1 Proof that $H|0\rangle \otimes H|1\rangle = (H \otimes H)(|0\rangle \otimes |1\rangle)$

Let's calculate the left and right parts of the equation separately.

Left part:

$$\begin{aligned}
H &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\
|0\rangle &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
\implies H|0\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \\
\implies H|1\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \\
\implies H|0\rangle \otimes H|1\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \end{bmatrix} \\
\implies H|0\rangle \otimes H|1\rangle &= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}
\end{aligned} \tag{1}$$

Right part:

$$
\begin{aligned}
(H \otimes H)(|0\rangle \otimes |1\rangle) &= (\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix})(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} & \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} & -\frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
\implies (H \otimes H)(|0\rangle \otimes |1\rangle) &= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}
\end{aligned}
\tag{2}
$$

$$
\implies H|0\rangle \otimes H|1\rangle = (H \otimes H)(|0\rangle \otimes |1\rangle) \tag{3}
$$

2 Proof that $H|\phi\rangle \otimes H|\psi\rangle = (H \otimes H)(|\phi\rangle \otimes |\psi\rangle)$

Let's calculate the left and right parts of the equation separately.

Left part:

$$
\begin{aligned}
H &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\
|\phi\rangle &= \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, |\psi\rangle = \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \\
H|\phi\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\alpha + \frac{1}{\sqrt{2}}\beta \\ \frac{1}{\sqrt{2}}\alpha - \frac{1}{\sqrt{2}}\beta \end{bmatrix} \\
H|\psi\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\gamma + \frac{1}{\sqrt{2}}\delta \\ \frac{1}{\sqrt{2}}\gamma - \frac{1}{\sqrt{2}}\delta \end{bmatrix} \\
\Longrightarrow H|\phi\rangle \otimes H|\psi\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}}\alpha + \frac{1}{\sqrt{2}}\beta \\ \frac{1}{\sqrt{2}}\alpha - \frac{1}{\sqrt{2}}\beta \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}}\gamma + \frac{1}{\sqrt{2}}\delta \\ \frac{1}{\sqrt{2}}\gamma - \frac{1}{\sqrt{2}}\delta \end{bmatrix} \\
&= \begin{bmatrix} (\frac{1}{\sqrt{2}}\alpha + \frac{1}{\sqrt{2}}\beta) \begin{bmatrix} \frac{1}{\sqrt{2}}\gamma + \frac{1}{\sqrt{2}}\delta \\ \frac{1}{\sqrt{2}}\gamma - \frac{1}{\sqrt{2}}\delta \end{bmatrix} \\ (\frac{1}{\sqrt{2}}\alpha - \frac{1}{\sqrt{2}}\beta) \begin{bmatrix} \frac{1}{\sqrt{2}}\gamma + \frac{1}{\sqrt{2}}\delta \\ \frac{1}{\sqrt{2}}\gamma - \frac{1}{\sqrt{2}}\delta \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}\alpha\gamma + \frac{1}{2}\alpha\delta + \frac{1}{2}\beta\gamma + \frac{1}{2}\beta\delta \\ \frac{1}{2}\alpha\gamma - \frac{1}{2}\alpha\delta + \frac{1}{2}\beta\gamma - \frac{1}{2}\beta\delta \\ \frac{1}{2}\alpha\gamma + \frac{1}{2}\alpha\delta - \frac{1}{2}\beta\gamma - \frac{1}{2}\beta\delta \\ \frac{1}{2}\alpha\gamma - \frac{1}{2}\alpha\delta - \frac{1}{2}\beta\gamma + \frac{1}{2}\beta\delta \end{bmatrix} \\
\Longrightarrow H|\phi\rangle \otimes H|\psi\rangle &= \begin{bmatrix} \frac{1}{2}(\alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta) \\ \frac{1}{2}(\alpha\gamma - \alpha\delta + \beta\gamma - \beta\delta) \\ \frac{1}{2}(\alpha\gamma + \alpha\delta - \beta\gamma - \beta\delta) \\ \frac{1}{2}(\alpha\gamma - \alpha\delta - \beta\gamma + \beta\delta) \end{bmatrix}
\end{aligned} \tag{4}
$$

Right part:

$$
\begin{aligned}
(H \otimes H)(|\phi\rangle \otimes |\psi\rangle) &= (\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix})(\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \otimes \begin{bmatrix} \gamma \\ \delta \end{bmatrix}) \\
&= (\begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} & \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\ \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} & -\frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \end{bmatrix})(\begin{bmatrix} \alpha \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \\ \beta \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \end{bmatrix}) \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}\alpha\gamma + \frac{1}{2}\alpha\delta + \frac{1}{2}\beta\gamma + \frac{1}{2}\beta\delta \\ \frac{1}{2}\alpha\gamma - \frac{1}{2}\alpha\delta + \frac{1}{2}\beta\gamma - \frac{1}{2}\beta\delta \\ \frac{1}{2}\alpha\gamma + \frac{1}{2}\alpha\delta - \frac{1}{2}\beta\gamma - \frac{1}{2}\beta\delta \\ \frac{1}{2}\alpha\gamma - \frac{1}{2}\alpha\delta - \frac{1}{2}\beta\gamma + \frac{1}{2}\beta\delta \end{bmatrix} \\
\implies (H \otimes H)(|\phi\rangle \otimes |\psi\rangle) &= \begin{bmatrix} \frac{1}{2}(\alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta) \\ \frac{1}{2}(\alpha\gamma - \alpha\delta + \beta\gamma - \beta\delta) \\ \frac{1}{2}(\alpha\gamma + \alpha\delta - \beta\gamma - \beta\delta) \\ \frac{1}{2}(\alpha\gamma - \alpha\delta - \beta\gamma + \beta\delta) \end{bmatrix}
\end{aligned} \tag{5}
$$

Conclusion:

$$
\implies H|\phi\rangle \otimes H|\psi\rangle = (H \otimes H)(|\phi\rangle \otimes |\psi\rangle) \tag{6}
$$