End Semester Examination: MA220 (Numerical Linear Algebra)
Time: 3 Hours 15 Minutes

Maximum Marks: 60

Roll No.: **Name:**

There are **13** questions in this paper. Attempt all questions. Give precise and brief answers. Standard results/formulae may be used. Symbols/notations have their usual meaning as per Lecture's discussion. **No electronic device, excluding a simple and non-programmable calculator, is allowed**. Do not write anything on the question paper except the Roll no. and Name at the top.

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1. Compute $||A||_\infty$, $||A||_1$, and $\kappa(B)$ (condition number of the matrix B with respect to 2−norm), where $A = \begin{bmatrix} 3 & -1 \\ 2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$. **[1+1+2 =4]**

2. If $A \in \mathbb{R}^{n\times n}$ is a matrix of rank r, then prove using the SVD that
$$||A||_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \cdots + \sigma_r^2},$$
where σ_i are the singular values of A and F stands for the Frobenius norm. **[3]**

3. Take $A = \begin{bmatrix} 2 & -2 \\ 2 & 4 \\ -1 & 4 \end{bmatrix}$. State the Eckart-Young Theorem (EYT) in 2−norm. Find a matrix $C \in \mathbb{R}^{3\times 2}$ that solves the minimization problem
$$\min_{B\in\mathbb{R}^{3\times 2},\ \text{rank}\, B=1} ||B - A||_2$$
What is $||C - A||_2$? Is EYT correct in the norm $||\cdot||_F$ and $||\cdot||_\infty$. **[2+3+1+1 =7]**

4. Write the Computational steps for Conjugate Gradient (CG) method. Take initial guess $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and solve the following system by CG: **[2+3 =5]**
$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} x = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}.$$

5. Write computational steps for general framework of any projection-based method for solution of $Ax = b$. Show that, in any projection method, If subspace K is invariant under A and the initial residual $r_0 \in K$, then method gives exact solution in one iteration only. Use this result to show that MR method gives exact solution in only one iteration if you solve $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ by taking initial guess $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find the solution also. **[2+2+1+2 =7]**

6. Apply SD method (only two iterations) to the system $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by taking initial guess $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Show that after two iteration, you get the same solution as the solution after one iteration of Gauss-Seidel. Can you say from it that Gauess-Seidel is always a better choice than SD for any Symmetric Positive Definite (SPD) matrix. **[3+3+1 =7]**

7. Write the Computational steps for Residual Norm Steepest Descent ($RNSD$) and Generalized Minimum Residual ($GMRES$) methods for solution of $Ax = b$. **[3+3 =6]**

8. Let A be any $n \times n$ matrix. Let K and L be two $m-$dimensional subspaces of R^n $(m < n)$. Let W and V be two $n \times m$ matrices such that their columns make bases of subspaces L and K, respectively. Let A be nonsingular and $L = AK$. Then, prove that the matrix $W^T AV$ is nonsingular. [3]

9. Prove that the similar matrices have the same set of eigenvalues. [3]

10. Let $A = \begin{bmatrix} 7 & 1 \\ 1 & 4 \end{bmatrix}$. Find the approximate eigenvalues of A by performing one step of the QR iterative method with the shift parameter $\mu = 4$. It is given that [4]

$$\begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \sqrt{10} & \frac{3}{\sqrt{10}} \\ 0 & \frac{1}{\sqrt{10}} \end{bmatrix}.$$

11. Find Incomplete LU (ILU) preconditioner P for $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$. [4]

12. Write three different ways (formulae only) to compute the pseudoinverse of any matrix $A \in \mathbb{R}^{m \times n}$ of rank n (Do not write standard commands like `pinv` in MATLAB). [3]

13. Let $x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 0 \\ \sqrt{2} \\ 1 \end{bmatrix}$. By using the Gram-Schmidt algorithm, find two orthonormal vectors $q_1, q_2 \in \mathbb{R}^3$ such that $\text{Span}\{q_1, q_2\} = \text{Span}\{x_1, x_2\}$. [4]

Ques 1

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 7 \end{bmatrix}$$

Remember to take absolute values.

(1) - $\|A\|_\infty = \max\{3+1, 2+7\} = 9$ max of sum of rows

(1) - $\|A\|_1 = \max\{3+2, 1+7\} = 8$ max of sum of columns

$$B = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 7 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= U \Sigma V^T$$

i.e. $\sigma_1 = 7$ and $\sigma_2 = 3$ $\frac{1}{2}$

Hence $\|B\|_2 = 7$

$$B^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/7 \end{bmatrix}$$

$\sigma_1 = \frac{1}{3}$, $\sigma_2 = \frac{1}{7}$

so

$\|B^{-1}\|_2 = \frac{1}{3}$

$1/2$

Condition number $K(B) = \|B\|_2 \, \|B^{-1}\|_2$ — (1) mark

$$= \frac{7}{3}$$

Ques 2

We know that

$$\|A\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}$$

$$= \sqrt{\text{Trace}(A^T A)} \quad \text{— (1)}$$

overall (+1)

Let $A = U\Sigma V^T$

Then $A^T A = (U\Sigma V^T)^T U\Sigma V^T$

$= V\Sigma^T U^T U\Sigma V^T$

$= V\Sigma^T\Sigma V^T$

So, $\text{Trace}(A^T A) = \text{Trace}(V\Sigma^T\Sigma V^T)$ — (1)

$= \text{Trace}\ \Sigma^T\Sigma$ $\quad$ [$\because V\Sigma^T\Sigma V^T$ and $\Sigma^T\Sigma$ are similar]

$= \sigma_1^2 + \cdots + \sigma_r^2$

Ques3

$$A = \begin{bmatrix} 2 & -2 \\ 2 & 4 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -2/6 \\ 2/3 & 4/6 \\ -1/3 & 4/6 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

SVD $1\frac{1}{2}$ marks

$$= \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = U \Sigma V^T$$

(SVD of A)

$$\Rightarrow A = 6 \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{— (1)}$$

2 marks

Eckart-Young Theorem Let $A \in \mathbb{R}^{m \times n}$ has rank r. Suppose $A = U\Sigma V^T$ is the SVD of A, i.e.

$$A = U\Sigma V^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T \quad \text{— (2)}$$

Take $A_k = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_k u_k v_k^T \quad (k \le r)$ — (3)

If $B \in \mathbb{R}^{m \times n}$ has rank k, then

$$\|B - A\| \ge \|A_k - A\| \quad \text{— (4)}$$

i.e. A_k solves the optimization problem

$$\min_{B \in \mathbb{R}^{m \times n},\ \text{rank}(B) = k} \|B - A\| \quad \text{— (5)}$$

Recall: (4) is correct in $\|\cdot\|_2$ and $\|\cdot\|_F$ norm.

and $\|A_k - A\|_2 = \sigma_{k+1}$ — (6)

$$\|A_k - A\|_F = \sqrt{\sigma_{k+1}^2 + \sigma_{k+2}^2 + \cdots + \sigma_r^2}$$

Thus, in this problem, by comparing ① and ②, we obtain from ④ that

$$C = 6\begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$= 6\begin{bmatrix} 0 & -1/3 \\ 0 & 2/3 \\ 0 & 2/3 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 4 \\ 0 & 4 \end{bmatrix}$$

$1\frac{1}{2}$ mark

and ⑥ provides that

$$\|C - A\|_2 = \sigma_2 = 3$$ — 1 mark

Eckart-Young Theorem is not valid in $\|\cdot\|_\infty$.

1 mark

2 mark

Ques 4

CG $Ax = b.$

Compute $r_0 = b - Ax_0$

$p_0 = r_0$ [STOP if $r_0 = 0$]

for $j = 0, 1, 2, \dots$ Do

$$\alpha_j = \frac{\langle r_j \; r_j \rangle}{\langle Ap_j \; p_j \rangle}$$

$$x_{j+1} = x_j + \alpha_j p_j$$

$$r_{j+1} = r_j - \alpha_j A p_j \quad [\text{STOP, if } r_{j+1} = 0]$$

$$\beta_j = \frac{\langle r_{j+1} \; r_{j+1} \rangle}{\langle r_j \; r_j \rangle}$$

$$p_{j+1} = r_{j+1} + \beta_j p_j$$

End Do.

3rd iteration

$$\cancel{\alpha_2 = \frac{\langle r_2 \; r_2 \rangle}{\langle Ap_2 , p_2 \rangle} = \frac{3}{-\frac{3}{4} + \frac{35}{4} + \frac{35}{4}} = \frac{3}{67/4}}$$

$$\cancel{\left(Ap_2 = \begin{bmatrix} \frac{2}{4} - \frac{7}{4} - \frac{7}{4} \\ \frac{1}{4} - \frac{11}{4} - \frac{7}{4} \\ \frac{1}{2} - \frac{7}{4} - \frac{11}{4} \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ -5 \end{bmatrix} \right.}$$

$$Ap_1 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r_0 = b - Ax_0 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$p_0 = r_0 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

1st iteration: $\left(Ap_0 = \begin{bmatrix} 8 \\ 4 \\ 4 \end{bmatrix} \right)$

$$\alpha_0 = \frac{\langle r_0 \; r_0 \rangle}{\langle Ap_0 \; p_0 \rangle} = \frac{16}{32} = \frac{1}{2}$$

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$r_1 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 8 \\ 4 \\ 4 \end{bmatrix} = - \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$$= r_0 - \alpha_0 A p_0$$

$$\beta_0 = \frac{8}{16} = \frac{1}{2}$$

$$p_1 = r_1 - \beta_0 p_0$$

$$= \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = + \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$$

$1\frac{1}{2}$ mark each

2nd iteration $\left(Ap_1 = \begin{bmatrix} 0 \\ -4 \\ -4 \end{bmatrix} \right)$

$$\alpha_1 = \frac{\langle r_1 \; r_1 \rangle}{\langle Ap_1 \; p_1 \rangle} = \frac{8}{16} = \frac{1}{2}$$

$$x_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} +2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

$$= x_1 + \alpha_1 p_1$$

$$r_2 = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(STOP)}$$

$$\cancel{\beta_1 = \frac{3}{8}}$$

$$\cancel{p_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \frac{3}{8} \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{3}{4} \\ -1 - 3/4 \\ -1 - 3/4 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -7/4 \\ -7/4 \end{bmatrix}}$$

Hence Solution is $x = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$

Ques 5

General framework for projection based method

2 mark

Given $Ax = b$ ($A \in \mathbb{R}^{n \times n}$)

Take $x := x_0$ (initial guess)

Until convergence DO

1. Take two subspaces K and L of $\mathbb{R}^n$ such that $\dim K = \dim L = m$

2. choose $V, W \in \mathbb{R}^{n \times m}$ such that columns of V make a basis of K and columns of W make a basis of L.

3. $r := b - Ax$

4. $y := (W^T A V)^{-1} W^T r$

5. $x := x + Vy$

End DO.

Ques:- Show that in projection method, If subspace K is invariant under A and initial residual $r_0 \in K$. Then method gives exact solution (in one step only)

Also remember: Let $\tilde{x}$ be approximate solution in K, then, we have

2 mark

$$\left.\begin{array}{l} \tilde{x} \in x_0 + K \\ b - A\tilde{x} \perp L \end{array}\right\} - (1) \Rightarrow \left.\begin{array}{l} \tilde{x} = x_0 + Vy \\ b - A\tilde{x} \perp L \end{array}\right\} - (2)$$

Define a projection Q_K^L in $\mathbb{R}^n$ s.t.

$$\left.\begin{array}{l} \text{Range}(Q_K^L) = K \\ \text{and } \text{Null}(Q_K^L) = L^{\perp} \end{array}\right] - (3)$$

This is possible because here $K \oplus L^{\perp} = \mathbb{R}^n$

From (2) & (3), we obtain

$$Q_K^L(b - A\tilde{x}) = 0 \quad - (4) \Rightarrow Q_K^L(b - Ax_0 - AVy) = 0$$
$$\Rightarrow Q_K^L(r_0 - AVy) = 0 \quad — (5)$$

Since $Vy \in K$ and it is given that K is invariant under A

we have $AVy \in K$

Also, it is given $r_0 \in K$ $\Big) - (6)$

Since Q_K^L is a projection on K (i.e. range of $Q_K^L = K$),

using ⑥ in ④, we obtain

$$b - A\tilde{x} = 0$$

$$\Rightarrow \quad A\tilde{x} = b$$

Hence $\tilde{x}$ is exact solution.

$$\left[\because \; Q_K^L(b - A\tilde{x}) = b - A\tilde{x} \right]$$

Rember! Let P be a projection on K then $\forall\, x \in K$ $\quad Px = x$

<u>In MR method</u>

(1 mark)

$$\tilde{x} = x_0 + Vy \in x_0 + K$$

$$b - A\tilde{x} \perp L$$

where $K = \{r_0\}$ and $L = \{Ar_0\}$

Hence, if r_0 is an eigenvector of A, then K is an eigenspace of A. Thus, in this situation

K is invariant and A

and obviously $r_0 \in K$

Hence MR will give exact solution in one iteration only if r_0 is an eigenvector of A.

In the given problem $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Then $r_0 = b - Ax_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Notable! $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = Ar_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = r_0 \Rightarrow r_0$ is an eigenvector of A

Hence MR will give exact solution in one iteration only.

MR iterative method

Given $Ax = b$ (A is PD)
$A \in \mathbb{R}^{n \times n}$

Take $x = x_0$ (initial guess) compute $r = b - Ax$
$p = Ar$

Until convergence Do

$$\alpha = \frac{\langle p \; r \rangle}{\langle p \; p \rangle}$$

$x = x + \alpha r$

$r = r - \alpha p$

$p = Ar$

End Do

2 marks

Given: $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$r = b - Ax = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$p = Ar = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Iteration 1:

$$\alpha = \frac{\langle p \; r \rangle}{\langle p \; p \rangle} = \frac{1}{1} = 1$$

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$r = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$ Hence the exact solution is $x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Ques 6

SD

Given $Ax = b$ — ① marks

($A \equiv n \times n$ is SPD)

Take $x = x_0$

$r = b - Ax$

$p = Ar$

Until convergence DO

$$\alpha = \frac{\langle r\ r \rangle}{\langle r\ p \rangle}$$

$x = x + \alpha r$

$r = r - \alpha p$

$p = Ar$

End Do.

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

iteration 1 $\left[r = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ p = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right.$

$\alpha = \frac{1}{2}$

$x = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$ — $1\frac{1}{2}$ marks

$r = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 1-1 \\ 0+1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$

$p = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

iteration 2 $\left[Ap = \begin{bmatrix} -2 \\ 5/2 \end{bmatrix} \right]$ — $\frac{1}{2}$ marks

$\alpha = \frac{1/4}{1/2} = \frac{2}{4} = \frac{1}{2}$

$x = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 1/2 \end{bmatrix} = \begin{pmatrix} 1/2 \\ 1/4 \end{pmatrix}$

Gauss - Seidel

$P x_{n+1} = (P - A)x_n + b$ — ① mark

$P = D - E = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (put $n = 0$, $x_0 = 0$) — ② mark

$\Rightarrow x_1 = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix}$; which is the same ~~x~~ as obtained in the second iteration of SD.

In general, GS is not better than SD. — ① mark

Ques 7

RNSD (3 marks)

Given $Ax = b$
$(A \equiv n \times n$ invertible$)$

Take $x = x_0$

$r = b - Ax$

$v = A^T r$

Until convergence Do

$$\alpha = \frac{\langle v \; v \rangle}{\langle Av \; Av \rangle}$$

$x = x + \alpha v$

$r = r - \alpha A v$

$v = A^T r$

End DO

GMRES (3 marks)

Given $Ax = b$
$(A \equiv n \times n$ invertible$)$

Take initial guess x_0

Compute $r_0 = b - Ax_0$

$\beta = \|r_0\|$

$$v_1 = \frac{r_0}{\beta}$$

for $j = 1, 2, \cdots m$ DO

$w_j = A v_j$

for $i = 1, 2, \cdots j$ Do

$h_{i,j} = \langle w_j \; v_i \rangle$

$w_j = w_j - h_{i,j} v_i$

End DO

$h_{j+1,j} = \|w_j\|$ [Break if $h_{j+1,j} = 0$ Set $m = j$]

$$v_{j+1} = \frac{w_j}{h_{j+1,j}}$$

End DO

Compute $y_m = \underset{y}{\arg\min} \|\beta e_1 - \bar{H}_m y\|$

$x_m = x_0 + V_m y_m$

In GMRES

Set $\bar{H}_m = [h_{i,j}]_{\substack{1 \le i \le m+1 \\ 1 \le j \le m}}$

$V_m = [v_1 \; v_2 \cdots v_m]$

Ques 8

Let $A \in \mathbb{R}^{n\times n}$

Given (i) K and L be two m-dim. subspaces of $\mathbb{R}^n$ $(m<n)$.

(ii) V and W be two matrices such that

columns of V make a basis of K
columns of W make a basis of L $\Big\}$ $V, W \in \mathbb{R}^{n\times m}$

(iii). A^{-1} exists

(iv) $L = AK$

Due to the above (ii) & (iv), there exists an invertible matrix $G \in \mathbb{R}^{m\times m}$ such that

$$W = AVG$$

1 mark

overall +1

So, $$W^T AV = G^T V^T A^T AV$$
$$= G^T (AV)^T AV$$
$$= G^T C$$

1 mark

where $C = (AV)^T AV$

Since A^{-1} exists

$\text{rank}(AV) = \text{rank } V$
$= m$

Thus AV is a full column rank matrix, hence the matrix $C = (AV)^T (AV)$ is invertible

i.e. C^{-1} exists

Hence $$(W^T AV)^{-1} = C^{-1} (G^T)^{-1}$$

G is invertible
$(G^T)^{-1} = (G^{-1})^T$

□

Ques 9.

Let $A, B \in \mathbb{C}^{n\times n}$ be two similar matrices.

Then $\exists$ an invertible matrix P such that

$$A = PBP^{-1} \quad \text{--- } (1)$$ | Def. – (1 marks)

Let $\lambda \in \sigma(A)$. Then

$$Ax = \lambda x \qquad (x \neq 0)$$

$$\Rightarrow PBP^{-1}x = \lambda x \qquad (\text{By } (1))$$ – (1 mark)

$$\Rightarrow B(P^{-1}x) = \lambda(P^{-1}x)$$

$$\Rightarrow \lambda \in \sigma(B)$$

Hence $\sigma(A) \subseteq \sigma(B)$

Since the cardinality of both of the sets $\sigma(A)$ and $\sigma(B)$ is n, hence $\sigma(A) = \sigma(B)$. | (1 mark)

Ques 10

$$A = \begin{bmatrix} 7 & 1 \\ 1 & 4 \end{bmatrix}$$

Algorithm - 2 marks.
Calculation - 2 marks.

$$A_0 = A - 4I = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 3/\sqrt{10} \\ 0 & 1/\sqrt{10} \end{bmatrix} = QR$$

$$A_1 = RQ + 4I$$

$$= \begin{bmatrix} \sqrt{10} & \frac{3}{\sqrt{10}} \\ 0 & \frac{1}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{33}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{-3}{10} \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} \frac{73}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{37}{10} \end{bmatrix}$$

$$\mathrm{eig}(A) = \begin{bmatrix} 3.6972 \\ 7.3028 \end{bmatrix}$$

approximate eigenvalues by QR with shift

$$= \begin{bmatrix} 3.7000 \\ 7.3000 \end{bmatrix}.$$

Ques 11)

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

Thus

$$P = ILU(A) = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\parallel$$

$$U^T D U$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1/2 \\ 1 & 1/2 & 2 \end{bmatrix}$$ ✓

2 marks - idea of ILU
2 marks - calculations.

Ques 12

1 mark each

Recall three methods to solve LSP $Ax = b$

$$A^T A \hat{x} = A^T b \quad — ①$$

$$R \hat{x} = Q^T b \quad — ②, \text{ where } A = QR$$

$$\hat{x} = A^+ b \quad — ③, \text{ where } A^+ \text{ stands for pseudoinverse (MP-inverse) of } A$$

Since A has full column rank, the LSS $\hat{x}$ is unique

Hence we obtain

$$\boxed{\begin{aligned} A^+ &= (A^T A)^{-1} A^T \\ A^+ &= R^{-1} Q^T \end{aligned}}$$

from ① & ③

from ② & ③

Moreover, other results are

$A^+ = V \Sigma^+ U^T$ (where $A = U \Sigma V^T$ is the SVD)

$A^+ = R^+ C^+$ (where $A = CR$ is the full rank factorization)

check ??

Ques 13

Let $x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 0 \\ \sqrt{2} \\ 1 \end{bmatrix}$. Then by using the Gram Schmidt algorithm find two orthonormal vectors $q_1, q_2 \in \mathbb{R}^3$ such that $\text{span}\{q_1, q_2\} = \text{span}\{x_1, x_2\}$.

Sol.

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \|x_1\| = \sqrt{2}$$

$$q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{x_1}{\|x_1\|}$$

$$w = x_2 - \langle x_2, q_1 \rangle q_1$$

$$= \begin{bmatrix} 0 \\ \sqrt{2} \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 1 \end{bmatrix}$$

$$\|w\| = \sqrt{\frac{1}{2} + \frac{1}{2} + 1} = \sqrt{2}$$

$$q_2 = \frac{w}{\|w\|} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}$$

2 marks - Algorithm

2 marks - calculation.