1 Definition

Notation:

$$(|\psi\rangle, |\phi\rangle) \tag{1}$$

Where $|\psi\rangle$ and $|\phi\rangle$ are vectors in a Vector space. in C^n:

$$\begin{aligned}
|\psi\rangle &= \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix} \\
|\phi\rangle &= \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix} \\
\implies (|\psi\rangle, |\phi\rangle) &= \sum_{i=1}^{n} \psi_i^* \phi_i
\end{aligned} \tag{2}$$

2 Properties

2.0.1 Right linearity

$$(|\psi\rangle, \alpha|\phi\rangle) = \alpha(|\psi\rangle, |\phi\rangle) \tag{3}$$

Proof:

$$\begin{aligned}
(|\psi\rangle, \alpha|\phi\rangle) &= \sum_{i=1}^{n} \psi_i^* \alpha \phi_i \\
&= \alpha \sum_{i=1}^{n} \psi_i^* \phi_i \\
&= \alpha(|\psi\rangle, |\phi\rangle) \\
\implies (|\psi\rangle, \alpha|\phi\rangle) &= \alpha(|\psi\rangle, |\phi\rangle)
\end{aligned} \tag{4}$$

2.0.2 Conjugate symmetry

$$(|\psi\rangle, |\phi\rangle) = (|\phi\rangle, |\psi\rangle)^* \tag{5}$$

Proof:

$$\begin{aligned}
(|\psi\rangle, |\phi\rangle) &= \sum_i \psi_i^* \phi_i \\
&= \sum_i \phi_i^* \psi_i, \\
&= (|\phi\rangle, |\psi\rangle)^* \\
\implies (|\psi\rangle, |\phi\rangle) &= (|\phi\rangle, |\psi\rangle)^*
\end{aligned} \tag{6}$$

Explanation why $\psi^*\phi = \phi^*\psi$:

$$\begin{aligned}
\psi &= \alpha + i\beta, \phi = \gamma + i\delta \\
\psi^*\phi &= (\alpha - i\beta)(\gamma + i\delta) \\
&= \alpha\gamma + \beta\delta + \alpha\delta i - \beta\gamma i \\
&= (\gamma - i\delta)(\alpha + i\beta) \\
&= \phi^*\psi \\
\implies \psi^*\phi &= \phi^*\psi
\end{aligned} \tag{7}$$

2.0.3 Inequality

$$(|\psi\rangle, |\psi\rangle) \geq 0, \iff |\psi\rangle = 0 \tag{8}$$

Proof:

$$\begin{aligned}
|\psi\rangle = 0 &= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\
(|\psi\rangle, |\psi\rangle) &= \sum_{i=1}^{n} \psi_i^* \psi_i \\
&= \sum_{i=1}^{n} 0 \\
&= 0 \\
\implies (|\psi\rangle, |\psi\rangle) &\geq 0
\end{aligned} \tag{9}$$

2.0.4 Left conjugate linearity

$$(\alpha|\psi\rangle, |\phi\rangle) = \alpha^*(|\psi\rangle, |\phi\rangle) \tag{10}$$

Proof:

$$\begin{aligned}
(\alpha|\psi\rangle, |\phi\rangle) &= \sum_{i=1}^{n} \alpha \psi_i^* \phi_i \\
&= \alpha^* \sum_{i=1}^{n} \psi_i^* \phi_i \\
&= \alpha^*(|\psi\rangle, |\phi\rangle) \\
\implies (\alpha|\psi\rangle, |\phi\rangle) &= \alpha^*(|\psi\rangle, |\phi\rangle)
\end{aligned} \tag{11}$$