Indian Institute of Technology Patna

MA-225: B.Tech. II year

Spring Semester: 2022-23 (End Semester Examination)

Maximum Marks: 50 | Total Time: 3 Hours

Note: This question paper contains eight questions. Answer all questions.

1. Let X be a random variable with probability density function (pdf) given by $f_X(x) = (x/2),\ 0 < x < 2;$ $f_X(x) = 0, otherwise$. Find cumulative distribution function of Y where $Y = 2X - 3$. Find pdf of Y. Evaluate the probability $P(Y < 0.5 \mid Y > -1)$. Also evaluate pdf of $Z = e^{-Y}$. [2+1+1+2]

2. Suppose that X and Y are jointly distributed random variables with pdf given by $f_{X,Y}(x,y) = 3(x^2y + xy^2),\ 0 < x < 1,\ 0 < y < 1;\ \ f_{X,Y}(x,y) = 0, otherwise$. Find correlation coefficient between X and Y. [6]

   -0.36

3. Suppose that X and Y are jointly distributed random variables with probability density function given by $f_{X,Y}(x,y) = c(x+y),\ 0 < x < y < 1;\ \ f_{X,Y}(x,y) = 0, otherwise$. Find the value of constant c. Compute expectations $E(Y \mid X = (1/2))$ and $E(e^{3Y} \mid X = (2/3))$. Find $m\ (> (1/3))$ so that $P(Y > m \mid X = (1/3)) = 0.5$. [1+3+2+3]

4. (i) Let X and Y be jointly distributed continuous random variables with pdf $f_{X,Y}(x,y),\ x > 0,\ y > 0;\ \ f_{X,Y}(x,y) = 0, otherwise$. Using the cumulative distribution function approach derive a formula for computing the pdf of $Z = X + Y$. [2]
   (ii) Suppose that X and Y are independent random variables with probability density functions $f_X(x) = ((\alpha+1)/2)e^{-\frac{\alpha+1}{2}x},\ x > 0,\ \alpha > 0$ and $f_Y(y) = ((\alpha+1)/2)e^{-\frac{\alpha+1}{2}y},\ y > 0,\ \alpha > 0$. Find pdf of $Z = X + Y$. Compute the expected value of Z. Find mode of Z. [2+1+1]

5. Suppose that scores X and Y of two tests are jointly distributed random variables where $(X, Y) \sim BVN(85, 90, 100, 16, 0.8)$. If a randomly selected student's score on test X is given as 80 then compute the probability that her score on test Y is more than 90. Find the probability that sum of her scores on the two tests will be more than 190. Further find c so that probability $P(2X > c) = 0.3$. [2+3+2]

6. Suppose X and Y are independent random variables such that $X \sim gamma(3,1)$ distribution and $Y \sim gamma(4,1)$ distribution. Consider the transformations $U = X + Y$ and $V = X/(X+Y)$. Find joint pdf of (U, V). Then evaluate marginal density functions of U and V. [2+2+2]

7. (i) State and prove weak law of large numbers. [1+1]
   (ii) Define convergence in probability. Consider sequence of random variables $X_n,\ n = 1, 2, 3, \ldots$, with probability mass function $P(X_n = 1) = (1/n)$ and $P(X_n = 0) = 1 - (1/n)$. Show that this sequence converges to zero in probability. [1+1]

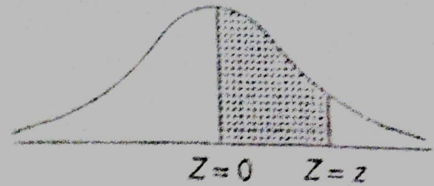
8. Let $X_n,\ n = 1, 2, 3, \ldots$, be sequence of independent and identically distributed (iid) Bernoulli random variables with probability mass function (pmf) $P(X_n = 1) = p,\ P(X_n = 0) = 1 - p,\ 0 < p < 1$. Find the waiting time distribution for the first arrival in this process. Then find the waiting time distribution of kth arrival where k is finite positive integer and $k > 1$. Further compute the expected value of k. [2+2+2]

Normal probability curve is given by :

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, -\infty < x < \infty$$

and standard normal probability curve is given by :

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right), -\infty < z < \infty$$



where $Z = \frac{X - E(X)}{\sigma_X} \sim N(0, 1)$

The following table gives the shaded area in the diagram, *viz.*, $P(0 < Z < z)$ for different values of z.

TABLE OF AREAS

↓Z→	0	1	2	3	4	5	6	7	8	9
·0	·0000	·0040	·0080	·0120	·0160	·0199	·0239	·0279	·0319	·0359
·1	·0398	·0438	·0478	·0517	·0557	·0596	·0636	·0675	·0714	·0759
·2	·0793	·0832	·0871	·0910	·0948	·0987	·1026	·1064	·1103	·1141
·3	·1179	·1217	·1255	·1293	·1331	·1368	·1406	·1443	·1480	·1517
·4	·1554	·1591	·1628	·1664	·1700	·1736	·1772	·1808	·1844	.1879
·5	·1915	·1950	·1985	·2019	·2054	·2088	·2123	·2157	·2190	·2224
·6	·2257	·2291	·2324	·2357	·2389	·2422	·2454	·2486	·2517	·2549
·7	·2580	·2611	·2642	·2673	·2703	·2734	·2764	·2794	·2823	·2852
·8	·2881	·2910	·2939	·2967	·2995	·3023	·3051	·3078	·3106	·3133
·9	·3159	·3186	·3212	·3238	·3264	·3289	·3315	·3340	·3365	·3389
1·0	·3413	·3438	·3461	·3485	·3508	·3531	·3554	·3577	·3599	·3621
1·1	·3643	·3655	·3686	·3708	·3729	·3749	·3770	·3790	·3810	·3830
1·2	·3849	·3869	·3888	·3907	·3925	·3944	·3962	·3980	·3997	·4015
1·3	·4032	·4049	·4066	·4082	·4099	·4115	·4131	·4147	·4162	·4177
1·4	·4192	·4207	·4222	·4236	·4251	·4265	·4279	·4292	·4306	·4319
1·5	·4332	·4345	·4357	·4370	·4382	·4394	·4406	·4418	·4429	·4441
1·6	·4452	·4463	·4474	·4484	·4495	·4505	·4515	·4525	·4535	·4545
1·7	·4554	·4564	·4573	·4582	·4591	·4599	·4608	·4616	·4625	·4633
1·8	·4641	·4649	·4656	·4664	·4671	·4678	·4686	·4693	·4699	·4706
1·9	·4713	·4719	·4726	·4732	·4738	·4744	·4750	·4756	·4761	·4767
2·0	·4772	·4778	·4783	·4788	·4793	·4798	·4803	·4808	·4812	·4817
2·1	·4821	·4826	·4830	·4834	·4838	·4842	·4846	·4850	·4854	·4857
2·2	·4861	·4864	·4868	·4871	·4875	·4678	·4881	·4884	·4887	·4890
2·3	·4893	·4896	·4898	·4901	·4904	·4906	·4909	·4911	·4913	·4916
2·4	·4918	·4920	·4922	·4925	·4927	·4929	·4931	·4932	·4934	·4936
2·5	·4938	·4940	·4941	·4943	·4945	·4946	·4948	·4959	·4951	·4952
2·6	·4953	·4955	·4956	·4957	·4959	·4960	·4961	·4962	·4963	·4964
2·7	·4965	·4966	·4967	·4968	·4969	·4970	·4971	·4972	·4973	·4974
2·8	·4974	·4975	·4976	·4977	·4977	·4978	·4979	·4979	·4980	·4981
2·9	·4981	·4982	·4982	·4983	·4984	·4984	·4985	·4985	·4986	·4986
3·0	·4987	·4987	·4987	·4988	·4988	·4989	·4989	·4989	·4990	·4990
3·1	·4990	·4991	·4991	·4991	·4992	·4992	·4992	·4992	·4993	·4993
3·2	·4993	·4993	·4994	·4994	·4994	·4994	·4994	·4995	·4995	·4995
3·3	·4995	·4995	·4995	·4996	·4996	·4996	·4996	·4996	·4996	·4997
3·4	·4997	·4997	·4997	·4997	·4997	·4997	·4997	·4997	·4997	·4998
3·5	·4998	·4998	·4998	·4998	·4998	·4998	·4998	·4998	·4998	·4998
3·6	·4998	·4998	·4999	·4999	·4999	·4999	·4999	·4999	·4999	·4999
3·7	·4999	·4999	·4999	·4999	·4999	·4999	·4999	·4999	·4999	·4999
3·9	·5000	·5000	·5000	·5000	·5000	5000	·5000	·5000	·5000	·5000