MAT 215

Topic 2

Functions of complex Variables:

$$w = f(z)$$

w ↓ output ↓ dependent variable

z ↳ input ↓ independent variable

Example

① $f(z) = z^2, \quad z = x+iy$

$$f(x+iy) = (x+iy)^2 = x^2 + 2xiy + i^2y^2$$

$$= (x^2-y^2) + i(2xy) \qquad \because i^2 = -1$$

$$= u(x,y) + i\,v(x,y)$$

$$u = \underbrace{x^2-y^2}_{\text{Real part of } f}, \qquad v(x,y) = \underbrace{2xy}_{\text{Imaginary part of } f}$$

② $f(z) = |z|^2 = x^2+y^2 = u(x,y) + iv(x,y)$

$$\Rightarrow u(x,y) = x^2+y^2 \quad \& \quad v(x,y) = 0$$

Domain:

Example: Find the domain for each of the functions:

a) $f(z) = \dfrac{1}{z^2+1}$

$z^2+1 \neq 0$

$z^2 \neq -1$

$z^2 \neq i^2$

$z \neq \pm i \rightarrow$ domain

b) $f(z) = \dfrac{z}{z+\bar{z}}$

$= \dfrac{x+iy}{x+iy+x-iy}$

$= \dfrac{x+iy}{2x} = \dfrac{x+iy}{2\,\mathrm{Re}(z)}$

domain: $\mathrm{Re}(z) \neq 0$

(3) Prove that $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist.

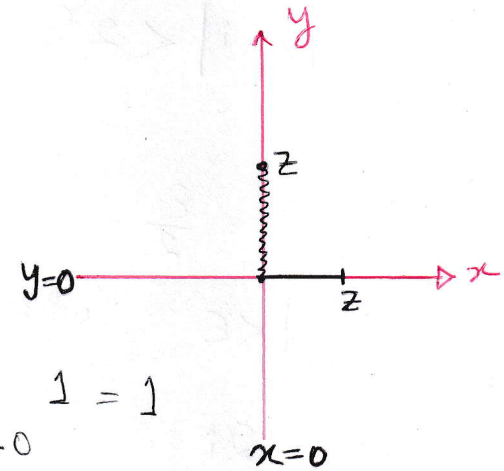
Let $z \to 0$, along x-axis, then $y = 0$

$$z = x + iy = x$$

$$\bar{z} = x - iy = x$$

$$\frac{\bar{z}}{z} = \frac{x}{x} = 1$$

$$\therefore \lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{z \to 0} 1 = 1$$



Note

If the limit exists it must be independent of the manner in which z approaches 0.

z approaches '0' at x-axis $\Rightarrow y = 0$

z approaches '0' at y-axis $\Rightarrow x = 0$

Let $z \to 0$ along y-axis, then $x = 0$

$$z = x + iy = iy \qquad \frac{\bar{z}}{z} = \frac{iy}{-iy} = -1$$

$$\bar{z} = x - iy = -iy$$

$$\therefore \lim_{z \to 0} \frac{\bar{z}}{z} = -1$$

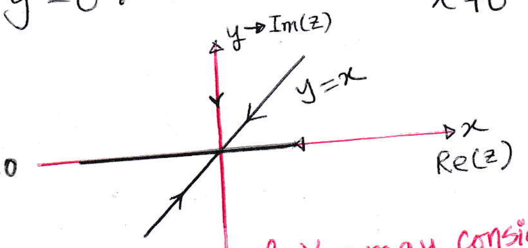
$\therefore$ the two sided approaches do not provide same result, the limit does not exist.

~~Example~~ Show that $f(z) = \dfrac{xy}{x^2+y^2}$ is not continuous

at $z = 0$.
$\hookrightarrow z_0$

NOTE: In order to prove something is not continuous, simply find two lines going through the desired limit point & show that along these two lines, the limits you get are different. On the other hand it is much harder to prove that something is continuous.

First, take the limit along the line $y = 0$. i.e. $f(z) = \dfrac{0}{x^2+0} = 0$



$$\lim_{z \to 0} \frac{xy}{x^2+y^2} = \lim_{z \to 0} 0 = 0$$

But now take the limit along the line $y = x$
i.e. $f(z) = \dfrac{x^2}{x^2+x^2} = \dfrac{1}{2}$

You may consider any one of the following graphs as they are going through $(0,0)$
$y = 0$
$y = x$

$$\lim_{z \to 0} \frac{xy}{x^2+y^2} = \lim_{z \to 0} \frac{1}{2} = \frac{1}{2}$$

$0 \neq \frac{1}{2}$

$\therefore f(z)$ is not continuous at $z = 0$, since the results are different

L' Hôspital's Rule : (Indeterminant form $\frac{0}{0}, \frac{\infty}{\infty}$)

The L' Hôspital's rule states that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$ with Inderminant form such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Note:

$0 \cdot \infty,\ \infty^0,\ 0^0,\ 1^\infty,\ \infty \pm \infty \Big\} \Rightarrow$ All imply ∞.

0^0 is a mathematical expression that is defined as 1 or left undefined, depending on context.

In algebra and combinatorics one typically defined defines $0^0 = 1$.

In mathematical analysis $0^0 = 0^{1-1} = 0^1 0^{-1} = \frac{0^1}{0^1} = \frac{0}{0} = undefined.$

In the case of limit it is considered as an indeterminate form.

Example:

$$\lim_{z \to i} \frac{z^{10}+1}{z^6+1} = \lim_{z \to i} \frac{10z^9}{6z^5}$$

$$\frac{i^{10}+1}{i^6+1} = \frac{(i^2)^5+1}{(i^2)^3+1} = \frac{-1+1}{-1+1} = \frac{0}{0}$$

$$= \lim_{z \to i} \frac{5}{3} z^4 = \frac{5}{3} i^4 = \frac{5}{3}(i^2)^2 = \frac{5}{3}(-1)^2 = \frac{5}{3}$$

Theorems on Limits

$w = f(z)$, $\downarrow (u, v)$ $\downarrow (x, y)$

Theorem 1 suppose $f(z) = u(x,y) + iv(x,y)$

$$\begin{cases} z_0 = x_0 + iy_0 \\ w_0 = u_0 + iv_0 \end{cases}$$

$u_0(x,y) + iv_0(x,y)$

then $\lim_{z \to z_0} f(z) = w_0$ —— (i)

iff $\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$ & $\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$ —— (ii)

Theorem 2

Suppose $\lim_{z \to z_0} f(z) = w_0$ & $\lim_{z \to z_0} F(z) = W_0$

Then $\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0$

$\lim_{z \to z_0} [f(z) \cdot F(z)] = w_0 \cdot W_0$

$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}$, if $W_0 \neq 0$

Examples:

① $\lim_{z \to 1+i} (z^2 - 5z + 10)$

$= \lim_{z \to 1+i} z^2 - 5 \lim_{z \to 1+i} z + \lim_{z \to 1+i} 10$

$= (1+i)^2 - 5(1+i) + 10$

$= 1 + 2i + i^2 - 5 - 5i + 10$

$= -3i + 5$

② $\lim_{z \to -2i} \dfrac{(2z+3)(z-1)}{z^2-2z+4}$

$$= \frac{\{2(-2i)+3\}\{(-2i)-1\}}{(-2i)^2-2(-2i)+4}$$

$$= \frac{(-4i+3)(-2i-1)}{-4+4i+4}$$

$$= \frac{8i^2-6i+4i-3}{4i} = \frac{-8-2i-3}{4i} \quad \because i^2=-1$$

$$= \frac{-8-2i-3}{(0+4i)} \cdot \frac{(0-4i)}{(0-4i)}$$

$$= \frac{-(2i+11)(-4i)}{(4i)(-4i)}$$

$$= \frac{(2i+11)(4i)}{-16i^2}$$

$$= \frac{8i^2+44i}{16}$$

$$= \frac{-8+44i}{16}$$

$$= -\frac{1}{2} + \frac{11}{4}i$$

(Exercise Sheet # 2)

1. Evaluate the following limits:

(i) $\lim_{z \to 1+i} \dfrac{z^2 - z + 1 - i}{z^2 - 2z + 2}$

$\dfrac{(1+i)^2 - \cancel{1} - i + \cancel{1} - i}{(1+i)^2 - \cancel{2} - 2i + \cancel{2}} = \dfrac{1+2i-1-2i}{1+2i-1-2i} = \dfrac{0}{0}$

$= \lim_{z \to 1+i} \dfrac{2z-1}{2z-2}$ Apply L'Hospital's rule

$= \dfrac{2(1+i)-1}{2(1+i)-2}$

$= \dfrac{2i+1}{2i}$

$= \dfrac{(2i+1)}{2i} \cdot \dfrac{(-2i)}{-2i}$

$= \dfrac{-4i^2 - 2i}{-4i^2}$

$= \dfrac{4-2i}{4}$

$= 1 - \frac{1}{2} i$

Continuity

$z = x + iy$

$w = f(z) = u + iv$

$f(z)$ is continuous at a point z_0 if $\lim_{z \to z_0} f(z) = f(z_0) \neq \infty$

(i) $\lim_{z \to z_0} f(z)$ exist

(ii) $f(z_0)$ exist

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$; $f(z)$ continuous iff $u(x,y)$ & $v(x,y)$ are continuous.

$f(z) = u(x,y) + i v(x,y)$

$= 3 + i \cdot \infty = \infty$

$= \infty + i(\sqrt{2}) = \infty$

(Exercise Sheet #2) $\frac{1}{x}$ $x \neq 0$, $x = 0$

3. Let $f(z) = \frac{z^2+4}{z-2i}$ if $z \neq 2i$, while $\underbrace{f(2i)}_{f(z_0)} = 3+4i$. Is $f(z)$ continuous at $\underset{z_0}{z = 2i}$?

$z < 2i$: $\lim_{z \to 2i^-} f(z) = ? = 4i$

$\boxed{z = 2i,\ f(z) = 3+4i}$

$z > 2i$: $\lim_{z \to 2i^+} f(z) = ? = 4i$

$z < 2i$ or $z > 2i$

$$f(z) = \begin{cases} \frac{z^2+4}{z-2i}, & z \neq 2i \\ 3+4i, & z = 2i \end{cases}$$ Given in the question

Algebric manipulation

$-2^2 i^2 = -4(-1) = 4$

(I) $$\lim_{z \to 2i} f(z) = \lim_{z \to 2i} \frac{z^2+4}{z-2i} = \lim_{z \to 2i} \frac{z^2 - 2^2 i^2}{z - 2i}$$

$$= \lim_{z \to 2i} \frac{(z+2i)(z-2i)}{(z-2i)}$$

$$= \lim_{z \to 2i} z + 2i$$

$$= 2i + 2i = 4i$$

(II) $f(2i) = 3 + 4i$ (Given)

(III) $\lim_{\underbrace{z \to 2i}_{z_0}} f(z) \neq \underbrace{f(2i)}_{z_0}$ $(\Rightarrow 4i \neq 3+4i)$

$f(z)$ is not continuous at $z = 2i$

Theorem: *If z_0 and w_0 are points in the z and w planes, respectively, then*

- $\lim_{z\to z_0} f(z) = \infty$ *if and only if* $\lim_{z\to z_0} \frac{1}{f(z)} = 0$
- $\lim_{z\to \infty} f(z) = w_0$ *if and only if* $\lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$
- $\lim_{z\to \infty} f(z) = \infty$ *if and only if* $\lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0.$

Limits:

$$\lim_{z\to z_0} f(z) = w_0 \quad \text{--- (1)}$$

$w = f(z)$, $w \to \underbrace{(u,v)}_{\text{image}}$, $z \to \underbrace{(x,y)}_{\text{original}}$

eqn (1) means for each +ve no. ε, there is a +ve no. δ such that $|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

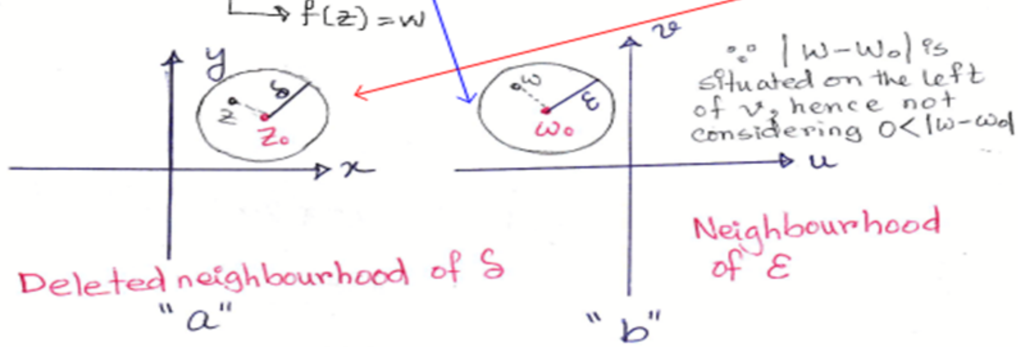
$f(z) = w$

or $|w - w_0| < \varepsilon$

$\lim_{z\to z_0} \underbrace{f(z)}_{w} = w_0$

$\therefore |w - w_0| < \varepsilon$

$0 < |z - z_0| < \delta$



"b" is the image of "a"

For each ε-neighbourhood $|w - w_0| < \varepsilon$ of w_0 there is a deleted δ-neighbourhood of $0 < |z - z_0| < \delta$ of z_0 such that

limit point

every point 'z' in it has an image 'w' lying in the ε-neighbourhood.

② Show that $\lim_{z \to i} \underbrace{2z}_{w} = \underbrace{2i}_{w_0}$

$|w - w_0| < \varepsilon \Rightarrow |2z - 2i| < \varepsilon$

$0 < |z - z_0| < \delta \Rightarrow 0 < |z - i| < \delta$

z_0

$|2z - 2i| < \varepsilon$ (i) whenever $0 < |z - i| < \delta$ for given any $\varepsilon > 0$, $\delta > 0$

$f(z) = w$

$|f(z) - w_0| < \varepsilon$

$0 < |z - z_0| < \delta$

Now $|2z - 2i| = 2|z - i| < 2\delta$ (ii) $\because |z - z_0| < \delta$ when $z_0 = i$

let $2\delta = \varepsilon \Rightarrow \delta = \frac{\varepsilon}{2}$

we have $|2z - 2i| < \varepsilon$ whenever $0 < |z - i| < \varepsilon/2$

Hence $\lim_{z \to i} 2z = 2i$

Show that if $f(z)=\frac{i\bar{z}}{2}$ then $\lim_{z\to 1} f(z)=\frac{i}{2}$

Show that $|w-w_0|<\varepsilon$ whenever $0<|z-z_0|<\delta$
while $\varepsilon>0$, $\delta>0$

$$\lim_{\underbrace{z\to 1}_{z_0}} \underbrace{f(z)}_{w}=\underbrace{\frac{i}{2}}_{w_0}$$

$\Rightarrow \left|f(z)-\frac{i}{2}\right|<\varepsilon$ whenever $0<|z-1|<\delta$

$\Rightarrow \left|\frac{i\bar{z}}{2}-\frac{i}{2}\right|<\varepsilon$ whenever $0<|z-1|<\delta$ → (i)

Now
$$\left|\frac{i\bar{z}}{2}-\frac{i}{2}\right| = \left|\frac{i\bar{z}-i}{2}\right| = \frac{|i\bar{z}-i|}{|2|} = \frac{1}{2}|z-1| < \frac{1}{2}\delta$$

$$|i\bar{z}-i| = |i(x-iy)-i| = |ix+y-i| = \sqrt{(x-1)^2+y^2}$$

$$|z-1| = |x+iy-1| = \sqrt{(x-1)^2+y^2}$$

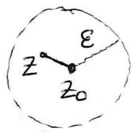
$\therefore \left|\frac{i\bar{z}}{2}-\frac{i}{2}\right|<\frac{1}{2}\delta$ → (ii)

Compare (i) & (ii) we have $\varepsilon=\frac{1}{2}\delta$ or $\delta=2\varepsilon$

Thus the condition of the limits is satisfied by points in the region $|z-1|<\delta$ when $\delta=2\varepsilon$

Derivatives

Let $f(z)$ be defined in a neighbourhood around z_0.



$0<|z-z_0|<\epsilon$

The derivative of $f(z)$ at z_0 is defined by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

Recall 1^{st} Principle of derivative MAT 110

$$f'(x) = \lim_{x \to w} \frac{f(x)-f(w)}{x-w}$$

Alternatively

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

The function $f(z)$ is said to be differentiable at z_0 if the derivative $\underbrace{f'(z) = \frac{dw}{dz}}$ is defined as:

$\therefore f(z) = w$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$y = f(x)$

$\frac{dy}{dx} = f'(x)$

Rules of differentiation in complex numbers are same as real numbers.

(Exercise Sheet #2)

5. Using the definitions, find the derivative of each function at the indicated points.

$x \to z$

$\Delta x = h \to \Delta z$

$\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

(i) $f(z) = \dfrac{2z-i}{z+2i}$ at $z=-i$

$f(z) = \dfrac{2z-i}{z+2i}$

$\therefore f(-i+\Delta z) = \dfrac{2(-i+\Delta z)-i}{-i+\Delta z+2i}$

& $f(-i) = \dfrac{2(-i)-i}{-i+2i}$

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$\therefore f'(-i) = \lim_{\Delta z\to 0} \frac{f(-i+\Delta z)-f(-i)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{1}{\Delta z}\left[f(-i+\Delta z)-f(-i)\right]$$

$$= \lim_{\Delta z\to 0} \frac{1}{\Delta z}\left[\frac{2(-i+\Delta z)-i}{-i+\Delta z+2i} - \frac{2(-i)-i}{-i+2i}\right]$$

$$= \lim_{\Delta z\to 0} \frac{1}{\Delta z}\left[\frac{-3i+2\Delta z}{i+\Delta z} + \frac{3\cancel{i}}{\cancel{i}}\right]$$

$$= \lim_{\Delta z\to 0} \frac{1}{\Delta z}\left[\frac{-3i+2\Delta z+3(i+\Delta z)}{(i+\Delta z)}\right]$$

$$= \lim_{\Delta z\to 0} \frac{1}{\Delta z}\left[\frac{\cancel{-3i}+2\Delta z+\cancel{3i}+3\Delta z}{(i+\Delta z)}\right]$$

$$= \lim_{\Delta z\to 0} \frac{1}{\cancel{\Delta z}}\cdot\frac{5\cancel{\Delta z}}{i+\Delta z}$$

$$= \frac{5}{i} = \frac{5}{i}\cdot\frac{(-i)}{(-i)} = \frac{-5i}{-i^2} = \frac{-5i}{1} = -5i$$

Examples:

1. Using the definition of derivative, find the limit of $f(z)=z^2$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$\cdot\ x \to z$

$\Delta x = h \to \Delta z$

$\cdot\ f(z)=z^2$

$f(z+\Delta z)=(z+\Delta z)^2$

$$f(z)=z^2$$

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{(z+\Delta z)^2-z^2}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{\cancel{z^2}+2z\Delta z+\Delta z^2-\cancel{z^2}}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{(2z+\Delta z)\cancel{\Delta z}}{\cancel{\Delta z}}$$

$$=\lim_{\Delta z\to 0} 2z+\Delta z$$

$$=2z$$

2. Let $f(z)=|z|^2$. Show that the derivative $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ of $f(z)=|z|^2$ exists only at $z=0$.

by definition

$\checkmark\ f(z)=|z|^2$

$f(z+\Delta z)=|z+\Delta z|^2$

$$\begin{cases}|z|^2=x^2+y^2\\ \therefore |z|^2=z\bar{z}\end{cases}$$

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{|z+\Delta z|^2-|z|^2}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{(z+\Delta z)(\overline{z+\Delta z})-z\bar{z}}{\Delta z}$$

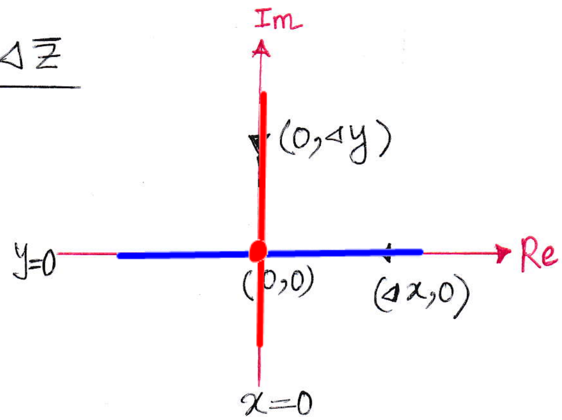
$\therefore\ \overline{z_1+z_2}=\bar{z}_1+\bar{z}_2$

$$=\lim_{\Delta z\to 0}\frac{(z+\Delta z)(\bar{z}+\Delta\bar{z})-z\bar{z}}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{\cancel{z\bar{z}}+\bar{z}\Delta z+z\Delta\bar{z}+\Delta z\Delta\bar{z}-\cancel{z\bar{z}}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z}\Delta z + z\Delta\bar{z} + \Delta z \Delta \bar{z}}{\Delta z}$$

$$= \lim_{\underset{0+0i}{\Delta z \to 0}} \left(\bar{z} + z\frac{\Delta\bar{z}}{\Delta z} + \Delta\bar{z}\right)$$



Case I: Approaching along real line, $\Delta z \to 0$ implies $\Delta y = 0$, $\Delta x \to 0$

Then $\displaystyle\lim_{\Delta z \to 0}\left(\bar{z} + z\cancel{\frac{\Delta\bar{z}}{\Delta z}} + \cancel{\Delta\bar{z}}\right) = \bar{z} + z$ —— (i)

$\bar{z} + z(1)\cancel{\frac{\Delta\bar{z}}{\Delta z}} + \Delta\bar{z}$

$\frac{\Delta\bar{z}}{\Delta z} = 1$

$y = 0$

$\because \Delta\bar{z} = \Delta z$ & $\Delta z = 0 \therefore \Delta\bar{z} = 0$

$z = x + iy = x + i\cdot 0 = x$

$\bar{z} = x - iy = x - i\cdot 0 = x$

when $y = 0$ then $z = \bar{z}$

$\Rightarrow \Delta z = \Delta\bar{z}$

Case II: Approaching along imaginary line, $\Delta z \to 0$ implies $\Delta x = 0$, $\Delta y \to 0$

then $\displaystyle\lim_{\Delta z \to 0}\left(\bar{z} + z\cancel{\frac{\Delta\bar{z}}{\Delta z}} + \Delta\bar{z}\right) = \bar{z} - z$ —— (ii)

$\bar{z} + z(-1)\cancel{\frac{\Delta\bar{z}}{\Delta z}} + \Delta\bar{z}$

$\frac{\Delta\bar{z}}{\Delta z} = (-1)$

$x = 0$

$\because \Delta\bar{z} = -\Delta z$ & $\Delta z = 0 \therefore \Delta\bar{z} = 0$

$z = x + iy = 0 + iy = iy$

$\bar{z} = x - iy = 0 - iy = -iy$

when $x = 0$ then $z = -\bar{z}$

$\Rightarrow \Delta z = -\Delta\bar{z}$

$\because$ the limits are unique, then compare (i) & (ii)

$z + \cancel{\bar{z}} = \cancel{\bar{z}} - z$

$2z = 0$

$\therefore z = 0 \Rightarrow \bar{z} = 0$ as well

$z = \bar{z}$ $\begin{cases} z = 0 = 0 + 0i = 0 \\ \bar{z} = \bar{0} = 0 - 0i = 0 \end{cases}$

as $\Delta z \to 0$
$\Rightarrow z = 0 + 0i$
$\bar{z} = 0 - 0i$

$$\therefore \lim_{\Delta z \to 0}\left(\bar{z} + z\frac{\Delta\bar{z}}{\Delta z} + \Delta\bar{z}\right)$$

$0 + 0.(\pm 1)$

$$= \lim_{\Delta z \to 0}(0 + 0 + \Delta\bar{z}) = \lim_{\Delta z \to 0}\Delta\bar{z} = \bar{0} = 0.$$

Hence $f'(z)$ exists only at $z = 0$.

(Exercise Sheet #2)

4. Find all points of discontinuity for the function $f(z) = \dfrac{2z-3}{z^2+2z+2}$.

$$f(z) = \frac{2z-3}{z^2+2z+2} \qquad z^2+2z+2 \neq 0$$

consider $z^2+2z+2=0$

$$z = \frac{-2 \pm \sqrt{2^2 - 4\times1\times2}}{2\times1}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

$\therefore f(z)$ is discontinuous at $z = -1 \pm i$.

Cauchy Riemann Equations

Named after French Mathematician Augustin Cauchy (1789-1857) and German Mathematician Bernhard Riemann (1826-66).

$U_x = V_y$, $U_y = -V_x$ (Partial derivative)

Every single function which is complex differentiable while $f(z) = u(x,y) + iv(x,y)$ must satisfy these equations

and $z_0 = (x_0, y_0)$ be a fixed point

$$\begin{matrix} U_x & V_x \\ U_y & V_y \end{matrix}$$

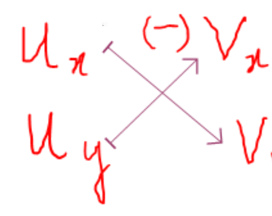
Analytic function:

If $f'(z)$ exists at all points 'z' of a region R, then $f(z)$ is said to be analytic in R.

Necessary & Sufficient Condition

Suppose $f(z) = u(x,y) + iv(x,y)$ and $f'(z)$ exists at $z_0 = x_0 + iy_0$. Then the 1^{st} order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy the Cauchy-Riemann equations:

$$\boxed{u_x = v_y \quad ; \quad u_y = -v_x} \longrightarrow$$ This is known as Necessary Condition



Also $f'(z_0)$ can be written as:

$$\boxed{f'(z_0) = u_x + iv_x} \longrightarrow$$ Sufficient Condition

Alternatively

$$\boxed{f'(z_0) = v_y - iu_y}$$

where these partial derivatives are to be evaluated at (x_0, y_0)

U_{xx} $\overset{(-)}{}$ V_{xx}
U_{xy} V_{xy}

For 2nd Partial Derivative

$$\boxed{U_{xx} = V_{xy} \quad \& \quad U_{xy} = -V_{xx}} \longrightarrow \text{Necessary condition}$$

$$\boxed{f''(z) = U_{xx} + iV_{xx}} \longrightarrow \text{sufficient condition}$$

or $\boxed{f''(z) = V_{yy} - iU_{yy}}$

Examples:

① Show that if $f(z) = z - \bar{z}$ then $f'(z)$ does not exist.

Given $f(z) = z - \bar{z} = x + iy - (x - iy) = 2iy$

$\therefore u(x,y) + iv(x,y) = 0 + i2y$

$u(x,y) = 0 \quad ; \quad v(x,y) = 2y$

$U_x = 0 \quad , \quad V_x = 0$

$U_y = 0 \quad , \quad V_y = 2$

$U_x \neq V_y \quad , \quad U_y \neq -V_x$

$\therefore f'(z)$ does not exist

by Necessary condition/Cauchy-Riemann Eqn.

$$\left.\begin{array}{l} U_x = V_y \\ U_y = -V_x \end{array}\right\} \text{Necessary condition}$$

$$\left.\begin{array}{l} f'(z_0) = U_x + iV_x \\ f'(z_0) = V_y - iU_y \end{array}\right\} \text{Sufficient condition}$$

② Show that $f'(z)$ and $f''(z)$ exist everywhere and find $f''(z)$ where $f(z)=z^3$.

$$f(z)=z^3=(x+iy)^3=x^3+3x^2iy+3xi^2y^2+i^3y^3$$

$$=x^3+i\,3x^2y-3xy^2-iy^3 \qquad \begin{cases} i^2=-1 \\ i^3=-i \end{cases}$$

$$=(x^3-3xy^2)+i(3x^2y-y^3)$$

$u(x,y)=x^3-3xy^2$; $v(x,y)=3x^2y-y^3$

$u_x=3x^2-3y^2$; (–) $v_x=6xy$

$u_y=-6xy$; $v_y=3x^2-3y^2$

Here $u_x=v_y$; $u_y=-v_x$

Also, u_x, u_y, v_x, v_y are continuous everywhere

Hence $f'(z)$ exist and $f'(z)=u_x+iv_x$ → Sufficient Condition

$$=(3x^2-3y^2)+i\,6xy$$

$$=3(x^2-y^2)+i\,6xy$$

$u_{yx}=u_{xy}$ $\quad u_y \to u_{yx} \quad u_x \to u_{xy}$

Similarly $u_{xx}=6x$; (–) $v_{xx}=6y$

$u_{xy}=-6y$; $v_{xy}=6x$

Hence $u_{xx}=v_{xy}$ & $u_{xy}=-v_{xx}$

→ Necessary Condition

Also $u_{xx}, u_{xy}, v_{xx}, v_{xy}$ are continuous everywhere, hence $f''(z)$ exist and $f''(z)=u_{xx}+iv_{xx}=6x+i6y=6(x+iy)$

Harmonic Function:

A real valued function u of two real variables 'x' and 'y' is said to be harmonic in a given domain of the xy-plane if throughout that domain, it has continuous partial derivatives of the 1st and 2nd order and satisfies the partial differential eqn:

$$\boxed{u_{xx}(x,y) + u_{yy}(x,y) = 0}$$

$f(z) = u(x,y) + iv(x,y)$

(Exercise Sheet #2)

7. Determine which of the following functions u are harmonic. For each harmonic function. Find the conjugate harmonic function v and express $u+iv$ as an analytic function of z:

(ii) $u = xe^x \cos y - ye^x \sin y$

$(uv)' = u'v + v'u$

$u = xe^x \cos y - ye^x \sin y$

$u_x = \cos y\,(xe^x + e^x) - ye^x \sin y$

$u_{xx} = \cos y\,(xe^x + e^x + e^x) - ye^x \sin y$

$= xe^x \cos y + 2e^x \cos y - ye^x \sin y$

$u_{xx} + u_{yy} = 0$

$u_y = -xe^x \sin y - e^x(y\cos y + \sin y)$

$= -e^x x \sin y - ye^x \cos y - e^x \sin y$

$u_{yy} = -xe^x \cos y - e^x(-y\sin y + \cos y) - e^x \cos y$

$= -xe^x \cos y + e^x y \sin y - e^x \cos y - e^x \cos y$

$= -xe^x \cos y - 2e^x \cos y + ye^x \sin y$

$$u_{xx} + u_{yy} = 0$$

$\therefore$ It is harmonic

$u_x \quad -V_x$

$u \quad\quad y$

Now we have to obtain $V(x,y)$

by Necessary condition

$$u_x(x,y) = xe^x \cos y + e^x \cos y - ye^x \sin y = V_y(x,y) \quad \text{— (i)}$$

$$u_y(x,y) = -e^x x \sin y - ye^x \cos y - e^x \sin y = -V_x(x,y) \quad \text{— (ii)}$$

holding 'x' constant, integrate eqn (i) with respect to 'y'.

$$\int V_y(x,y)\,dy = \int (xe^x \cos y + e^x \cos y - ye^x \sin y)\,dy$$

$\int uv = u\int v - \int [u' \int v]$

$u = y,\ v = \sin y$

$$V(x,y) = xe^x \sin y + e^x \sin y - e^x \int y \sin y\,dy$$

$$= xe^x \sin y + e^x \sin y - e^x\left[-y\cos y - \int [1(-\cos y)]\,dy\right]$$

$$= xe^x \sin y + e^x \sin y - e^x\left[-y\cos y + \sin y\right] + \phi(x)$$

Constant from integration $\because$ x-fixed, y-variable.

$$= xe^x \sin y + \cancel{e^x \sin y} + e^x y \cos y - \cancel{e^x \sin y} + \phi(x)$$

$$V(x,y) = xe^x \sin y + e^x y \cos y + \phi(x) \quad \text{— (iii)}$$

Now differentiate (iii) w.r.t. x (so that we can compare with eqn (ii) and evaluate $\phi(x)$)

$$V_x(x,y) = \sin y(xe^x + e^x) + e^x y\cos y + \phi'(x)$$

$$V_x(x,y) = xe^x \sin y + e^x \sin y + e^x y \cos y + \phi'(x) \quad \text{—(iv)}$$

Compare (ii) & (iv) {multiply (ii) with (-1)}

$$e^x x\sin y + e^x y\cos y + e^x \sin y$$
$$= e^x x\sin y + e^x \sin y + e^x y\cos y + \phi'(x)$$

$$\Rightarrow \phi'(x) = 0$$

$$\Rightarrow \int \phi'(x)\,dx = \int 0\,dx$$

$$\Rightarrow \phi(x) = c$$

Substitute $\phi(x)$ into (iii)

$$\therefore V(x,y) = xe^x \sin y + e^x y\cos y + c$$

$$\therefore u + iv = xe^x \cos y - ye^x \sin y + i(xe^x \sin y + e^x y \cos y + c)$$

Example (Analytic function)

[i.e. $f(z) = \bar{z}$ is non-analytic everywhere]

$\checkmark\ f(z) = \bar{z}$

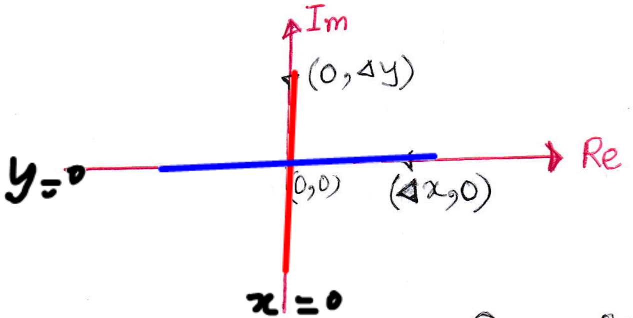
$$f'(z) = \frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad \text{by definition}$$

$f(z+\Delta z) = \overline{z+\Delta z}$

$$\frac{d}{dz}\bar{z} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \frac{\overline{x+iy+\Delta x+i\Delta y} - \overline{x+iy}}{\Delta z}$$

$$= \frac{x-iy+\Delta x-i\Delta y-(x-iy)}{\Delta x+i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$



Δz approaches Re-axis $\Rightarrow \Delta y = 0$ & $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta z \to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta z \to 0} 1 = 1$$

Δz approaches Im-axis $\Rightarrow \Delta x = 0$ & $y \to 0$

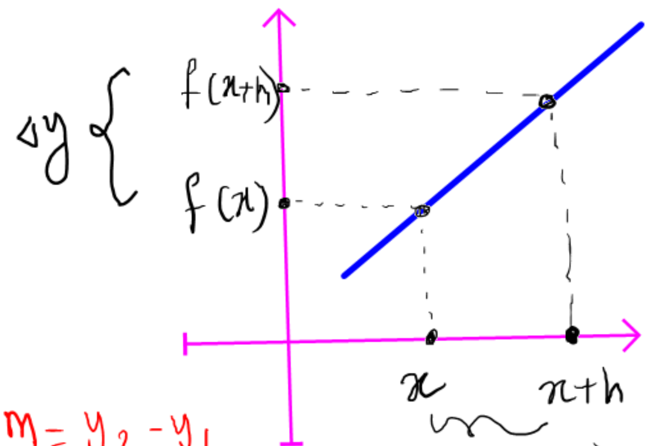
$$\lim_{\Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta z \to 0} \frac{-i\Delta y}{i\Delta y} = \lim_{\Delta z \to 0} -1 = -1$$

$\therefore$ Two sided limits are not equal, the derivative does not exist.

$\Rightarrow f(z) = \bar{z}$ is non-analytic everywhere

Derivatives

$$f'(x) = \lim_{h \to 0} \frac{\overbrace{f(x+h) - f(x)}^{\Delta y}}{\underbrace{h}_{\Delta x}} \quad \text{def of slope}$$



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

$$\Delta x = (x+h) - x$$

$$\therefore \Delta x = h$$

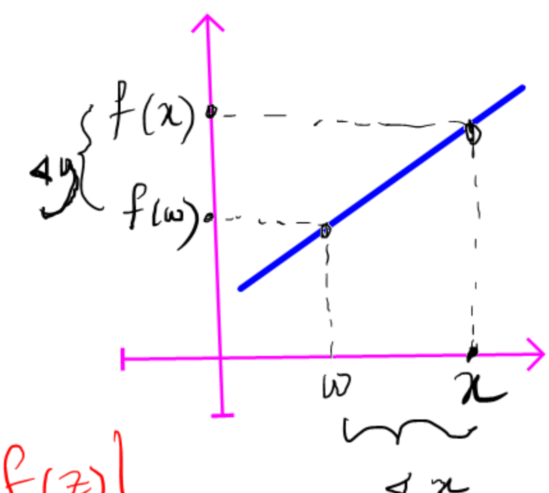
For Complex

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\boxed{\begin{array}{c} x \to z \\ \Delta x = h \to \Delta z \end{array}}$$

Alternatively,

$$\left\{ \begin{array}{l} y = f(x) \\ \frac{dy}{dx} = f'(x) \end{array} \right. \qquad f'(x) = \lim_{x \to w} \frac{\overbrace{f(x) - f(w)}^{\Delta y}}{\underbrace{x - w}_{\Delta x}}$$



for Complex

$$f'(z_0) = \lim_{z \to z_0} \frac{\overbrace{f(z) - f(z_0)}^{\Delta w}}{\underbrace{z - z_0}_{\Delta z}}$$

$$\left. \begin{array}{l} w = f(z) \\ \frac{dw}{dz} = f'(z) \end{array} \right\}$$

ii) $\lim_{z \to \infty} f(z) = \infty$ iff $\lim_{z \to 0} \frac{1}{f(\frac{1}{z})} = 0$

$\Rightarrow f(\infty) = \infty$

$$= \frac{1}{f(\frac{1}{0})}$$

$$= \frac{1}{f(\infty)}$$

$$= \frac{1}{\infty} = 0$$

iii) $\lim_{z \to \infty} f(z) = w_0$ iff $\lim_{z \to 0} f(\frac{1}{z}) = w_0$

$\Rightarrow f(\infty) = w_0$

$$= f\left(\frac{1}{0}\right)$$

$$= f(\infty)$$

$$= w_0$$

i) $\lim_{z \to z_0} f(z) = \infty$ iff $\lim_{z \to z_0} \frac{1}{f(z)} = 0$

$\Rightarrow f(z_0) = \infty$

$$= \frac{1}{f(z_0)}$$

$$= \frac{1}{\infty} = 0$$

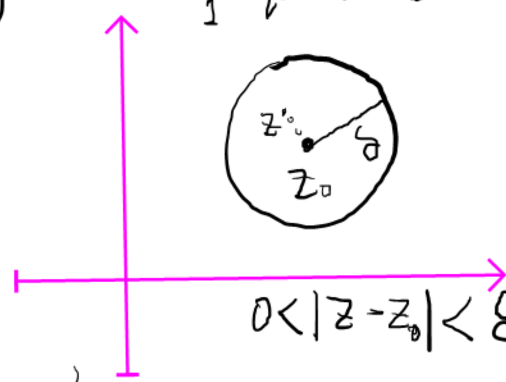
$$f(z) = \omega$$

$z \to x+iy$, $\omega \to u+iv$, $u(x,y)+iv(x,y)$

(x,y) (u,v)

1^{st} quadrant

deleted Neighbourhood of δ



$0<|z-z_0|<\delta$

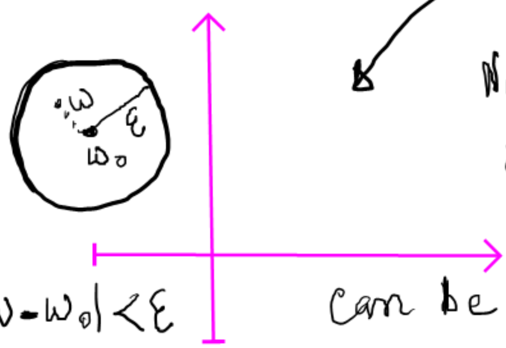
Transform

$|f(z)-\omega_0| \Rightarrow |\omega-\omega_0|$

$\lim_{z\to z_0} f(z) = \omega_0 \quad \because f(z)=\omega$

$\Rightarrow \lim_{z\to z_0} \omega = \omega_0$

Neighbourhood of ε



$|\omega-\omega_0|<\varepsilon$

image

(receiving)

Can be present at any quadrant

$$\lim_{x\to\infty} 1^x = ?$$

let $y = 1^x$

$\ln y = \ln 1^x$

$= x \ln 1$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln 1$

$\ln y = \infty \cdot 0$

$\log_e y = \infty$

$y = e^\infty = \infty$

$\lim_{x\to\infty} 1^x$

$= \lim_{x\to\infty} y$

$= \lim_{x\to\infty} \infty$

$= \infty$

$\therefore \lim_{x\to\infty} 1^x = \infty$