Lecture 12: Discrete Probability Distributions

GEOMETRIC DISTRIBUTION:

If we repeat an experiment until success occurs where probability to succeed in a single experiment is p, then the require number of trials to be n has probability

$$P(n) = (1-p)^{n-1} \times p$$

n = total number of trials

n-1 = number of unsuccessful trials

p = each experiment has a probability of success

(1-p) = probability fail on experiment

Examples:

Q1: Suppose you are rolling a die repetitively. Let's say rolling a die 6 is called success and you stop rolling the die right after the first success. Find the probability that it'd take you exactly 7 turns to be successful.

Solution: P(n) = $(1-p)^{n-1} \times p$

$=(1-\frac{1}{6})^{7-1} \times \frac{1}{6}$

=$(1-\frac{1}{6})^{6} \times \frac{1}{6}$

Here, P = probability of getting 6 is $\frac{1}{6}$

(1-P) = probability of not getting 6 is (1-$\frac{1}{6}$)

n = total number of trials 7

Q2: You are appearing at 6 consecutive exams. Probability for you to pass each of the exam, p = $\frac{4}{5}$. Once you pass an exam, there is no need to sit for the rest. What is the probability that you pass having sat for all 6 exams?

Ans: $P(n) \; = (1-p)^{n-1} \times p$

$$= (1-\frac{4}{5})^{6-1} \times \frac{4}{5}$$

$$= (1-\frac{4}{5})^{5} \times \frac{4}{5}$$

Here, n = number of trials 6

p = probability of passing in exam $\frac{4}{5}$

(1-p) = probability of failing in exam $(1-\frac{4}{5})$

Q3. Suppose the probability of success in an experiment = p. you are running the experiment repetitively. X= the number of trials required. find the probability distribution for X.

Ans. We know that the probability corresponding to X=x can be found using a geometric distribution formula because for any number of trials n required for one success, we need n-1 failure at first and then a success at last. So, the probability gets first success at nth turn =$(1-P)^{n-1} \times P$

X	1	2	3	4	…….	n	…
P(X=x)	p	$(1-p) \times p$	$(1-p)^2 \times p$	$(1-p)^3 \times p$		$(1-p)^{n-1} \times p$	

Now for n=1, P(X=1) = $(1-p)^{1-1} \times p = p$

n=2, P(X=2) = $(1-p)^{2-1} \times \; p \; = (1-\mathrm{p}) \times p$

n=3, P(X=3) = $(1-p)^{3-1} \times \; p \; = \; (1-p)^2 \times p$

n=4, P(X=4) = $(1-p)^{4-1} \times \; p \; = \; (1-p)^3 \times p$

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n=n, P(X=n) = $(1-p)^{n-1} \times p$

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upto n = ∞

Conditions

GEOMETRIC DISTRIBUTION is a type of distribution that occurs when the following 4 conditions are met

1. Each trial has only two outcomes

2. The trials are independent

3. The probability of outcomes does not change

4. The variables of interest are the number of trials until the first success

Expectation of geometric random variable :

E(x) = $\frac{1}{p}$

Variance of Geometric Random Variable :

var(x) = $\sigma^2 = \frac{1-p}{p^2} = \frac{q}{p^2}$

Standard deviation of Geometric Random Variable :

$\sigma(x) = \frac{\sqrt{1-p}}{p} = \frac{\sqrt{q}}{p}$

Example: 2% of all tires produced by company ABC has a defect. A random sample of hundred tires is tested for quality assurance.

a. What is the probability that the 6th tire selected is the first to have a defect?
b. What is the probability that the first defect is identified among the first 3 samples?
c. How many tires would you expect to test until you find the first defective one?
d. Calculate the variance and ~~standard deviation~~?

Ans: **a**. probability that a tire is defective, p=2% = $\frac{2}{100}$ = 0.02

P(x=6) = q^{x-1}p = $(1-0.02)^{6-1} \times 0.02 = 0.0181$

b. P (X$\leq 3) = P(X=1) + P(X=2) + P(X=3)$

$=0.98^0 \times .02 + 0.98^1 \times .02 + 0.98^2 \times .02$

=0.0588

c. E(x)=μ = $\frac{1}{.02}$ =50

d. var(x) = $\frac{q}{p^2} = \frac{0.98}{.02^2} = 2450$

BINOMIAL DISTRIBUTION:

If we repeat an experiment with probability of success at a single trial P, then the probability of exactly r successes out of n trials is

$$P(n) = n_{C_r} \times p^r \times (1-p)^{n-r}$$

r = number of success

n = total number of trials

p = probability of success in a single experiment

(1-p) = probability of failure in a single experiment

Examples:

Q1. Suppose you are rolling a die repetitively. Let's say rolling a die 6 is called success. Find the probability that you'd be successful exactly twice after rolling the die 7 times?

Ans:

$P(n) = n_{C_r} \times p^r \times (1-p)^{n-r}$

$= 7_{C_2} \times (\frac{1}{6})^2 \times (1 - \frac{1}{6})^{7-2}$

$= 7_{C_2} \times (\frac{1}{6})^2 \times (1 - \frac{1}{6})^5$

n = total number of trials 7

r = number of success 2

p = probability of getting 6 is $\frac{1}{6}$

(1-p) = probability of not getting 6 is $(1-\frac{1}{6})$

Q2: You are appearing at 6 consecutive exams. Probability for you to pass each of the exam, P = $\frac{4}{5}$. What is the probability that you pass exactly 4 of total 6 exams?

Ans: $P(n) = n_{C_r} \times p^r \times (1-p)^{n-r}$

$= 6_{C_4} \times (\frac{4}{5})^4 \times (1 - \frac{4}{5})^{6-4}$

$= 6_{C_4} \times (\frac{4}{5})^4 \times (1 - \frac{4}{5})^2$

n = total number of trials 6

r = number of success 4

p = probability of pass is $\frac{4}{5}$

(1-p) = probability of fail is (1-$\frac{4}{5}$)

Q3. Suppose, A and B are two wrestlers. In a single match both of them have an equal chance of winning and there is no tie in this form of wrestling. They are fighting each other in a 6-match tournament, Y = the number of matches won by A. find the distribution of Y

Ans:
p = probability of A winning a match
(1-p) = probability of B winning a match

Since chances are both equal for A&B,

p = 1-p

2p = 1

p = $\frac{1}{2}$

Y	0	1	2	3	4	5	6
P(Y)	$\frac{1}{2^6}$	$\frac{3}{2^5}$	$\frac{15}{2^6}$	$\frac{5}{2^4}$	$\frac{15}{2^6}$	$\frac{3}{2^5}$	$\frac{1}{2^6}$

From binomial distribution we know that, probability of A to win exactly 'r' matches in a 'n' match tournament,

$$n_{C_r} \times p^r \times (1-p)^{n-r}$$

Here n = 6, r = y = 0, 1, 2, 3, 4, 5, 6

P(Y=0) = $6_{C_0} \times (\frac{1}{2})^0 \times (1-\frac{1}{2})^{6-0}$ = $\frac{1}{2^6}$

P(Y=1) = $6_{C_1} \times (\frac{1}{2})^1 \times (1-\frac{1}{2})^{6-1}$ = $\frac{3}{2^5}$

P(Y=2) = $6_{C_2} \times (\frac{1}{2})^2 \times (1-\frac{1}{2})^{6-2}$ = $\frac{15}{2^6}$

P(Y=3) = $6_{C_3} \times (\frac{1}{2})^3 \times (1-\frac{1}{2})^{6-3}$ = $\frac{5}{2^4}$

P(Y=4) = $6_{C_4} \times (\frac{1}{2})^4 \times (1-\frac{1}{2})^{6-4}$ = $\frac{15}{2^6}$

P(Y=5) = $6_{C_5} \times (\frac{1}{2})^5 \times (1-\frac{1}{2})^{6-5}$ = $\frac{3}{2^5}$

P(Y=6) = $6_{C_6} \times (\frac{1}{2})^6 \times (1-\frac{1}{2})^{6-6}$ = $\frac{1}{2^6}$

Conditions

BINOMIAL DISTRIBUTION is a type of distribution that occurs when the following conditions are met-

1. Each trial has only two outcomes

2. The trials are independent

3. The probability of outcomes does not change

4. The number of trials is fixed

5. The variable of interest is the number of successful trials(r) among (n) trials

Mean of binomial random variable:

Mean = E(x) = np expected value

Variance of binomial random variable:

Var(x) = np(1-p) =npq

Standard deviation of binomial random variable:

$\sigma(x)$= $\sqrt{np(1-p)} = \sqrt{npq}$

Example: let the probability of a student taking STA 201 = 0.25. 30 students are randomly chosen from the students of BRACU

a. Find the probability that exactly 8 students out of 30 took STA201
b. Probability that fewer than 5 students out of 30 took STA201=?
c. Calculate the mean and standard deviation of this binomial distribution

Ans: **a**. p = 0.25, n = 30, r = 8

b(8; 30; 0.25) = $30_{C_8} \times (.25)^8 \times (1-0.25)^{30-8} = 0.1593$

b.

$\sum_{x=0}^{4}$ b(x < 5; 30; 0.25)

$= 30_{C_0} \times (.25)^0 \times (1-0.25)^{30-0} + 30_{C_1} \times (.25)^1 \times (1-0.25)^{30-1}$

$+30_{C_2} \times (.25)^2 \times (1-0.25)^{30-2} + 30_{C_3} \times (.25)^3 \times (1-0.25)^{30-3} +$

$30_{C_4} \times (.25)^4 \times (1-0.25)^{30-4}$

$= 0.0979$

c.

Mean = μ = np = $30\times\ 0.25 = \ 7.5$

Var(x)= np(1-p) = $30\times\ 0.25\ \times\ 0.75 = 5.625$

Standard deviation = $\sqrt{5.625}\ = 2.372$

POISSON DISTRIBUTION:

A random variable is said to have Poisson distribution if the following condition are satisfied:

1. The event can occur any number of times during a time period
2. The events occur independently

3. The rate of occurrence is constant
4. The probability of an event occurring is proportional to the length of the time period.

Formula:

$$P\ (X\ = x)\ = \frac{e^{-\lambda}\ \lambda^{x}}{x!}$$

Examples:

Q1. A small business receives, on average, 12 customers per day. What is the probability that the business will receive exactly 8 customers in a day?

Ans: λ = 12

$$\text{P (X =x)} = \frac{e^{-\lambda}\ \lambda^{x}}{x!}$$

$$\text{P (X =8)} = \frac{e^{-12}\ 12^{8}}{8!} = 0.065523 = 6.55\%$$

Q2. A student receives on average 7 text message in a two hours period

a. What is the probability that the student will receive exactly 9 text messages in two hours?

b. What's the probability that the student will receive exactly 24 text messages in 8 hours?

Ans.

a.

λ = 7

x = 9

P (X =9) = $\frac{e^{-7}\, 7^9}{9!}$ = 0.1014

$$\approx\ 10.14\%$$

b.

λ = 28

x = 24

$P\ (X\ = 24) = \frac{e^{-28}\, 28^{24}}{24!}$

= 0.060095

$\approx\ 6.01\%$

Q3. A small business receives on average 8 cells per hour

a. probability that it will receive 7 cells in an hour =?

b. what is the probability that the business will receive, at most 5 calls in an hour?

c. What is the probability that the business will receive more than 6 calls in an hour?

Ans:

a. λ = 8, x= 7

P (X =7) = $\frac{e^{-8}\, 8^7}{7!}$ = 0.1395865 ≈ 13.96%

b. $P\ (X\ \leq\ 5) = P(X = 0) + P(X = 1) + P(X = 2) + \ldots + P(X = 5)$

$$\square\ (\square\ \leq\ 5) = e^{-8} \times [\frac{8^0}{0!} + \frac{8^1}{1!} + \cdots + \frac{8^5}{5!}]$$

=0.191236

$\approx 19.12\%$

c. P(X > 6) = 1 – P (X≤ 6)

$$=1 - e^{-8} \times [\frac{8^0}{0!} + \frac{8^1}{1!} + \cdots + \frac{8^6}{6!}]$$

=.686626$\approx$ 68.66%

Exercise: From Jay L. Davore's Probability & Statistics

Section 3.4 - 46, 47 (a, b, c) , 49, 50, 51, 52

Section 3.6 - 81 (a, b, c), 82, 83, 84