

On Normality and Equidistribution for Separator Enumerators

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Abstract

A separator is a countable dense subset of $[0, 1)$, and a separator enumerator is a naming scheme that assigns a real number in $[0, 1)$ to each finite word so that the set of all named values is a separator. Mayordomo introduced separator enumerators to define f -normality and a relativized finite-state dimension $\dim_{\text{FS}}^f(x)$, where finite-state dimension measures the asymptotic lower rate of finite-state information needed to approximate x through its f -names. This framework extends classical base- k normality, and Mayordomo showed that it supports a point-to-set principle for finite-state dimension. This representation-based viewpoint has since been developed further in follow-up work, including by Calvert et al., yielding strengthened randomness notions such as supernormal and highly normal numbers.

Mayordomo posed the following open question: can f -normality be characterized via equidistribution properties of the sequence $(|\Sigma|^n a_n^f(x))_{n=0}^\infty$, where $a_n^f(x)$ is the sequence of best approximations to x from below induced by f ? We give a strong negative answer: we construct computable separator enumerators f_0, f_1 and a point x such that $a_n^{f_0}(x) = a_n^{f_1}(x)$ for all n , yet $\dim_{\text{FS}}^{f_0}(x) = 0$ while $\dim_{\text{FS}}^{f_1}(x) = 1$. Consequently, no criterion depending only on the sequence $(|\Sigma|^n a_n^f(x))_{n=0}^\infty$ - in particular, no equidistribution property of this sequence - can characterize f -normality uniformly over all separator enumerators. On the other hand, for a natural finite-state coherent class of separator enumerators we recover a complete equidistribution characterization of f -normality. We also show that beyond finite-state coherence, this characterization can fail even for a separator enumerator computable in nearly linear time.

1 Introduction

Finite-state dimension is a quantitative notion of the rate of randomness in an individual infinite sequence, as measured by finite automata. It was introduced by Dai, Lathrop, Lutz, and Mayordomo [3] as a finite-state analogue of effective Hausdorff dimension [7, 6]. It admits several equivalent characterizations, including formulations via finite-state gambling, finite-state compression, and block entropy rates [3, 4, 1]. It also has an important connection with the theory of normal numbers: a real number is normal to base k if and only if its base- k digit sequence has finite-state dimension equal to 1 [1].

An important connection between effective dimension and classical fractal dimension is provided by point-to-set principles. Lutz and Lutz [8] proved that the Hausdorff dimension in Euclidean spaces of a set can be obtained by minimizing, over oracles, the supremum of the relativized effective dimensions of its points. This reduces many lower-bound questions in geometric measure theory to analyzing the information density of carefully chosen points, and it has led to several new results and new proofs across classical fractal geometry; see [9]. In a recent work, Mayordomo [11] established

an analogous point-to-set principle for finite-state dimension. A key feature of this development is that it is representation-based: instead of fixing a base- k expansion, it uses *separator enumerators*, i.e. naming schemes that assign reals in $[0, 1)$ to finite words so that the range forms a countable dense subset. With such an enumerator $f : \Sigma^* \rightarrow [0, 1)$, one can measure how efficiently a finite-state transducer can output an f -name approximating a real x to a given precision, leading to a relativized finite-state dimension $\dim_{\text{FS}}^f(x)$ and the induced notion of *f -normality*, defined by the condition $\dim_{\text{FS}}^f(x) = 1$ [11]. Calvert et al. [2] develop this framework further, introducing strengthened notions such as supernormal and highly normal numbers under broad classes of representations.

In this paper we study the relationships between f -normality and equidistribution. In the classical base- k setting, normality has a sharp equidistribution characterization: x is normal to base k if and only if the sequence $(k^n x)_{n \geq 1}$ is uniformly distributed modulo 1 [5]. Motivated by the classical equivalence between normality and equidistribution, Mayordomo [11] asked whether an analogous equidistribution criterion holds in the setting of f -normality. Let $(a_n^f(x))_{n \geq 0}$ denote the best-approximation-from-below sequence associated with f and x , i.e. $a_n^f(x) = \max\{f(w) : |w| \leq n, f(w) \leq x\}$. Mayordomo posed the following open question:

Can f -normality be characterized via equidistribution properties of the sequence $(|\Sigma|^n a_n^f(x))_{n \geq 0}$?

We show that the answer is negative in a strong way. We construct two total computable rational-valued separator enumerators f_0, f_1 and a point $x \in [0, 1)$ such that $a_n^{f_0}(x) = a_n^{f_1}(x)$ for all n , yet $\dim_{\text{FS}}^{f_0}(x) = 0$ while $\dim_{\text{FS}}^{f_1}(x) = 1$. Consequently, no criterion depending only on the single numeric sequence $(|\Sigma|^n a_n^f(x))_{n \geq 0}$ —in particular, no equidistribution property of that sequence—can characterize f -normality uniformly over all separator enumerators.

At the same time, an equidistribution characterization does hold under a natural structural restriction on the representation. We identify a class of *finite-state coherent* enumerators obtained from the standard base- k grid by an invertible synchronous Mealy-machine relabeling, and we prove that in this regime f -normality is equivalent to a k -adic equidistribution property of the integer sequence $(k^n a_n^f(x))_{n \geq 1}$ (uniform distribution of residues modulo k^m for every fixed m). Beyond finite-state coherence, we show that this equivalence can already break for a separator enumerator f computable in $O(n \log n)$ time: there is a point x whose scaled best-from-below approximations are k -adically equidistributed, but x is not f -normal.

Section 2 defines separator enumerators, finite-state transducers, \dim_{FS}^f , and f -normality. Section 3 proves the negative result via a pair of computable enumerators with identical best-from-below chains but sharply different relativized finite-state dimension. Section 4 establishes the k -adic equidistribution characterization for finite-state coherent enumerators, and we then show that this correspondence can already fail for a separator enumerator computable in nearly linear time.

2 Preliminaries

Let Σ be a finite alphabet with $k = |\Sigma| \geq 2$ and write Σ^* for the set of finite strings over Σ . For $w \in \Sigma^*$, $|w|$ denotes its length, and for an infinite sequence $X \in \Sigma^\infty$ we write $X \upharpoonright n$ for its length- n prefix (using the operator \upharpoonright defined in the preamble). We use $\lfloor \cdot \rfloor$ for the floor function and interpret congruences $b_n \equiv r \pmod{k^m}$ in the usual sense. Throughout this section we fix an identification $\Sigma = \{0, 1, \dots, k-1\}$. We view finite words over Σ as base- k numerals and use the associated k -adic grid in $[0, 1)$. For a word $u = u_1 u_2 \cdots u_n \in \Sigma^n$, define its base- k value

$$\text{val}(u) = \sum_{i=1}^n u_i k^{n-i} \in \{0, 1, \dots, k^n - 1\}, \quad \text{and} \quad \text{grid}(u) = \frac{\text{val}(u)}{k^n} \in [0, 1).$$

Thus $\text{grid}(\Sigma^n) = \{j/k^n : 0 \leq j < k^n\}$. Let $\text{seq}_k(x) \in \Sigma^\infty$ denote the (canonical) base- k expansion of $x \in [0, 1)$ chosen so as *not* to end in $(k-1)^\infty$.

Finite-state transducers model one-pass, constant-memory transformations on words and are used to measure finite-state description length.

Definition 1 (Finite-state transducer (FST)). *A Σ -finite-state transducer (briefly, Σ -FST) is a tuple $T = (Q, \delta, \nu, q_0)$ where Q is a finite nonempty set of states, $\delta : Q \times \Sigma \rightarrow Q$ is a transition function, $\nu : Q \times \Sigma \rightarrow \Sigma^*$ is an output function, and $q_0 \in Q$ is the start state.*

The transition function extends to words by $\delta(q, \lambda) = q$ and $\delta(q, wa) = \delta(\delta(q, w), a)$. For $q \in Q$ and $w \in \Sigma^*$, define the output $\nu(q, w) \in \Sigma^*$ by $\nu(q, \lambda) = \lambda$ and $\nu(q, wa) = \nu(q, w)\nu(\delta(q, w), a)$ for $a \in \Sigma$. The overall output of T on input w is $T(w) = \nu(q_0, w)$.

We measure how concisely a transducer can generate a given target string.

Definition 2 (T -information content [11]). *Let T be a Σ -FST and $w \in \Sigma^*$. Define*

$$K^T(w) = \min\{|\pi| : \pi \in \Sigma^* \text{ and } T(\pi) = w\},$$

with the convention $K^T(w) = \infty$ if w is not in the range of T .

A separator enumerator is a naming scheme for a countable dense subset of $[0, 1)$, assigning a real in $[0, 1)$ to each finite word.

Definition 3 (Separator, separator enumerator [11]). *A set $S \subseteq [0, 1)$ is a separator if it is countable and dense in $[0, 1)$. A function $f : \Sigma^* \rightarrow [0, 1)$ is a separator enumerator (SE) if $\text{Im}(f)$ is a separator.*

We call an SE $f : \Sigma^* \rightarrow [0, 1)$ *total computable* if there is an algorithm that, on input (w, t) with $w \in \Sigma^*$ and $t \in \mathbb{N}$, outputs a rational q such that $|q - f(w)| \leq 2^{-t}$, and halts on every input. Given an enumerator f , we quantify how many input symbols a transducer needs in order to produce some f -name that δ -approximates a target point.

Definition 4 (Relativized approximation complexity [11]). *Let f be an SE, T a Σ -FST, $\delta > 0$, and $x \in [0, 1)$. Define*

$$K_\delta^{T,f}(x) = \min\{K^T(w) : w \in \Sigma^* \text{ and } |f(w) - x| < \delta\}.$$

The induced finite-state dimension is the optimal asymptotic approximation rate achievable by finite-state transducers.

Definition 5 (Relativized finite-state dimension and f -normality [11]). *Let f be an SE and $x \in [0, 1)$. Define*

$$\dim_{\text{FS}}^f(x) = \inf_{T \text{ } \Sigma\text{-FST}} \liminf_{\delta \rightarrow 0^+} \frac{K_\delta^{T,f}(x)}{\log_k(1/\delta)}.$$

We say x is f -normal if $\dim_{\text{FS}}^f(x) = 1$.

The standard base- k naming map will serve as the baseline enumerator throughout. Let $f_{\text{std}} : \Sigma^* \rightarrow [0, 1)$ be the standard base- k enumerator

$$f_{\text{std}}(u) = \text{grid}(u) = \frac{\text{val}(u)}{k^{|u|}} \quad (u \neq \lambda),$$

with $f_{\text{std}}(\lambda) = 0$. We now define the induced *best approximation from below* chain associated with a separator enumerator.

Definition 6 (Best approximation from below [11]). *Let f be an SE and $x \in [0, 1]$. For each $n \in \mathbb{N}$, define $a_n^f(x)$ to be any value $f(w)$ with $|w| \leq n$ such that $f(w) \leq x$ and $x - f(w)$ is minimum among all u with $|u| \leq n$ and $f(u) \leq x$. Equivalently,*

$$a_n^f(x) = \max\{f(w) : |w| \leq n, f(w) \leq x\}.$$

3 Equidistribution cannot characterize normality for separator enumerators

We now address Mayordomo's question on equidistribution criteria for f -normality. Fix a separator enumerator $f : \Sigma^* \rightarrow [0, 1]$ and $x \in [0, 1]$, and let $(a_n^f(x))_{n \geq 0}$ be the best approximation from below sequence corresponding to f and x (see Definition 6).

Question 1 ([11]). *Can f -normality be characterized via equidistribution properties of the sequence $(|\Sigma|^n a_n^f(x))_{n \geq 0}$?*

We answer this question in the negative by constructing two total computable rational-valued separator enumerators f_0, f_1 and a point $x \in [0, 1]$ such that $a_n^{f_0}(x) = a_n^{f_1}(x)$ for all n , yet $\dim_{\text{FS}}^{f_0}(x) = 0$ while $\dim_{\text{FS}}^{f_1}(x) = 1$.

3.1 Preliminary lemmas

From the finite-state transducer characterization of finite-state dimension due to Doty and Moser (see Theorem 3.11 from [4]), we obtain the following corollary.

Lemma 1. *There exists an infinite sequence $Z \in \Sigma^\infty$ such that for every Σ -FST T ,*

$$\liminf_{n \rightarrow \infty} \frac{K^T(z_n)}{n} = 1,$$

where $z_n := Z \upharpoonright n$.

Proof. Fix any Σ -normal sequence Z . By the transducer characterization of finite-state dimension and the fact that normal sequences have finite-state dimension equal to 1,

$$1 = \dim_{\text{FS}}(Z) = \inf_T \liminf_{n \rightarrow \infty} \frac{K^T(z_n)}{n}.$$

Therefore, for every Σ -FST T one must have $\liminf_n K^T(z_n)/n = 1$; otherwise the infimum would be < 1 . \square

We also need a second sequence whose prefixes are distinct from the first but have the same property. This is achieved by a fixed symbol permutation.

Lemma 2. *Let $\pi : \Sigma \rightarrow \Sigma$ be a bijection and extend it letterwise to $\pi : \Sigma^* \rightarrow \Sigma^*$. If $Z \in \Sigma^\infty$ satisfies Lemma 1, then so does $\pi(Z)$, i.e. the prefixes $t_n := \pi(Z) \upharpoonright n = \pi(z_n)$ satisfy $\liminf_n K^T(t_n)/n = 1$ for every Σ -FST T .*

Proof. Fix a Σ -FST T . Let P be the 1-state transducer that maps each input symbol a to the single output symbol $\pi(a)$. Then $P(w) = \pi(w)$ for all w , and similarly there is a 1-state transducer P^{-1} with $P^{-1}(w) = \pi^{-1}(w)$ for all w . For any string u ,

$$K^T(\pi(u)) = K^T(P(u)) \geq K^{P^{-1} \circ T}(u),$$

because any input producing $\pi(u)$ under T yields an input producing u under $P^{-1} \circ T$. Thus, for $t_n = \pi(z_n)$,

$$\frac{K^T(t_n)}{n} \geq \frac{K^{P^{-1} \circ T}(z_n)}{n}.$$

Taking \liminf and applying Lemma 1 for the transducer $P^{-1} \circ T$ we obtain $\liminf_n K^T(t_n)/n = 1$. \square

Fix a bijection $\pi : \Sigma \rightarrow \Sigma$ with no fixed points (such a derangement exists for all $k \geq 2$) and with $\pi(0) \neq 0$. Let Z be as in Lemma 1, let $Y = \pi(Z)$, and write $z_n := Z \upharpoonright n$ and $y_n := Y \upharpoonright n = \pi(z_n)$. Then $z_n \neq y_n$ for all n because π has no fixed points, and $y_n \neq 0^n$ for all n because $\pi(0) \neq 0$.

3.2 Construction of the two separator enumerators

Fix $x := \frac{1}{2} \in (0, 1)$. For each $n \in \mathbb{N}$, define $r_n := k^{-(n+2)}$ and

$$A_n := (x - r_n, x - r_{n+1}) \cup (x + r_{n+1}, x + r_n).$$

Then A_n is a nonempty open set and $(A_n)_{n \geq 0}$ partition the punctured neighborhood

$$(x - r_0, x + r_0) \setminus \{x\} = \bigsqcup_{n \geq 0} A_n$$

, and $\log_k(1/r_n) = n + 2$. Define the target *best-from-below* values

$$c_n := x - \frac{r_n + r_{n+1}}{2} \in (x - r_n, x - r_{n+1}) \subseteq A_n$$

. These values will later be enforced as the canonical length- n approximants to x for both enumerators, i.e., we will arrange $a_n^{f_i}(x) = c_n$ for each n . Observe that $c_n < x$ and $c_n \uparrow x$ as $n \rightarrow \infty$.

For each n , fix a computable listing $(d_{n,t})_{t \in \mathbb{N}}$ of *distinct* rationals in $A_n \cap \mathbb{Q} \setminus \{c_n\}$ whose range is dense in A_n , and set $D_n := \{d_{n,t} : t \in \mathbb{N}\}$. Also fix a computable listing $(q_t)_{t \in \mathbb{N}}$ of *distinct* rationals in $([0, 1] \setminus (x - r_0, x + r_0)) \cap \mathbb{Q}$ whose range is dense there, and set $D_{\text{far}} := \{q_t : t \in \mathbb{N}\}$. We now define two functions $f_0, f_1 : \Sigma^* \rightarrow [0, 1]$.

Step 1 (fixing the best-from-below chain for lengths $\leq n$). For each $n \geq 0$, define

$$f_0(0^n) := c_n, \quad f_1(z_n) := c_n.$$

(Here $0 \in \Sigma$ is a fixed symbol, and 0^n is the all-0 word of length n .)

Step 2 (ensuring density near x using incompressible prefixes). Partition \mathbb{N} into pairwise disjoint infinite sets $(L_n)_{n \geq 0}$ such that $m \in L_n$ implies $m \geq n + 1$. For concreteness, one may take $L_n = \{2^n(2t + 1) : t \in \mathbb{N}\}$; then $m \in L_n$ implies $m \geq 2^n \geq n + 1$ for $n \geq 0$. Fix computable bijections $\varphi_n : L_n \rightarrow D_n$ by setting, for $t \in \mathbb{N}$,

$$\varphi_n(2^n(2t + 1)) := d_{n,t}.$$

Now set, for each $n \geq 0$ and each $m \in L_n$,

$$f_0(y_m) = f_1(y_m) := \varphi_n(m) \in A_n.$$

Note that there is no conflict with Step 1, since $y_m \neq 0^m$ and $y_m \neq z_m$ for all m . Thus, for each n , the set $\{f_i(y_m) : m \in L_n\}$ is dense in A_n (for both $i = 0, 1$).

Step 3 (defining all remaining values far from x while maintaining density). Let W be the set of all remaining strings not yet assigned a value by Steps 1–2:

$$W := \Sigma^* \setminus \left(\{0^n : n \in \mathbb{N}\} \cup \{z_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\} \right).$$

Enumerate Σ^* in the length lexicographic order, then all words of length 2 in lexicographic order, and so on. Call this enumeration $(u_j)_{j \in \mathbb{N}}$. Define w_j to be the j th word in this list that lies in W , i.e., the j th u_t such that $u_t \notin \{0^n : n \in \mathbb{N}\} \cup \{z_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$. Also enumerate $D_{\text{far}} = \{q_0, q_1, q_2, \dots\}$ according to the fixed computable listing above. Define $f_0(w_j) = f_1(w_j) := q_j$ for all j .

Lemma 3. *The functions f_0, f_1 defined above are total computable rational-valued separator enumerators: each $\text{Im}(f_i)$ is countable and dense in $[0, 1]$.*

Proof. Countability is immediate since Σ^* is countable.

We also note that f_0 and f_1 are total computable (with rational outputs). Assume the fixed dense sets D_n and D_{far} come with fixed computable enumerations, and that each $\varphi_n : L_n \rightarrow D_n \setminus \{c_n\}$ is a fixed computable bijection. Also fix Z to be a computable Σ -normal sequence, so $n \mapsto z_n = Z \upharpoonright n$ is computable; then $Y = \pi(Z)$ is computable and $n \mapsto y_n = Y \upharpoonright n$ is computable.

On input $w \in \Sigma^*$ of length ℓ , we compute $f_i(w)$ as follows. First check whether $w = 0^\ell$ (all symbols equal 0), whether $w = z_\ell$ (compute z_ℓ and compare), and whether $w = y_\ell$ (compute y_ℓ and compare). If $w = 0^\ell$ output $f_0(w) = c_\ell$; if $w = z_\ell$ output $f_1(w) = c_\ell$; and if $w = y_\ell$ then find the unique n with $\ell \in L_n$ (the L_n are decidable and disjoint) and output $f_0(w) = f_1(w) = \varphi_n(\ell)$. Otherwise $w \in W$, and W is decidable because $w \notin W$ iff one of the three checks above holds. In this case, compute the index j such that $w = w_j$ in Step 3 by enumerating all words of Σ^* by increasing length, and within each fixed length in lexicographic order, skipping exactly those words that are not in W , until w is reached; then output $f_0(w) = f_1(w) = q_j$ (where (q_j) is the fixed computable enumeration of D_{far}). This procedure halts for every input w , so f_0, f_1 are total computable.

For density, let $I \subseteq [0, 1)$ be a nonempty open interval. If I intersects $[0, 1) \setminus (x - r_0, x + r_0)$, then I contains a rational in D_{far} , hence an image point of f_i . Otherwise, $I \subseteq (x - r_0, x + r_0)$, so I intersects A_n for some $n \geq 0$ (because the annuli partition $(x - r_0, x + r_0) \setminus \{x\}$ and I is open, hence cannot be $\{x\}$). Since D_n is dense in A_n and $\{f_i(y_m) : m \in L_n\} = \varphi_n(L_n)$ is dense in A_n , the interval I contains some $f_i(y_m)$. Thus $\text{Im}(f_i)$ meets every nonempty open interval, so it is dense. \square

Since f_0 and f_1 are rational-valued and the above procedure computes the defining case and the corresponding index effectively, it in fact yields exact computation: there is a Turing machine that, on input $w \in \Sigma^*$, halts and outputs the rational value $f_i(w)$ itself (rather than merely producing 2^{-t} -approximations). The next lemma shows that f_0 and f_1 induce exactly the same sequence of length-bounded approximations of x from below (and hence the same associated numeric scaling sequence).

Lemma 4. *For every $n \in \mathbb{N}$, $a_n^{f_0}(x) = a_n^{f_1}(x) = c_n$. Consequently, the numeric sequences $(k^n a_n^{f_0}(x))$ and $(k^n a_n^{f_1}(x))$ are identical.*

Proof. Fix n . We first show $a_n^{f_0}(x) = c_n$. By definition, $f_0(0^n) = c_n \leq x$, so $a_n^{f_0}(x) \geq c_n$. Now consider any string w with $|w| \leq n$ and $f_0(w) \leq x$. If $w = 0^m$ for some $m \leq n$, then $f_0(w) = c_m \leq c_n$ because (c_m) is increasing. If $w = y_m$ for some m , then $f_0(w) \in A_j$ for some j with $m \in L_j$. In particular, $f_0(w) \leq x - r_{j+1} < x$. Moreover, since $m \in L_j$ implies $m \geq j + 1$,

we have $j \leq m - 1 \leq n - 1$ whenever $m \leq n$. Thus $f_0(w) \leq x - r_{j+1} \leq x - r_n < c_n$ (because $c_n > x - r_n$ by construction). Finally, if $w \in W$ then $f_0(w) \in D_{\text{far}} \subseteq [0, 1] \setminus (x - r_0, x + r_0)$, so either $f_0(w) \leq x - r_0 < x - r_n < c_n$ or $f_0(w) \geq x + r_0 > x$. In either case it cannot exceed c_n while staying $\leq x$. Therefore, among all $|w| \leq n$ with $f_0(w) \leq x$, the maximum is attained at $w = 0^n$ with value c_n . Hence $a_n^{f_0}(x) = c_n$.

The proof for f_1 is identical, replacing the witness 0^n by z_n (since $f_1(z_n) = c_n$) and noting that no other string of length $\leq n$ attains a value in $(c_n, x]$ by the same case analysis. Thus $a_n^{f_1}(x) = c_n$. \square

We now complete the construction by showing that, for the fixed point $x = \frac{1}{2}$, the two separator enumerators f_0 and f_1 constructed above induce different relativized finite-state dimensions: $\dim_{\text{FS}}^{f_0}(x) = 0$ while $\dim_{\text{FS}}^{f_1}(x) = 1$. Together with Lemma 4, this yields a negative answer to Mayordomo's open question [11].

Theorem 1. *There exist total computable rational-valued separator enumerators $f_0, f_1 : \Sigma^* \rightarrow [0, 1]$ and a point $x \in [0, 1)$ such that*

$$a_n^{f_0}(x) = a_n^{f_1}(x) \quad \text{for all } n,$$

yet

$$\dim_{\text{FS}}^{f_0}(x) = 0 \quad \text{and} \quad \dim_{\text{FS}}^{f_1}(x) = 1.$$

In particular, x is not f_0 -normal, while x is f_1 -normal.

Proof. We use the constructions above with the fixed $x = \frac{1}{2}$. By Lemma 4, the sequences $a_n^{f_0}(x)$ and $a_n^{f_1}(x)$ coincide (both equal c_n), so $(k^n a_n(x))$ is identical for f_0 and f_1 . It remains to compute the relativized finite-state dimensions.

Part 1: We show $\dim_{\text{FS}}^{f_0}(x) = 0$. For each integer $L \geq 1$, let T_L be the 1-state transducer that outputs 0^L on every input symbol. Then $T_L(\pi) = 0^{L|\pi|}$ for all π , hence

$$K^{T_L}(0^n) \leq \left\lceil \frac{n}{L} \right\rceil.$$

Fix n and consider $\delta_n := r_n/2$ (so $\log_k(1/\delta_n) = n + 2 + \log_k 2$). Since $f_0(0^n) = c_n$ and $|x - c_n| = |x - (x - (r_n + r_{n+1})/2)| = (r_n + r_{n+1})/2 < r_n = 2\delta_n$, we have $|f_0(0^n) - x| < 2\delta_n$. Therefore

$$K_{2\delta_n}^{T_L, f_0}(x) \leq K^{T_L}(0^n) \leq \left\lceil \frac{n}{L} \right\rceil.$$

Hence

$$\liminf_{n \rightarrow \infty} \frac{K_{2\delta_n}^{T_L, f_0}(x)}{\log_k(1/(2\delta_n))} \leq \liminf_{n \rightarrow \infty} \frac{\lceil n/L \rceil}{(n+2) + O(1)} = \frac{1}{L}.$$

To justify that this controls the full $\liminf_{\delta \rightarrow 0^+}$ (and not only the subsequence $2\delta_n$), note that $K_\delta^{T_L, f_0}(x)$ is monotone non-increasing in δ . Hence for any $\delta \in (2\delta_{n+1}, 2\delta_n]$,

$$K_\delta^{T_L, f_0}(x) \leq K_{2\delta_{n+1}}^{T_L, f_0}(x).$$

Moreover, for such δ we have $\log_k(1/\delta) \geq \log_k(1/(2\delta_n))$. Therefore,

$$\frac{K_\delta^{T_L, f_0}(x)}{\log_k(1/\delta)} \leq \frac{K_{2\delta_{n+1}}^{T_L, f_0}(x)}{\log_k(1/(2\delta_n))}.$$

Taking \liminf over all $\delta \rightarrow 0^+$ and using that $n \rightarrow \infty$ along the corresponding intervals yields

$$\begin{aligned} \liminf_{\delta \rightarrow 0^+} \frac{K_\delta^{T_L, f_0}(x)}{\log_k(1/\delta)} &\leq \liminf_{n \rightarrow \infty} \frac{K_{2\delta_{n+1}}^{T_L, f_0}(x)}{\log_k(1/(2\delta_{n+1}))} \cdot \frac{\log_k(1/(2\delta_{n+1}))}{\log_k(1/(2\delta_n))} \\ &\leq \liminf_{n \rightarrow \infty} \frac{K_{2\delta_{n+1}}^{T_L, f_0}(x)}{\log_k(1/(2\delta_{n+1}))} \cdot \lim_{n \rightarrow \infty} \frac{n+3+O(1)}{n+2+O(1)} \\ &\leq \frac{1}{L}. \end{aligned}$$

Since the above holds for all T_L , taking the infimum over all transducers T and then letting $L \rightarrow \infty$ yields $\dim_{\text{FS}}^{f_0}(x) = 0$.

Part 2: We show $\dim_{\text{FS}}^{f_1}(x) = 1$. Fix an arbitrary Σ -FST T . We prove that $\liminf_{\delta \rightarrow 0^+} \frac{K_\delta^{T, f_1}(x)}{\log_k(1/\delta)} \geq 1$, which implies $\dim_{\text{FS}}^{f_1}(x) = 1$ after taking the infimum over T .

Consider the scale $\delta_n := r_n$. Any value $f_1(w)$ within distance $\delta_n = r_n$ of x must lie in $(x - r_n, x + r_n) = \{x\} \cup \bigcup_{j \geq n} A_j$, so w must be either (i) one of the special words z_j with $j \geq n$ (since $f_1(z_j) = c_j \in A_j$), or (ii) one of the special words y_m with $f_1(y_m) \in A_j$ for some $j \geq n$ (these are the values placed densely in the annuli), because by construction all other words are mapped into D_{far} outside $(x - r_0, x + r_0)$. Therefore,

$$K_{r_n}^{T, f_1}(x) \geq \min \left(\min_{j \geq n} K^T(z_j), \min_{m: f_1(y_m) \in \bigcup_{j \geq n} A_j} K^T(y_m) \right).$$

We now lower bound each term asymptotically by n . By Lemma 1, we have $\liminf_{j \rightarrow \infty} K^T(z_j)/j = 1$. Hence for every $\varepsilon > 0$ there exists J such that for all $j \geq J$, $K^T(z_j) \geq (1-\varepsilon)j$; consequently, for all $n \geq J$, $\min_{j \geq n} K^T(z_j) \geq (1-\varepsilon)n$. Similarly, by Lemma 2 applied to Y , $\liminf_{m \rightarrow \infty} K^T(y_m)/m = 1$, so for every $\varepsilon > 0$ there exists M such that for all $m \geq M$, $K^T(y_m) \geq (1-\varepsilon)m$. In our construction, if $f_1(y_m) \in A_j$, then $m \in L_j$, and by design $m \geq j+1$; therefore, whenever $f_1(y_m) \in \bigcup_{j \geq n} A_j$, we have $m \geq n+1$. For all $n \geq M$,

$$\min_{m: f_1(y_m) \in \bigcup_{j \geq n} A_j} K^T(y_m) \geq (1-\varepsilon)(n+1) \geq (1-\varepsilon)n.$$

Putting the two bounds together, for all sufficiently large n , $K_{r_n}^{T, f_1}(x) \geq (1-\varepsilon)n$. Since $\log_k(1/r_n) = n+2$, we obtain

$$\liminf_{n \rightarrow \infty} \frac{K_{r_n}^{T, f_1}(x)}{\log_k(1/r_n)} \geq \liminf_{n \rightarrow \infty} \frac{(1-\varepsilon)n}{n+2} = 1-\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this yields $\liminf_{n \rightarrow \infty} \frac{K_{r_n}^{T, f_1}(x)}{\log_k(1/r_n)} \geq 1$. By monotonicity of $K_\delta^{T, f_1}(x)$ in δ and the fact that any $\delta \in (r_{n+1}, r_n]$ satisfies $\log_k(1/\delta) \in [n+2, n+3]$, it follows that the full $\liminf_{\delta \rightarrow 0^+}$ is also at least 1. Thus

$$\liminf_{\delta \rightarrow 0^+} \frac{K_\delta^{T, f_1}(x)}{\log_k(1/\delta)} \geq 1$$

for every T , which implies that $\dim_{\text{FS}}^{f_1}(x) = 1$. □

As an immediate consequence, f -normality cannot be characterized by any property of the single numerical sequence $(k^n a_n^f(x))$.

Corollary 1. *There is no property P of the numeric sequence $(k^n a_n^f(x))_{n \in \mathbb{N}}$ (in particular, no equidistribution property of this sequence) such that for all separator enumerators f and all $x \in [0, 1]$,*

$$x \text{ is } f\text{-normal} \iff (k^n a_n^f(x)) \text{ has property } P.$$

Proof. Take f_0, f_1, x from Theorem 1. By Lemma 4, the sequences $(k^n a_n^{f_0}(x))$ and $(k^n a_n^{f_1}(x))$ are identical, so they either both satisfy P or both fail P . But by Theorem 1, x is f_1 -normal and not f_0 -normal, so no such P can exist. \square

4 Finite-State Coherent Enumerators and an Equidistribution Characterization of f -Normality

From the previous section, we know that no equidistribution (or other distributional) criterion depending only on the sequence $(k^n a_n^f(x))_{n \geq 1}$ can characterize f -normality uniformly over all separator enumerators. The goal of this section is to isolate a natural structural regime in which such a characterization *does* hold. We do this by restricting to *finite-state coherent enumerators*, i.e. naming schemes obtained from the standard base- k grid by an invertible synchronous Mealy-machine relabeling.

To state an equidistribution characterization in this setting, we must use a notion that remains meaningful for the standard enumerator. For the standard base- k naming map $f_{\text{std}}(w) = \text{val}(w)/k^{|w|}$, the scaled approximation sequence $b_n(x) := k^n a_n^{f_{\text{std}}}(x)$ is integer-valued (indeed $b_n(x) = \lfloor k^n x \rfloor$), so the usual notion of equidistribution modulo 1 becomes trivial. The natural replacement is to ask for uniform distribution of the residues of $b_n(x)$ at every finite base- k resolution, i.e. modulo k^m for each fixed m . This leads to the following notion of k -adic equidistribution, which we adopt in the rest of the paper.

Definition 7 (k -adic equidistribution). *A sequence $(b_n)_{n \geq 1}$ of integers is k -adically equidistributed if for every $m \geq 1$ and every residue $r \in \{0, 1, \dots, k^m - 1\}$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{1 \leq n \leq N : b_n \equiv r \pmod{k^m}\} = \frac{1}{k^m}.$$

For finite-state coherent enumerators, the best-from-below approximants admit an explicit closed form, and the induced f -dimension is invariant under finite-state coherent relabelings. These two facts combine to yield a clean equidistribution characterization of f -normality in terms of k -adic equidistribution of the integer sequence $(k^n a_n^f(x))_{n \geq 1}$.

4.1 Finite-state coherent enumerators

We begin by defining invertible synchronous Mealy machines.

Definition 8 (Invertible synchronous Mealy machine). *An invertible synchronous Mealy machine is a tuple $M = (Q, \delta, \lambda, q_0)$ where Q is a finite nonempty set of states, $\delta : Q \times \Sigma \rightarrow Q$ is a transition function, and $\lambda : Q \times \Sigma \rightarrow \Sigma$ is an output function such that for every state $q \in Q$, the map $a \mapsto \lambda(q, a)$ is a permutation of Σ .*

The induced map $M : \Sigma^ \rightarrow \Sigma^*$ is defined by reading left-to-right: set $M(\lambda) = \lambda$, and for $w = w_1 \cdots w_n$ define $u = M(w) = u_1 \cdots u_n$ by the recursion*

$$q_i = \delta(q_{i-1}, w_i), \quad u_i = \lambda(q_{i-1}, w_i) \quad (i = 1, \dots, n),$$

with q_0 as the initial state. In particular, $|M(w)| = |w|$ for all w .

We state the basic properties of invertible synchronous Mealy machines.

Lemma 5. *Let M be an invertible synchronous Mealy machine.*

1. *For every n , the restriction $M : \Sigma^n \rightarrow \Sigma^n$ is a bijection.*
2. *There exists an invertible synchronous Mealy machine M^{-1} such that for all $w \in \Sigma^*$, we have $M^{-1}(M(w)) = w$.*
3. *M extends letter-by-letter (via the same recursion as in the definition of M) to a bijection $M : \Sigma^\infty \rightarrow \Sigma^\infty$ with inverse M^{-1} .*

Proof. (1) Fix n . Because at each step the output letter is obtained by applying a permutation depending on the current state, distinct inputs cannot merge: if $w \neq w'$ then at the first position i where they differ, the machine is in the same state (because it has read the same prefix) and applies a permutation to two different letters, hence outputs different letters at position i . Thus M is injective on Σ^n , hence bijective because Σ^n is finite.

(2) One constructs M^{-1} by reversing the per-state permutations: in state q output $\lambda(q, \cdot)^{-1}(a)$ on input a , and update the state consistently (standard Mealy-machine inversion). Because all per-state maps are permutations, this is well-defined.

(3) The same recursion as in definition of M works on infinite inputs; bijectivity follows from (1) on all finite prefixes. \square

Now we formalize the finite-state coherence condition, which captures length-preserving, bounded-memory relabelings of the standard base- k grid.

Definition 9 (Finite-state coherent enumerator). *A function $f : \Sigma^* \rightarrow [0, 1)$ is finite-state coherent if there exists an invertible synchronous Mealy machine M such that for every nonempty $w \in \Sigma^n$, $f(w) = \text{grid}(M(w)) = \frac{\text{val}(M(w))}{k^n}$. (For definiteness, set $f(\lambda) = 0$.)*

Finite-state coherence immediately forces f to enumerate the entire standard k -adic grid at each length.

Lemma 6. *If f is finite-state coherent, then $\text{Im}(f) = \bigcup_{n \geq 1} \{j/k^n : 0 \leq j < k^n\}$, hence f is a separator enumerator.*

Proof. Fix $n \geq 1$. By Lemma 5(1), $M(\Sigma^n) = \Sigma^n$. Therefore

$$f(\Sigma^n) = \text{grid}(M(\Sigma^n)) = \text{grid}(\Sigma^n) = \left\{ \frac{j}{k^n} : 0 \leq j < k^n \right\}.$$

Taking the union over n gives the claimed image, which is countable and dense in $[0, 1)$. \square

For finite-state coherent enumerators, the best-from-below approximation sequence coincides with the usual base- k truncations.

Lemma 7. *Let f be finite-state coherent and $x \in [0, 1)$. Then for every $n \geq 1$, $a_n^f(x) = \frac{\lfloor k^n x \rfloor}{k^n}$. In particular, the scaled sequence is integer-valued: $k^n a_n^f(x) = \lfloor k^n x \rfloor \in \{0, 1, \dots, k^n - 1\}$.*

Proof. By Lemma 6, for each $m \leq n$ the set $f(\Sigma^m)$ equals the full grid $\{j/k^m : 0 \leq j < k^m\}$. Hence the set of all values $f(w)$ with $|w| \leq n$ is exactly $\bigcup_{m=1}^n \{j/k^m : 0 \leq j < k^m\}$. Among the level- n grid points $\{j/k^n\}$, the largest one $\leq x$ is $\lfloor k^n x \rfloor / k^n$. It remains to check that no coarser grid point (denominator k^m with $m < n$) can exceed this value while staying $\leq x$. But for each $m < n$, $\frac{\lfloor k^m x \rfloor}{k^m} \leq \frac{\lfloor k^n x \rfloor}{k^n}$, because multiplying both sides by k^n gives $k^{n-m} \lfloor k^m x \rfloor \leq \lfloor k^n x \rfloor$, which holds since $k^{n-m} \lfloor k^m x \rfloor \leq k^n x$ and the left-hand side is an integer. \square

Next we relate approximation complexity under a finite-state coherent f to approximation complexity under the standard base- k enumerator. Recall that $f_{\text{std}} : \Sigma^* \rightarrow [0, 1]$ is the standard base- k enumerator.

Lemma 8. *Let f be finite-state coherent via a Mealy machine M , i.e. $f(w) = \text{grid}(M(w))$ for all w . Then for every Σ -FST T , every $x \in [0, 1]$, and every $\delta > 0$, $K_\delta^{T,f}(x) = K_\delta^{M \circ T, f_{\text{std}}}(x)$, where $M \circ T$ denotes the output-composition transducer $\pi \mapsto M(T(\pi))$.*

Proof. Since $\text{grid}(u) = f_{\text{std}}(u)$ for every $u \in \Sigma^*$, we obtain $K_\delta^{T,f}(x) = \min\{K^T(w) : |f_{\text{std}}(M(w)) - x| < \delta\}$. Substitute $u = M(w)$. Since $M : \Sigma^{|w|} \rightarrow \Sigma^{|w|}$ is bijective for each length, this is equivalent to $K_\delta^{T,f}(x) = \min\{K^T(M^{-1}(u)) : |f_{\text{std}}(u) - x| < \delta\}$. For every $u \in \Sigma^*$, $K^T(M^{-1}(u)) = \min\{|\pi| : T(\pi) = M^{-1}(u)\} = \min\{|\pi| : M(T(\pi)) = u\} = K^{M \circ T}(u)$, and substituting yields the claim. \square

The next proposition formalizes the key robustness property of finite-state coherence: composing the naming map with an invertible synchronous Mealy relabeling does not change the relativized finite-state approximation complexity, and hence does not change the induced f -dimension.

Proposition 1. *If f is finite-state coherent, then for every $x \in [0, 1]$, $\dim_{\text{FS}}^f(x) = \dim_{\text{FS}}^{f_{\text{std}}}(x)$. In particular, x is f -normal if and only if x is f_{std} -normal.*

Proof. Let f be finite-state coherent via M . By Lemma 8, for every T and every δ , $K_\delta^{T,f}(x) = K_\delta^{M \circ T, f_{\text{std}}}(x)$, hence the corresponding \liminf ratios coincide. Taking \inf_T over all FSTs on the left equals taking \inf_S over all FSTs on the right, because $T \mapsto M \circ T$ is a bijection on FSTs (with inverse $S \mapsto M^{-1} \circ S$). Therefore the two infima coincide. \square

The next theorem states that, for the *standard* base- k naming map f_{std} , the paper's notion of f -normality (i.e. $\dim_{\text{FS}}^f(x) = 1$) coincides exactly with the classical notion of base- k normality of x .

Theorem 2. *For $x \in [0, 1]$, one has $\dim_{\text{FS}}^{f_{\text{std}}}(x) = 1$ if and only if x is base- k normal.*

Proof. By Theorem 3.3 of [11], for every $x \in [0, 1]$, $\dim_{\text{FS}}^{f_{\text{std}}}(x) = \dim_{\text{FS}}(\text{seq}_k(x))$. By the standard characterization of normality via finite-state dimension (e.g. [1]), one has $\dim_{\text{FS}}(\text{seq}_k(x)) = 1$ if and only if $\text{seq}_k(x)$ is base- k normal. Finally, by definition, $\text{seq}_k(x)$ is base- k normal if and only if x is base- k normal. Combining these equivalences yields the claim. \square

It is straightforward to verify that k -adic equidistribution of the sequence $(\lfloor k^n x \rfloor)_{n \geq 1}$ coincides with base- k normality [5].

Theorem 3. *Let $x \in [0, 1]$ and set $b_n = \lfloor k^n x \rfloor$. Then x is base- k normal if and only if $(b_n)_{n \geq 1}$ is k -adically equidistributed.*

Finally we combine finite-state coherent invariance with the explicit form of $a_n^f(x)$ to obtain the equidistribution characterization in terms of the scaled approximation sequence.

Theorem 4. *Let f be a finite-state coherent separator enumerator over Σ and let $x \in [0, 1]$. Then x is f -normal if and only if the integer sequence $(k^n a_n^f(x))_{n \geq 1}$ is k -adically equidistributed.*

Proof. Let $b_n(x) := k^n a_n^f(x)$. Since f is finite-state coherent, Proposition 1 gives x is f -normal if and only if x is f_{std} -normal. By Theorem 2, f_{std} -normality is equivalent to base- k normality of x . On the other hand, Theorem 3 states that base- k normality of x is equivalent to k -adic equidistribution of the integer sequence $(\lfloor k^n x \rfloor)_{n \geq 1}$. Finally, Lemma 7 identifies the scaled approximation sequence for finite-state coherent f with this canonical sequence, namely $b_n(x) = \lfloor k^n x \rfloor$ for all $n \geq 1$. Substituting this identity into the previous equivalence yields x is f -normal if and only if $(b_n(x))_{n \geq 1}$ is k -adically equidistributed, which is exactly the claim. \square

4.2 Beyond finite-state coherence: a nearly linear time computable counterexample

This subsection shows that once one moves beyond finite-state coherent enumerators, a k -adic *equidistribution*-based criterion for f -normality can fail even under very low resource bounds. Concretely, we construct a nearly linear time computable separator enumerator f and a point $x \in [0, 1)$ such that $b_n(x) := k^n a_n^f(x)$ is k -adically equidistributed, yet $\dim_{\text{FS}}^f(x) = 0$.

Towards defining the enumerator, fix $\Sigma = \{0, 1, \dots, k - 1\}$ with $k = |\Sigma| \geq 2$ and set $x := 1/k$. For $n \geq 1$ define $J_n := k^{n-1} - n$ and $c_n := J_n/k^n = 1/k - n/k^n$. Then $0 \leq c_n < x$ for all n and (c_n) is nondecreasing since $c_{n+1} - c_n = ((k-1)n-1)/k^{n+1} \geq 0$.

Definition of the enumerator. Define $f : \Sigma^* \rightarrow [0, 1)$ as follows, where $n = |w|$.

$$f(w) := \begin{cases} c_n, & \text{if } w = 0^n, \\ \frac{\text{val}(w)}{k^n}, & \text{if } w \text{ starts with } 0, w \neq 0^n, \text{ val}(w) \leq J_n, \\ 0, & \text{if } w \text{ starts with } 0, w \neq 0^n, \text{ val}(w) > J_n, \\ x + \frac{1}{k^{n+1}}, & \text{if } w = 10^{n-1}, \\ \frac{\text{val}(w)}{k^n}, & \text{otherwise.} \end{cases}$$

In particular, clauses with leading digit 0 satisfy $f(w) \leq c_n < x$, while clauses with leading digit $\neq 0$ satisfy $f(w) > x$.

Lemma 9. *The function f is a rational-valued separator enumerator, and f is $O(n \log n)$ time computable.*

Proof. For density in $[0, x)$, fix $n \geq 1$. For each integer $1 \leq j \leq J_n$ there exists a length- n word w beginning with 0 with $\text{val}(w) = j$, and then $f(w) = j/k^n$ by clause (ii). Also $0 \in \text{Im}(f)$: for any n choose a length- n word w beginning with 0 with $\text{val}(w) > J_n$ (e.g. $w = 0(k-1)^{n-1}$), so clause (iii) gives $f(w) = 0$. Since $c_n = J_n/k^n \uparrow x$, it follows that $\bigcup_{n \geq 1} \{j/k^n : 0 \leq j \leq J_n\}$ is dense in $[0, x)$.

For density in $(x, 1)$, clause (v) realizes all grid points $\text{val}(w)/k^n$ with first digit nonzero, except that the single point $x = \text{val}(10^{n-1})/k^n$ is replaced by $x + 1/k^{n+1} > x$ by clause (iv). To see that the realized points are dense in $(x, 1)$, fix $\varepsilon > 0$ and choose $n \geq 2$ with $k^{-n} < \varepsilon$. Then $w := 10^{n-2}1$ has length n , begins with 1, and satisfies $f(w) = \text{val}(w)/k^n = x + 1/k^n \in (x, x + \varepsilon)$, so every interval $(x, x + \varepsilon)$ meets $\text{Im}(f)$. Since the grid points are dense in $(x, 1)$ and we only removed/perturbed one point on each level, it follows that $\text{Im}(f)$ meets every nonempty open interval in $(x, 1)$. Thus $\text{Im}(f)$ is dense in $[0, 1]$, so f is a separator enumerator.

We use the log-cost RAM model (see [10]), in which registers store nonnegative integers in binary and each instruction has cost proportional to the bit-lengths of the operands and addresses used; in particular, random access to the i -th input symbol costs $O(\log i)$. We represent the output rational $f(w)$ as a pair (p, m) meaning p/k^m , where p is written explicitly in base k as a digit string (so the output length is $O(|w|)$). In this model, f is computable in time $O(|w| \log |w|)$ by a direct case analysis. One first reads the input via accesses to $b_1, \dots, b_{|w|+1}$ in order to determine $|w|$ and a constant number of flags (whether $w = 0^{|w|}$, whether $w = 10^{|w|-1}$, and whether the first digit is 0); this costs $\sum_{i \leq |w|} O(\log i) = O(|w| \log |w|)$, and the auxiliary bookkeeping uses only $O(\log |w|)$ -bit counters. In the easy cases, the output (p, m) is produced by copying w or by writing the required base- k digit string for p (namely, the explicit strings for $k^{|w|} + 1$ or $J_{|w|}$), which takes $O(|w|)$ digit steps and hence $O(|w| \log |w|)$ time accounting for log-cost indexing/output. In the remaining case $w = 0u$ with $|u| = |w| - 1$, one computes the base- k digit string of $J_{|w|} = k^{|w|-1} - |w|$ and compares it

lexicographically with u ; both tasks take $O(|w|)$ digit steps, each incurring only $O(\log |w|)$ overhead from log-cost addressing/counters. Thus the total running time is $O(|w| \log |w|)$. \square

We next identify the induced best-from-below approximation chain at the special point $x = 1/k$.

Lemma 10. *For every $n \geq 1$, one has $a_n^f(x) = c_n$. Consequently, $b_n(x) = k^n a_n^f(x) = J_n = k^{n-1} - n$.*

Proof. Since $f(0^n) = c_n$ by clause (i) and $c_n = 1/k - n/k^n < x = 1/k$, we have $a_n^f(x) \geq c_n$. Now let $w \in \Sigma^n$ with $f(w) < x$. If w begins with a nonzero digit, then $f(w) > x$ by clause (iv) or (v), a contradiction; hence w begins with 0. If $w = 0^n$ then $f(w) = c_n$. If $w \neq 0^n$, then by clause (ii) or (iii) either $f(w) = \text{val}(w)/k^n$ with $\text{val}(w) \leq J_n$, or $f(w) = 0$; in the first case

$$f(w) = \frac{\text{val}(w)}{k^n} \leq \frac{J_n}{k^n} = c_n,$$

and in the second case $f(w) = 0 \leq c_n$. Thus every $w \in \Sigma^n$ with $f(w) < x$ satisfies $f(w) \leq c_n$, so $a_n^f(x) \leq c_n$. Therefore $a_n^f(x) = c_n$, and multiplying by k^n yields $b_n(x) = k^n c_n = J_n = k^{n-1} - n$. \square

We now verify that the sequence $(b_n(x))_{n \geq 1}$ is k -adically equidistributed.

Lemma 11. *The integer sequence $(b_n(x))_{n \geq 1}$ is k -adically equidistributed.*

Proof. Fix $m \geq 1$. For all $n \geq m + 1$ one has $k^m \mid k^{n-1}$, hence $b_n(x) = k^{n-1} - n \equiv -n \pmod{k^m}$ for all $n \geq m + 1$. As n ranges over $\{1, 2, \dots, N\}$, the residues of $-n \pmod{k^m}$ are asymptotically uniform, and omitting the finite initial segment $n \leq m$ does not affect limiting frequencies. Therefore, for every $r \in \{0, 1, \dots, k^m - 1\}$, $\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : b_n(x) \equiv r \pmod{k^m}\} = \frac{1}{k^m}$. \square

Finally, despite this equidistribution property, the point x has vanishing relativized finite-state dimension with respect to f .

Proposition 2. *For the above f and $x = 1/k$, one has $\dim_{\text{FS}}^f(x) = 0$.*

Proof. For each $L \geq 1$ let T_L be the 1-state Σ -FST that outputs 0^L on every input symbol. Then $T_L(\pi) = 0^{L|\pi|}$, so $K^{T_L}(0^n) \leq \lceil n/L \rceil$. Let $\delta_n := 2n/k^n$. Since $f(0^n) = c_n$ and $|x - c_n| = \frac{n}{k^n} < \delta_n$ we have $K_{\delta_n}^{T_L, f}(x) \leq K^{T_L}(0^n) \leq \lceil n/L \rceil$. Since $\log_k(1/\delta_n) = \log_k(k^n/(2n)) = n - \log_k(2n)$, we obtain

$$\liminf_{n \rightarrow \infty} \frac{K_{\delta_n}^{T_L, f}(x)}{\log_k(1/\delta_n)} \leq \liminf_{n \rightarrow \infty} \frac{\lceil n/L \rceil}{n - \log_k(2n)} = \frac{1}{L}.$$

Taking the infimum over transducers (in particular over T_L) and letting $L \rightarrow \infty$ yields $\dim_{\text{FS}}^f(x) = 0$. \square

Taking $x = 1/k$ and f as defined above we obtain the following.

Theorem 5. *There exist a rational-valued $O(n \log n)$ -time computable separator enumerator $f : \Sigma^* \rightarrow [0, 1)$ and a point $x \in [0, 1)$ such that the integer sequence $(k^n a_n^f(x))_{n \geq 1}$ is k -adically equidistributed, while $\dim_{\text{FS}}^f(x) = 0$.*

Proof. Take $x = 1/k$ and f as defined above. Lemma 10 identifies $k^n a_n^f(x) = b_n(x)$, Lemma 11 shows $(b_n(x))$ is k -adically equidistributed, and Proposition 2 gives $\dim_{\text{FS}}^f(x) = 0$. \square

5 Discussion and open questions

We show that, in general, no distributional property of the scaled best-from-below approximation sequence $(k^n a_n^f(x))_{n \geq 1}$ can characterize f -normality uniformly over all separator enumerators (indeed, we construct computable enumerators f_0, f_1 for which the associated scaled approximation sequences coincide while the corresponding f -normality behavior diverges). At the same time, we identify a structured regime—finite-state coherent enumerators—in which f -normality is equivalent to k -adic equidistribution of $(k^n a_n^f(x))_{n \geq 1}$. We also show that this correspondence can already fail for an efficiently computable separator enumerator, in fact for an $O(n \log n)$ -time computable f . This raises the natural question of how far this correspondence persists beyond finite-state coherent relabelings. One concrete setting is when the naming map is computable by a deterministic push-down transducer. In particular, does there exist such a separator enumerator f and a point $x \in [0, 1]$ for which the integer sequence $(k^n a_n^f(x))_{n \geq 1}$ is k -adically equidistributed while $\dim_{\text{FS}}^f(x) < 1$? It would also be interesting to identify the widest natural family of separator enumerators for which an equidistribution characterization of f -normality remains valid.

A related direction concerns the role of invertibility in the defining Mealy-machine relabeling: invertibility is used to transport approximation complexity via the substitution $u = M(w)$ and to ensure that $T \mapsto M \circ T$ is a bijection on finite-state transducers, yielding invariance of the induced f -dimension. It would be interesting to determine whether some weaker condition (e.g. levelwise surjectivity or bounded-to-one behavior on each Σ^n) suffices for the same equidistribution characterization, or whether non-invertible finite-state relabelings can already break it.

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