BRAC UNIVERSITY

Inspiring Excellence

Probability

Introduction: In our daily life very often we use the term “probability”. Probability or a tendency “uncertainty” or “chance” refers to the probable movements or to occurring an event. Every-day, we express our thinking using the sentence like:

i) Everyone who lives will must die.
ii) We can live without breathing.
iii) It may rain today

The idea of probability is expressed in the above three sentences. In the first sentence we see a event, that is a certain. Every certain event is an obvious proclamation of probability. Here value of the probability is 1.

In the second sentence, the probability of living without breathing is zero. Now if we consider the third sentence, we realize that there lies a probability to rain today. Here we do not know the exact probability but the probability lies between 0 to 1.

❖ **Probability vs Possibility**

Sometimes a distinction is made between probability & possibility. Possibility precedes probability. We cannot make educated guess or prediction unless we know at least “what is possible”. We can hardly predict who is going to win the next world cup football, for example, unless we know who the possible participants are. Possibility thus refers to exhaustiveness of outcomes, while probability refers to their predictive behavior.

Probability

Probability measures the likelihood of occurring an event. Numerical or quantitative measure of uncertainty or chance of an even of a random experiment is called probability.

Examples:

❑ Weather Forecasting

❑ Predicting batting average in cricket matches

❑ Politics

❑ Choosing appropriate insurance strategies

❑ Selecting sport strategies etc.

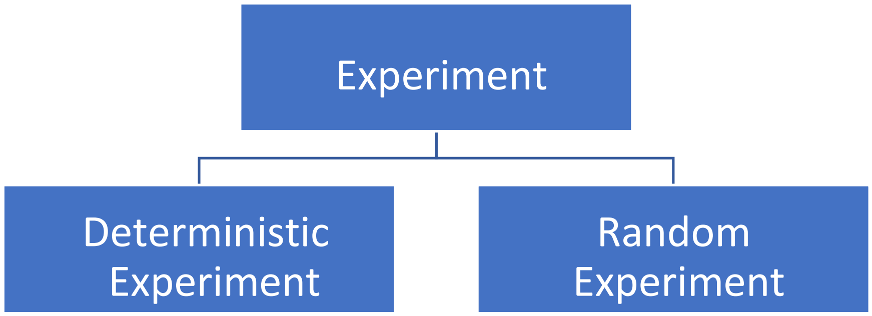
Basic Terminologies: Three keywords are used while studying probability:

➢ Experiment

➢ Outcome

➢ Event

Experiment: Experiment is an act that can be repeated under given conditions. An experiment is a procedure which can repeated infinitely but it has a well-defined set of possible outcomes. Unit experiment is called trial. Experiment may be Deterministic or predictable and Random or un predictable.



Tossing a coin is an example of experiment since it can be infinitely repeated and it has defined set of possible outcomes which are “heads” & “tails”.

Rolling a dice is another example of experiment which can be repeated several times and an infinite set of possible outcomes, S = {1, 2, 3, 4, 5, 6}

Measuring the lifetime of an automobile is also an example of experiment and it’s sample space consists of all nonnegative real numbers. That is, $S = [0, \infty)$

❑ **Deterministic or predictable experiment:** An experiment is called deterministic when the outcome or result is unique or certain. Everyone conducting the experiment will get the same result or outcome. The results of these experiments are known with certainty and is known prior to its conduct.

Examples:

➢ Predicting the amount of money in a bank account if you know the initial deposit and the interest rate.
➢ $2\ H_2 + O_2 = 2\ H_2O$
➢ An experiment conducted to verify the Newton’s law of motion and an experiment conducted to verify the Economic Law of Demand are examples of deterministic or predictable experiment.

Random Experiment: A Random Experiment is an experiment, trial, or observation that can be repeated numerous times under the same conditions. The outcome of an individual random experiment must be independent and identically distributed. It must in no way be affected by any previous outcome and cannot be predicted with certainty.

Examples-

- Tossing a coin.
- Rolling a dice.
- The selection of a numbered ball (1-50) in an urn.
- The time difference between two messages arriving at a message center.
- The time difference between two different voice calls over a particular network.
- The number of calls to a communication system during a fixed length interval of time.
- Number of defected items produced by a machine by an hour
- Drawing a card from a pack

Outcomes: Result of an experiment are known as outcomes. In a tossing coin experiment "Head" is an outcome of the experiment. There are 2 types of outcome-

❑ **Sample Space & sample point:** A set or collection of all possible outcomes of a random experiment is called sample space of that random experiment and it is denoted by $'\Omega'$, S. Each outcome of an experiment is a sample point or element in the sample space.

- A discrete sample space in in which there is a finite number of outcomes or a countably infinite number of outcomes.
- A continuous sample space has uncountable outcomes.
- If every sample point has equal probability than it is called simple sample space otherwise it is compound sample space.

For example,

➢ Tossing a coin: S = {H, T}

➢ Throwing a dice: S = {1, 2, 3, 4, 5, 6}

➢ Lifetime of an electric bulb: S= $\{x|0 \leq x < \infty\} = [0,\infty)$

❑ Consider an experiment that consists of rolling two balanced dice, one white and one red are thrown and number of dots on their upper faces are noted, also if b be the outcomes of the white die and r be the outcomes of the red die. If we let denote the outcome in which white dice has value w and red dice has value r, then the sample space of this experiment is:

		White Die					
		1	2	3	4	5	6
Red Die	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Event: One or more outcomes of a random experiment constitute an event. Events are generally denoted by capital letter. An event, E can be defined as a subset of a sample space, S. For example, in throwing a die experiment, the sample space is S = {1, 2, 3, 4, 5, 6}. E = {2, 4, 6} is an event, which can be described in words as "the number is even".

Types of events: There are several types of events. Events can be-

Sure event: An event is called sure event when it must occur. The probability of a sure event is one. This means that this kind of event is certain to happen.

Impossible event: An event is called impossible event when it never happens. This means that probability of an impossible event is always zero and it will never happen.

Uncertain event: When the outcomes of an event may or may not happen, then this type of event is called uncertain event. These kinds of events are important. In most of the cases, we need to find the probabilities of these kinds of events. The probabilities of these kinds of events lie between 0 and 1.

Mutually exclusive events: Two events are called mutually exclusive if the occurrence of one event means that none of the other events can occur simultaneously in a single trial. In other words, if one of those events occur, then the other evets will not occur.

For example, In tossing a coin experiment, event E_1 = {Head} and event, E_2 = {Tail} are mutually exclusive events as both of the events E_1 & E_2 cannot occur at the same time.

On a day, Event E_1= {Rain} & event E_2 = {Sunny} may occur simultaneously. These are not mutually exclusive events.

Mutually Exclusive (mutually exclusive events mean both the events cannot occur at the same time. For example, in a coin tossing experiment both head and tail cannot occur at the same time so the occurrences of head or tail is mutually exclusive events, turning left or right at the same time are also mutually exclusive events.)

Event-01: Muztafiz get 6 wickets in a match

Event-02: Taskin get 6 wickets in the same match

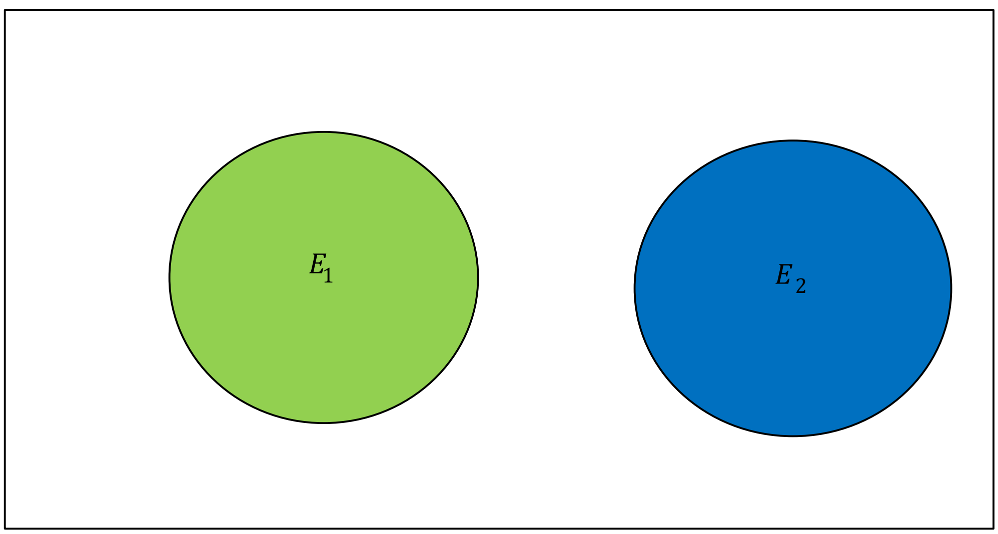
Equally likely events: Equally likely events are events that have the same probability or likelihood of occurring. Each numeral on a die is equally likely to occur when the die is tossed.

The sample space of throwing a die is, S = { 1, 2, 3, 4, 5, 6 } and the probability of getting a chosen numeral = $\frac{1}{6}$. Here the chance of occurring each numeral is the same and so they are equally likely events.

Joint events (joint events have common elements. For example, hearts and kings are joint events.)

Dependent (dependent events indicate that they can be influenced by the previous events. Example: After taking a card from a deck of card the probability changes since there are less cards available then)

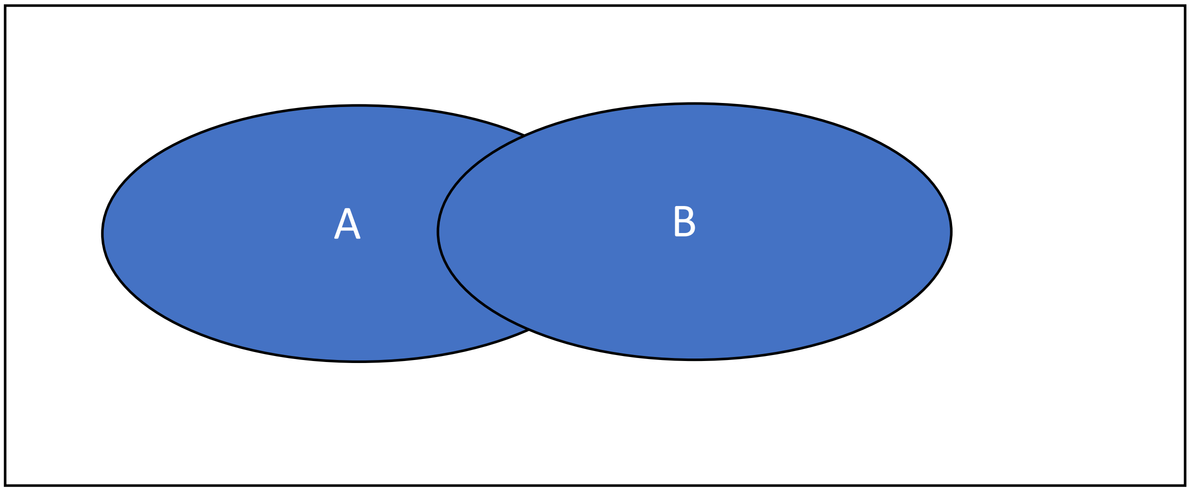
Disjoint events: Two events are called disjoint, if they have no common elements between them. Mutually exclusive events are disjoint events.



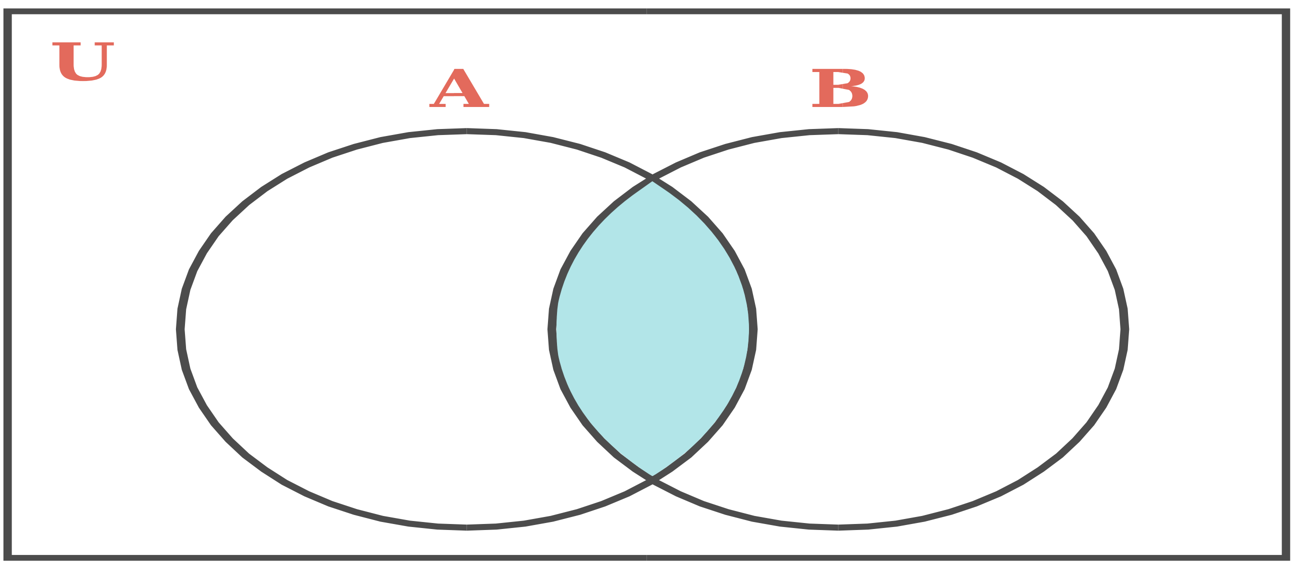
Dependent Events: If two events have some common elements then both of these events are referred as dependent events. For example, in a deck of 52 cards, if E_1 be an event of selecting 'Red' card and E_2 be another event of selecting 'Queen' card then these events are joint events since there are two 'Queen' cards in a set of 'Red' cards.

- **Union**: The union of two sets contains all the elements contained in either set (or both sets). The union is notated $A \cup B$, *where A and B are two sets.*
- **Intersection**: The intersection of two sets contains only the elements that are in both sets. The intersection is notated $A \cap B$.

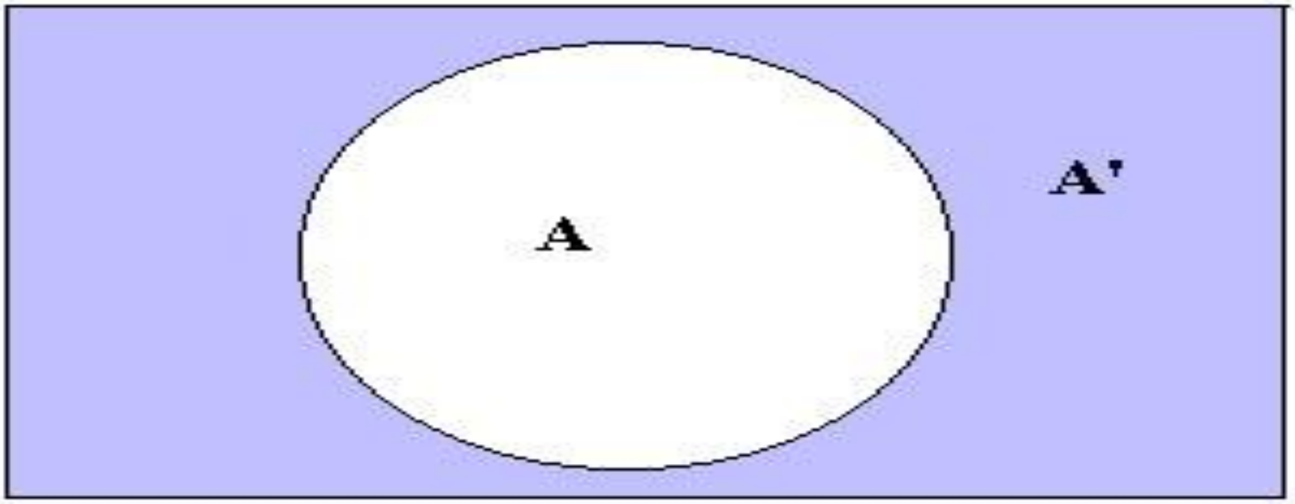
- **Compliment**: The complement of a set *A* contains everything that is *not* in the set *A*. The complement is notated *A'*, or A^c, or sometimes $\bar{A}$.



A union B Elements that belong to either A, B or both



A intersect B Elements that belong to both A & B



Elements that don't belong to A

Event-01: Score of Sakib AL Hasan in an innings

Event-02: Score of Virat Kohli in another innings

Independent Events: Two events are known as independent events if the occurrence of one event does not affect the probability of occurring another event. For two independent events A and B, the probability that A and B will both occur is found by multiplying the two probabilities. P (A and B) = P (A) P(B). For example; In a coin tossing experiment events, E_1 = {Head} and event E_2 ={Tail} are independent events because in this case occurrence of event, E_1 does not affect event, E_2.

Independent (independent event means each event is not affected by other events.
Example: Tossing a coin, throwing a die etc.)

Basic concepts of probability

Probability:

The probability of an event measures the likelihood of the occurrence of that event.

The classical probability approach is,

$$Probability\ of\ an\ event = \frac{Number\ of\ favorable\ outcomes}{Total\ number\ of\ possible\ outcomes}$$

- The probability of event A is denoted by P(A)
- Probability is quantified as a number between 0 and 1, where, loosely speaking, 0 indicates impossibility and 1 indicates certainty.
- The higher the probability of an event, the more likely it is that the event will occur.
- The sum of probabilities of all sample points in a sample space is equal to 1.
- The probability of event A is the sum of the probabilities of all the sample points in event A.

Example 1:

Suppose we draw a card from a deck of playing cards. What is the probability that we draw a spade?

Solution: The sample space of this experiment consists of 52 cards, and the probability of each sample point is 1/52. Since there are 13 spades in the deck, the probability of drawing a spade is,

$$P(\text{Spade}) = {}^{13}/_{52} = {}^{1}/_{4}$$

Example 2: Suppose a coin is flipped 3 times. What is the probability of getting two tails and one head?

Solution: For this experiment, the sample space consists of 8 sample points.

S = {TTT, TTH, THT, THH, HTT, HTH, HHT, HHH}

Each sample point is equally likely to occur, so the probability of getting any particular sample point is 1/8. The event "getting two tails and one head" consists of the following subset of the sample space.

A = {TTH, THT, HTT}

The probability of Event A is the sum of the probabilities of the sample points in A. Therefore,

P(A) = 1/8 + 1/8 + 1/8 = 3/8

Approaches of Assigning Probability

- At first we identify the sample space S of the random experiment.
- We then define our favorable event and assign a probability to the event using one of the following 2 basic approaches:
- Classical or mathematical or a priori probability approach
- Frequentist or empirical or statistical approach

Classical approach: If a random experiment has a total of 'n' possible outcomes, all of which are mutually exclusive, equally likely and collectively exhaustive, such that m of the outcomes are favorable to the event A, then the probability of the event is defined by,

$$P(A) = \frac{m}{n}$$

The definition was given by James Bernoulli. Let $\bar{A}$ be the complementary event to A. The favorable

Statistical approach: If an experiment is repeated n times under the same conditions and event E occurs m times out of n times, then $P(E) = \lim_{n\to\infty} \frac{m}{n}$

That is, when n is very large, P(E) is very close to the relative frequency of event E. For example; In a dice throwing experiment- S= {1, 2, 3, 4, 5, 6}. And our favorable event is E= {2} Let, 2 occurred a total of 998 times out of total 6000 trials. Therefore

$$P(E) = \lim_{n\to\infty} \frac{998}{6000}$$

Axioms of Probability: Valid probabilities will follow 3 axioms-

Axiom 1: (Axiom of positivizes): $0 \le P(E) \le 1$

Axiom 2: (Axiom of certainty): P(S) = 1

Axiom 3: (Axiom of additivity): For a sequence of disjoint events $E_1, E_2, \ldots, E_n$

$$P\left(\cup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

Example 3: A person holds ticket in a lottery that offers 10 prizes and sells 120 tickets. What is the probability that the person will not win a prize?

Solution: Let A be the event of winning a prize. Here, P(A) = $\frac{n(A)}{n(S)} = \frac{10}{120} = \frac{1}{12}$

Thus, $P(A^c) = 1 - \frac{1}{12} = \frac{11}{12}$

So, the probability that the person will not win a prize is $\frac{11}{12}$

Example 4: A dice is thrown in an experiment. What is the probability that an even no will occur?

Solution: The sample space for the experiment is, S = { 1, 2, 3, 4, 5, 6 }

Let the event is E = {2, 4, 6 }

Here, n(E) = 3 and n(S) = 6

Therefore, the probability of occurring the event, E

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2} = 0.5$$

Addition Laws

- **For disjoint events A and B**- The probability that, either event A or event B will occur is,

  $$P(A \cup B) = P(A) + P(B)$$

- **For disjoint events A, B, C, … , and Z**- The probability that, either event A or event B or event C or … or event Z will occur is, $P(A \cup B \cup C \cup \cdots \cup Z) = P(A) + P(B) + P(C) + \ldots + P(Z)$
- **For joint events A and B**- The probability that, either event A or event B or both will occur is,

  $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **For joint events A, B, and C**- The probability that, either event A or event B or event C or any two of them or all will occur is,

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

Example 5: In a company, 60% of the employees have motorcycle, 40% has private car and 20% has both. If an employee is selected randomly from that company, then

a) What is the probability that the employee has either motorcycle or private car?

b) What is the probability that the employee has neither motorcycle nor private car?

Solution: Let, M= the randomly selected employ has motorcycle

C= the randomly selected employee has car

$$\text{Here, } P\,(M) = \frac{60}{100} = 0.6,$$

$$P(C) = \frac{40}{100} = 0.4$$

$$P(M \cap C) = \frac{20}{100} = 0.2$$

a. Probability that the person has either motorcycle or private car is,

$$\begin{aligned} P(M \cup C) &= P(M) + P(C) - P(M \cap C) \\ &= 0.6 + 0.4 - 0.2 \\ &= 0.8 \end{aligned}$$

b. Probability that the person has neither motorcycle nor private car is

$$\begin{aligned} P(M \cup C)^c &= 1 - P(M \cup C) \\ &= 1 - 0.8 \\ &= 0.2 \end{aligned}$$

Conditional probability: Conditional probability is defined as the likelihood of an event or outcome occurring, based on the occurrence of a previous event or outcome. The probability that event A occurs, given that event B has occurred, is called a conditional probability.

The conditional probability of A, given B, is denoted by the symbol P(A|B).

$$P(\mathrm{A|B}) = \frac{p(\mathrm{A \cap B})}{\mathrm{P(B)}};\ \ for\ P(\mathrm{B}) > 0$$

So, we can write, $P(\mathrm{A \cap B}) = P(\mathrm{A|B})\,P(\mathrm{B})$ (Product Rule)

Or, $P(\mathrm{A \cap B}) = \mathrm{P(B|A)\,P(A)}$ ((Product Rule)

Example 6: In a class of 120 students, 60 are studying English, 50 are studying French and 20 are studying both English and French. If a student is selected at random from this class, what is the probability that he or she is studying English given that he is studying French.

Solution:

$$\text{Here, P(E)} = \frac{60}{120} = 0.5$$

$$\text{P(F)} = \frac{50}{120} = 0.42$$

$$P(E \cap F) = \frac{20}{120} = 0.17$$

The probability that s/he is studying English given that s/he is studying French is,

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{20}{50} = 0.4$$

Conditional Probability: Chain Rule

- The chain rule permits the calculation of any member of the joint distribution of a set of random variables using only conditional probabilities.
- The chain rule is useful in the study of Bayesian networks which describe a probability distribution in terms of conditional probabilities.

The conditional probability of A, given B is,

$$P(A|B) = \frac{P(A \cap B)}{P(B)};\ \ for\ P(B) > 0$$

So we can write,

$$P(A \cap B) = P(A|B)\ P(B)$$

Similarly, for events A, B & C,

$$P(A \cap B \cap C) = P(A)\ P(B|A)\ P(C|A \cap B)$$

Multiplication Laws

- **For two dependent events A and B**

The probability that, both event A and event B will occur simultaneously is,

$$P(A \cap B) = P(A|B)\ P(B)$$

Here, occurrence of event A depends on occurrence of event B.

- **For two independent events A and B**

The probability that, both event A and event B will occur simultaneously is,

$$P(A \cap B) = P(A)\ P(B)$$

Example 7: In rainy season, it rains 70% of the days in Bangladesh. When it rains, 80% times it makes thunderstorms. What is the probability that, in a particular day of rainy season, it will rain and it will thunderstorm?

Solution: Let, R= it will rain on that particular day

T= it will thunderstorm on that particular day

Here, given that, $P(R) = \frac{70}{100} = 0.7$ and

$$P(T|R) = \frac{80}{100} = 0.8$$

Therefore, the probability that, on that particular day of rainy season, it will rain and it will thunderstorm is-

$$P(R \cap T) = P(T|R)\ P(R)$$
$$= 0.8 * 0.7$$
$$= 0.56$$

Example 8: A jar contains 3 red, 5 green, 2 blue and 6 yellow marbles. A marble is chosen at random from the jar. After replacing it, a second marble is chosen. What is the probability of choosing a green and then a yellow marble?

Solution:

Let,

G = Green marble will be chosen

Y = Yellow marble will be chosen

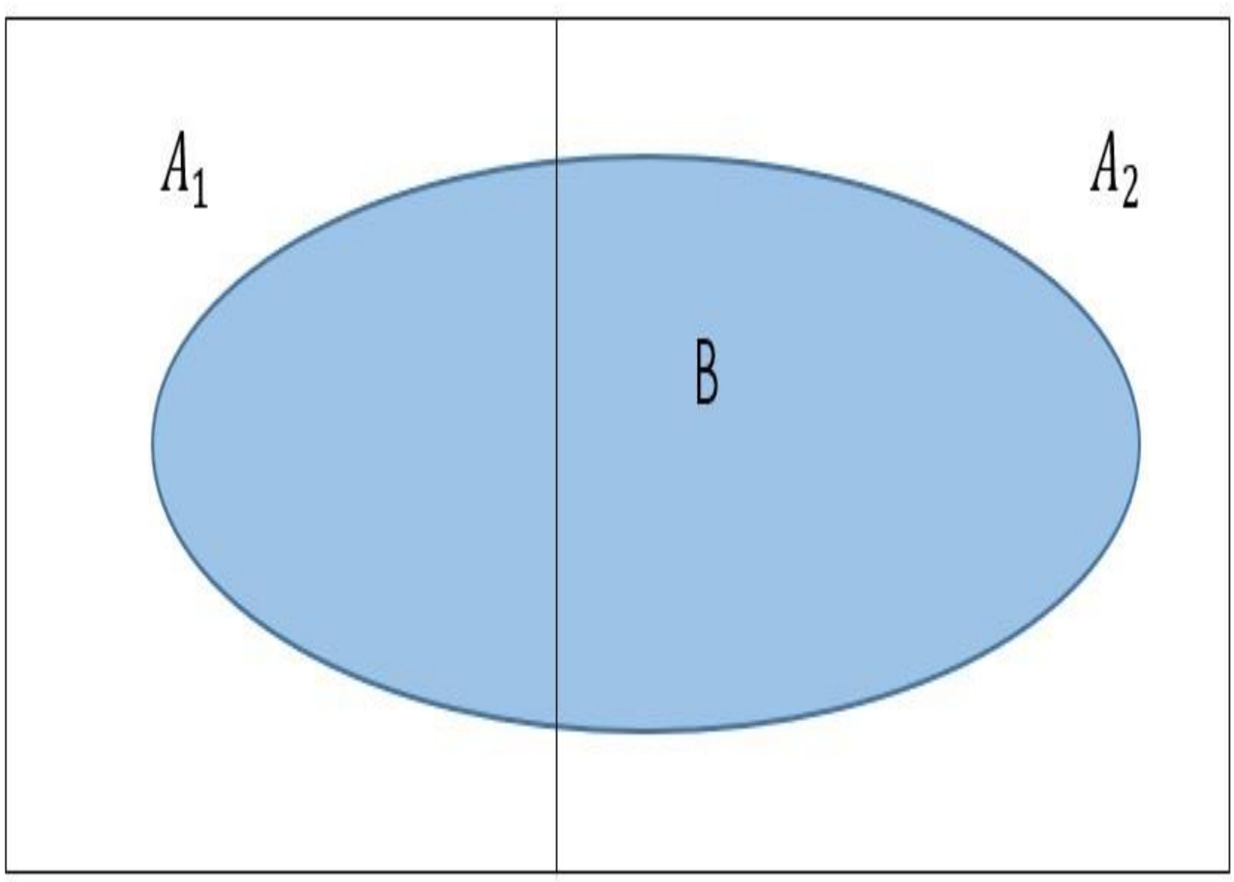
Here,

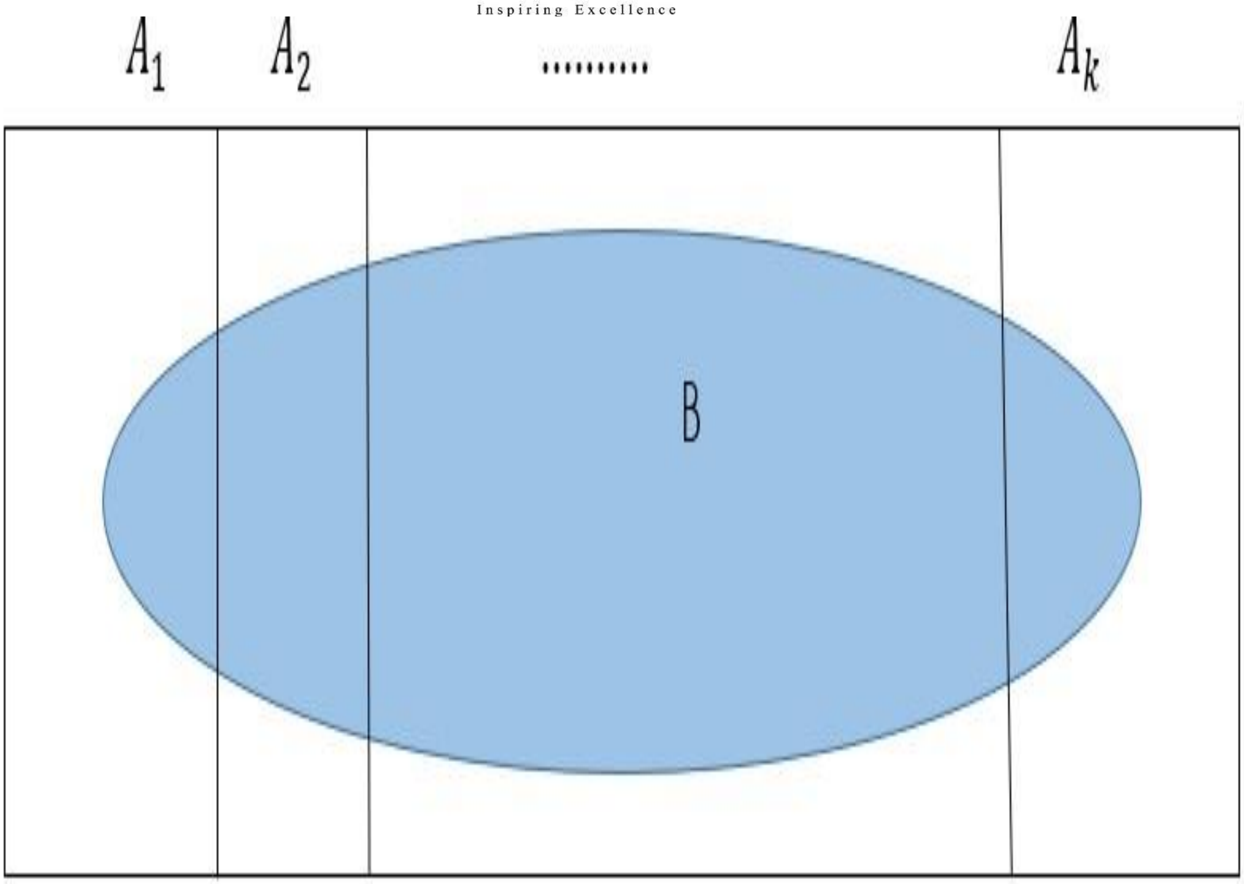
$P(G) = \frac{5}{16}$ & $P(Y) = \frac{6}{16}$

Then, the probability of choosing a green and then a yellow marble is,

$$P(G \cap Y) = P(G) \times P(Y)$$
$$= \frac{5}{16} \times \frac{6}{16}$$
$$= \frac{30}{256}$$
$$= \frac{15}{128}$$

Law of Total Probability Let, events A_1 and A_2 form partition of S. Let B be an event with P(B)>0. Then, $P(B) = P(A_1 \cap B) + P(A_2 \cap B) = P(A_1)\, P(B|A_1) + P(A_2)\, P(B|A_2)$





Let, events A_1, A_2, …, A_k form partition of S. Let B be an event with P(B)>0. Then,

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots\ldots + P(A_k \cap B)$$
$$= P(A_1)\, P(B|A_1) + P(A_2)\, P(B|A_2) + \ldots.. + P(A_k)\, P(B|A_k)$$
$$= \sum_{i=1}^{k} P(A_i)\, P(B|A_i)$$

Example 9: A person has undertaken a mining job. The probabilities of completion of job on time with and without rain are 0.42 and 0.90 respectively. If the probability that it will rain is 0.45, then determine the probability that the mining job will be completed on time.

Solution: Let A be the event that the mining job will be completed on time and B be the event that it rains. We have,

$$P(B) = 0.45,$$
$$P\text{ (no rain)} = P(B') = 1 - P(B) = 1 - 0.45 = 0.55$$

By multiplication law of probability,

$$P(A|B) = 0.42$$
$$P(A|B') = 0.90$$

Since, events B and B′ form partitions of the sample space S, by total probability theorem, we have,

$$P(A) = P(B)\, P(A|B) + P(B')\, P(A|B')$$
$$=0.45 \times 0.42 + 0.55 \times 0.90$$
$$= 0.189 + 0.495 = 0.684$$

So, the probability that the job will be completed on time is 0.684.

Probability using contingency table:

Contingency table:

Contingency table is a power tool in data analysis for comparing categorical variables. Although it is designed for analyzing categorical variables, this approach can also be applied to other discrete variables and even continuous variables.

A general 2×2 contingency table will be like the follows:

X \ Y	Y_1	Y_2	Total
X_1	a	b	a + b
X_2	c	d	c + d
Total	a + c	b + d	a + b + c + d

Here the two variables are X and Y and each of them have two possible categories.

Example:10. Suppose a study of speeding violations and drivers who use cell phones produced the following fictional data:

	Speeding violation in the last year	No speeding violation in the last year	Total
Cell phone user	25	280	305
Not a cell phone user	45	405	450
Total	70	685	755

a. Find P (Person is a car phone user).

Ans. P (person is a car phone user) $= \dfrac{\text{number of car phone users}}{\text{Total number of users in the study}} = \dfrac{305}{755}$

b. Find P (person had no violation in the last year)

Ans. P (person had no violation in the last year) $= \dfrac{\text{number of car phone users that had no violation}}{\text{Total number of users in the study}} = \dfrac{685}{755}$

c. Find P (Person is a car phone user | person had a violation in the last year)

Ans. P (Person is a car phone user | person had a violation in the last year)

$$= \frac{\text{number of car phone users that had violation in the last year}}{\text{Total number of users in the study that had violation in the last year}} = \frac{25}{70}$$