Indian Institute of Technology Patna
Department of Mathematics
Probability Theory and Random Processes (MA225)
Mid-Semester Exam
Full Marks: 30 Time: 2 Hrs

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1. Suppose that X and Y are independent, identically distributed, geometric random variables with parameter p. Show that [4]

$$P(X = i|X + Y = n) = \frac{1}{n-1}, \qquad for\ i = 1, 2, \ldots, (n-1)$$

2. Consider 10 independent tosses of a biased coin with a probability of heads of p. [1+1+2+2]

   (a) Let A be the event that there are 6 heads in the first 8 tosses. Let B be the event that the 9^{th} toss results in heads. Are these events independent?

   (b) Find the probability that there are 3 heads in the first 4 tosses and 2 neads in the last 3 tosses.

   (c) Given that there were 4 heads in the first 7 tosses, find the probability that the 2^{nd} head occurred during the 4^{th} toss.

   (d) Find the probability that there are 5 heads in the first 8 tosses and 3 heads in the last 5 tosses.

3. Give an example of two continuous random variables X and Y which are not jointly continuous. [2]

4. Let X be a random variable that takes non-negative integer values. Show that [4]

$$E[X] = \sum_{k=1}^{\infty} P(X \geq k).$$

   [Hint: Express the RHS as double summation]

5. Suppose that it is known that life time of a light bulb is modelled by an exponential random variable with parameter λ. In a store, there are two options for buying a light bulb; a new light bulb and an used light bulb, which is offered at a discount. Which bulb would you prefer buying? Justify your answer mathematically. [4]

6. What is the memoryless property of geometric random variable? Using this property, find the variance of a geometric random variable. [1 + 3]

7. The random variables X and Y are described by a joint PDF which is constant within the unit area quadrilateral with vertices $(0, 0)$, $(0, 1)$, $(1, 2)$, and $(1, 1)$. [2+1+3]

   (a) Find the marginal PDFs of X and Y.

   (b) Find the expected value of $X + Y$.

   (c) Find the variance of $X + Y$.

MA225 - Probability Theory & Random Processes.

Midsem Exam Solutions.

Q.1.

$$P(X=i \mid X+Y=n) = \frac{P(\{X=i\} \cap \{X+Y=n\})}{P(X+Y=n)} \quad [\text{Def}$$

$$= \frac{P(\{X=i\} \cap \{Y=n-i\})}{P(X+Y=n)}$$

$$= \frac{P(X=i) \cdot P(Y=n-i)}{P(X+Y=n)} \quad [\text{Since ind}$$

$$\text{Now, } P(X+Y=n) = \sum_{i=1}^{n-1} P(X=i) \cdot P(X+Y=n \mid X=i) \quad [\text{total prob law}]$$

$$= \sum_{i=1}^{n-1} P(X=i) \cdot P(Y=n-i \mid X=i)$$

$$= \sum_{i=1}^{n-1} P(X=i) \cdot P(Y=n-i) \quad [\text{Sine ind}]$$

2

$$\text{Therefore, } P(X=i \mid X+Y=n) = \frac{P(X=i) \cdot P(Y=n-i)}{\sum_{i=1}^{n-1} P(X=i) \cdot P(Y=n-i)}$$

$$= \frac{(1-p)^{i-1} \cdot p \, (1-p)^{n-i-1} \cdot p}{\sum_{i=1}^{n-1} (1-p)^{i-1} \cdot p \, (1-p)^{n-i-1} \, p} = \frac{(1-p)^{n-2} \cdot p^2}{\sum_{i=1}^{n-1} (1-p)^{n-2} \cdot p^2} = \frac{1}{n-1}$$

2

Q. 2. a) $P(B|A) = \dfrac{P(B\cap A)}{P(A)} = \dfrac{p\binom{8}{6}(1-p)^2 p^6}{\binom{8}{6}(1-p)^2 p^6}$

$= p = P(B)$

Therefore, $P(B|A) = P(B)$, that is, conditional prob is same as unconditional prob. Hence A & B are ind. — 1

b) C = event that 3 heads in first 4 tosses

D = " " 2 " " last 3 tosses

Since there is no overlap in tosses, they are ind. ·5

$\therefore P(C\cap D) = P(C)\cdot P(D) = \binom{4}{3} p^3(1-p)\cdot\binom{3}{2}p^2(1-p)$

$= \binom{4}{3}\binom{3}{2} p^5 (1-p)^2$ — ·5

c) E = 4 heads in first 7 tosses

F = 2nd head occurred in 4th toss.

$P(F|E) = \dfrac{P(F\cap E)}{P(E)} = \dfrac{\binom{3}{1}p(1-p)^2\cdot p\cdot\binom{3}{2}p^2(1-p)}{\binom{7}{4}p^4(1-p)^3}$

$= \dfrac{\binom{3}{1}\binom{3}{2}}{\binom{7}{4}}$

— 2

d) G = 5 heads in first 8 tosses.

H = 3 " " last 5 tosses

Since, there is some overlap, G & H are not ind

We calculate the prob of $G \cap H$ by total prob law, conditioned on no of heads occurred during toss 6, 7 & 8. 1

$$P(G\cap H) = P(G\cap H \mid 1 \text{ head in tosses } 6 \text{ to } 8) \cdot P(1 \text{ head in } 6 \text{ to } 8)$$
$$+ P(G\cap H \mid 2 \text{ heads in tosses } 6 \text{ to } 8) \cdot P(2 \text{ heads in } 6 \text{ to } 8)$$
$$+ P(G\cap H \mid 3 \text{ heads in tosses } 6 \text{ to } 8) \cdot P(3 \text{ tosses in } 6 \text{ to } 8$$

$$= \binom{5}{4} \cdot p^4(1-p) \cdot \binom{3}{1} p(1-p)^2 \cdot p^2 + \binom{5}{3} p^3(1-p)^2 \cdot \binom{3}{2} p^2(1-p) \binom{2}{1} p(1-$$
$$+ \binom{5}{2} p^2(1-p)^3 \cdot p^3 \cdot (1-p)^2$$
$$= \binom{5}{4}\binom{3}{1} p^7(1-p)^3 + \binom{5}{3}\binom{3}{2}\binom{2}{1} \cdot p^6(1-p)^4 + \binom{5}{2} p^5(1-p)^5$$

1

toss1 toss2 . . . . . . [toss6 toss7 toss8] . . . . .

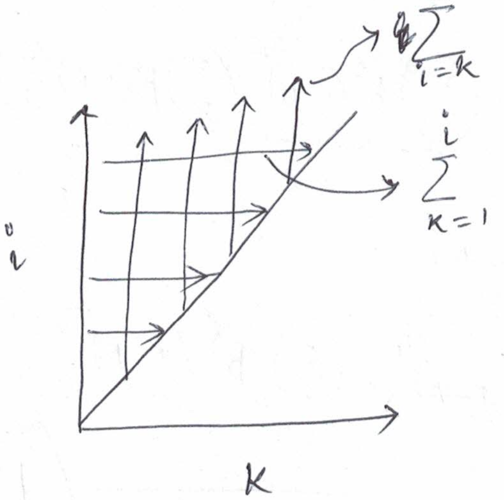
4

3. X and Y ~~be~~ is any continuous rvs s.t. $X = Y$. (Any function with zero area)

All the prob mass is on the line $x=y$ in this case. Which implies that X & Y are not jointly continuous. If they were jointly continuous, then prob on any line would be 0. —— 1

4. We know $P(X \geq k) = \sum_{i=k}^{\infty} p_X(i)$

$$\therefore \sum_{k=1}^{\infty} P(X \geq k) = \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} p_X(i) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} p_X(i)$$ —— 2



$$= \sum_{i=1}^{\infty} i \cdot p_X(i)$$ [Since $p_X(i)$ ind of k]

$$= E[X]$$ —— 2

Q.5. Let T = lifetime of a new bulb

Given that $T \sim \exp(\lambda)$

$\therefore$ We know, $P(T > x) = e^{-\lambda x}$ for $x \geq 0$.

1

This means that prob that the new bulb lasts x time unit is $e^{-\lambda x}$.

1

Let us assume that the used lightbulb has been used for t time unit, therefor $T > t$.

Let X be the remaining lifetime of the used bulb.

$\therefore X = T - t$.

Now, $$P(X > x \mid T > t) = \frac{P(T - t > x \cap T > t)}{P(T > t)}$$

$$= \frac{P(T > t + x)}{P(T > t)} = \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}}$$

$$= e^{-\lambda x}$$

This means that prob that the used bulb lasts for x time unit is same as the prob that the new bulb lasts for the same time unit.

Since, the used buld is offered at a discount, We should prefer buying the used bulb.

2

6

Q.6 Memoryless Property of Geometric r.v.

Given that the first toss is a Tail, the number of tosses required to get the first head is still a geometric rv with ~~p~~ the same parameter. — 1

Let us divide (partition) the sample space into two cases; 1st toss is H (A_1) and 1st toss is T (A_2). Now apply total expectation Law.

$$E[X] = P(A_1)\cdot E[X|A_1] + P(A_2)\cdot E[X|A_2]$$

$$= P(X=1)\cdot E[X|X=1] + P(X>1)\cdot E[X|X>1]$$

$$= p\cdot 1 + (1-p)\cdot[E[X]+1] \quad \text{(By memoryless property)}$$

— 1

$$\Rightarrow E[X] = \cancel{b}\ 1/p$$

Similarly, $E[X^2] = \cancel{p}\ P(X=1)\cdot E[X^2|X=1] + P(X>1)\cdot E[X^2|X>1]$

$$= p\cdot 1 + (1-p)\,E[(1+X)^2] \quad \text{(By memoryless property)}$$

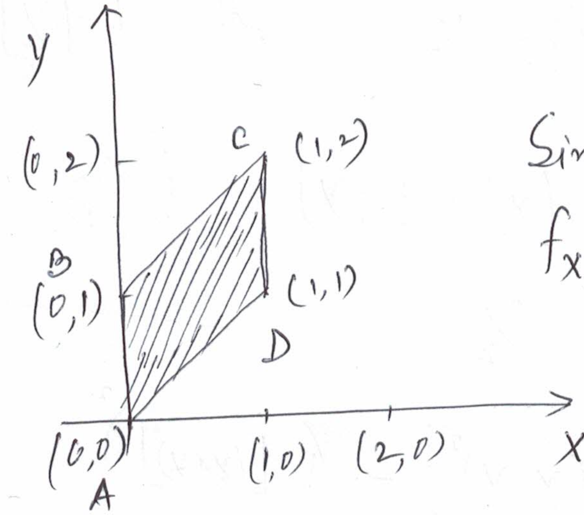
$$= p + (1-p)[1 + 2E[X] + E[X^2]] \quad \text{(By linearity of exp}$$

$$\Rightarrow E[X^2] = \frac{2}{p^2} - \frac{1}{p}$$

$$\therefore Var(X) = E[X^2] - (E[X])^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

— 2

Q. 7.



Since the area is 1 unit^2,

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{over } ABCD \\ 0 & o/w \end{cases}$$ ——— 1

a) for $0 \le x \le 1$

$$f_X(x) = \int_y f_{X,Y}(x,y)\,dy = \int_x^{x+1} 1 \cdot dy = 1$$

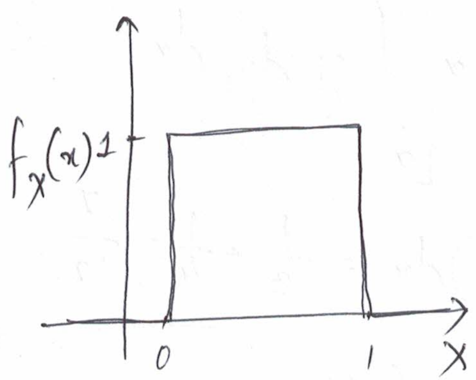
for $0 \le y \le 1$

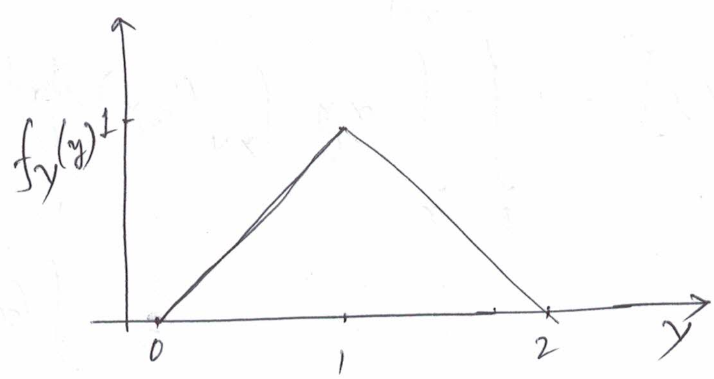
$$f_Y(y) = \int_x f_{X,Y}(x,y)\,dx = \int_0^y 1 \cdot dx = y$$

and

for $1 \le y \le 2$

$$f_Y(y) = \int_{y-1}^{1} 1 \cdot dx = 1 - (y-1) = 2 - y$$





——— 1

b) Clearly $E[X] = 1/2$ and $E[Y] = 1$

$\therefore E[X+Y] = E[X] + E[Y]$ (By linearity of expectation)

$= 3/2$ 1

c) $Var(X+Y) = E\left[(X+Y)^2\right] - \left(E[(X+Y)]\right)^2$

$= E[X^2] + 2E[XY] + E[Y^2] - \left(E[X+Y]\right)^2$

$\therefore E[X^2] = \int_0^1 x^2 . 1\, dx = \left[\frac{x^3}{3}\right]_0^1 = 1/3$ 1

$E[Y^2] = \int_0^2 y^2 f_Y(y)\, dy = \int_0^1 y^2 . y\, dy + \int_1^2 y^2 (2-y)\, dy$

$= \left[\frac{y^4}{4}\right]_0^1 + \left[2\frac{y^3}{3} - \frac{y^4}{4}\right]_1^2$

$= 1/4 + \left[\left(\frac{2\times 8}{3} - 4\right) - \left(\frac{2}{3} - \frac{1}{4}\right)\right] = \frac{1}{4} + \frac{14}{3} - \frac{15}{4} = \frac{14}{3} - \frac{1}{}$

$= 7/6$ 1

$E[XY] = \int_0^1 \int_x^{x+1} xy\, f_{X,Y}(x,y)\, dy\, dx = \int_0^1 x \left[\int_x^{x+1} y\, dy\right] dx$

$= \int_0^1 \left(x^2 + \frac{x}{2}\right) dx = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ 1

$\therefore Var(X+Y) = \frac{1}{3} + 2 . \frac{7}{12} + \frac{7}{6} - \left(\frac{3}{2}\right)^2 = 5/12$

⊛ Alternate way of solving 7(b) and (c)

Let $U = Y - X$.

Note for every value of $X \in [0,1]$, U takes value between 0 and 1; and it is uniformly distributed. —— 1

This means that U is ind of X. —— 1

$\therefore \; X + Y = 2X + U$

$\therefore \; E[X+Y] = E[2X+U] = 2 \cdot E[X] + E[U]$

$= 2 \cdot \frac{1}{2} + \frac{1}{2}$ [Since both X & U are uniform over $[0,1]$]

$= \frac{3}{2}$ —— 1

$Var(X+Y) = Var(2X+U) = Var(2X) + Var(U)$

$= 4\, Var(X) + Var(U)$

$= 4 \cdot \frac{1}{12} + \frac{1}{12} = \frac{5}{12}$

—— 1