$$\bigcap_{n\in\mathbb{N}} \left(-\tfrac{1}{n}, \tfrac{1}{n}\right) = \{0\}$$ closed

$$\bigcup_{n\in\mathbb{N}} \left[\tfrac{1}{n}, 1\right] = (0,1]$$ not closed

From the desk of :

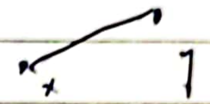
Date:

Topology of $\mathbb{R}^n$:

Let $x, y \in \mathbb{R}^n$, then

$$d(x,y) := \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\dots+(x_n-y_n)^2} = \|x-y\|$$

length of the line segment in $\mathbb{R}^n$



$d: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^+$ satisfy

- $d(x,y) > 0$ if $x \neq y$ & $d(x,x) = 0$
- $d(x,y) = d(y,x)$
- $d(x,y) + d(y,z) \geq d(x,z)$ (triangle inequality)

$(\mathbb{R}^n, d)$ – a metric space with the distance fn. d.

Then this d defines a topology on $\mathbb{R}^n$, τ_d

$(\mathbb{R}^n, \tau_d)$ – top. space.

$$B(x,\varepsilon) = \{ y \in \mathbb{R}^n \mid d(x,y) < \varepsilon \} \quad (x \in \mathbb{R}^n, \varepsilon > 0)$$

ε-neighborhood of the point x.

Def: $U \subseteq \mathbb{R}^n$ is open, if $\forall x \in U$,
$\exists\, \varepsilon > 0$ s.t. $B(x,\varepsilon) \subseteq U$.

Remark: All ε-neighborhoods are open sets.

- An open set containing x is also called an open neighborhood of x.

$\tau_{\mathbb{R}^n}$ = all open sets of $(\mathbb{R}^n, d)$. $\phi, \mathbb{R}^n$ are open set,

- $\phi, \mathbb{R}^n \in \tau_{\mathbb{R}^n}$
- $U_1, \ldots, U_k \in \tau_{\mathbb{R}^n}$, then $U_1 \cap \ldots \cap U_k \in \tau_{\mathbb{R}^n}$.
  (i.e. intersection of finite no. of open sets is open.)
- Arbitrary union of open sets is open, i.e. if $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq \tau_{\mathbb{R}^n}$, $\forall \alpha$, then $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau_{\mathbb{R}^n}$

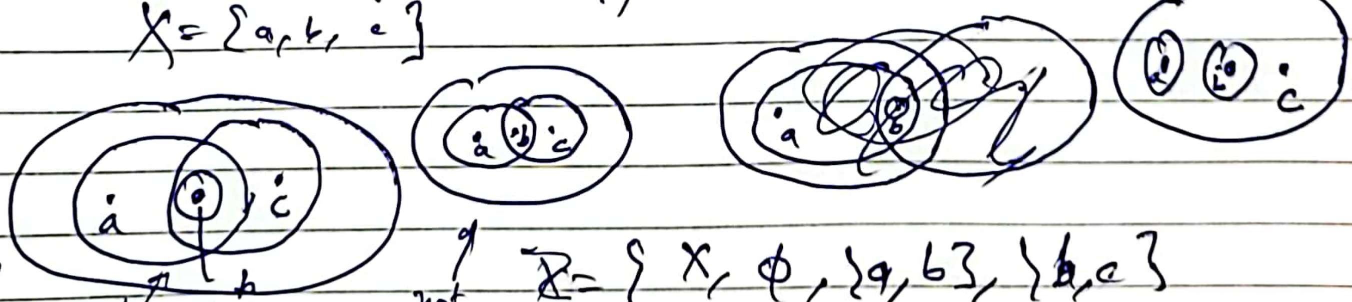
<u>Topology on a set $X \neq \phi$.</u> Let τ be a collection of subsets of X $\ni$

- $\phi, X \in \tau$
- ~~finite~~ if $U_1, \ldots, U_k \in \tau \Rightarrow \cap U_k \in \tau$
- $\{U_\alpha\}_{\alpha \in \Lambda}$; Λ arbitrary index set, then $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$.

Then (X, τ) is a topological space & τ is known as a topology on X.

<u>Note:</u> Elements of a topology are known as open sets.

<u>Eg:</u> $X = \{a, b, c\}$



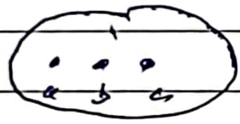
X topology not topology $\tau = \{X, \phi, \{a, b\}, \{b, c\}, \{b\}\}$

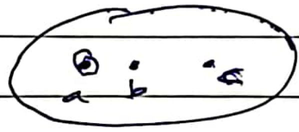
Topology - by James Munkres (Second edition)

Basic Topology (UTM)

by Armstrong.

$X = \{a, b, c\}$



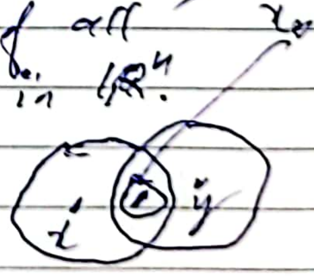


$\tau_1 = \{X, \phi\}$

$\tau_2 = \{\phi, \{a\}, \{a, b, c\}\}$

Basis for a Topology:

- Take $\mathbb{R}^n$, $\mathcal{B} = \{ B(x, \varepsilon) \mid x \in \mathbb{R}^n, \varepsilon > 0 \}$ ← collection of all open balls in $\mathbb{R}^n$.
- $\forall x \in \mathbb{R}^n \ \exists$ $B(x, \varepsilon) \ni x \in B(x, \varepsilon)$



- If $x_0 \in B(x, \varepsilon_1) \cap B(y, \varepsilon_2)$, then there is a $B(x_0, \varepsilon_3) \ni x_0 \in B(x_0, \varepsilon_3)$

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Then we can generate a topology on $\mathbb{R}^n$:

A set $S \subseteq \mathbb{R}^n$ is open if $\forall x \in S$ $\exists$ an element $B(x_0, \varepsilon) \in \mathcal{B}$ $\ni$ $x \in B(x_0, \varepsilon)$ & $B(x_0, \varepsilon) \subseteq S$

~~$\mathcal{T}_d = \{ S \mid S$ is an open~~

Then $\mathcal{B}$ generates the topology $\mathcal{T}_d$.

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<u>Examples</u>: 1.) $X \neq \phi$ set.

$\tau_f = \{ U \subseteq X \mid X - U$ is finite or is all of $X\}$

↑ finite complement topology. Then τ_f defines a topology on X.

• $X \in \tau_f \because X - X = \phi$ - finite & $\phi \in \tau_f \because X - \phi = X \in \tau$

• If $U_\alpha \in \tau_\alpha$ non-empty, $\alpha \in \Lambda$ (some indexed set)

$X - \bigcup_\alpha U_\alpha = \bigcap (X - U_\alpha)$ (each $X - U_\alpha$ is either finite or all of X)

$\|$ $\Rightarrow$ is a finite set

$X \cap (\bigcup_\alpha U_\alpha)^c$

$= X \cap (\bigcap U_\alpha^c)$

$= \bigcap (X \cap U_\alpha^c)$ • If $U_1, \dots U_n \neq$ nonempty $\in \tau_f$.

$= \bigcap U_\alpha^c$ $X - \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n X - U_i$

$= \bigcap (X - U_\alpha)$ finite union of finite set

$\|$

$X \cap (\bigcap_{i=1}^n U_i)^c$

$\bigcap_{i=1}^n U_i = (X - U_i^c)$ is finite

From the desk of : Date:

Eg: $\tau_c = \{ U \subseteq X \mid X - U$ is either countable or is all of $X \}$

To show that τ_c defines a topology on X.

Defn:

$\mathcal{B}$ — Collection of sets of X is a basis if

- ~~whenever~~ for all $x \in X$, $\exists\ B \in \mathcal{B} \ni$ $x \in B$ — (1)
- whenever $x \in B_1 \cap B_2$, $B_1, B_2 \in \mathcal{B}$, then $\exists\ B_3 \in \mathcal{B}$ $\ni x \in B_3 \subseteq B_1 \cap B_2$ — (2)

Consider the following collection of ~~X~~ subsets of X

$\tau = \{ U \subseteq X \mid \forall x \in U, \exists\ B \in \mathcal{B} \ni x \in B \subseteq U \}$

Claim: τ defines a topology on X

- $\phi \in \tau$

  $X \in \tau$ (follows from (1))

  $\because x \in X, \exists\ B \in \mathcal{B} \ni x \in B \subseteq X$

- For $U_\alpha \in \tau$, $\alpha \in \Lambda$ — arbitrary index set

Then

i. $\bigcup_{\alpha\in\Delta} U_\alpha \in \mathcal{T}$, this is because

if $x \in \bigcup_{\alpha\in\Delta} U_\alpha$ this imply

$x \in U_\alpha$ for some α ②

this again imply

$\exists\, B \in \mathcal{B} \ni x \in B \subseteq U_\alpha \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$.

• $\bigcap_{\beta=1}^{k} U_\beta \in \mathcal{T}$, if $y \in \bigcap_{\beta=1}^{k} U_\beta$

~~$\Rightarrow y \in U_\beta, \forall \beta = 1 \text{ to } k$~~

We can show this by induction

For $k=1$, $U_1 \in \mathcal{T}$, (obvious)

For $k=2$, $U_1 \cap U_2 \in \mathcal{T}$ — ③

If we take $y \in U_1 \cap U_2$ then

$y \in U_1$ & $y \in U_2$

$\Rightarrow \exists\, B_1 \in \mathcal{B}$ & $B_2 \in \mathcal{B} \ni$

$y \in B_1 \subseteq U_1$ & $y \in B_2 \subseteq U_2$

$\Rightarrow y \in B_1 \cap B_2$

Now by ②, $\exists\, B_3 \in \mathcal{B} \ni$

$y \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$

Assume for $k-1$, we have

$$U_1 \cap \cdots \cap U_{k-1} \in \mathcal{Z}$$

then for k,

$$\underbrace{(U_1 \cap U_2 \cdots \cap U_{k-1})}_{\in \mathcal{Z}} \cap \underbrace{U_k}_{\in \mathcal{Z}} \Rightarrow U_1 \cap \cdots \cap U_k \in \mathcal{Z}$$

(By ③)

Hence, we have shown that $\mathcal{Z}$ defines a topology on X & we say that the collection of sets $\mathcal{B}$ generates $\mathcal{Z}$ & $\mathcal{B}$ is called a basis for the topology ~~of~~ $\mathcal{Z}$.

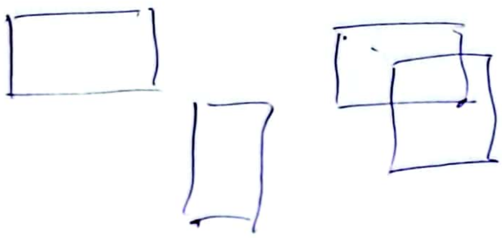
Note: 1. Now by defn, every element of $\mathcal{Z}$ defines an open subset of X in the topology $\mathcal{Z}$.

2. $\mathcal{B} \subseteq \mathcal{Z}$.

Defn: A subset U of X is ~~said~~ called open set (w.r.t some basis $\mathcal{B}$) if for $x \in U$, $\exists$ $B \in \mathcal{B}$ $\ni$ $x \in B \subseteq U$.

Examples: 1.) $\mathcal{B}_B$ = collection of interior of all balls (circles) in $\mathbb{R}^2$

2.) $\mathcal{B}_R$ = collection of interior of all rectangles in $\mathbb{R}^2$. (sides parallel to the axes)



Fact 1 Let (X, τ) be a topological space & let $\mathcal{B}$ be a basis for τ. Then $\tau = \{$ all possible unions of elements of $\mathcal{B} \} = \eta$

Proof: ~~given a collection~~
Let $U \in \eta \Rightarrow U = \bigcup_{\alpha \in \Lambda} B_\alpha$, $\alpha \in \Lambda$ & $B_\alpha \in \mathcal{B}$

As $B_\alpha \in \mathcal{B} \subseteq \tau$ & τ is a topology

$$U = \bigcup_{\alpha \in \Lambda} B_\alpha \in \tau \Rightarrow \eta \subseteq \tau$$

Conversely, if $V \in \tau$, then $\forall x \in V$
$\exists B_x \in \mathcal{B} \ni x \in B_x \subseteq V$. Then

$$V = \bigcup_{x \in V} B_x \in \eta \Rightarrow \tau \subseteq \eta$$

Examples:

Discrete topology on a set $X \neq \phi$

$X = \{1, 2\}$; $\mathcal{P}(X) = \{\{1\}, \{2\}, \{1, 2\}\}$

$\mathcal{B} = \{\{1\}, \{2\}\}$

Defn: Suppose τ_1 & τ_2 are two topologies on X. If

$\tau_1 \subseteq \tau_2$, we say τ_2 is finer than τ_1.

$\tau_1 \supseteq \tau_2$, we say τ_2 is coarser than τ_1

Eg: $X = \{a, b, c\}$

$\tau_1 = \{\{b\}, \{a, b\}, \{b, c\}, X, \phi\}$

$\tau_2 = \{\{a, b\}, X, \phi\}$; $\tau_2 \subseteq \tau_1$

$\tau_3 = \{\{c\}, \{b\}, \{a, b\}, \{b, c\}, X, \phi\}$; $\tau_3 \supseteq \tau_1$

$\tau_4 = \{\{a\}, \{b, c\}, X, \phi\}$. ; τ_1 is not comparable to τ_4.

Topologies on $\mathbb{R}$

The collection $\mathcal{B}_1 = \{\{ \cancel{(a,b)}\ x \in \mathbb{R} \mid a < x < b\} ; a, b \in \mathbb{R}\}$

generates the standard topology on $\mathbb{R}$.

$\mathcal{B}_2 = \{ [a,b) \mid a,b \in \mathbb{R} \}$

generates lower limit topology on $\mathbb{R}$

Product Topology $(X \times Y)$

If (X, τ_X) & (Y, τ_Y) are two topological space, we define

$\mathcal{B}_{X\times Y} := \tau_X \times \tau_Y := \{ U \times V \mid U \in \tau_X, V \in \tau_Y \}$

Then $\mathcal{B}_{X\times Y}$ defines a basis for the topology on $X \times Y$.

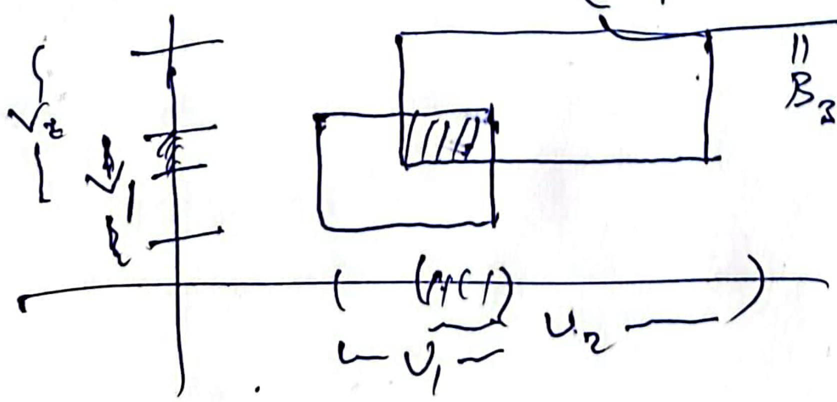
<u>Check:</u> (1) $\forall (x,y) \in X \times Y$, $\exists\, X \times Y \in \mathcal{B}_{X\times Y}$ ($\because X \in \tau_X$ & $Y \in \tau_Y$)

& $(x,y) \in X \times Y$

(2) Let $U_1 \times V_1$ & $U_2 \times V_2 \in \mathcal{B}_{X\times Y}$, then

For $(x,y) \in (\overset{B_1}{U_1 \times V_1}) \cap (\overset{B_2}{U_2 \times V_2})$

$= \underbrace{(\overset{\in \tau_X}{U_1 \cap U_2}) \times (\overset{\in \tau_Y}{V_1 \cap V_2})}_{= B_3}$



Fact: $\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$

is a basis for the topology of $X \times Y$, where $\mathcal{B}$ is a basis for τ_X & $\mathcal{C}$ is a basis for τ_Y.

Eg: $X = \mathbb{R}$, $Y = \mathbb{R}$, $\tau_X = \tau_Y =$ topology generated by all open intervals of $\mathbb{R}$.

$\tau_{X \times Y} =$ topology generated by all open intervals of the form $(a,b) \times (c,d)$ {interior of ~~of~~ rectangles}

The Subspace topology:

Let (X, τ_X) be a topological space & let $Y \subseteq X$. Then there is an induced topology on Y coming from X.

It is defined as follows:

$$\tau_Y = \{Y \cap U \mid U \in \tau_X\}$$

τ_Y is called the subspace topology.

Note: All open sets in Y (in τ_Y) are created by intersecting open sets in X (in τ_X) ~~with~~ ~~by~~ with Y.

Check: $\phi \in \tau_Y$

$\because \phi \in \tau_X$ & $\phi \cap Y = \phi \in \tau_Y$

$X \in \tau_Y$ & $Y \cap X = Y \in \tau_Y$.

$\because$ If $U_1 \cap Y, \dots, U_k \cap Y \in \tau_Y$

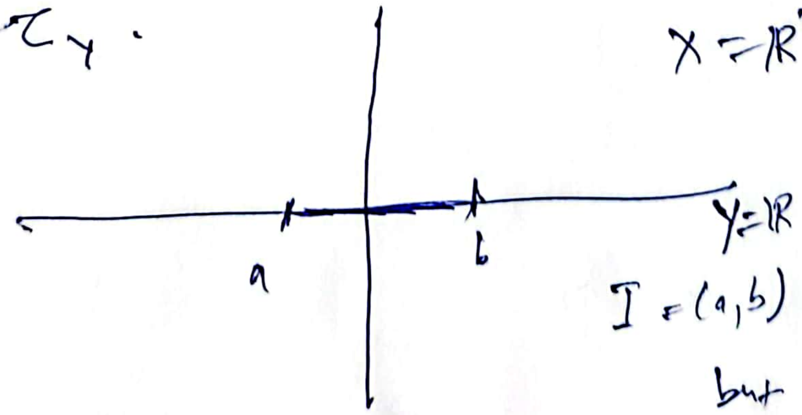
then $(U_1 \cap Y) \cap \dots \cap (U_k \cap Y) = (\underbrace{U_1 \cap \dots \cap U_k}_{\in \tau_X}) \cap Y \in \tau_Y$

$\because$ If $U_\alpha \cap Y \in \tau_Y : \alpha \in \Lambda$ $\Rightarrow$

then $\bigcup_{\alpha \in \Lambda} (U_\alpha \cap Y) = (\underbrace{\bigcup_{\alpha \in \Lambda} U_\alpha}_{\in \tau_X}) \cap Y \in \tau_Y$.

Fact: If $\mathcal{B}_X$ is a basis for τ_X (a topology on X). Then $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}_X\}$ defines a basis for the subspace topology τ_Y.

$X = \mathbb{R}^2$



$I = (a, b)$ is open in Y but it is not open in X.

FACT: Let (Y, τ_Y) be a subspace of (X, τ_X)

If U is open in Y & Y is open in X, then U is open in X.

Proof: Since U is open in Y;

$U = Y \cap V$; V open in X.

Now Y & V are both open in X

$\Rightarrow U = Y \cap V$ is open in X

($\because Y \cap V \in \tau_X$ & $Y \in \tau_X$ & $V \in \tau_X$)

Defn: We say that a subset $U \subseteq (X, \tau_X)$ is open if $U \in \tau_X$.

Fact: If (A, τ_A) is a subspace of (X, τ_X) & (B, τ_B) is a subspace of (Y, τ_Y) then the product topology on $A \times B$, i.e. $\tau_A \times \tau_B$ is same as the topology induced on $A \times B$ from $X \times Y$, i.e. $\tau_{A \times B} = \{U \cap A \times B \mid U \in \tau_{X \times Y}\}$

The order topology

simply ordered set

- A relⁿ '$<$' on a set S is called a simple order if following is satisfied:

- $\forall x, y \in S$, $\ni x \neq y$, either $x < y$ or $y < x$
- For no x in S, $x < x$ holds
- If $x < y$, & $y < z$, then $x < z$

Then $(S, <)$ is called a simply ordered set.

We can talk about intervals in a simply ordered set

$$(a, b) = \{ x \mid a < x < b \}$$

$$(a, b] = \{ x \mid a < x \leq b \}$$

$$[a, b) = \{ x \mid a \leq x < b \}$$

$$[a, b] = \{ x \mid a \leq x \leq b \}$$

Defn: Let X be a set with '$<$' as a simple order defined on it. Also, assume $\mathrm{card}(X) > 1$.

Consider

$\mathcal{B}$ = the collection consisting of all sets of the form

(1) all ~~open~~ intervals of the form (a,b)

(2) $[a_0, b)$, a_0 is the smallest element (if there exists one) of X

(3) $(a, b_0]$, b_0 is the largest element (if any) of X.

Then $\mathcal{B}$ defines a basis for a topology on X, & the topology τ_0 so obtained is called the order topology. (check!)

Eg: 1.) Standard topology on $\mathbb{R}$ is the order topology on $\mathbb{R}$ (coming from the simple order '$<$' on $\mathbb{R}$)

2.) $\mathbb{Z}_+ = \{1, 2, \ldots\}$, the order topology on $\mathbb{Z}_+$ is the discrete topology

For if $n > 1$, we have $(n-1, n+1) = \{n\}$ — a basis element which is by def^n open

& when $n = 1$, we can take $[1, 2) = \{1\}$, again a basis element

~~The~~ Therefore, all the singletons are open in order topology & hence, it is discrete.

Let $(X, <)$ be an ordered set.

& let $Y \subseteq X$, then $<|_Y$ gives a simple order on Y, making Y an ordered set, so there is an order topology τ_o^Y on Y (coming from $<|_Y$) & there is a subspace topology τ_s^Y on Y coming from X.

In general, τ_o^Y & τ_s^Y need not be same.

Eg: i) $Y = [0,1]$: $\mathcal{B}_Y = \{(a,b) \cap [0,1] \mid (a,b) \in \mathcal{B}_{\mathbb{R}}\}$

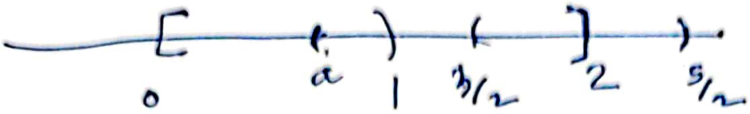
$$(a,b) \cap [0,1] = \begin{cases} (a,b) & ; \text{ if } a,b \in Y \\ [0,b) & ; \text{ if only } b \text{ is in } Y \\ (a,1] & ; \text{ if only } a \text{ is in } Y \\ Y \text{ or } \phi & ; \text{ if neither } a \text{ nor } b \text{ is in } Y. \end{cases}$$

these sets forms a basis for the order topology on Y

$[0,b)$ & $(a,1]$ are open in Y.
but they are not open in $\mathbb{R} = X$.

Here, the subspace topology coming from $\mathbb{R}$ agrees with the order topology.

Eg 2: $Y = [0,1) \cup \{2\}$



The set $\{2\}$ is open in Y in subspace topology coming from $\mathbb{R}$ $(Y \subseteq \mathbb{R})$. Since, the interval $(3/2, 5/2)$ is open in $\mathbb{R}$ & hence the set $(3/2, 5/2) \cap Y = \{2\}$ is open in $\mathbb{R}$ (by defn of subspace topology. Therefore, the singleton $\{2\}$ is open in the subspace topology.

Since $\{2\}$ is the Largest element of Y any basis element of the order topology which contains $\{2\}$ must be of the form $(a, 2]$ for some $a \in Y$. & this ~~set conta~~ basis contains the points x which are less than 1 & greater than a ~~& clean~~ i.e, points $\in (a,1)$ & $(a,1) \not\subseteq \{2\}$

$\therefore (a,2] \not\subseteq \{2\}$

$\therefore \{2\}$ is not open.

Closed sets:

A subset C of (X, τ) is said to be closed if the set $X - C \in \tau$, i.e., $X - C$ is open.

$X - C = C^c$

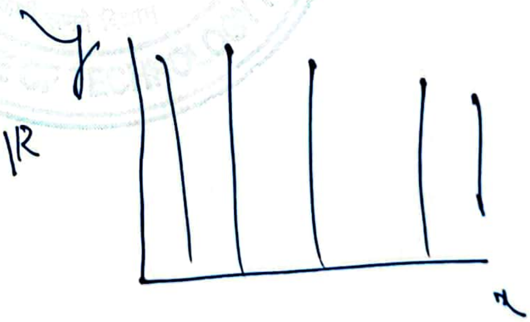
Eg: 1.) $[a, b]$ is closed in $\mathbb{R}$. This is because $\mathbb{R} - [a, b] = (-\infty, a) \cup (b, +\infty)$

open, open

$\Rightarrow \mathbb{R} - [a, b]$ is open in $\mathbb{R}$

& hence $[a, b]$ is closed in $\mathbb{R}$.

2) $(-\infty, 0) \times \mathbb{R}$



(B) $\mathbb{R} \times (-\infty, 0)$

3) In a discrete topological space, every set is open as well as closed.

Eg 4.)

$$Y = [0, 1] \cup (2, 3)$$

In the subspace topology (coming from $\mathbb{R}$), the subset $[0,1]$ is open in Y. This is because $[0,1] = (-\frac{1}{2}, \frac{3}{2}) \cap Y$ & $(-\frac{1}{2}, \frac{3}{2})$ is open in $\mathbb{R}$. Same way we can see $(2,3)$ is open in Y $\left((2,3) = \underbrace{(1.8, 3.8)}_{\text{open in } \mathbb{R}} \cap Y \right)$. But here $(2,3)$ is open in $\mathbb{R}$ also.

Now $Y - [0,1] = (2,3) \Rightarrow [0,1]$ is closed in Y (subs. top.)

& $Y - (2,3) = [0,1] \Rightarrow (2,3)$ is closed in Y (subs. top.)

Thus the sets $[0,1]$ & $(2,3)$ are open as well as closed in Y.

<u>Thm:</u> Let $(X, \mathcal{T})$ be a top. space. Then following holds:

(1) ϕ & X are closed,

(2) arbitrary intersection of closed sets are closed.

(3) finite union of closed sets are closed.

~~Fact~~ Defn: Let $Y \subseteq X$, be a topological space (subspace topology). Then we say a set C is closed in Y if and only if $C = A \cap Y$, A is a closed subset of X, X is a top. space.

Remark: 1.) A set C which is closed in Y need not be closed in X.

2.) If $Y \subseteq X$ (subspace top'), C is closed in Y & Y is closed in X, then C is closed in X.

Let $C \subseteq X$. We define, the interior of C, $\operatorname{int}(C) := \{ \bigcup_\alpha U_\alpha \mid \alpha$ runs over all open sets $U_\alpha \ni U_\alpha \subseteq C \}$

& the closure of C,

$\bar{C}$ or $Cl(C) := \{ \bigcap_\beta C_\beta \mid \beta$ runs over all closed sets $C_\beta \ni C \subseteq C_\beta \}$

Remarks: 1) By defn, interior of a set in a topological set is an open set & closure of a set is a closed set

2) For $A \subseteq X$, $\text{int } A \subseteq A \subseteq \overline{A}$
& we say A is open if $A = \text{int } A$
& A is closed if $A = \overline{A}$.

3) Let $Y \subseteq X$ be a topological subspace of X. & let $A \subseteq Y$
& let $\overline{A}_X$ denotes the closure of subset A in X
& $\overline{A}_Y$ denotes the closure of A in Y
then $\overline{A}_Y = \overline{A}_X \cap Y$.

Facts:

Defn: Let $A \subseteq X$. Then
(1) $x \in \overline{A}$ if and only if every open set U containing x intersects A, (i.e, $U \cap A \neq \emptyset$ if $x \in U$)

(2) Let $\mathcal{B}$ be a basis for the topology on X. Then $x \in \overline{A}$ if and only if every $B \in \mathcal{B} \ni x \in B$ intersects A.

Proof: (1) We show the equivalence for contrapositive stmts.

($\because P \Leftrightarrow Q \sim (\text{not } P) \Leftrightarrow (\text{not } Q)$)

$x \notin \bar{A}$ iff $\exists$ an open set U containing x that does not intersects A.
$\Leftrightarrow$

$\Rightarrow$: Assume $x \notin \bar{A} \Rightarrow x \in X - \bar{A}$

$X - \bar{A}$ is an open set of X

Take $U = X - \bar{A}$ & $A \cap X - \bar{A} = \phi$

($\because \bar{A} \cap X - \bar{A} = \phi$ & $A \subseteq \bar{A}$)

hence, the implication.

$\Leftarrow$: Suppose U is an open set which contains x & $U \cap A = \phi$ (~~$A \subseteq X - U$~~)

Now $X - U$ is closed set & $x \notin X - U$, by defn, $\bar{A} \subseteq X - U$

(~~$\because A \subseteq X - U$~~)

Therefore, $x \notin \bar{A}$ ($\because X - U$ is a closed set which contains A)

Proof of (2): exercise.

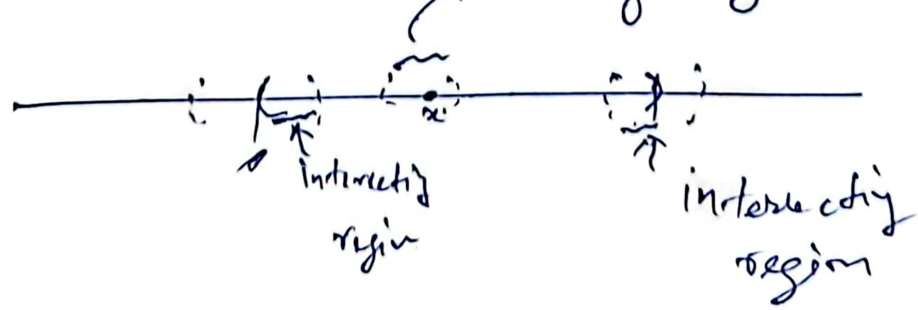
Convention: "U is an open set containing x" $\Leftrightarrow$ "U is a neighborhood of x".

Note: If $A \subseteq X$, then $x \in \bar{A}$ if and only if every neighborhood of x intersects A.

Examples: 1) Let $X = (\mathbb{R}, \tau_{std.})$ & let $A = (0,1)$, then

$\bar{A} = [0,1]$, $\because$ ~~every~~ clearly every nbhd U_x of every point $x \in (0,1)$ intersects $(0,1)$; $(U_x \cap (0,1) \neq \phi)$

Now, if $x = 0$ or 1 (although $x \notin (0,1)$) every nbhd of 0 or 1 has a non-empty intersection with $(0,1)$



2) Let $A = \{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \}$. Then $\bar{A} = \{0\} \cup A$.

Clearly, $\frac{1}{n} \in \bar{A}$, $\forall n \in \mathbb{Z}_+$

$\because$ every neighborhood of $\frac{1}{n}$ atleast contains $\frac{1}{n}$ & hence the intersection is non-empty (i.e. $U \cap A = \{\frac{1}{n}\}$ atleast) $(U \cap A \neq \phi)$.

$0 \in \bar{A}$; This is because every neighborhood of 0 in $\mathbb{R}$, ~~conta~~ however small, contains infinitely many points of the form $\frac{1}{m}$; $m \in \mathbb{Z}_+$

($\because$ if $I = (-\varepsilon, \varepsilon)$ is ~~such~~ a neighborhood ~~of 0 then any~~ $x \in I$, will ~~satisfy~~ $|x| < \varepsilon$ & or $x < \varepsilon$ (if ~~x~~ is on the right of 0).

~~Now Using the~~

of 0 & ~~the interval $(0, \varepsilon)$. contains~~

$0 < \varepsilon$, $\exists N_0 \in \mathbb{N} \ni$

$\frac{1}{N} < \varepsilon$, $\forall N \geq N_0$.

(using Archimedean property of $\mathbb{R}$;

3) & $C = \{0\} \cup (1, 2)$. Then $\bar{C} = \{0\} \cup [1, 2]$

4)

CamScanner

4) Let $A = \mathbb{Q} \subseteq \mathbb{R} = X$. Then $\bar{A} = X$ or $\overline{\mathbb{Q}} = \mathbb{R}$.

Clearly, any $x \in \mathbb{Q} \subseteq \overline{\mathbb{Q}}$
[$\because$ every nbhd U of x in $\mathbb{R}$ contains ~~...~~ countably many rationals & hence
or $U \cap \mathbb{Q} \neq \phi$]

Now if $x \in \mathbb{Q}$ irrational no., then ~~also~~ ~~there~~ any nbhd V of x contains countably many rationals.
$V \cap \mathbb{Q} \neq \phi$.

5) $\overline{\mathbb{N}} = \mathbb{N}$ or $\overline{\mathbb{Z}_+} = \mathbb{Z}$.

6) $\mathbb{R}^+ = \{x > 0 \mid x \in \mathbb{R}\}$. Then $\overline{\mathbb{R}^+} = \mathbb{R}^+ \cup \{0\}$.

7) $Y = [0,1] \subseteq \mathbb{R} = X$; $A = (0, \frac{1}{2}) \subseteq Y$, its closure in $\mathbb{R}$ is $[0, \frac{1}{2}] = \bar{A}_{\mathbb{R}}$ → closed in $\mathbb{R}$

whereas its closure in Y is $(0, \frac{1}{2}]$ closed set in Y.
$= [0, \frac{1}{2}] \cap Y$

Limit Points

Defn: Let $A \subseteq X$, a set. A point $x \in X$ is a limit point of A if every nbhd of x in X, intersects A in some point different than x. In other words, x is a limit point of A if $x \in \overline{A - \{x\}}$

$X = \mathbb{R}$

Examples 1.) $A = (0,1]$. Then 0 is a limit point of A. Because any nbhd around 0 will contain an ~~look like~~ open interval of the form $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ & $(-\varepsilon, \varepsilon) \cap (0,1] = (0, \varepsilon] \neq \phi$

Hence, we have shown any nbhd of 0 intersects $(0,1]$ in points different from 0. A similar argument will show any other point $x \in (0,1]$ is also a limit point of $(0,1]$.

$\therefore$ the limit point set of A is $[0,1]$.

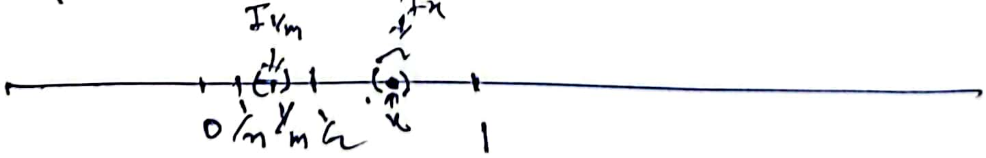
(If $x \in \mathbb{R} \setminus [0,1]$, then x is not a limit point of A. This is because given x, we can always choose an interval I around x $\ni I \cap A = \phi$. (small enough).

$\text{dist}(x, 1) = a$

$I = (x - a/2, x + a/2)$)

2.) $B = \{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \} \subsetneq \mathbb{R}$

0 is the only limit point of B & no other point of $\mathbb{R}$ is a limit point of B.



$I_x \cap B = \phi$; $I_{1/m} \cap B = \{1/m\}$

$\Downarrow$

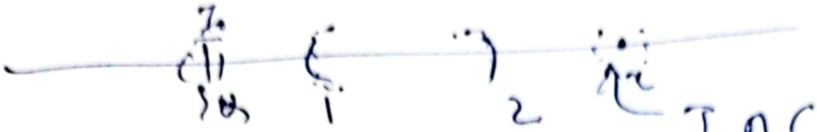
x is not a limit point

$\Downarrow$

$1/m$ is not a limit point

3) $C = \{0\} \cup (1,2) \subseteq \mathbb{R}$

The set of limit points of C is $[1,2]$, no other point of $\mathbb{R}$ is a limit point of C.



$I_1 \cap C = \phi$ $\Rightarrow$ not a limit point.

$I_0 \cap C = \{0\}$ $\Rightarrow$ 0 is not a limit point

4) The set of limit points of $\mathbb{Q}$ (rational numbers) is $\mathbb{R}$.

5) The set of limit points of $\mathbb{Z}_+$ (set of natural no.s) in $\mathbb{R}$ is an empty set.

~~Fact~~ **Fact:** Let $A \subseteq X$; X - top space & let A' be the set of all limit points of A. Then $\bar{A} = A \cup A'$.

Def: A set $A \subseteq X$ is closed in X iff it contains all its limit points.

(See eg. 5.)

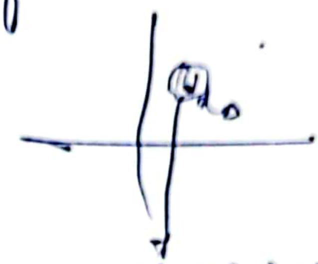
Fact: Singletons are closed in $\mathbb{R}^n$.

Proof: Let $x_0 \in \mathbb{R}^n$, ~~then~~ consider the set $\{x_0\} \subseteq \mathbb{R}^n$, then the set of limit points of $\{x_0\}$ is an empty set;

$\therefore$ by previous fact,

$$\overline{\{x_0\}} = \{x_0\} \cup \phi = \{x_0\}$$

$\Rightarrow$ $\{x_0\}$ is closed in $\mathbb{R}^n$.



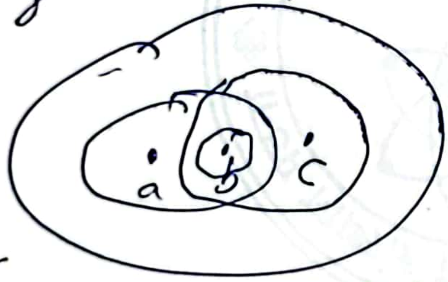
$U_{x_0} \cap \{x_0\} = \{x_0\}$

$\Downarrow$

x_0 is not a limit point

Note: In an arbitrary topological space, singletons may not be closed.

Eg 1:



$X = \{a, b, c\}$

$\tau = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}$

$\{b\}$ is not closed, as its complement

$\{b\}^c = \{a, c\} \notin \tau$.

is not open.

Defn: ~~For a topol.~~ Let (X, τ) be a topological sp. We say a sequence of elements $\{x_i\}_{i \in \mathbb{N}}$ converges to a point x in X ($x_i \longrightarrow x$), if $\forall$ neighborhood U of x in X, $\exists$ a natural number $N \in \mathbb{N}$ $\ni$ $x_n \in U$, $\forall$ $n \geq N$.

Example: X)

Note: In an arbitrary topological space a sequence may converge to more than one point. For eg., in the Eg. 1 (last page) if we take $x_n = b$, $\forall n \in \mathbb{N}$, then $\{x_n\}$ ~~defines a seq~~ defines a sequence in X. & we $x_n \to a$, $x_n \to b$, $x_n \to c$

Neighborhoods of a in τ : $\{a,b\}, X$

" " b in τ : $\{b\}, \{a,b\}, \{b,c\}, X$.

" " c in τ : $\{b,c\}, X$.

In each of these cases we can take $N=1$.

To avoid these kind of situations, we are going to define an important ~~class~~ kind of topology ~~known as~~ (where the above things won't happen) known as Hausdorff topology. (this is due to Felix Hausdorff)

Def^n: Let (X, τ) be a topological space. We say (X, τ) is Hausdorff (or τ is Hausdorff topology on X) if whenever $x \neq y$; $x, y \in X$, $\exists\, U_x \in \tau$ & $U_y \in \tau$ $\ni$ $x \in U_x$ & $y \in U_y$ $U_x \cap U_y = \phi$.

[this also means that distinct points in a Hausdorff space can be separated by disjoint open sets]

Some important Theorems about Hausdorff spaces:

1.) Finite point sets are closed in a Hausdorff space.

Note: The condition finite point sets being closed is a weaker condition than the Hausdorff condition.

Eg $(\mathbb{R}, \tau_f)$ is not a Hausdorff space, but finite point sets are closed in fin. topology.

Def^n: If in a topological space (X, τ), every finite point sets are closed, we say (X, τ) is a T_1- space.

Fact: Let (X, τ) be a T_1- space & let $A \subseteq X$. Then $x \in X$ is a limit point of A if and only if every neighborhood of x contains infinitely many elements of A.

~~Fact~~ Theorem: Let (X, τ) be a Hausdorff space, ~~Then~~ & Let $\{x_n\}$ be a sequence of points of X, then this sequence can converge at most one point of X. In other words, in a Hausdorff space every sequence has a unique limit.

Proof: Let $x_n \to x$ & let $y \neq x$, then we can find $U_x, U_y \ni U_x \cap U_y = \phi \ni x \in U_x \& y \in U_y$

As U_x contains infinitely many x_n's, there are only finitely many x_n's left which may go inside U_y, hence $x_n \not\to y$.

Fact: Every simply ordered set is a Hausdorff space (order topology). Product of two Hausdorff space is Hausdorff. A Subspace of Hausdorff space is Hausdorff.

Continuous functions: (

Defn: To talk about continuous functions we need topological spaces.

Defn: Let (X, τ) & (Y, η) be two topological spaces. & Let $f: X \longrightarrow Y$ be a function. We say f is continuous if $V \in \eta \Rightarrow f^{-1}(V) \in \tau$, this means, if inverse image of every open set in (Y, η) is open in (X, τ).

Note: 1. $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$

so if $V \cap f(X) = \phi$. then $f^{-1}(V) = \phi$.

2. Since any open set V can be expressed

as $V = \bigcup_{\alpha} B_\alpha$ ($B_\alpha \in \mathcal{B}_Y$ & basis for η)

& we know $f^{-1}(V) = \bigcup_{\alpha} f^{-1}(B_\alpha)$

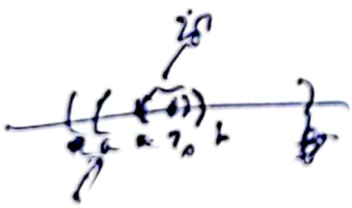
$f^{-1}(B_\alpha) \in \tau$, ($\because B_\alpha \in \mathcal{B}_Y \subseteq \eta$ open)

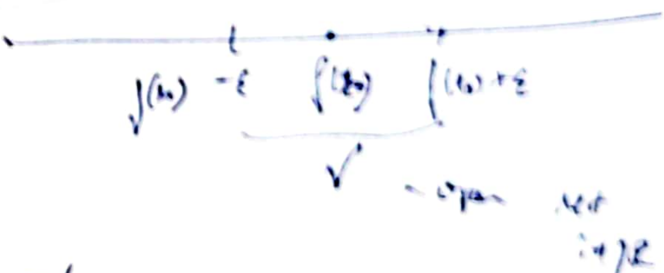
& τ is a topology, $\therefore$ $f^{-1}(V) = \bigcup_{\alpha} f^{-1}(B_\alpha) \in \tau$

& hence $f^{-1}(V)$ is open

($f^{-1}(V) \in \tau$)

3. The above def^n imply the ε-δ def^n for continuity for a function $f : \mathbb{R} \to \mathbb{R}$.





$f^{-1}(V)$ is open in $\mathbb{R}$,

& $x_0 \in f^{-1}(V)$, $\exists$ an interval

I (basis elt) s.t. $x_0 \in I = (a, b)$

Take $\delta = \min(x_0 - a, b - x_0)$

then if $x \in (x_0 - \delta, x_0 + \delta) \subseteq (a,b) \subseteq f^{-1}(V)$

then $f(x) \in V$ & so hence, i.e. $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$

$\Rightarrow |f(x) - f(x_0)| < \varepsilon$

~~Charac~~ A useful characterization of continuous functions:

Let (X,τ) & (Y,η) be topological spaces & let $f: X \longrightarrow Y$ be a function. Then the following statements are equivalent.

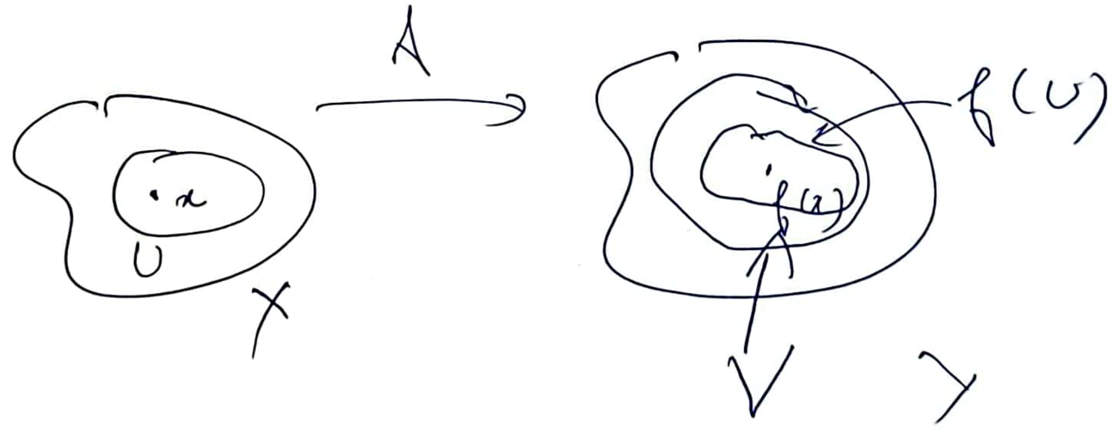
1) f is continuous

~~2) If $A \subseteq X$ is closed &~~

2) For every $A \subseteq X$, $\Rightarrow f(\bar{A}) \subseteq \overline{f(A)}$

3) If $B \subseteq X$ is closed in (Y,η) $\Rightarrow f^{-1}(B)$ is closed in (X,τ).

4) $\forall x \in X$ & $\forall$ neighborhood V of $f(x)$, $\exists$ a nbhd U of $x \ni f(U) \subseteq V$.



Homeomorphisms

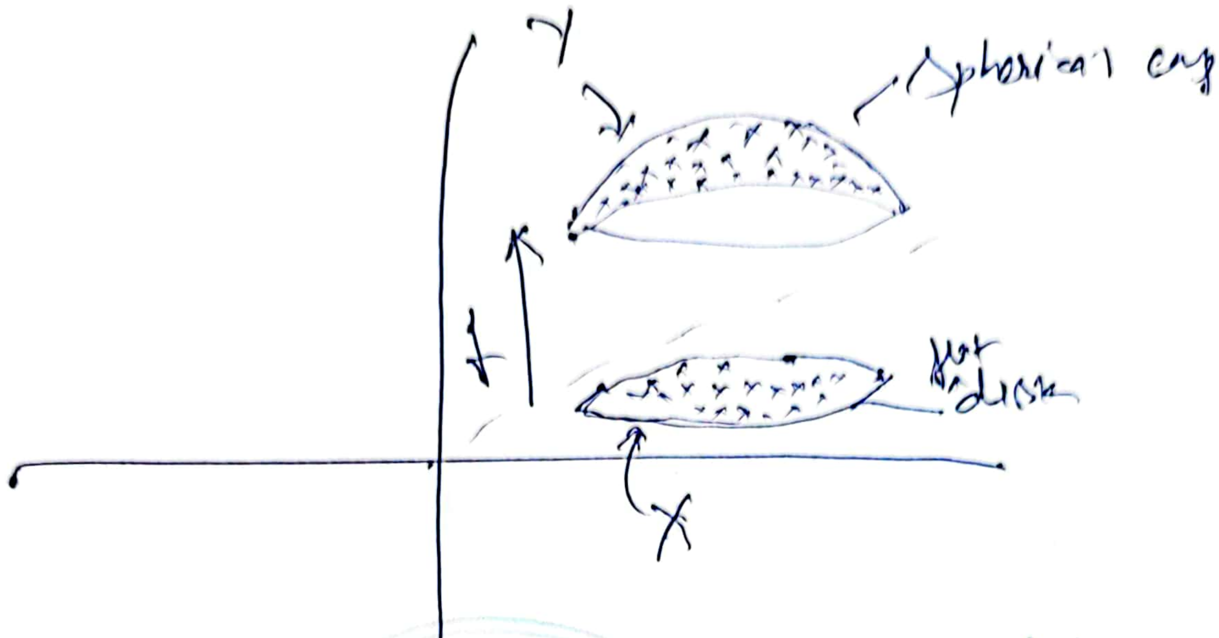
A bijective map $f: X \to Y$ is said to be a homeomorphism, if both the map f and its inverse map $f^{-1}: Y \to X$ are continuous.

Note: 1. If $f: X \to Y$ is a bijective map. Then f is homeomorphism, ~~if and only if $f(U)$~~ if the following is satisfied: U is open in X if and only if $f(U)$ is open in Y. — (1)

This ~~def~~ (1) also reflects the fact that a homeomorphism also gives a bijective correspondence between the topologies on X and Y. This means if τ is a topology on X & η is a topology on Y, then τ is equivalent to η in the sense "τ & η have the same cardinality".

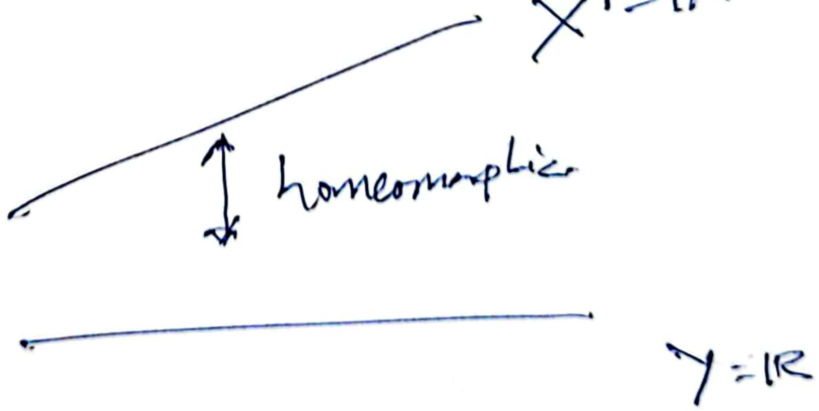
Examples

1)

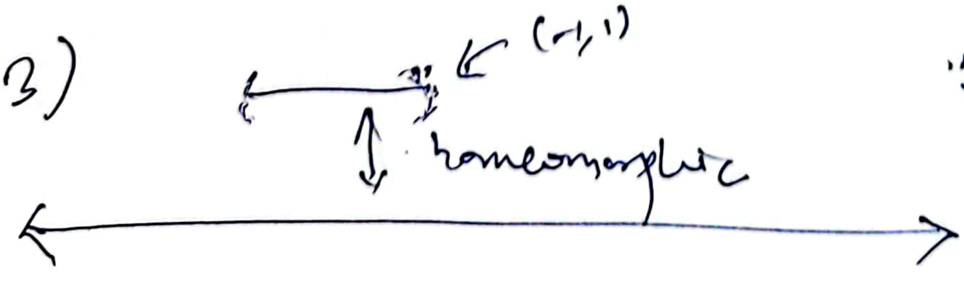


$f : X \to Y$ is a homeomorphism & we say that the spaces X and Y are homeomorphic.

2) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3x+1$ is a homeomorphism



3)

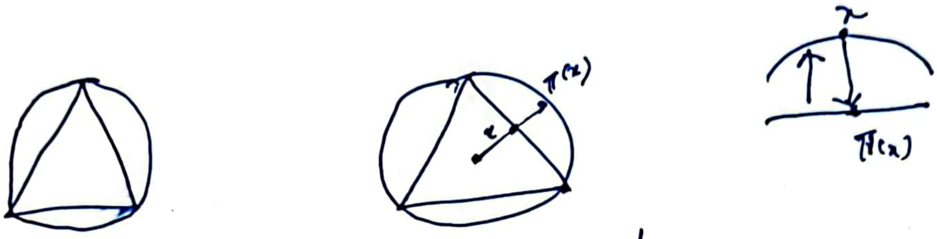


$f : (-1,1) \to \mathbb{R}$

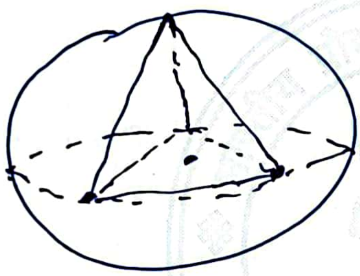
$f(x) = \frac{x}{1-x^2}$

What is its inverse?

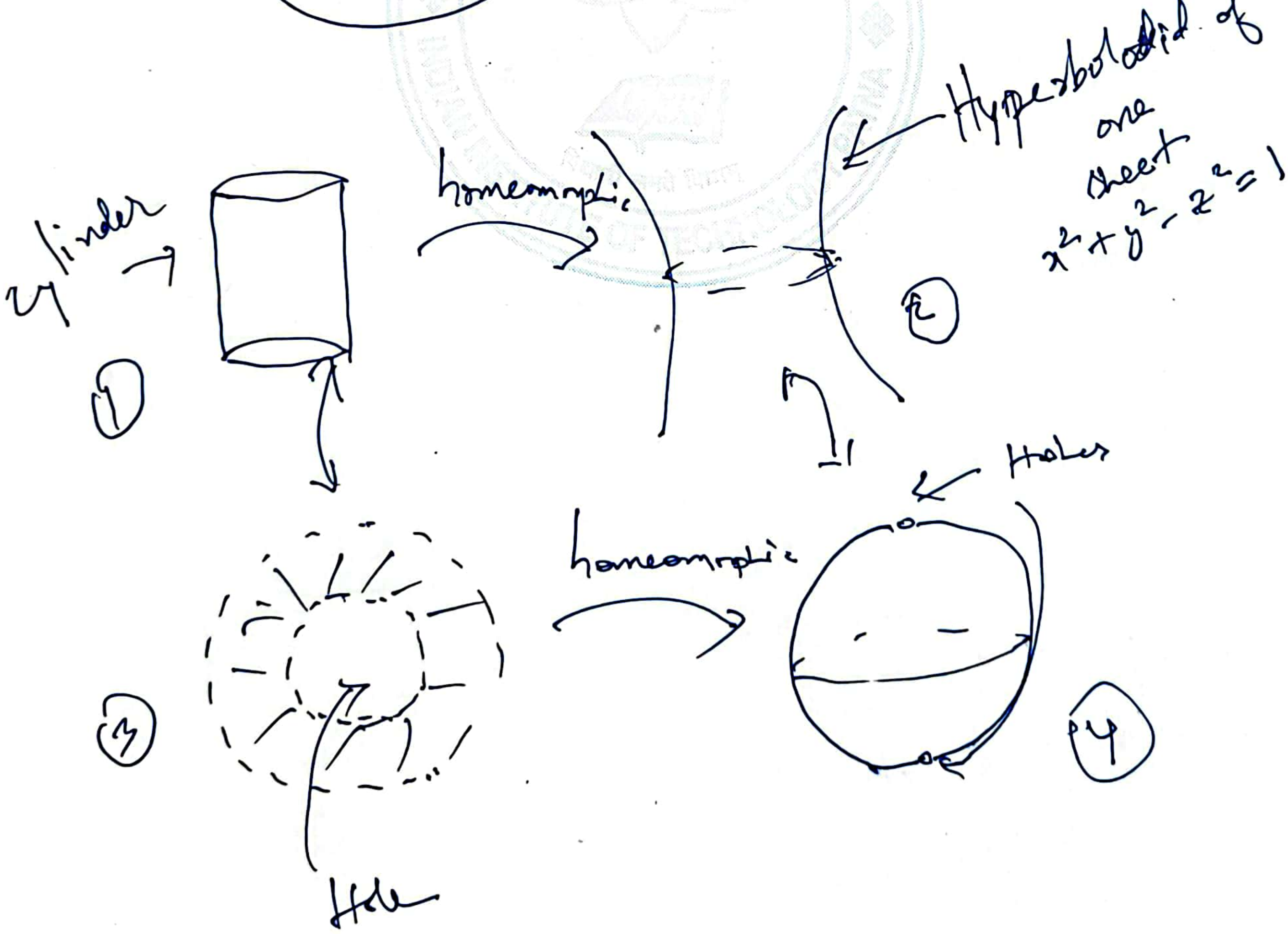
Examples



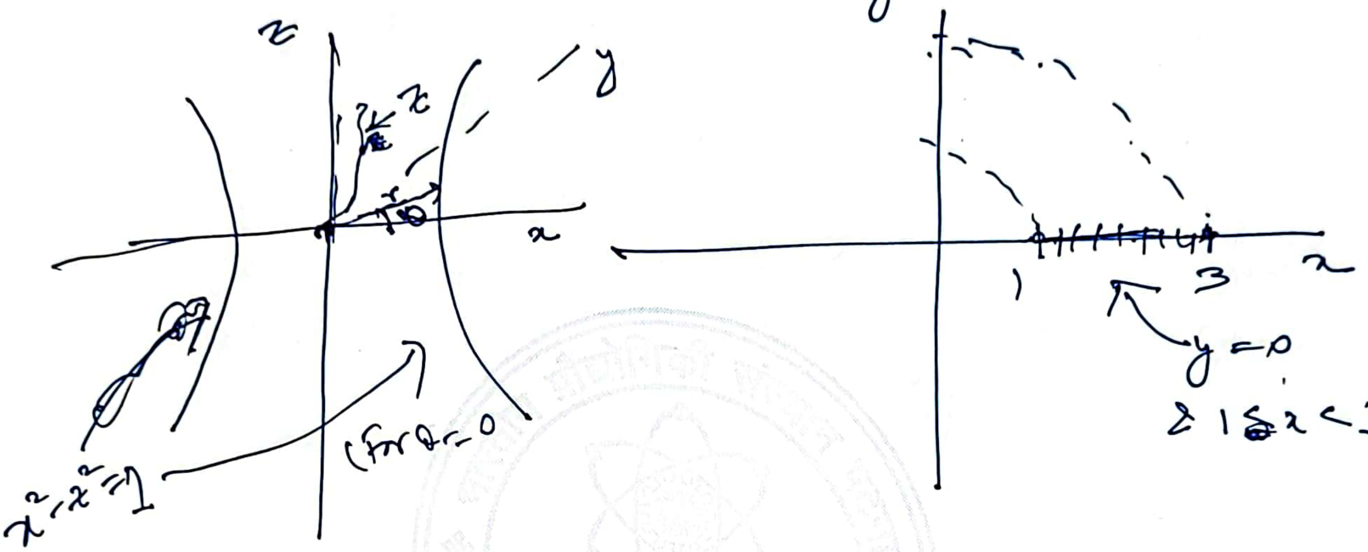
A triangle and a circle are homeomorphic.
(topologically same)



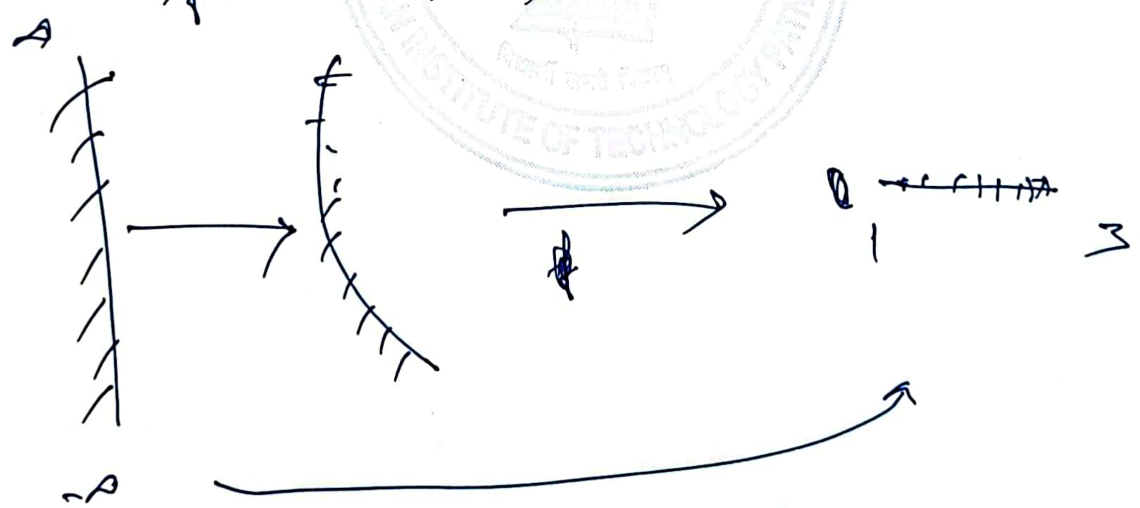
Tetrahedron and sphere are topologically same



① We use cylindrical / polar coordinates to assign coordinates on the ~~cylinder~~ hyperboloid of one sheet (r, θ, z)



$f : (-\infty, \infty) \to (1, 3)$



$$f(x) = \frac{x}{(1+|x|)} + 2.$$

Then f is a bijection f $\mathbb{R}$ & $(1, 3)$ is also continuous & its inverse f is also continuous

Find f^{-1}

Now when $\theta \in [0, 2\pi]$, we get the full parabola & for $\theta = 0$ & $\theta = 2\pi$, the branches of hyperbola which we get, coincides.

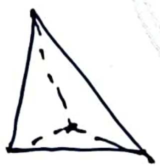
Now we can define a map

f : Hyperboloid of one sheet $\longrightarrow$ Annulus

$(r, \theta, z) \xmapsto[\text{homeomorphism}]{} (f(z), \theta)$

Remark: Homeomorphism defines an equivalence relation on the class of topological spaces.

Tetrahedron (T)

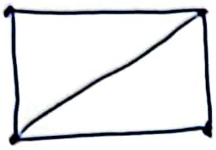


$\rightarrow$ # vertices $\overset{V}{=} 4$, # edges $\overset{E}{=} 6$

faces $\overset{F}{=} 4$,

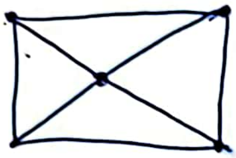
$V - E + F = 4 - 6 + 4 = 2$

or $\chi(T) =$ Euler's no. $= 2$



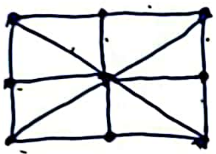
$\sim \#V=4, \#E=5, \#F=2$

$V-E+F = 4-5+2 = 1$



$\sim \#V=5, \#E=8, \#F=4$

$V-E+F = 5-8+4 = 1$



$\sim \#V=9, \#E=16, \#F=8$

$V-E+F = 9-16+8 = 1$

Fact: Homeomorphic topological spaces have the same Euler number.

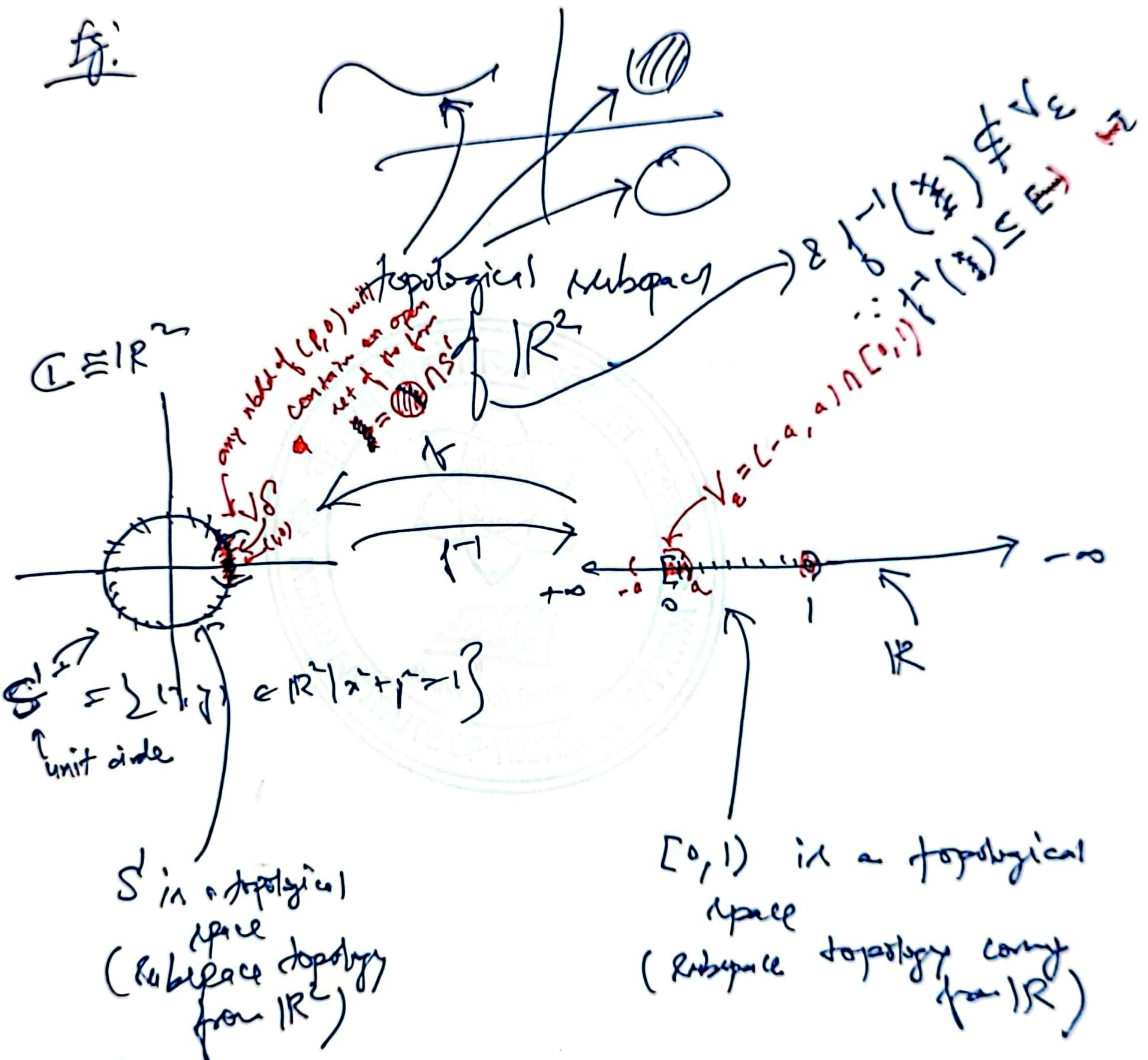
- Euler no. of a polyhedra is 2.

Defn: A map $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $x_0 \in \mathbb{R}^n$ if $\forall\, \varepsilon > 0$ $\exists\, \delta > 0 \ni \|f(x) - f(x_0)\|_{\mathbb{R}^m} < \varepsilon$ whenever $\|x - x_0\|_{\mathbb{R}^n} < \delta$.

- $\mathbb{R}^n$ & $\mathbb{R}^m$ are not homeomorphic, if $n \neq m$. (difficult to prove)

• Any subset of a Euclidean space ($\mathbb{R}^n$), automatically becomes a topological space under the subspace topology coming from $\mathbb{R}^n$.

Eg:



Define $f : [0,1) \longrightarrow S^1$ by

$$x \longmapsto e^{2\pi i x} \text{ or } (\cos 2\pi x, \sin 2\pi x)$$

Then f is continuous, one-one & onto, but its inverse f^{-1} is not continuous.