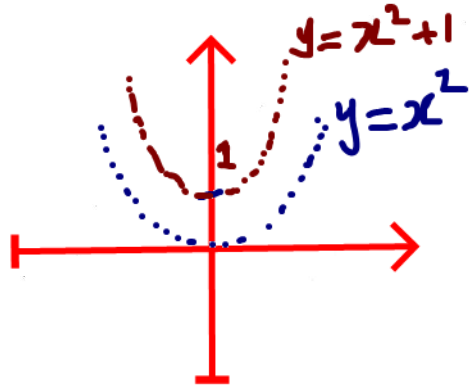
(a) the curve $y = x^2 + 1$

(b) the straight line joining $(0,1)$ & $(2,5)$

(c) the straight lines from $(0,1)$ to $(0,5)$ and then from $(0,5)$ to $(2,5)$.

(d) the straight lines from $(0,1)$ to $(2,1)$ and then from $(2,1)$ to $(2,5)$.

(a)



$$y = x^2+1$$
$$dy = 2x\,dx$$

$$\int_{(0,1)}^{(2,5)} \left((3x+y)\,dx + (2y-x)\,dy\right)$$

$$= \int_0^2 \left((3x+x^2+1)\,dx + (2(x^2+1)-x)\,2x\,dx\right)$$

$$= \int_0^2 \left((3x+x^2+1)\,dx + (2x^2+2-x)\,2x\,dx\right)$$

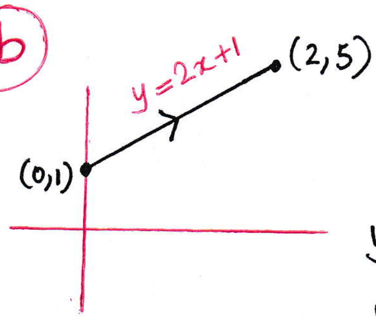
$$= \int_0^2 \left(3x+x^2+1+4x^3+4x-2x^2\right)dx$$

$$= \int_0^2 \left(4x^3-x^2+7x+1\right)dx$$

$$= \left[\frac{4x^4}{4} - \frac{x^3}{3} + \frac{7x^2}{2} + x\right]_0^2$$

$$= 16 - \frac{8}{3} + 7\cdot\frac{4}{2} + 2 = 16 - \frac{8}{3} + 14 + 2 = \frac{88}{3}$$

(b)



$$m = \frac{\Delta y}{\Delta x} = \frac{4}{2} = 2$$

$$y-y_1 = m(x-x_1)$$
$$y-1 = 2(x-0)$$
$$y = 2x+1$$
$$dy = 2\,dx$$

$$\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$$

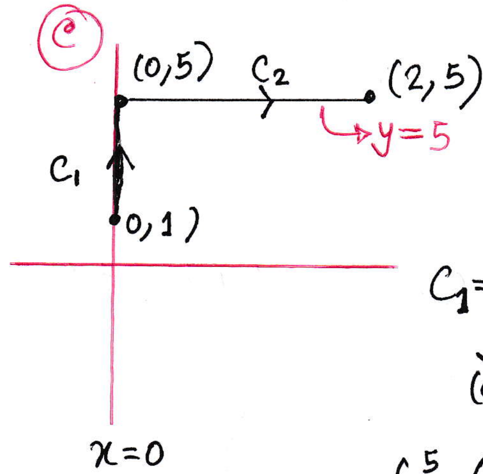
$$= \int_0^2 \left((3x+2x+1)\,dx + (2(2x+1)-x)\,2\,dx\right)$$

$$= \int_0^2 \left((5x+1)+(6x+4)\right)dx$$

$$= \int_0^2 (11x+5)\,dx = \left[11\cdot\frac{x^2}{2} + 5x\right]_0^2$$

$$= 11\cdot\frac{4}{2} + 5(2) - 0 = 32$$

(c)



The line joining (0,1) to (0,5) along y-axis $\Rightarrow x=0 \longrightarrow C_1$

$dx=0$

$$C_1=\int_{(0,1)}^{(2,5)} (3x+y)\,dx+(2y-x)\,dy$$

$$=\int_1^5 (0+y)\cdot 0+(2y-0)\,dy$$

$$=\int_1^5 2y\,dy = 2\left[\frac{y^2}{2}\right]_1^5 = 25-1=24$$

The line joining (0,5) to (2,5) is:

x-axis : $y=5 \longrightarrow C_2$

$dy=0$

$$C_2 : \int_{(0,1)}^{(2,5)} (3x+y)\,dx+(2y-x)\,dy$$

$$=\int_0^2 (3x+5)\,dx+(10-x)(0)$$

$$=\int_0^2 (3x+5)\,dx$$

$$=\left[\frac{3x^2}{2}+5x\right]_0^2 = \frac{3}{2}\cdot 4+5(2)-0=16$$

$\therefore$ The required value $=\oint_C f(z)\,dz = \int_{C_1}\dots dz+\int_{C_2}\dots dz$

$= 24+16=40$

(Exercise Sheet # 4)

6. Evaluate $\int_{i}^{2-i} (3xy+iy^2)\,dz$

a) along the straight line joining $z=i$ and $z=2-i$

b) along the parabola $x=2t-2,\ y=1+t-t^2$

a) $z=i \Rightarrow (0,1)$

$z=2-i \Rightarrow (2,-1)$

$z=x+iy$

$dz=dx+i\,dy$

$m=\dfrac{\Delta y}{\Delta x}$

$=-\dfrac{2}{2}=-1$

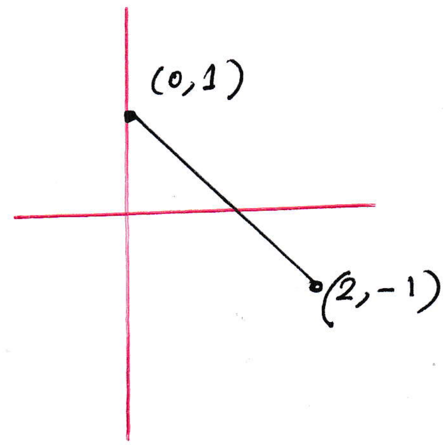
$y-y_1=m(x-x_1)$

$y-1=-1(x-0)$

$y-1=-x$

$y=-x+1$

$dy=-dx$



$$\int_{i}^{2-i} (3xy+iy^2)\,dz$$

$$\int_{(0,1)}^{(2,-1)} (3xy+iy^2)(dx+i\,dy)$$

$$=\int_0^2 \left\{3x(-x+1)+i(-x+1)^2\right\}\left\{dx+i(-dx)\right\}$$

$$=\int_0^2 \left\{-3x^2+3x+i(x^2-2x+1)\right\}(1-i)\,dx$$

$$=\int_0^2 (-3x^2+3x+ix^2-2ix+i)(1-i)\,dx$$

$$=(1-i)\left[-\frac{\cancel{3}x^3}{\cancel{3}}+\frac{3x^2}{2}+i\frac{x^3}{3}-\cancel{2}i\frac{x^2}{\cancel{2}}+ix\right]_0^2$$

$$= (1-i)\left[-8 + 6 + i\frac{8}{3} - 4i + 2i - 0\right]$$

$$= (1-i)\left(-8+6+i\left(\frac{8-12+6}{3}\right)\right)$$

$$= (1-i)\left(-2+\frac{2i}{3}\right) = -2+2i+\frac{2}{3}i-\frac{2}{3}i^2 = \frac{8i}{3}-\frac{4}{3}$$

$$= \frac{4}{3}(-1+2i)$$

(b) $x = 2t-2$ where $0 \le x \le 2$

$\therefore\ 0 = 2t-2$, $t = 1$; $2 = 2t-2$, $t = 2$

$z = x+iy$

$z(t) = (2t-1)+i(-t^2+t+1)$

$z'(t) = 2+i(1-2t)$

$(2,-1)$

$$\int_{i}^{2-i} (3xy+iy^2)\,dz$$

$(0,1)$

$\therefore\ x \in [0,2]$

Given:

$x = 2t-1$

$y = 1+t-t^2$

$f(z) = 3xy+iy^2$

$f(z(t)) = 3(2t-1)(1+t-t^2) + i(1+t-t^2)^2$

$= (6t-3)(1+t-t^2)+i(1+t-t^2)^2$

$= (-6t^3+9t^2+3t-3)+i(1+t-t^2)^2$

$$\oint_c f(z)dz$$

$$= \int_1^2 \underbrace{f(z(t))}\,\underbrace{z'(t)\,dt}$$

$$= \int_1^2 \left\{ \underbrace{(-6t^3 - 9t^2 + 3t - 3) + i(1+t-t^2)^2} \right\} \underbrace{(2 + i(1-2t))}\,dt$$

$$= \int_1^2 \left\{ (-6t^3 - 9t^2 + 3t - 3) + i\left[(1+t)^2 - 2(1+t)t^2 + (t^2)^2\right] \right\} (2 + i - 2ti)\,dt$$

$$= \int_1^2 \left\{ (-6t^3 - 9t^2 + 3t - 3) + i\left[1 + 2t + t^2 - 2t^2 - 2t^3 + t^4\right] \right\}$$
$$(2 + i - 2ti)\,dt$$

$$= \int_1^2 (-6t^3 - 9t^2 + 3t - 3 + i + 2ti + \underbrace{it^2 - i2t^2}_{-it^2} - i2t^3 + it^4)(2 + i - 2ti)\,dt$$

$$= \int_1^2 \Big[-12t^3 + 24t^2 - 12 + 2i + 4ti - 2it^2 - i4t^3 + 2it^4$$
$$- 6it^3 + 12it^2 - 6i + \underbrace{i^2}_{(-1)} + 2t\underbrace{i^2}_{(-1)} - \underbrace{i^2}_{(-1)}t^2 - \underbrace{i^2}_{(-1)}2t^3 + \underbrace{i^2}_{(-1)}t^4$$
$$- 12t^4 i - 24t^3 i + 12ti - 2t\underbrace{i^2}_{(-1)} - 4t^2\underbrace{i^2}_{(-1)} + 2t^3\underbrace{i^2}_{(-1)} + 4t^4\underbrace{i^2}_{(-1)} - 2t^5\underbrace{i^2}_{(1)} \Big]\,dt$$

$$= \int_1^2 \Big[-12t^3 + 29t^2 - 13 - 4i + 16ti + 10t^2 i - 34t^3 i - 10t^4 i - 5t^4 i$$
$$- 5t^4 + 2t^5 \Big]\,dt$$

$$= \left[-\overset{3}{\cancel{12}}\frac{t^4}{\cancel{4}} + 29\frac{t^3}{3} - 13t - 4it + \overset{8}{\cancel{16}}\,i\frac{t^2}{\cancel{2}} + 10i\frac{t^3}{3} \right.$$

$$\left. - \overset{17}{\cancel{34}}\,i\frac{t^4}{\underset{2}{\cancel{4}}} - \overset{2}{\cancel{10}}\,i\frac{t^5}{\cancel{5}} - \frac{\cancel{5}t^5}{\cancel{5}} + \cancel{2}\frac{t^6}{\underset{3}{\cancel{6}}} \right]_1^2$$

$$= \left[-3t^4 + \frac{29}{3}t^3 - 13t - 4it - 8it^2 + \frac{10}{3}it^3 - \frac{17}{2}it^4 - 2it^5 \right.$$

$$\left. - t^5 + \frac{1}{3}t^6 \right]_1^2$$

$$= (-48 + 77.33 - 26 - 8i - 32i + 26.67i - 136i - 64i - 32$$

$$+ 21.33) - (-3 + \frac{29}{3} - 13 - 4i - 8i + \frac{10}{3}i - \frac{17}{2}i - 2i - 1 - \frac{1}{3})$$

$$= -7.34 - 213.33i - (-7.67 - 19.17i)$$

$$= -7.34 - 213.33i + 7.67 + 19.17i$$

$$= 0.33 - 194.16i$$

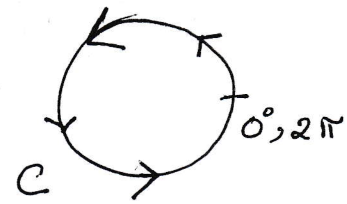
There might be some minor calculation error in the process of basic integration but the concept of the application is correct in this sum.

(Exercise Sheet # 4)

8. Evaluate $\int_c \frac{dz}{z-2}$ around (a) the circle $|z-2|=4$

(b) the circle $|z-1|=9$

(a) $|z-2|=4$ $\qquad |z|=R$, $z=Re^{i\theta}$



$z-2=4e^{i\theta}$

$z=2+4e^{i\theta}$

$dz=4ie^{i\theta}\,d\theta$

$$\int \frac{dz}{z-2} = \int_0^{2\pi} \frac{4ie^{i\theta}}{4e^{i\theta}}\,d\theta = i\int_0^{2\pi} e^{0}\,d\theta = i\int_0^{2\pi} 1\,d\theta = i\left[\theta\right]_0^{2\pi}$$

$$= 2\pi i$$

(b) $|z-1|=9$

$z-1=9e^{i\theta}$ $\qquad \left\{ \begin{array}{l} |z|=R \\ z=re^{i\theta} \end{array} \right.$

$z=1+9e^{i\theta}$

$dz=9ie^{i\theta}\,d\theta$

$$\int_c \frac{dz}{z-2} = \int_c \frac{dz}{z-1-1}$$

$$= \int_0^{2\pi} \frac{9ie^{i\theta}}{9e^{i\theta}-1}\,d\theta$$

Let $u=9e^{i\theta}-1$

$du=9ie^{i\theta}\,d\theta$

limits:

$\theta=0 \Rightarrow u=8$

$\theta=2\pi \Rightarrow u=8$

$$= \int_8^8 \frac{du}{u}$$

$$= 0$$

$u=9e^{i2\pi}-1$

$=9(\cos 2\pi + i\sin 2\pi)-1$

$=9(1+i\cdot 0)-1=8$

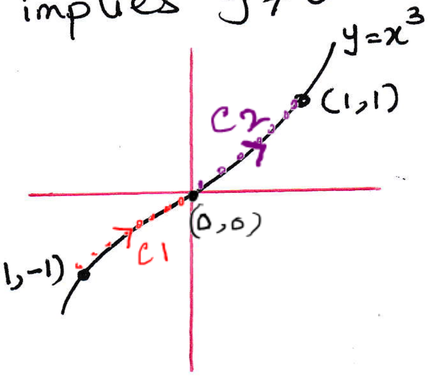
Example $f(z)$ is defined by equations

$$f(z) = \begin{cases} 1, & y<0 \\ 4y, & y>0 \end{cases}$$

C is the arc from $z=-1-i$ to $z=1+i$ along the curve $y=x^3$

$f(z)$ is changing the concavity at $y=0$ $\{\because$ the conditions are provided as $y>0$ or $y<0$ which implies $y\neq 0$

$z=-1,-i \Rightarrow (-1,-1)$; $z=1+i \Rightarrow (1,1)$



$$\therefore \int_{C\,(-1,-1)}^{(1,1)} f(z)\,dz = \int_{C_1\,(-1,-1)}^{(0,0)} \underbrace{f(z)}_{1}dz + \int_{C_2\,(0,0)}^{(1,1)} \underbrace{f(z)}_{4y}dz$$

C_1 : $y=x^3$, $z = x+iy = x+ix^3$ $\quad \because y=x^3$

$(-1,-1)$ to $(0,0)$ $\quad dz = (1+3x^2 i)\,dx$ $\quad$ over $-1\leq x\leq 0$. when $y<0$

$$\int_{C_1} f(z)dz = \int_{C_1} 1\,dz = \int_{-1}^{0} (1+3x^2 i)\,dx$$

$$= \left[x + 3\frac{x^3}{3} i\right]_{-1}^{0}$$

$$= -(-1-i) = 1+i$$

$C_2 : (0,0)$ to $(1,1)$

$$\int_{C_2} f(z)dz = \int_{C_2} 4y \, dz$$

$y = x^3$

$z = x + iy$

$= x + ix^3$

$dz = (1+3x^2 i)dx$

$$= \int_0^1 4x^3 (1+3x^2 i)dx$$

$$= \int_0^1 (4x^3 + 12x^5 i)dx$$

$$= \left[\frac{\cancel{4}x^4}{\cancel{4}} + i\,\cancel{12}^{\,2}\frac{x^6}{\cancel{6}}\right]_0^1$$

$$= \left[x^4 + 2ix^6\right]_0^1$$

$$= 1 + 2i$$

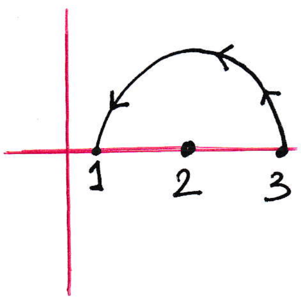
$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

$$= 1 + i + 1 + 2i$$

$$= 2 + 3i$$

Example Evaluate the contour integral $\int_c \frac{dz}{z-2}$ where c is the $\underbrace{\text{upper semicircle}}$ with radius 1 centered at $z=2$.

$\rightarrow 0^\circ$ to π



$|z-z_0|=r$

$z-z_0=re^{i\theta}$

$z=z_0+re^{i\theta}$

$=2+1e^{i\theta}$

$z=2+e^{i\theta}$, $0\le\theta\le\pi$

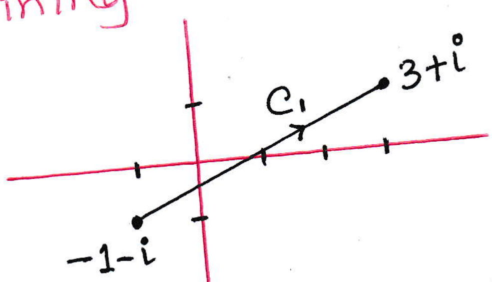
$dz=ie^{i\theta}d\theta$

$$\int_c \frac{1}{z-2}dz = \int_0^\pi \frac{ie^{i\theta}}{2+e^{i\theta}-2}d\theta$$

$$=\int_0^\pi i\,d\theta$$

$$=i\left[\theta\right]_0^\pi = \pi i$$

Example Show that $\int_{c_1} z\,dz = \int_{c_2} z\,dz = 4+2i$, where c_1 is the line segment from $-1-i$ to $3+i$, and c_2 is the portion of the parabola $x=y^2+2y$ joining $-1-i$ to $3+i$



a) The line segment that contains the points $(-1,-1)$, $(3,1)$

$$m = \frac{\Delta y}{\Delta x} = \frac{2}{4} = \frac{1}{2}$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{1}{2}(x - 3)$$

$$y = 1 + \frac{x}{2} - \frac{3}{2}$$

$$y = \frac{1}{2}x - \frac{1}{2}$$

OR $\quad x = 2y + 1$

along C_1: $\quad z = x + iy = 2y + 1 + iy$

$\qquad dz = (2+i)\,dy$

$$\int_{C_1} z\,dz = \int_{-1}^{1} (2y+1+iy)(2+i)\,dy$$

$$= \int_{-1}^{1} (4y + 2 + 2iy + 2iy + i - y)\,dy$$

$$= \int_{-1}^{1} (3y + 4iy + i + 2)\,dy$$

$$= \left[\frac{3y^2}{2} + \frac{4iy^2}{2} + iy + 2y\right]_{-1}^{1}$$

$$= \frac{3}{2} + 2i + i + 2 - \left(\frac{3}{2} + 2i - i - 2\right)$$

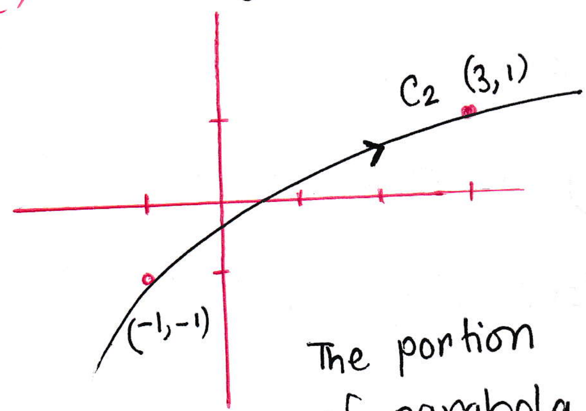
$$= 4 + 2i$$

(b) similarly along C_2

$z = x + iy$

$= (y^2+2y) + iy$ given $x = y^2+2y$

$dz = (2y + 2 + i)\,dy$



The portion of parabola

$$\int_{C_2} z\,dz = \int_{-1}^{1} \left[(y^2+2y) + iy\right](2y+2+i)\,dy$$

$$= \int_{-1}^{1} \left(2y^3 + 4y^2 + 2iy^2 + 2y^2 + 4y + 2iy + iy^2 + 2iy - y\right)dy$$

$$= \int_{-1}^{1} \left(2y^3 + 6y^2 + 3iy^2 + 3y + 4iy\right)dy$$

$$= \left[\frac{\cancel{2}y^4}{\cancel{4}_{2}} + \frac{\cancel{6}^{2}y^3}{\cancel{3}} + \cancel{3}i\,\frac{y^3}{\cancel{3}} + \frac{3y^2}{2} + \cancel{4}^{2}i\,\frac{y^2}{\cancel{2}}\right]_{-1}^{1}$$

$$= \frac{1}{2} + 2 + i + \frac{3}{2} + 2i - \left(\frac{1}{2} - 2 - i + \frac{3}{2} + 2i\right)$$

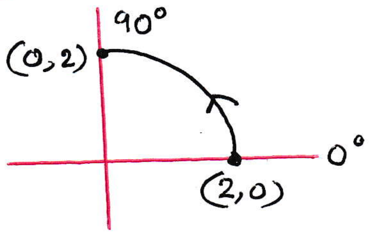
$$= 4 + 2i$$

This outcome does not hold in general for arc length while C_1, C_2 are independent of paths.

(Exercise Sheet #4)

5. Evaluate $\oint_C (z^2+3z)dz$

a) along the circle $|z|=2$ from $(2,0)$ to $(0,2)$ in a counter clockwise direction.



$|z|=2 \Rightarrow z = 2e^{i\theta}$

$\therefore dz = 2ie^{i\theta}\,d\theta$

$$\oint_C (z^2+3z)\,dz = \int_0^{\pi/2} (4e^{2i\theta}+6e^{i\theta})2ie^{i\theta}\,d\theta$$

$$= 2i\int_0^{\pi/2} (4e^{3i\theta}+6e^{2i\theta})\,d\theta$$

$$= 2i\left[\frac{4e^{3i\theta}}{3i}+\frac{6e^{2i\theta}}{2i}\right]_0^{\pi/2}$$

$$= 2\left[\frac{4}{3}e^{3i\theta}+3e^{2i\theta}\right]_0^{\pi/2}$$

$$= 2\left[\left(\frac{4}{3}e^{i\frac{3\pi}{2}}+3e^{i\pi}\right)-\left(\frac{4}{3}e^0+3e^0\right)\right]$$

$$= 2\left[\left\{\frac{4}{3}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)+3(\cos\pi+i\sin\pi)\right\} - \left\{\frac{4}{3}+3\right\}\right]$$

$$= 2\left[\frac{4}{3}(0-i)+3(-1-0)-\frac{4}{3}-3\right]$$

$$= 2\left[-\frac{4}{3}i-3-\frac{13}{3}\right]$$

$$= 2\left[-\frac{4}{3}i-\frac{22}{3}\right] = -\frac{44}{3}-\frac{8}{3}i$$