The Gram-Schmidt method is a method to transform a set of vectors into an orthogonal set of vectors.

1 Definition

Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$ and $S = \{v_1, v_2, \ldots, v_n\}$ a set of vectors of V. By definition the projection of V on U is:

$$P_u(v) = \frac{v \cdot u}{u \cdot u} u \tag{1}$$

The Gram-Schmidt method is defined as:

$$\begin{aligned} w_1 &= v_1 \\ w_2 &= v_2 - P_{w_1}(v_2) \\ w_3 &= v_3 - P_{w_1}(v_3) - P_{w_2}(v_3) \\ &\vdots \\ w_n &= v_n - P_{w_1}(v_n) - P_{w_2}(v_n) - \cdots - P_{w_{n-1}}(v_n) \end{aligned} \tag{2}$$

The process can be summarized as:

$$w_n = v_n - \sum_{i=1}^{n-1} P_{w_i}(v_n) \tag{3}$$

2 Proof

2.1 Two vector orthogonal

Two vector A and B are orthogonal if $A \cdot B = 0$.

$$\begin{gathered} A = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{pmatrix} \\ A \cdot B = \sum_{i=1}^{n} x_i y_i \end{gathered} \tag{4}$$

If the two vectors are orthogonal then $\sum_{i=1}^{n} x_i y_i = 0$.

2.2 Gram-Schmidt vectors

$$
\begin{aligned}
w_n &= v_n - \sum_{i=1}^{n-1} P_{w_i}(v_n) \\
w_{n+1} &= v_{n+1} - \sum_{i=1}^{n} P_{w_i}(v_{n+1}) \\
\implies w_n \cdot w_{n+1} &= \left(v_n - \sum_{i=1}^{n-1} P_{w_i}(v_n)\right) \cdot \left(v_{n+1} - \sum_{i=1}^{n} P_{w_i}(v_{n+1})\right) \\
&= v_n \cdot v_{n+1} - \sum_{i=1}^{n-1} v_n \cdot P_{w_i}(v_{n+1}) - \sum_{i=1}^{n} P_{w_i}(v_n) \cdot v_{n+1} + \sum_{i=1}^{n-1} \sum_{j=1}^{n} P_{w_i}(v_n) \cdot P_{w_j}(v_{n+1})
\end{aligned}
\tag{5}
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