Lecture Note

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Contents

List of Figures

Chapter 1

Coordinate system, scalars and vectors

"Philosophy is written in this grand book, the universe which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth."[1]
By-Galileo Galilee.

1.1 Coordinate systems

In physics and engineering, we deal with some physical phenomena. In order to describe these phenomena in terms of mathematics, we first need to introduce the concept of a coordinate system.
There are three commonly used coordinate systems: Cartesian, cylindrical and spherical. In this chapter we will describe a Cartesian coordinate system and a

[1] 1 Galileo Galilei, The Assayer, tr. Stillman Drake (1957), Discoveries and Opinions of Galileo pp. 237-8.

cylindrical coordinate system. There are basis for any coordinate system.

- Choice a origin.
- Choice of axes.
- Choice of positive direction for each axis.

1.1.1 Cartesian Coordinate System

Let's start with a very simple example: We want to locate a point in a plane. In order to locate the point, first we choose a point as the origin and two perpendicular axes, the x and y axis. While setting the axes we are also choosing their positive directions. This type of coordinate system is called Cartesian coordinates.

- **Choice of Origin:** First choose any point as Origin O that is the most convenient.
- **Choice of axes:** In Cartesian coordinate system, the simplest set of axes are x -axis, y-axis (in case of 2 dimensions) and x -axis, y-axis, z-axis (in case of 3 dimension). All of them are at right angles with respect to each other. The range of them are- $-\infty < x < +\infty$, $-\infty < y < +\infty$, $-\infty < z < +\infty$. Any point P in space can be assigned with a triplet value of (x_p, y_P, z_p) as the coordinate value of the point.

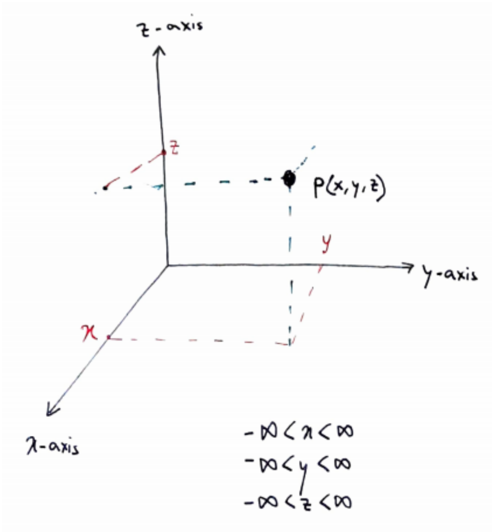

Figure 1.1: Cartesian Coordinate system in 3D

- **Choice of positive direction for each axis:** For each coordinate axis, we choose a positive direction and denote it with a + sign along the positive direction.In physics problems we are free to choose our axes and positive directions any way that we decide best fits a given problem.

1.1.2 Cylindrical Coordinate System

There are infinitely many ways of selecting a coordinate systems for a given problems. Most of the time the symmetry and constraints in a particular problem can make the problem much easier to solve with a clever choice of coordinate system. Consider an example, if you rotate a uniform cylinder about the longitudinal axis (symmetry axis), the cylinder appears unchanged. The operation of rotating the cylinder is called a symmetry operation, and the object undergoing the operation, the cylinder, is exactly the same as before the operation was performed. This symmetry property of cylinders suggests a coordinate system, called cylindrical coordinate system.

First choose an origin O and axis through O , which we call the z -axis. The number z represents the familiar coordinate of the point P along the z -axis. Here,

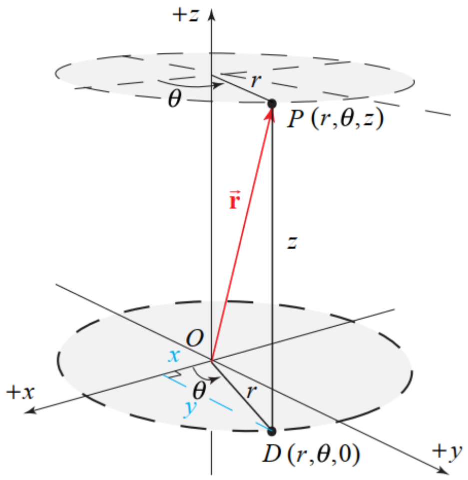

Figure 1.2: Cylindrical coordinate system

the **cylindrical coordinates** of the point P are (r, θ, z) (Figure-1.2). The non-negative number r represents the distance from the z -axis to the point P . The points in space corresponding to a constant positive value of r lie on a circular cylinder. The locus of points corresponding to $r = 0$ is the z -axis. In the plane $z = 0$, define a reference ray through O , which we shall refer to as the positive x -axis. Draw a line through the point P that is parallel to the z -axis. Let D denote the point of intersection between that line PD and the plane $z = 0$. Draw a ray OD from the origin to the point D . Let θ denote the directed angle from the reference ray to the ray OD . The angle θ is positive when measured counter-clockwise and negative when measured clockwise.

Here, The coordinates (r, θ) are also called polar coordinates. we can also transform polar coordinates (r, θ) to Cartesian coordinates (x, y) by following the coordinates transformation rules.

$$x = r\cos(\theta),$$

$$y = r\sin(\theta)$$

Conversely, if we are given the Cartesian coordinates (x, y) , the coordinates (r, θ)

can be determined from the coordinate transformations,

$$r = \sqrt{x^2 + y^2},$$
$$\theta = \tan^{-1} \frac{y}{x}$$

1.2 Introduction to vectors

1.2.1 Scalars and Vectors

In Physics and Engineering, we are interested in things that we can measure. After setting standard unit, we see certain physical quantity such as mass or the absolute temperature at some point in space require just some number (magnitude) to measure.

A single number can represent each of these quantities, with appropriate units, which are called **Scalar** quantities.Examples: Mass, Temperature, Work, Energy, distance etc.

However, There are other physical quantities which we can not measure just by some value (magnitude). In order to define those quantity, we require both magnitude and directions. Those are called **Vector** quantities. For example, Force is a vector quantity. In order to describe forces acting on a body, we need both magnitude and direction along which the force is acting on the body.

1.2.2 Properties of Vectors

Any Physical quantity that requires both magnitude and direction in order to describe them is called a vector quantity. Let denote a vector A as $\vec{\mathbf{A}}$. Then the magnitude of $\vec{\mathbf{A}}$ vector is $|\vec{\mathbf{A}}| = A$. Geometrically we can represent vectors using arrays, Where the length of the array represents the magnitude of the vector. The array also points in the direction of the vector.

(**Remarks** : Vector doesn't means an array, it's the geometric representation)

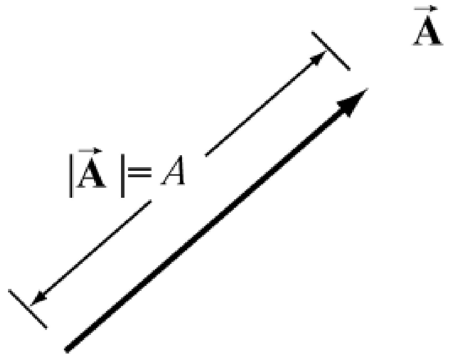

Figure 1.3: Vectors as arrows.

For vectors, there are two operations.

- Vector Addition
- Scalar Multiplication of Vectors

(1)Vector Addition

Vectors can be added. There is a geometric way to add vectors. Lets consider two vector $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. We can define a new vector $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ " the vector addition" of the two vectors in a geometric way, **"head-to-tail addition rule for vectors".** Draw the arrow that represents $\vec{\mathbf{A}}$. Place the tail of the arrow that represents $\vec{\mathbf{B}}$ at the tip of the arrow for $\vec{\mathbf{A}}$ as shown in Figure 1.4a. The arrow that starts at the tail of $\vec{\mathbf{A}}$ and goes to the tip of $\vec{\mathbf{B}}$ is defined to be the "vector addition" $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$. We can also make an equivalent construction for vector addition. The vector $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ can be drawn with their tail at the same point forming the sides of a parallelogram. Then the diagonal represents the addition of the two vector $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$. (Figure 1.4b)

It follows following properties

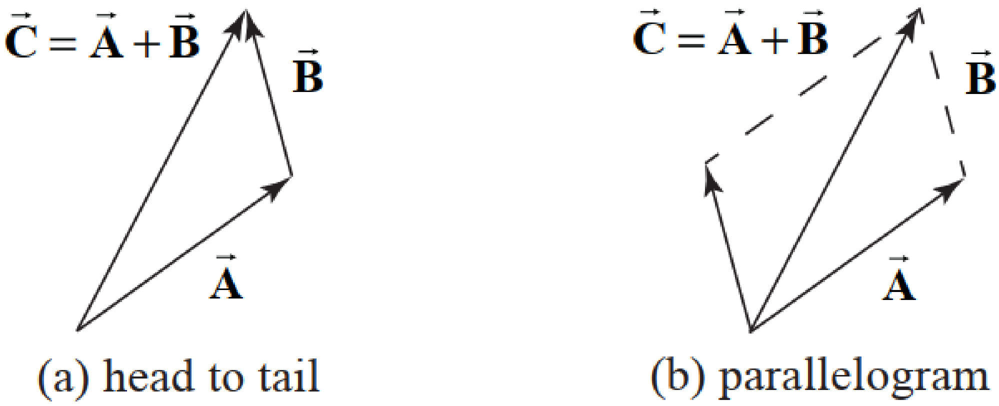

Figure 1.4: Vector addition.

Commutativity:

Order in vector addition doesn't matter.

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}}$$

Vector additions follow commutativity. As shown in (Figure 1.5), it doesn't matter whether we start from vector $\vec{\mathbf{A}}$ or vector $\vec{\mathbf{B}}$, the sum if always the same vector $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$.

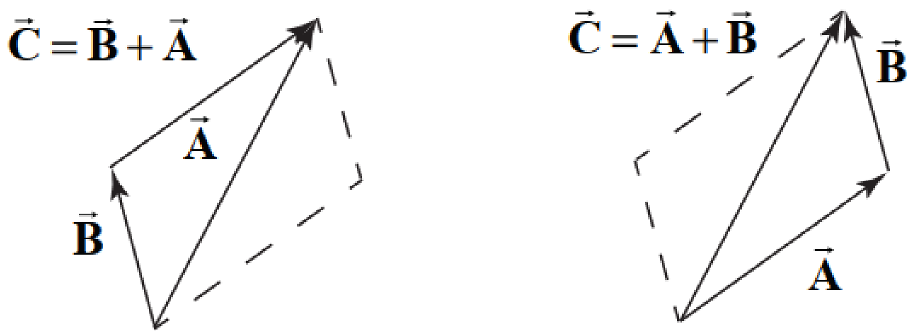

Figure 1.5: Commutative property of Vector addition.

Associativity:

In case of vector addition of three vectors, it doesn't matter with which two with start with,

$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}} = \vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}})$$

Here in the first diagram in Figure 1.6, we can add $(\vec{\mathbf{B}} + \vec{\mathbf{C}})$ first, then add them with $\vec{\mathbf{A}}$ to get $(\vec{\mathbf{B}}+\vec{\mathbf{C}})+\vec{\mathbf{A}}$, then using the commutative property of vector addition, we have $\vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}})$. Whereas, in the second diagram, we add $(\vec{\mathbf{A}} + \vec{\mathbf{B}})$ first and then with $\vec{\mathbf{C}}$ to arrive at the same vector $(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}}$ as before.

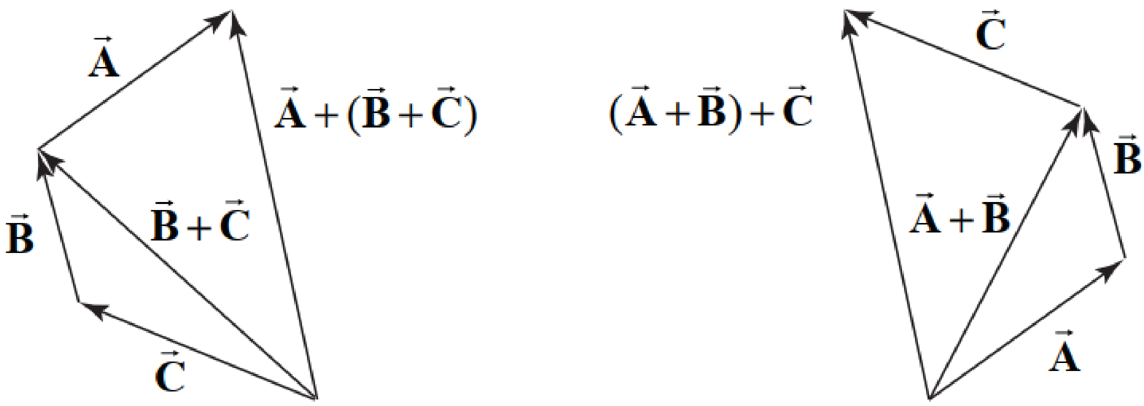

Figure 1.6: Associative property of Vector addition.

Identity Element for Vector Addition:

There is a unique identity element known as null vector $\vec{\mathbf{0}}$ for vector addition. For all vectors $\vec{\mathbf{A}}$,

$$\vec{\mathbf{A}} + \vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\mathbf{A}} = \vec{\mathbf{A}}$$

Inverse Element for Vector Addition:

For every vectors $\vec{\mathbf{A}}$, there is a unique vector $-\vec{\mathbf{A}}$, such that,

$$\vec{\mathbf{A}} + (-\vec{\mathbf{A}}) = \vec{\mathbf{0}}$$

Therefore, vector $-\vec{\mathbf{A}}$ is called the inverse vector of $\vec{\mathbf{A}}$. Both have the same magnitude but opposite in direction.

Figure 1.7: Additive Inverse

(2) Scalar Multiplication of Vectors

For any vector $\vec{\mathbf{A}}$, we can multiply it with any real number. You can think of it as stretching the length of the vector by some number. Let c be a real positive number. Then the multiplication of $\vec{\mathbf{A}}$ by c is a new vector, which we denote by the symbol $c\vec{\mathbf{A}}$. The magnitude of this vector is,

$$|c\vec{\mathbf{A}}| = c|\vec{\mathbf{A}}|$$

The direction of the new vector depends the value of c (either positive or negative). Let c $>$ 0 , then the direction of $c\vec{\mathbf{A}}$ is the same as the direction of $\vec{\mathbf{A}}$. However, the direction of $-c\vec{\mathbf{A}}$ is opposite of $\vec{\mathbf{A}}$.

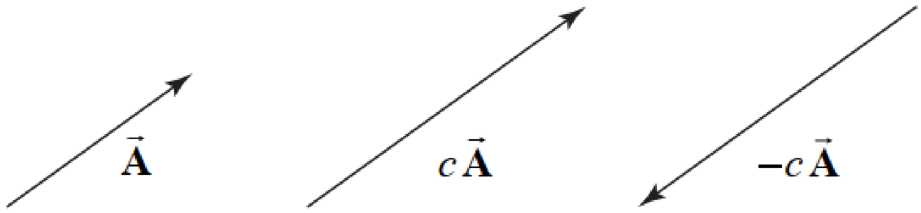

Figure 1.8: Multiplication of vector $\vec{\mathbf{A}}$ by $c > 0$, and $c < 0$

Identity Element for Scalar Multiplication:

The number 1 acts as an identity element for multiplication,

$$1\,\vec{\mathbf{A}} = \vec{\mathbf{A}}\,1 = \vec{\mathbf{A}}$$

Others additional properties of scalar multiplication of vectors are following:

Associative Law for Scalar Multiplication:

Just like vector addition, the order of scalar multiplication also doesn't matter. Let b and c are two scalar number, then

$$b(c\vec{\mathbf{A}}) = c(b\vec{\mathbf{A}}) = (bc)\vec{\mathbf{A}} = (cb\vec{\mathbf{A}})$$

Distributive Law for Vector Addition:

Let c is a real number. for any vector summation $\vec{\mathbf{A}} + \vec{\mathbf{B}}$, it satisfy a distributive law.

$$c(\vec{\mathbf{A}} + \vec{\mathbf{B}}) = (c\vec{\mathbf{A}} + c\vec{\mathbf{B}})$$

Distributive Law for Scalar Addition:

Vectors also satisfy a distributive law for scalar addition. Let b and c are two scalar. then,

$$(b+c)\vec{\mathbf{A}} = b\vec{\mathbf{A}} + c\vec{\mathbf{A}}$$

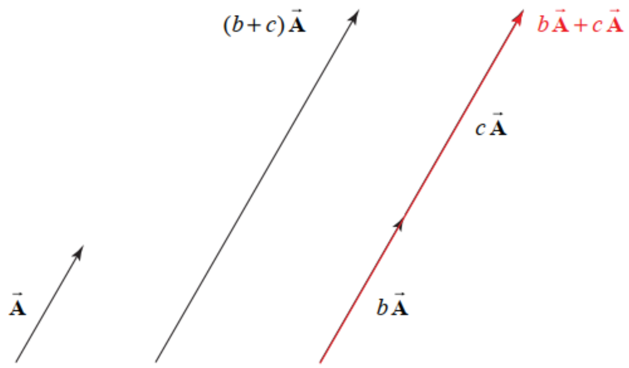

Figure 1.9: Distributive law for scalar multiplication.

1.3 Unit vector

If the value of the vector is unity, this is called unit vector. We can divide a vector by it's own value to get a unit vector. Let $\vec{\mathbf{A}}$ is a vector, we can denote the unit vector for $\vec{\mathbf{A}}$ as,

$$\hat{A} = \frac{\vec{\mathbf{A}}}{|\vec{\mathbf{A}}|} \tag{1.1}$$

Note that, $|\hat{A}| = \frac{\vec{\mathbf{A}}}{|\vec{\mathbf{A}}|} = 1$, and the direction of it is along or parallel to $\vec{\mathbf{A}}$.

1.4 Vectors in Cartesian Coordinates

As we have seen, Vectors and their operations make physical sense without referring to any coordinate system at all. But, it is often helpful to refer to a coordinate system in order to calculate using vectors. Sometimes, clever choice of coordinates could make calculations very simple.

First choose your coordinate system with an origin, axes. In Cartesian coordinate system we denote unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ along x, y and z axis. Now, we can decompose any vector as component vectors along each coordinate axis. As shown

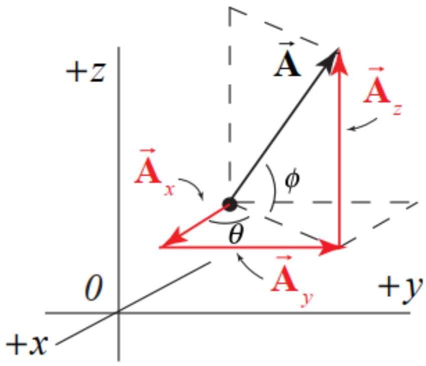

Figure 1.10: Component vectors in Cartesian coordinates.

in Figure (1.10), using the vector addition rule, we can decompose $\vec{\mathbf{A}}$ along it's three components.

$$\vec{\mathbf{A}} = \vec{\mathbf{A_x}} + \vec{\mathbf{A_y}} + \vec{\mathbf{A_z}} \tag{1.2}$$

Here, $\vec{\mathbf{A_x}}$, $\vec{\mathbf{A_y}}$ and $\vec{\mathbf{A_z}}$ are called component vectors $\vec{\mathbf{A}}$ pointing along x-axis, y-axis and z-axis. Once we have chosen unit vectors $(\hat{\imath}, \hat{\jmath}, \hat{k})$, we can define,

$$\vec{\mathbf{A_x}} = A_x\hat{\imath},$$

$$\vec{\mathbf{A_y}} = A_y\hat{\jmath},$$

$$\vec{\mathbf{A_z}} = A_z\hat{k}$$

In this expression the term A_x, A_y,A_z , (without the arrow above) are called the x, y and z -component of the vector A. Remember they are all scalar quantity not vectors.

A vector $\vec{\mathbf{A}}$ is represented by its three components (A_x, A_y, A_z) . Thus we need three numbers to describe a vector in three-dimensional space. We write the vector $\vec{\mathbf{A}}$ as,

$$\vec{\mathbf{A}} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \tag{1.3}$$

Using the Pythagorean theorem, we can define the magnitude of $\vec{\mathbf{A}}$ as,

$$|\vec{\mathbf{A}}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{1.4}$$

1.4.1 Unit Vector

We can denote a unit vector in the direction of $\vec{\mathbf{A}}$ as,

$$\hat{A} = \frac{A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \tag{1.5}$$

1.4.2 Direction

Lets consider a vector $\vec{\mathbf{A}} = (A_x, A_y)$ in 2D plane. Let θ denote the angle that the vector $\vec{\mathbf{A}}$ makes in the counterclockwise direction with the positive x -axis. Then the x-component and y-component of the vector are,

$$A_x = A\cos(\theta),$$
$$A_y = A\sin(\theta)$$

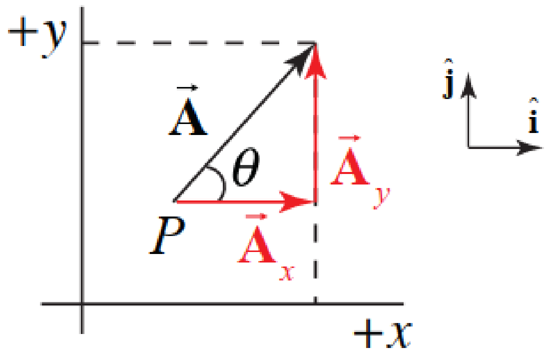

Figure 1.11: Components of a vector in the xy -plane.

We can write as,

$$\vec{\mathbf{A}} = A\cos(\theta)\hat{\imath} + A\sin(\theta)\hat{\jmath} \tag{1.6}$$

the tangent of the angle θ can be determined by,

$$\frac{A_y}{A_x} = \frac{A\sin(\theta)}{A\cos(\theta)} = \tan(\theta)$$
$$\implies \theta = \tan^{-1}(\frac{A_y}{A_x})$$

1.5 Adding vectors in a coordinate system

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are two vectors in the x-y plane. In the coordinate system, we can denote both $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ as,

$$\vec{\mathbf{A}} = A_x\hat{\imath} + A_y\hat{\jmath}$$
$$\vec{\mathbf{B}} = B_x\hat{\imath} + B_y\hat{\jmath}$$

We can write the vector addition of them as, $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$

$$\begin{aligned}\vec{\mathbf{C}} =& (A_x\hat{\imath} + A_y\hat{\jmath}) + (B_x\hat{\imath} + B_y\hat{\jmath}) \\ &= (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} \\ &= C_x\hat{\imath} + C_y\hat{\jmath}\end{aligned}$$

Here, the components of $\vec{\mathbf{C}}$ are,

$$\begin{aligned}C_x &= A_x + B_x \\ C_y &= A_y + B_y\end{aligned}$$

1.6 Example Problem 1

The displacement vector two particles are $\vec{a} = 5\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$ and $\vec{b} = 6\hat{\imath} + 2\hat{\jmath} - 6\hat{k}$.

- Find the value of $\vec{a} - \vec{b}$ in unit vector notation.
- Find the value of $\vec{a} + \vec{b}$ in unit vector notation..

1.7 Example Problem 2

Consider two points, P_1 with coordinates (x_1, y_1) and P_2 with coordinates (x_2, y_2), that are separated by distance d . Find a vector $\vec{\mathbf{A}}$ from the origin to the point on the line connecting P_1 and P_2 that is located a distance a from the point P_1.

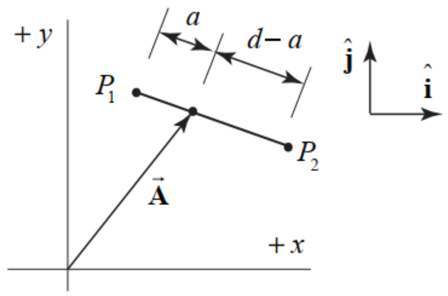

Figure 1.12: Example Problem 2