Mid Term Examination: MA220 (Numerical Linear Algebra)

Time: 2 Hours Maximum Marks: 30

Roll No.: **Name:**

There are **11** questions in this paper. Attempt all questions. Give precise and brief answers. Standard results/formulae may be used. Symbols/notations have their usual meaning as per Lecture's discussion. **No electronic device, excluding a simple and non-programmable calculator, is allowed**. Do not write anything on the question paper except the Roll no. and Name at the top.

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1. Let $||\cdot||_2$ be standard vector norm on $\mathbb{R}^n$. The same notation $||\cdot||_2$ also indicates the standard matrix norm on $\mathbb{R}^{n\times n}$. Let $P \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. It is given that $||x||_* = ||Px||_2$ is a vector norm on $\mathbb{R}^n$. Take $||A||_* = ||PAP^{-1}||_2$. Show that $||\cdot||_*$ is a norm on $\mathbb{R}^{n\times n}$ that is induced by the vector norm $||\cdot||_*$. [3]

2. Let $A = \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}$, where $\epsilon = 0.3 \times 10^{-2}$. Use Gauss elimination with the 4−decimal-digit floating point arithmetic and find LU decomposition of A without partial pivoting. Use this LU decomposition to solve $Ax = b$, where $b = \begin{bmatrix} 0.7 \\ 0.9 \end{bmatrix}$, and denote the solution by $\hat{x}$. Repeat this problem with partial pivoting, and by using the corresponding PLU decomposition of A, find solution of the given system and denote it by $\tilde{x}$. Which one from $\hat{x}$ and $\tilde{x}$ is closer to the exact solution of the system? What will happen if $\epsilon = 0.3 \times 10^{-15}$ and computation is done on a machine that supports 16−decimal-digit floating point arithmetic? [4]

NOTE: For the following Questions 3 - 7: Take $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$.

3. Find matrices $Q \in \mathbb{R}^{3\times 2}$ and $R \in \mathbb{R}^{2\times 2}$ such that $Q^TQ = I_2$, R is diagonal and invertible, and $A = QR$. [Hint: The columns of A are orthogonal - you can find Q and R just by using the rules of matrix-matrix multiplication.] [1.5]

4. Using the QR decomposition obtained in Question 3, find the MP-inverse of A. [Warning: No marks will be awarded if you find the MP-inverse by some other formula - you have to use the QR decomposition obtained in Question 3.] [1.5]

5. Find SVD: $A = U\Sigma V^T$, where $U \in \mathbb{R}^{3\times 2}$, $V \in \mathbb{R}^{2\times 2}$ and $\Sigma \in \mathbb{R}^{2\times 2}$ such that $U^TU = I_2$, $V^TV = I_2$, $\Sigma = \begin{bmatrix} \sqrt{14} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$. [Hint: Just convert the QR decomposition obtained in Question 3 into the SVD - again the rules of matrix-matrix multiplication may be useful.] [1.5]

6. Using SVD obtained in Question 5, find the MP-inverse of A. Is the answer exactly same as in Question 4? Why? [1.5]

7. By finding an appropriate basis of the left null space of A, extend the economic size SVD of A obtained in Question 5 into the full SVD. Explain all steps of your answer. Note that we have obtained the economic size (reduced order) SVD of A in Question 5. [3]

$\underline{\hat{x} = A^{+} b}$

$Ax = b$

$A^T A \hat{x} = A^T b$

$R^T \cancel{Q^T} \cancel{Q} R \hat{x} = R^T Q^T b$

$\boxed{R\hat{x} = Q^T b}$

8. An experiment has been performed where the following data has been collected

x_i	-1	0	1	2
y_i	1	0	1	3

Find a best fit curve $y = f(x) := c_0 + c_1 x + c_2 x^2$ that approximates the data by finding the Choleski decomposition of the matrix associated to the least square solution normal equation. Is the solution unique? [3]

9. Judge the convergence of Jacobi and Gauss-Seidel iterative methods for the system $Ax = b$, where $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ [4]

10. Find Householder QR decomposition of $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ [4]

11. Using RREF, find the basis of the row space, column space, and null space of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$

Further, find a full-column rank matrix C and a full-row rank matrix R such that $A = CR$ and then express the matrix A as a sum of rank-one matrices. [3]

Ques 1

First, we show that

$\|A\|_* = \|PAP^{-1}\|_2$ is a norm

(i) $\|A\|_* = \|PAP^{-1}\|_2 \geq 0$ ($\because$ $\|\cdot\|_2$ is a norm)

(ii) $\|A\|_* = 0 \Leftrightarrow \|PAP^{-1}\|_2 = 0 \Leftrightarrow PAP^{-1} = 0$ ($\because$ $\|\cdot\|_2$ is a norm)

$\Updownarrow$

$A = 0$ ($\because$ P is nonsingular)

(iii) $\|\alpha A\|_* = \|\alpha PAP^{-1}\|_2 = |\alpha|\,\|PAP^{-1}\|_2 = |\alpha|\,\|A\|_*$

(iv) $\|A+B\|_* = \|P(A+B)P^{-1}\|_2 = \|PAP^{-1} + PBP^{-1}\|_2$

$\leq$

$\|PAP^{-1}\|_2 + \|PBP^{-1}\|_2$ ($\because$ $\|\cdot\|_2$ is a norm)

$=$

$\|A\|_* + \|B\|_*$

Now

$$\|A\|_* = \|PAP^{-1}\|_2 = \sup_{x \neq 0} \frac{\|PAP^{-1}x\|_2}{\|x\|_2}$$

$$= \sup_{y \neq 0} \frac{\|PAy\|_2}{\|Py\|_2} \quad (\text{Take } P^{-1}x = y)$$

$$= \sup_{y \neq 0} \frac{\|Ay\|_*}{\|y\|_*}$$

Given that $\|z\|_* = \|Pz\|_2$ is a vector norm.

Hence $\|Ay\|_* \leq \|A\|_* \|y\|_*$

and clearly, matrix norm $\|\cdot\|_*$ is induced by vector norm $\|\cdot\|_*$.

Ques 2

$$A = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{E_1} \begin{bmatrix} \varepsilon & 1 \\ 0 & \beta \end{bmatrix} = U \qquad \beta = 1 - \frac{1}{\varepsilon}$$

$$E_1 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{\varepsilon} & 1 \end{bmatrix}$$

So

$$A = LU = E_1^{-1} U$$

where $L = \begin{bmatrix} 1 & 0 \\ \frac{1}{\varepsilon} & 1 \end{bmatrix}$ $\quad U = \begin{bmatrix} \varepsilon & 1 \\ 0 & \beta \end{bmatrix}$

Solve $\quad Ax = b = \begin{bmatrix} 0.7 \\ 0.9 \end{bmatrix}$

$\Updownarrow$

$LUx = b$

$Lc = b \Rightarrow c = \begin{bmatrix} 0.7 \\ 0.9 - \frac{0.7}{\varepsilon} \end{bmatrix} = \begin{bmatrix} 0.7000 \\ \alpha \end{bmatrix}$ (forward subs)

$U\hat{x} = c \Rightarrow \hat{x} = \begin{bmatrix} \left(0.7 - \frac{\alpha}{\beta}\right)/\varepsilon \\ \frac{\alpha}{\beta} \end{bmatrix} = \begin{bmatrix} 0.2006 \\ 0.6994 \end{bmatrix}$ (Backward subs)

$$A = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{P_1} \begin{bmatrix} 1 & 1 \\ \varepsilon & 1 \end{bmatrix} \xrightarrow{E_2} \begin{bmatrix} 1 & 1 \\ 0 & \beta_1 \end{bmatrix}$$

$\varepsilon = 0.3 \times 10^{-2} = 0.003$

$\beta_1 = 1 - \varepsilon$

$E_2 = \begin{bmatrix} 1 & 0 \\ -\varepsilon & 1 \end{bmatrix}$ $\quad P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\boxed{\begin{array}{l} \text{where} \\ P = P_1 \\ L = \begin{bmatrix} 1 & 0 \\ \varepsilon & 1 \end{bmatrix} \& \ U = \begin{bmatrix} 1 & 1 \\ 0 & \beta_1 \end{bmatrix} \end{array}}$$

$PAx = Pb = \begin{bmatrix} 0.9 \\ 0.7 \end{bmatrix}$

$\Updownarrow$

$LUx = \begin{bmatrix} 0.9 \\ 0.7 \end{bmatrix}$

$Lc = \begin{bmatrix} 0.9 \\ 0.7 \end{bmatrix} \Rightarrow c = \begin{bmatrix} 0.9 \\ 0.7 - 0.9 \times \varepsilon \end{bmatrix} = \begin{bmatrix} 0.9 \\ \alpha_1 \end{bmatrix}$

$Ux = \begin{bmatrix} 0.9 \\ \alpha_1 \end{bmatrix} \Rightarrow \tilde{x} = \begin{bmatrix} 0.9 - \frac{\alpha_1}{\beta_1} \\ \frac{\alpha_1}{\beta_1} \end{bmatrix} = \begin{bmatrix} 0.2006 \\ 0.6994 \end{bmatrix}$

Thus $\hat{x} = \tilde{x} = \begin{bmatrix} 0.2006 \\ 0.6994 \end{bmatrix}$. In the given situation, $\tilde{x}$ will be more close to exact solution as pivoting reduces the numerical error.

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$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{14} \\ 1/\sqrt{3} & 1/\sqrt{14} \\ 1/\sqrt{3} & -3/\sqrt{14} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{14} \end{bmatrix}$$

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(least squares problem)

Recall three methods to solve LSP: $Ax = b$

ⓐ $- A^T A \hat{x} = A^T b$

ⓑ $- R\hat{x} = Q^T b$

ⓒ $- \hat{x} = A^+ b$

From ⓑ & ⓒ : $A^+ = R^{-1} Q^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{14}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 2/\sqrt{14} & 1/\sqrt{14} & -3/\sqrt{14} \end{bmatrix}$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 2/14 & 1/14 & -3/14 \end{bmatrix}$$

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From

$$A = \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{14} \\ 1/\sqrt{3} & 1/\sqrt{14} \\ 1/\sqrt{3} & -3/\sqrt{14} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{14} \end{bmatrix} = QR$$

$$= \underbrace{Q \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{=} \overbrace{\begin{bmatrix} \sqrt{14} & 0 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{= R}$$

$$= \begin{bmatrix} 2/\sqrt{14} & 1/\sqrt{3} \\ 1/\sqrt{14} & 1/\sqrt{3} \\ -3/\sqrt{14} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{14} & 0 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= U \Sigma V^T$$

Q6

From part (iii)

$$A^+ = V \Sigma^+ U^T$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{14} & 0 \\ 0 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2/\sqrt{14} & 1/\sqrt{14} & -3/\sqrt{14} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2/14 & 1/14 & -3/14 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 2/14 & 1/14 & -3/14 \end{bmatrix}$$

Yes the answer is same, because, A^+ is unique.

~~(2)~~ $\cancel{B = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}}$

~~Take~~ $\cancel{P = \frac{1}{\sqrt{2}} \begin{bmatrix} V & -V \\ U & U \end{bmatrix}}$ ~~Then~~

$\cancel{P^T B P = \frac{1}{2} \begin{bmatrix} V & U \\ -V & U \end{bmatrix} \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} V & -V \\ U & U \end{bmatrix}}$

$\cancel{= \frac{1}{2} \begin{bmatrix} V & U \\ -V & U \end{bmatrix} \begin{bmatrix} A^T U & A^T U \\ AV & -AV \end{bmatrix} = \frac{1}{2} \begin{bmatrix} V^T A^T U + U^T A V \end{bmatrix}}$

Question 7

Reduced order SVD

$$A = V \Sigma U^T \quad (\text{from (iii)})$$

Full SVD

$$A = \left[V \,\middle|\, \text{basis of } Null(A^T) \right]_{3\times 3} \begin{bmatrix} \Sigma \\ \hline 0 \quad 0 \end{bmatrix}_{3\times 2} \left[U^T \right]_{2\times 2} \quad \text{— } (*)$$

$\hookrightarrow$ It is already 2×2, because null(A) is trivial.

$$= V_{new} \Sigma_{new} U^T$$

Now find a basis of $Null(A^T)$ such that new $V_{new}^T V = I$

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\sim$$

$$RREF(A^T) = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 5 \end{bmatrix}$$

so $Null(A^T) = \left\{ \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix} \right\}$ which is orthogonal to the columns of V.

So

$$V_{new} = \left[\begin{array}{c|c} V_{old} & 4/\sqrt{42} \\ \shortparallel & -5/\sqrt{42} \\ V & 1/\sqrt{42} \end{array} \right]$$

Thus

full SVD of A =

$$\begin{bmatrix} 2/\sqrt{14} & 1/\sqrt{3} & 4/\sqrt{42} \\ 1/\sqrt{14} & 1/\sqrt{3} & -5/\sqrt{42} \\ -3/\sqrt{14} & 1/\sqrt{3} & 1/\sqrt{42} \end{bmatrix} \begin{bmatrix} \sqrt{14} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Ques 8

$x_i = -1 \quad 0 \quad 1 \quad 2$

$y_i = 1 \quad 0 \quad 1 \quad 3$

curve to fit: $y = c_0 + c_1 x + c_2 x^2$

Substitute data in curve; we obtain

$$\left.\begin{aligned} c_0 - c_1 + c_2 &= 1 \\ c_0 &= 0 \\ c_0 + c_1 + c_2 &= 1 \\ c_0 + 2c_1 + 4c_2 &= 3 \end{aligned}\right) \equiv \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

$$\equiv \quad Ax = b$$

Normal equation for LS solution $\hat{x}$: $A^T A \hat{x} = A^T b$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \qquad A^T b = \begin{bmatrix} 5 \\ 6 \\ 14 \end{bmatrix}$$

thus $$A^T A = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 & 6 \\ 0 & 5 & 5 \\ 0 & 5 & 9 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 & 6 \\ 0 & 5 & 5 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

So $\underline{A^T A = LU}$, where $U = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & 2 \end{bmatrix}$

Choleski Decomp

and $L = U^T$

$A^T A \hat{x} = A^T b$

$LU\hat{x} = \begin{bmatrix} 5 \\ 6 \\ 14 \end{bmatrix}$

$$Lc = \begin{bmatrix} 5 \\ 6 \\ 14 \end{bmatrix} \equiv \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{5} & 0 \\ 3 & \sqrt{5} & 2 \end{bmatrix} c = \begin{bmatrix} 5 \\ 6 \\ 14 \end{bmatrix} \Rightarrow c = \begin{bmatrix} 5/2 \\ \frac{7}{2\sqrt{5}} \\ 3/2 \end{bmatrix}$$

$$U\hat{x} = c \equiv \begin{bmatrix} 2 & 1 & 3 \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & 2 \end{bmatrix}\hat{x} = \begin{bmatrix} 5/2 \\ \frac{7}{2\sqrt{5}} \\ \frac{3}{2} \end{bmatrix} \Rightarrow \hat{x} = \begin{bmatrix} 3/20 \\ -1/20 \\ 3/4 \end{bmatrix}$$

So curve is $f(x) = \frac{3}{20} - \frac{1}{20}x + \frac{3}{4}x^2 = \boxed{0.15 - 0.05x + 0.75x^2}$ Ans.

This sol. is unique as A has full col. rank.

Ques 9

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad -E = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

NOTE :- $A = D - E - F$

$$-F = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

P for Jacobi $= D$

P for Gauss-Seidel $= D - E$

Error matrix for Jacobi:

$$E_J = P^{-1}(P - A)$$

$$= D^{-1}(D - A) = D^{-1}(E+F) = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

eigenvalues of $E_J = -1.2808,\ 0.5,\ 0.7808$

spectral radius of $E_J = 1.2808$. Hence Jacobi is not convergent.

Error matrix for Gauss Seidel:

$$E_{GS} = P^{-1}(P - A)$$

$$= (D-E)^{-1}(D - E - A) = (D-E)^{-1}F$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

eigenvalues of $E_{GS} = 0,\ \frac{1}{2},\ \frac{1}{2}$

spectral radius of $E_{GS} = \frac{1}{2}$. Hence Gauss-Seidel is convergent for the given system.

Q10

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$\hat{x}_1$ is the first compt of x_1

$$H_1 = P_1 = I_3 - 2\omega_1\omega_1^T$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \beta = -\mathrm{Sign}(\hat{x}_1) = -\text{ive}$$

$$\|x_1\| = \sqrt{3}$$

$$\omega_1 = \frac{x_1 - \beta\|x_1\| e_1}{\|x_1 - \beta\|x_1\| e_1\|}$$

$$= \frac{\begin{bmatrix}1\\1\\1\end{bmatrix} + \sqrt{3}\begin{bmatrix}1\\0\\0\end{bmatrix}}{\left\| \begin{bmatrix}1\\1\\1\end{bmatrix} + \sqrt{3}\begin{bmatrix}1\\0\\0\end{bmatrix} \right\|} = \frac{\begin{bmatrix}1+\sqrt{3}\\1\\1\end{bmatrix}}{\left\|\begin{bmatrix}1+\sqrt{3}\\1\\1\end{bmatrix}\right\|} = \frac{1}{\sqrt{6+2\sqrt{3}}}\begin{bmatrix}1+\sqrt{3}\\1\\1\end{bmatrix}$$

$1+3+2\sqrt{3}+1+1$

$\sqrt{6+2\sqrt{3}}$

$$= \begin{bmatrix} 0.8881 \\ 0.3251 \\ 0.3251 \end{bmatrix}$$

Knowledge of HH 2 marks
1st iteration 1 mark
2nd iteration 1 mark

$$H_1 = \begin{bmatrix} -0.5774 & -0.5774 & -0.5774 \\ -0.5774 & 0.7887 & -0.2113 \\ -0.5774 & -0.2113 & 0.7887 \end{bmatrix}$$

$$H_1 * A = \begin{bmatrix} -1.7321 & -3.4641 \\ 0 & 0.3660 \\ 0 & 1.3660 \end{bmatrix}$$

$$P_2 = I_2 - 2\omega_2\omega_2^T$$

$$x_2 = \begin{bmatrix} 0.3660 \\ 1.3660 \end{bmatrix} \qquad \beta = -\mathrm{Sig}(\hat{x}_2) = -\text{ive}$$

$$\omega_2 = \frac{x_2 - \beta\|x_2\| e_1}{\|x_2 - \beta\|x_2\| e_1\|}$$

$$\|x_2\| = 1.4142$$

$$= \begin{bmatrix} 0.7934 \\ 0.6088 \end{bmatrix} \qquad P_2 = \begin{bmatrix} -0.2588 & -0.9659 \\ -0.9659 & 0.2588 \end{bmatrix}$$

$$H_2 = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P_2 \end{array}\right]$$

Thus

$$H_2 H_1 A = \begin{bmatrix} -1.7321 & -3.4641 \\ 0 & -1.4142 \\ 0 & 0 \end{bmatrix} = R$$

Thus $A = QR$
where $Q = H_1^T H_2^T = \begin{bmatrix} -0.5774 & 0.7071 & 0.4082 \\ -0.5774 & 0 & -0.8165 \\ -0.5774 & -0.7071 & 0.4082 \end{bmatrix}$

Ques 11

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

1 mark

$$I_3 = RREF(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Col space} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} \right\}$$ $\frac{1}{2}$ marks

$$\text{row space} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$ $\frac{1}{2}$ marks

$$\text{Null space} = \{0\}$$ $\frac{1}{2}$ marks

$$A = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} [1 \ 0 \ 0]} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} [0 \ 1 \ 0]} + \underbrace{\begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} [0 \ 0 \ 1]}$$ $\frac{1}{2}$ mark

each matrix has rank 1