1 Introduction

This document explain the notations and diverses things about them

2 Notations

Here is the list of the differents notations that you can find in this document

$$\begin{cases} z^* & \text{Complex conjugate} \\ |\psi\rangle & \text{Vector notation named "ket"} \\ \langle\psi| & \text{Dual vector notation named "bra"} \\ \langle\psi|\phi\rangle & \text{Inner product} \\ |\psi\rangle\langle\phi| & \text{Outer product} \\ |\psi\rangle \otimes |\phi\rangle & \text{Tensor product} \\ A* & \text{Complex conjugate of the matrix A} \\ A^T & \text{Transposition of matrix A} \\ A^\dagger = (A^T)^* = (A^*)^T & \text{Hermitian conjugate of matrix A} \\ \langle\phi|A|\psi\rangle & \text{Inner product between } |\phi\rangle \text{ and } A|\psi\rangle \end{cases} \tag{1}$$

2.1 Complex conjugate

$$\begin{aligned} z &= a + ib \\ z^* &= a - ib \end{aligned} \tag{2}$$

With a and b real numbers.

2.2 Vector notation

$$\begin{aligned} |\psi\rangle &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \langle\psi| &= \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \end{aligned} \tag{3}$$

See how $|\psi\rangle$ is in column and $\langle\psi|$ is in line. It's because $\langle\psi|$ is the dual vector of $|\psi\rangle$.

2.3 Inner product

Inner product of two vectors $|\psi\rangle$ and $|\phi\rangle$ is defined as follows:

$$\begin{aligned} \langle\psi|\phi\rangle &= \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \\ \langle\psi|\phi\rangle &= \alpha^*\gamma + \beta^*\delta \end{aligned} \tag{4}$$

2.4 Outer product

Outer product of two vector is defined as follow

$$\begin{aligned} |\psi\rangle\langle\phi| &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \gamma^* & \delta^* \end{pmatrix} \\ &= \begin{pmatrix} \alpha\gamma^* & \alpha\delta^* \\ \beta\gamma^* & \beta\delta^* \end{pmatrix} \end{aligned} \tag{5}$$

2.5 Complex conjugate of matrix

$$\begin{aligned} A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ A^* &= \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \end{aligned} \tag{6}$$

2.6 Transposition of matrix

$$\begin{aligned} A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ A^T &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \end{aligned} \tag{7}$$

2.7 Hermitian conjugate of matrix

$$\begin{aligned} A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ A^\dagger &= \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \end{aligned} \tag{8}$$

2.8 Inner product between $|\phi\rangle$ and $A|\psi\rangle$

$$
\begin{aligned}
|\phi\rangle &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
|\psi\rangle &= \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \\
A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
(|\phi\rangle, A|\psi\rangle) &= \langle\phi|A|\psi\rangle \\
&= \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \\
&= \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} a\gamma + b\delta \\ c\gamma + d\delta \end{pmatrix} \\
&= \alpha^*(a\gamma + b\delta) + \beta^*(c\gamma + d\delta) \\
&= \alpha^* a\gamma + \alpha^* b\delta + \beta^* c\gamma + \beta^* d\delta
\end{aligned} \tag{9}
$$

Proof that $(|\phi\rangle, A|\psi\rangle) = (A^\dagger|\phi\rangle, |\psi\rangle)$

$$
\begin{aligned}
(A^\dagger|\phi\rangle, |\psi\rangle) &= \langle\phi|A^\dagger|\psi\rangle \\
&= \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \\
&= \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} a^*\gamma + c^*\delta \\ b^*\gamma + d^*\delta \end{pmatrix} \\
&= \alpha^* a^*\gamma + \alpha^* c^*\delta + \beta^* b^*\gamma + \beta^* d^*\delta \\
&= \alpha^* a^*\gamma + \beta^* b^*\gamma + \alpha^* c^*\delta + \beta^* d^*\delta \\
&= \alpha^*(a^*\gamma + b^*\gamma) + \beta^*(c^*\delta + d^*\delta) \\
&= \alpha^*(a^* + b^*)\gamma + \beta^*(c^* + d^*)\delta \\
&= \alpha^*(a + b)^*\gamma + \beta^*(c + d)^*\delta \\
&= \alpha^*(a\gamma + b\gamma)^* + \beta^*(c\delta + d\delta)^* \\
&= \alpha^*(a\gamma + b\delta)^* + \beta^*(c\gamma + d\delta)^*
\end{aligned} \tag{10}
$$

3 Exercices

3.1 Show that (1, -1), (1, 2) and (2, 1) are linearly dependant

A set of vectors $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$ is linearly dependant if there exist a set of complex numbers $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1|\psi_1\rangle + c_2|\psi_2\rangle + \cdots + c_n|\psi_n\rangle = 0 \tag{11}$$

Let's set

$$\begin{aligned}
|\psi\rangle &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
|\phi\rangle &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
|\chi\rangle &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
\implies |\psi\rangle + |\phi\rangle - 2|\chi\rangle &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
&= 0
\end{aligned} \tag{12}$$