

# On Normality and Equidistribution for Separator Enumerators

Subin Pulari<sup>1</sup>

<sup>1</sup> National Research University Higher School of Economics Moscow

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## Abstract

A separator is a countable dense subset of  $[0, 1)$ , and a separator enumerator is a naming scheme that assigns a real number in  $[0, 1)$  to each finite word so that the set of all named values is a separator. Mayordomo introduced separator enumerators to define  $f$ -normality and a relativized finite-state dimension  $\dim_{\text{FS}}^f(x)$ , where finite-state dimension measures the asymptotic lower rate of finite-state information needed to approximate  $x$  through its  $f$ -names. This framework extends classical base- $k$  normality, and Mayordomo showed that it supports a point-to-set principle for finite-state dimension. This representation-based viewpoint has since been developed further in follow-up work, including by Calvert et al., yielding strengthened randomness notions such as supernormal and highly normal numbers.

Mayordomo posed the following open question: can  $f$ -normality be characterized via equidistribution properties of the sequence  $(|\Sigma|^n a_n^f(x))_{n=0}^\infty$ , where  $a_n^f(x)$  is the sequence of best approximations to  $x$  from below induced by  $f$ ? We give a strong negative answer: we construct computable separator enumerators  $f_0, f_1$  and a point  $x$  such that  $a_n^{f_0}(x) = a_n^{f_1}(x)$  for all  $n$ , yet  $\dim_{\text{FS}}^{f_0}(x) = 0$  while  $\dim_{\text{FS}}^{f_1}(x) = 1$ . Consequently, no criterion depending only on the sequence  $(|\Sigma|^n a_n^f(x))_{n=0}^\infty$  - in particular, no equidistribution property of this sequence - can characterize  $f$ -normality uniformly over all separator enumerators. On the other hand, for a natural finite-state coherent class of separator enumerators we recover a complete equidistribution characterization of  $f$ -normality. We also show that beyond finite-state coherence, this characterization can fail even for a separator enumerator computable in nearly linear time.

## 1 Introduction

Finite-state dimension is a quantitative notion of the rate of randomness in an individual infinite sequence, as measured by finite automata. It was introduced by Dai, Lathrop, Lutz, and Mayordomo [3] as a finite-state analogue of effective Hausdorff dimension [7, 6]. It admits several equivalent characterizations, including formulations via finite-state gambling, finite-state compression, and block entropy rates [3, 4, 1]. It also has an important connection with the theory of normal numbers: a real number is normal to base  $k$  if and only if its base- $k$  digit sequence has finite-state dimension equal to 1 [1].

An important connection between effective dimension and classical fractal dimension is provided by point-to-set principles. Lutz and Lutz [8] proved that the Hausdorff dimension in Euclidean spaces of a set can be obtained by minimizing, over oracles, the supremum of the relativized effective dimensions of its points. This reduces many lower-bound questions in geometric measure theory to analyzing the information density of carefully chosen points, and it has led to several new results and new proofs across classical fractal geometry; see [9]. In a recent work, Mayordomo [11] established

an analogous point-to-set principle for finite-state dimension. A key feature of this development is that it is representation-based: instead of fixing a base- $k$  expansion, it uses *separator enumerators*, i.e. naming schemes that assign reals in  $[0, 1)$  to finite words so that the range forms a countable dense subset. With such an enumerator  $f : \Sigma^* \rightarrow [0, 1)$ , one can measure how efficiently a finite-state transducer can output an  $f$ -name approximating a real  $x$  to a given precision, leading to a relativized finite-state dimension  $\dim_{\text{FS}}^f(x)$  and the induced notion of  $f$ -normality, defined by the condition  $\dim_{\text{FS}}^f(x) = 1$  [11]. Calvert et al. [2] develop this framework further, introducing strengthened notions such as supernormal and highly normal numbers under broad classes of representations.

In this paper we study the relationships between  $f$ -normality and equidistribution. In the classical base- $k$  setting, normality has a sharp equidistribution characterization:  $x$  is normal to base  $k$  if and only if the sequence  $(k^n x)_{n \geq 1}$  is uniformly distributed modulo 1 [5]. Motivated by the classical equivalence between normality and equidistribution, Mayordomo [11] asked whether an analogous equidistribution criterion holds in the setting of  $f$ -normality. Let  $(a_n^f(x))_{n \geq 0}$  denote the best-approximation-from-below sequence associated with  $f$  and  $x$ , i.e.  $a_n^f(x) = \max\{f(w) : |w| \leq n, f(w) \leq x\}$ . Mayordomo posed the following open question:

*Can  $f$ -normality be characterized via equidistribution properties of the sequence  $(|\Sigma|^n a_n^f(x))_{n \geq 0}$ ?*

We show that the answer is negative in a strong way. We construct two total computable rational-valued separator enumerators  $f_0, f_1$  and a point  $x \in [0, 1)$  such that  $a_n^{f_0}(x) = a_n^{f_1}(x)$  for all  $n$ , yet  $\dim_{\text{FS}}^{f_0}(x) = 0$  while  $\dim_{\text{FS}}^{f_1}(x) = 1$ . Consequently, no criterion depending only on the single numeric sequence  $(|\Sigma|^n a_n^f(x))_{n \geq 0}$ —in particular, no equidistribution property of that sequence—can characterize  $f$ -normality uniformly over all separator enumerators.

At the same time, an equidistribution characterization does hold under a natural structural restriction on the representation. We identify a class of *finite-state coherent* enumerators obtained from the standard base- $k$  grid by an invertible synchronous Mealy-machine relabeling, and we prove that in this regime  $f$ -normality is equivalent to a  $k$ -adic equidistribution property of the integer sequence  $(k^n a_n^f(x))_{n \geq 1}$  (uniform distribution of residues modulo  $k^m$  for every fixed  $m$ ). Beyond finite-state coherence, we show that this equivalence can already break for a separator enumerator  $f$  computable in  $O(n \log n)$  time: there is a point  $x$  whose scaled best-from-below approximations are  $k$ -adically equidistributed, but  $x$  is not  $f$ -normal.

Section 2 defines separator enumerators, finite-state transducers,  $\dim_{\text{FS}}^f$ , and  $f$ -normality. Section 3 proves the negative result via a pair of computable enumerators with identical best-from-below chains but sharply different relativized finite-state dimension. Section 4 establishes the  $k$ -adic equidistribution characterization for finite-state coherent enumerators, and we then show that this correspondence can already fail for a separator enumerator computable in nearly linear time.

## 2 Preliminaries

Let  $\Sigma$  be a finite alphabet with  $k = |\Sigma| \geq 2$  and write  $\Sigma^*$  for the set of finite strings over  $\Sigma$ . For  $w \in \Sigma^*$ ,  $|w|$  denotes its length, and for an infinite sequence  $X \in \Sigma^\infty$  we write  $X \upharpoonright n$  for its length- $n$  prefix (using the operator  $\upharpoonright$  defined in the preamble). We use  $\lfloor \cdot \rfloor$  for the floor function and interpret congruences  $b_n \equiv r \pmod{k^m}$  in the usual sense. Throughout this section we fix an identification  $\Sigma = \{0, 1, \dots, k-1\}$ . We view finite words over  $\Sigma$  as base- $k$  numerals and use the associated  $k$ -adic grid in  $[0, 1)$ . For a word  $u = u_1 u_2 \dots u_n \in \Sigma^n$ , define its base- $k$  value

$$\text{val}(u) = \sum_{i=1}^n u_i k^{n-i} \in \{0, 1, \dots, k^n - 1\}, \quad \text{and} \quad \text{grid}(u) = \frac{\text{val}(u)}{k^n} \in [0, 1).$$

Thus  $\text{grid}(\Sigma^n) = \{j/k^n : 0 \leq j < k^n\}$ . Let  $\text{seq}_k(x) \in \Sigma^\infty$  denote the (canonical) base- $k$  expansion of  $x \in [0, 1)$  chosen so as *not* to end in  $(k-1)^\infty$ .

Finite-state transducers model one-pass, constant-memory transformations on words and are used to measure finite-state description length.

**Definition 1** (Finite-state transducer (FST)). A  $\Sigma$ -finite-state transducer (briefly,  $\Sigma$ -FST) is a tuple  $T = (Q, \delta, \nu, q_0)$  where  $Q$  is a finite nonempty set of states,  $\delta : Q \times \Sigma \rightarrow Q$  is a transition function,  $\nu : Q \times \Sigma \rightarrow \Sigma^*$  is an output function, and  $q_0 \in Q$  is the start state.

The transition function extends to words by  $\delta(q, \lambda) = q$  and  $\delta(q, wa) = \delta(\delta(q, w), a)$ . For  $q \in Q$  and  $w \in \Sigma^*$ , define the output  $\nu(q, w) \in \Sigma^*$  by  $\nu(q, \lambda) = \lambda$  and  $\nu(q, wa) = \nu(q, w) \nu(\delta(q, w), a)$  for  $a \in \Sigma$ . The overall output of  $T$  on input  $w$  is  $T(w) = \nu(q_0, w)$ .

We measure how concisely a transducer can generate a given target string.

**Definition 2** ( $T$ -information content [11]). Let  $T$  be a  $\Sigma$ -FST and  $w \in \Sigma^*$ . Define

$$K^T(w) = \min\{|\pi| : \pi \in \Sigma^* \text{ and } T(\pi) = w\},$$

with the convention  $K^T(w) = \infty$  if  $w$  is not in the range of  $T$ .

A separator enumerator is a naming scheme for a countable dense subset of  $[0, 1)$ , assigning a real in  $[0, 1)$  to each finite word.

**Definition 3** (Separator, separator enumerator [11]). A set  $S \subseteq [0, 1)$  is a separator if it is countable and dense in  $[0, 1)$ . A function  $f : \Sigma^* \rightarrow [0, 1)$  is a separator enumerator (SE) if  $\text{Im}(f)$  is a separator.

We call an SE  $f : \Sigma^* \rightarrow [0, 1)$  *total computable* if there is an algorithm that, on input  $(w, t)$  with  $w \in \Sigma^*$  and  $t \in \mathbb{N}$ , outputs a rational  $q$  such that  $|q - f(w)| \leq 2^{-t}$ , and halts on every input. Given an enumerator  $f$ , we quantify how many input symbols a transducer needs in order to produce some  $f$ -name that  $\delta$ -approximates a target point.

**Definition 4** (Relativized approximation complexity [11]). Let  $f$  be an SE,  $T$  a  $\Sigma$ -FST,  $\delta > 0$ , and  $x \in [0, 1)$ . Define

$$K_\delta^{T,f}(x) = \min\{K^T(w) : w \in \Sigma^* \text{ and } |f(w) - x| < \delta\}.$$

The induced finite-state dimension is the optimal asymptotic approximation rate achievable by finite-state transducers.

**Definition 5** (Relativized finite-state dimension and  $f$ -normality [11]). Let  $f$  be an SE and  $x \in [0, 1)$ . Define

$$\dim_{\text{FS}}^f(x) = \inf_{T \in \Sigma\text{-FST}} \liminf_{\delta \rightarrow 0^+} \frac{K_\delta^{T,f}(x)}{\log_k(1/\delta)}.$$

We say  $x$  is  $f$ -normal if  $\dim_{\text{FS}}^f(x) = 1$ .

The standard base- $k$  naming map will serve as the baseline enumerator throughout. Let  $f_{\text{std}} : \Sigma^* \rightarrow [0, 1)$  be the standard base- $k$  enumerator

$$f_{\text{std}}(u) = \text{grid}(u) = \frac{\text{val}(u)}{k^{|u|}} \quad (u \neq \lambda),$$

with  $f_{\text{std}}(\lambda) = 0$ . We now define the induced *best approximation from below* chain associated with a separator enumerator.

**Definition 6** (Best approximation from below [11]). *Let  $f$  be an SE and  $x \in [0, 1]$ . For each  $n \in \mathbb{N}$ , define  $a_n^f(x)$  to be any value  $f(w)$  with  $|w| \leq n$  such that  $f(w) \leq x$  and  $x - f(w)$  is minimum among all  $u$  with  $|u| \leq n$  and  $f(u) \leq x$ . Equivalently,*

$$a_n^f(x) = \max\{f(w) : |w| \leq n, f(w) \leq x\}.$$

### 3 Equidistribution cannot characterize normality for separator enumerators

We now address Mayordomo's question on equidistribution criteria for  $f$ -normality. Fix a separator enumerator  $f : \Sigma^* \rightarrow [0, 1]$  and  $x \in [0, 1]$ , and let  $(a_n^f(x))_{n \geq 0}$  be the best approximation from below sequence corresponding to  $f$  and  $x$  (see Definition 6).

**Question 1** ([11]). *Can  $f$ -normality be characterized via equidistribution properties of the sequence  $(|\Sigma|^n a_n^f(x))_{n \geq 0}$ ?*

We answer this question in the negative by constructing two total computable rational-valued separator enumerators  $f_0, f_1$  and a point  $x \in [0, 1]$  such that  $a_n^{f_0}(x) = a_n^{f_1}(x)$  for all  $n$ , yet  $\dim_{\text{FS}}^{f_0}(x) = 0$  while  $\dim_{\text{FS}}^{f_1}(x) = 1$ .

#### 3.1 Preliminary lemmas

From the finite-state transducer characterization of finite-state dimension due to Doty and Moser (see Theorem 3.11 from [4]), we obtain the following corollary.

**Lemma 1.** *There exists an infinite sequence  $Z \in \Sigma^\infty$  such that for every  $\Sigma$ -FST  $T$ ,*

$$\liminf_{n \rightarrow \infty} \frac{K^T(z_n)}{n} = 1,$$

where  $z_n := Z \upharpoonright n$ .

*Proof.* Fix any  $\Sigma$ -normal sequence  $Z$ . By the transducer characterization of finite-state dimension and the fact that normal sequences have finite-state dimension equal to 1,

$$1 = \dim_{\text{FS}}(Z) = \inf_T \liminf_{n \rightarrow \infty} \frac{K^T(z_n)}{n}.$$

Therefore, for every  $\Sigma$ -FST  $T$  one must have  $\liminf_n K^T(z_n)/n = 1$ ; otherwise the infimum would be  $< 1$ .  $\square$

We also need a second sequence whose prefixes are distinct from the first but have the same property. This is achieved by a fixed symbol permutation.

**Lemma 2.** *Let  $\pi : \Sigma \rightarrow \Sigma$  be a bijection and extend it letterwise to  $\pi : \Sigma^* \rightarrow \Sigma^*$ . If  $Z \in \Sigma^\infty$  satisfies Lemma 1, then so does  $\pi(Z)$ , i.e. the prefixes  $t_n := \pi(Z) \upharpoonright n = \pi(z_n)$  satisfy  $\liminf_n K^T(t_n)/n = 1$  for every  $\Sigma$ -FST  $T$ .*

*Proof.* Fix a  $\Sigma$ -FST  $T$ . Let  $P$  be the 1-state transducer that maps each input symbol  $a$  to the single output symbol  $\pi(a)$ . Then  $P(w) = \pi(w)$  for all  $w$ , and similarly there is a 1-state transducer  $P^{-1}$  with  $P^{-1}(w) = \pi^{-1}(w)$  for all  $w$ . For any string  $u$ ,

$$K^T(\pi(u)) = K^T(P(u)) \geq K^{P^{-1} \circ T}(u),$$

because any input producing  $\pi(u)$  under  $T$  yields an input producing  $u$  under  $P^{-1} \circ T$ . Thus, for  $t_n = \pi(z_n)$ ,

$$\frac{K^T(t_n)}{n} \geq \frac{K^{P^{-1} \circ T}(z_n)}{n}.$$

Taking  $\liminf$  and applying Lemma 1 for the transducer  $P^{-1} \circ T$  we obtain  $\liminf_n K^T(t_n)/n = 1$ .  $\square$

Fix a bijection  $\pi : \Sigma \rightarrow \Sigma$  with no fixed points (such a derangement exists for all  $k \geq 2$ ) and with  $\pi(0) \neq 0$ . Let  $Z$  be as in Lemma 1, let  $Y = \pi(Z)$ , and write  $z_n := Z \upharpoonright n$  and  $y_n := Y \upharpoonright n = \pi(z_n)$ . Then  $z_n \neq y_n$  for all  $n$  because  $\pi$  has no fixed points, and  $y_n \neq 0^n$  for all  $n$  because  $\pi(0) \neq 0$ .

### 3.2 Construction of the two separator enumerators

Fix  $x := \frac{1}{2} \in (0, 1)$ . For each  $n \in \mathbb{N}$ , define  $r_n := k^{-(n+2)}$  and

$$A_n := (x - r_n, x - r_{n+1}) \cup (x + r_{n+1}, x + r_n).$$

Then  $A_n$  is a nonempty open set and  $(A_n)_{n \geq 0}$  partition the punctured neighborhood

$$(x - r_0, x + r_0) \setminus \{x\} = \bigsqcup_{n \geq 0} A_n$$

, and  $\log_k(1/r_n) = n + 2$ . Define the target *best-from-below* values

$$c_n := x - \frac{r_n + r_{n+1}}{2} \in (x - r_n, x - r_{n+1}) \subseteq A_n$$

. These values will later be enforced as the canonical length- $n$  approximants to  $x$  for both enumerators, i.e., we will arrange  $a_n^{f_i}(x) = c_n$  for each  $n$ . Observe that  $c_n < x$  and  $c_n \uparrow x$  as  $n \rightarrow \infty$ .

For each  $n$ , fix a computable listing  $(d_{n,t})_{t \in \mathbb{N}}$  of *distinct* rationals in  $A_n \cap \mathbb{Q} \setminus \{c_n\}$  whose range is dense in  $A_n$ , and set  $D_n := \{d_{n,t} : t \in \mathbb{N}\}$ . Also fix a computable listing  $(q_t)_{t \in \mathbb{N}}$  of *distinct* rationals in  $([0, 1] \setminus (x - r_0, x + r_0)) \cap \mathbb{Q}$  whose range is dense there, and set  $D_{\text{far}} := \{q_t : t \in \mathbb{N}\}$ . We now define two functions  $f_0, f_1 : \Sigma^* \rightarrow [0, 1]$ .

**Step 1 (fixing the best-from-below chain for lengths  $\leq n$ ).** For each  $n \geq 0$ , define

$$f_0(0^n) := c_n, \quad f_1(z_n) := c_n.$$

(Here  $0 \in \Sigma$  is a fixed symbol, and  $0^n$  is the all-0 word of length  $n$ .)

**Step 2 (ensuring density near  $x$  using incompressible prefixes).** Partition  $\mathbb{N}$  into pairwise disjoint infinite sets  $(L_n)_{n \geq 0}$  such that  $m \in L_n$  implies  $m \geq n + 1$ . For concreteness, one may take  $L_n = \{2^n(2t + 1) : t \in \mathbb{N}\}$ ; then  $m \in L_n$  implies  $m \geq 2^n \geq n + 1$  for  $n \geq 0$ . Fix computable bijections  $\varphi_n : L_n \rightarrow D_n$  by setting, for  $t \in \mathbb{N}$ ,

$$\varphi_n(2^n(2t + 1)) := d_{n,t}.$$

Now set, for each  $n \geq 0$  and each  $m \in L_n$ ,

$$f_0(y_m) = f_1(y_m) := \varphi_n(m) \in A_n.$$

Note that there is no conflict with Step 1, since  $y_m \neq 0^m$  and  $y_m \neq z_m$  for all  $m$ . Thus, for each  $n$ , the set  $\{f_i(y_m) : m \in L_n\}$  is dense in  $A_n$  (for both  $i = 0, 1$ ).

**Step 3 (defining all remaining values far from  $x$  while maintaining density).** Let  $W$  be the set of all remaining strings not yet assigned a value by Steps 1–2:

$$W := \Sigma^* \setminus \left( \{0^n : n \in \mathbb{N}\} \cup \{z_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\} \right).$$

Enumerate  $\Sigma^*$  in the length lexicographic order, then all words of length 2 in lexicographic order, and so on. Call this enumeration  $(u_j)_{j \in \mathbb{N}}$ . Define  $w_j$  to be the  $j$ th word in this list that lies in  $W$ , i.e., the  $j$ th  $u_t$  such that  $u_t \notin \{0^n : n \in \mathbb{N}\} \cup \{z_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$ . Also enumerate  $D_{\text{far}} = \{q_0, q_1, q_2, \dots\}$  according to the fixed computable listing above. Define  $f_0(w_j) = f_1(w_j) := q_j$  for all  $j$ .

**Lemma 3.** *The functions  $f_0, f_1$  defined above are total computable rational-valued separator enumerators: each  $\text{Im}(f_i)$  is countable and dense in  $[0, 1)$ .*

*Proof.* Countability is immediate since  $\Sigma^*$  is countable.

We also note that  $f_0$  and  $f_1$  are total computable (with rational outputs). Assume the fixed dense sets  $D_n$  and  $D_{\text{far}}$  come with fixed computable enumerations, and that each  $\varphi_n : L_n \rightarrow D_n \setminus \{c_n\}$  is a fixed computable bijection. Also fix  $Z$  to be a computable  $\Sigma$ -normal sequence, so  $n \mapsto z_n = Z \upharpoonright n$  is computable; then  $Y = \pi(Z)$  is computable and  $n \mapsto y_n = Y \upharpoonright n$  is computable.

On input  $w \in \Sigma^*$  of length  $\ell$ , we compute  $f_i(w)$  as follows. First check whether  $w = 0^\ell$  (all symbols equal 0), whether  $w = z_\ell$  (compute  $z_\ell$  and compare), and whether  $w = y_\ell$  (compute  $y_\ell$  and compare). If  $w = 0^\ell$  output  $f_0(w) = c_\ell$ ; if  $w = z_\ell$  output  $f_1(w) = c_\ell$ ; and if  $w = y_\ell$  then find the unique  $n$  with  $\ell \in L_n$  (the  $L_n$  are decidable and disjoint) and output  $f_0(w) = f_1(w) = \varphi_n(\ell)$ . Otherwise  $w \in W$ , and  $W$  is decidable because  $w \notin W$  iff one of the three checks above holds. In this case, compute the index  $j$  such that  $w = w_j$  in Step 3 by enumerating all words of  $\Sigma^*$  by increasing length, and within each fixed length in lexicographic order, skipping exactly those words that are not in  $W$ , until  $w$  is reached; then output  $f_0(w) = f_1(w) = q_j$  (where  $(q_j)$  is the fixed computable enumeration of  $D_{\text{far}}$ ). This procedure halts for every input  $w$ , so  $f_0, f_1$  are total computable.

For density, let  $I \subseteq [0, 1)$  be a nonempty open interval. If  $I$  intersects  $[0, 1) \setminus (x - r_0, x + r_0)$ , then  $I$  contains a rational in  $D_{\text{far}}$ , hence an image point of  $f_i$ . Otherwise,  $I \subseteq (x - r_0, x + r_0)$ , so  $I$  intersects  $A_n$  for some  $n \geq 0$  (because the annuli partition  $(x - r_0, x + r_0) \setminus \{x\}$  and  $I$  is open, hence cannot be  $\{x\}$ ). Since  $D_n$  is dense in  $A_n$  and  $\{f_i(y_m) : m \in L_n\} = \varphi_n(L_n)$  is dense in  $A_n$ , the interval  $I$  contains some  $f_i(y_m)$ . Thus  $\text{Im}(f_i)$  meets every nonempty open interval, so it is dense.  $\square$

Since  $f_0$  and  $f_1$  are rational-valued and the above procedure computes the defining case and the corresponding index effectively, it in fact yields exact computation: there is a Turing machine that, on input  $w \in \Sigma^*$ , halts and outputs the rational value  $f_i(w)$  itself (rather than merely producing  $2^{-t}$ -approximations). The next lemma shows that  $f_0$  and  $f_1$  induce exactly the same sequence of length-bounded approximations of  $x$  from below (and hence the same associated numeric scaling sequence).

**Lemma 4.** *For every  $n \in \mathbb{N}$ ,  $a_n^{f_0}(x) = a_n^{f_1}(x) = c_n$ . Consequently, the numeric sequences  $(k^n a_n^{f_0}(x))$  and  $(k^n a_n^{f_1}(x))$  are identical.*

*Proof.* Fix  $n$ . We first show  $a_n^{f_0}(x) = c_n$ . By definition,  $f_0(0^n) = c_n \leq x$ , so  $a_n^{f_0}(x) \geq c_n$ . Now consider any string  $w$  with  $|w| \leq n$  and  $f_0(w) \leq x$ . If  $w = 0^m$  for some  $m \leq n$ , then  $f_0(w) = c_m \leq c_n$  because  $(c_m)$  is increasing. If  $w = y_m$  for some  $m$ , then  $f_0(w) \in A_j$  for some  $j$  with  $m \in L_j$ . In particular,  $f_0(w) \leq x - r_{j+1} < x$ . Moreover, since  $m \in L_j$  implies  $m \geq j + 1$ ,

we have  $j \leq m - 1 \leq n - 1$  whenever  $m \leq n$ . Thus  $f_0(w) \leq x - r_{j+1} \leq x - r_n < c_n$  (because  $c_n > x - r_n$  by construction). Finally, if  $w \in W$  then  $f_0(w) \in D_{\text{far}} \subseteq [0, 1) \setminus (x - r_0, x + r_0)$ , so either  $f_0(w) \leq x - r_0 < x - r_n < c_n$  or  $f_0(w) \geq x + r_0 > x$ . In either case it cannot exceed  $c_n$  while staying  $\leq x$ . Therefore, among all  $|w| \leq n$  with  $f_0(w) \leq x$ , the maximum is attained at  $w = 0^n$  with value  $c_n$ . Hence  $a_n^{f_0}(x) = c_n$ .

The proof for  $f_1$  is identical, replacing the witness  $0^n$  by  $z_n$  (since  $f_1(z_n) = c_n$ ) and noting that no other string of length  $\leq n$  attains a value in  $(c_n, x]$  by the same case analysis. Thus  $a_n^{f_1}(x) = c_n$ .  $\square$

We now complete the construction by showing that, for the fixed point  $x = \frac{1}{2}$ , the two separator enumerators  $f_0$  and  $f_1$  constructed above induce different relativized finite-state dimensions:  $\dim_{\text{FS}}^{f_0}(x) = 0$  while  $\dim_{\text{FS}}^{f_1}(x) = 1$ . Together with Lemma 4, this yields a negative answer to Mayordomo's open question [11].

**Theorem 1.** *There exist total computable rational-valued separator enumerators  $f_0, f_1 : \Sigma^* \rightarrow [0, 1)$  and a point  $x \in [0, 1)$  such that*

$$a_n^{f_0}(x) = a_n^{f_1}(x) \text{ for all } n,$$

yet

$$\dim_{\text{FS}}^{f_0}(x) = 0 \quad \text{and} \quad \dim_{\text{FS}}^{f_1}(x) = 1.$$

In particular,  $x$  is not  $f_0$ -normal, while  $x$  is  $f_1$ -normal.

*Proof.* We use the constructions above with the fixed  $x = \frac{1}{2}$ . By Lemma 4, the sequences  $a_n^{f_0}(x)$  and  $a_n^{f_1}(x)$  coincide (both equal  $c_n$ ), so  $(k^n a_n(x))$  is identical for  $f_0$  and  $f_1$ . It remains to compute the relativized finite-state dimensions.

**Part 1: We show  $\dim_{\text{FS}}^{f_0}(x) = 0$ .** For each integer  $L \geq 1$ , let  $T_L$  be the 1-state transducer that outputs  $0^L$  on every input symbol. Then  $T_L(\pi) = 0^{L|\pi|}$  for all  $\pi$ , hence

$$K^{T_L}(0^n) \leq \left\lceil \frac{n}{L} \right\rceil.$$

Fix  $n$  and consider  $\delta_n := r_n/2$  (so  $\log_k(1/\delta_n) = n + 2 + \log_k 2$ ). Since  $f_0(0^n) = c_n$  and  $|x - c_n| = |x - (x - (r_n + r_{n+1})/2)| = (r_n + r_{n+1})/2 < r_n = 2\delta_n$ , we have  $|f_0(0^n) - x| < 2\delta_n$ . Therefore

$$K_{2\delta_n}^{T_L, f_0}(x) \leq K^{T_L}(0^n) \leq \left\lceil \frac{n}{L} \right\rceil.$$

Hence

$$\liminf_{n \rightarrow \infty} \frac{K_{2\delta_n}^{T_L, f_0}(x)}{\log_k(1/(2\delta_n))} \leq \liminf_{n \rightarrow \infty} \frac{\lceil n/L \rceil}{(n+2) + O(1)} = \frac{1}{L}.$$

To justify that this controls the full  $\liminf_{\delta \rightarrow 0+}$  (and not only the subsequence  $2\delta_n$ ), note that  $K_{\delta}^{T_L, f_0}(x)$  is monotone non-increasing in  $\delta$ . Hence for any  $\delta \in (2\delta_{n+1}, 2\delta_n]$ ,

$$K_{\delta}^{T_L, f_0}(x) \leq K_{2\delta_{n+1}}^{T_L, f_0}(x).$$

Moreover, for such  $\delta$  we have  $\log_k(1/\delta) \geq \log_k(1/(2\delta_n))$ . Therefore,

$$\frac{K_{\delta}^{T_L, f_0}(x)}{\log_k(1/\delta)} \leq \frac{K_{2\delta_{n+1}}^{T_L, f_0}(x)}{\log_k(1/(2\delta_n))}.$$

Taking  $\liminf$  over all  $\delta \rightarrow 0^+$  and using that  $n \rightarrow \infty$  along the corresponding intervals yields

$$\begin{aligned} \liminf_{\delta \rightarrow 0^+} \frac{K_\delta^{T_L, f_0}(x)}{\log_k(1/\delta)} &\leq \liminf_{n \rightarrow \infty} \frac{K_{2\delta_{n+1}}^{T_L, f_0}(x)}{\log_k(1/(2\delta_{n+1}))} \cdot \frac{\log_k(1/(2\delta_{n+1}))}{\log_k(1/(2\delta_n))} \\ &\leq \liminf_{n \rightarrow \infty} \frac{K_{2\delta_{n+1}}^{T_L, f_0}(x)}{\log_k(1/(2\delta_{n+1}))} \cdot \lim_{n \rightarrow \infty} \frac{n+3+O(1)}{n+2+O(1)} \\ &\leq \frac{1}{L}. \end{aligned}$$

Since the above holds for all  $T_L$ , taking the infimum over all transducers  $T$  and then letting  $L \rightarrow \infty$  yields  $\dim_{\text{FS}}^{f_0}(x) = 0$ .

**Part 2: We show  $\dim_{\text{FS}}^{f_1}(x) = 1$ .** Fix an arbitrary  $\Sigma$ -FST  $T$ . We prove that  $\liminf_{\delta \rightarrow 0^+} \frac{K_\delta^{T, f_1}(x)}{\log_k(1/\delta)} \geq 1$ , which implies  $\dim_{\text{FS}}^{f_1}(x) = 1$  after taking the infimum over  $T$ .

Consider the scale  $\delta_n := r_n$ . Any value  $f_1(w)$  within distance  $\delta_n = r_n$  of  $x$  must lie in  $(x - r_n, x + r_n) = \{x\} \cup \bigsqcup_{j \geq n} A_j$ , so  $w$  must be either (i) one of the special words  $z_j$  with  $j \geq n$  (since  $f_1(z_j) = c_j \in A_j$ ), or (ii) one of the special words  $y_m$  with  $f_1(y_m) \in A_j$  for some  $j \geq n$  (these are the values placed densely in the annuli), because by construction all other words are mapped into  $D_{\text{far}}$  outside  $(x - r_0, x + r_0)$ . Therefore,

$$K_{r_n}^{T, f_1}(x) \geq \min \left( \min_{j \geq n} K^T(z_j), \min_{m: f_1(y_m) \in \bigcup_{j \geq n} A_j} K^T(y_m) \right).$$

We now lower bound each term asymptotically by  $n$ . By Lemma 1, we have  $\liminf_{j \rightarrow \infty} K^T(z_j)/j = 1$ . Hence for every  $\varepsilon > 0$  there exists  $J$  such that for all  $j \geq J$ ,  $K^T(z_j) \geq (1-\varepsilon)j$ ; consequently, for all  $n \geq J$ ,  $\min_{j \geq n} K^T(z_j) \geq (1-\varepsilon)n$ . Similarly, by Lemma 2 applied to  $Y$ ,  $\liminf_{m \rightarrow \infty} K^T(y_m)/m = 1$ , so for every  $\varepsilon > 0$  there exists  $M$  such that for all  $m \geq M$ ,  $K^T(y_m) \geq (1-\varepsilon)m$ . In our construction, if  $f_1(y_m) \in A_j$ , then  $m \in L_j$ , and by design  $m \geq j+1$ ; therefore, whenever  $f_1(y_m) \in \bigcup_{j \geq n} A_j$ , we have  $m \geq n+1$ . For all  $n \geq M$ ,

$$\min_{m: f_1(y_m) \in \bigcup_{j \geq n} A_j} K^T(y_m) \geq (1-\varepsilon)(n+1) \geq (1-\varepsilon)n.$$

Putting the two bounds together, for all sufficiently large  $n$ ,  $K_{r_n}^{T, f_1}(x) \geq (1-\varepsilon)n$ . Since  $\log_k(1/r_n) = n+2$ , we obtain

$$\liminf_{n \rightarrow \infty} \frac{K_{r_n}^{T, f_1}(x)}{\log_k(1/r_n)} \geq \liminf_{n \rightarrow \infty} \frac{(1-\varepsilon)n}{n+2} = 1-\varepsilon.$$

As  $\varepsilon > 0$  was arbitrary, this yields  $\liminf_{n \rightarrow \infty} \frac{K_{r_n}^{T, f_1}(x)}{\log_k(1/r_n)} \geq 1$ . By monotonicity of  $K_\delta^{T, f_1}(x)$  in  $\delta$  and the fact that any  $\delta \in (r_{n+1}, r_n]$  satisfies  $\log_k(1/\delta) \in [n+2, n+3)$ , it follows that the full  $\liminf_{\delta \rightarrow 0^+}$  is also at least 1. Thus

$$\liminf_{\delta \rightarrow 0^+} \frac{K_\delta^{T, f_1}(x)}{\log_k(1/\delta)} \geq 1$$

for every  $T$ , which implies that  $\dim_{\text{FS}}^{f_1}(x) = 1$ . □

As an immediate consequence,  $f$ -normality cannot be characterized by any property of the single numerical sequence  $(k^n a_n^f(x))$ .



**Corollary 1.** *There is no property  $P$  of the numeric sequence  $(k^n a_n^f(x))_{n \in \mathbb{N}}$  (in particular, no equidistribution property of this sequence) such that for all separator enumerators  $f$  and all  $x \in [0, 1)$ ,*

$$x \text{ is } f\text{-normal} \iff (k^n a_n^f(x)) \text{ has property } P.$$

*Proof.* Take  $f_0, f_1, x$  from Theorem 1. By Lemma 4, the sequences  $(k^n a_n^{f_0}(x))$  and  $(k^n a_n^{f_1}(x))$  are identical, so they either both satisfy  $P$  or both fail  $P$ . But by Theorem 1,  $x$  is  $f_1$ -normal and not  $f_0$ -normal, so no such  $P$  can exist.  $\square$

## 4 Finite-State Coherent Enumerators and an Equidistribution Characterization of $f$ -Normality

From the previous section, we know that no equidistribution (or other distributional) criterion depending only on the sequence  $(k^n a_n^f(x))_{n \geq 1}$  can characterize  $f$ -normality uniformly over all separator enumerators. The goal of this section is to isolate a natural structural regime in which such a characterization *does* hold. We do this by restricting to *finite-state coherent enumerators*, i.e. naming schemes obtained from the standard base- $k$  grid by an invertible synchronous Mealy-machine relabeling.

To state an equidistribution characterization in this setting, we must use a notion that remains meaningful for the standard enumerator. For the standard base- $k$  naming map  $f_{\text{std}}(w) = \text{val}(w)/k^{|w|}$ , the scaled approximation sequence  $b_n(x) := k^n a_n^{f_{\text{std}}}(x)$  is integer-valued (indeed  $b_n(x) = \lfloor k^n x \rfloor$ ), so the usual notion of equidistribution modulo 1 becomes trivial. The natural replacement is to ask for uniform distribution of the residues of  $b_n(x)$  at every finite base- $k$  resolution, i.e. modulo  $k^m$  for each fixed  $m$ . This leads to the following notion of  $k$ -adic equidistribution, which we adopt in the rest of the paper.

**Definition 7** ( $k$ -adic equidistribution). *A sequence  $(b_n)_{n \geq 1}$  of integers is  $k$ -adically equidistributed if for every  $m \geq 1$  and every residue  $r \in \{0, 1, \dots, k^m - 1\}$ ,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : b_n \equiv r \pmod{k^m}\} = \frac{1}{k^m}.$$

For finite-state coherent enumerators, the best-from-below approximants admit an explicit closed form, and the induced  $f$ -dimension is invariant under finite-state coherent relabelings. These two facts combine to yield a clean equidistribution characterization of  $f$ -normality in terms of  $k$ -adic equidistribution of the integer sequence  $(k^n a_n^f(x))_{n \geq 1}$ .

### 4.1 Finite-state coherent enumerators

We begin by defining invertible synchronous Mealy machines.

**Definition 8** (Invertible synchronous Mealy machine). *An invertible synchronous Mealy machine is a tuple  $M = (Q, \delta, \lambda, q_0)$  where  $Q$  is a finite nonempty set of states,  $\delta : Q \times \Sigma \rightarrow Q$  is a transition function, and  $\lambda : Q \times \Sigma \rightarrow \Sigma$  is an output function such that for every state  $q \in Q$ , the map  $a \mapsto \lambda(q, a)$  is a permutation of  $\Sigma$ .*

*The induced map  $M : \Sigma^* \rightarrow \Sigma^*$  is defined by reading left-to-right: set  $M(\lambda) = \lambda$ , and for  $w = w_1 \cdots w_n$  define  $u = M(w) = u_1 \cdots u_n$  by the recursion*

$$q_i = \delta(q_{i-1}, w_i), \quad u_i = \lambda(q_{i-1}, w_i) \quad (i = 1, \dots, n),$$

*with  $q_0$  as the initial state. In particular,  $|M(w)| = |w|$  for all  $w$ .*

We state the basic properties of invertible synchronous Mealy machines.

**Lemma 5.** *Let  $M$  be an invertible synchronous Mealy machine.*

1. *For every  $n$ , the restriction  $M : \Sigma^n \rightarrow \Sigma^n$  is a bijection.*
2. *There exists an invertible synchronous Mealy machine  $M^{-1}$  such that for all  $w \in \Sigma^*$ , we have  $M^{-1}(M(w)) = w$ .*
3.  *$M$  extends letter-by-letter (via the same recursion as in the definition of  $M$ ) to a bijection  $M : \Sigma^\infty \rightarrow \Sigma^\infty$  with inverse  $M^{-1}$ .*

*Proof.* (1) Fix  $n$ . Because at each step the output letter is obtained by applying a permutation depending on the current state, distinct inputs cannot merge: if  $w \neq w'$  then at the first position  $i$  where they differ, the machine is in the same state (because it has read the same prefix) and applies a permutation to two different letters, hence outputs different letters at position  $i$ . Thus  $M$  is injective on  $\Sigma^n$ , hence bijective because  $\Sigma^n$  is finite.

(2) One constructs  $M^{-1}$  by reversing the per-state permutations: in state  $q$  output  $\lambda(q, \cdot)^{-1}(a)$  on input  $a$ , and update the state consistently (standard Mealy-machine inversion). Because all per-state maps are permutations, this is well-defined.

(3) The same recursion as in definition of  $M$  works on infinite inputs; bijectivity follows from (1) on all finite prefixes.  $\square$

Now we formalize the finite-state coherence condition, which captures length-preserving, bounded-memory relabelings of the standard base- $k$  grid.

**Definition 9** (Finite-state coherent enumerator). *A function  $f : \Sigma^* \rightarrow [0, 1)$  is finite-state coherent if there exists an invertible synchronous Mealy machine  $M$  such that for every nonempty  $w \in \Sigma^n$ ,  $f(w) = \text{grid}(M(w)) = \frac{\text{val}(M(w))}{k^n}$ . (For definiteness, set  $f(\lambda) = 0$ .)*

Finite-state coherence immediately forces  $f$  to enumerate the entire standard  $k$ -adic grid at each length.

**Lemma 6.** *If  $f$  is finite-state coherent, then  $\text{Im}(f) = \bigcup_{n \geq 1} \{j/k^n : 0 \leq j < k^n\}$ , hence  $f$  is a separator enumerator.*

*Proof.* Fix  $n \geq 1$ . By Lemma 5(1),  $M(\Sigma^n) = \Sigma^n$ . Therefore

$$f(\Sigma^n) = \text{grid}(M(\Sigma^n)) = \text{grid}(\Sigma^n) = \left\{ \frac{j}{k^n} : 0 \leq j < k^n \right\}.$$

Taking the union over  $n$  gives the claimed image, which is countable and dense in  $[0, 1)$ .  $\square$

For finite-state coherent enumerators, the best-from-below approximation sequence coincides with the usual base- $k$  truncations.

**Lemma 7.** *Let  $f$  be finite-state coherent and  $x \in [0, 1)$ . Then for every  $n \geq 1$ ,  $a_n^f(x) = \frac{\lfloor k^n x \rfloor}{k^n}$ . In particular, the scaled sequence is integer-valued:  $k^n a_n^f(x) = \lfloor k^n x \rfloor \in \{0, 1, \dots, k^n - 1\}$ .*

*Proof.* By Lemma 6, for each  $m \leq n$  the set  $f(\Sigma^m)$  equals the full grid  $\{j/k^m : 0 \leq j < k^m\}$ . Hence the set of all values  $f(w)$  with  $|w| \leq n$  is exactly  $\bigcup_{m=1}^n \{j/k^m : 0 \leq j < k^m\}$ . Among the level- $n$  grid points  $\{j/k^n\}$ , the largest one  $\leq x$  is  $\lfloor k^n x \rfloor / k^n$ . It remains to check that no coarser grid point (denominator  $k^m$  with  $m < n$ ) can exceed this value while staying  $\leq x$ . But for each  $m < n$ ,  $\frac{\lfloor k^m x \rfloor}{k^m} \leq \frac{\lfloor k^n x \rfloor}{k^n}$ , because multiplying both sides by  $k^n$  gives  $k^{n-m} \lfloor k^m x \rfloor \leq \lfloor k^n x \rfloor$ , which holds since  $k^{n-m} \lfloor k^m x \rfloor \leq k^n x$  and the left-hand side is an integer.  $\square$

Next we relate approximation complexity under a finite-state coherent  $f$  to approximation complexity under the standard base- $k$  enumerator. Recall that  $f_{\text{std}} : \Sigma^* \rightarrow [0, 1)$  is the standard base- $k$  enumerator.

**Lemma 8.** *Let  $f$  be finite-state coherent via a Mealy machine  $M$ , i.e.  $f(w) = \text{grid}(M(w))$  for all  $w$ . Then for every  $\Sigma$ -FST  $T$ , every  $x \in [0, 1)$ , and every  $\delta > 0$ ,  $K_\delta^{T,f}(x) = K_\delta^{M \circ T, f_{\text{std}}}(x)$ , where  $M \circ T$  denotes the output-composition transducer  $\pi \mapsto M(T(\pi))$ .*

*Proof.* Since  $\text{grid}(u) = f_{\text{std}}(u)$  for every  $u \in \Sigma^*$ , we obtain  $K_\delta^{T,f}(x) = \min\{K^T(w) : |f_{\text{std}}(M(w)) - x| < \delta\}$ . Substitute  $u = M(w)$ . Since  $M : \Sigma^{|w|} \rightarrow \Sigma^{|w|}$  is bijective for each length, this is equivalent to  $K_\delta^{T,f}(x) = \min\{K^T(M^{-1}(u)) : |f_{\text{std}}(u) - x| < \delta\}$ . For every  $u \in \Sigma^*$ ,  $K^T(M^{-1}(u)) = \min\{|\pi| : T(\pi) = M^{-1}(u)\} = \min\{|\pi| : M(T(\pi)) = u\} = K^{M \circ T}(u)$ , and substituting yields the claim.  $\square$

The next proposition formalizes the key robustness property of finite-state coherence: composing the naming map with an invertible synchronous Mealy relabeling does not change the relativized finite-state approximation complexity, and hence does not change the induced  $f$ -dimension.

**Proposition 1.** *If  $f$  is finite-state coherent, then for every  $x \in [0, 1)$ ,  $\dim_{\text{FS}}^f(x) = \dim_{\text{FS}}^{f_{\text{std}}}(x)$ . In particular,  $x$  is  $f$ -normal if and only if  $x$  is  $f_{\text{std}}$ -normal.*

*Proof.* Let  $f$  be finite-state coherent via  $M$ . By Lemma 8, for every  $T$  and every  $\delta$ ,  $K_\delta^{T,f}(x) = K_\delta^{M \circ T, f_{\text{std}}}(x)$ , hence the corresponding  $\liminf$  ratios coincide. Taking  $\inf_T$  over all FSTs on the left equals taking  $\inf_S$  over all FSTs on the right, because  $T \mapsto M \circ T$  is a bijection on FSTs (with inverse  $S \mapsto M^{-1} \circ S$ ). Therefore the two infima coincide.  $\square$

The next theorem states that, for the *standard* base- $k$  naming map  $f_{\text{std}}$ , the paper's notion of  $f$ -normality (i.e.  $\dim_{\text{FS}}^f(x) = 1$ ) coincides exactly with the classical notion of base- $k$  normality of  $x$ .

**Theorem 2.** *For  $x \in [0, 1)$ , one has  $\dim_{\text{FS}}^{f_{\text{std}}}(x) = 1$  if and only if  $x$  is base- $k$  normal.*

*Proof.* By Theorem 3.3 of [11], for every  $x \in [0, 1)$ ,  $\dim_{\text{FS}}^{f_{\text{std}}}(x) = \dim_{\text{FS}}(\text{seq}_k(x))$ . By the standard characterization of normality via finite-state dimension (e.g. [1]), one has  $\dim_{\text{FS}}(\text{seq}_k(x)) = 1$  if and only if  $\text{seq}_k(x)$  is base- $k$  normal. Finally, by definition,  $\text{seq}_k(x)$  is base- $k$  normal if and only if  $x$  is base- $k$  normal. Combining these equivalences yields the claim.  $\square$

It is straightforward to verify that  $k$ -adic equidistribution of the sequence  $(\lfloor k^n x \rfloor)_{n \geq 1}$  coincides with base- $k$  normality [5].

**Theorem 3.** *Let  $x \in [0, 1)$  and set  $b_n = \lfloor k^n x \rfloor$ . Then  $x$  is base- $k$  normal if and only if  $(b_n)_{n \geq 1}$  is  $k$ -adically equidistributed.*

Finally we combine finite-state coherent invariance with the explicit form of  $a_n^f(x)$  to obtain the equidistribution characterization in terms of the scaled approximation sequence.

**Theorem 4.** *Let  $f$  be a finite-state coherent separator enumerator over  $\Sigma$  and let  $x \in [0, 1)$ . Then  $x$  is  $f$ -normal if and only if the integer sequence  $(k^n a_n^f(x))_{n \geq 1}$  is  $k$ -adically equidistributed.*

*Proof.* Let  $b_n(x) := k^n a_n^f(x)$ . Since  $f$  is finite-state coherent, Proposition 1 gives  $x$  is  $f$ -normal if and only if  $x$  is  $f_{\text{std}}$ -normal. By Theorem 2,  $f_{\text{std}}$ -normality is equivalent to base- $k$  normality of  $x$ . On the other hand, Theorem 3 states that base- $k$  normality of  $x$  is equivalent to  $k$ -adic equidistribution of the integer sequence  $(\lfloor k^n x \rfloor)_{n \geq 1}$ . Finally, Lemma 7 identifies the scaled approximation sequence for finite-state coherent  $f$  with this canonical sequence, namely  $b_n(x) = \lfloor k^n x \rfloor$  for all  $n \geq 1$ . Substituting this identity into the previous equivalence yields  $x$  is  $f$ -normal if and only if  $(b_n(x))_{n \geq 1}$  is  $k$ -adically equidistributed, which is exactly the claim.  $\square$

## 4.2 Beyond finite-state coherence: a nearly linear time computable counterexample

This subsection shows that once one moves beyond finite-state coherent enumerators, a *k-adic equidistribution*-based criterion for *f*-normality can fail even under very low resource bounds. Concretely, we construct a nearly linear time computable separator enumerator *f* and a point  $x \in [0, 1)$  such that  $b_n(x) := k^n a_n^f(x)$  is *k*-adically equidistributed, yet  $\dim_{\text{FS}}^f(x) = 0$ .

Towards defining the enumerator, fix  $\Sigma = \{0, 1, \dots, k-1\}$  with  $k = |\Sigma| \geq 2$  and set  $x := 1/k$ . For  $n \geq 1$  define  $J_n := k^{n-1} - n$  and  $c_n := J_n/k^n = 1/k - n/k^n$ . Then  $0 \leq c_n < x$  for all  $n$  and  $(c_n)$  is nondecreasing since  $c_{n+1} - c_n = ((k-1)n - 1)/k^{n+1} \geq 0$ .

**Definition of the enumerator.** Define  $f : \Sigma^* \rightarrow [0, 1)$  as follows, where  $n = |w|$ .

$$f(w) := \begin{cases} c_n, & \text{if } w = 0^n, \\ \frac{\text{val}(w)}{k^n}, & \text{if } w \text{ starts with } 0, w \neq 0^n, \text{val}(w) \leq J_n, \\ 0, & \text{if } w \text{ starts with } 0, w \neq 0^n, \text{val}(w) > J_n, \\ x + \frac{1}{k^{n+1}}, & \text{if } w = 10^{n-1}, \\ \frac{\text{val}(w)}{k^n}, & \text{otherwise.} \end{cases}$$

In particular, clauses with leading digit 0 satisfy  $f(w) \leq c_n < x$ , while clauses with leading digit  $\neq 0$  satisfy  $f(w) > x$ .

**Lemma 9.** *The function  $f$  is a rational-valued separator enumerator, and  $f$  is  $O(n \log n)$  time computable.*

*Proof.* For density in  $[0, x)$ , fix  $n \geq 1$ . For each integer  $1 \leq j \leq J_n$  there exists a length- $n$  word  $w$  beginning with 0 with  $\text{val}(w) = j$ , and then  $f(w) = j/k^n$  by clause (ii). Also  $0 \in \text{Im}(f)$ : for any  $n$  choose a length- $n$  word  $w$  beginning with 0 with  $\text{val}(w) > J_n$  (e.g.  $w = 0(k-1)^{n-1}$ ), so clause (iii) gives  $f(w) = 0$ . Since  $c_n = J_n/k^n \uparrow x$ , it follows that  $\bigcup_{n \geq 1} \{j/k^n : 0 \leq j \leq J_n\}$  is dense in  $[0, x)$ .

For density in  $(x, 1)$ , clause (v) realizes all grid points  $\text{val}(w)/k^n$  with first digit nonzero, except that the single point  $x = \text{val}(10^{n-1})/k^n$  is replaced by  $x + 1/k^{n+1} > x$  by clause (iv). To see that the realized points are dense in  $(x, 1)$ , fix  $\varepsilon > 0$  and choose  $n \geq 2$  with  $k^{-n} < \varepsilon$ . Then  $w := 10^{n-2}1$  has length  $n$ , begins with 1, and satisfies  $f(w) = \text{val}(w)/k^n = x + 1/k^n \in (x, x + \varepsilon)$ , so every interval  $(x, x + \varepsilon)$  meets  $\text{Im}(f)$ . Since the grid points are dense in  $(x, 1)$  and we only removed/perturbed one point on each level, it follows that  $\text{Im}(f)$  meets every nonempty open interval in  $(x, 1)$ . Thus  $\text{Im}(f)$  is dense in  $[0, 1)$ , so  $f$  is a separator enumerator.

We use the log-cost RAM model (see [10]), in which registers store nonnegative integers in binary and each instruction has cost proportional to the bit-lengths of the operands and addresses used; in particular, random access to the  $i$ -th input symbol costs  $O(\log i)$ . We represent the output rational  $f(w)$  as a pair  $(p, m)$  meaning  $p/k^m$ , where  $p$  is written explicitly in base  $k$  as a digit string (so the output length is  $O(|w|)$ ). In this model,  $f$  is computable in time  $O(|w| \log |w|)$  by a direct case analysis. One first reads the input via accesses to  $b_1, \dots, b_{|w|+1}$  in order to determine  $|w|$  and a constant number of flags (whether  $w = 0^{|w|}$ , whether  $w = 10^{|w|-1}$ , and whether the first digit is 0); this costs  $\sum_{i \leq |w|} O(\log i) = O(|w| \log |w|)$ , and the auxiliary bookkeeping uses only  $O(\log |w|)$ -bit counters. In the easy cases, the output  $(p, m)$  is produced by copying  $w$  or by writing the required base- $k$  digit string for  $p$  (namely, the explicit strings for  $k^{|w|} + 1$  or  $J_{|w|}$ ), which takes  $O(|w|)$  digit steps and hence  $O(|w| \log |w|)$  time accounting for log-cost indexing/output. In the remaining case  $w = 0u$  with  $|u| = |w| - 1$ , one computes the base- $k$  digit string of  $J_{|w|} = k^{|w|-1} - |w|$  and compares it

lexicographically with  $u$ ; both tasks take  $O(|w|)$  digit steps, each incurring only  $O(\log |w|)$  overhead from log-cost addressing/counters. Thus the total running time is  $O(|w| \log |w|)$ .  $\square$

We next identify the induced best-from-below approximation chain at the special point  $x = 1/k$ .

**Lemma 10.** *For every  $n \geq 1$ , one has  $a_n^f(x) = c_n$ . Consequently,  $b_n(x) = k^n a_n^f(x) = J_n = k^{n-1} - n$ .*

*Proof.* Since  $f(0^n) = c_n$  by clause (i) and  $c_n = 1/k - n/k^n < x = 1/k$ , we have  $a_n^f(x) \geq c_n$ . Now let  $w \in \Sigma^n$  with  $f(w) < x$ . If  $w$  begins with a nonzero digit, then  $f(w) > x$  by clause (iv) or (v), a contradiction; hence  $w$  begins with 0. If  $w = 0^n$  then  $f(w) = c_n$ . If  $w \neq 0^n$ , then by clause (ii) or (iii) either  $f(w) = \text{val}(w)/k^n$  with  $\text{val}(w) \leq J_n$ , or  $f(w) = 0$ ; in the first case

$$f(w) = \frac{\text{val}(w)}{k^n} \leq \frac{J_n}{k^n} = c_n,$$

and in the second case  $f(w) = 0 \leq c_n$ . Thus every  $w \in \Sigma^n$  with  $f(w) < x$  satisfies  $f(w) \leq c_n$ , so  $a_n^f(x) \leq c_n$ . Therefore  $a_n^f(x) = c_n$ , and multiplying by  $k^n$  yields  $b_n(x) = k^n c_n = J_n = k^{n-1} - n$ .  $\square$

We now verify that the sequence  $(b_n(x))_{n \geq 1}$  is  $k$ -adically equidistributed.

**Lemma 11.** *The integer sequence  $(b_n(x))_{n \geq 1}$  is  $k$ -adically equidistributed.*

*Proof.* Fix  $m \geq 1$ . For all  $n \geq m + 1$  one has  $k^m \mid k^{n-1}$ , hence  $b_n(x) = k^{n-1} - n \equiv -n \pmod{k^m}$  for all  $n \geq m + 1$ . As  $n$  ranges over  $\{1, 2, \dots, N\}$ , the residues of  $-n \pmod{k^m}$  are asymptotically uniform, and omitting the finite initial segment  $n \leq m$  does not affect limiting frequencies. Therefore, for every  $r \in \{0, 1, \dots, k^m - 1\}$ ,  $\lim_{N \rightarrow \infty} \frac{1}{N} \# \{1 \leq n \leq N : b_n(x) \equiv r \pmod{k^m}\} = \frac{1}{k^m}$ .  $\square$

Finally, despite this equidistribution property, the point  $x$  has vanishing relativized finite-state dimension with respect to  $f$ .

**Proposition 2.** *For the above  $f$  and  $x = 1/k$ , one has  $\dim_{\text{FS}}^f(x) = 0$ .*

*Proof.* For each  $L \geq 1$  let  $T_L$  be the 1-state  $\Sigma$ -FST that outputs  $0^L$  on every input symbol. Then  $T_L(\pi) = 0^{L|\pi|}$ , so  $K^{T_L}(0^n) \leq \lceil n/L \rceil$ . Let  $\delta_n := 2n/k^n$ . Since  $f(0^n) = c_n$  and  $|x - c_n| = \frac{n}{k^n} < \delta_n$  we have  $K_{\delta_n}^{T_L, f}(x) \leq K^{T_L}(0^n) \leq \lceil n/L \rceil$ . Since  $\log_k(1/\delta_n) = \log_k(k^n/(2n)) = n - \log_k(2n)$ , we obtain

$$\liminf_{n \rightarrow \infty} \frac{K_{\delta_n}^{T_L, f}(x)}{\log_k(1/\delta_n)} \leq \liminf_{n \rightarrow \infty} \frac{\lceil n/L \rceil}{n - \log_k(2n)} = \frac{1}{L}.$$

Taking the infimum over transducers (in particular over  $T_L$ ) and letting  $L \rightarrow \infty$  yields  $\dim_{\text{FS}}^f(x) = 0$ .  $\square$

Taking  $x = 1/k$  and  $f$  as defined above we obtain the following.

**Theorem 5.** *There exist a rational-valued  $O(n \log n)$ -time computable separator enumerator  $f : \Sigma^* \rightarrow [0, 1)$  and a point  $x \in [0, 1)$  such that the integer sequence  $(k^n a_n^f(x))_{n \geq 1}$  is  $k$ -adically equidistributed, while  $\dim_{\text{FS}}^f(x) = 0$ .*

*Proof.* Take  $x = 1/k$  and  $f$  as defined above. Lemma 10 identifies  $k^n a_n^f(x) = b_n(x)$ , Lemma 11 shows  $(b_n(x))$  is  $k$ -adically equidistributed, and Proposition 2 gives  $\dim_{\text{FS}}^f(x) = 0$ .  $\square$

## 5 Discussion and open questions

We show that, in general, no distributional property of the scaled best-from-below approximation sequence  $(k^n a_n^f(x))_{n \geq 1}$  can characterize  $f$ -normality uniformly over all separator enumerators (indeed, we construct computable enumerators  $f_0, f_1$  for which the associated scaled approximation sequences coincide while the corresponding  $f$ -normality behavior diverges). At the same time, we identify a structured regime—finite-state coherent enumerators—in which  $f$ -normality is equivalent to  $k$ -adic equidistribution of  $(k^n a_n^f(x))_{n \geq 1}$ . We also show that this correspondence can already fail for an efficiently computable separator enumerator, in fact for an  $O(n \log n)$ -time computable  $f$ . This raises the natural question of how far this correspondence persists beyond finite-state coherent relabelings. One concrete setting is when the naming map is computable by a deterministic push-down transducer. In particular, does there exist such a separator enumerator  $f$  and a point  $x \in [0, 1)$  for which the integer sequence  $(k^n a_n^f(x))_{n \geq 1}$  is  $k$ -adically equidistributed while  $\dim_{\text{FS}}^f(x) < 1$ ? It would also be interesting to identify the widest natural family of separator enumerators for which an equidistribution characterization of  $f$ -normality remains valid.

A related direction concerns the role of invertibility in the defining Mealy-machine relabeling: invertibility is used to transport approximation complexity via the substitution  $u = M(w)$  and to ensure that  $T \mapsto M \circ T$  is a bijection on finite-state transducers, yielding invariance of the induced  $f$ -dimension. It would be interesting to determine whether some weaker condition (e.g. levelwise surjectivity or bounded-to-one behavior on each  $\Sigma^n$ ) suffices for the same equidistribution characterization, or whether non-invertible finite-state relabelings can already break it.

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