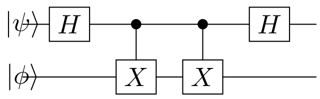
Figure 1: A circuit with two Hadamard gates



$$
\begin{aligned}
|\Psi_0\rangle &= |\psi\rangle * |\phi\rangle \\
|\psi\rangle &= \alpha|0\rangle + \beta|1\rangle \\
|\phi\rangle &= \gamma|0\rangle + \delta|1\rangle
\end{aligned} \tag{1}
$$

Step 1

$$
\begin{aligned}
|\Psi_1\rangle &= H|\psi\rangle * |\phi\rangle \\
&= H(\alpha|0\rangle + \beta|1\rangle) * |\phi\rangle \\
&= (\alpha H|0\rangle + \beta H|1\rangle) * |\phi\rangle \\
H|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
H|1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\
\implies |\Psi_1\rangle &= (\alpha\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + \beta\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) * |\phi\rangle \\
&= (\frac{\alpha}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{\beta}{\sqrt{2}}(|0\rangle - |1\rangle)) * |\phi\rangle \\
&= (\frac{\alpha}{\sqrt{2}}|0\rangle + \frac{\alpha}{\sqrt{2}}|1\rangle + \frac{\beta}{\sqrt{2}}|0\rangle - \frac{\beta}{\sqrt{2}}|1\rangle) * |\phi\rangle \\
&= (\frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle) * |\phi\rangle
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
\implies |\Psi_1\rangle &= (\frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle) * |\phi\rangle \\
&= \frac{\alpha+\beta}{\sqrt{2}}|0\rangle * |\phi\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle * |\phi\rangle \\
&= \frac{\alpha+\beta}{\sqrt{2}}|0\rangle * (\gamma|0\rangle + \delta|1\rangle) + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle * (\gamma|0\rangle + \delta|1\rangle) \\
&= \frac{\alpha+\beta}{\sqrt{2}}|0\rangle\gamma|0\rangle + \frac{\alpha+\beta}{\sqrt{2}}|0\rangle\delta|1\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle\gamma|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle\delta|1\rangle \\
&= \gamma\frac{\alpha+\beta}{\sqrt{2}}|00\rangle + \delta\frac{\alpha+\beta}{\sqrt{2}}|01\rangle + \gamma\frac{\alpha-\beta}{\sqrt{2}}|10\rangle + \delta\frac{\alpha-\beta}{\sqrt{2}}|11\rangle
\end{aligned} \tag{3}
$$

Step 2

$$
\begin{aligned}
|\Psi_2\rangle &= CNOT(|\Psi_1\rangle) \\
&= CNOT(\gamma\frac{\alpha+\beta}{\sqrt{2}}|00\rangle + \delta\frac{\alpha+\beta}{\sqrt{2}}|01\rangle + \gamma\frac{\alpha-\beta}{\sqrt{2}}|10\rangle + \delta\frac{\alpha-\beta}{\sqrt{2}}|11\rangle) \\
&= \gamma\frac{\alpha+\beta}{\sqrt{2}}CNOT(|00\rangle) + \delta\frac{\alpha+\beta}{\sqrt{2}}CNOT(|01\rangle) + \gamma\frac{\alpha-\beta}{\sqrt{2}}CNOT(|10\rangle) + \delta\frac{\alpha-\beta}{\sqrt{2}}CNOT(|11\rangle)
\end{aligned} \tag{4}
$$

$$
\implies |\Psi_2\rangle = \gamma\frac{\alpha+\beta}{\sqrt{2}}|00\rangle + \delta\frac{\alpha+\beta}{\sqrt{2}}|01\rangle + \gamma\frac{\alpha-\beta}{\sqrt{2}}|11\rangle + \delta\frac{\alpha-\beta}{\sqrt{2}}|10\rangle \tag{5}
$$

Step 3

$$
\begin{aligned}
|\Psi_3\rangle &= CNOT(|\Psi_2\rangle) \\
&= CNOT(\gamma\frac{\alpha+\beta}{\sqrt{2}}|00\rangle + \delta\frac{\alpha+\beta}{\sqrt{2}}|01\rangle + \gamma\frac{\alpha-\beta}{\sqrt{2}}|11\rangle + \delta\frac{\alpha-\beta}{\sqrt{2}}|10\rangle) \\
&= \gamma\frac{\alpha+\beta}{\sqrt{2}}CNOT(|00\rangle) + \delta\frac{\alpha+\beta}{\sqrt{2}}CNOT(|01\rangle) + \gamma\frac{\alpha-\beta}{\sqrt{2}}CNOT(|11\rangle) + \delta\frac{\alpha-\beta}{\sqrt{2}}CNOT(|10\rangle)
\end{aligned} \tag{6}
$$

$$
\implies |\Psi_3\rangle = \gamma\frac{\alpha+\beta}{\sqrt{2}}|00\rangle + \delta\frac{\alpha+\beta}{\sqrt{2}}|01\rangle + \gamma\frac{\alpha-\beta}{\sqrt{2}}|10\rangle + \delta\frac{\alpha-\beta}{\sqrt{2}}|11\rangle \tag{7}
$$

Step 4

$$
\begin{aligned}
|\Psi_4\rangle &= \gamma\frac{\alpha+\beta}{\sqrt{2}}H|0\rangle|0\rangle + \delta\frac{\alpha+\beta}{\sqrt{2}}H|0\rangle|1\rangle + \gamma\frac{\alpha-\beta}{\sqrt{2}}H|1\rangle|0\rangle + \delta\frac{\alpha-\beta}{\sqrt{2}}H|1\rangle|1\rangle \\
H|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
H|1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\
&= \gamma\frac{\alpha+\beta}{\sqrt{2}} * \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle + \delta\frac{\alpha+\beta}{\sqrt{2}} * \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|1\rangle + \gamma\frac{\alpha-\beta}{\sqrt{2}} * \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|0\rangle \\
&+ \delta\frac{\alpha-\beta}{\sqrt{2}} * \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|1\rangle \\
&= \gamma\frac{\alpha+\beta}{2}(|00\rangle + |10\rangle) + \delta\frac{\alpha+\beta}{2}(|01\rangle + |11\rangle) + \gamma\frac{\alpha-\beta}{2}(|00\rangle - |10\rangle) + \delta\frac{\alpha-\beta}{2}(|01\rangle - |11\rangle) \\
&= \gamma\frac{\alpha+\beta}{2}|00\rangle + \gamma\frac{\alpha+\beta}{2}|10\rangle + \delta\frac{\alpha+\beta}{2}|01\rangle + \delta\frac{\alpha+\beta}{2}|11\rangle + \gamma\frac{\alpha-\beta}{2}|00\rangle - \gamma\frac{\alpha-\beta}{2}|10\rangle \\
&+ \delta\frac{\alpha-\beta}{2}|01\rangle - \delta\frac{\alpha-\beta}{2}|11\rangle \\
&= (\gamma\frac{\alpha+\beta}{2} + \gamma\frac{\alpha-\beta}{2})|00\rangle + (\gamma\frac{\alpha+\beta}{2} - \gamma\frac{\alpha-\beta}{2})|10\rangle \\
&+ (\delta\frac{\alpha+\beta}{2} + \delta\frac{\alpha-\beta}{2})|01\rangle + (\delta\frac{\alpha+\beta}{2} - \delta\frac{\alpha-\beta}{2})|11\rangle \\
&= \gamma\frac{2\alpha}{2}|00\rangle + \gamma\frac{2\beta}{2}|10\rangle + \delta\frac{2\alpha}{2}|01\rangle + \delta\frac{2\beta}{2}|11\rangle \\
&= \gamma\alpha|00\rangle + \gamma\beta|10\rangle + \delta\alpha|01\rangle + \delta\beta|11\rangle \\
&= \gamma\alpha|0\rangle|0\rangle + \gamma\beta|1\rangle|0\rangle + \delta\alpha|0\rangle|1\rangle + \delta\beta|1\rangle|1\rangle \\
&= \alpha|0\rangle(\gamma|0\rangle + \delta|1\rangle) + \beta|1\rangle(\gamma|0\rangle + \delta|1\rangle) \\
&= \alpha|0\rangle|\phi\rangle + \beta|1\rangle|\phi\rangle \\
&= (\alpha|0\rangle + \beta|1\rangle)|\phi\rangle \\
&= |\psi\rangle * |\phi\rangle
\end{aligned}
\tag{8}
$$

$$
\implies |\Psi_4\rangle = |\psi\rangle * |\phi\rangle = |\Psi_0\rangle \tag{9}
$$