Transform of Piecewise Continuous Function

Evaluate $\mathcal{L}\{F(t)\}$ where $F(t) = \begin{cases} 0, & 0 \leq t < 3 \\ 2, & t \geq 3 \end{cases}$

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^3 e^{-st}(0)\, dt + \int_3^\infty e^{-st}(2)\, dt$$

$$= 0 + \lim_{l \to \infty} \int_3^l 2e^{-st}\, dt$$

$$= 2 \lim_{l \to \infty} \left[\frac{e^{-st}}{-s}\right]_3^l$$

$$= -\frac{2}{s} \lim_{l \to \infty} \left[e^{-sl} - e^{-s3}\right]$$

$$= -\frac{2}{s} \lim_{l \to \infty} \left[\frac{1}{e^{sl}} - \frac{1}{e^{3s}}\right]$$

$$= -\frac{2}{s}\left[\frac{1}{e^{\infty}} - \frac{1}{e^{3s}}\right]$$

$$= \frac{2}{se^{3s}}, \quad s > 0$$

The following piecewise-defined function

$$f(t) = \begin{cases} g(t) ; & 0 \leq t < a \\ h(t) ; & t \geqslant a \end{cases}$$

Can be written as a Unit Step function:

$$f(t) = g(t)\left[u(t-0) - u(t-a)\right] + h(t)\left[u(t-a)\right]$$

$$= g(t)\,u(t) - g(t)\,u(t-a) + h(t)\,u(t-a)$$

The following piecewise-defined function

$$f(t) = \begin{cases} 0 , & 0 \leq t < a \\ g(t) , & a \leq t < b \\ 0 , & t \geqslant b \end{cases}$$

can be written as a Unit Step function:

$$f(t) = 0\left[u(t-0) - u(t-a)\right] + g(t)\left[u(t-a) - u(t-b)\right] + 0\left[u(t-b)\right]$$

$$\therefore f(t) = g(t)\left[u(t-a) - u(t-b)\right]$$

Express $f(t) = \begin{cases} 20t, & 0 \leq t < 5 \\ 0, & t \geq 5 \end{cases}$ in terms of unit step function.

$$f(t) = 20t\,[u(t-0) - u(t-5)] + 0\,[u(t-5)]$$
$$= 20t\,(u(t) - u(t-5))$$

Introducing Laplate into unit step function:

Refer page 6 Part B Lec Note:

Known as 2nd Translation Theorem

$$\mathcal{L}(u(t-a)f(t)) = e^{-sa}\,\mathcal{L}\{F(t+a)\}$$
$$= e^{-sa} f(s+a)$$

OR

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}\,\mathcal{L}\{f(t)\}$$
$$= e^{-as} F(s)$$

Find the laplace transformation of

$f(t)=1$ a $a=3$

$$f(t) = 2 - 3u(t-2) + u(t-3)$$

$f(t)=1$ $f(t)=1$

$f(t+a) = f(t+2) = 1$ $\cancel{f(t+a)}\ f(t+3) = 1$

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{1\} - 3\mathcal{L}\{u(t-2)\cdot 1\} + \mathcal{L}\{u(t-3)\cdot 1\}$$
$$= 2\mathcal{L}\{1\} - 3e^{-2s}\mathcal{L}\{1\} + e^{-3s}\mathcal{L}\{1\}$$
$$= 2\frac{1}{s} - 3e^{-2s}\frac{1}{s} + e^{-3s}\frac{1}{s}$$
$$= \frac{2}{s} - \frac{3e^{-2s}}{s} + \frac{e^{-3s}}{s}$$

$$\mathcal{L}\{u(t-a)f(t)\} = e^{-sa}\,\mathcal{L}\{F(t+a)\}$$

$$\therefore \mathcal{L}\{u(t-a)f(t)\} = e^{-sa} f(s+a)$$

$$\therefore u(t-a)f(t) = \mathcal{L}^{-1}\{e^{-sa} f(s+a)\}$$

$$\Rightarrow u(t-a)f(t-a) = \mathcal{L}^{-1}\{e^{-sa} f(s)\}$$

<u>Example:</u> $\mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\}$ $\therefore a = 2$

$$f(s) = \frac{1}{s-4}$$

$$= u(t-2)f(t-2)$$

$$= u(t-2)e^{4(t-2)}$$

$$\mathcal{L}\{F(t)\} = f(s)$$

$$F(t) = \mathcal{L}^{-1}\{f(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$= e^{4t}$$

$$F(t-2) = e^{4(t-2)}$$

Example: $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+9}e^{-\frac{\pi s}{2}}\right\}$

$$u(t-a)f(t-a) = \mathcal{L}^{-1}\{e^{-sa}f(s)\}$$

$a = \frac{\pi}{2}$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\frac{\pi}{2}s}\right\}$$

$$= u\left(t-\frac{\pi}{2}\right)f\left(t-\frac{\pi}{2}\right)$$

$$= u\left(t-\frac{\pi}{2}\right)\cos 3\left(t-\frac{\pi}{2}\right)$$

$$f(s) = \frac{s}{s^2+9}$$

$$\mathcal{L}\{F(t)\} = f(s)$$

$$F(t) = \mathcal{L}^{-1}\{f(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$\therefore F(t) = \cos 3t$$

$$F\left(t-\frac{\pi}{2}\right) = \cos 3\left(t-\frac{\pi}{2}\right)$$

Example Solve $Y' + Y = f(t)$, $Y(0) = 5$

where $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$

$$\mathcal{L}\{Y'\} + \mathcal{L}\{Y\} = \mathcal{L}\left\{\begin{matrix} 0[u(t-0) - u(t-\pi)] \\ + 3\cos t\,[u(t-\pi)] \end{matrix}\right\}$$

$$\Rightarrow \left[S^1\mathcal{L}\{Y\} - S^0 Y(0)\right] + \mathcal{L}\{Y\} = 3\mathcal{L}\{\cos t\,(u(t-\pi))\}$$

$$sy - 5 + y = 3\mathcal{L}\{\cos t\, u(t-\pi)\}$$

$$= -3\frac{s}{s^2+1} \cdot e^{-\pi s}$$

$$\mathcal{L}\{u(t-a)F(t)\}$$
$$= e^{-sa}\mathcal{L}\{F(t+a)\}$$
$$= e^{-s\pi}\mathcal{L}\{F(t+\pi)\}$$

$$y(s+1) = 5 - \frac{3s}{s^2+1} e^{-\pi s}$$

$$F(t) = \cos t$$
$$F(t+\pi) = \cos(t+\pi)$$
$$F(t+\pi) = -\cos t$$

$$y = \frac{1}{s+1}\left\{5 - \frac{3s}{s^2+1}e^{-\pi s}\right\}$$

$$\therefore \mathcal{L}\{F(t+\pi)\} = \mathcal{L}\{-\cos t\} = \frac{-s}{s^2+1}$$

$$y = \frac{5}{s+1} - \frac{3se^{-\pi s}}{(s+1)(s^2+1)}$$

$$\mathcal{L}\{Y\} = \frac{5}{s+1} - \frac{3se^{-\pi s}}{(s+1)(s^2+1)}$$

$$Y = \mathcal{L}^{-1}\left\{\frac{5}{s+1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{se^{-\pi s}}{(s+1)(s^2+1)}\right\}$$

$$\frac{se^{-\pi s}}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$se^{-\pi s} = A(s^2+1) + (Bs+C)(s+1)$$

$$se^{-\pi s} = As^2 + A + Bs^2 + Cs + Bs + C$$

"s^2"

$0 = A + B$

$B = -A$

"s"

$e^{-\pi s} = C + B$

$e^{-\pi s} = C - A$ — (ii)

"constant"

$0 = A + C$ — (iii)

$\therefore C = -A$

(ii) + (iii) $2C = e^{-\pi s}$

$C = \frac{1}{2}e^{-\pi s}$ → $A = -\frac{1}{2}e^{-\pi s}$

Also $B = C$ $\because B = -A$, $C = -A$

$\therefore B = \frac{1}{2}e^{-\pi s}$

$$Y = 5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{se^{-\pi s}}{(s+1)(s^2+1)}\right\}$$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{A}{s+1} + \frac{Bs+C}{s^2+1}\right\}$$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 3\left[\mathcal{L}^{-1}\left\{\frac{A}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{Bs}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{C}{s^2+1}\right\}\right]$$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 3\left[\mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}e^{-\pi s}}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\frac{1}{2}e^{-\pi s}\right\}\right.$$

$$\left. + \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}e^{-\pi s}}{s^2+1}\right\}\right]$$

$$= 5e^{-t} + \frac{3}{2}\left[\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s+1}}_{e^{-t}} e^{-\pi s}\right\} - \mathcal{L}^{-1}\left\{\underbrace{\frac{s}{s^2+1}}_{\cos t} e^{-\pi s}\right\}\right.$$

$\cos t$; $f(t-\pi) = \cos(t-\pi)$

$\boxed{a = \pi}$ $\quad f(t-\pi) = e^{-(t-\pi)}$

$$\left. - \mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s^2+1}}_{\sin t} e^{-\pi s}\right\}\right]$$

$\sin t$; $f(t-\pi) = \sin(t-\pi)$

find $f(t-a) = f(t-\pi)$

$$\boxed{u(t-a)f(t-a) = \mathcal{L}^{-1}\{e^{-sa}f(s)\}}$$

$$= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)}u(t-\pi) - \cos(t-\pi)u(t-\pi) - \sin(t-\pi)u(t-\pi)\right]$$

$$= 5e^{-t} + \frac{3}{2}\left(e^{-(t-\pi)} - \cos(t-\pi) - \sin(t-\pi)\right)u(t-\pi).$$

$$= 5e^{-t} + \frac{3}{2}\left(e^{-(t-\pi)} + \cos t + \sin t\right)u(t-\pi).$$

$\cos x \cos(-\pi) - \sin x \sin(-\pi)$

$\cos(x-\pi) = -\cos x$

$\sin(x-\pi) = -\sin x$

$\sin x \cos(-\pi) + \cos x \sin(-\pi)$