Discrete Mathematical Structures (UCS405)

Lecture 1: Introduction to Discrete Mathematics

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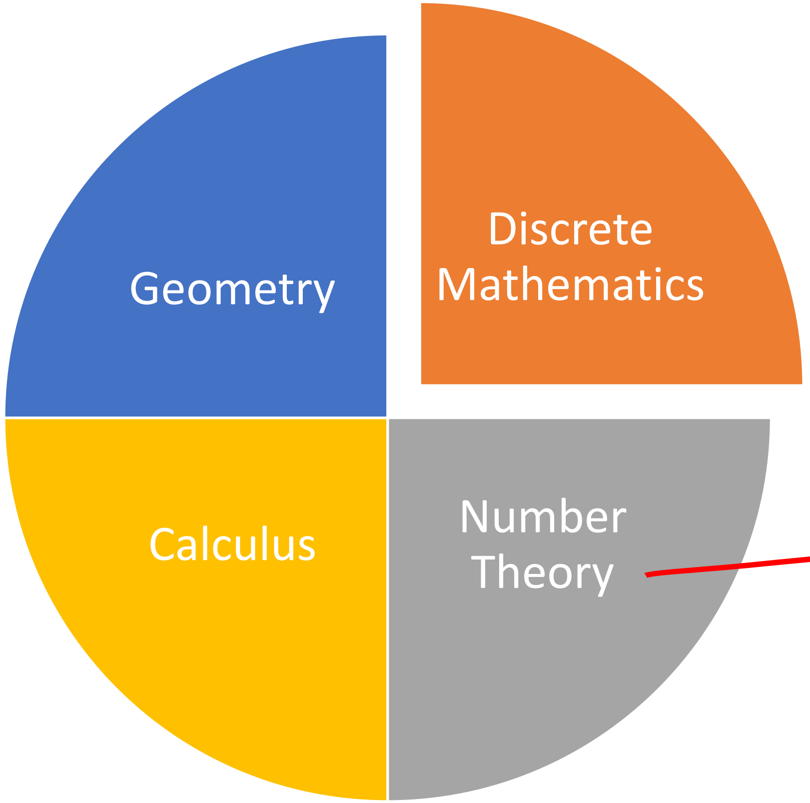
Agenda for Today

- Fundamental questions about discrete mathematics
  - What is Discrete Mathematics?
  - Why to study discrete Mathematics?
  - How it is different than other flavours of mathematics?
  - How it is relevant to computer engineers?
- Introduction to Discrete Mathematics
- Goal of the subject
- Introduction to Set Theory

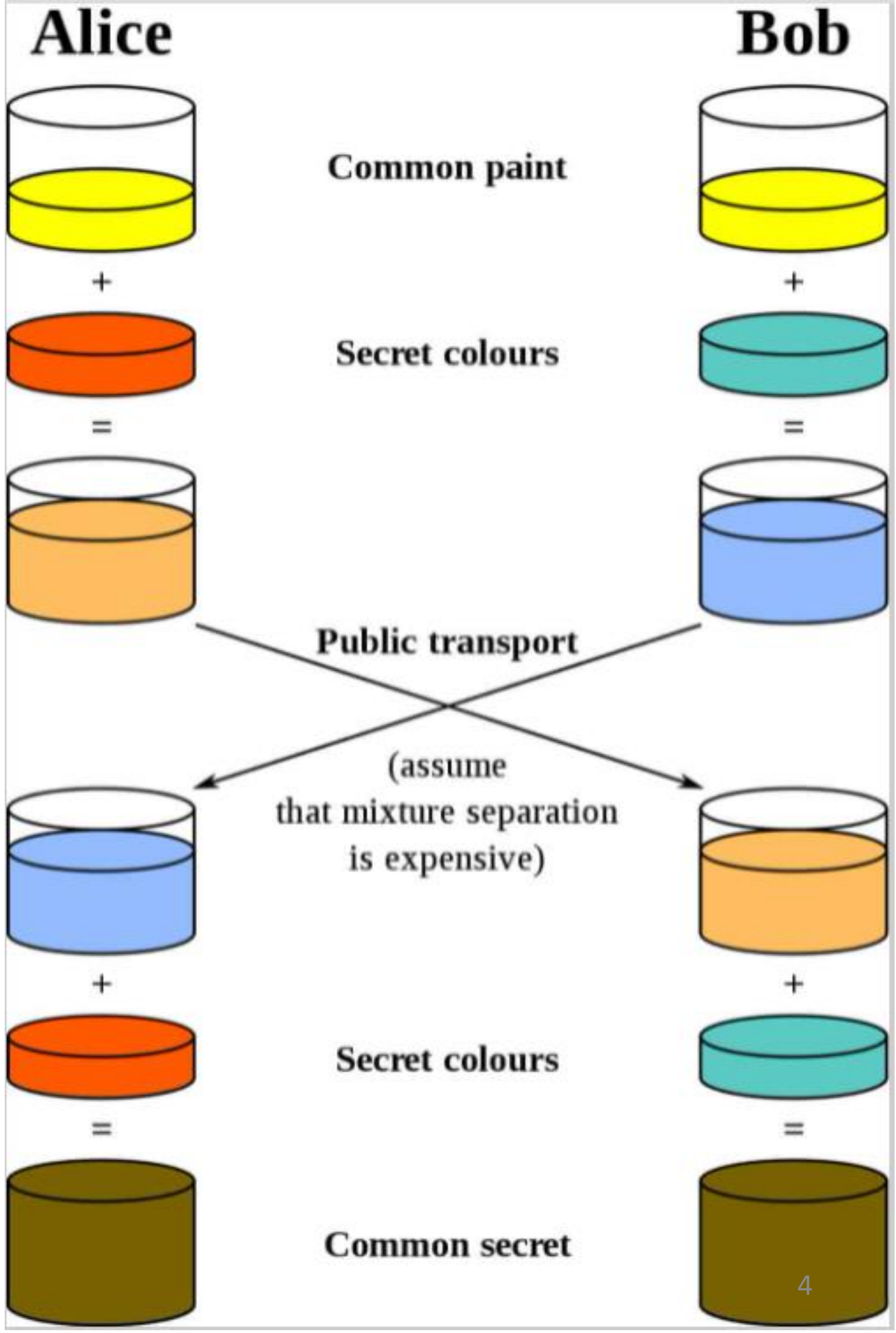
Flavours of Mathematics



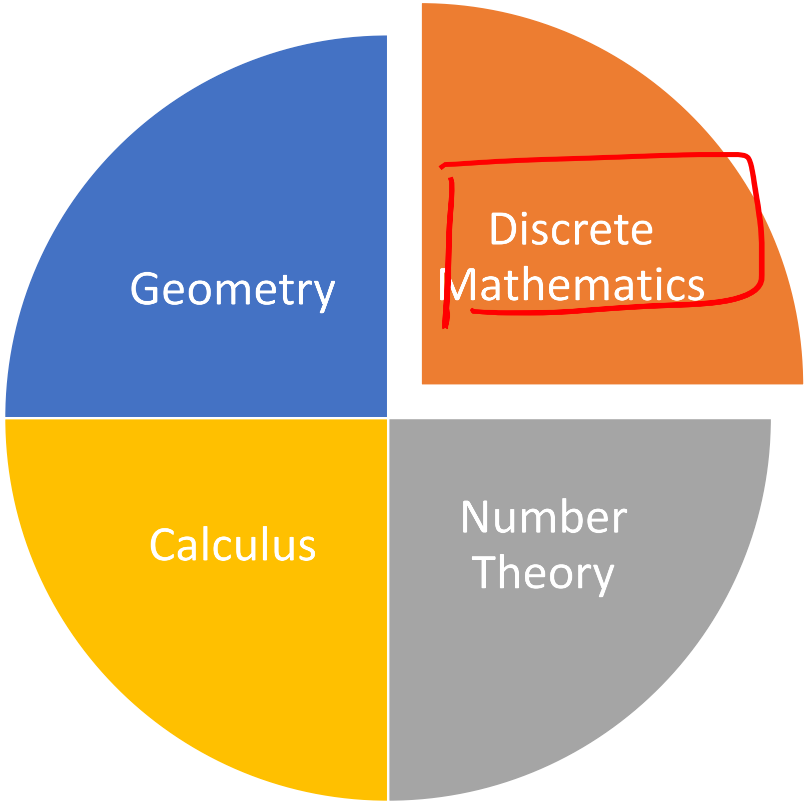
1976

Number Theory


Whitfield Diffie and Martin Hellman



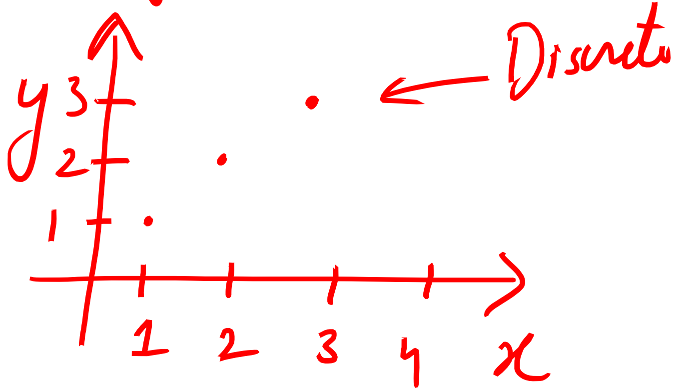
Flavours of Mathematics



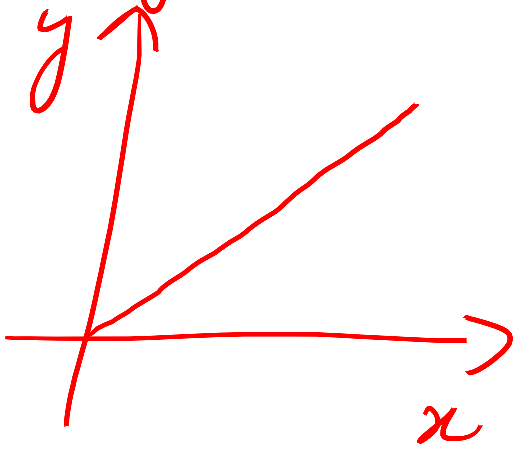
Discrete Mathematics

- Study of discrete objects
- Discrete Object?
  - Things like people, chair, table etc.
  - Natural Numbers
  - Integers

$y = x \in \mathbb{N}$



$y = x \in \mathbb{R}$



Continuous

Relevance to Computer Science

- Application Oriented
  - Operating System
  - Database Management System
  - Computer Networks
  - Complier
- Discrete Mathematics
  - Data Structures/Algorithms – Time Complexity (Recurrence Relations), Hash functions, Randomized algorithms
  - Computer Networks – Graph theory
  - Cryptography – Group Theory, Lattice Structures

Why to Study Discrete Mathematics

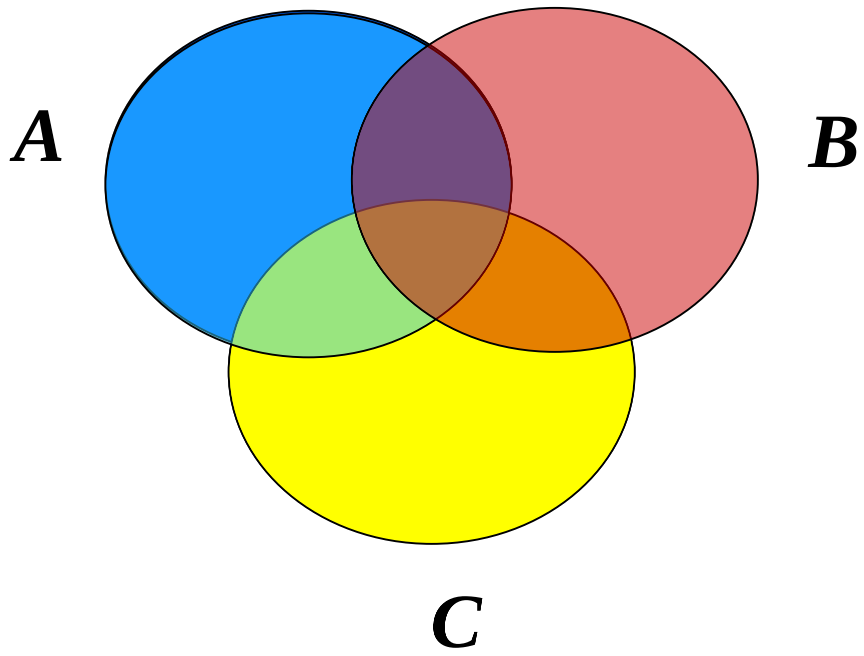
- Discrete Objects – Common characteristics
- Develop common tools
  - Combinatorics
  - Finite Set Theory
  - Finite Group Theory
  - Discrete Probability
  - Graph Theory
- Discrete mathematics is the foundation course for computer science

Goal

- How to give formal mathematical proofs
- Learn different techniques that will help us model problems in a mathematical way.
- Learn different tools that can be used to attack the problems.

Syllabus

1. Set Theory
2. Function
3. Relation
4. Propositional Logic
5. Proof Techniques and Counting
6. Graph Theory
7. Group Theory

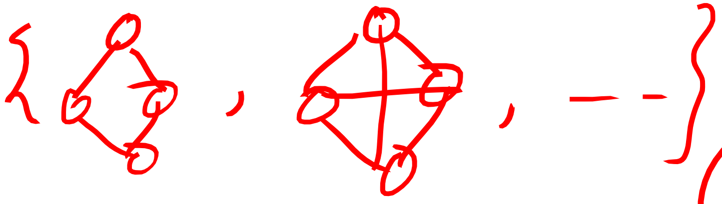


Set Theory

Content

1. Definition of a set
2. Cardinality of a set
3. Equality of the sets
4. Subset and Proper Subset
5. Power set
6. Representation of a set
7. Different types of set
8. Disjoint set
9. Venn Diagrams
10. Cartesian product of a set
11. Assignment/Quiz/Exercise questions

Set Definition



$\{1, 1, 2, 3, 4, 4\}$

$\{1, 2, 3, 4\} = A$

Distinct

- A set is an **unordered** **collection** of objects.
- The objects in a set are called the **elements** or **members** of the set S
- Notation:
  - $a \in A$ - If a is an element of the set A
  - $a \notin A$ - If a is not a member of A

$2 \in A$

$\{1, 2, 3, 4\}$

$5 \notin A$

- Example:
  - S = {2, 3, 5, 7, 11, 13, 17, 19}
  - S = {CSC1130, CSC2110, ERG2020, MAT2510}
- Set is a single mathematical object, and it can be an element of another set.
  - S = {{1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4}}

Defining Sets by Properties

$A = \{1, 2, 3, 4, 5\}$

$A = \{x \mid x \text{ is a positive}\}$

$= \{2, 4, 6, 8, \ldots\}$ even no.

- Define a set by
  - listing all its elements – Roster
  - describing the *properties* that its elements should satisfy – $\boxed{\text{Set builder}}$ notation
    - Notation: $\{x \in A \mid P(x)\}$ ← condition
    - *x* is in set *A* such that *x* satisfies property *P*.
    - *Example:*

$$\{x \mid x \text{ is a prime number and } x < 1000\}$$

Example of Sets

{1, 2, 3, 4}
{4, 2, 1, 3}

$2 \cdot 2 \cdot 2 \ldots \cdot 2$
1 2 3 — n
0 1 0 1 0 1 0 1

Well known sets:

- the set of all real numbers, $\mathbb{R}$
- the set of all complex numbers, $\mathbb{C}$
- the set of all integers, $\mathbb{Z}$
- the set of all positive integers, $\mathbb{Z}^+$
- empty set, ∅/{}, the set with no elements.

$\mathbb{N}$

$|A| = 1 \rightarrow$ Singleton set

$A = \{1, 2, 3, 4\}$

$|A| = 4$

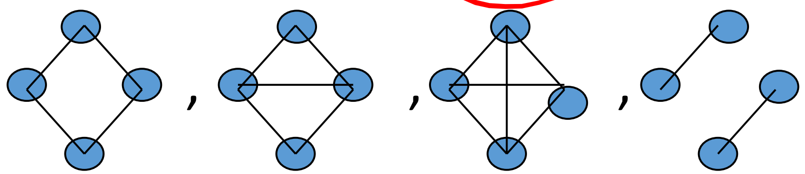
ϕ, { }

Other examples:

The set of all polynomials with degree at most three: {1, x, x^2, x^3, 2x+3x^2,…}.

n bit

The set of all n-bit strings: {000…0, 000…1, …, 111…1} $\rightarrow 2^n$

The set of all graphs with four nodes: {  }

Membership

Order, number of occurrence is not important.

e.g. {a,b,c} = {c,b,a} = {a,a,b,c,b}

$x \in R$

$x \in Z$

The most basic question in set theory is whether an element is in a set.

$x \in A$ x is an element of A

x is in A

$x \notin A$ x is not an element of A

x is not in A

e.g.

Z, R, C

Z

Recall that Z is the set of all integers. $7 \in Z$ $2/3 \notin Z$

Let P be the set of all prime numbers. $97 \in P$ $321 \notin P$

Let Q be the set of all rational numbers. $0.5 \in Q$ $\sqrt{2} \notin Q$

Cardinality

Finite set

0 element → Null set

1 element → Singleton set

$A =$

Infinite set

Cardinality of Sets

- The cardinality of a set S, denoted |S|, is the number of elements in S. If the set has an infinite number of elements, then its cardinality is ∞.
- The cardinality of a set A is denoted by | A |.
- 1: If $A = \phi$, then $|A| = 0$.
- 2: If A has exactly n elements, then
- $|A| = n$.
- Note that n is a nonnegative number.
- 3: If A is an infinite set, then $|A| = \infty$.

$|A| = 0$

$A = \phi = \{\ \}$

Equal Sets

$T \wedge T = T$

$X, Y \Rightarrow$

if $\forall x (x \in X \to x \in Y) \wedge$

$(x \in Y \to x \in X)$

$\hookrightarrow$ True

then $X = Y$

$\boxed{A = B}$?

$A = \{x \mid x^2 + 2x - 8 = 0\}$

$B = \{-4, 2\}$

$x^2 + 2x - 8 = 0$

$x^2 + 4x - 2x - 8 = 0$

$x(x+4) - 2(x+4) = 0$

$x = -4, 2$

i) $\forall x (x \in A \to x \in B)$ - T

$-4 \in A$

$2 \in A$

ii) $\forall x (x \in B \to x \in A) \to T$

$x = -4$
$x = 2$ $\Rightarrow [x^2 + 2x - 8 = 0]$

Example: Equal Set

X
Y $\Big]$ $X = Y$

$\forall x \left(x \in X \to x \in Y\right) \wedge \left(x \in Y \to x \in X\right)$

True $\to$ $X = Y$

$\exists \to$ There exists

$\forall \to$ for all

$x^2 + 2x - 8 = 0$

$x^2 + 4x - 2x - 8 = 0$

$(x+4)(x-2) = 0$

$x = -4 | 2$

$X = \{x \mid x^2 + 2x - 8 = 0\} \leftarrow$ Set Builder

$Y = \{-4, 2\}$

Part 1 $\quad x \in X \to x \in Y$

$x^2 + 2x - 8 = 0 \Leftarrow -4$

Part 2

Subset

$\neg(\forall) = \exists$

X

Y

$\underbrace{\forall x(x \in A \to x \in B)}_{\Downarrow} \wedge (\underline{x \in B \to x \in A})$ $\hookrightarrow$ $\cancel{A} = \cancel{Y}B$

$A \subseteq B \wedge B \subseteq A \Rightarrow \underline{A = B}$

- Suppose X and Y are two sets, If every element of X is an element of Y. We say X is a subset of Y, $X \subseteq Y$

$X = \{1, 5\}$ $\quad Y = \{1, 4, 5, 7, 9\}$ $\quad \underline{X = Y}$

$X \subseteq Y$ $\quad \notin$

$Z = \{1, 2, 5\}$

$\forall x(x \in A \to x \in B) \Rightarrow \underline{\text{True}}$

$\downarrow \quad A \subseteq B$

$\neg(\forall x(\underbrace{x \in A}_{p} \to \underbrace{x \in B}_{q})) \Rightarrow \underline{\text{True}}$

$\hookrightarrow A \notin B$

$p \to q \equiv \neg p \vee q$

$\neg(p \to q) \equiv p \wedge \neg q$

$x \in B \Rightarrow q$

$x \notin B \Rightarrow \neg q$

$\exists x \neg(p \to q)$

$\exists x(p \wedge \neg q) \Rightarrow \underline{\exists x(x \in A \wedge x \notin B)}$

$A \notin B$

Proof

Proper Subset

- A is a subset of B and $A \neq B$, then $A \subset B$

$A = \{1, 5\}$

$C = \{1, 3, 5\}$

$B = \{1, 3, 5\}$

$A \subset B$

$C \subseteq B$

$C \not\subset B$

Power Set

$n = |A| = 3$

$2^n = |P(A)| = 8 = 2^3$

- It is the set of all subsets (proper or not) of a set
- Notation: P(A)

$A = \{a, b, c\}$

$P(A) = \{ \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, \phi \}$

"a" $\Rightarrow$ 8

Contains a | do not contain a

Contains a	do not contain a
$\{a\}$	ϕ
$\{a, b\}$	$\{b\}$
$\{a, c\}$	$\{c\}$
$\{a, b, c\}$	$\{b, c\}$

4   4   B

$P(B) = \frac{P(A)}{2}$

Theorem

$^nC_k = 0 \text{ to } n$

- If $|A| = n$ then $|P(A)| = 2^n$

Basic $n = 0$ $A = \{\ \}$

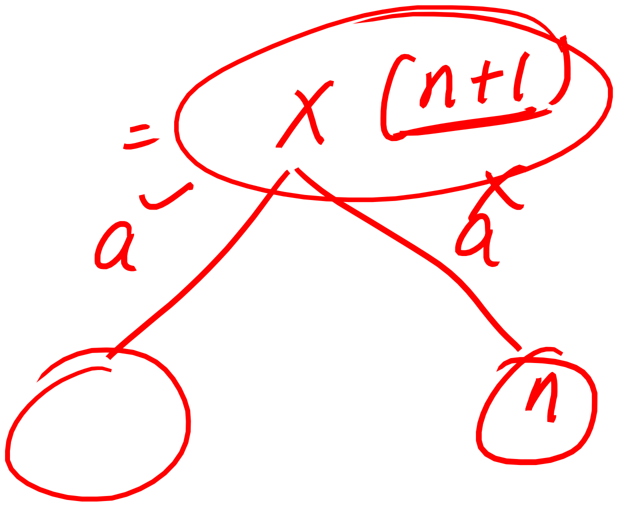
$|P(A)| = 1$ $P(A) = \{\{\ \}\}$ or $\{\phi\}$

Assume $\Rightarrow |A| = n$

$|P(A)| = 2^n$

To Prove $n+1$

$|A| = n + \text{'}a\text{'}$ =



$\Rightarrow Y$ $|P(Y)| = 2^n$

$|P(Y)| = \dfrac{|P(X)|}{2}$

$|P(X)| = |P(Y)| \cdot 2 = 2^n \cdot 2 = 2^{n+1}$

Theorem Cont.

$|A| = n$ $|P(A)| = 2^n$

Basic Step - For $n = 0$ - $\{\}\,|\,\phi$ $\rightarrow |P(A)| = 1$ $2^0 = 1$ $= \{\{\}\} = \{\phi\}$

Inductive Step - Assume that if $|A| = n$ then $|P(A)| = 2^n \rightarrow$ True

Now, we add a new element 'a' to the set $A \Rightarrow n+1$

$|A| = n+1$

P(x) we have 2 classes of subsets -

i) Contains the element 'a' in all the subsets

ii) Donot contain 'a' - $P(Y)$ - all the subsets that donot contain a

If $P(X)$ - represent all the subsets of A having $n+1$ element

$|P(Y)| = \frac{|P(X)|}{2} \Rightarrow |P(X)| = 2 \times |P(Y)| = 2 \times 2^n = 2^{n+1}$

Agenda for Today

- Set operations
  - Union
  - Intersection
  - Difference
  - Cartesian Product
- Representation of sets
  - Venn Diagram
    - Example
- Properties/Laws of sets

Set Operations

- Binary Operations:
  - Union: contain elements in either A or in B or both
  - Intersection: contain elements which are in A and B
  - Difference: elements that are in A but not in B
  - Cartesian Product
- Unary Operation $\rightarrow$
  - Complement

$A \cup B \Rightarrow \forall x \in A \vee x \in B$

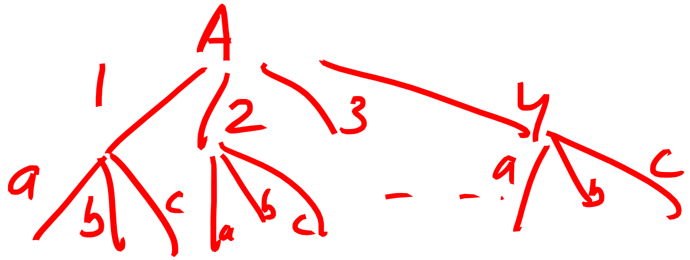
$A \cap B \Rightarrow x \in A \wedge x \in B$

$A - B = x \in A \rightarrow x \notin B$

$\underline{A \times B}$

$\overline{A} = U - A$

Cartesian Product — $A \times B$ — Order pairs



$A = \{1, 2, 3, 4\}$

$B = \{a, b, c\}$

$\underline{A} \times B = \{(1,a)\ (2,a)\ (3,a)\ (4,a)\ (1,b)\ (2,b) \ldots$

$\ldots (4,c)\}$

$A \times B = |A| \times |B|$

$A \times B \neq B \times A$

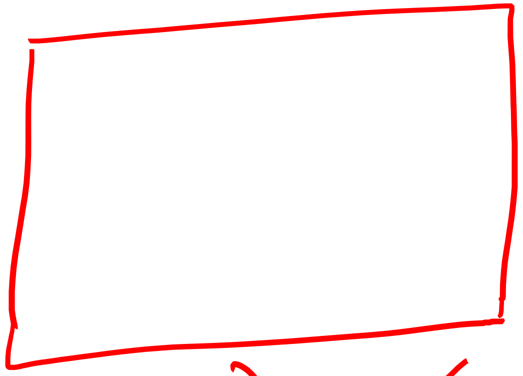
$B \times A = \{(a,1)\ (a,2) \ldots$

$|A \times B| = |B \times A|$

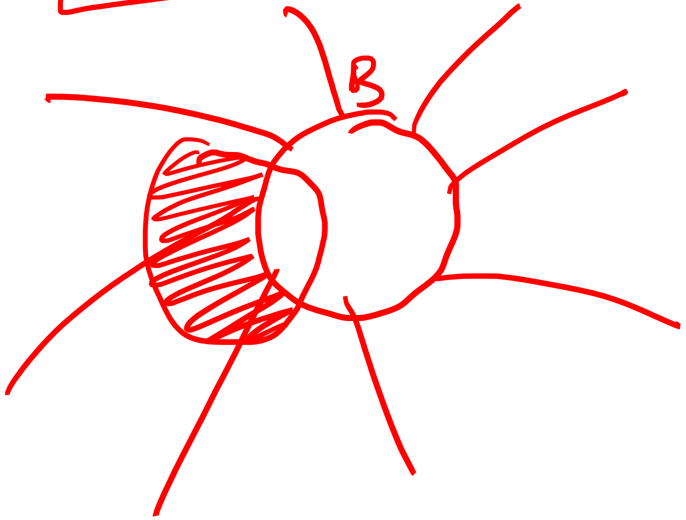
Cardinality of Cartesian Product

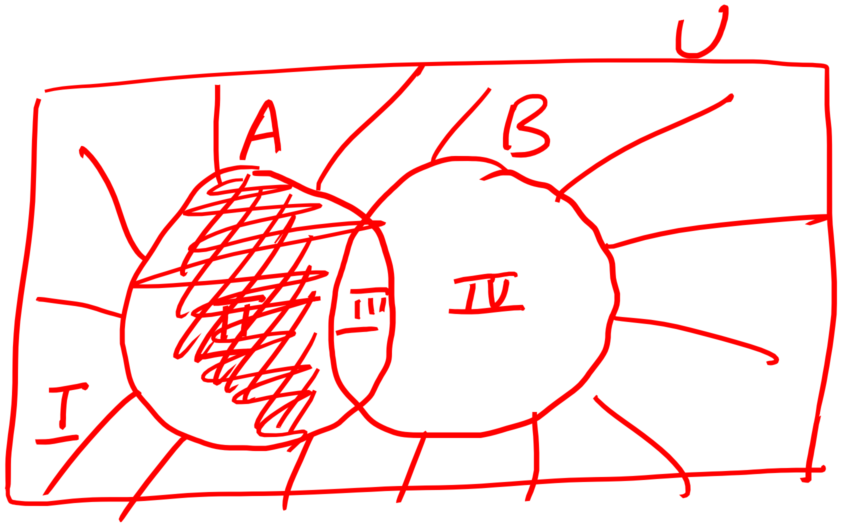
Representation of Sets – Venn Diagrams

Universal Set (U)



Set A, B





$A = \{1, 2, 3, 4, 5\}$

$B = \{3, 4, 6, 7, 8\}$

$A - B = \{1, 2, 5\}$

$A \cap B = \{3, 4\}$

$A - (A \cap B) = \{1, 2, 5\}$

$\mathrm{I} = \overline{A \cup B}$

$\mathrm{II} = A - B = A \cap \overline{B} = A$

$\mathrm{III} = \cancel{A \cap B} \quad A \cap B$

$\mathrm{IV} = B - A = B \cap \overline{A}$

Venn Diagram

Example

$U = 165$

$|M| = 79$

$|P| = 83$

$|C| = 63$

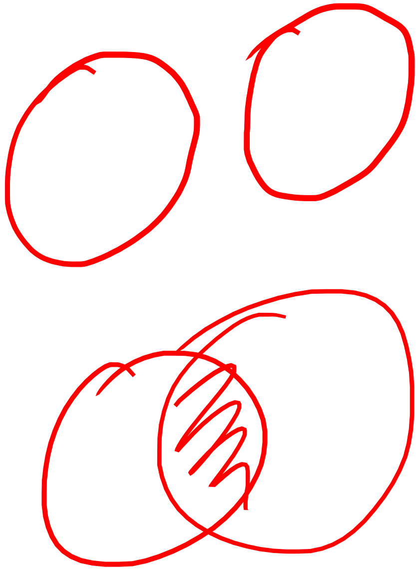
$|M \cap C| = 33$

$|M \cap P| = 20$

$|P \cap C| = 24$

$|M \cap P \cap C| = 8$

- Among a group of 165 students, 33 students took mathematics and computer science, 20 took mathematics and physics, and 24 students took physics and computer science. 8 students took physics, mathematics and computer science. 79 students took mathematics, 83 students took physics and 63 students took computer science. How many students took none of these 3 subjects?

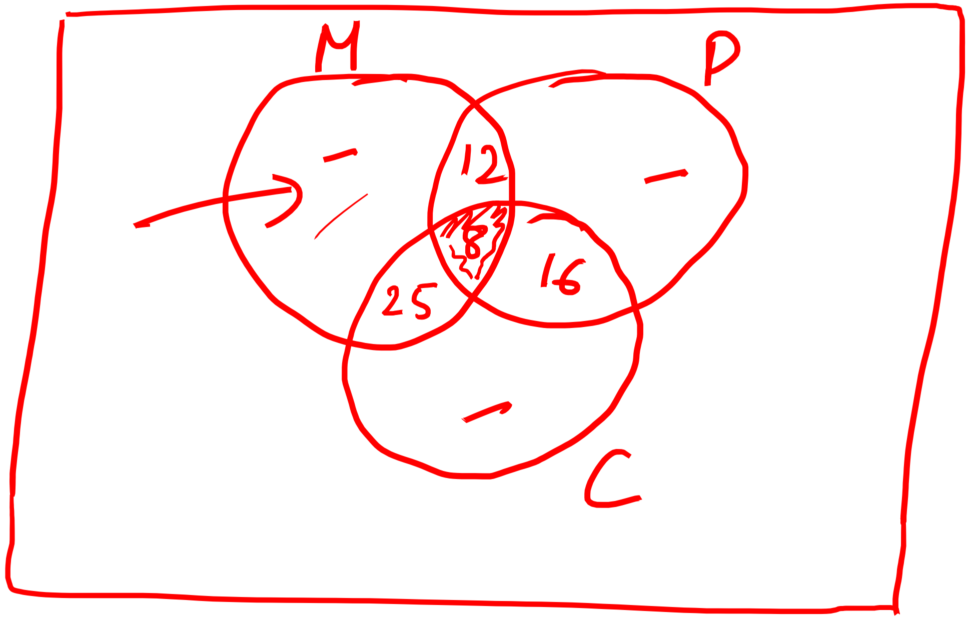


$79 - 12 - 8 - 25$

$|M \cup P \cup C| = \_\_\_\_\_\_$

$|P \cup M| = 79 + 83 -$

$|P| + |M| - |P \cap M|$



Representation of Sets

Tabular Form	Descriptive Form
• **Tabular Form:** • Listing all the elements of a set, separated by commas and enclosed within braces or curly brackets{}. • **Examples:** • In the following examples we write the sets in Tabular Form. • A = {1, 2, 3, 4, 5} is the set of first five **Natural Numbers**. • B = {2, 4, 6, 8, …, 50} is the set of **Even numbers** up to 50 • C = {1, 3, 5, 7, 9, …} is the set of **positive odd numbers.**	• **Descriptive Form:** • Stating in words the elements of a set. • **Examples**: • Now we will write the same examples which we write in Tabular Form ,in the Descriptive Form. • A = set of first five Natural Numbers.(is the Descriptive Form) • B = set of positive even integers less or equal to fifty. (is the Descriptive Form) • C = {1, 3, 5, 7, 9, …} (is the Tabular Form) • C = set of positive odd integers. (is the Descriptive Form)

Representation of Sets

$\{x \mid x \in N,\ x \text{ is even no} < 10 \ \le 8\}$

Set Builder Form

- **Set Builder Form:**
  - Writing in symbolic form the common characteristics shared by all the elements of the set.
- **Examples:**
  - Now we will write the same examples which we write in Tabular as well as Descriptive Form ,in Set Builder Form .
  - A = {x ∈N | x<=5} (is the Set Builder Form)
  - B = {x ∈ E | 0 < x <=50} (is the Set Builder Form)
  - C = {x ∈O | 0 < x } (is the Set Builder Form)

Your Task☺

- **Write the following sets in the set builder form.**
- (a) A = {2, 4, 6, 8} $\Rightarrow \{x \mid x \in I \text{ Even } < 10,\ \le 8\}$
- (b) B = {3, 9, 27, 81} $\Rightarrow \{x \mid x = 3^n,\ n \in N \ \le 4\}$
- (c) C = {1, 4, 9, 16, 25}
- (d) D = {1, 3, 5, ……}
- (e) E = {a, e, i, o, u}

Representation of Sets

Your Task☺

- **Write the following sets in the roster form.**

- (a) A = {x : x ∈ W, x ≤ 5}
- (b) B = {The set all even numbers less than 12)
- (c) C = {x : x is divisible by 12}
- (d) D = {The set of first seven natural numbers}
- (e) E = {The set of whole numbers less than 5}

Representation of Sets

Answers of the Previous Questions

- **Write the following sets in the roster form.**

- (a) {0, 1, 2, 3, 4, 5}
- (b) {2, 4, 6, 8. 10}
- (c) {12, 24, 36, ...}
- (d) {1, 2, 3, 4, 5, 6, 7}
- (e) {0, 1, 2, 3, 4}

Well-Defined Set

Numerical Sets (Well-Defined)

- Set of even numbers:

  {..., -4, -2, 0, 2, 4, ...}
- Set of odd numbers:

  {..., -3, -1, 1, 3, ...}
- Set of prime numbers:

  {2, 3, 5, 7, 11, 13, 17, ...}
- Positive multiples of 3 that are less than 10:

  {3, 6, 9}

Not Well-Defined Set

Numerical Sets (Not Well-Defined)

- There can also be sets of numbers that have no common property, they are just defined that way. For example:

- {2, 3, 6, 828, 3839, 8827}
- {4, 5, 6, 10, 21}
- {2, 949, 48282, 428859, 119484}
- {111, 8888, 001922, 98373773}

Exercise

Your Task☺

- **Which of the following are well-defined sets?**

1. All the colors in the rainbow.
2. All the points that lie on a straight line.
3. All the honest members in the family.
4. All the consonants of the English alphabet.
5. All the tall boys of the school.
6. All the hardworking teachers in a school.
7. All the prime numbers less than 100.
8. All the letters in the word GEOMETRY.

Your Task☺

- **Which of the following are well-defined sets?**

1. All the colors in the rainbow.
2. All the points that lie on a straight line.
3. All the honest members in the family.
4. All the consonants of the English alphabet.
5. All the tall boys of the school.
6. All the hardworking teachers in a school.
7. All the prime numbers less than 100.
8. All the letters in the word GEOMETRY.

Answers: 1, 2, 4, 7, 8 are well-defined.

Basic Concepts

$A = \{x, \underline{\{x\}}\}$

$\{x\} \in A$ ✓

$A = \{x\}$

$x \in A$

$\{x\} \notin A$

$\{x\} \in A$

Your Task☺

- **Determine whether each of the following statements is true or false.**
  - $x \in \{x\}$ — T
  - $\{x\} \subseteq \{x\}$ — T
  - $\underline{\{x\} \in} \{x\}$ — F
  - $\{x\} \in \{\{x\}\}$ — T
  - $\varnothing \subseteq \{x\}$ — T
  - $\varnothing \in \{x\}$ — F

$\{\{x\}\}$ $\{x\}$

$\{x\}$ x

$\{x\}$

x $\underline{\{x\}} \in$

Solution

- **Determine whether each of the following statements is true or false.**
  - $x \in \{x\}$ TRUE
    - (Because x is the member of the singleton set { x })
  - $\{x\} \subseteq \{x\}$ TRUE
    - (Because Every set is the subset of itself. Note that every Set has necessarily two subsets $\varnothing$ and the Set itself, these two subset are known as Improper subsets and any other subset is called Proper Subset)
  - $\{x\} \in \{x\}$ FALSE
    - (Because { x} is not the member of {x}) Similarly other
  - $\{x\} \in \{\{x\}\}$ TRUE
  - $\varnothing \subseteq \{x\}$ TRUE
  - $\varnothing \in \{x\}$ FALSE

Finite and Infinite Sets

Your Task☺

- **Classify the following as finite and infinite sets.**

a. A = {x : x ∈ N and x is even} – ∞

b. B = {x : x ∈ N and x is composite} – ∞

c. C = {x : x ∈ N and $3x - 2 = 0$} – Finite | Null

d. D = {x : x ∈ N and $x^2 = 9$ } – Finite | Singleton

e. E = {The set of numbers which are multiple of 3} – ∞

f. F = {The set of letters in English alphabets} – Finite

g. G = {The set of persons living in a house} – Finite

h. H = {x : x ∈ P, P is a number} – ∞

i. I = {The set of fractions with numerator 3} – ∞

Finite and Infinite Sets

Answer of the Previous Questions
a. Infinite
b. Infinite
c. Finite
d. Finite
e. Infinite
f. Finite
g. Finite
h. Infinite
i. Infinite

Empty Set/Null Set

Empty Set / Null Set

- An empty set is a set which has no members.
- **Example:**
- If, H = {the number of dinosaurs on earth}
- Then, H is an empty set.
- That is, H = {}
- **Note:** An empty set is denoted by the symbol {} or ϕ.
- **Note** the subtlety in $\boxed{\phi \neq \{\phi\}}$
  - The left-hand side is the empty set
  - The right hand-side is a singleton set, and a set containing a set

$\{\} = \phi$

$\{\phi\}$

$=$

$\{\{\}\}$

Singleton/Unit Set

Singleton Set or Unit Set

- A singleton is a set that contains exactly one element.
- Sometimes, it is known as unit set.
- The singleton containing only the element a can be written {a}.
- Note that ϕ is empty set and {ϕ} is not empty set but it is a singleton set.

- Singleton set or unit set contains only
- one element. A singleton set is denoted
- by {s}.
- **Example** : $S = \{x \mid x \in N, 7 < x < 9\}$

Singleton or Null?

Your Task☺

- **Identify the following as null set or singleton set.**

a. $A = \{x : x \in N, 1 < x < 2\}$ – Null

b. P = {Point of intersection of two lines} – S

c. C = {x : x is an even prime number greater than 2} – N

d. Q = {x : x is an even prime number} – S

e. $E = \{x : x^2 = 9, x \text{ is even}\}$ – ~~S~~ Null

f. $B = \{0\}$ – S

g. D = {The set of largest 1 digit number} – S {9}

h. F = {The set of triangles having 4 sides} – N

i. H = {The set of even numbers not divisible by 2} – N

Answers:

a. Null
b. Singleton
c. Null
d. Singleton
e. Null
f. Singleton
g. Singleton
h. Null
i. Null

Equal and Equivalent Sets

- **Equal Set:** Two sets are equal if they both have the same members.
- **Example 1:** if $A = \{1, 2, 3\}$
- And $B = \{1, 2, 3\}$
- Then $A = B$, that is both sets are equal.
- **Example 2:** if $C = \{1, 2, 5\}$
- And $D = \{5, 1, 2\}$
- Then $C = D$, that is both sets are equal.
- **Note:** The order in which the members of a set are written does not matter.

Equal and Equivalent Sets

- **Equivalent Set:** Two sets are equivalent if they have the same number of elements.
- **Example 1:** if F = {2, 4, 6, 8, 10}
- And G = {10, 20, 30, 40, 50}

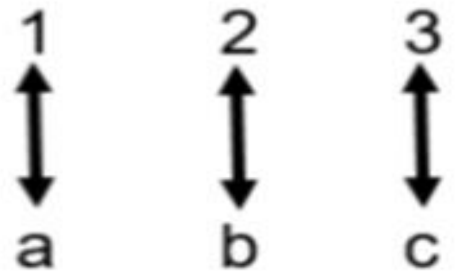
Equivalent

- Then n(F)=n(G) =5, that is, sets F and G are equivalent.

- **Example 2:** if A = {1, 2, 3}
- And B = {a, b, c}
- Then n(A)=n(B), that is, sets A and B are equivalent.

Equal and Equivalent Sets

- **Note:** When each member of a set matches one and only one member of the other set, there is a 1-1 correspondence between the two sets.
- For Example:



- Sets that cannot be paired in a 1-1 correspondence are called **non-equivalents sets**.

Your Task☺

- **Which of the following pairs of sets are equivalent or equal?**

a. $A = \{x : x \in N, x \leq 6\}$ → $\{1, 2, 3, 4, 5, 6\}$

$B = \{x : x \in W, 1 \leq x \leq 6\}$

b. P = {The set of letters in word "plane"}

Q = {The set of letters in word "plain"} Equivalent

c. X = {The set of color in the rainbow}

Y = {The set of days in a week} Eq.

d. $M = \{4, 8, 12, 16\}$

$N = \{8, 12, 4, 16\}$ =

Answers:

a. Equal Sets
b. Equivalent Sets
c. Equivalent Sets
d. Equal Sets

One More Task☺

- **Is this is equal set?**
- {2,3,5,7} , {2,2,3,5,3,7}

{2, 3, 5, 7}

✓ Equal

- And-----------------:
- {2,3,5,7} , {2,3}

✓ Not Equal

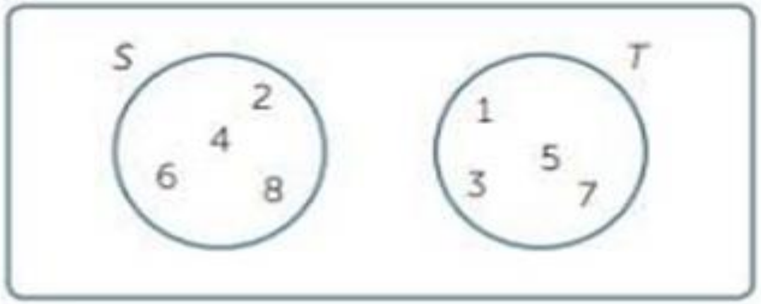
Note that:

- Equal sets are always equivalent.
- Equivalent sets may not be equal

Disjoint Sets

- **Disjoint sets:**
- Two sets are called disjoint if they have no elements in common.

- **For Example:**
- The sets S = {2, 4, 6, 8} and T = {1, 3, 5, 7} are disjoint.



Your Task ☺

- **Which of the following sets are disjoint or overlapping:**

a. A = {The set of boys in the school}
   B = {The set of girls in the school} Disjoint

b. P = {The set of letters in the English alphabets}
   Q = {The set of vowels in the English alphabets}

c. $X = \{x : x \text{ is an odd number}, x < 9\}$
   $Y = \{x : x \text{ is an even number}, x < 10\}$ D

d. E = {9, 99, 999}
   F = {1, 10, 100} D

Answers:

a. Disjoint Sets
b. Overlapping Sets
c. Disjoint Sets
d. Disjoint Sets

Union of Sets

- **Examples of Union of Sets**
- A = {x / x is a number bigger than 4 and smaller than 8}
- B = {x / x is a positive number smaller than 7}

- A = {5, 6, 7} and B = {1, 2, 3, 4, 5, 6}
- A ∪ B = {1, 2, 3, 4, 5, 6, 7}
- Or A ∪ B = {x / x is a number bigger than 0 and smaller than 8}

Union of Sets

- **Examples of Union of Sets**
- A = {#, %, $}
- B = { } $= \phi$
- Then, A ∪ B = {#, %, $}

$A \cup \phi = A$

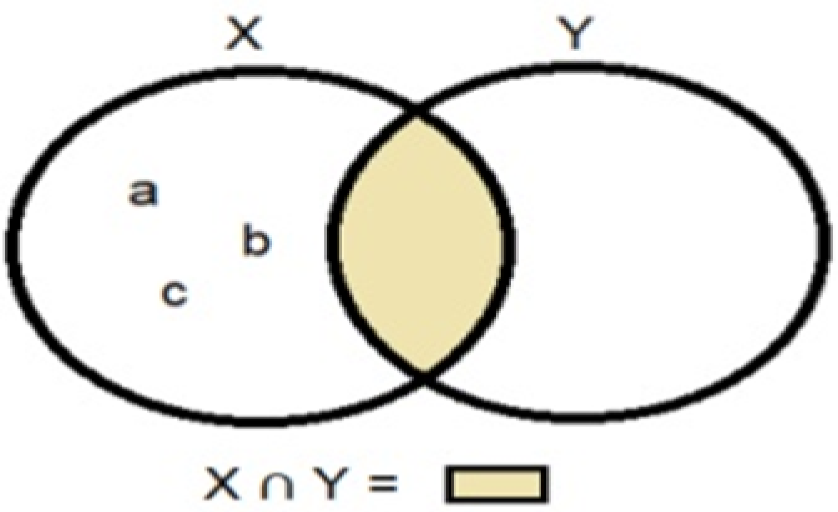
- **Examples of Union of Sets**
- N = {-5, -4, 0, 6, 8}   and O = {-4, 0, 8, 9}
- Then, N ∪ O = {-5, -4, 0, 6, 8, 9}

Intersection of Sets

- **Example of Intersection of Sets**
- If A = {a, b, c, d} and B = {1, a, 2, b}.
- A n B is all the common elements of the set A and B.
- Therefore, $A \cap B = \{a, b\}$.

More Example

- If $X = \{a, b, c\}$ and $Y = \{\phi\}$. Find intersection of two given sets X and Y.
- **Solution:**
- $X \cap Y = \{\ \}$



$X \cap Y = \{\ \}$ or ϕ (an empty set). Non empty sets which have no members in common are called "**disjoint sets**".

Your Task☺

- If set A = {4, 6, 8, 10, 12}, set B = {3, 6, 9, 12, 15, 18} and set C = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

- (i) Find the intersection of sets A and B.
- (ii) Find the intersection of two set B and C.
- (iii) Find the intersection of the given sets A and C.

Solution

- (i) Intersection of sets A and B is $A \cap B$
- Set of all the elements which are common to both set A and set B is {6, 12}.

- (ii) Intersection of two set B and C is $B \cap C$
- Set of all the elements which are common to both set B and set C is {3, 6, 9}.

- (iii) Intersection of the given sets A and C is $A \cap C$
- Set of all the elements which are common to both set A and set C is {4, 6, 8, 10}.

Your Task☺

- A = {1, 2, 3} and B = {4, 5, 6}.

- Find the difference between the two sets:
- (i) A and B
- (ii) B and A

Solution

- The two sets are disjoint as they do not have any elements in common.
- (i) A - B = {1, 2, 3} = A
- (ii) B - A = {4, 5, 6} = B

More Task☺

- **Given three sets P, Q and R such that:** {11, 12, 13, 14, 15}
- P = {x : x is a natural number between 10 and 16},
- Q = {y : y is a even number between 8 and 20} and
- R = {7, 9, 11, 14, 18, 20} {10, 12, 14, 16, 18}
-
- (i) Find the difference of two sets P and Q
- (ii) Find Q - R
- (iii) Find R - P
- (iv) Find Q – P

P-Q = {11, 13, 15}

Solution

- According to the given statements:
- P = {11, 12, 13, 14, 15}
- Q = {10, 12, 14, 16, 18}
- R = {7, 9, 11, 14, 18, 20}
- (i) P – Q = {Those elements of set P which are not in set Q}
  = {11, 13, 15}
- (ii) Q – R = {Those elements of set Q not belonging to set R}
  = {10, 12, 16}
- (iii) R – P = {Those elements of set R which are not in set P}
  = {7, 9, 18, 20}
- (iv) Q – P = {Those elements of set Q not belonging to set P}
  = {10, 16, 18}

Universal Set

- The universal set is the set of all elements that are considered in a specific theory. We'll note the universal set with U.
- We'll choose as universal set: U = {6,7,8,9,15,16,17,18,20,21}.
- We have to determine the sets:
- M = {x / x are the multiple of 3}
- N = {x / x are the multiple of 5}
- The elements of M and N have to be chosen from the universal set U.
- To determine M, we'll identify the multiples of 3 from U: {6,9,15,18,21}
- M = {6,9,15,18,21}
- To determine N, we'll identify the multiples of 5 from U: {15,20}.
- N = {15,20}

Complement of a Set

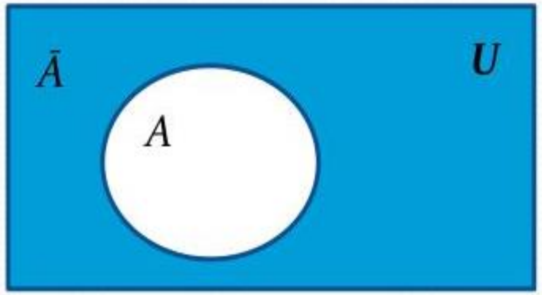
- If a set A is a subset of a given Universal Se U, Then the difference U – A or U \ A id the complement of A.
- We write U – A or U \ A = A′. We say A complement.
- **Example:**
- If, U = {3, 5, 7, 9, 11, 13, 15, 17, 19}
- And, A = {5, 11, 17, 19}
- Then, U — A = A′ = {3, 7, 9, 13, 15}

Complement of a Set

- Where, A' is "the complement of A".
- The union of A and A' is the Universal set.
- $U = A \cup A' = \{5, 11, 17, 19\} \cup \{3, 7, 9, 13, 15\}$
- $U = A \cup A' = \{3, 5, 7, 9, 11, 13, 15, 17, 19\}$

- The intersection of A and A' is an empty set.
- $A \cap A' = \{\,\}$ or ϕ

Venn Diagram for Complement



Your Task ☺

- **Find the complement of A in U**
- A = { x / x is a number bigger than 4 and smaller than 8}
- U = { x / x is a positive number smaller than 7}
- A = { 5, 6, 7} and U = { 1, 2, 3, 4, 5, 6}
- A′ = { 1, 2, 3, 4}
- Or A′ = { x / x is a number bigger than 1 and smaller than 5 }

U $\quad$ A, B, C

Properties of Sets ϕ

- Associative Law - $(A \cup B) \cup C = A \cup (B \cup C)$
  $(A \cap B) \cap C = A \cap (B \cap C)$
- Commutative Law - $A \cap B = B \cap A$ or $A \cup B = B \cup A$
- Distributive Law - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- Identity Laws - $A \cup \phi = A$, $A \cap U = A$
- Complement Laws - $A \cup \bar{A} = U$, $A \cap \bar{A} = \phi$

Properties of Sets

- Idempotent Laws – $A \cup A = A$ $\quad A \cap A = A$
- Bound Laws – $A \cup U = U$, $\quad A \cap \phi = \phi$
- Absorption Laws – $A \cup (A \cap B) = A$
  $A \cap (A \cup B) = A$
- Involution Laws – $\overline{\overline{A}} = A$
- D' Morgan's Laws for set – $\overline{A \cup B} = \overline{A} \cap \overline{B}$, $\quad \overline{A \cap B} = \overline{A} \cup \overline{B}$
- 0/1 Laws – $\overline{\phi} = U$, $\quad \overline{U} = \phi$

$\Rightarrow (A \cap A) \cup (A \cap B)$

$A \cup (A \cap B) = A$

$\Rightarrow (A \cup A) \cap (A \cup B)$

$A \cap (A \cup B) \Rightarrow A$

Commutative ~~Complement~~ Law

① $\underline{A \cup B} = \underline{B \cup A}$

$x \in A \cup B$

$x \in A$ or $x \in B$

$x \in B$ or $x \in A$

$x \in B \cup A$

$\forall x \, (x \in A \cup B \rightarrow x \in B \cup A)$ ①

$x \in B \cup A$

$x \in B$ or $x \in A$

$x \in A$ or $x \in B$

$x \in A \cup B$

$\forall x \, (x \in B \cup A \rightarrow x \in A \cup B)$ — ②

$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\{x \mid x \underline{\qquad}\}$

$x \in \overline{A}$

$\underline{x \notin A}$

<u>Part 1</u>

$x \in \overline{A \cap B} \leftarrow$

$x \notin A \cap B$ — Def. of complement

~ or $\neg(x \in A \cap B)$ – Def. of negation

$\neg(x \in A \wedge x \in B)$ – Def. of intersection

$\neg(x \in A) \vee \neg(x \in B)$

$x \notin A \vee x \notin B$

$x \in \overline{A} \vee x \in \overline{B}$

$x \in \overline{A} \cup \overline{B}$

<u>Part 2</u>

$x \in \overline{A} \cup \overline{B}$

$x \in \overline{A}$ or $x \in \overline{B}$

$x \notin A$ or $x \notin B$

$\neg(x \in A)$ or $\neg(x \in B)$

$\neg(x \in A \wedge x \in B)$

$\neg(x \in A \cap B) \Rightarrow x \notin A \cap B$

$x \in \overline{A \cap B}$

Proof – Commutative Law $\Rightarrow$ Set Diff $\boxed{A - B = B - A}$ ✗

1) $A \cup B = B \cup A$ OR $A \cap B = B \cap A$

$\forall x \left(\underbrace{(x \in A \rightarrow x \in B)}_{A \subseteq B} \right) \wedge \left(\underbrace{x \in B \rightarrow x \in A}_{B \subseteq A} \right) \Rightarrow A = B$

$x \in A \cup B$

$x \in A$ or $x \in B$

$x \in B$ or $x \in A$

$x \in B \cup A$

$(x \in A \cup B \rightarrow x \in B \cup A)$

Let $x \in B \cup A$

$x \in B$ or $x \in A$

$x \in A$ or $x \in B$

$x \in A \cup B$

$(x \in B \cup A \rightarrow x \in A \cup B)$

$A \cup B = B \cup A$

Proof – De Morgan's Law

① $\overline{A \cap B} = \bar{A} \cup \bar{B}$ ② $\overline{A \cup B} = \bar{A} \cap \bar{B}$

$x \in \overline{A \cap B}$

$x \notin A \cap B$

$\neg(x \in (A \cap B))$

$\neg(x \in A \text{ and } x \in B)$

$\neg x \in A \text{ or } \neg x \in B$

$x \notin A \text{ or } x \notin B$

$x \in \bar{A} \text{ or } x \in \bar{B}$

$x \in \bar{A} \cup \bar{B}$

$\forall x (x \in \overline{A \cap B} \rightarrow x \in \bar{A} \cup \bar{B})$ — ①

$x \in \bar{A} \cup \bar{B}$

$x \in \bar{A} \text{ or } x \in \bar{B}$

$x \notin A \text{ or } x \notin B$

$\neg(x \in A) \text{ or } \neg(x \in B)$

$\neg(x \in A \text{ and } x \in B)$

$\neg(x \in A \cap B)$

$x \notin A \cap B \rightarrow \underline{x \in \overline{A \cap B}}$

$\forall x (x \in \bar{A} \cup \bar{B} \rightarrow x \in \overline{A \cap B})$ — ②

$\underline{\overline{A \cap B} = \bar{A} \cup \bar{B}}$

$\{1, 2, 3\} \Leftrightarrow \{2, 2, 2, 1, 3, 1, 3, 3\}$

Classical set $\neq$ Multiset

Multi-set

Definition: These are unordered collection of elements where an element can occur as a member more than once. The notation $\{m_1.a_1, m_2.a_2 \ldots\ldots m_r.a_r\}$ denotes the multi-set with element a_1 occurring m_1 times, element a_2 occurring m_2 times and so on. The numbers m_i , *i = {1, 2,.....r} are called multiplicities of the element* a_i , *i = {1, 2,.....r}*

Multi-set of prime factors of a number n

$$120 = 2^3 3^1 5^1$$

which gives the multiset {2, 2, 2, 3, 5}.

$A \subseteq B \Rightarrow$ if $\nu_A \leq \nu_B$

$\{a, a, a$

$\nu_a = 3$

$\{1, 1, 2, 2, 2, 3, 3\} \subseteq \{1, 1, 1, 2, 2, 2, 2, 3, 3\}$

$A = B \Rightarrow \nu_A = \nu_B$

$\{1, 1, 2, 3, 3, 3\} \not\subseteq$

$A = \{1, 1, 2\} = 3$

$P(A) = \{\phi, \{1\}, \{2\}, \{1, 1\}, \{1, 2\}, \{1, 1, 2\}\}$

Operations on Multi-set

ν_A

$\{3, 3, 4\} \neq \{3, 4\}$

$(1, 0)$

Multiset

1. Union: - $\max(\nu_A, \nu_B)$

For example, if A = {2, 3, 4, 4}, B = {1, 4, 3, 3}
then A ∪ B = {1, 2, 3, 3, 4, 4}.

$= \{2, \underline{3, 3, 4, 4}, 1\}$

2. Intersection: $\min(\nu_A, \nu_B) = \{3, 3, 4\}$

For example, if A = {3, 3, 3, 4, 4}, B = {1, 4, 3, 3}
then A ∩ B = {3, 3, 4}.

Operations on Multi-set

3. Addition/Sum/Merge:

$\nu_A + \nu_B$

For example, if A = {1, 1, 2, 2, 4, 4, 4}, B = {1, 2, 3, 3}
then A + B = {1, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4}.

4. Difference:

$max(\nu_A - \nu_B, 0)$

For example, if A = {3, 3, 3, 4, 4}, B = {1, 4, 3, 3}
then A - B = {3, 4}.

$B - A = \{1\}$ 0−1

−1,

Ram is Honest

[0, 1] Fuzzy

0 —

0.4 —

0.8 —

1 —

$A = \{(a, 0.8), (b, 0.4), (c, 1)\}$

Lotfi Zadeh

1967

Fuzzy Set Operation

Given X to be the universe of discourse and Ã and Ḃ to be fuzzy sets with $\mu_A(x)$ and $\mu_B(x)$ are their respective membership function, the fuzzy set operations are as follows:

$\forall x \; \mu_A(x) : [0, 1]$

A

$\tilde{A}$

Union:

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

$x \in A$

Intersection:

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)), \forall x \in \tilde{A}\}$

Complement:

$$\mu_A(x) = 1 - \mu_A(x)$$

$\mu_A(x) : x \to [0,1]$

Fuzzy Set Operation (Continue)

$A \cap B = \{(x_1, 0.5), (x_2, 0.2), (x_3, 0)\}$

Example:

A = {(x_1,0.5),(x_2,0.7),(x_3,0)} B = {(x_1,0.8),(x_2,0.2),(x_3,1)}

1

Union: max

A U B = {(x_1,0.8),(x_2,0.7),(x_3,1)}

$\bar{B} = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0)\}$

Because

$\bar{A} = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1)\}$

$\mu_{A \cup B}(x_1) = \max(\mu_A(x_1), \mu_B(x_1))$

$= \max(0.5, 0.8)$

$= 0.8$

$\mu_{A \cup B}(x_2) = 0.7$ and $\mu_{A \cup B}(x_3) = 1$

Fuzzy Set Operation (Continue)

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$ $B = \{(x_1,0.8),(x_2,0.2),(x_3,1)\}$

<u>Intersection:</u>

$$A \cap B = \{(x_1,0.5),(x_2,0.2),(x_3,0)\}$$

Because

$$\mu_{A \cap B}(x_1) = \min\ (\mu_A(x_1),\ \mu_B(x_1))$$

$$= \max(0.5,0.8)$$

$$= 0.5$$

$$\mu_{A \cap B}(x_2) = 0.2 \text{ and } \mu_{A \cap B}(x_3) = 0$$

Fuzzy Set Operation (Continue)

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$

<u>Complement:</u>

$$A^c = \{(x_1,0.5),(x_2,0.3),(x_3,1)\}$$

Because

$$\mu_A(x_1) = 1 - \mu_A(x_1)$$

$$= 1 - 0.5$$

$$= 0.5$$

$$\mu_A(x_2) = 0.3 \ \text{ and } \ \mu_A(x_3) = 1$$

Consider the fuzzy set A

[0, 1]

A = {(1, 0), (2, 0), (3, 0.2), (4, 0.5), (5, 0.8), (6, 1)}

1) There is a chance of 95% rain this evening

Ans:

2) It is going to rain heavily in the evening

Card (A) = |A| = 0 + 0 + 0.2 + 0.5 + 0.8 + 1 = 2.5

Relcard (A) = ||A|| = $\frac{2.5}{6} \approx 0.417$

45

$\boxed{20}$

Fuzzy - AI

Temp - I/P ⇒ 3 Fuzzy sets → Cold, Warm, Hot

Fan Speed - O/P ⇒ " → Low Med. High

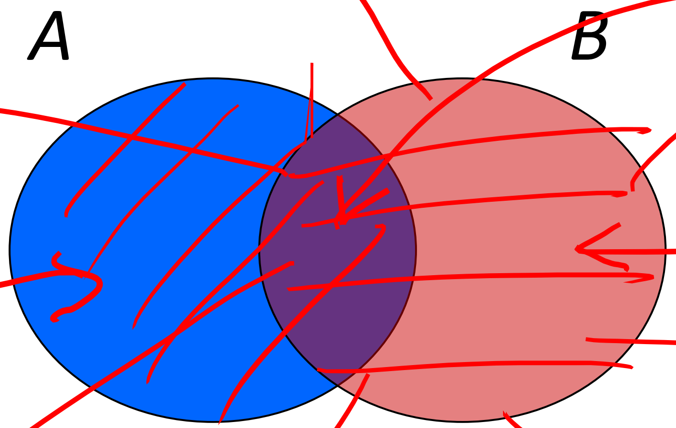
Principle of Inclusion-Exclusion (2 sets) (PIE) / Subtraction Principle

For two arbitrary sets A and B

$$|A \cup B| = |\underline{A}| + |\underline{B}| - |A \cap B|$$

$|A| = |A \cap \bar{B}| + |A \cap B|$

$|B| = |\bar{A} \cap B| + |A \cap B|$



$A \cap B$

$\bar{A} \cap B$

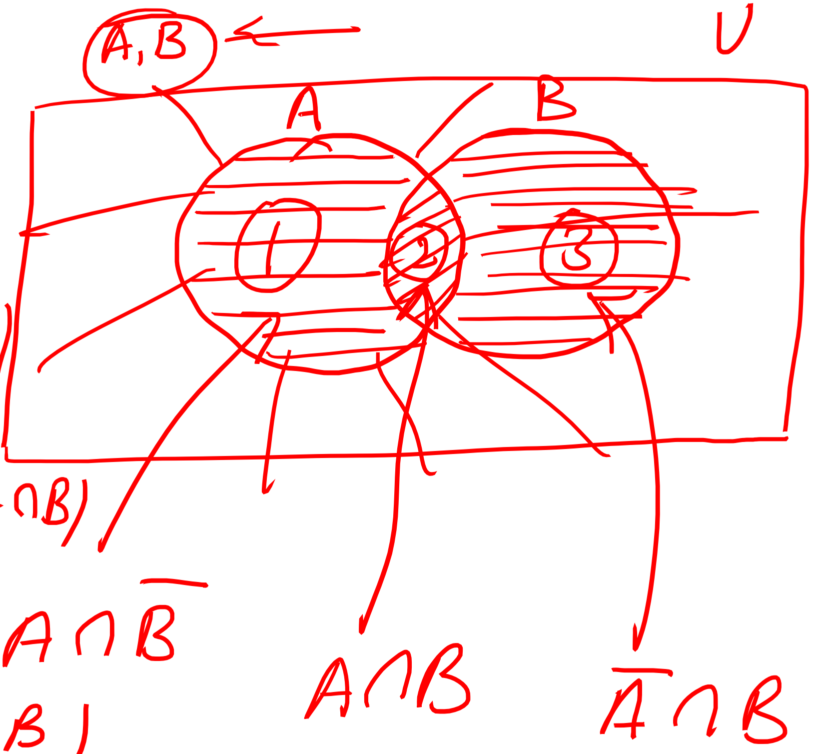
$A \cap \bar{B}$

Proof

$$|A \cup B| = |A \cap \bar{B}| + |A \cap B| + |\bar{A} \cap B| \quad \text{— (1)}$$

$$|A| = |A \cap \bar{B}| + |A \cap B| \Rightarrow |A \cap \bar{B}| = |A| - |A \cap B|$$

$$|B| = |A \cap B| + |\bar{A} \cap B| \Rightarrow |\bar{A} \cap B| = |B| - |A \cap B|$$



$$|A \cup B| = |A| - \cancel{|A \cap B|} + \cancel{|A \cap B|} + |B| - |A \cap B|$$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Eg - $\underline{100}$ — F

50 — R

20 — $F \cap R$

$$|F \cup R| = 100 + 50 - 20$$

$$= \underline{130}$$

Inclusion-Exclusion (2 sets)

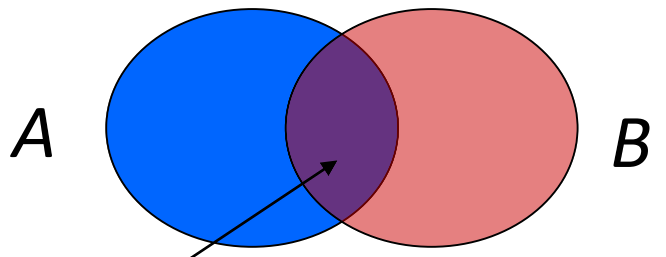
Let S be the set of integers from 1 through 1000 that are multiples of 3 or multiples of 5.

Let A be the set of integers from 1 to 1000 that are multiples of 3.

Let B be the set of integers from 1 to 1000 that are multiples of 5.

It is clear that S is the union of A and B,

but notice that A and B are not disjoint.



$|A| = 1000/3 = 333$ $|B| = 1000/5 = 200$

$A \cap B$ is the set of integers that are multiples of 15, and so $|A \cap B| = 1000/15 = 66$

So, by the inclusion-exclusion principle, we have $|S| = |A| + |B| - |A \cap B| = 467$.

Inclusion-Exclusion (3 sets)

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$|A|+|B|+|C| = |A \cup B \cup C| + |A \cap B| + \boxed{|A \cap B \cap \bar{C}| + |A \cap B \cap C|}$

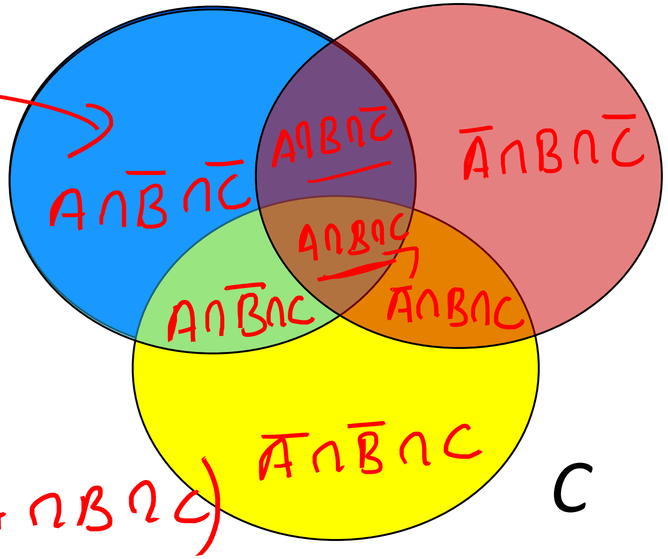
$+ |A \cap \bar{B} \cap C| + |\bar{A} \cap B \cap C| + |A \cap B \cap C|$

$|A \cap C|$

$|\bar{A} \cap B \cap C| + |A \cap B \cap C| = |B \cap C|$

$|B \cap C| - |A \cap B \cap C|$

$|A \cup B \cup C| = |A \cap \bar{B} \cap \bar{C}| + |A \cap B \cap \bar{C}| + |\bar{A} \cap B \cap \bar{C}| + |A \cap B \cap C| + |A \cap \bar{B} \cap C| + |\bar{A} \cap B \cap C| + |\bar{A} \cap \bar{B} \cap C|$



$|A| = |A \cap \bar{B} \cap \bar{C}| + |A \cap B \cap \bar{C}| + |A \cap B \cap C| + |A \cap \bar{B} \cap C|$

$+$

$|B| =$

$|C| =$

Proof

$\forall$ A_i, $i = 1, 2, \ldots, n$

$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$

$|A| + |B| + |C| = |A \cup B \cup C| + |A \cap B| + |A \cap C| + |\bar{A} \cap B \cap C|$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |\bar{A} \cap B \cap C|$

$-(|B \cap C| - |A \cap B \cap C|)$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$\boxed{(1-1)^m}$

Proof

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| -$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |A_1 \cap A_2 \cdots \cap A_n|$$

Inclusion-Exclusion (3 sets)

U

From a total of 50 students:

$|A| \rightarrow$ 30 know Java $= |J|$

$|B| \rightarrow$ 18 know C++ $= |C+|$

$|C| \rightarrow$ 26 know C# $= |C\#|$

$|A \cap B| \rightarrow$ 9 know both Java and C++ $= |J \cap C+|$

$|A \cap C| \rightarrow$ 16 know both Java and C# $= |J \cap C\#|$

$|B \cap C| \rightarrow$ 8 know both C++ and C# $= |C+ \cap C\#|$

$|A \cup B \cup C| \rightarrow$ 47 know at least one language.

How many know none?

How many know all?

$|J \cap C+ \cap C\#|$ $|A \cap B \cap C|$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

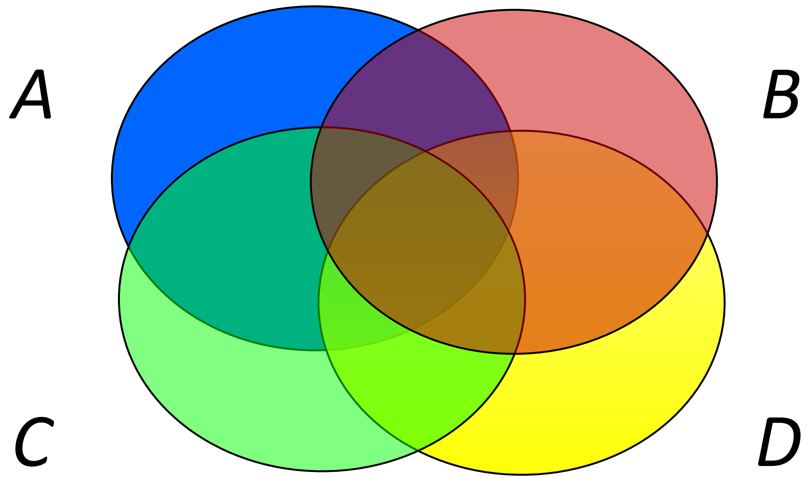
$47 = 30 + 18 + 26 - 9 - 16 - 8 + \underline{|J \cap C \cap C\#|}$ $|J \cup C+ \cup C\#|$

$$47 = 30 + 18 + 26 - 9 - 16 - 8 + |A \cap B \cap C|$$

$$|A \cap B \cap C| = 6$$

Inclusion-Exclusion (4 sets)

$$
\begin{aligned}
|A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\
& - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\
& + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\
& - |A \cap B \cap C \cap D|
\end{aligned}
$$



Bell Number

Partitioning of a Set

Partition of a set, say S, is a collection of n disjoint subsets, say $P_1, P_2, \ldots P_n$ that satisfies the following three conditions –

A = {1, 2, 3, 4, 5, 6}

Valid partition ← S_1 = {{1, 4, 5}, {2}, {3,6}}

S_2 = {{ϕ}, {1, 2}, {3, 4, 5, 6}}

S_3 = {{1, 2, 3, 4}, {4, 5, 6}}

S_4 = {{1, 2}, {4, 5, 6}}

	1	2	3
S_1	✓	✓	✓
S_2	✗	✓	✓
S_3	✓	✓	✗
S_4	✓	✗	✓

1. P_i does not contain the empty set.

$$[P_i \neq \{\emptyset\} \ for \ all \ 0 < i \leq n]$$

2. The union of the subsets must equal the entire original set.

$$[P_1 \cup P_2 \cup \cdots \cup P_n = S]$$

3. The intersection of any two distinct sets is empty.

$$[P_a \cap P_b = \{\emptyset\}, \ for \ a \neq b \ where \ n \geq a, \ b \geq 0]$$

Covering on Set A

It is defined as a set on non-empty subsets Ai, whose union leads to the original set A and which are need not be pairwise disjoint. Here are the two conditions that are to be satisfied:

$$\bigcup_{i \in n} A_i = A \quad , \quad A_i \neq \phi$$

$$\Rightarrow \left[A_i \cap A_j \neq \phi \; for \; each \; (i, j) \in n ; i \neq j \right]$$

30 40 50 60 70 80

$$= \{ A_1, A_2, A_3, \cancel{A_4}, A_5, A_6 \} \rightarrow \{ \{A_1, A_2\}, \{A_3\}, \{A_4, A_5\}, \{A_6\} \}$$

start 1 2

finish 2 3

$|S| = 7$, How many subsets of S have atmost 3 elements?

Total no. of subsets $= 2^7$

$= 128$

0, 1, 2, 3

$= {}^7C_0 + {}^7C_1 + {}^7C_2 + {}^7C_3$

$\Rightarrow$ 64

$\boxed{{}^nC_k}$

No. of subsets containing k elements

${}^4C_2$

$\{1, 2, 3, 4\}$

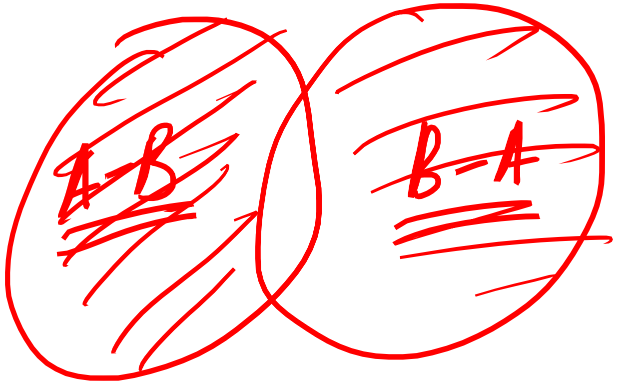
$\{\{1, 2\}, \{1, 3\}\ \{1, 4\}\ \{2, 3\}$

$\{2, 4\}\ \}$

1. Set → 256 subsets

2. <u>Symmetrical Difference</u> - Either in A or B but not in both

$$(A-B) \cup (B-A)$$



3. A set contains $2n+1$ elements. Find the no. of subsets of this set containing more than n elements.

$\{\}$ Bell Number – Eric Temple Bell (1976)

$\Rightarrow B_0 = 1$

$B_1 = \{1\} \Rightarrow 1$

$\{\{1\}\}$

$B_2 = \{1, 2\} \Rightarrow 2$

$\{\{1\}, \{2\}\}$

$\{\{1, 2\}\}$

$B_3 = \{1, 2, 3\} \Rightarrow 5$

$\{\{1\}, \{2\}, \{3\}\}$

$\{\{1, 2\}, \{3\}\}$

23 1

13 2

123

$B_4 = \{1, 2, 3, 4\} = 15$

1 2 3 4

$\{\{1, 2\}, \{3\}, \{4\}\}$

13 2 4

14 2 3

23 1 4

24 1 3

$\{\{3, 4\}, \{1\}, \{2\}\}$

123 4

134 2

234 1

124 3

$n \rightarrow 2^n$

Closed Form Formula

12 34

13 24

23 14

1234

$$\left[B_{n+1} = \sum_{k=0}^{n} {}^{n}C_k \, B_k \right]$$

$${}^{n}C_k \Rightarrow$$

$$\boxed{B_0 = 1}$$

$$B_1 = {}^{0}C_0 B_0 = 1$$

$$B_2 = {}^{1}C_0 B_0 + {}^{1}C_1 B_1 = 1 + 1 = 2$$

$$B_3 = {}^{2}C_0 B_0 + {}^{2}C_1 B_1 + {}^{2}C_2 B_2 = 1 + 2 + 2 = 5$$

$$B_4 = \underbrace{{}^{3}C_0 B_0 + {}^{3}C_1 B_1 + {}^{3}C_2 B_2}_{} + {}^{3}C_3 B_3 = 15$$

$$B_5 =$$

1 $\quad B_1$

$1 \rightarrow 2$ $\quad B_2$

$2 \rightarrow 3 \rightarrow 5$ $\quad B_3$

$5 \rightarrow 7 \rightarrow 10 \rightarrow 15$ $\quad B_4$

$15 \rightarrow 20 \rightarrow 27 \rightarrow 37 \rightarrow 52$ $\quad B_5$

1 $\quad B_1$

$1 \rightarrow 2$ $\quad B_2$

$2 \rightarrow 3 \rightarrow 5$

$5 \rightarrow 7 \rightarrow 10 \rightarrow 15$

$15 \rightarrow 20 \rightarrow 27 \rightarrow 37 + 52$

1

1 2

2 3 5

5 7 10 15

15 20 27 37 52

① $|S| = 7$ How many subsets of S have atmost 3 elements

$$0 + 1 + 2 + 3$$

$$\boxed{{}^{n}C_{k}}$$

$$^7C_0 + {}^7C_1 + {}^7C_2 + {}^7C_3$$

② A set contains $2n+1$ elements. Find the no. of subsets of this set containing more than n elements.

$n+1$

$n+2$

$\vdots$

$2n+1$

2^{2n}

$$\left[\underline{{}^{2n+1}C_{n+1}} + \underline{{}^{2n+1}C_{n+2}} + \text{———} + \underline{{}^{2n+1}C_{2n+1}}\right]$$

X

$$(1+x)^{2n+1} = {}^{2n+1}C_0 + {}^{2n+1}C_1\, x + {}^{2n+1}C_2\, x^2 + \text{——} + \text{——} + {}^{2n+1}C_{2n+1}\, x^{2n+1}$$

$x = 1$

$$2^{2n+1} = {}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \text{———} + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + \underline{{}^{2n+1}C_{2n}} + {}^{2n+1}C_{2n+1}$$

$$\underline{{}^nC_r = {}^nC_{n-r}}$$

$$2^{2n+1} = \underline{{}^{2n+1}C_{2n+1}} + \underline{{}^{2n+1}C_{2n}} + \text{———} + \underline{{}^{2n+1}C_{n+1}} \;\Big|\; {}^{2n+1}C_{n+1} + \text{———}$$

$$\boxed{X = 2^{2n}}$$

$$2^{2n+1} = 2\cdot(X) = \text{——} + {}^{2n+1}C_{2n} + {}^{2n+1}C_{2n+1}$$