On Normality and Equidistribution for Separator Enumerators

Subin Pulari[1]

[1] National Research University Higher School of Economics Moscow

February 1, 2026

Abstract

A separator is a countable dense subset of $[0, 1)$, and a separator enumerator is a naming scheme that assigns a real number in $[0, 1)$ to each finite word so that the set of all named values is a separator. Mayordomo introduced separator enumerators to define f-normality and a relativized finite-state dimension $\dim_{\mathrm{FS}}^{f}(x)$, where finite-state dimension measures the asymptotic lower rate of finite-state information needed to approximate x through its f-names. This framework extends classical base-k normality, and Mayordomo showed that it supports a point-to-set principle for finite-state dimension. This representation-based viewpoint has since been developed further in follow-up work, including by Calvert et al., yielding strengthened randomness notions such as supernormal and highly normal numbers.

Mayordomo posed the following open question: can f-normality be characterized via equidistribution properties of the sequence $\left(|\Sigma|^n a_n^f(x)\right)_{n=0}^{\infty}$, where $a_n^f(x)$ is the sequence of best approximations to x from below induced by f? We give a strong negative answer: we construct computable separator enumerators f_0, f_1 and a point x such that $a_n^{f_0}(x) = a_n^{f_1}(x)$ for all n, yet $\dim_{\mathrm{FS}}^{f_0}(x) = 0$ while $\dim_{\mathrm{FS}}^{f_1}(x) = 1$. Consequently, no criterion depending only on the sequence $\left(|\Sigma|^n a_n^f(x)\right)_{n=0}^{\infty}$ - in particular, no equidistribution property of this sequence - can characterize f-normality uniformly over all separator enumerators. On the other hand, for a natural finite-state coherent class of separator enumerators we recover a complete equidistribution characterization of f-normality.

1 Introduction

Finite-state dimension is a quantitative notion of the rate of randomness in an individual infinite sequence, as measured by finite automata. It was introduced by Dai, Lathrop, Lutz, and Mayordomo [3] as a finite-state analogue of effective Hausdorff dimension [7, 6]. It admits several equivalent characterizations, including formulations via finite-state gambling, finite-state compression, and block entropy rates [3, 4, 1]. It also has an important connection with the theory of normal numbers: a real number is normal to base k if and only if its base-k digit sequence has finite-state dimension equal to 1 [1].

An important connection between effective dimension and classical fractal dimension is provided by point-to-set principles. Lutz and Lutz [8] proved that the Hausdorff dimension in Euclidean spaces of a set can be obtained by minimizing, over oracles, the supremum of the relativized effective dimensions of its points. This reduces many lower-bound questions in geometric measure theory to analyzing the information density of carefully chosen points, and it has led to several new results and new proofs across classical fractal geometry; see [9]. In a recent work, Mayordomo [10] established an analogous point-to-set principle for finite-state dimension. A key feature of this development is that it is representation-based: instead of fixing a base-k expansion, it uses *separator enumerators*, i.e. naming schemes that assign reals in $[0, 1)$ to finite words so that the range forms a countable dense subset. With such an enumerator $f : \Sigma^* \to [0, 1)$, one can measure how efficiently a finite-state transducer can output an f-name approximating a real x to a given precision, leading to a relativized finite-state dimension $\dim^f_{\mathrm{FS}}(x)$ and the induced notion of *f-normality*, defined by the condition $\dim^f_{\mathrm{FS}}(x) = 1$ [10]. Calvert et al. [2] develop this framework further, introducing strengthened notions such as supernormal and highly normal numbers under broad classes of representations.

In this paper we study the relationships between f-normality and equidistribution. In the classical base-k setting, normality has a sharp equidistribution characterization: x is normal to base k if and only if the sequence $(k^n x)_{n\geq 1}$ is uniformly distributed modulo 1 [5]. Motivated by the classical equivalence between normality and equidistribution, Mayordomo [10] asked whether an analogous equidistribution criterion holds in the setting of f-normality. Let $(a^f_n(x))_{n\geq 0}$ denote the best-approximation-from-below sequence associated with f and x, i.e. $a^f_n(x) = \max\{f(w) : |w| \leq n,\ f(w) \leq x\}$. Mayordomo posed the following open question:

Can f-normality be characterized via equidistribution properties of the sequence $(|\Sigma|^n a^f_n(x))_{n\geq 0}$?

We show that the answer is negative in a strong way. We construct two total computable rational-valued separator enumerators f_0, f_1 and a point $x \in [0, 1)$ such that $a^{f_0}_n(x) = a^{f_1}_n(x)$ for all n, yet $\dim^{f_0}_{\mathrm{FS}}(x) = 0$ while $\dim^{f_1}_{\mathrm{FS}}(x) = 1$. Consequently, no criterion depending only on the single numeric sequence $\big(|\Sigma|^n a^f_n(x)\big)_{n\geq 0}$—in particular, no equidistribution property of that sequence—can characterize f-normality uniformly over all separator enumerators.

At the same time, an equidistribution characterization does hold under a natural structural restriction on the representation. We identify a class of *finite-state coherent* enumerators obtained from the standard base-k grid by an invertible synchronous Mealy-machine relabeling, and we prove that in this regime f-normality is equivalent to a k-adic equidistribution property of the integer sequence $\big(k^n a^f_n(x)\big)_{n\geq 1}$ (uniform distribution of residues modulo k^m for every fixed m).

Section 2 defines separator enumerators, finite-state transducers, $\dim^f_{\mathrm{FS}}$, and f-normality. Section 3 proves the negative result via a pair of computable enumerators with identical best-from-below chains but sharply different relativized finite-state dimension. Section 4 establishes the k-adic equidistribution characterization for finite-state coherent enumerators.

2 Preliminaries

Let Σ be a finite alphabet with $k = |\Sigma| \geq 2$ and write Σ^* for the set of finite strings over Σ. For $w \in \Sigma^*$, $|w|$ denotes its length, and for an infinite sequence $X \in \Sigma^\infty$ we write $X \upharpoonright n$ for its length-n prefix (using the operator $\upharpoonright$ defined in the preamble). We use $\lfloor\cdot\rfloor$ for the floor function and interpret congruences $b_n \equiv r \pmod{k^m}$ in the usual sense. Throughout this section we fix an identification $\Sigma = \{0, 1, \ldots, k-1\}$. We view finite words over Σ as base-k numerals and use the associated k-adic grid in $[0, 1)$. For a word $u = u_1 u_2 \cdots u_n \in \Sigma^n$, define its base-$k$ value

$$\mathrm{val}(u) = \sum_{i=1}^{n} u_i\, k^{n-i} \in \{0, 1, \ldots, k^n - 1\}, \qquad \text{and} \qquad \mathrm{grid}(u) = \frac{\mathrm{val}(u)}{k^n} \in [0, 1).$$

Thus $\mathrm{grid}(\Sigma^n) = \{j/k^n : 0 \leq j < k^n\}$. Let $\mathrm{seq}_k(x) \in \Sigma^\infty$ denote the (canonical) base-k expansion of $x \in [0, 1)$ chosen so as *not* to end in $(k-1)^\infty$.

Finite-state transducers model one-pass, constant-memory transformations on words and are used to measure finite-state description length.

Definition 1 (Finite-state transducer (FST))**.** *A* Σ-finite-state transducer *(briefly,* Σ-FST*) is a tuple* $T = (Q, \delta, \nu, q_0)$ *where* Q *is a finite nonempty set of states,* $\delta : Q \times \Sigma \to Q$ *is a transition function,* $\nu : Q \times \Sigma \to \Sigma^*$ *is an output function, and* $q_0 \in Q$ *is the start state.*

The transition function extends to words by $\delta(q, \lambda) = q$ *and* $\delta(q, wa) = \delta(\delta(q, w), a)$*. For* $q \in Q$ *and* $w \in \Sigma^*$*, define the output* $\nu(q, w) \in \Sigma^*$ *by* $\nu(q, \lambda) = \lambda$ *and* $\nu(q, wa) = \nu(q, w)\, \nu(\delta(q, w), a)$ *for* $a \in \Sigma$*. The overall output of* T *on input* w *is* $T(w) = \nu(q_0, w)$*.*

We measure how concisely a transducer can generate a given target string.

Definition 2 (T-information content [10])**.** *Let* T *be a* Σ*-FST and* $w \in \Sigma^*$*. Define*

$$K^T(w) = \min\{|\pi| : \ \pi \in \Sigma^* \text{ and } T(\pi) = w\},$$

with the convention $K^T(w) = \infty$ *if* w *is not in the range of* T*.*

A separator enumerator is a naming scheme for a countable dense subset of $[0, 1)$, assigning a real in $[0, 1)$ to each finite word.

Definition 3 (Separator, separator enumerator [10])**.** *A set* $S \subseteq [0, 1)$ *is a* separator *if it is countable and dense in* $[0, 1)$*. A function* $f : \Sigma^* \to [0, 1)$ *is a* separator enumerator *(SE) if* $\mathrm{Im}(f)$ *is a separator.*

We call an SE $f : \Sigma^* \to [0, 1)$ *total computable* if there is an algorithm that, on input (w, t) with $w \in \Sigma^*$ and $t \in \mathbb{N}$, outputs a rational q such that $|q - f(w)| \leq 2^{-t}$, and halts on every input. Given an enumerator f, we quantify how many input symbols a transducer needs in order to produce some f-name that δ-approximates a target point.

Definition 4 (Relativized approximation complexity [10])**.** *Let* f *be an SE,* T *a* Σ*-FST,* $\delta > 0$*, and* $x \in [0, 1)$*. Define*

$$K^{T,f}_\delta(x) = \min\{K^T(w) : \ w \in \Sigma^* \text{ and } |f(w) - x| < \delta\}.$$

The induced finite-state dimension is the optimal asymptotic approximation rate achievable by finite-state transducers.

Definition 5 (Relativized finite-state dimension and f-normality [10])**.** *Let* f *be an SE and* $x \in [0, 1)$*. Define*

$$\dim^f_{\mathrm{FS}}(x) = \inf_{T \ \Sigma\text{-FST}} \ \liminf_{\delta \to 0^+} \ \frac{K^{T,f}_\delta(x)}{\log_k(1/\delta)}.$$

We say x *is* f-normal *if* $\dim^f_{\mathrm{FS}}(x) = 1$*.*

The standard base-k naming map will serve as the baseline enumerator throughout. Let $f_{\mathrm{std}} : \Sigma^* \to [0, 1)$ be the standard base-k enumerator

$$f_{\mathrm{std}}(u) = \mathrm{grid}(u) = \frac{\mathrm{val}(u)}{k^{|u|}} \qquad (u \neq \lambda),$$

with $f_{\mathrm{std}}(\lambda) = 0$. We now define the induced *best approximation from below* chain associated with a separator enumerator.

Definition 6 (Best approximation from below [10])**.** *Let* f *be an SE and* $x \in [0, 1)$*. For each* $n \in \mathbb{N}$*, define* $a^f_n(x)$ *to be any value* $f(w)$ *with* $|w| \leq n$ *such that* $f(w) \leq x$ *and* $x - f(w)$ *is minimum among all* u *with* $|u| \leq n$ *and* $f(u) \leq x$*. Equivalently,*

$$a^f_n(x) = \max\{f(w) : \ |w| \leq n, \ f(w) \leq x\}.$$

3 Equidistribution cannot characterize normality for separator enumerators

We now address Mayordomo's question on equidistribution criteria for f-normality. Fix a separator enumerator $f : \Sigma^* \to [0, 1)$ and $x \in [0, 1)$, and let $(a_n^f(x))_{n\geq 0}$ be the best approximation from below sequence corresponding to f and x (see Definition 6).

Question 1 ([10]). *Can f-normality be characterized via equidistribution properties of the sequence $(|\Sigma|^n\, a_n^f(x))_{n\geq 0}$?*

We answer this question in the negative by constructing two total computable rational-valued separator enumerators f_0, f_1 and a point $x \in [0, 1)$ such that $a_n^{f_0}(x) = a_n^{f_1}(x)$ for all n, yet $\dim_{\mathrm{FS}}^{f_0}(x) = 0$ while $\dim_{\mathrm{FS}}^{f_1}(x) = 1$.

3.1 Preliminary lemmas

From the finite-state transducer characterization of finite-state dimension due to Doty and Moser (see Theorem 3.11 from [4]), we obtain the following corollary.

Lemma 1. *There exists an infinite sequence $Z \in \Sigma^\infty$ such that for every Σ-FST T,*

$$\liminf_{n\to\infty} \frac{K^T(z_n)}{n} = 1,$$

where $z_n := Z{\upharpoonright}n$.

Proof. Fix any Σ-normal sequence Z. By the transducer characterization of finite-state dimension and the fact that normal sequences have finite-state dimension equal to 1,

$$1 = \dim_{\mathrm{FS}}(Z) = \inf_T \liminf_{n\to\infty} \frac{K^T(z_n)}{n}.$$

Therefore, for every Σ-FST T one must have $\liminf_n K^T(z_n)/n = 1$; otherwise the infimum would be < 1. □

We also need a second sequence whose prefixes are distinct from the first but have the same property. This is achieved by a fixed symbol permutation.

Lemma 2. *Let $\pi : \Sigma \to \Sigma$ be a bijection and extend it letterwise to $\pi : \Sigma^* \to \Sigma^*$. If $Z \in \Sigma^\infty$ satisfies Lemma 1, then so does $\pi(Z)$, i.e. the prefixes $t_n := \pi(Z){\upharpoonright}n = \pi(z_n)$ satisfy $\liminf_n K^T(t_n)/n = 1$ for every Σ-FST T.*

Proof. Fix a Σ-FST T. Let P be the 1-state transducer that maps each input symbol a to the single output symbol $\pi(a)$. Then $P(w) = \pi(w)$ for all w, and similarly there is a 1-state transducer P^{-1} with $P^{-1}(w) = \pi^{-1}(w)$ for all w. For any string u,

$$K^T(\pi(u)) = K^T(P(u)) \geq K^{P^{-1}\circ T}(u),$$

because any input producing $\pi(u)$ under T yields an input producing u under $P^{-1} \circ T$. Thus, for $t_n = \pi(z_n)$,

$$\frac{K^T(t_n)}{n} \geq \frac{K^{P^{-1}\circ T}(z_n)}{n}.$$

Taking $\liminf$ and applying Lemma 1 for the transducer $P^{-1} \circ T$ we obtain $\liminf_n K^T(t_n)/n = 1$. □

Fix a bijection $\pi : \Sigma \to \Sigma$ with no fixed points (such a derangement exists for all $k \geq 2$) and with $\pi(0) \neq 0$. Let Z be as in Lemma 1, let $Y = \pi(Z)$, and write $z_n := Z \upharpoonright n$ and $y_n := Y \upharpoonright n = \pi(z_n)$. Then $z_n \neq y_n$ for all n because π has no fixed points, and $y_n \neq 0^n$ for all n because $\pi(0) \neq 0$.

3.2 Construction of the two separator enumerators

Fix $x := \frac{1}{2} \in (0, 1)$. For each $n \in \mathbb{N}$, define $r_n := k^{-(n+2)}$ and

$$A_n := \big(x - r_n,\ x - r_{n+1}\big) \ \cup\ \big(x + r_{n+1},\ x + r_n\big).$$

Then A_n is a nonempty open set and $(A_n)_{n\geq 0}$ partition the punctured neighborhood

$$\big(x - r_0,\ x + r_0\big) \setminus \{x\} = \bigsqcup_{n\geq 0} A_n$$

, and $\log_k(1/r_n) = n + 2$. Define the target *best-from-below* values

$$c_n := x - \frac{r_n + r_{n+1}}{2} \in (x - r_n, x - r_{n+1}) \subseteq A_n$$

. These values will later be enforced as the canonical length-n approximants to x for both enumerators, i.e., we will arrange $a_n^{f_i}(x) = c_n$ for each n. Observe that $c_n < x$ and $c_n \uparrow x$ as $n \to \infty$.

For each n, fix a computable listing $(d_{n,t})_{t\in\mathbb{N}}$ of *distinct* rationals in $A_n \cap \mathbb{Q} \setminus \{c_n\}$ whose range is dense in A_n, and set $D_n := \{d_{n,t} : t \in \mathbb{N}\}$. Also fix a computable listing $(q_t)_{t\in\mathbb{N}}$ of *distinct* rationals in $\big([0, 1) \setminus (x - r_0, x + r_0)\big) \cap \mathbb{Q}$ whose range is dense there, and set $D_{\text{far}} := \{q_t : t \in \mathbb{N}\}$. We now define two functions $f_0, f_1 : \Sigma^* \to [0, 1)$.

Step 1 (fixing the best-from-below chain for lengths $\leq n$). For each $n \geq 0$, define

$$f_0(0^n) := c_n, \qquad f_1(z_n) := c_n.$$

(Here $0 \in \Sigma$ is a fixed symbol, and 0^n is the all-0 word of length n.)

Step 2 (ensuring density near x using incompressible prefixes). Partition $\mathbb{N}$ into pairwise disjoint infinite sets $(L_n)_{n\geq 0}$ such that $m \in L_n$ implies $m \geq n + 1$. For concreteness, one may take $L_n = \{2^n(2t + 1) : t \in \mathbb{N}\}$; then $m \in L_n$ implies $m \geq 2^n \geq n + 1$ for $n \geq 0$. Fix computable bijections $\varphi_n : L_n \to D_n$ by setting, for $t \in \mathbb{N}$,

$$\varphi_n\big(2^n(2t + 1)\big) := d_{n,t}.$$

Now set, for each $n \geq 0$ and each $m \in L_n$,

$$f_0(y_m) = f_1(y_m) := \varphi_n(m) \in A_n.$$

Note that there is no conflict with Step 1, since $y_m \neq 0^m$ and $y_m \neq z_m$ for all m. Thus, for each n, the set $\{f_i(y_m) : m \in L_n\}$ is dense in A_n (for both $i = 0, 1$).

Step 3 (defining all remaining values far from x while maintaining density). Let W be the set of all remaining strings not yet assigned a value by Steps 1–2:

$$W := \Sigma^* \setminus \Big(\{0^n : n \in \mathbb{N}\}\ \cup\ \{z_n : n \in \mathbb{N}\}\ \cup\ \{y_n : n \in \mathbb{N}\}\Big).$$

Enumerate Σ^* in the length lexicographic order, then all words of length 2 in lexicographic order, and so on. Call this enumeration $(u_j)_{j\in\mathbb{N}}$. Define w_j to be the jth word in this list that lies in W, i.e., the jth u_t such that $u_t \notin \{0^n : n \in \mathbb{N}\}\ \cup\ \{z_n : n \in \mathbb{N}\}\ \cup\ \{y_n : n \in \mathbb{N}\}$. Also enumerate $D_{\text{far}} = \{q_0, q_1, q_2, \dots\}$ according to the fixed computable listing above. Define $f_0(w_j) = f_1(w_j) := q_j$ for all j.

Lemma 3. *The functions f_0, f_1 defined above are total computable rational-valued separator enumerators: each* $\mathrm{Im}(f_i)$ *is countable and dense in* $[0, 1)$.

Proof. Countability is immediate since Σ^* is countable.

We also note that f_0 and f_1 are total computable (with rational outputs). Assume the fixed dense sets D_n and D_{far} come with fixed computable enumerations, and that each $\varphi_n : L_n \to D_n \setminus \{c_n\}$ is a fixed computable bijection. Also fix Z to be a computable Σ-normal sequence, so $n \mapsto z_n = Z \upharpoonright n$ is computable; then $Y = \pi(Z)$ is computable and $n \mapsto y_n = Y \upharpoonright n$ is computable.

On input $w \in \Sigma^*$ of length ℓ, we compute $f_i(w)$ as follows. First check whether $w = 0^\ell$ (all symbols equal 0), whether $w = z_\ell$ (compute z_ℓ and compare), and whether $w = y_\ell$ (compute y_ℓ and compare). If $w = 0^\ell$ output $f_0(w) = c_\ell$; if $w = z_\ell$ output $f_1(w) = c_\ell$; and if $w = y_\ell$ then find the unique n with $\ell \in L_n$ (the L_n are decidable and disjoint) and output $f_0(w) = f_1(w) = \varphi_n(\ell)$. Otherwise $w \in W$, and W is decidable because $w \notin W$ iff one of the three checks above holds. In this case, compute the index j such that $w = w_j$ in Step 3 by enumerating all words of Σ^* by increasing length, and within each fixed length in lexicographic order, skipping exactly those words that are not in W, until w is reached; then output $f_0(w) = f_1(w) = q_j$ (where (q_j) is the fixed computable enumeration of D_{far}). This procedure halts for every input w, so f_0, f_1 are total computable.

For density, let $I \subseteq [0, 1)$ be a nonempty open interval. If I intersects $[0, 1) \setminus (x - r_0, x + r_0)$, then I contains a rational in D_{far}, hence an image point of f_i. Otherwise, $I \subseteq (x - r_0, x + r_0)$, so I intersects A_n for some $n \geq 0$ (because the annuli partition $(x - r_0, x + r_0) \setminus \{x\}$ and I is open, hence cannot be $\{x\}$). Since D_n is dense in A_n and $\{f_i(y_m) : m \in L_n\} = \varphi_n(L_n)$ is dense in A_n, the interval I contains some $f_i(y_m)$. Thus $\mathrm{Im}(f_i)$ meets every nonempty open interval, so it is dense. $\square$

Since f_0 and f_1 are rational-valued and the above procedure computes the defining case and the corresponding index effectively, it in fact yields exact computation: there is a Turing machine that, on input $w \in \Sigma^*$, halts and outputs the rational value $f_i(w)$ itself (rather than merely producing 2^{-t}-approximations). The next lemma shows that f_0 and f_1 induce exactly the same sequence of length-bounded approximations of x from below (and hence the same associated numeric scaling sequence).

Lemma 4. *For every $n \in \mathbb{N}$, $a_n^{f_0}(x) = a_n^{f_1}(x) = c_n$. Consequently, the numeric sequences $(k^n a_n^{f_0}(x))$ and $(k^n a_n^{f_1}(x))$ are identical.*

Proof. Fix n. We first show $a_n^{f_0}(x) = c_n$. By definition, $f_0(0^n) = c_n \leq x$, so $a_n^{f_0}(x) \geq c_n$. Now consider any string w with $|w| \leq n$ and $f_0(w) \leq x$. If $w = 0^m$ for some $m \leq n$, then $f_0(w) = c_m \leq c_n$ because (c_m) is increasing. If $w = y_m$ for some m, then $f_0(w) \in A_j$ for some j with $m \in L_j$. In particular, $f_0(w) \leq x - r_{j+1} < x$. Moreover, since $m \in L_j$ implies $m \geq j + 1$, we have $j \leq m - 1 \leq n - 1$ whenever $m \leq n$. Thus $f_0(w) \leq x - r_{j+1} \leq x - r_n < c_n$ (because $c_n > x - r_n$ by construction). Finally, if $w \in W$ then $f_0(w) \in D_{\mathrm{far}} \subseteq [0, 1) \setminus (x - r_0, x + r_0)$, so either $f_0(w) \leq x - r_0 < x - r_n < c_n$ or $f_0(w) \geq x + r_0 > x$. In either case it cannot exceed c_n while staying $\leq x$. Therefore, among all $|w| \leq n$ with $f_0(w) \leq x$, the maximum is attained at $w = 0^n$ with value c_n. Hence $a_n^{f_0}(x) = c_n$.

The proof for f_1 is identical, replacing the witness 0^n by z_n (since $f_1(z_n) = c_n$) and noting that no other string of length $\leq n$ attains a value in $(c_n, x]$ by the same case analysis. Thus $a_n^{f_1}(x) = c_n$. $\square$

We now complete the construction by showing that, for the fixed point $x = \frac{1}{2}$, the two separator enumerators f_0 and f_1 constructed above induce different relativized finite-state dimensions:

$\dim_{\mathrm{FS}}^{f_0}(x) = 0$ while $\dim_{\mathrm{FS}}^{f_1}(x) = 1$. Together with Lemma 4, this yields a negative answer to Mayordomo's open question [10].

Theorem 1. *There exist total computable rational-valued separator enumerators $f_0, f_1 : \Sigma^* \to [0, 1)$ and a point $x \in [0, 1)$ such that*

$$a_n^{f_0}(x) = a_n^{f_1}(x) \quad \textit{for all } n,$$

yet

$$\dim_{\mathrm{FS}}^{f_0}(x) = 0 \qquad \textit{and} \qquad \dim_{\mathrm{FS}}^{f_1}(x) = 1.$$

In particular, x is not f_0-normal, while x is f_1-normal.

Proof. We use the constructions above with the fixed $x = \frac{1}{2}$. By Lemma 4, the sequences $a_n^{f_0}(x)$ and $a_n^{f_1}(x)$ coincide (both equal c_n), so $(k^n a_n(x))$ is identical for f_0 and f_1. It remains to compute the relativized finite-state dimensions.

Part 1: We show $\dim_{\mathrm{FS}}^{f_0}(x) = 0$. For each integer $L \geq 1$, let T_L be the 1-state transducer that outputs 0^L on every input symbol. Then $T_L(\pi) = 0^{L|\pi|}$ for all π, hence

$$K^{T_L}(0^n) \leq \left\lceil \frac{n}{L} \right\rceil.$$

Fix n and consider $\delta_n := r_n/2$ (so $\log_k(1/\delta_n) = n + 2 + \log_k 2$). Since $f_0(0^n) = c_n$ and $|x - c_n| = |x - (x - (r_n + r_{n+1})/2)| = (r_n + r_{n+1})/2 < r_n = 2\delta_n$, we have $|f_0(0^n) - x| < 2\delta_n$.Therefore

$$K_{2\delta_n}^{T_L, f_0}(x) \leq K^{T_L}(0^n) \leq \left\lceil \frac{n}{L} \right\rceil.$$

Hence

$$\liminf_{n\to\infty} \frac{K_{2\delta_n}^{T_L,f_0}(x)}{\log_k(1/(2\delta_n))} \leq \liminf_{n\to\infty} \frac{\lceil n/L \rceil}{(n+2) + O(1)} = \frac{1}{L}.$$

To justify that this controls the full $\liminf_{\delta\to 0^+}$ (and not only the subsequence $2\delta_n$), note that $K_\delta^{T_L,f_0}(x)$ is monotone non-increasing in δ. Hence for any $\delta \in (2\delta_{n+1}, 2\delta_n]$,

$$K_\delta^{T_L,f_0}(x) \;\leq\; K_{2\delta_{n+1}}^{T_L,f_0}(x).$$

Moreover, for such δ we have $\log_k(1/\delta) \geq \log_k(1/(2\delta_n))$. Therefore,

$$\frac{K_\delta^{T_L,f_0}(x)}{\log_k(1/\delta)} \;\leq\; \frac{K_{2\delta_{n+1}}^{T_L,f_0}(x)}{\log_k(1/(2\delta_n))}.$$

Taking $\liminf$ over all $\delta \to 0^+$ and using that $n \to \infty$ along the corresponding intervals yields

$$\begin{aligned}
\liminf_{\delta\to 0^+} \frac{K_\delta^{T_L,f_0}(x)}{\log_k(1/\delta)} &\leq \liminf_{n\to\infty} \frac{K_{2\delta_{n+1}}^{T_L,f_0}(x)}{\log_k(1/(2\delta_{n+1}))} \cdot \frac{\log_k(1/(2\delta_{n+1}))}{\log_k(1/(2\delta_n))} \\
&\leq \liminf_{n\to\infty} \frac{K_{2\delta_{n+1}}^{T_L,f_0}(x)}{\log_k(1/(2\delta_{n+1}))} \cdot \lim_{n\to\infty} \frac{n+3+O(1)}{n+2+O(1)} \\
&\leq \frac{1}{L}.
\end{aligned}$$

Since the above holds for all T_L, taking the infimum over all transducers T and then letting $L \to \infty$ yields $\dim_{\mathrm{FS}}^{f_0}(x) = 0$.

Part 2: We show $\dim_{\mathrm{FS}}^{f_1}(x) = 1$. Fix an arbitrary Σ-FST T. We prove that $\liminf_{\delta\to 0^+} \frac{K_\delta^{T,f_1}(x)}{\log_k(1/\delta)} \geq 1$, which implies $\dim_{\mathrm{FS}}^{f_1}(x) = 1$ after taking the infimum over T.

Consider the scale $\delta_n := r_n$. Any value $f_1(w)$ within distance $\delta_n = r_n$ of x must lie in $(x - r_n,\ x + r_n) = \{x\} \cup \bigsqcup_{j\geq n} A_j$, so w must be either (i) one of the special words z_j with $j \geq n$ (since $f_1(z_j) = c_j \in A_j$), or (ii) one of the special words y_m with $f_1(y_m) \in A_j$ for some $j \geq n$ (these are the values placed densely in the annuli), because by construction all other words are mapped into D_{far} outside $(x - r_0, x + r_0)$. Therefore,

$$K_{r_n}^{T,f_1}(x) \ \geq\ \min\Big(\min_{j\geq n} K^T(z_j),\ \min_{m:\ f_1(y_m)\in\bigcup_{j\geq n} A_j} K^T(y_m)\Big).$$

We now lower bound each term asymptotically by n. By Lemma 1, we have $\liminf_{j\to\infty} K^T(z_j)/j = 1$. Hence for every $\varepsilon > 0$ there exists J such that for all $j \geq J$, $K^T(z_j) \geq (1-\varepsilon)j$; consequently, for all $n \geq J$, $\min_{j\geq n} K^T(z_j) \geq (1-\varepsilon)n$. Similarly, by Lemma 2 applied to Y, $\liminf_{m\to\infty} K^T(y_m)/m = 1$, so for every $\varepsilon > 0$ there exists M such that for all $m \geq M$, $K^T(y_m) \geq (1-\varepsilon)m$. In our construction, if $f_1(y_m) \in A_j$, then $m \in L_j$, and by design $m \geq j+1$; therefore, whenever $f_1(y_m) \in \bigcup_{j\geq n} A_j$, we have $m \geq n+1$. For all $n \geq M$,

$$\min_{m:\ f_1(y_m)\in\bigcup_{j\geq n} A_j} K^T(y_m) \ \geq\ (1-\varepsilon)(n+1) \ \geq\ (1-\varepsilon)n.$$

Putting the two bounds together, for all sufficiently large n, $K_{r_n}^{T,f_1}(x) \geq (1-\varepsilon)n$. Since $\log_k(1/r_n) = n+2$, we obtain

$$\liminf_{n\to\infty} \frac{K_{r_n}^{T,f_1}(x)}{\log_k(1/r_n)} \ \geq\ \liminf_{n\to\infty} \frac{(1-\varepsilon)n}{n+2} \ =\ 1-\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this yields $\liminf_{n\to\infty} \frac{K_{r_n}^{T,f_1}(x)}{\log_k(1/r_n)} \geq 1$. By monotonicity of $K_\delta^{T,f_1}(x)$ in δ and the fact that any $\delta \in (r_{n+1}, r_n]$ satisfies $\log_k(1/\delta) \in [n+2, n+3)$, it follows that the full $\liminf_{\delta\to 0^+}$ is also at least 1. Thus

$$\liminf_{\delta\to 0^+} \frac{K_\delta^{T,f_1}(x)}{\log_k(1/\delta)} \geq 1$$

for every T, which implies that $\dim_{\mathrm{FS}}^{f_1}(x) = 1$.

$\square$

As an immediate consequence, f-normality cannot be characterized by any property of the single numerical sequence $(k^n a_n^f(x))$.

Corollary 1. *There is no property P of the numeric sequence $(k^n a_n^f(x))_{n\in\mathbb{N}}$ (in particular, no equidistribution property of this sequence) such that for all separator enumerators f and all $x \in [0,1)$,*

$$x \text{ is } f\text{-normal} \iff (k^n a_n^f(x)) \text{ has property } P.$$

Proof. Take f_0, f_1, x from Theorem 1. By Lemma 4, the sequences $(k^n a_n^{f_0}(x))$ and $(k^n a_n^{f_1}(x))$ are identical, so they either both satisfy P or both fail P. But by Theorem 1, x is f_1-normal and not f_0-normal, so no such P can exist. $\square$

4 Finite-State Coherent Enumerators and an Equidistribution Characterization of f-Normality

From the previous section, we know that no equidistribution (or other distributional) criterion depending only on the sequence $(k^n a_n^f(x))_{n\geq 1}$ can characterize f-normality uniformly over all separator enumerators. The goal of this section is to isolate a natural structural regime in which such a characterization *does* hold. We do this by restricting to *finite-state coherent enumerators*, i.e. naming schemes obtained from the standard base-k grid by an invertible synchronous Mealy-machine relabeling.

To state an equidistribution characterization in this setting, we must use a notion that remains meaningful for the standard enumerator. For the standard base-k naming map $f_{\text{std}}(w) = \text{val}(w)/k^{|w|}$, the scaled approximation sequence $b_n(x) := k^n a_n^{f_{\text{std}}}(x)$ is integer-valued (indeed $b_n(x) = \lfloor k^n x \rfloor$), so the usual notion of equidistribution modulo 1 becomes trivial. The natural replacement is to ask for uniform distribution of the residues of $b_n(x)$ at every finite base-k resolution, i.e. modulo k^m for each fixed m. This leads to the following notion of k-adic equidistribution, which we adopt in the rest of the paper.

Definition 7 (k-adic equidistribution)**.** *A sequence $(b_n)_{n\geq 1}$ of integers is* k-adically equidistributed *if for every $m \geq 1$ and every residue $r \in \{0, 1, \ldots, k^m - 1\}$,*

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \leq n \leq N : \ b_n \equiv r \pmod{k^m}\} = \frac{1}{k^m}.$$

For finite-state coherent enumerators, the best-from-below approximants admit an explicit closed form, and the induced f-dimension is invariant under finite-state coherent relabelings. These two facts combine to yield a clean equidistribution characterization of f-normality in terms of k-adic equidistribution of the integer sequence $(k^n a_n^f(x))_{n\geq 1}$.

4.1 Finite-state coherent enumerators

We begin by defining invertible synchronous Mealy machines.

Definition 8 (Invertible synchronous Mealy machine)**.** *An* invertible synchronous Mealy machine *is a tuple $M = (Q, \delta, \lambda, q_0)$ where Q is a finite nonempty set of states, $\delta : Q \times \Sigma \to Q$ is a transition function, and $\lambda : Q \times \Sigma \to \Sigma$ is an output function such that for every state $q \in Q$, the map $a \mapsto \lambda(q, a)$ is a permutation of Σ.*

The induced map $M : \Sigma^* \to \Sigma^*$ is defined by reading left-to-right: set $M(\lambda) = \lambda$, and for $w = w_1 \cdots w_n$ define $u = M(w) = u_1 \cdots u_n$ by the recursion

$$q_i = \delta(q_{i-1}, w_i), \qquad u_i = \lambda(q_{i-1}, w_i) \quad (i = 1, \ldots, n),$$

with q_0 as the initial state. In particular, $|M(w)| = |w|$ for all w.

We state the basic properties of invertible synchronous Mealy machines.

Lemma 5. *Let M be an invertible synchronous Mealy machine.*

1. *For every n, the restriction $M : \Sigma^n \to \Sigma^n$ is a bijection.*

2. *There exists an invertible synchronous Mealy machine M^{-1} such that for all $w \in \Sigma^*$, we have $M^{-1}(M(w)) = w$.*

3. *M extends letter-by-letter (via the same recursion as in the definition of M) to a bijection $M : \Sigma^\infty \to \Sigma^\infty$ with inverse M^{-1}.*

Proof. (1) Fix n. Because at each step the output letter is obtained by applying a permutation depending on the current state, distinct inputs cannot merge: if $w \neq w'$ then at the first position i where they differ, the machine is in the same state (because it has read the same prefix) and applies a permutation to two different letters, hence outputs different letters at position i. Thus M is injective on Σ^n, hence bijective because Σ^n is finite.

(2) One constructs M^{-1} by reversing the per-state permutations: in state q output $\lambda(q, \cdot)^{-1}(a)$ on input a, and update the state consistently (standard Mealy-machine inversion). Because all per-state maps are permutations, this is well-defined.

(3) The same recursion as in definition of M works on infinite inputs; bijectivity follows from (1) on all finite prefixes. □

Now we formalize the finite-state coherence condition, which captures length-preserving, bounded-memory relabelings of the standard base-k grid.

Definition 9 (Finite-state coherent enumerator)**.** *A function $f : \Sigma^* \to [0, 1)$ is* finite-state coherent *if there exists an invertible synchronous Mealy machine M such that for every nonempty $w \in \Sigma^n$, $f(w) = \mathrm{grid}(M(w)) = \frac{\mathrm{val}(M(w))}{k^n}$. (For definiteness, set $f(\lambda) = 0$.)*

Finite-state coherence immediately forces f to enumerate the entire standard k-adic grid at each length.

Lemma 6. *If f is finite-state coherent, then $\mathrm{Im}(f) = \bigcup_{n\geq 1}\{j/k^n : 0 \leq j < k^n\}$, hence f is a separator enumerator.*

Proof. Fix $n \geq 1$. By Lemma 5(1), $M(\Sigma^n) = \Sigma^n$. Therefore

$$f(\Sigma^n) = \mathrm{grid}(M(\Sigma^n)) = \mathrm{grid}(\Sigma^n) = \left\{ \frac{j}{k^n} : 0 \leq j < k^n \right\}.$$

Taking the union over n gives the claimed image, which is countable and dense in $[0, 1)$. □

For finite-state coherent enumerators, the best-from-below approximation sequence coincides with the usual base-k truncations.

Lemma 7. *Let f be finite-state coherent and $x \in [0, 1)$. Then for every $n \geq 1$, $a_n^f(x) = \frac{\lfloor k^n x \rfloor}{k^n}$. In particular, the scaled sequence is integer-valued: $k^n a_n^f(x) = \lfloor k^n x \rfloor \in \{0, 1, \ldots, k^n - 1\}$.*

Proof. By Lemma 6, for each $m \leq n$ the set $f(\Sigma^m)$ equals the full grid $\{j/k^m : 0 \leq j < k^m\}$. Hence the set of all values $f(w)$ with $|w| \leq n$ is exactly $\bigcup_{m=1}^n\{j/k^m : 0 \leq j < k^m\}$. Among the level-$n$ grid points $\{j/k^n\}$, the largest one $\leq x$ is $\lfloor k^n x\rfloor/k^n$. It remains to check that no coarser grid point (denominator k^m with $m < n$) can exceed this value while staying $\leq x$. But for each $m < n$, $\frac{\lfloor k^m x \rfloor}{k^m} \leq \frac{\lfloor k^n x \rfloor}{k^n}$, because multiplying both sides by k^n gives $k^{n-m}\lfloor k^m x\rfloor \leq \lfloor k^n x\rfloor$, which holds since $k^{n-m}\lfloor k^m x\rfloor \leq k^n x$ and the left-hand side is an integer.

□

Next we relate approximation complexity under a finite-state coherent f to approximation complexity under the standard base-k enumerator. Recall that $f_{\mathrm{std}} : \Sigma^* \to [0, 1)$ is the standard base-k enumerator.

Lemma 8. *Let f be finite-state coherent via a Mealy machine M, i.e. $f(w) = \mathrm{grid}(M(w))$ for all w. Then for every Σ-FST T, every $x \in [0,1)$, and every $\delta > 0$, $K_\delta^{T,f}(x) = K_\delta^{M\circ T,\, f_{\mathrm{std}}}(x)$, where $M \circ T$ denotes the output-composition transducer $\pi \mapsto M(T(\pi))$.*

Proof. Since $\mathrm{grid}(u) = f_{\mathrm{std}}(u)$ for every $u \in \Sigma^*$, we obtain $K_\delta^{T,f}(x) = \min\{K^T(w) : |f_{\mathrm{std}}(M(w)) - x| < \delta\}$. Substitute $u = M(w)$. Since $M : \Sigma^{|w|} \to \Sigma^{|w|}$ is bijective for each length, this is equivalent to $K_\delta^{T,f}(x) = \min\{K^T(M^{-1}(u)) : |f_{\mathrm{std}}(u) - x| < \delta\}$. For every $u \in \Sigma^*$, $K^T(M^{-1}(u)) = \min\{|\pi| : T(\pi) = M^{-1}(u)\} = \min\{|\pi| : M(T(\pi)) = u\} = K^{M\circ T}(u)$, and substituting yields the claim. ☐

The next proposition formalizes the key robustness property of finite-state coherence: composing the naming map with an invertible synchronous Mealy relabeling does not change the relativized finite-state approximation complexity, and hence does not change the induced f-dimension.

Proposition 1. *If f is finite-state coherent, then for every $x \in [0,1)$, $\dim_{\mathrm{FS}}^f(x) = \dim_{\mathrm{FS}}^{f_{\mathrm{std}}}(x)$. In particular, x is f-normal if and only if x is f_{std}-normal.*

Proof. Let f be finite-state coherent via M. By Lemma 8, for every T and every δ, $K_\delta^{T,f}(x) = K_\delta^{M\circ T, f_{\mathrm{std}}}(x)$, hence the corresponding $\liminf$ ratios coincide. Taking $\inf_T$ over all FSTs on the left equals taking $\inf_S$ over all FSTs on the right, because $T \mapsto M \circ T$ is a bijection on FSTs (with inverse $S \mapsto M^{-1} \circ S$). Therefore the two infima coincide. ☐

The next theorem states that, for the *standard* base-k naming map f_{std}, the paper's notion of f-normality (i.e. $\dim_{\mathrm{FS}}^f(x) = 1$) coincides exactly with the classical notion of base-k normality of x.

Theorem 2. *For $x \in [0,1)$, one has $\dim_{\mathrm{FS}}^{f_{\mathrm{std}}}(x) = 1$ if and only if x is base-k normal.*

Proof. By Theorem 3.3 of [10], for every $x \in [0,1)$, $\dim_{\mathrm{FS}}^{f_{\mathrm{std}}}(x) = \dim_{\mathrm{FS}}(\mathrm{seq}_k(x))$. By the standard characterization of normality via finite-state dimension (e.g. [1]), one has $\dim_{\mathrm{FS}}(\mathrm{seq}_k(x)) = 1$ if and only if $\mathrm{seq}_k(x)$ is base-k normal. Finally, by definition, $\mathrm{seq}_k(x)$ is base-k normal if and only if x is base-k normal. Combining these equivalences yields the claim. ☐

It is straightforward to verify that k-adic equidistribution of the sequence $(\lfloor k^n x \rfloor)_{n\ge1}$ coincides with base-k normality [5].

Theorem 3. *Let $x \in [0,1)$ and set $b_n = \lfloor k^n x \rfloor$. Then x is base-k normal if and only if $(b_n)_{n\ge1}$ is k-adically equidistributed.*

Finally we combine finite-state coherent invariance with the explicit form of $a_n^f(x)$ to obtain the equidistribution characterization in terms of the scaled approximation sequence.

Theorem 4. *Let f be a finite-state coherent separator enumerator over Σ and let $x \in [0,1)$. Then x is f-normal if and only if the integer sequence $(k^n a_n^f(x))_{n\ge1}$ is k-adically equidistributed.*

Proof. Let $b_n(x) := k^n a_n^f(x)$. Since f is finite-state coherent, Proposition 1 gives x is f-normal if and only if x is f_{std}-normal. By Theorem 2, f_{std}-normality is equivalent to base-k normality of x. On the other hand, Theorem 3 states that base-k normality of x is equivalent to k-adic equidistribution of the integer sequence $\big(\lfloor k^n x \rfloor\big)_{n\ge1}$. Finally, Lemma 7 identifies the scaled approximation sequence for finite-state coherent f with this canonical sequence, namely $b_n(x) = \lfloor k^n x \rfloor$ for all $n \ge 1$. Substituting this identity into the previous equivalence yields x is f-normal if and only if $(b_n(x))_{n\ge1}$ is k-adically equidistributed, which is exactly the claim. ☐

5 Discussion and open questions

We show that, in general, no distributional property of the scaled best-from-below approximation sequence $(k^n a_n^f(x))_{n\geq 1}$ can characterize f-normality uniformly over all separator enumerators (indeed, we construct computable enumerators f_0, f_1 for which the associated scaled approximation sequences coincide while the corresponding f-normality behavior diverges). At the same time, we identify a structured regime—finite-state coherent enumerators—in which f-normality is equivalent to k-adic equidistribution of $(k^n a_n^f(x))_{n\geq 1}$. This raises the natural question of how far this correspondence persists beyond finite-state coherent relabelings. One concrete setting is when the naming map is computable by a deterministic pushdown transducer. In particular, does there exist such a separator enumerator f and a point $x \in [0,1)$ for which the integer sequence $(k^n a_n^f(x))_{n\geq 1}$ is k-adically equidistributed while $\dim_{\mathrm{FS}}^f(x) < 1$?. It would also be interesting to identify the widest natural family of separator enumerators for which an equidistribution characterization of f-normality remains valid.

A related direction concerns the role of invertibility in the defining Mealy-machine relabeling: invertibility is used to transport approximation complexity via the substitution $u = M(w)$ and to ensure that $T \mapsto M \circ T$ is a bijection on finite-state transducers, yielding invariance of the induced f-dimension. It would be interesting to determine whether some weaker condition (e.g. levelwise surjectivity or bounded-to-one behavior on each Σ^n) suffices for the same equidistribution characterization, or whether non-invertible finite-state relabelings can already break it.

References

[1] Chris Bourke, John M Hitchcock, and NV Vinodchandran. Entropy rates and finite-state dimension. *Theoretical Computer Science*, 349(3):392–406, 2005.

[2] Wesley Calvert, Emma Gruner, Elvira Mayordomo, Daniel Turetsky, and Java Darleen Villano. Normality, relativization, and randomness. *Theory of Computing Systems*, 69(3):26, 2025.

[3] Jack J Dai, James I Lathrop, Jack H Lutz, and Elvira Mayordomo. Finite-state dimension. *Theoretical Computer Science*, 310(1-3):1–33, 2004.

[4] David Doty and Philippe Moser. Finite-state dimension and lossy decompressors, 2006.

[5] L. Kuipers and H. Niederreiter. *Uniform distribution of sequences*. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974.

[6] Jack H. Lutz. Dimension in complexity classes. *SIAM Journal on Computing*, 32(5):1236–1259, 2003.

[7] Jack H. Lutz. The dimensions of individual strings and sequences. *Information and Computation*, 187(1):49–79, 2003.

[8] Jack H. Lutz and Neil Lutz. Algorithmic information, plane kakeya sets, and conditional dimension. *ACM Trans. Comput. Theory*, 10(2), May 2018.

[9] Jack H Lutz and Neil Lutz. Who asked us? how the theory of computing answers questions about analysis. In *Complexity and Approximation: In Memory of Ker-I Ko*, pages 48–56. Springer, 2020.

[10] Elvira Mayordomo. A point to set principle for finite-state dimension. In Arnold Beckmann, Isabel Oitavem, and Florin Manea, editors, *Crossroads of Computability and Logic: Insights, Inspirations, and Innovations*, pages 299–304, Cham, 2025. Springer Nature Switzerland.