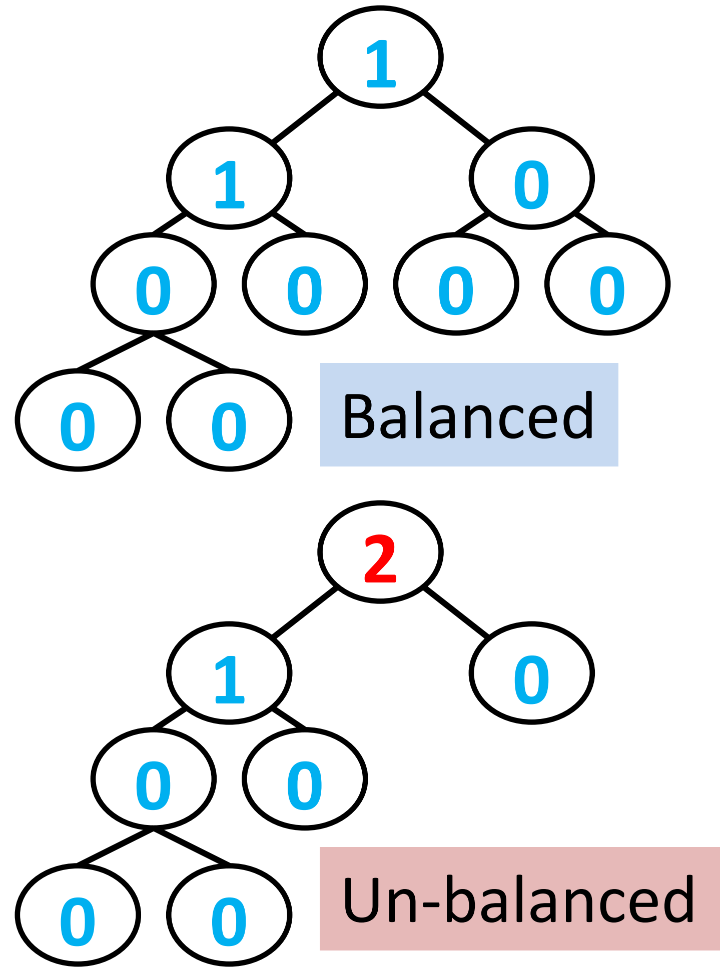
AVL Trees

Introduction

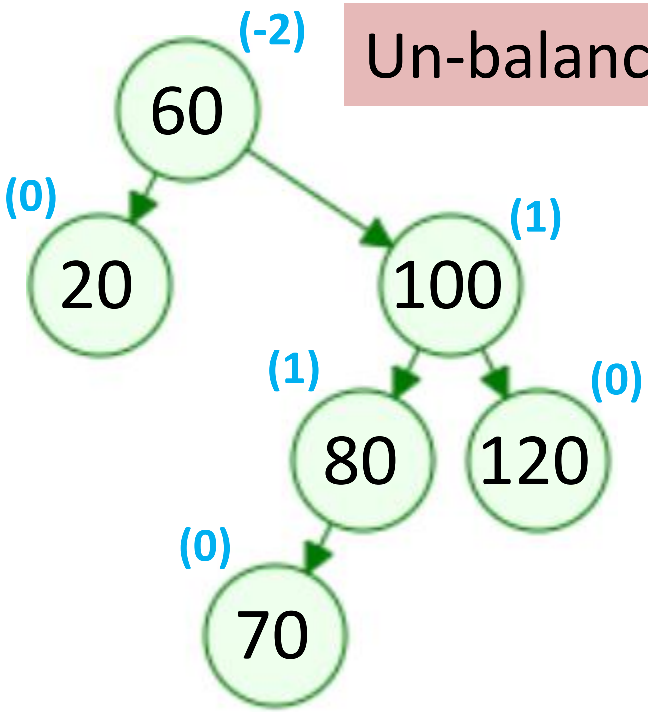
Note: Numbers within nodes represent height difference, i.e. **height of left sub-tree – height of right sub-tree**.

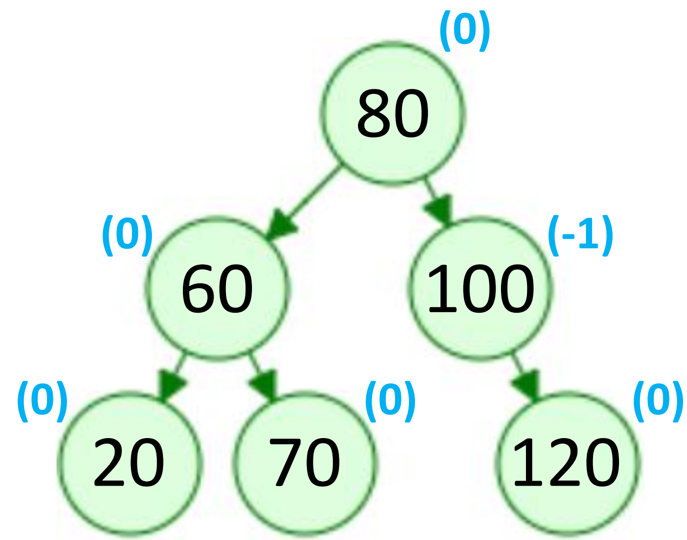
- In a Binary Search Tree with n nodes
  - Average case height is O(log n)
  - Worst case height is O(n)
- Thus it would be nice to be able to maintain a balanced tree during insertion.
  - A binary tree is said to be balanced if, for every node in the tree, the height of its two subtrees differs at most of one.



Balanced BST???

Un-balanced BST





Balanced BST

Contd…

- Invented by Georgy **A**delson-**V**elsky and Evgenii **L**andis in 1962.
- Height balanced binary search trees.
- Each node has a balance factor.
- Let **HL** and **HR** be the heights of left and right subtrees of any node, then

$$|\mathbf{HL} - \mathbf{HR}| <= 1$$

- Balance factor (**bal**) of a node **K** is **HL** – **HR**.
  - Left High (LH) = +1 (left sub-tree higher than right sub-tree)
  - Even High (EH) = 0 (left and right sub-trees have same height)
  - Right High (RH) = -1 (right sub-tree higher than left sub-tree)

Height of AVL Trees

- **Guaranteed to be in the order of lg_2n for a tree containing n nodes.**
- If an AVL tree has minimum number of nodes, then one of its subtrees is higher than the other by 1.
- Let, the left subtree is bigger than the right subtree, and
  - $N(h)$ = minimum number of nodes in an AVL tree of height h rooted at r.
  - $N(h-1)$ = minimum number of nodes in the left subtree of r.
  - $N(h-2)$ = minimum number of nodes in the right subtree of r.

Contd…

$N(h) = 1 + N(h-1) + N(h-2)$

As per assumption $N(h-1) > N(h-2)$, so

$N(h) > 1 + N(h-2) + N(h-2) = 1 + 2 \cdot N(h-2) > 2 \cdot N(h-2)$

That is,

$N(h) > 2 \cdot N(h-2)$

Knowing $N(0) = 1$, this recurrence can be solved.

$N(h) > 2 \cdot N(h-2) > 2 \cdot 2 \cdot N(h-4) > 2 \cdot 2 \cdot 2 \cdot N(h-6) > \cdots > 2^{h/2}$

To ensure it's $2^{h/2}$, lets check for a particular $h = 6$

$N(6) > 2 \cdot N(6-2) > 2 \cdot 2 \cdot N(4-2) > 2 \cdot 2 \cdot 2 \cdot N(2-2) > 2^3$

Contd…

Thus,

$$N(h) > 2^{h/2}$$

Taking log,

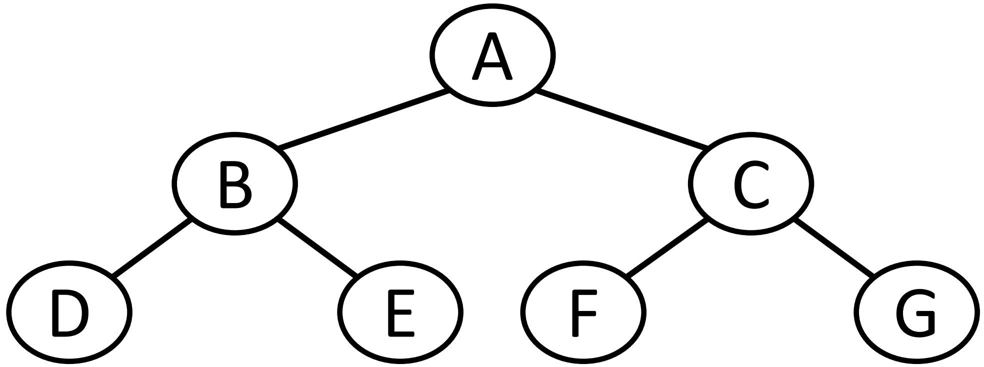
$$\log N(h) > \log 2^{h/2} \Leftrightarrow h < 2 \log N(h)$$

- Thus, in the worst-case AVL trees have height h = O(log n).
- This means that nicer/more balanced AVL trees will have the same bound on their height.

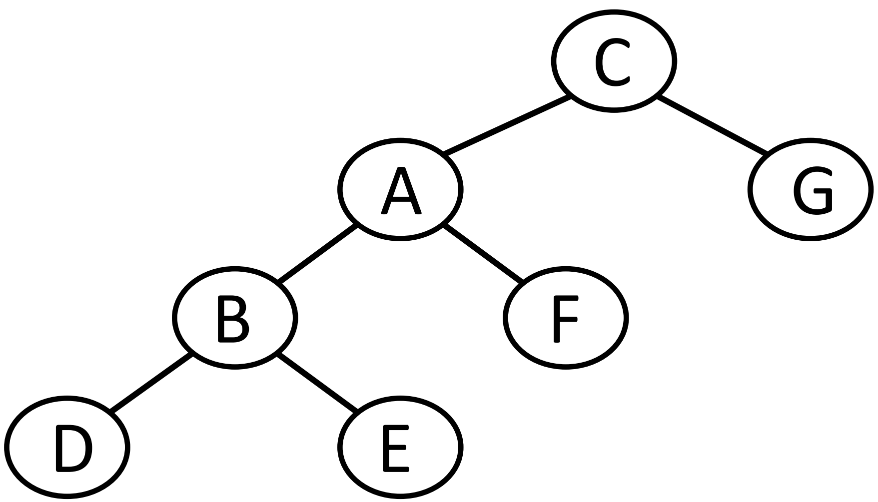
Operations on AVL Tree

- Search
  - Similar as in the case of binary search trees, since both are organized according to the same criteria.
  - Complexity O(lg n).
- Insertion and Deletion
  - Similar as in the case of binary search trees. But after insertion or deletion of a node, the tree might have lost its AVL property (i.e. balance factor becomes greater than 1).
  - To maintain the AVL structure, further modifications (known as **ROTATIONS**) are required.

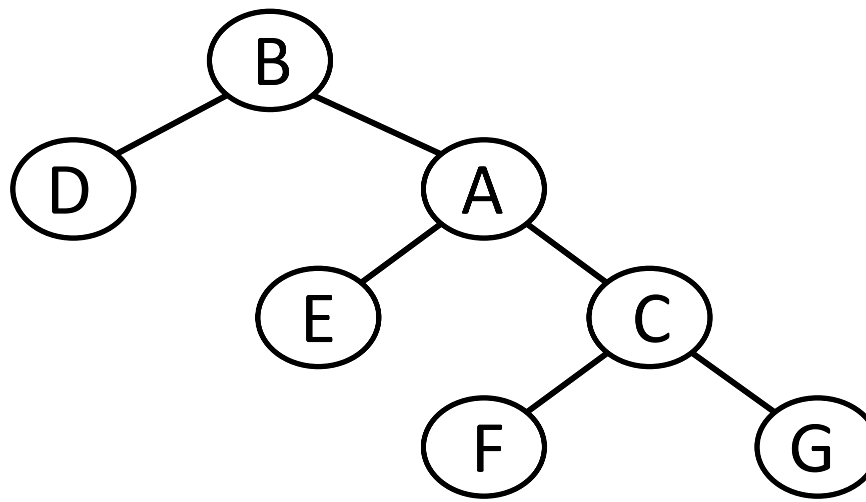
Rotation



- Left rotation

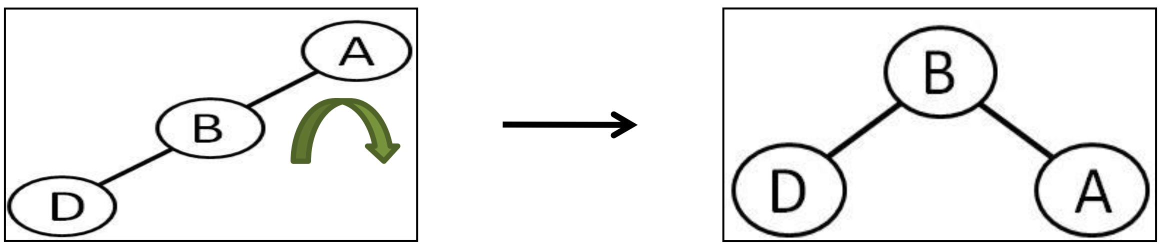


- Right rotation

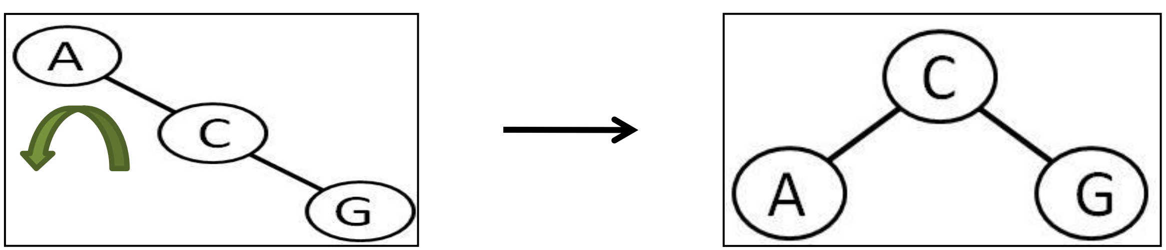


Unbalanced Cases

- Single rotation
  - **Left of Left**: insertion turned the left subtree of a left high AVL tree into a left high tree.

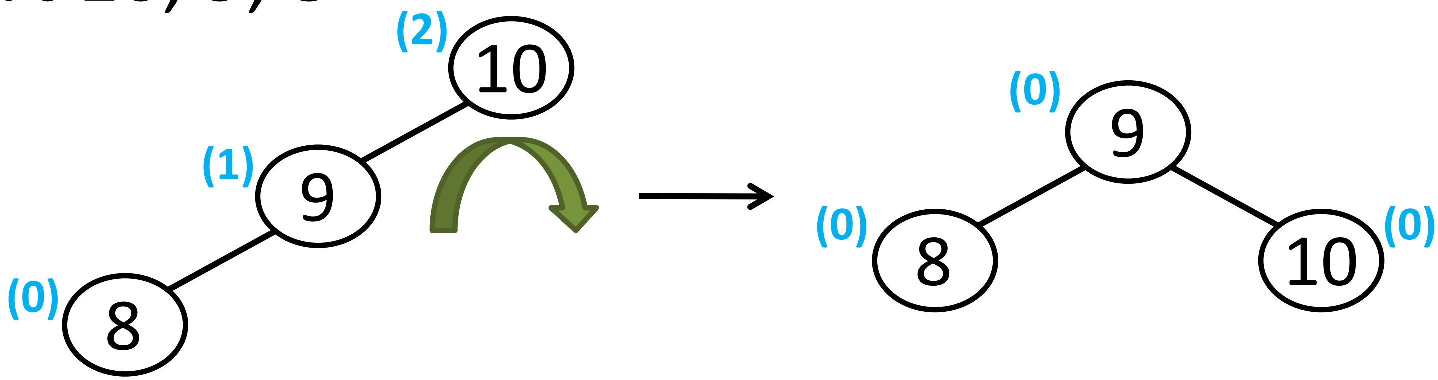


  - **Right of Right**: insertion turned the right subtree of a right high AVL tree into a right high tree.

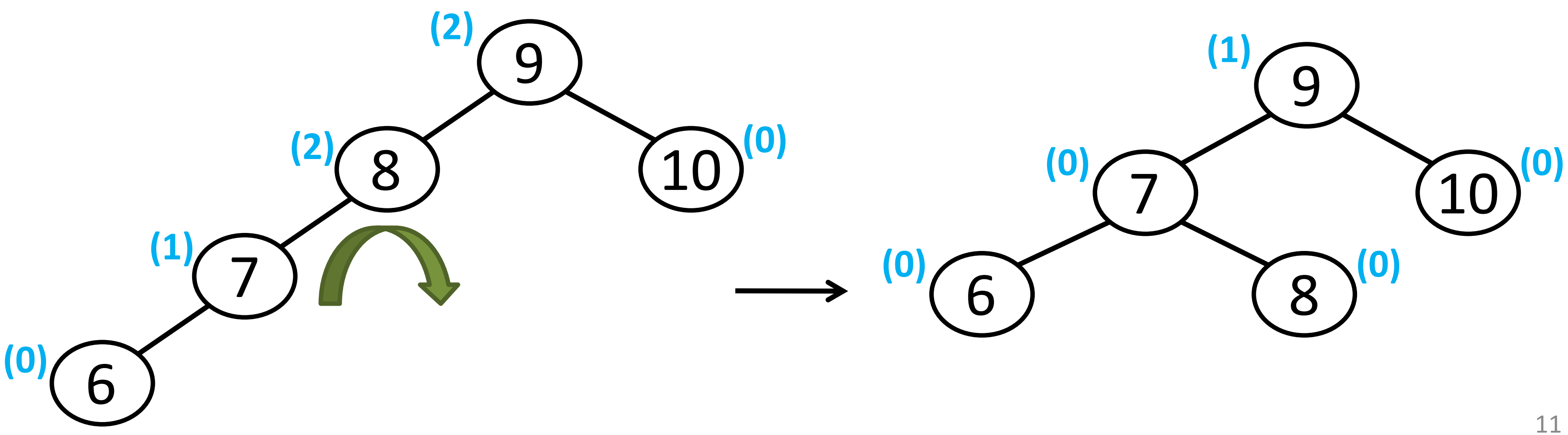


Example 1: Insert 10, 9, 8, 7, 6, 5, 4, 3, 2, 1

- Insert 10, 9, 8

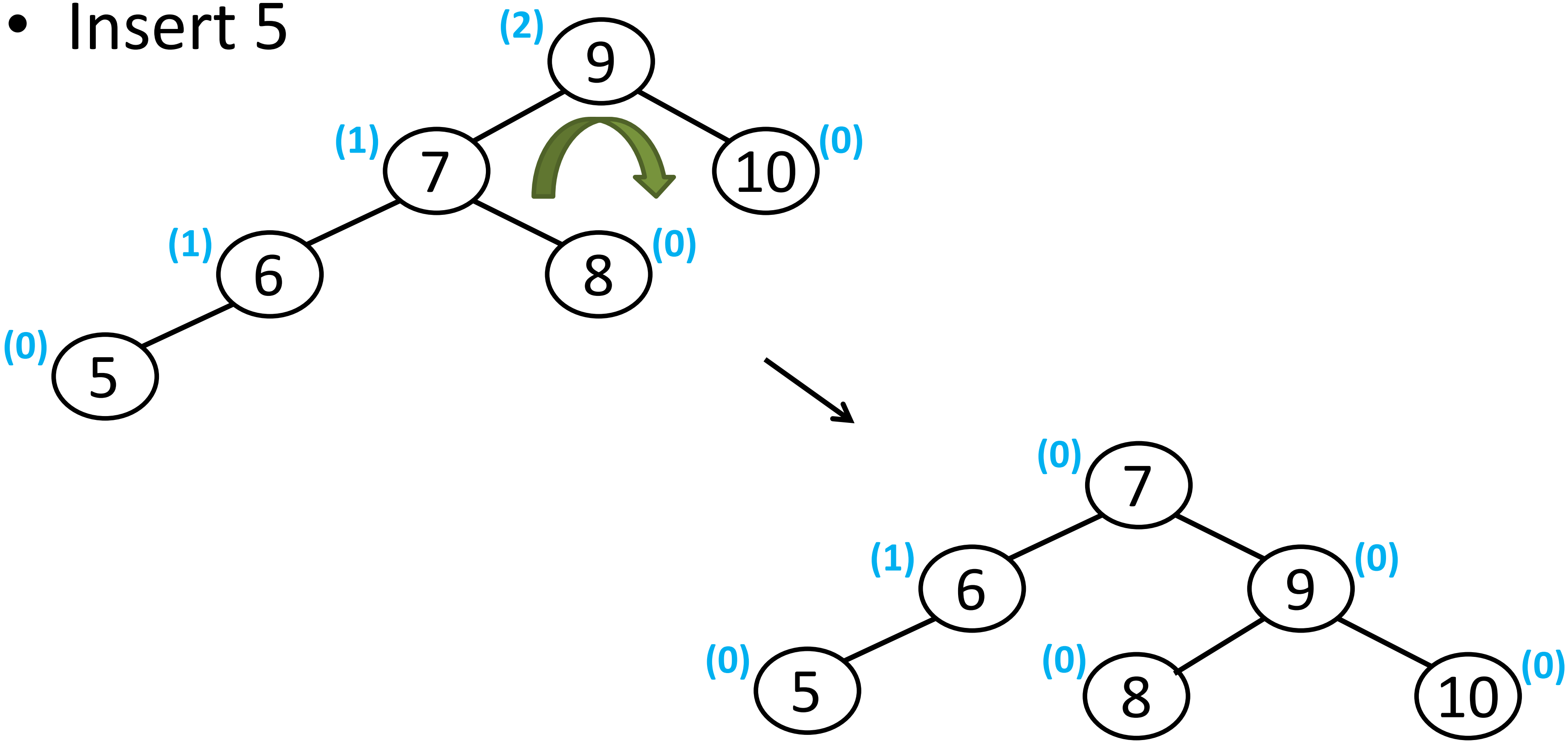


- Insert 7, 6



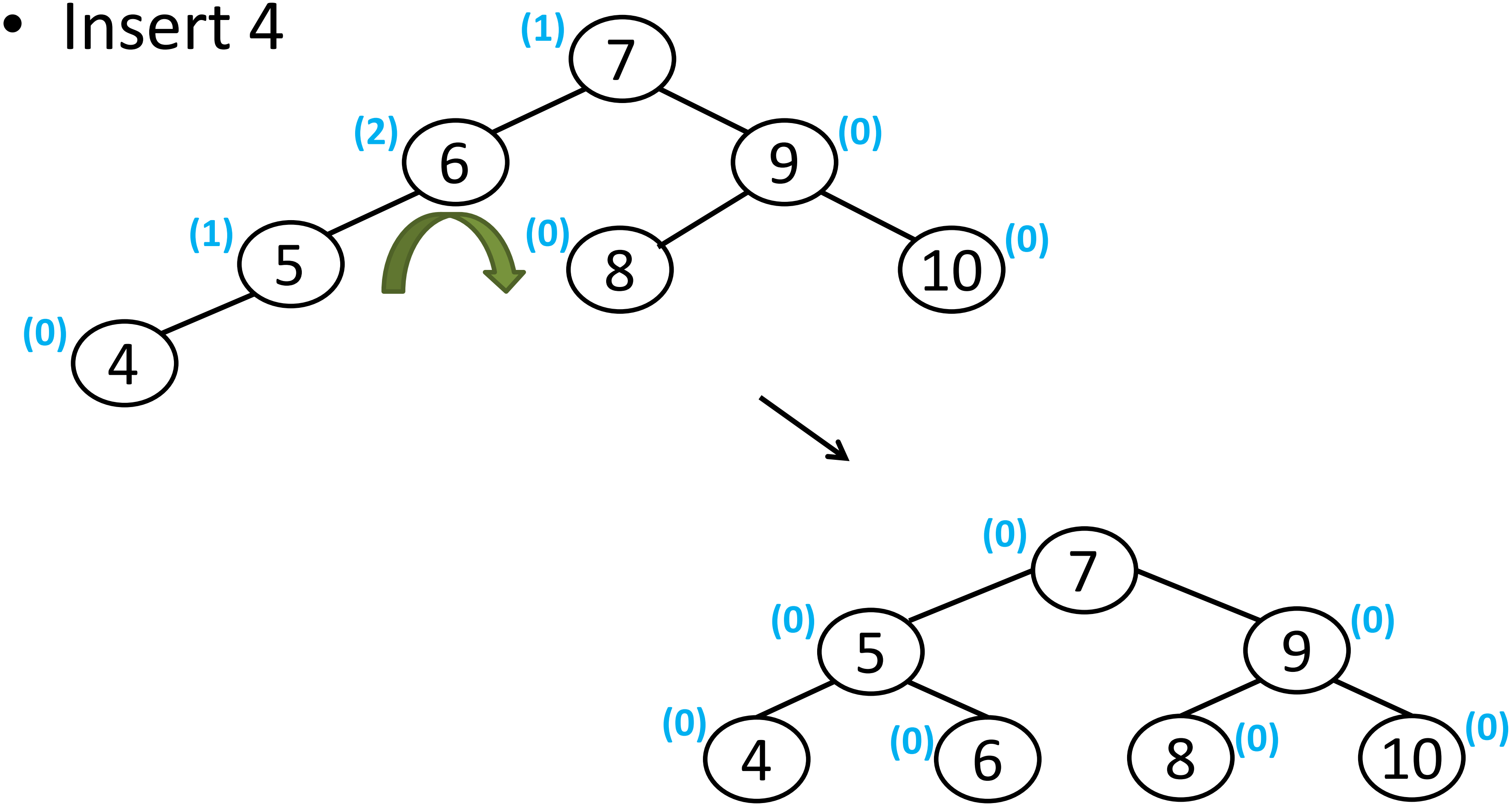
Contd…

- Insert 5



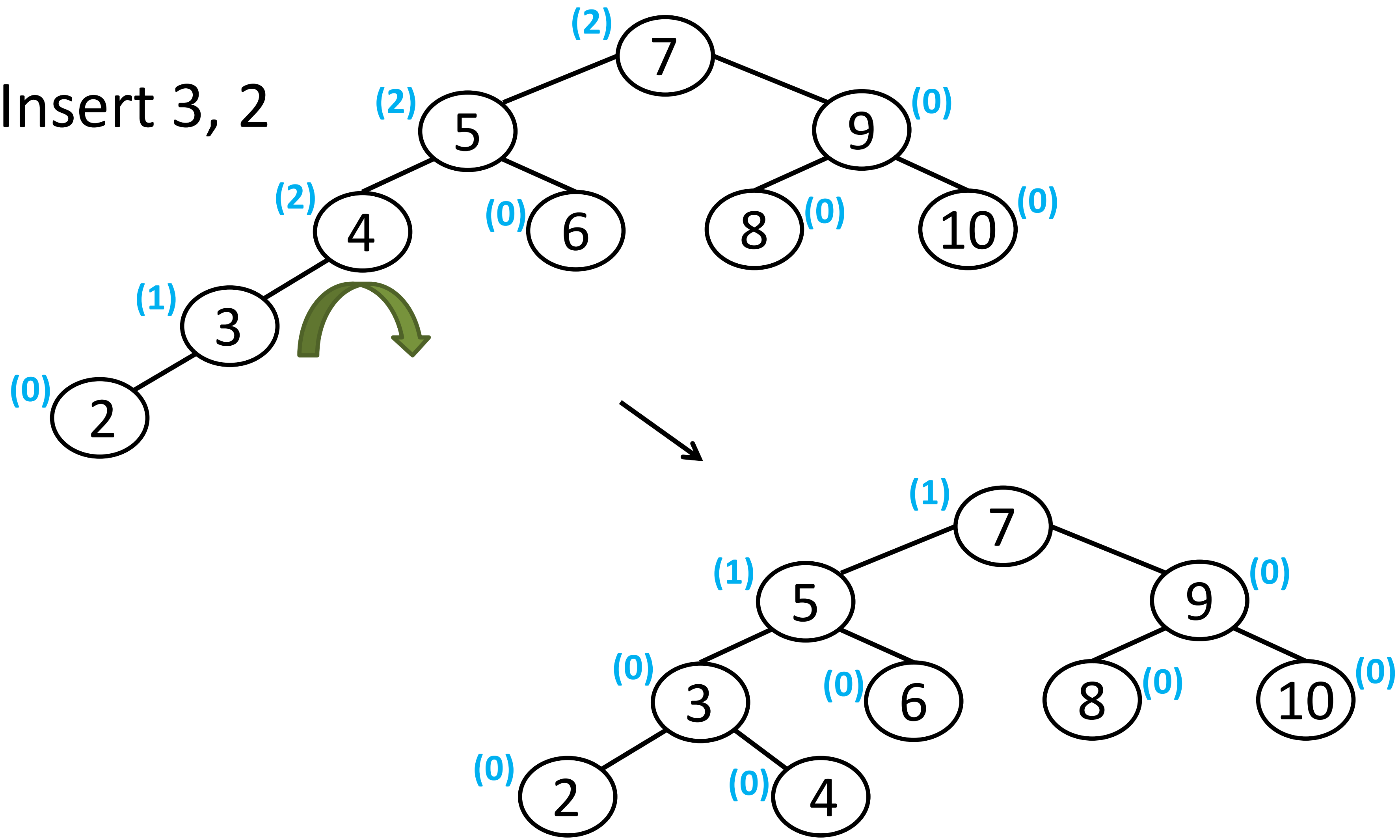
Contd…

- Insert 4



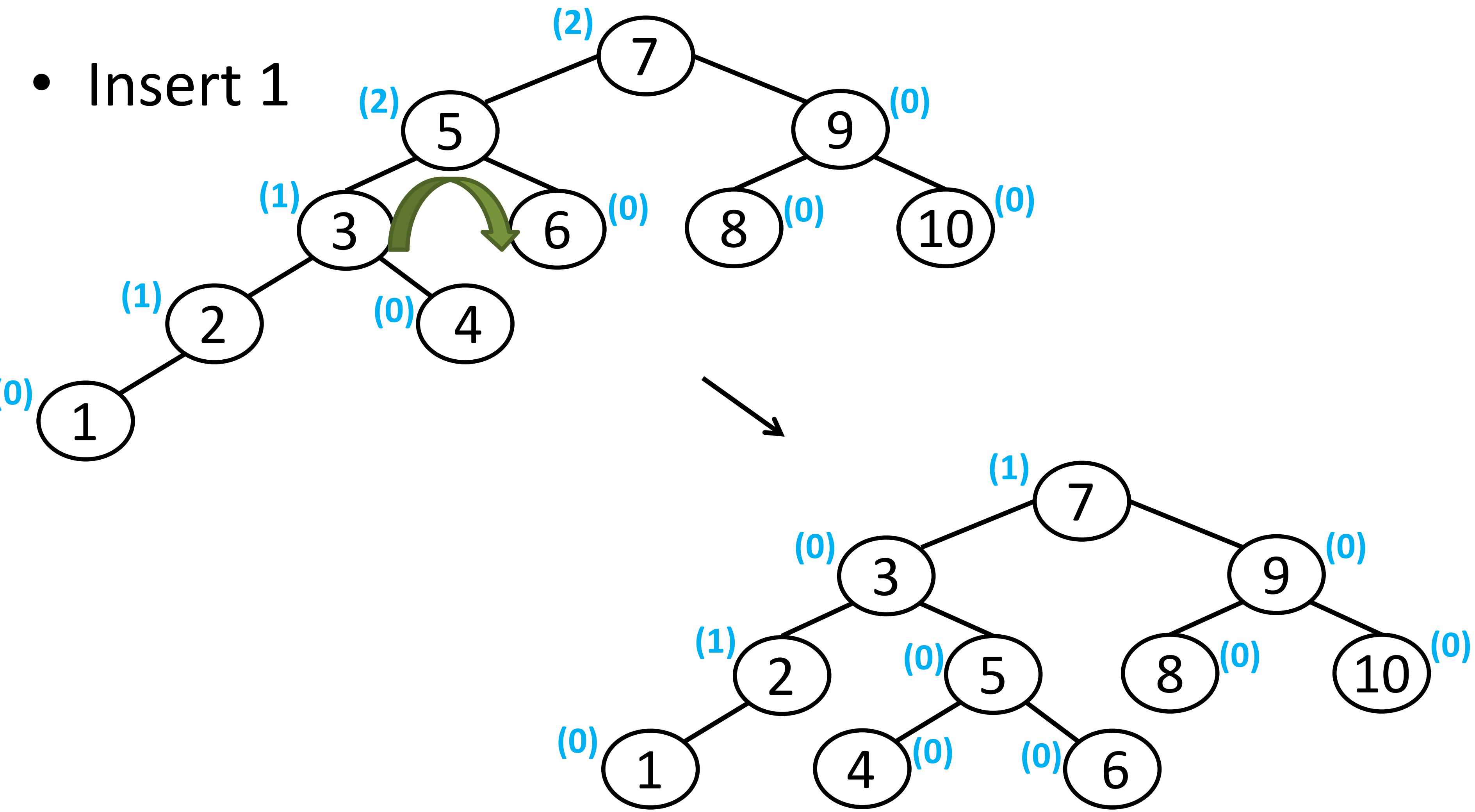
Contd…

- Insert 3, 2



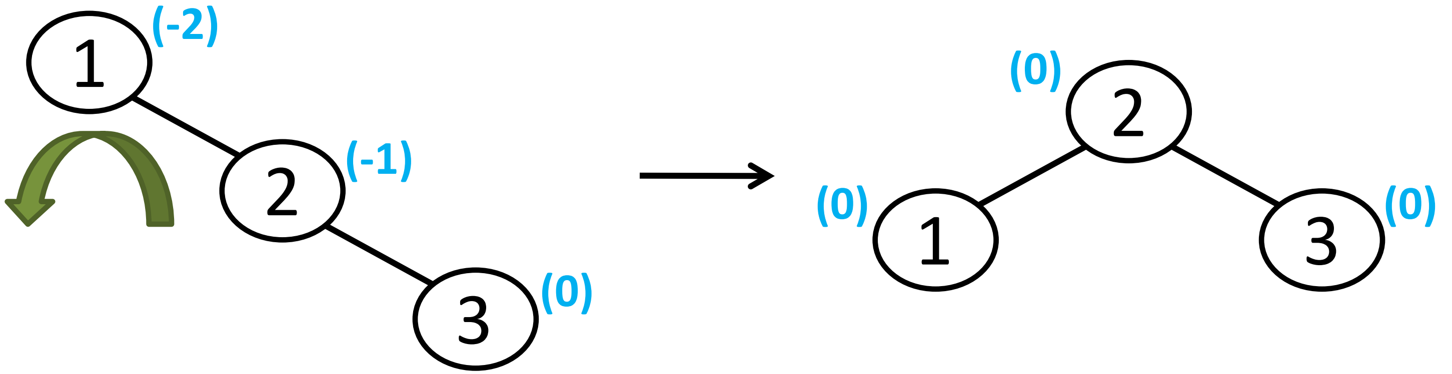
Contd…

- Insert 1

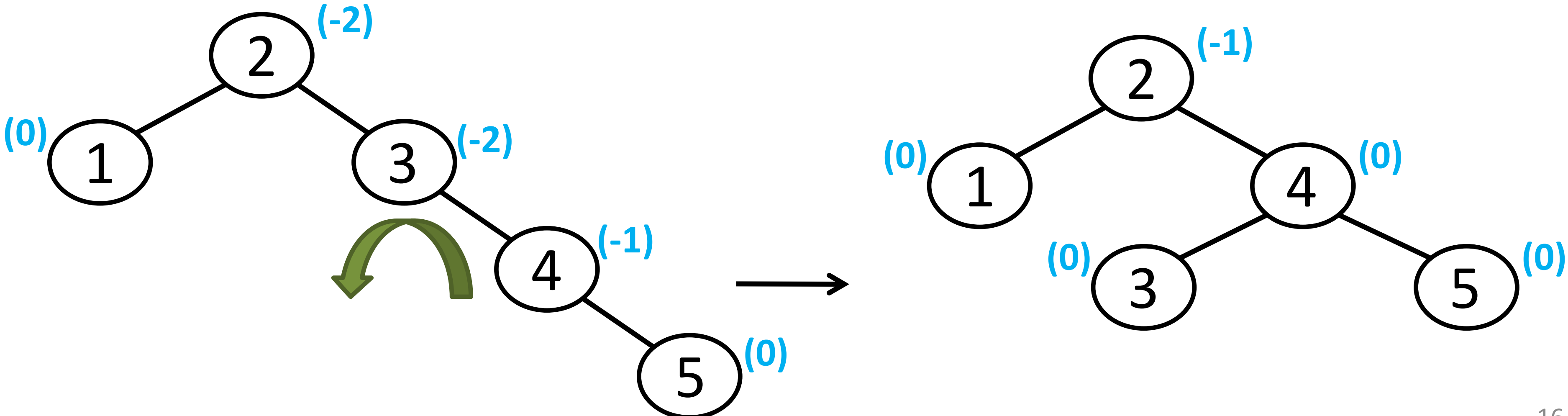


Example 2: Insert 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

- Insert 1, 2, 3

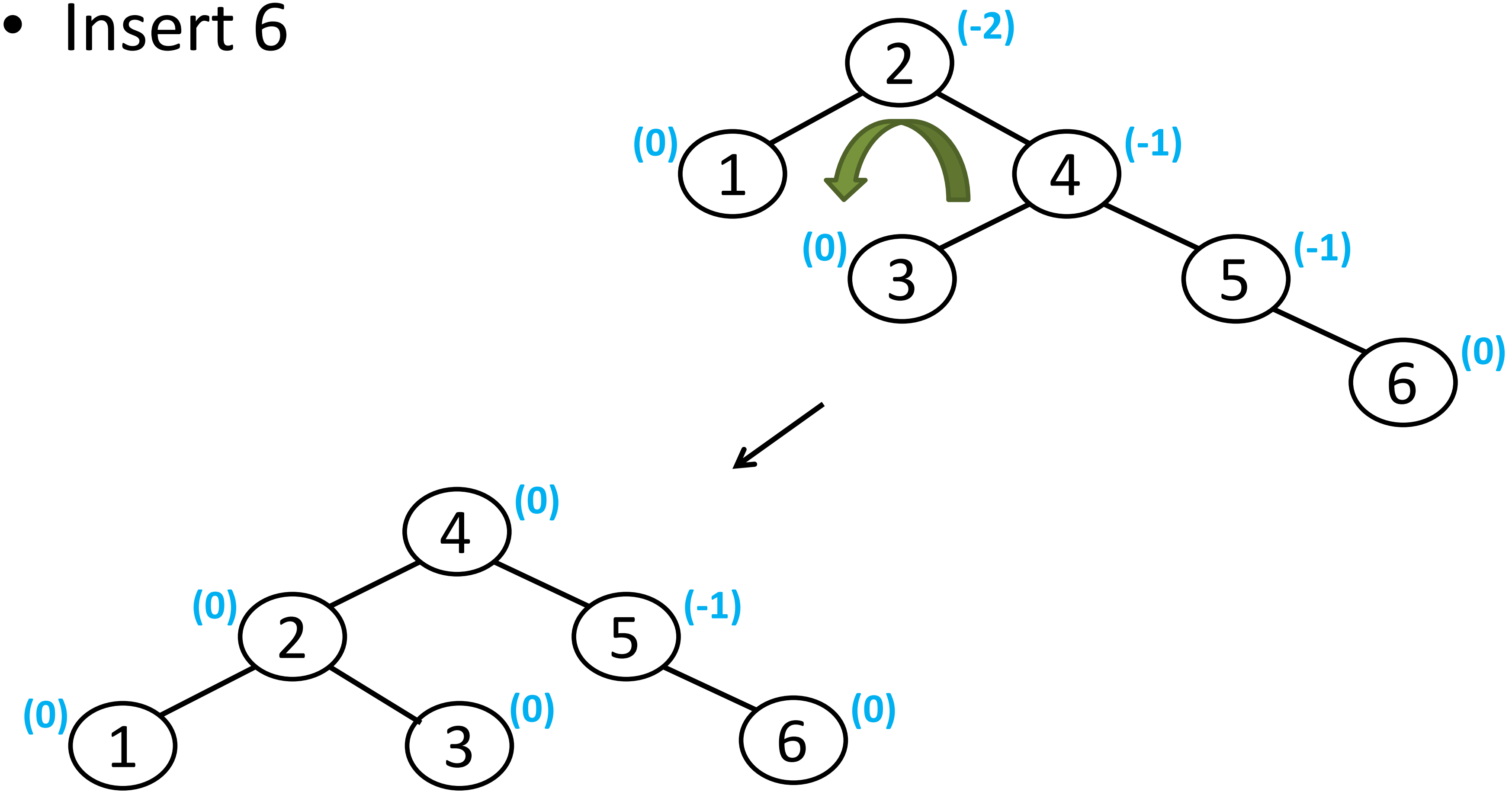


- Insert 4, 5



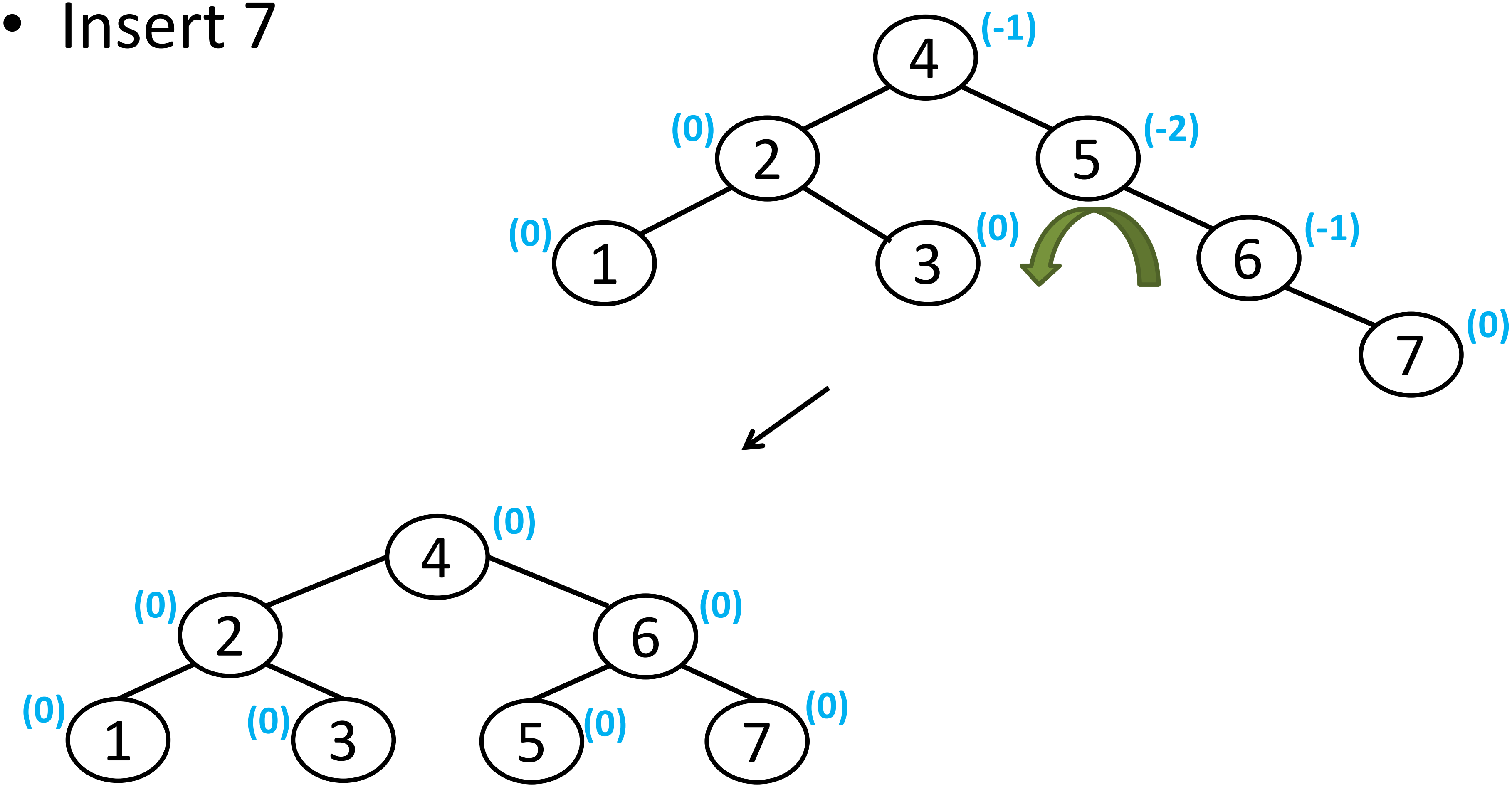
Contd…

- Insert 6



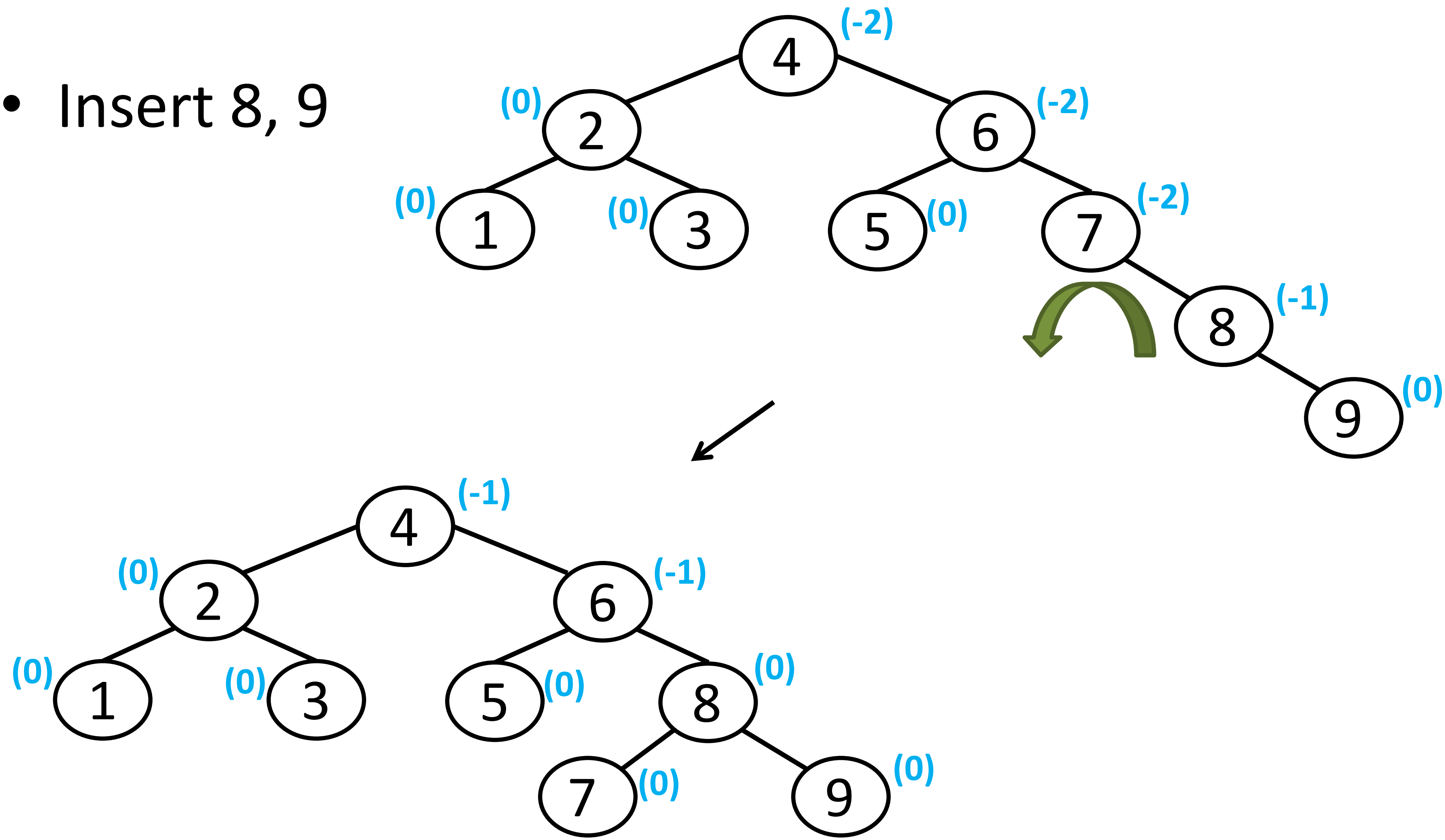
Contd…

- Insert 7



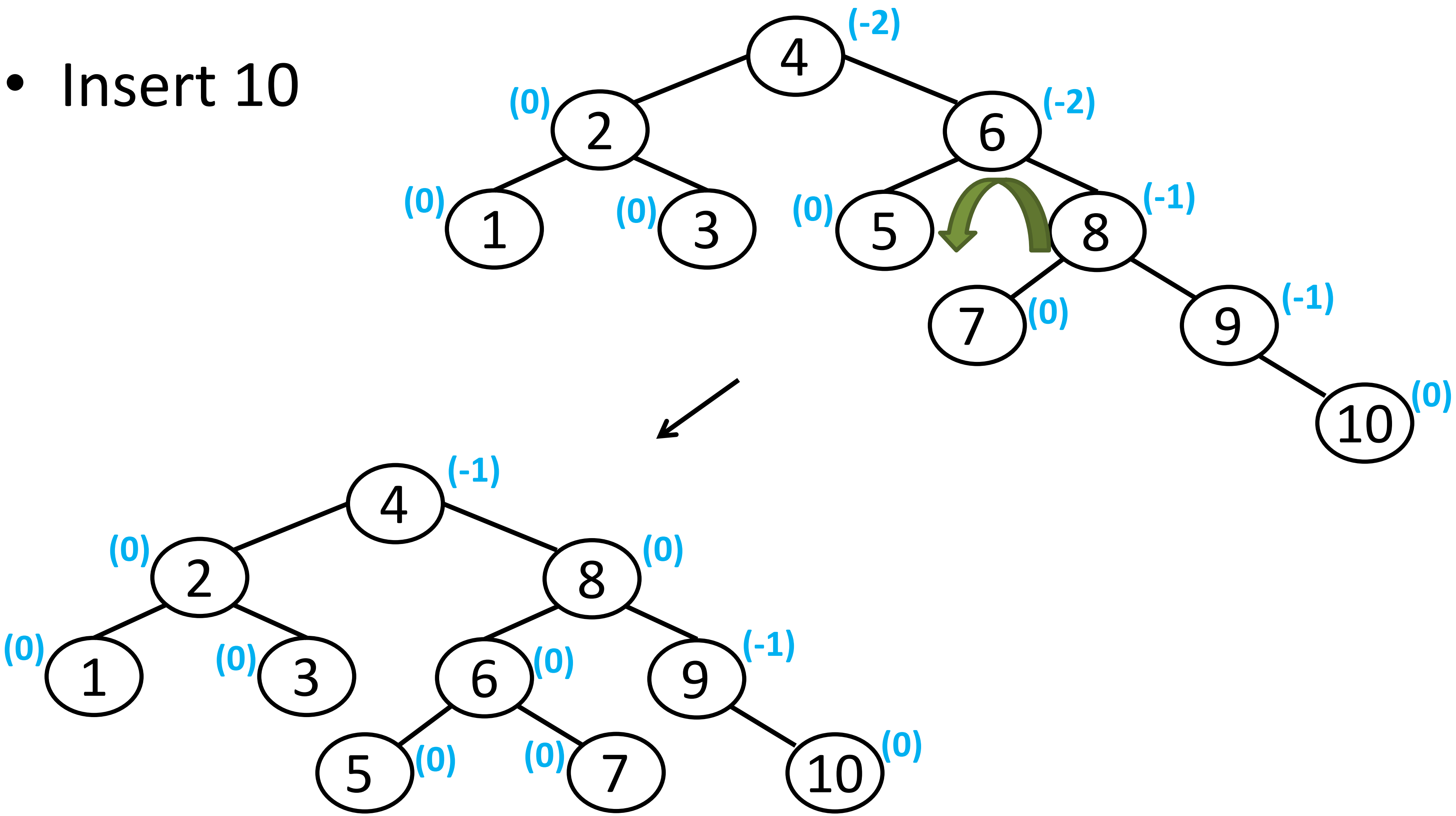
Contd…

- Insert 8, 9



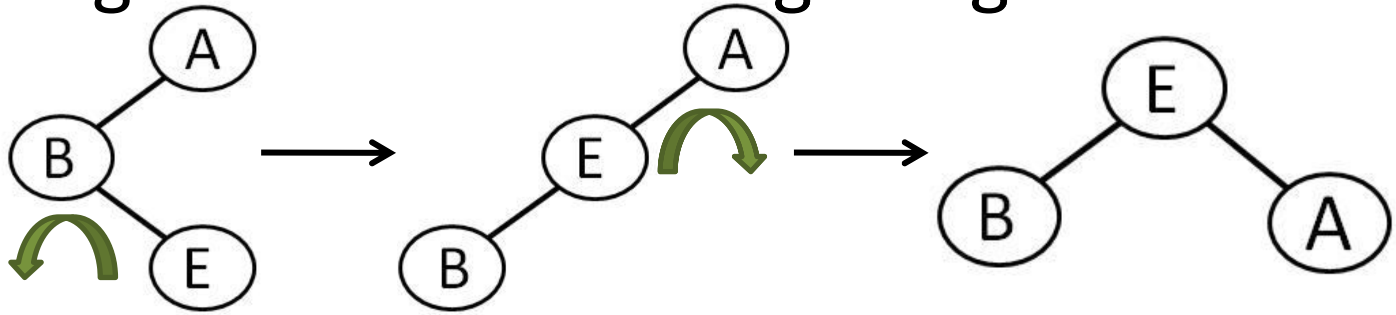
Contd…

- Insert 10

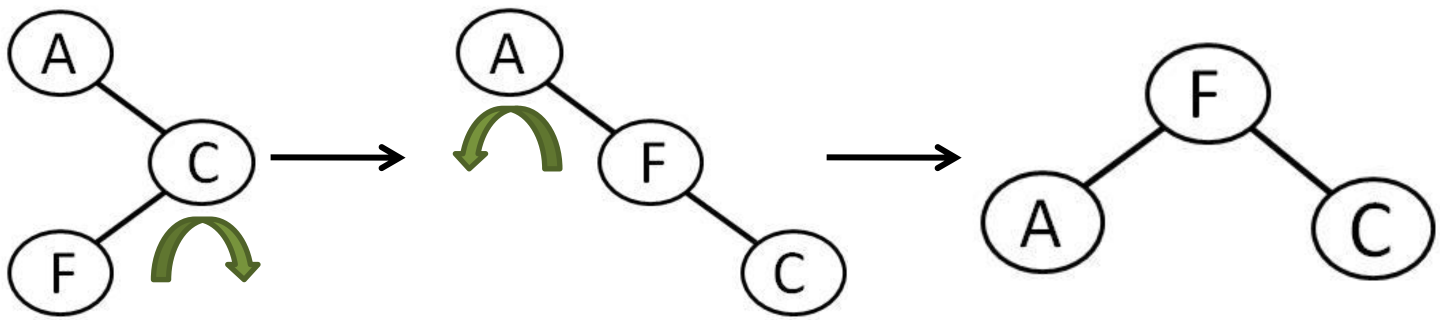


Unbalanced Cases

- Double rotation
  - **Right of Left**: insertion turned the left subtree of a left high AVL tree into a right high tree.

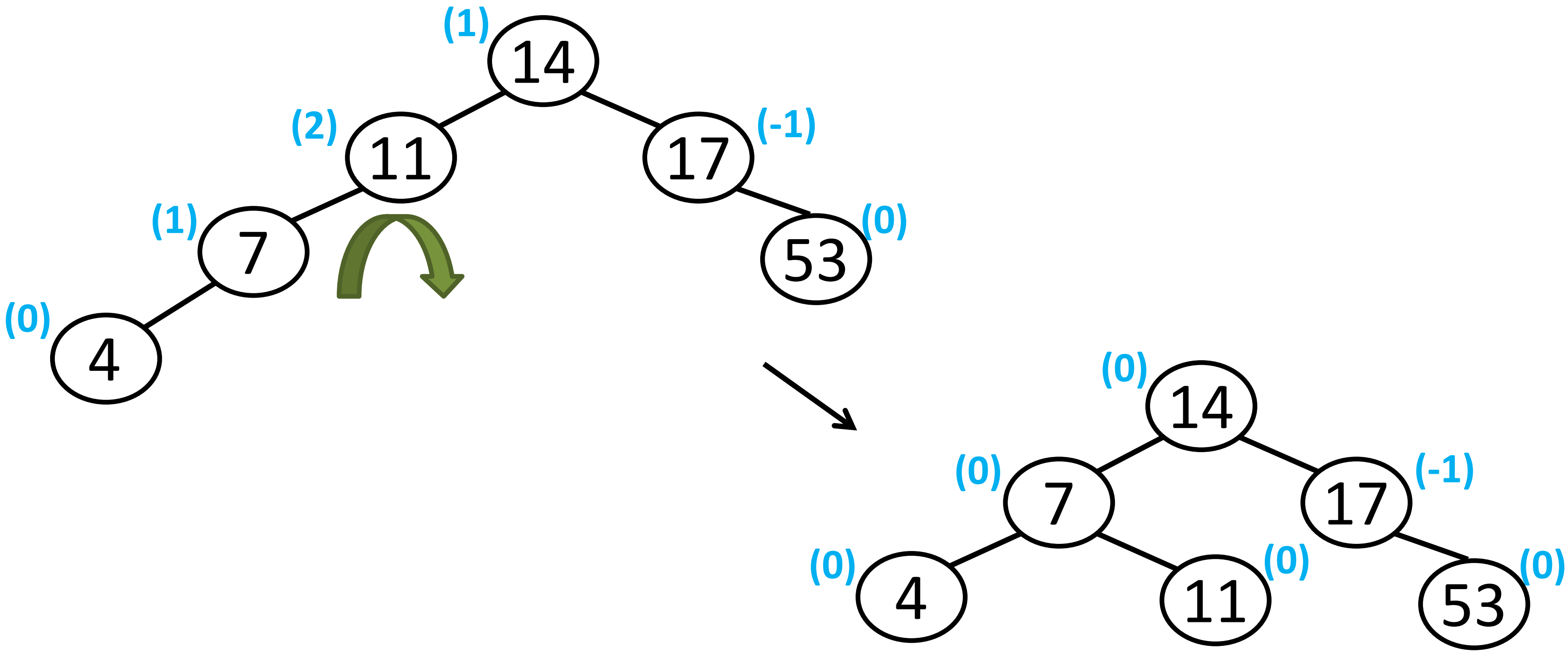


  - **Left of Right**: insertion turned the right subtree of a right high AVL tree into a left high tree.



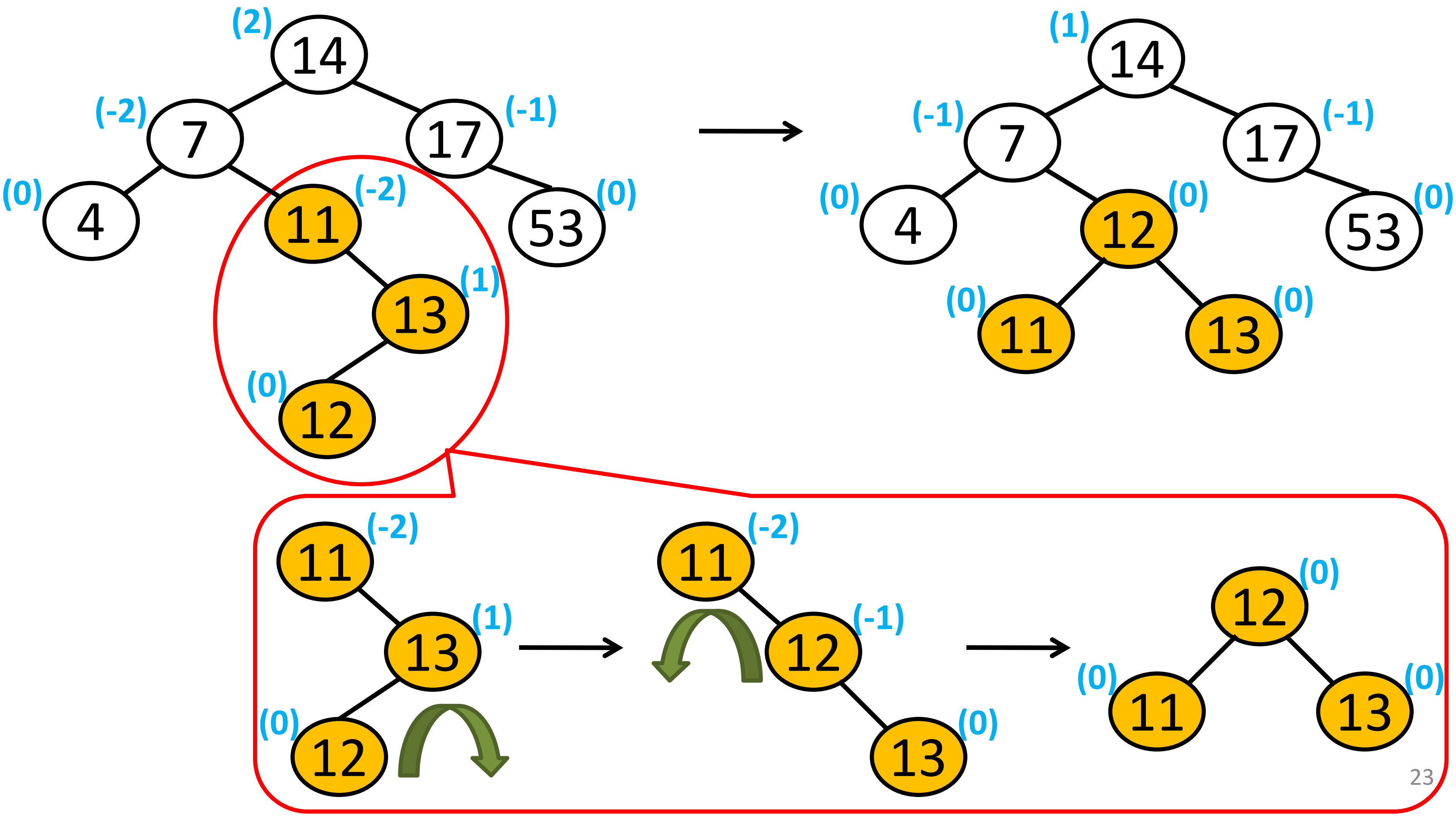
Example 3: Insert 14, 17, 11, 7, 53, 4, 13, 12, 8

- Insert 14, 17, 11,7, 53, 4



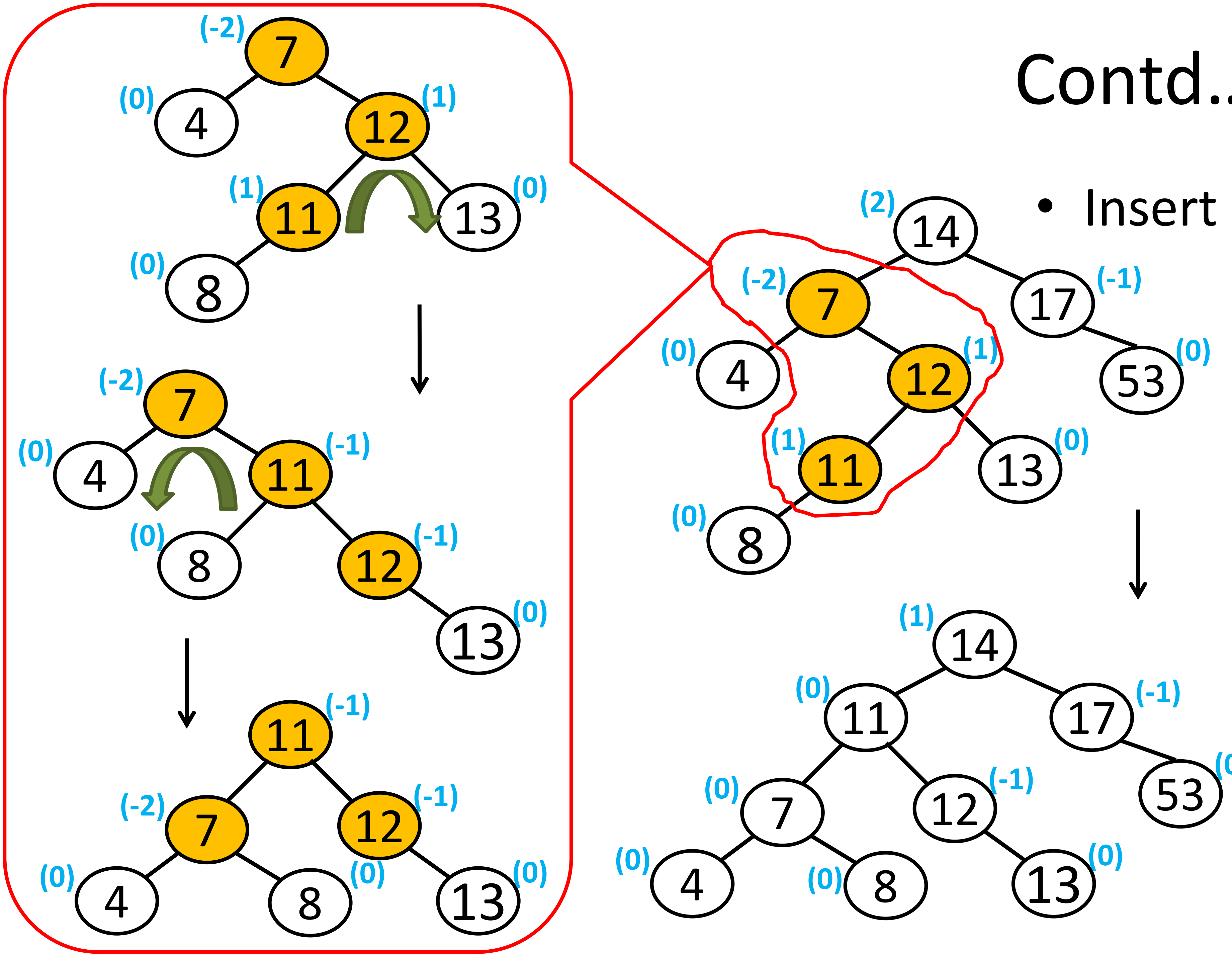
Contd…

- Insert 13, 12



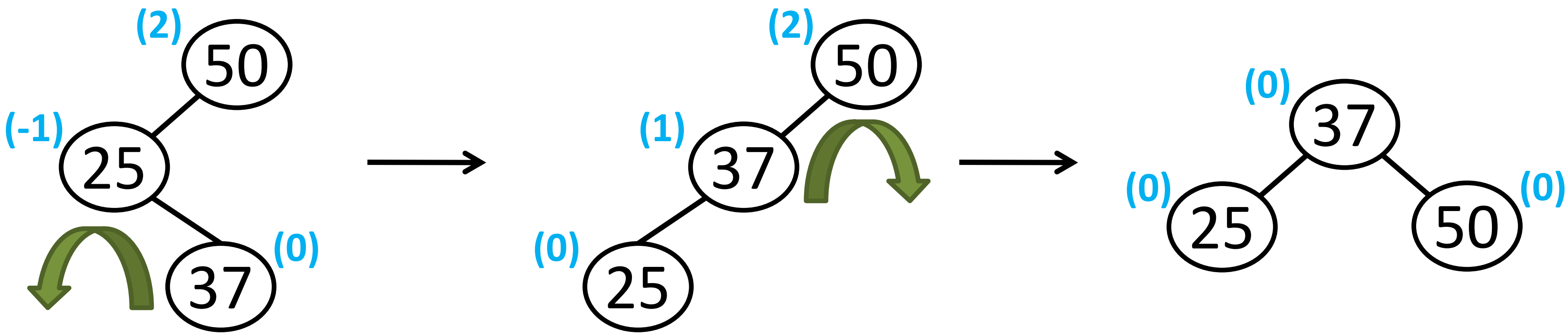
Contd...

- Insert 8

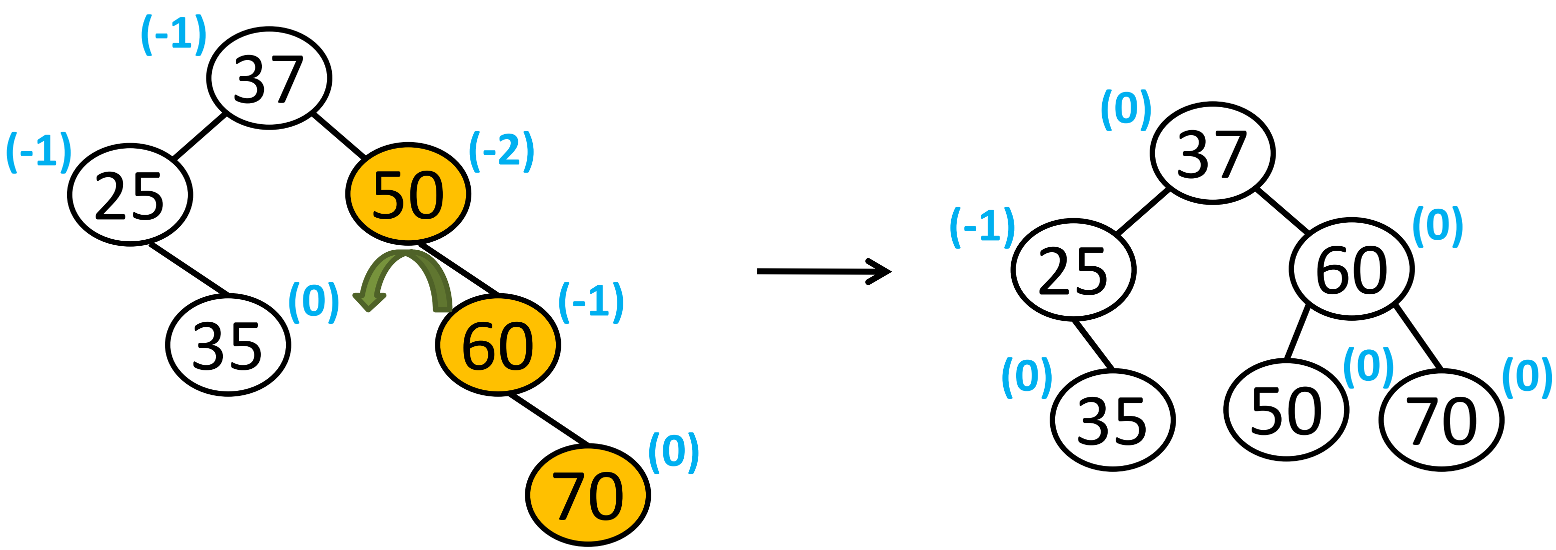


Example 4: 50, 25, 37, 35, 60, 70, 30, 45, 34, 40, 55

- Insert 50, 25, 37

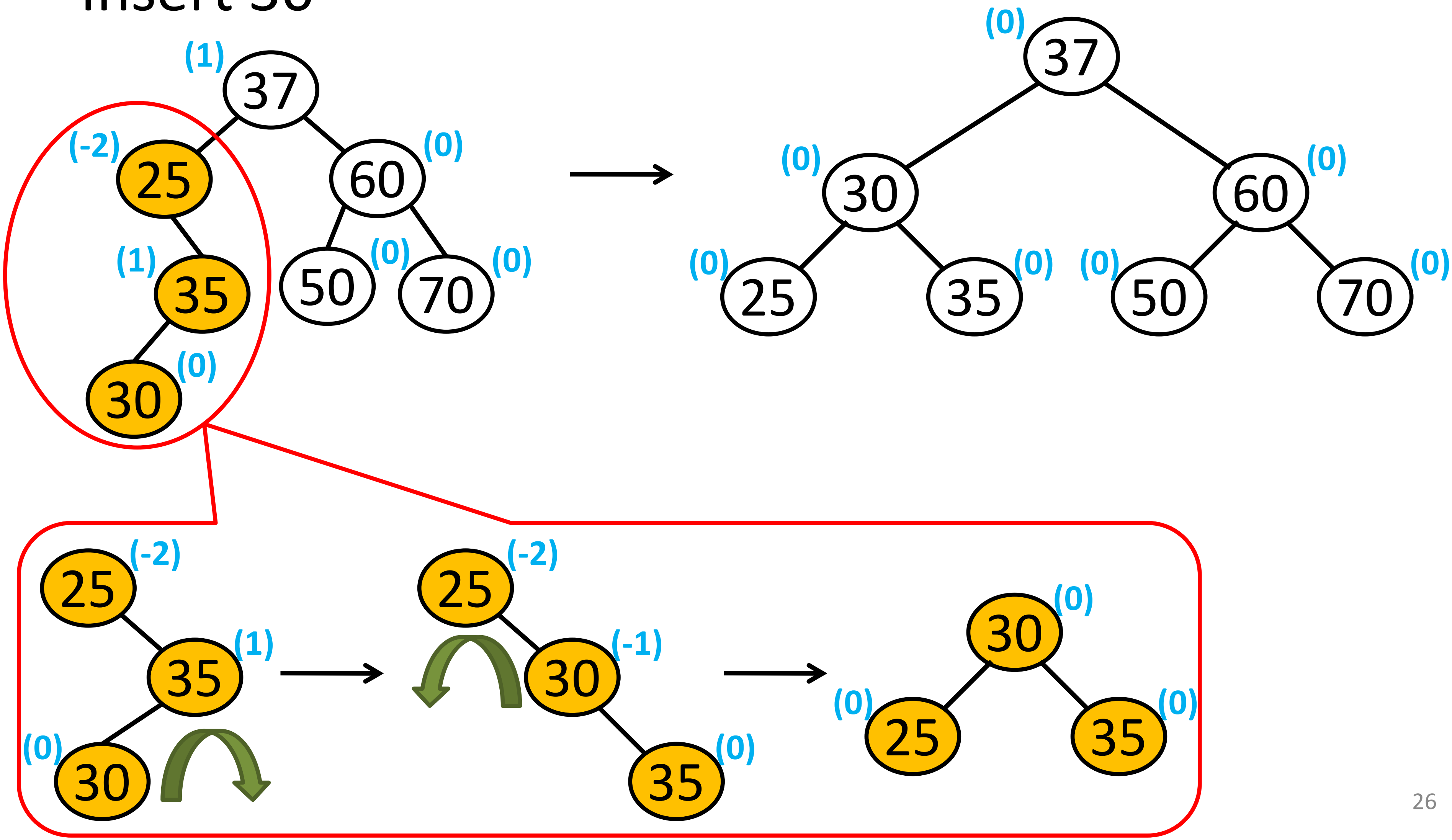


- Insert 35, 60, 70



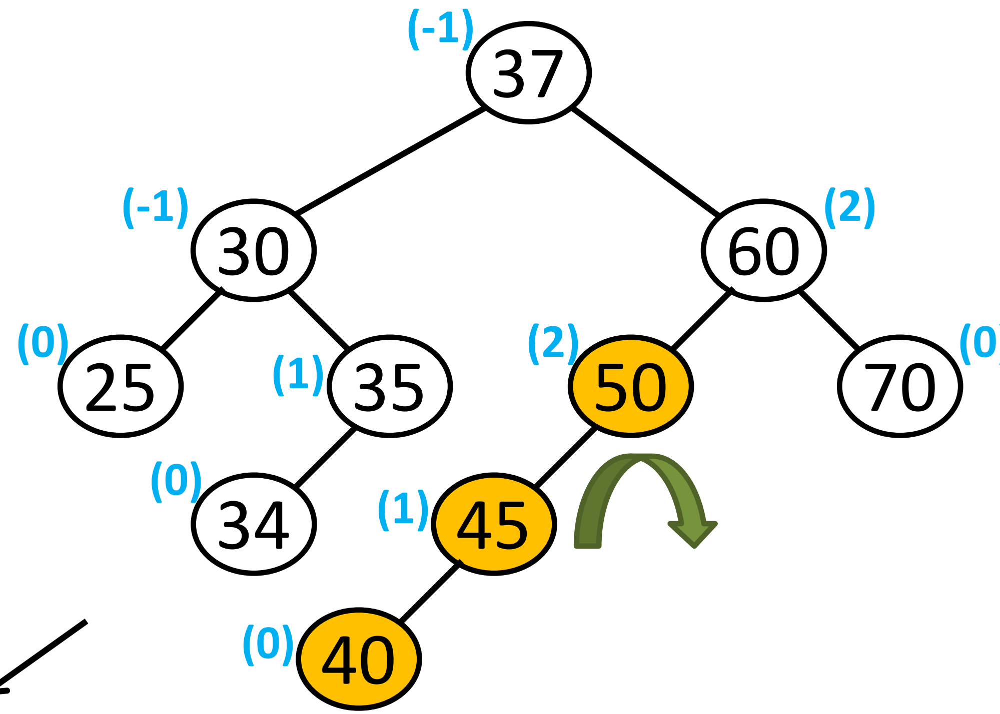
Contd…

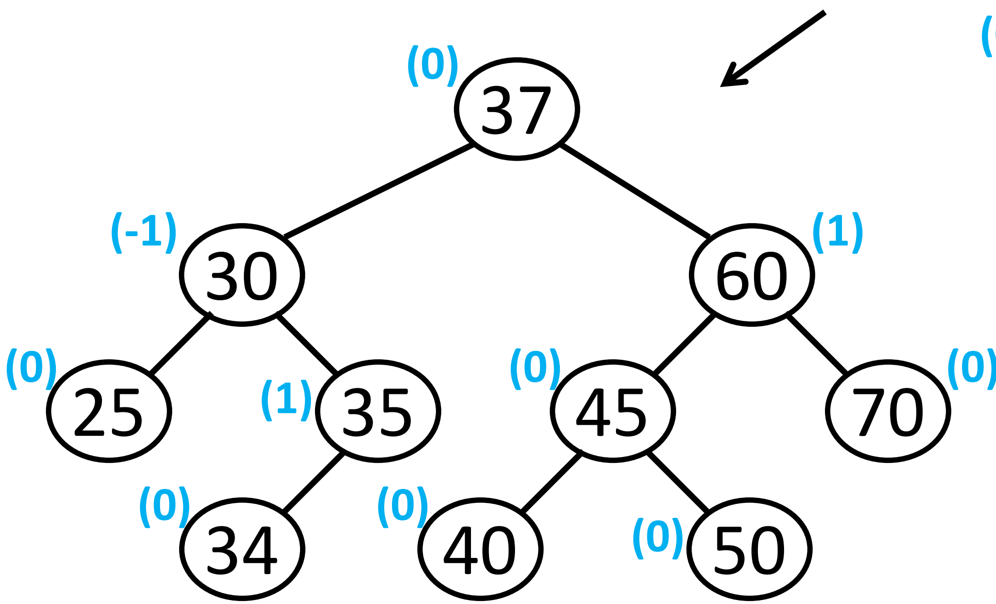
- Insert 30



Contd…

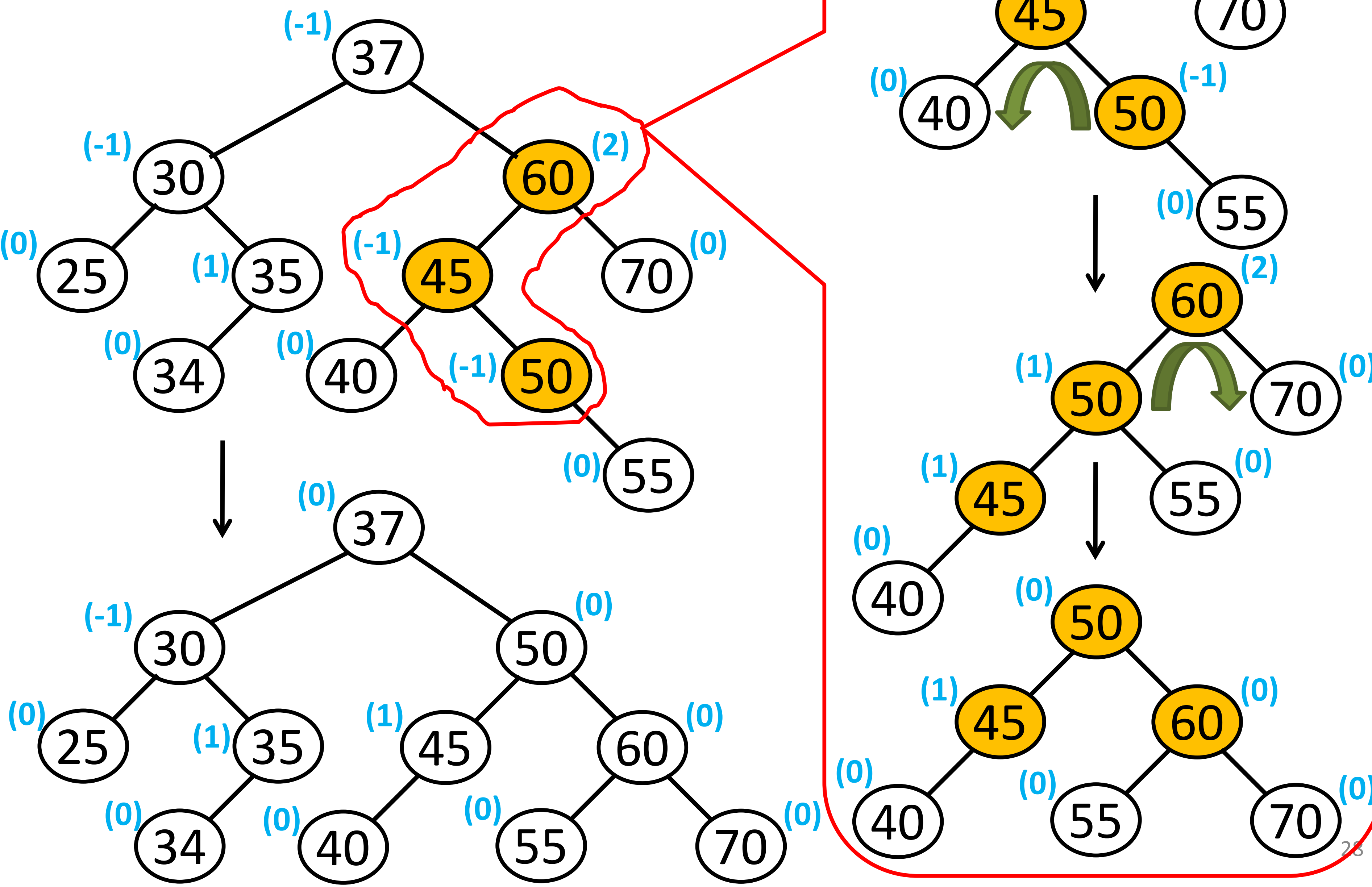
- Insert 45, 34, 40



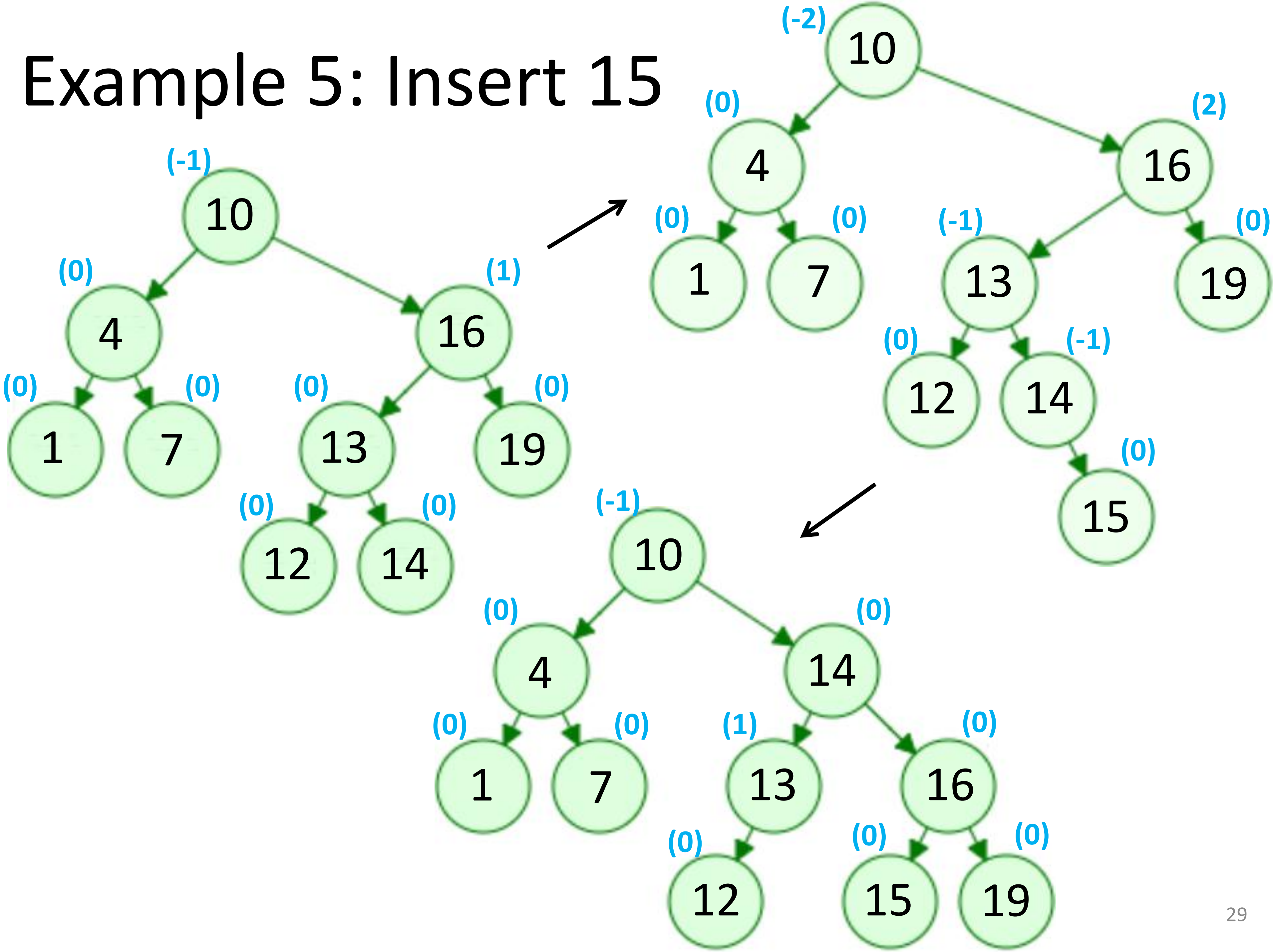


Contd...

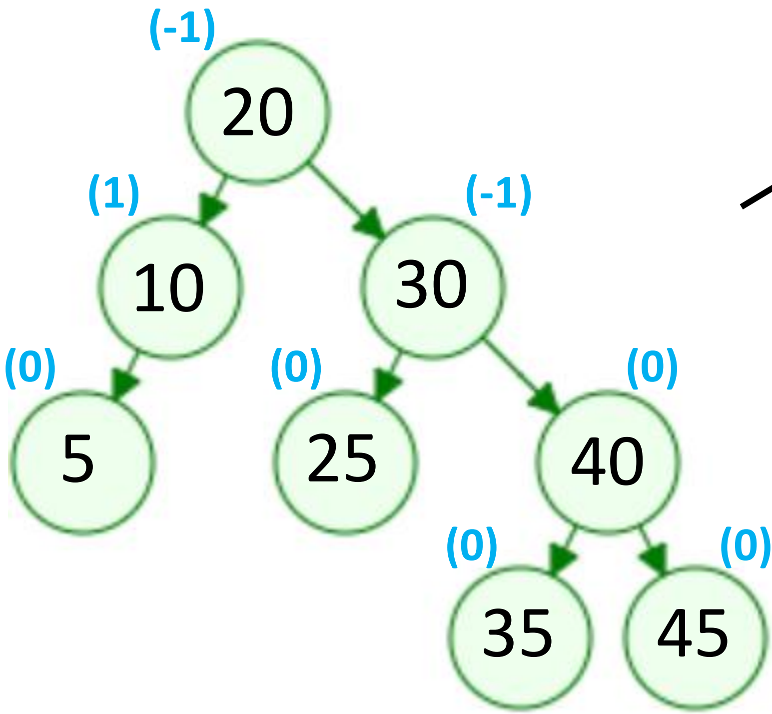
- Insert 55

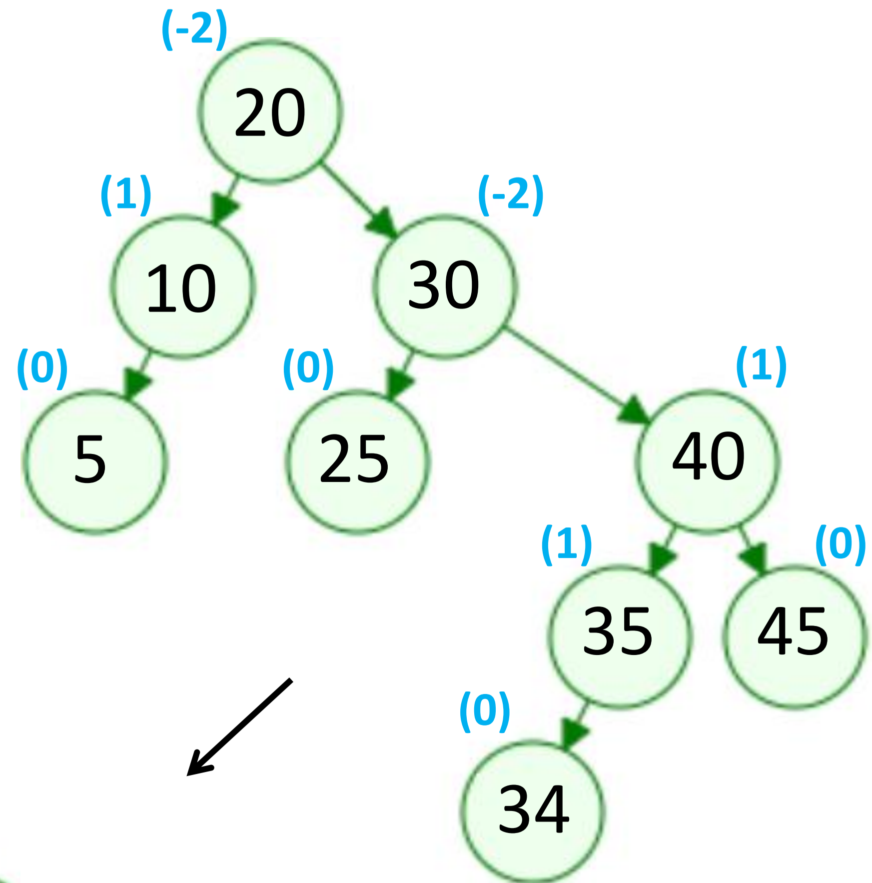


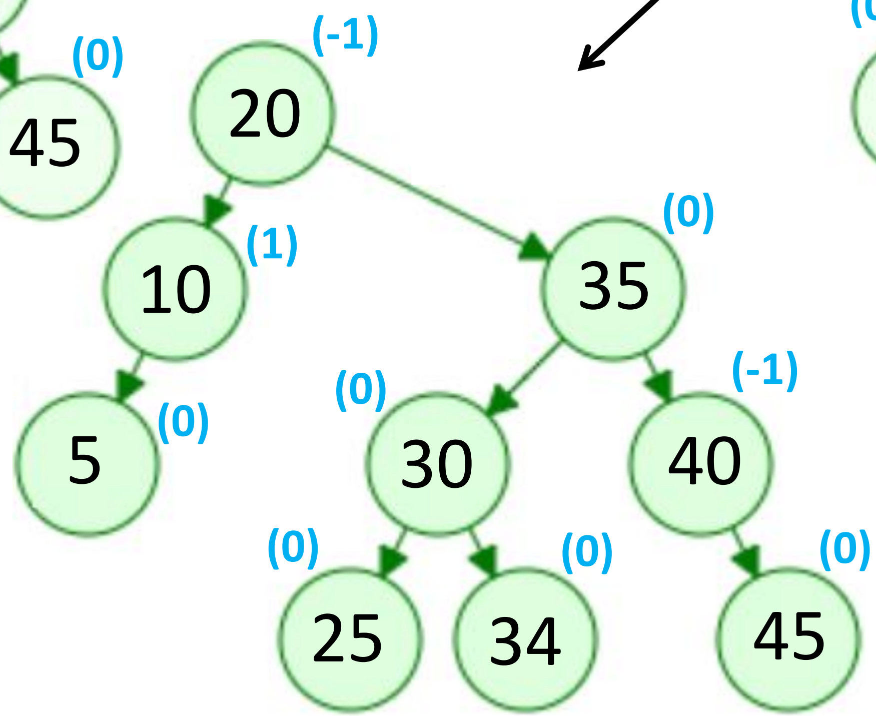
Example 5: Insert 15



Example 6: Insert 34

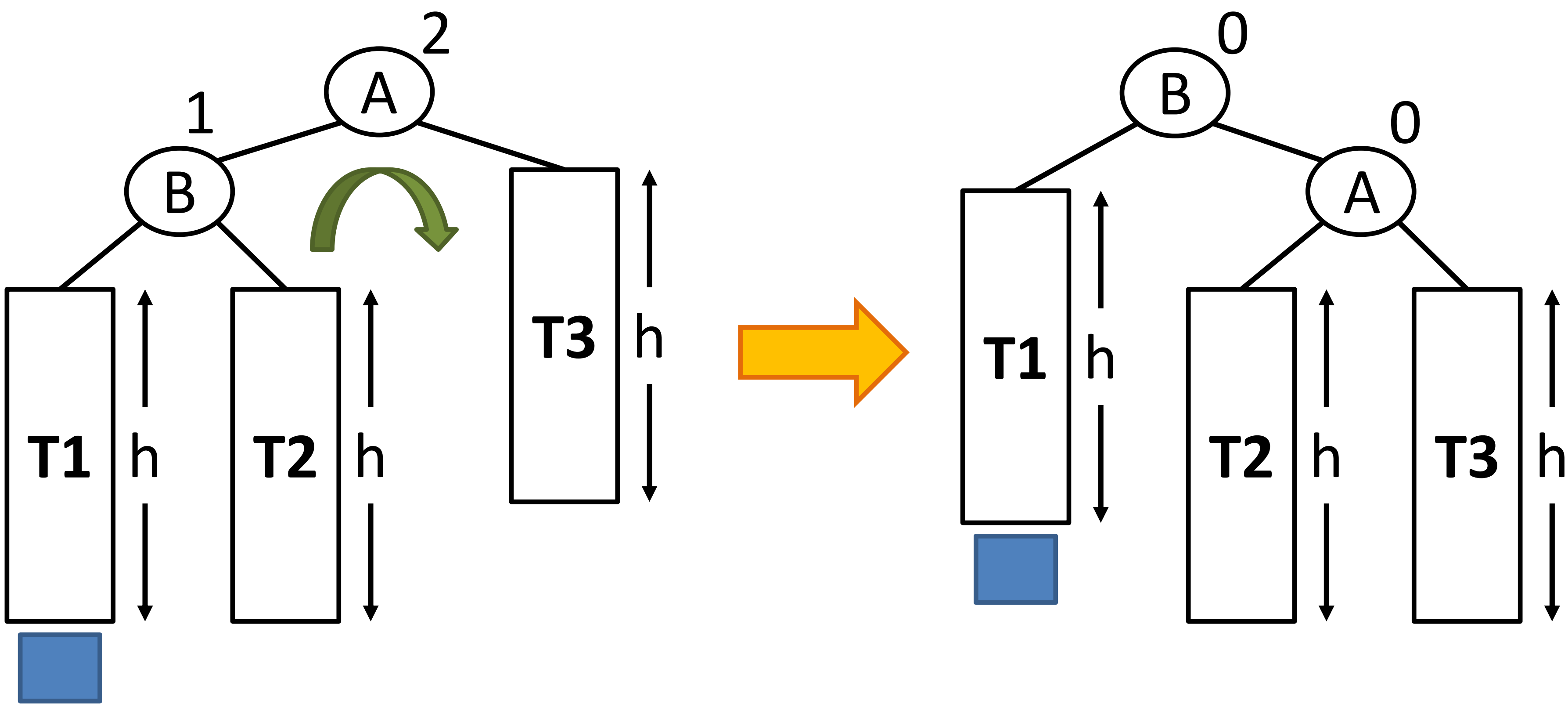






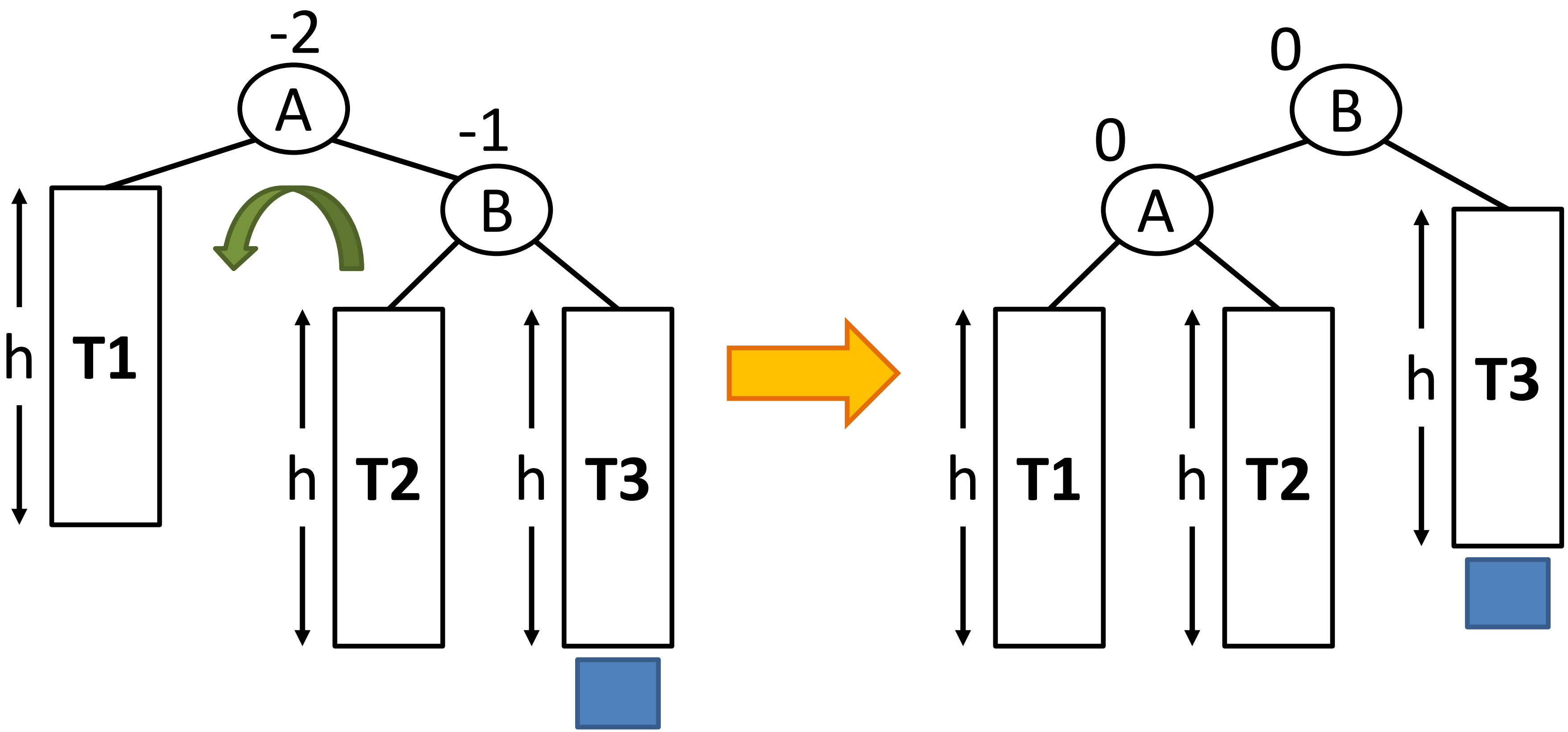
Case 1: Left of Left (Insertion)

$\boldsymbol{BalFactor} > 1$ and $\boldsymbol{Key} < \boldsymbol{Node{\rightarrow}left{\rightarrow}data}$

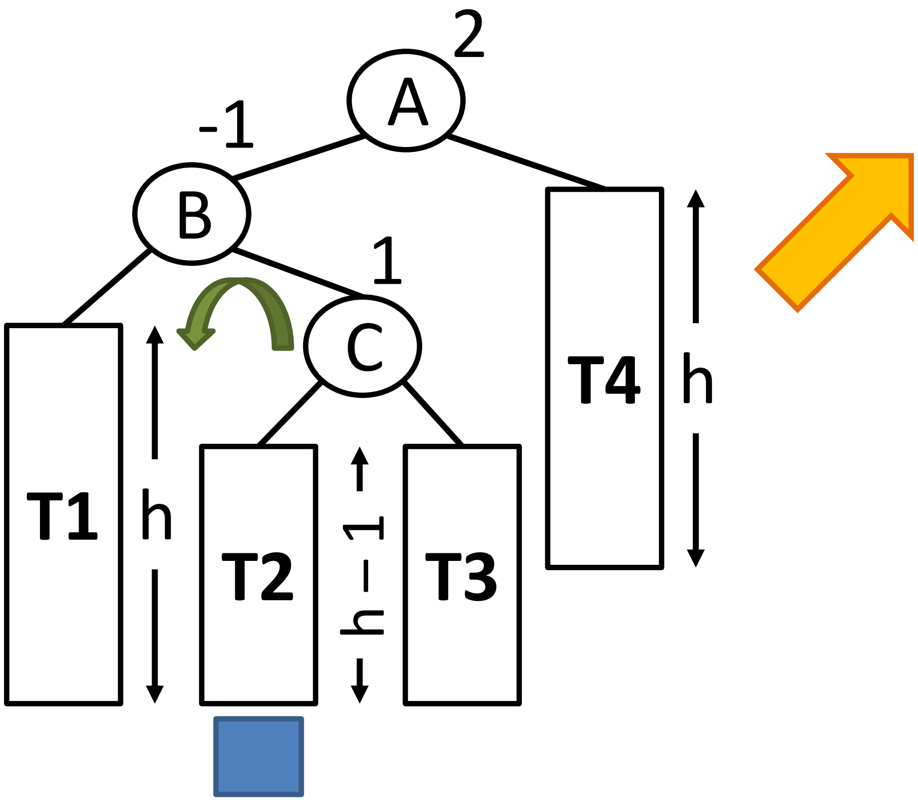


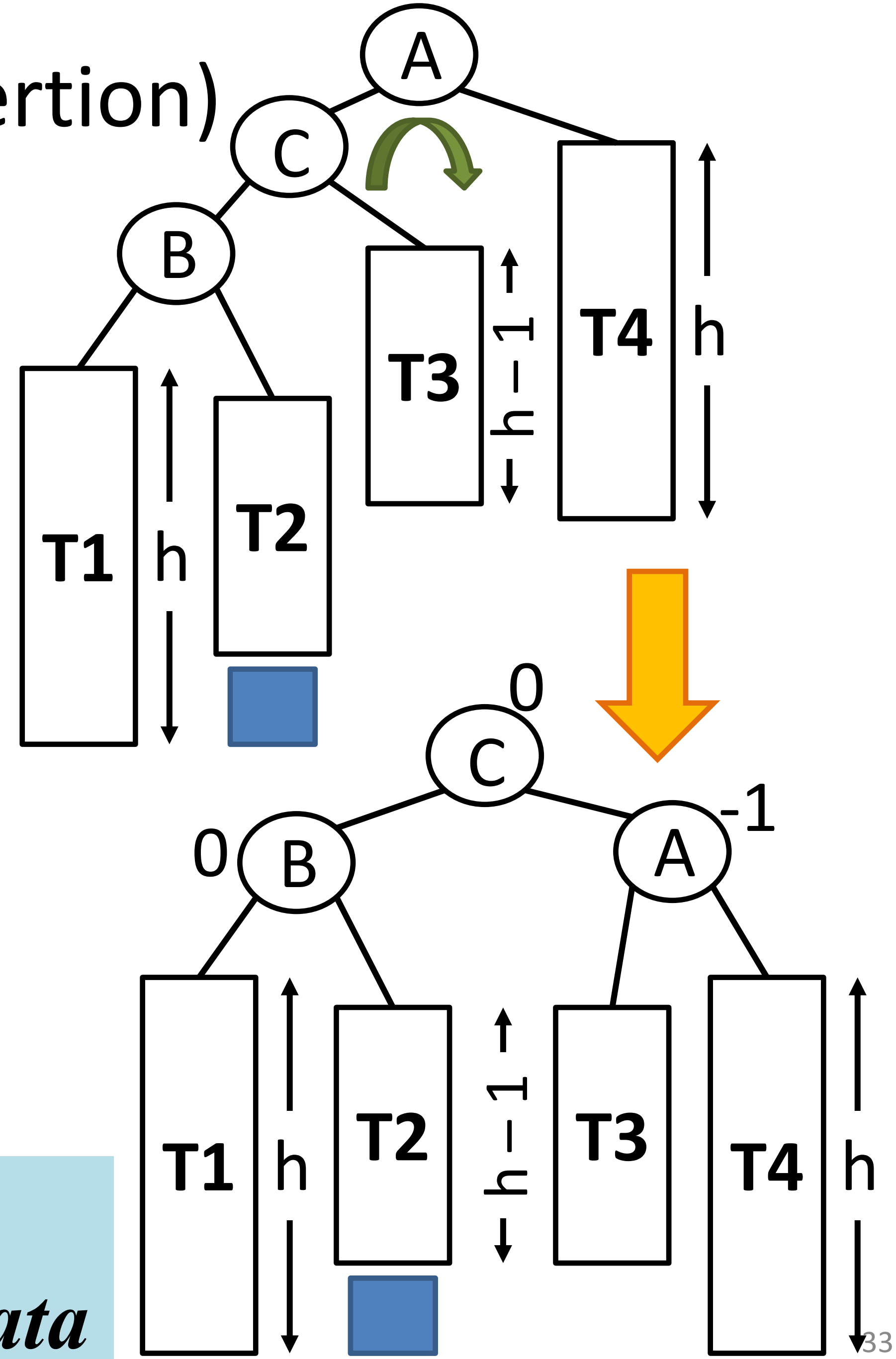
Case 2: Right of Right (Insertion)

BalFactor < -1 and ***Key*** > ***Node→right→data***



Case 3: Right of Left (Insertion)

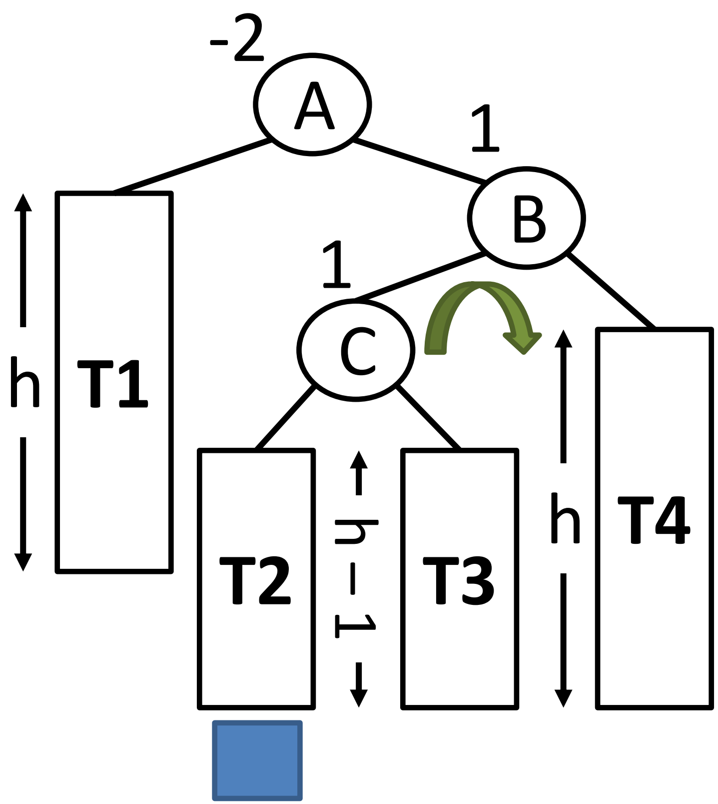


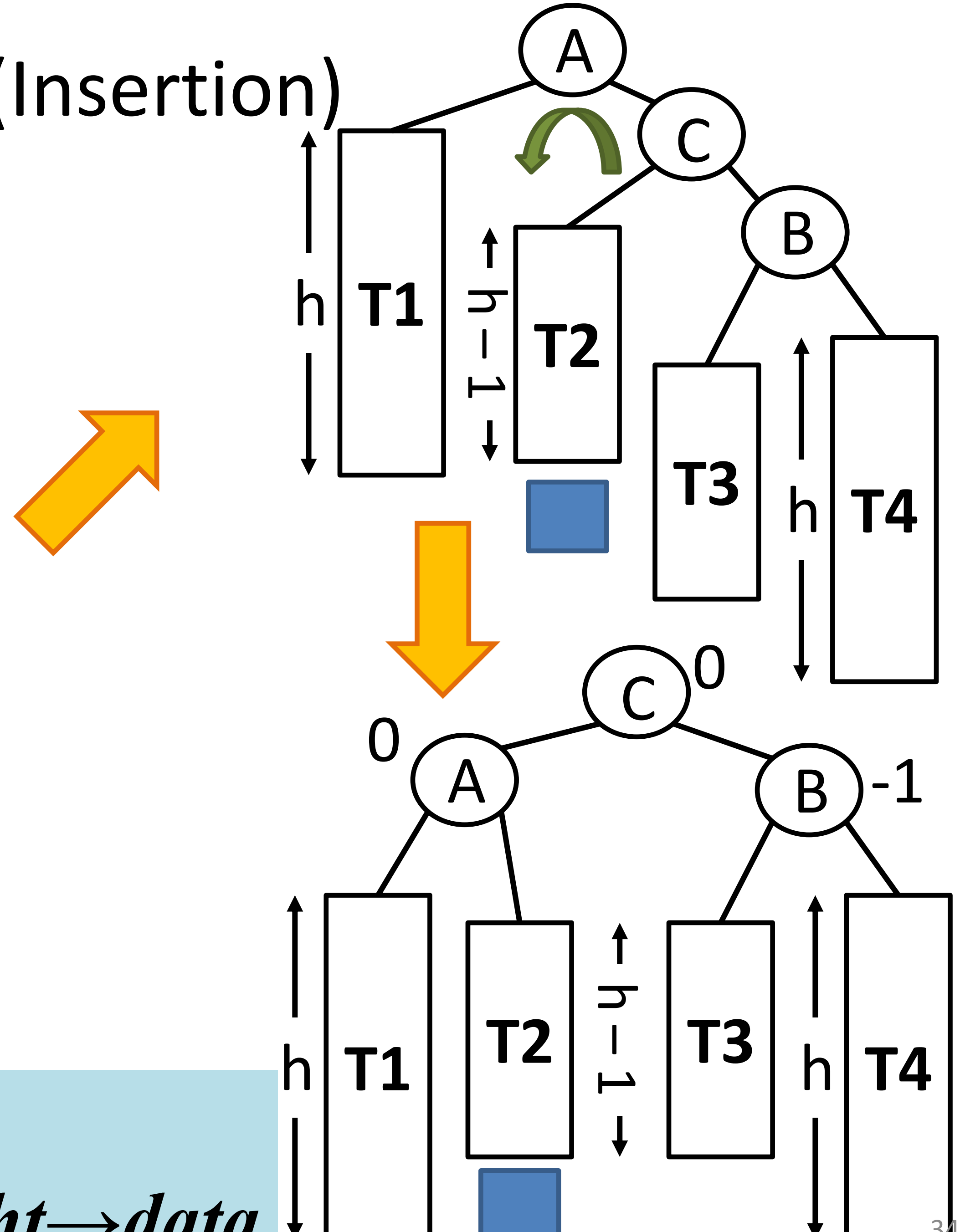


BalFactor > 1 and

Key* > *Node*→*left*→*data

Case 4: Left of Right (Insertion)





BalFactor < -1 and

Key < ***Node→right→data***

Node Structure

For a new node

- data = value
- left = right = NULL
- height = 1

- Four elements
  - data <dataType>
  - left <pointer to Node>
  - right <pointer to Node>
  - height <int> // Height of a node

    OR

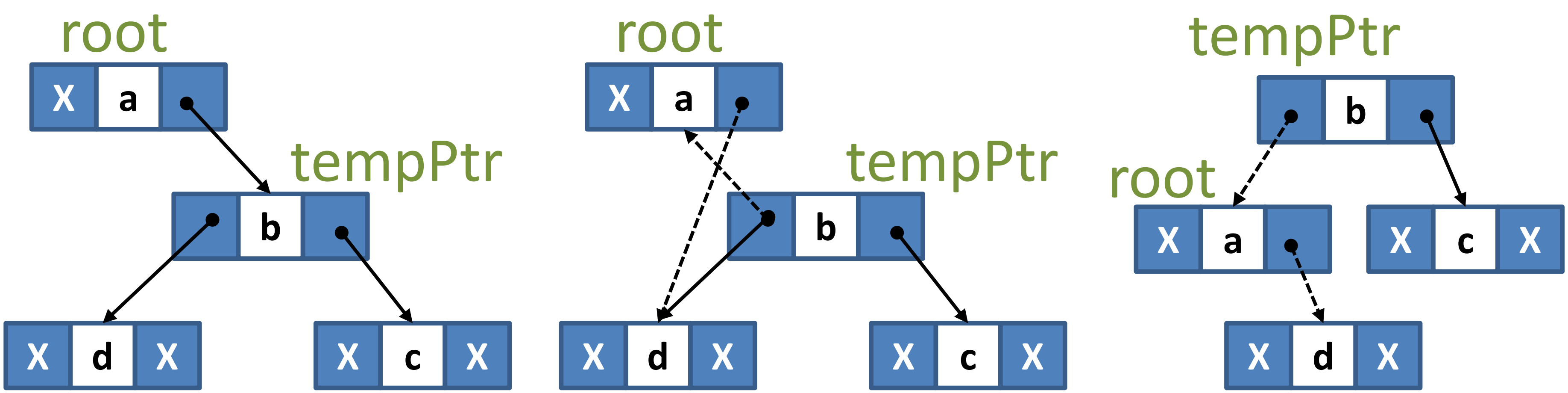
    bal <LH (= 1), EH (= 0), RH (= -1)> // Balance factor

Left Rotation

Algorithm rotateLeft(root)

1. Exchange right subtree of root with left subtree of its right subtree.
2. Make its right subtree new root.

```
NODE* rotateLeft(NODE *root)
{   NODE *tempPtr;
    tempPtr = root -> right;
    root->right = tempPtr -> left;
    tempPtr -> left = root;
    // Update height of root
    // Update height of tempPtr
    return tempPtr;    }
```

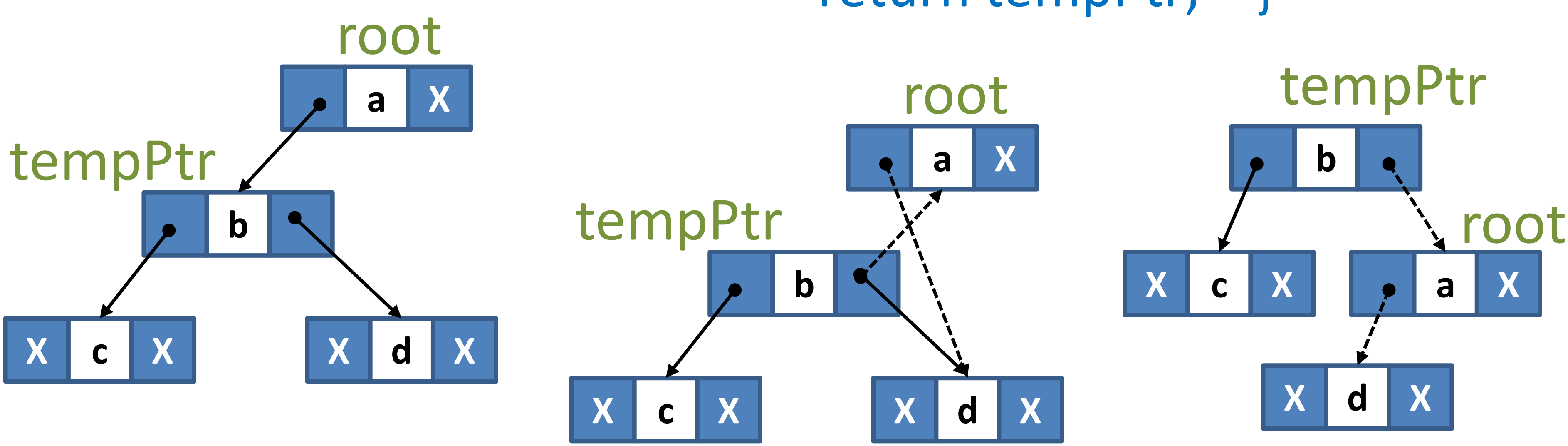


Right Rotation

Algorithm rotateRight(root)

1. Exchange left subtree of root with right subtree of its left subtree.
2. Make its left subtree new root.

```
NODE* rotateRight(NODE *root)
{   NODE *tempPtr;
    tempPtr = root -> left;
    root->left = tempPtr -> right;
    tempPtr -> right = root;
    // Update height of root
    // Update height of tempPtr
    return tempPtr;    }
```



Insertion (recursion)

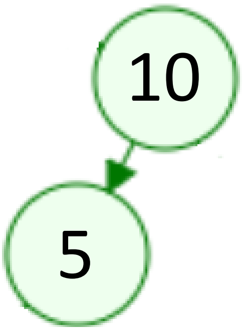
1. Insert the new-node with value $\boldsymbol{Key}$ using normal BST insertion.
2. Update height of the ancestor node, say $\boldsymbol{Node}$.
3. Get the balance factor of this ancestor node, i.e. $\boldsymbol{Node}$.
4. $\boldsymbol{BalFactor}$ = (height of $\boldsymbol{Node \to left}$ — height of $\boldsymbol{Node \to right}$).
5. If $\boldsymbol{BalFactor} > 1$ and $\boldsymbol{Key} < \boldsymbol{Node \to left \to data}$
6. Return $\boldsymbol{rotateRight}(\boldsymbol{Node})$. // Left of Left.

Example – 1

$\boldsymbol{BalFactor} > 1$ and
$\boldsymbol{Key} < \boldsymbol{Node} \rightarrow \boldsymbol{left} \rightarrow \boldsymbol{data}$

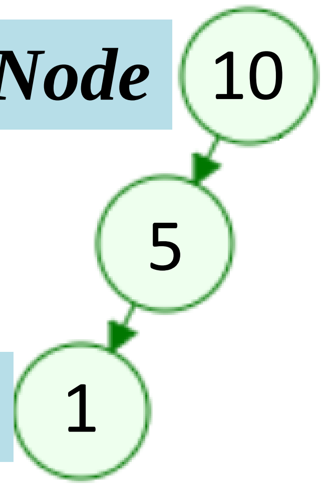
Height = 2

Height = 1



Insert ***Key*** = 1

Node



Key

Height = 3, ***BalFactor*** = 2

Height = 2, ***BalFactor*** = 1

Height = 1

rotateRight(Node)

Contd...

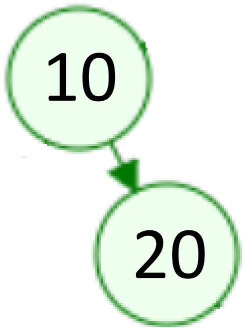
7. If $\boldsymbol{BalFactor} < -1$ and $\boldsymbol{Key} > \boldsymbol{Node \rightarrow right \rightarrow data}$
8. Return $\boldsymbol{rotateLeft}(\boldsymbol{Node})$; // Right of Right.

Example – 2

BalFactor < -1 and
Key > ***Node→right→data***

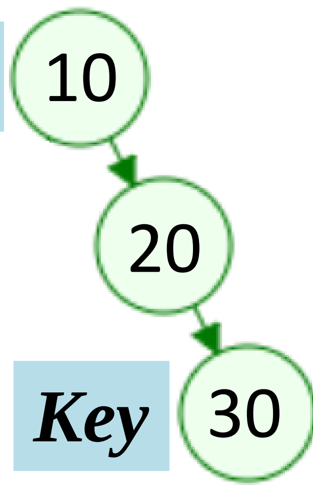
Height = 2

Height = 1



Insert ***Key*** = 30

Node



Height = 3, ***BalFactor*** = -2

rotateLeft(Node)

Height = 2, ***BalFactor*** = -1

Key

Height = 1

Contd…

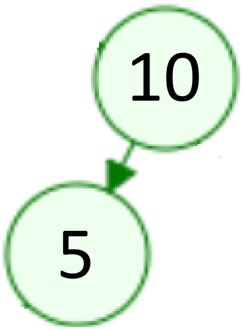
7. If $\boldsymbol{BalFactor}$ < -1 and $\boldsymbol{Key} > \boldsymbol{Node \rightarrow right \rightarrow data}$
8. Return $\boldsymbol{rotateLeft}(\boldsymbol{Node})$; // Right of Right.
9. If $\boldsymbol{BalFactor}$ > 1 and $\boldsymbol{Key} > \boldsymbol{Node \rightarrow left \rightarrow data}$
10. $\boldsymbol{Node \rightarrow left = rotateLeft}(\boldsymbol{Node \rightarrow left})$;
11. Return $\boldsymbol{rotateRight}(\boldsymbol{Node})$; // Right of Left.

Example – 3

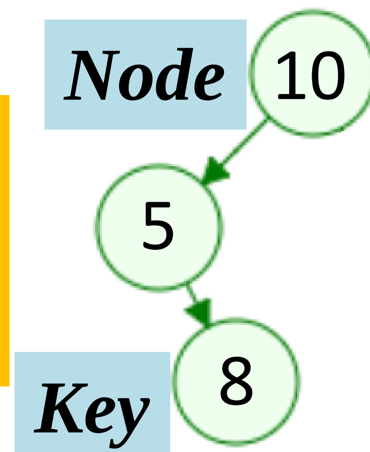
BalFactor > 1 and
Key* > *Node*→*left*→*data

Height = 2

Height = 1



Insert ***Key*** = 8



Height = 3, ***BalFactor*** = 2

Height = 2, ***BalFactor*** = -1

Height = 1

Node→left =
rotateLeft(Node→left);
rotateRight(Node);

Contd...

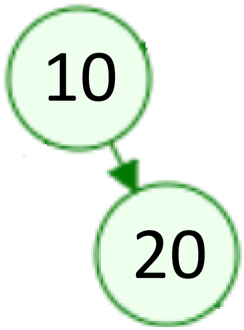
7. If $\boldsymbol{BalFactor}$ < -1 and $\boldsymbol{Key} > \boldsymbol{Node \rightarrow right \rightarrow data}$
8. Return $\boldsymbol{rotateLeft}(\boldsymbol{Node})$; // Right of Right.
9. If $\boldsymbol{BalFactor}$ > 1 and $\boldsymbol{Key} > \boldsymbol{Node \rightarrow left \rightarrow data}$
10. $\boldsymbol{Node \rightarrow left} = \boldsymbol{rotateLeft}(\boldsymbol{Node \rightarrow left})$;
11. Return $\boldsymbol{rotateRight}(\boldsymbol{Node})$; // Right of Left.
12. If $\boldsymbol{BalFactor}$ < -1 and $\boldsymbol{Key} < \boldsymbol{Node \rightarrow right \rightarrow data}$
13. $\boldsymbol{Node \rightarrow right} = \boldsymbol{rotateRight}(\boldsymbol{Node \rightarrow right})$;
14. Return $\boldsymbol{rotateLeft}(\boldsymbol{Node})$; // Left of Right.

Example – 4

BalFactor < -1 and ***Key*** < ***Node→right→data***

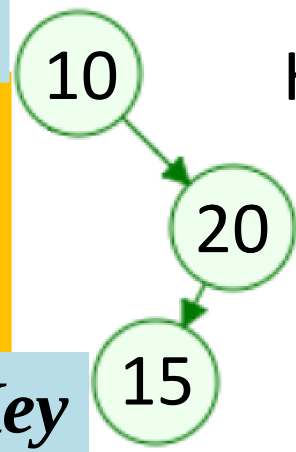
Height = 2

Height = 1



Insert ***Key*** = 15

Node



Height = 3, ***BalFactor*** = -2

Height = 2, ***BalFactor*** = 1

Height = 1

Key

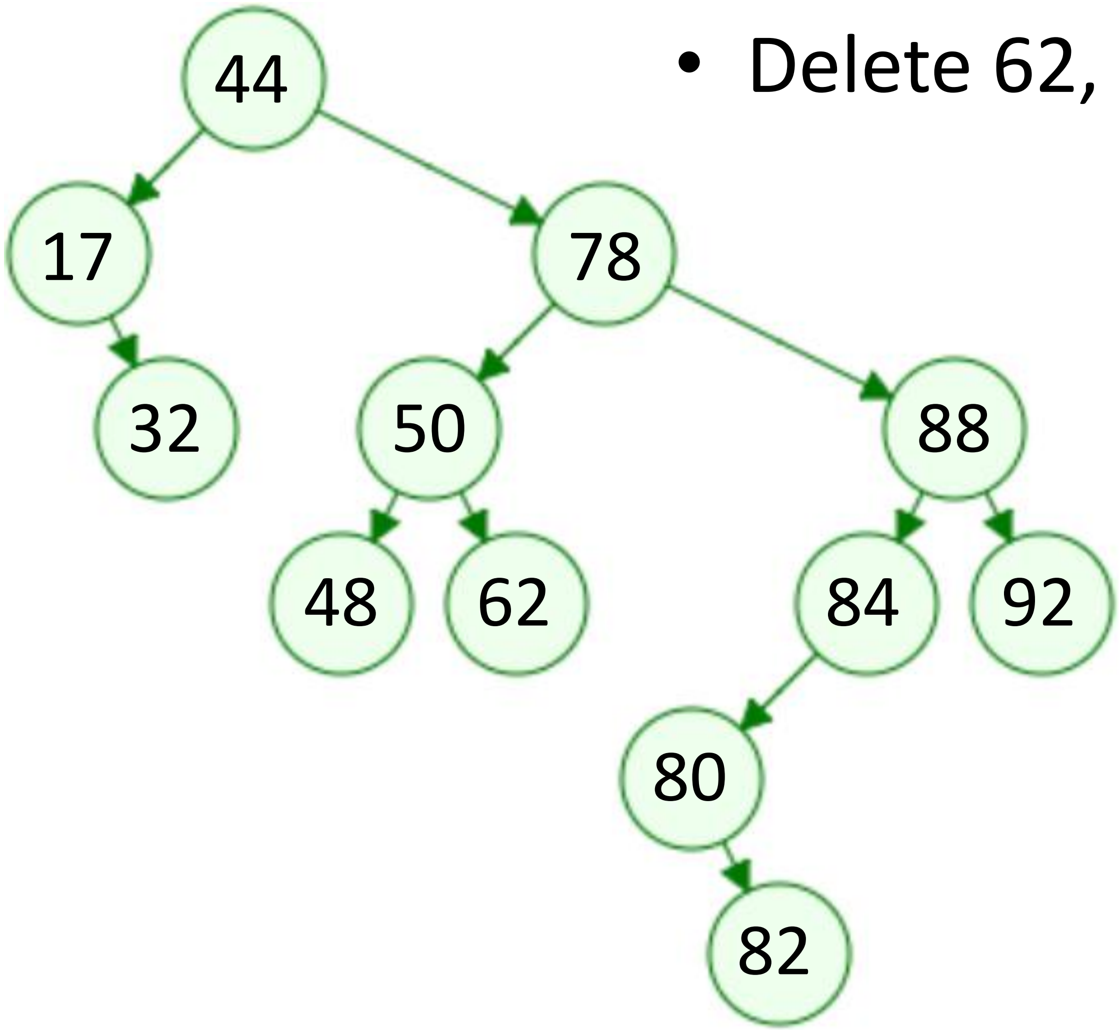
Node→right = rotateRight(Node→right);
rotateLeft(Node);

BST Deletion

- Search for a node to remove.
- If the node is found, then there are three cases:

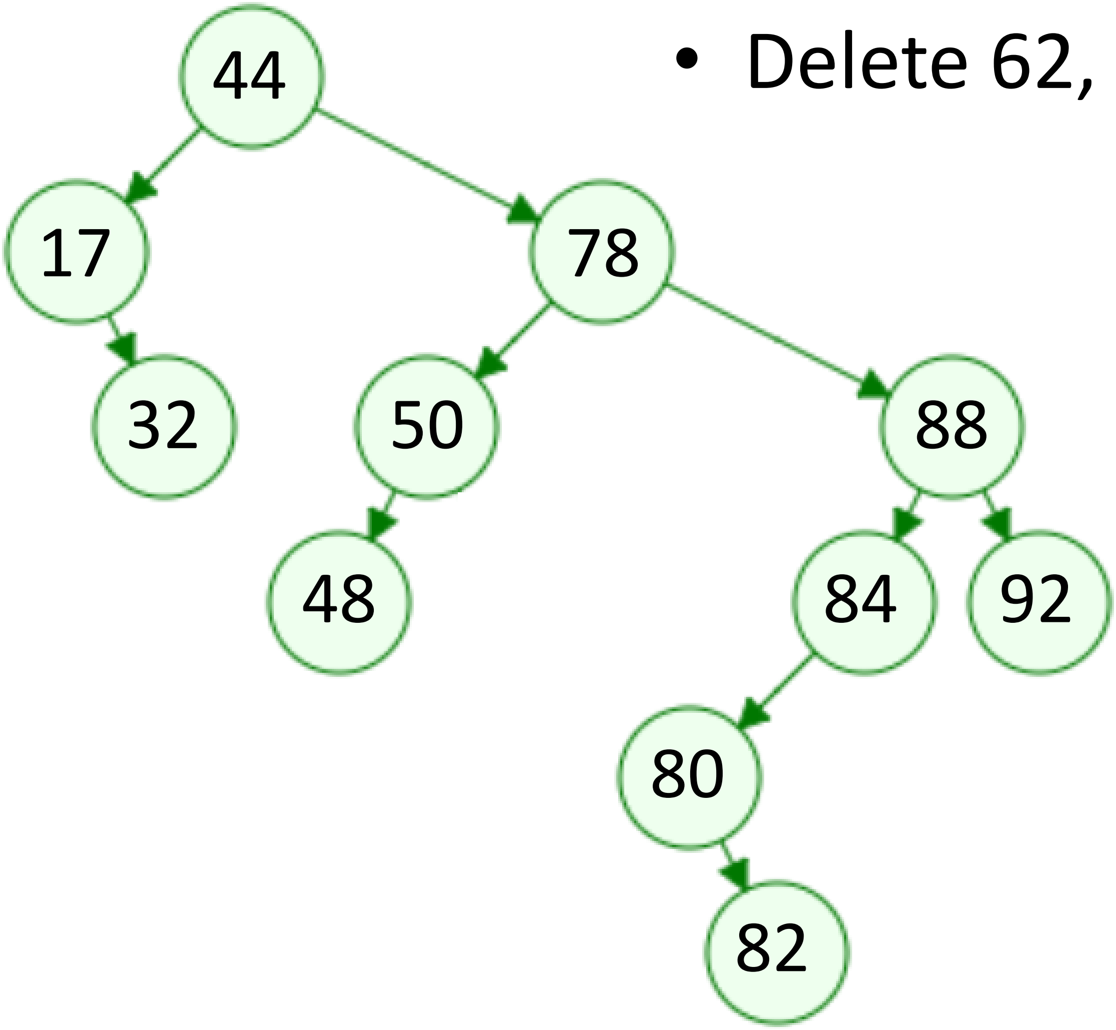
1. Node to be removed has no children.
   - Set corresponding link of the parent to NULL and dispose the node.
2. Node to be removed has one child.
   - Link single child (with it's subtree) directly to the parent of the removed node.
3. Node to be removed has two children.
   - Find inorder successor of the node.
   - Copy contents of the inorder successor to the node being removed.
   - Delete the inorder successor from the right subtree.

Example – BST Deletion



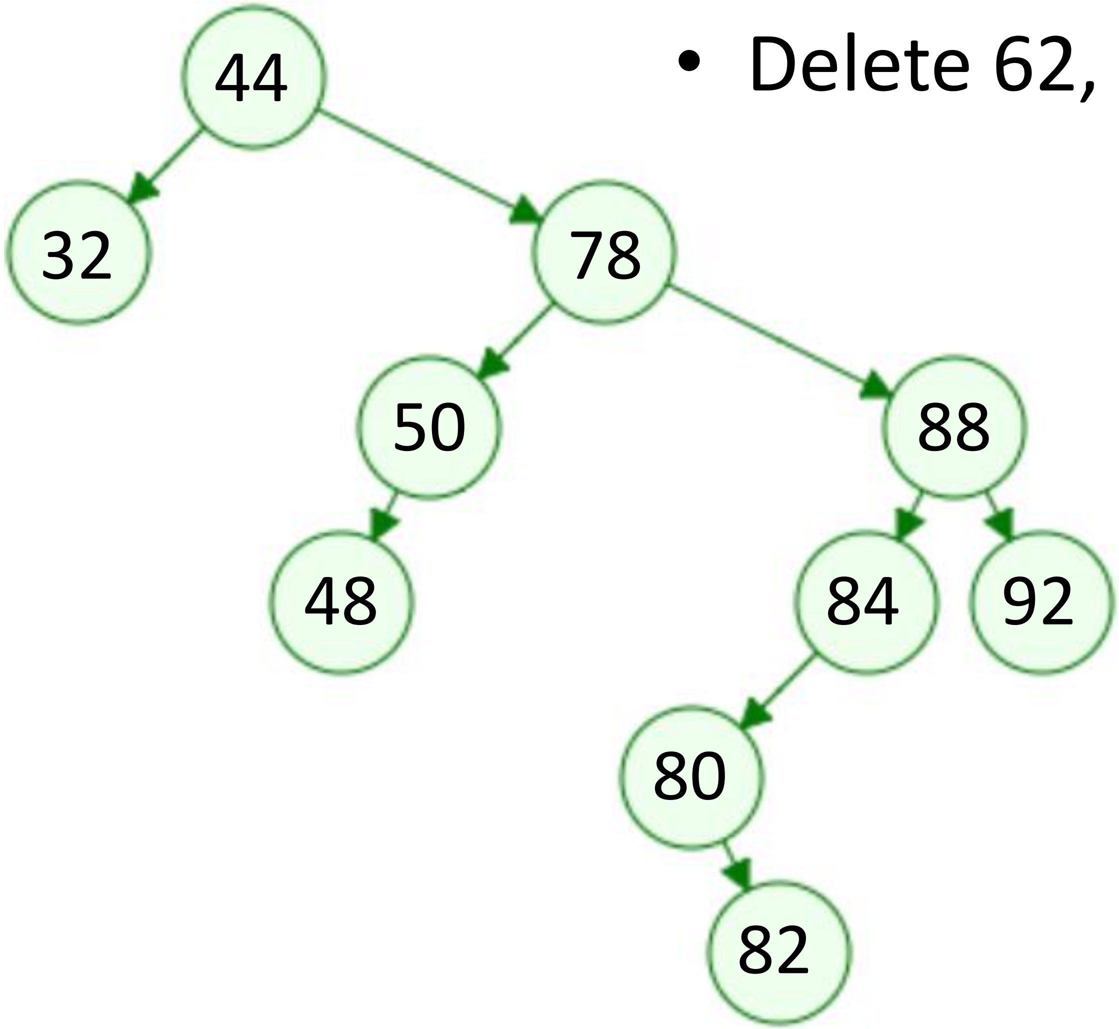
- Delete 62, 17, and 78

After deleting 62



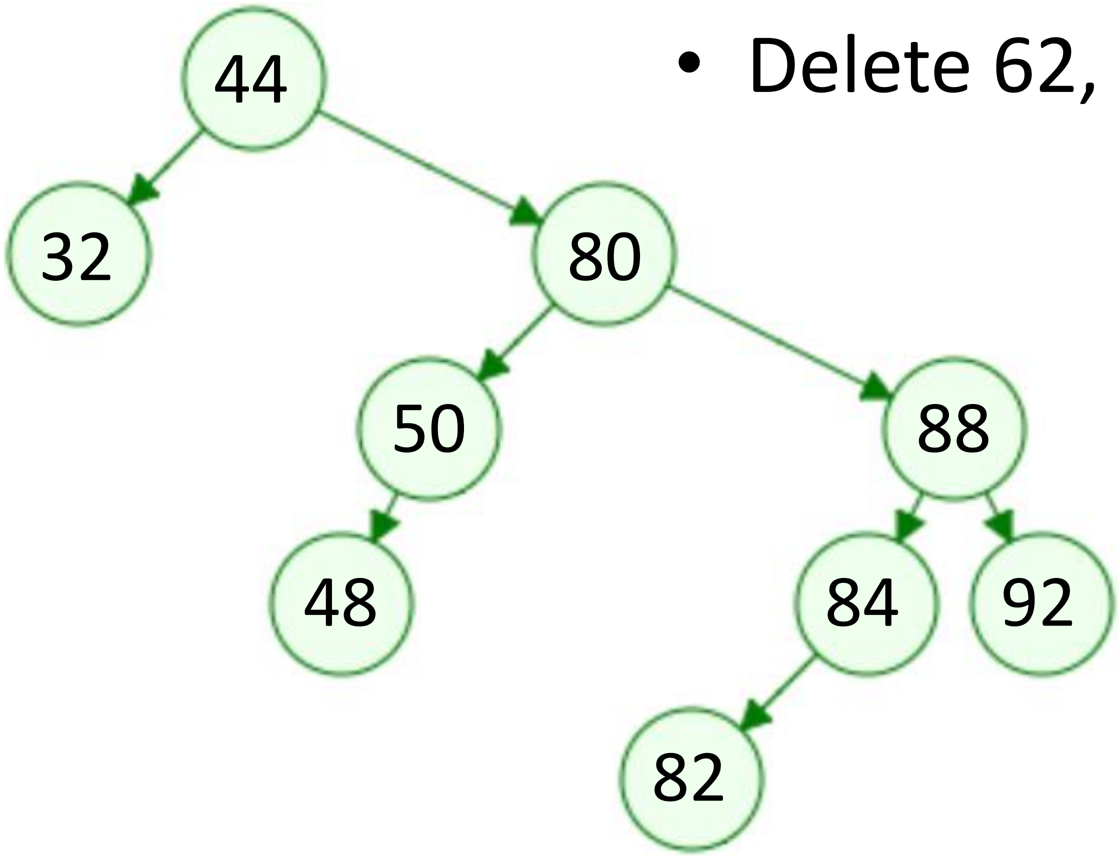
- Delete 62, 17, and 78

After deleting 17



- Delete 62, 17, and 78

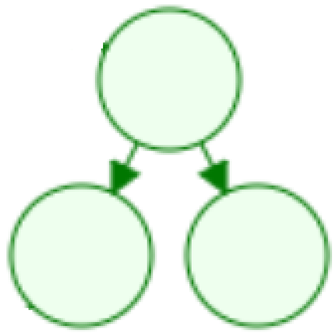
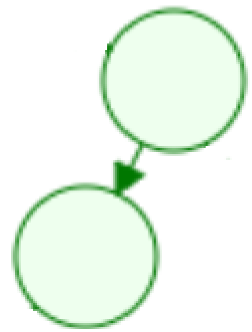
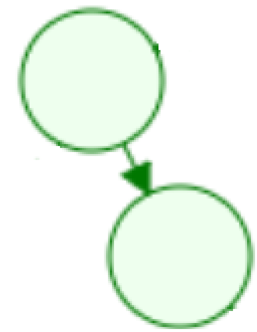
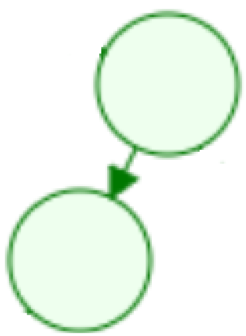
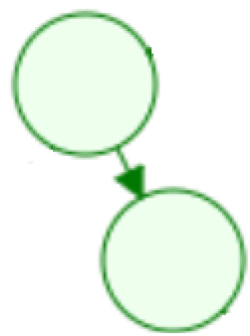
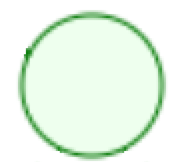
After deleting 78



- Delete 62, 17, and 78

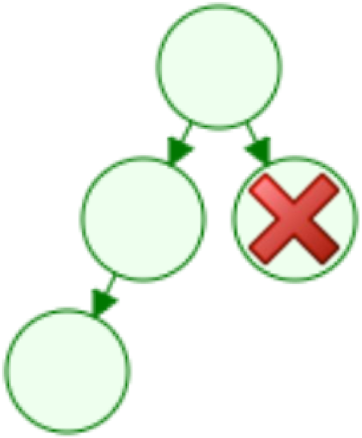
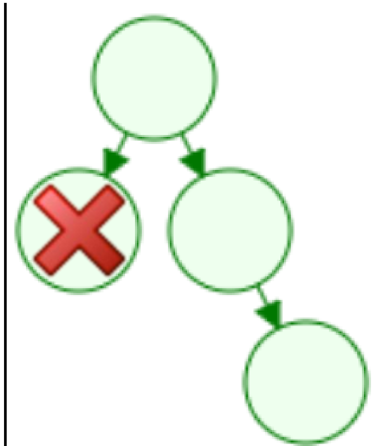
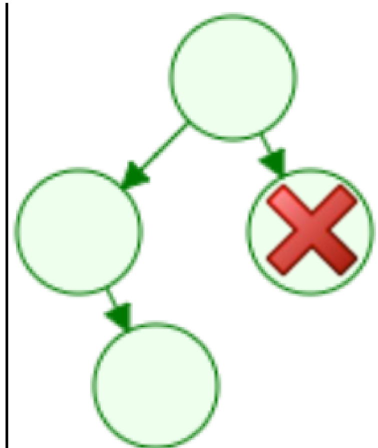
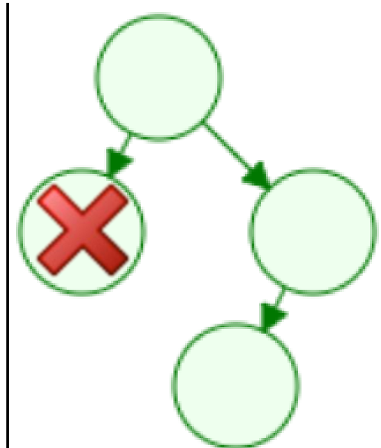
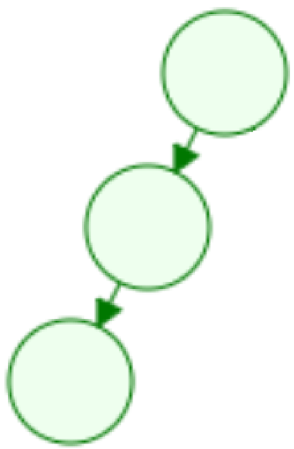
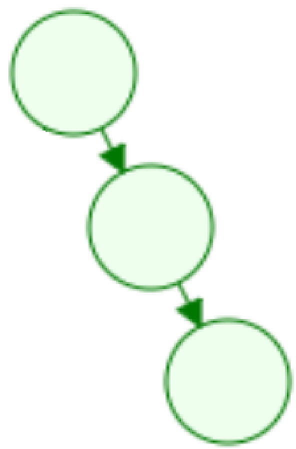
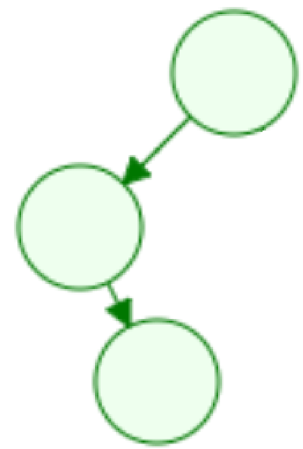
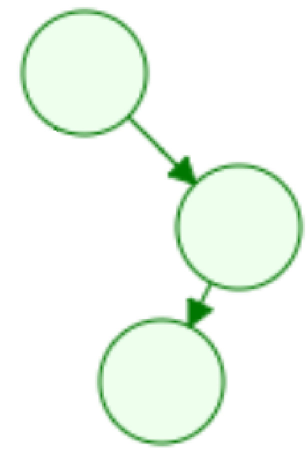
AVL Deletion

REQUIRES NO ROTATION

	Before deletion	After deletion
• Case 1		 OR 
• Case 2	 OR 	

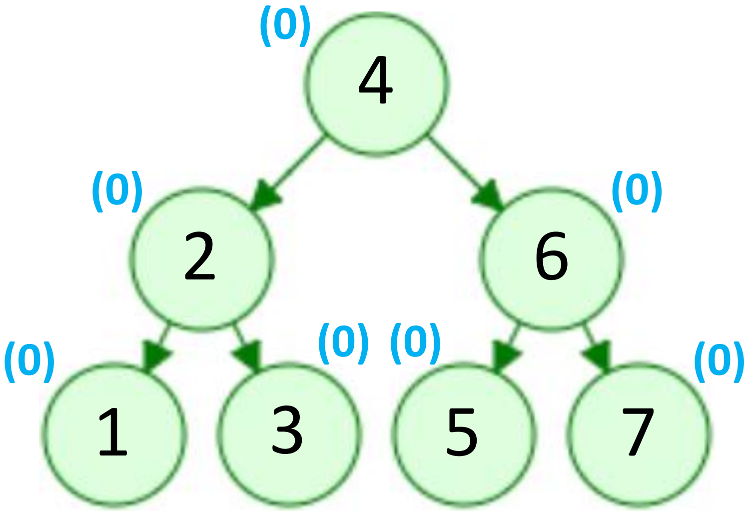
Contd…

- Case 3

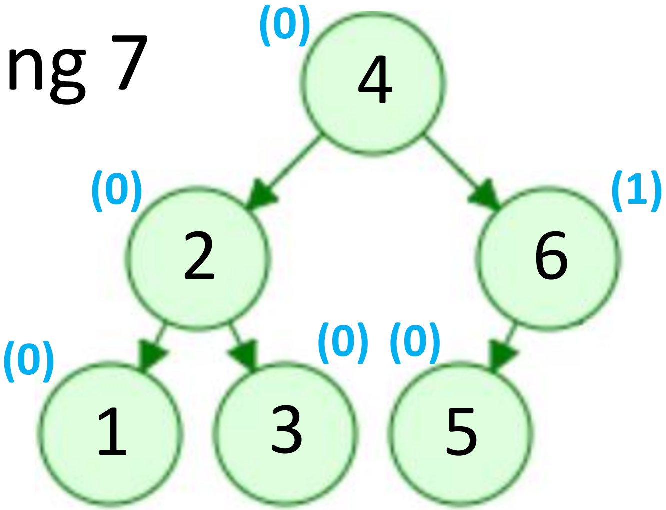
Before deletion				
After deletion				
Rotations	**Right**	**Left**	**Left then Right**	**Right then Left**

Example – 1

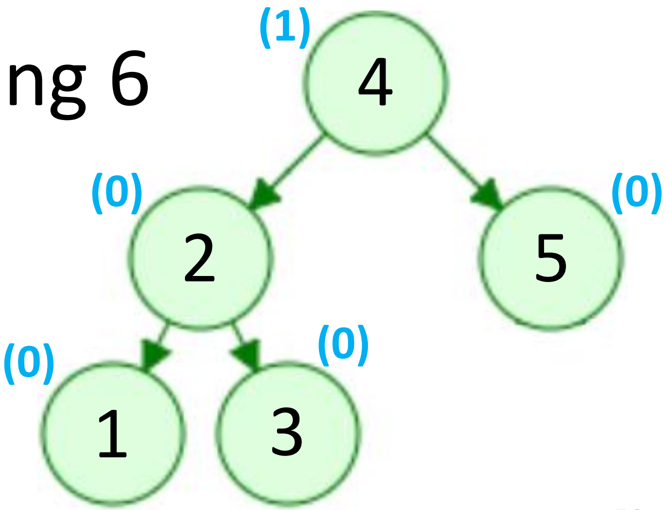
- Delete 7, 6, 5



After deleting 7

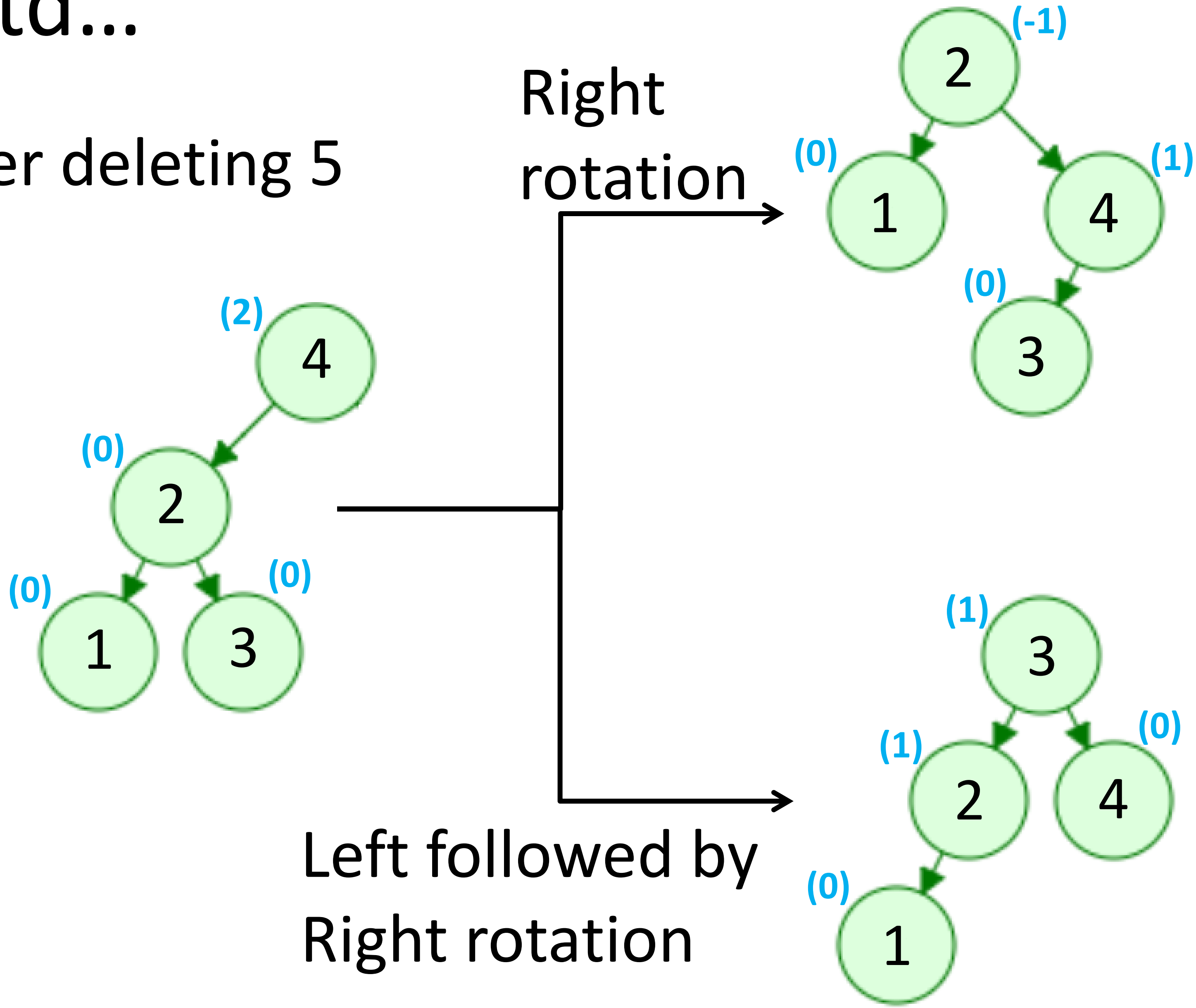


After deleting 6



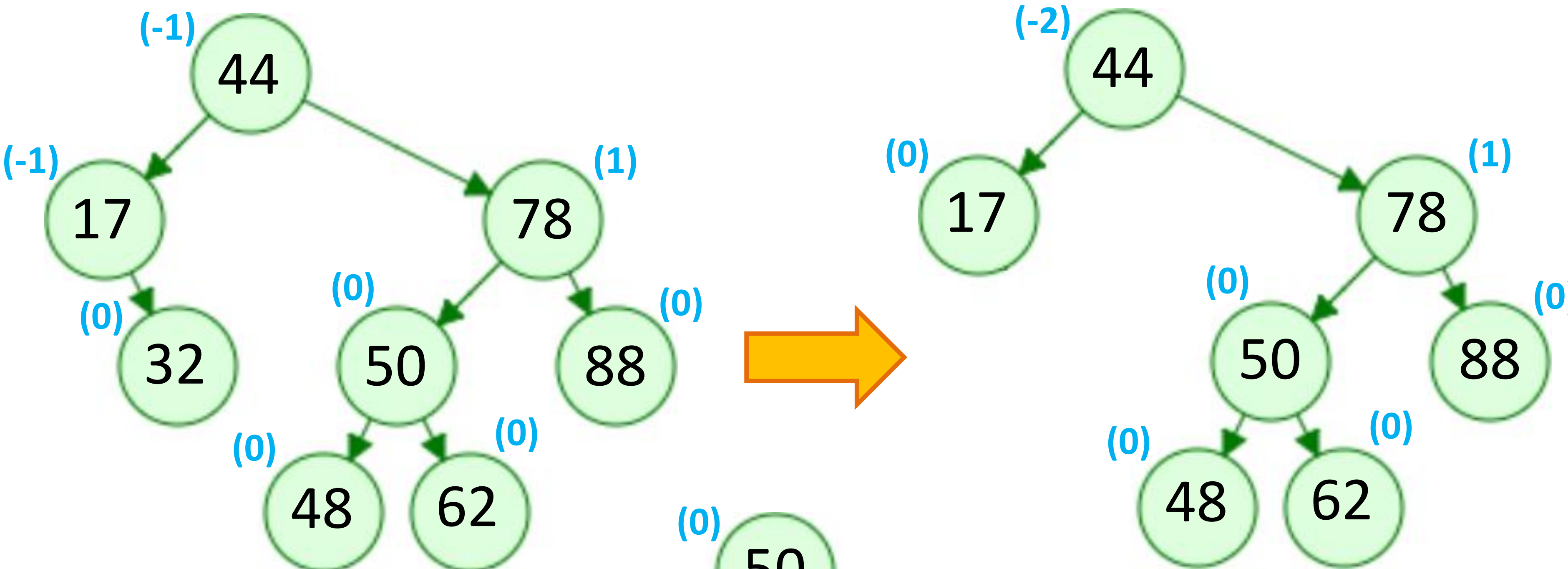
Contd…

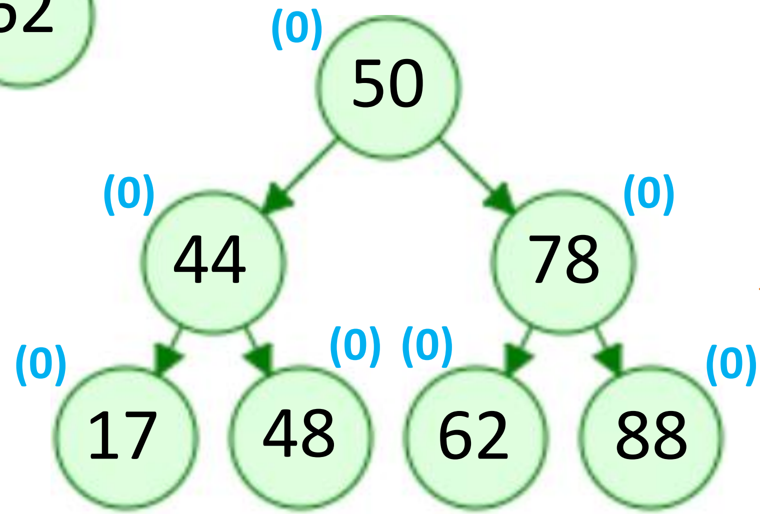
- After deleting 5



Example – 2

Delete 32

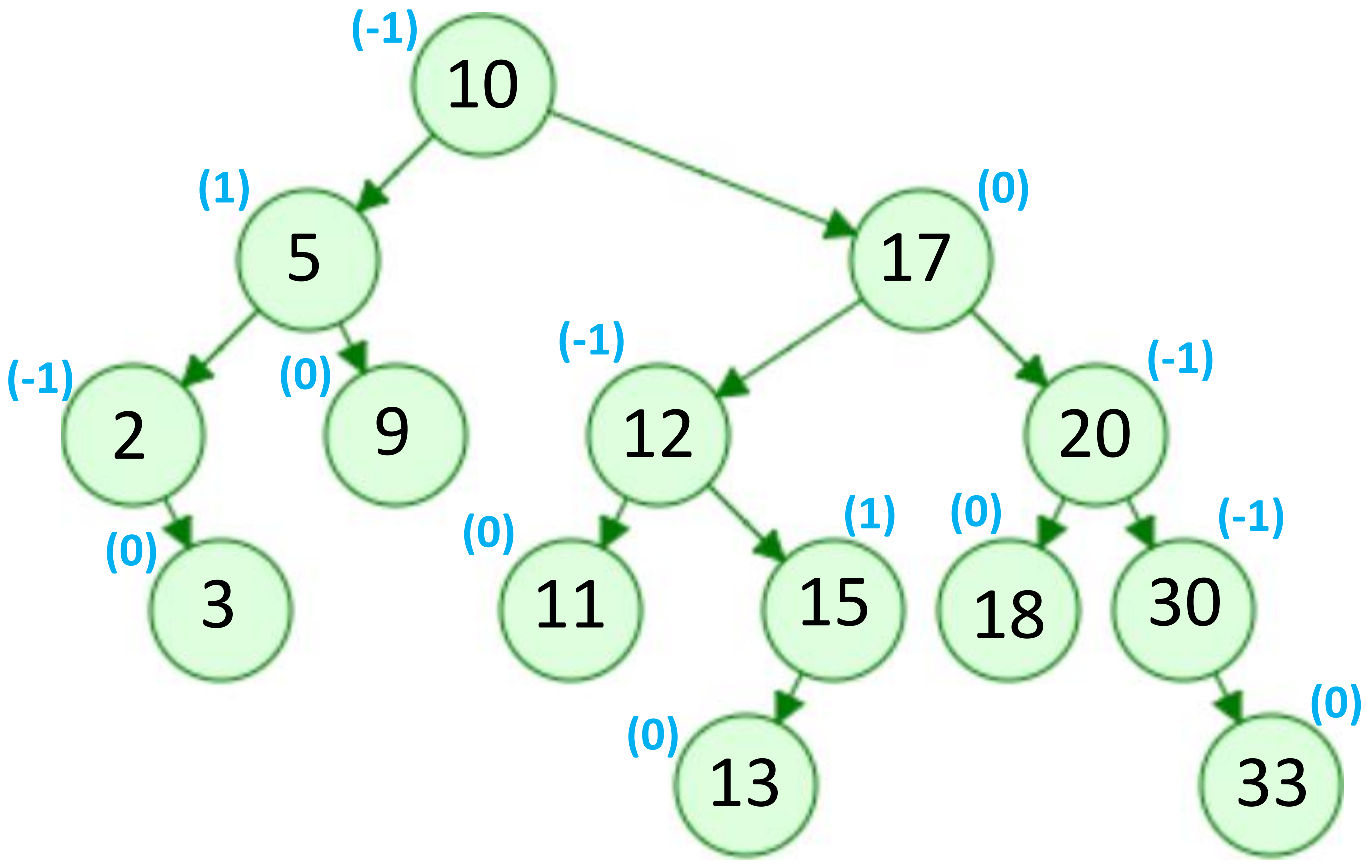




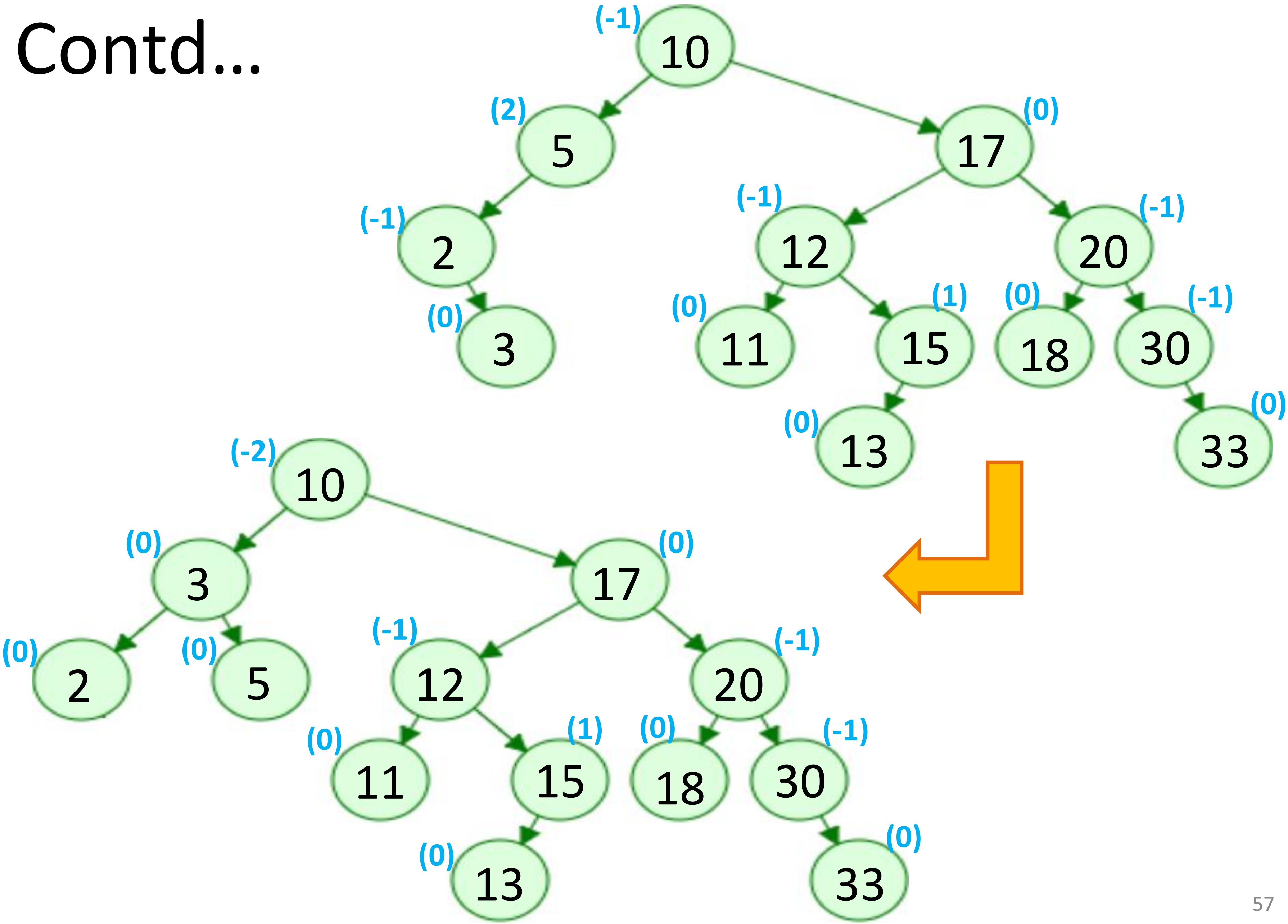
Right followed by left rotation

Example – 3

- Delete 9

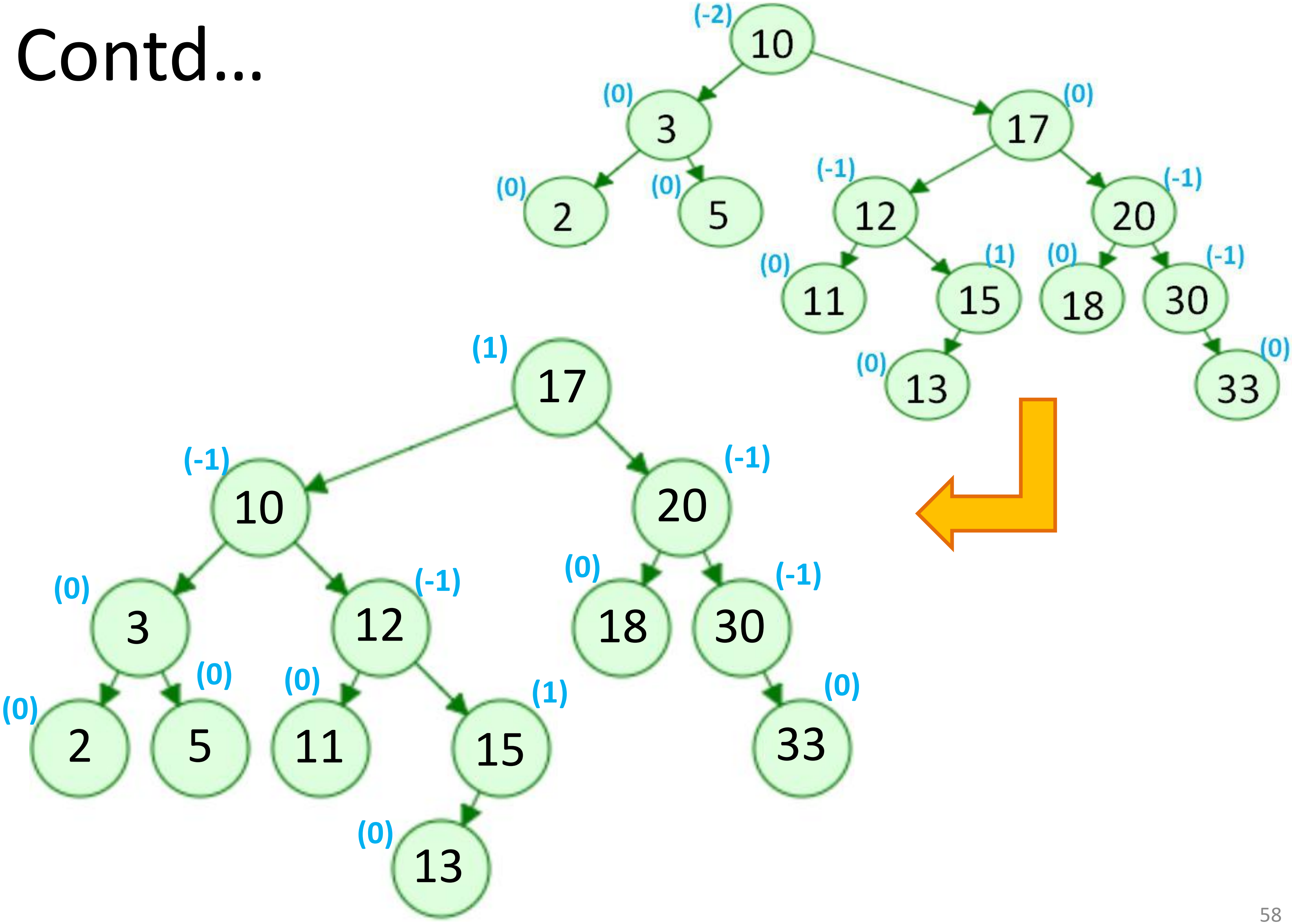


Contd…



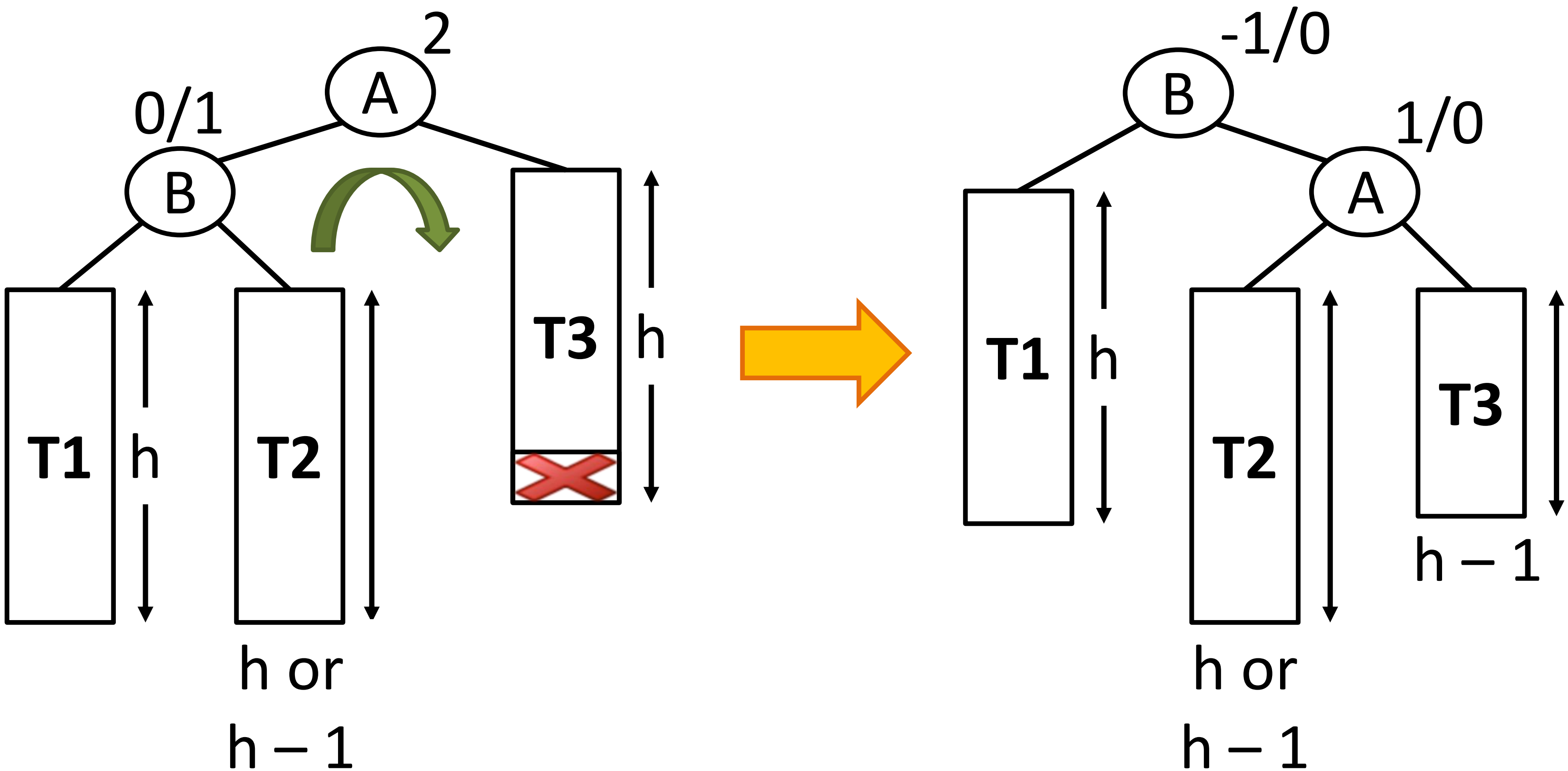
(-1)
10
(2)
5
(0)
17
(-1)
2
(-1)
12
(-1)
20
(0)
3
(0)
11
(1)
15
(0)
18
(-1)
30
(0)
13
(0)
33
(-2)
10
(0)
3
(0)
17
(0)
2
(0)
5
(-1)
12
(-1)
20
(0)
11
(1)
15
(0)
18
(-1)
30
(0)
13
(0)
33

Contd…


(-2) 10
(0) 3
(0) 17
(0) 2
(0) 5
(-1) 12
(-1) 20
(0) 11
(1) 15
(0) 18
(-1) 30
(0) 13
(0) 33
(1) 17
(-1) 10
(-1) 20
(0) 3
(-1) 12
(0) 18
(-1) 30
(0) 2
(0) 5
(0) 11
(1) 15
(0) 33
(0) 13

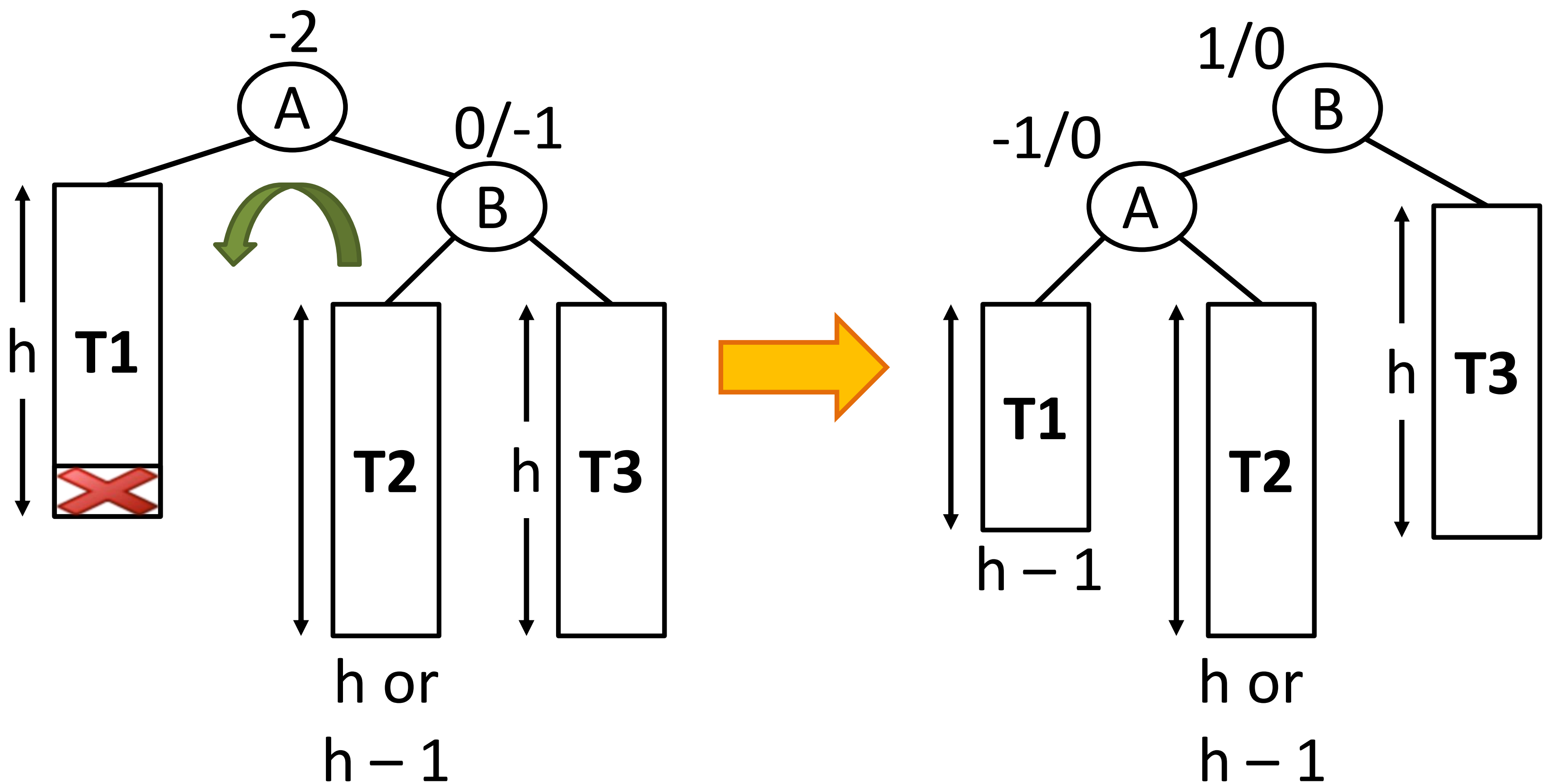
Case 1: Left of Left (Deletion)

BalFactor > 1 and ***balance*(*Node*→*left*)** >= 0

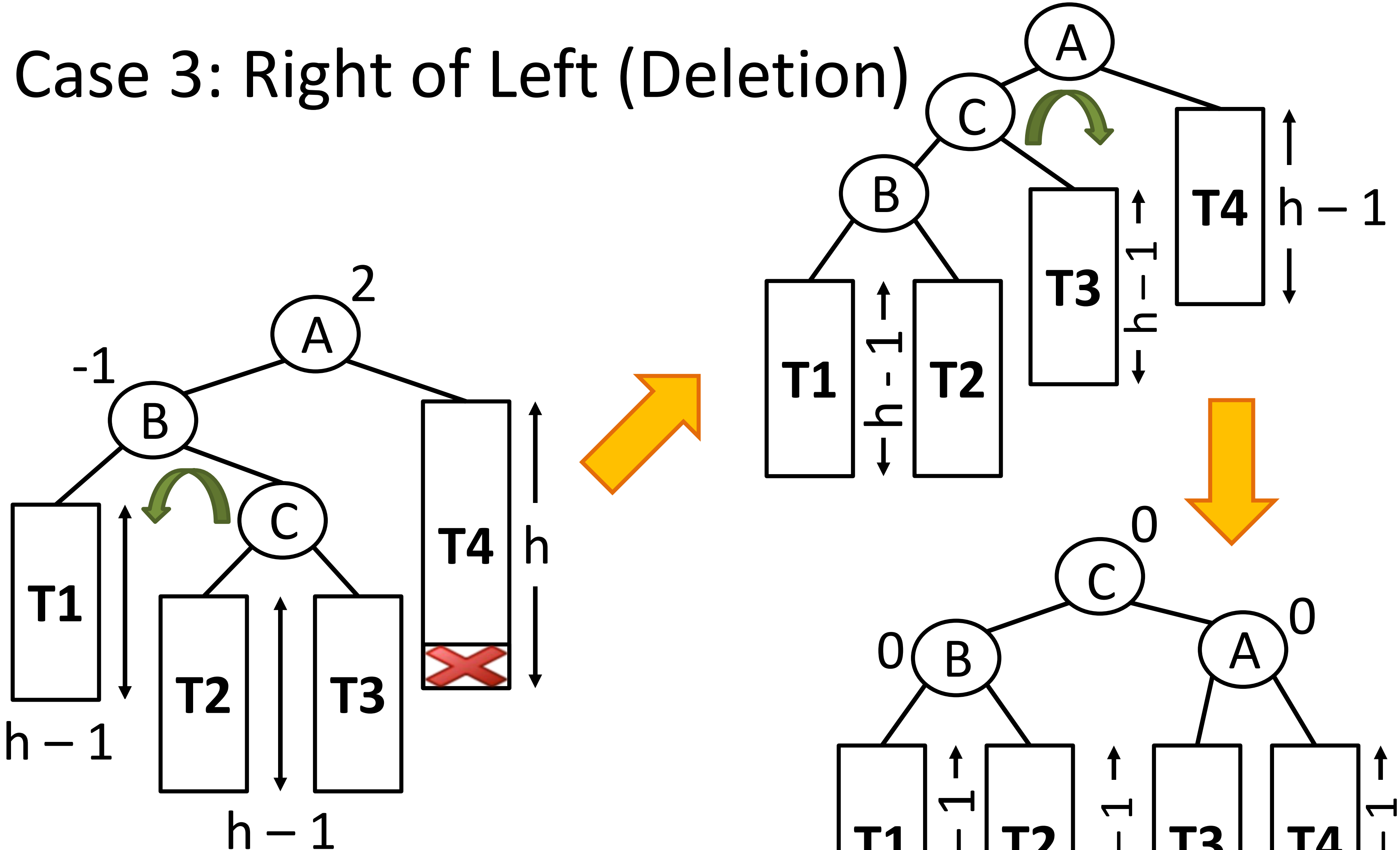


Case 2: Right of Right (Deletion)

BalFactor < -1 and ***balance(Node→right)*** <= 0

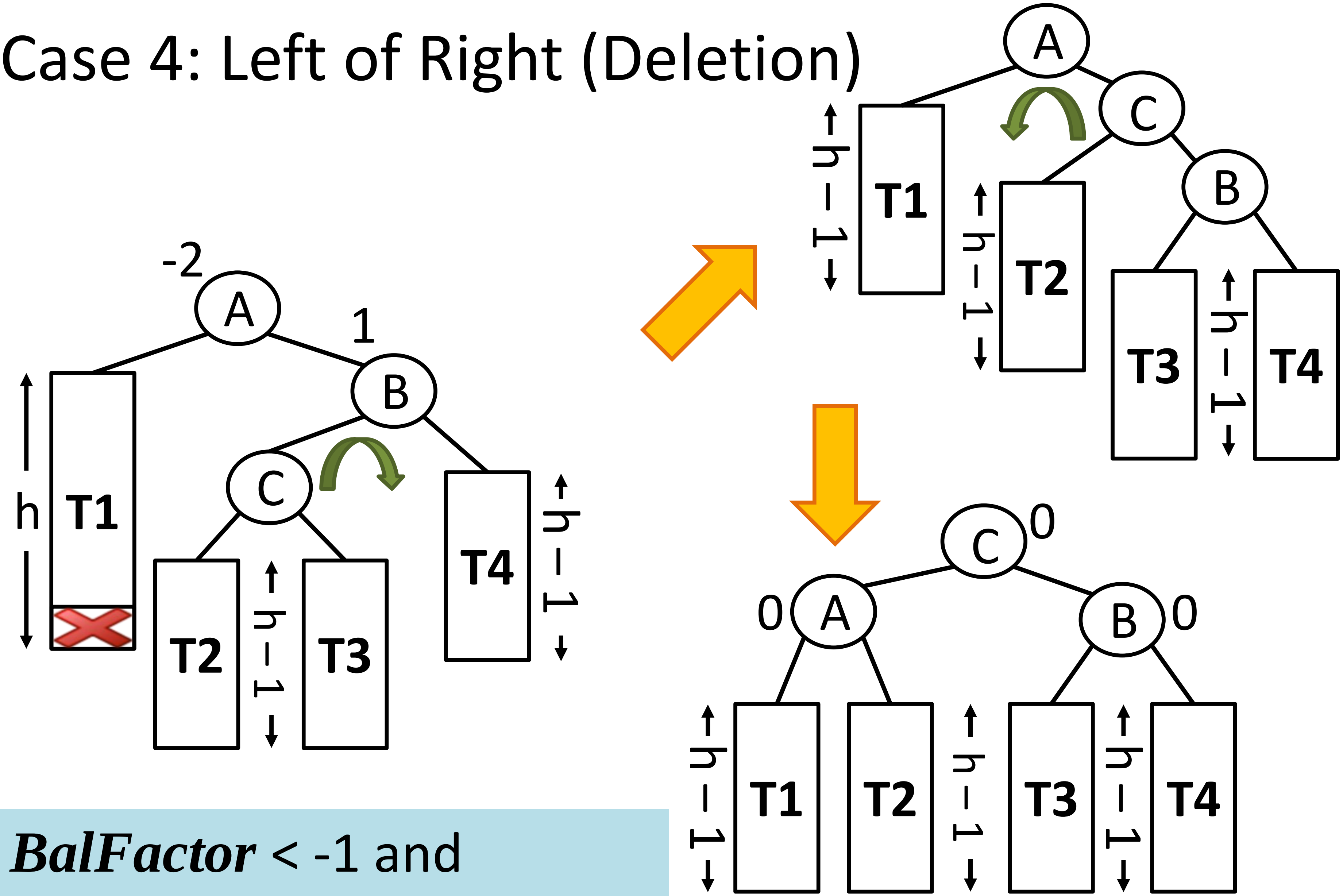


Case 3: Right of Left (Deletion)



$\boldsymbol{BalFactor} > 1$ and
$\boldsymbol{balance(Node \rightarrow left)} < 0$

Case 4: Left of Right (Deletion)



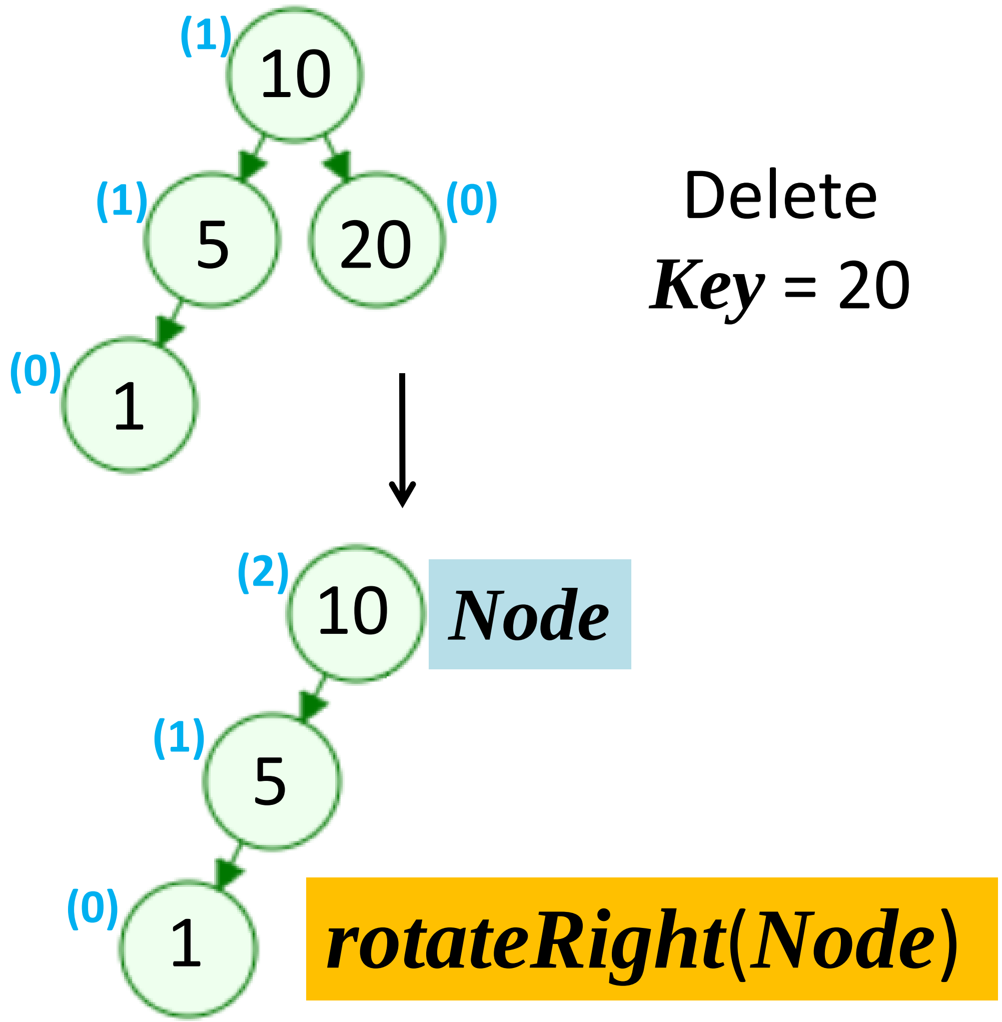
BalFactor < -1 and
***balance*(*Node*→*right*)** > 0

Deletion (recursion)

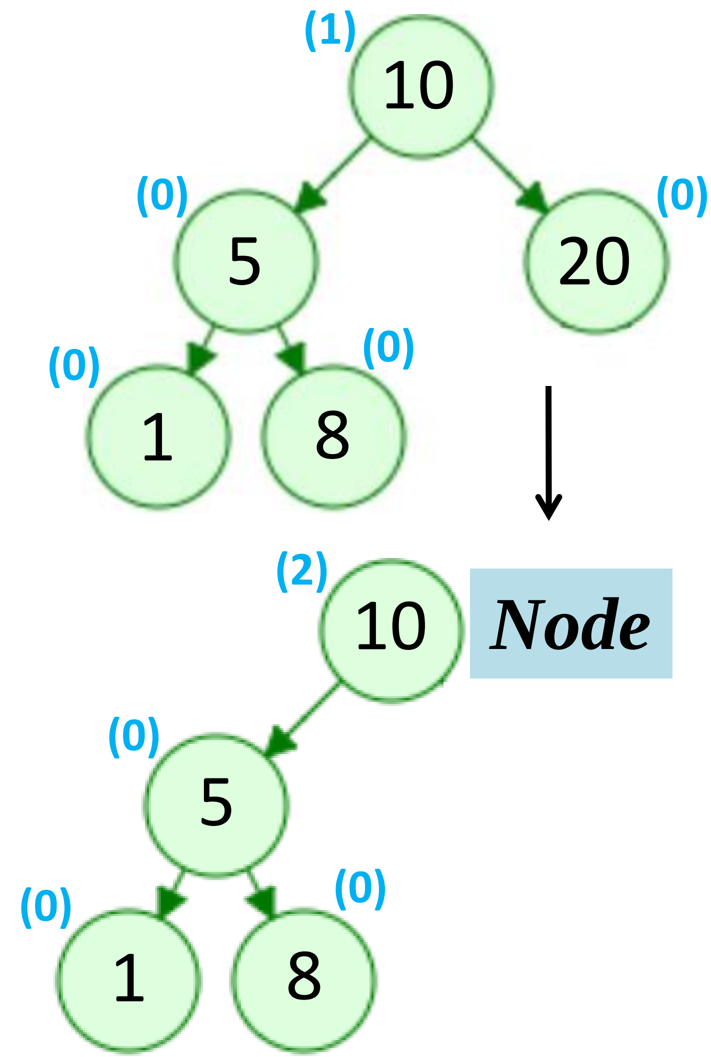
1. Delete the node with value ***Key*** using normal BST deletion.
2. Update height of the ancestor node, say ***Node***.
3. Get the balance factor of this ancestor node, i.e. ***Node***.
4. ***BalFactor*** = (height of ***Node→left*** — height of ***Node→right***).
5. If ***BalFactor*** > 1 and ***balance(Node→left)*** >= 0
6. Return ***rotateRight***(***Node***). // Left of Left.

Example – 1

BalFactor > 1 and *balance*(*Node*→*left*) >= 0



Delete
Key = 20

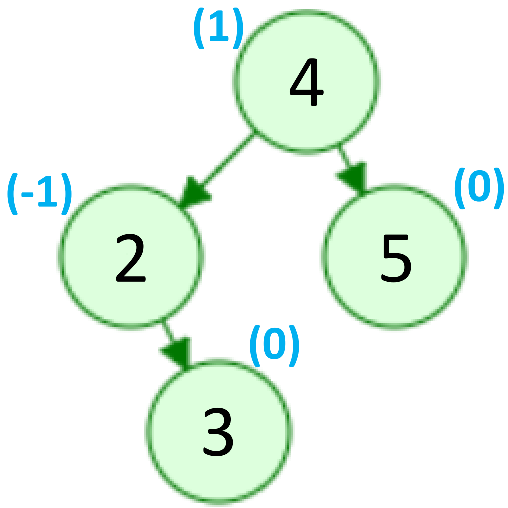


Contd...

7. If $\boldsymbol{BalFactor} > 1$ and $\boldsymbol{balance(Node \rightarrow left)} < 0$
8. $\boldsymbol{Node \rightarrow left} = \boldsymbol{rotateLeft}(\boldsymbol{Node \rightarrow left})$;
9. Return $\boldsymbol{rotateRight}(\boldsymbol{Node})$; // Right of Left.

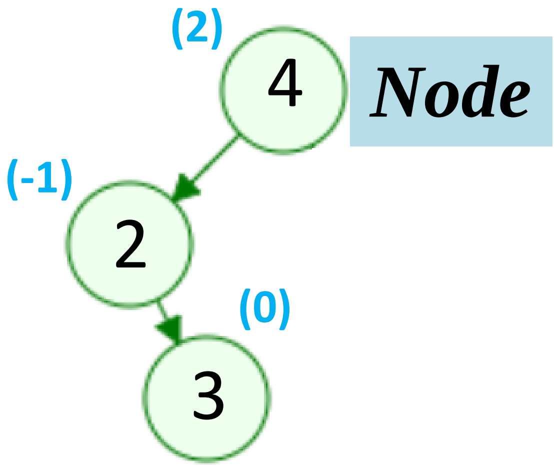
Example – 2

$\boldsymbol{BalFactor} > 1$ and
$\boldsymbol{balance(Node \rightarrow left)} < 0$



Delete ***Key*** = 5

$\longrightarrow$



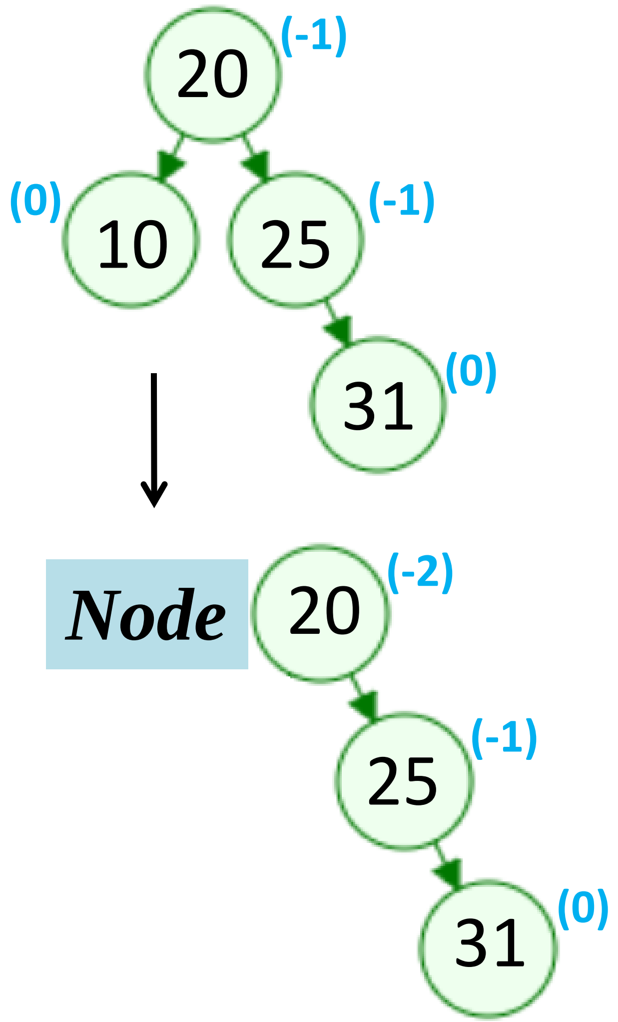
Node→left = rotateLeft(Node→left);
rotateRight(Node);

Contd...

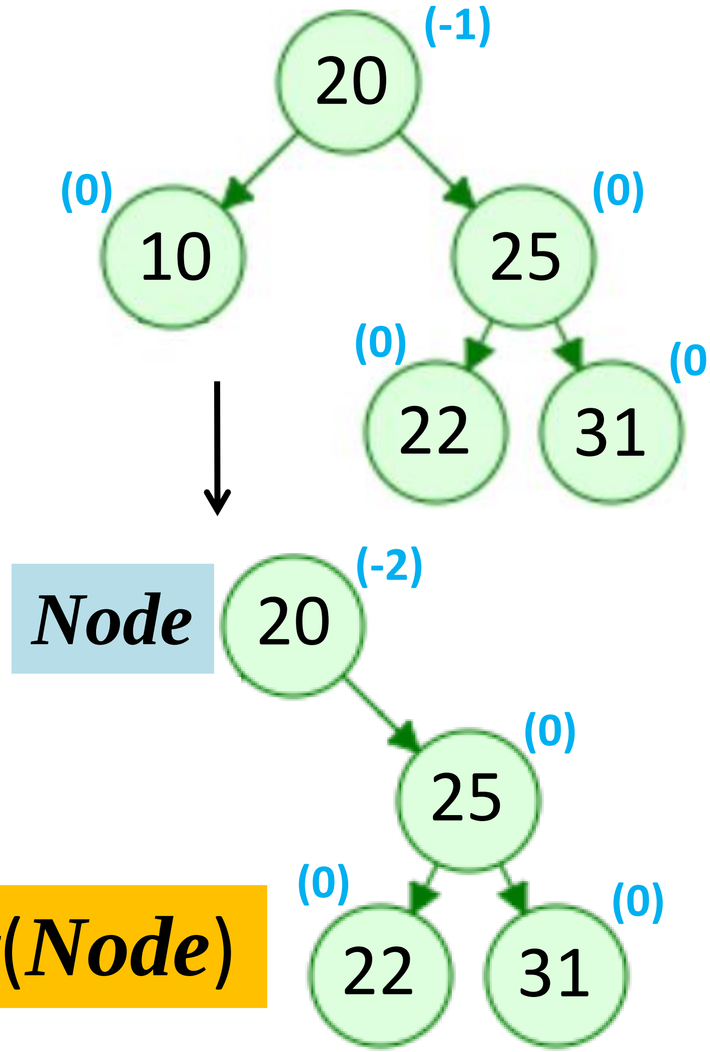
7. If $\boldsymbol{BalFactor} > 1$ and $\boldsymbol{balance(Node \rightarrow left)} < 0$
8. $\quad \boldsymbol{Node \rightarrow left} = \boldsymbol{rotateLeft}(\boldsymbol{Node \rightarrow left})$;
9. $\quad$ Return $\boldsymbol{rotateRight}(\boldsymbol{Node})$; // Right of Left.
10. If $\boldsymbol{BalFactor} < -1$ and $\boldsymbol{balance(Node \rightarrow right)} <= 0$
11. $\quad$ Return $\boldsymbol{rotateLeft}(\boldsymbol{Node})$; // Right of Right.

Example – 3

BalFactor < -1 and
***balance*(*Node*→*right*)** <= 0



Delete
Key = 10



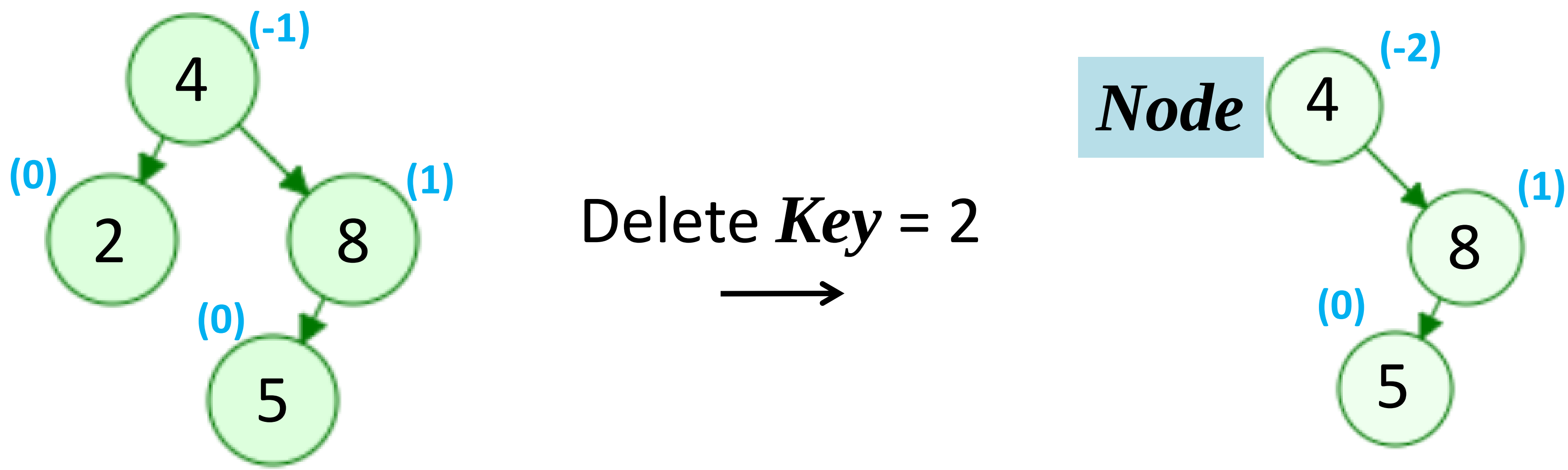
***rotateLeft*(*Node*)**

Contd...

7. If ***BalFactor*** > 1 and ***balance(Node→left)*** < 0
8. ***Node→left*** = ***rotateLeft***(***Node→left***);
9. Return ***rotateRight***(***Node***); // Right of Left.
10. If ***BalFactor*** < -1 and ***balance(Node→right)*** <= 0
11. Return ***rotateLeft***(***Node***); // Right of Right.
12. If ***BalFactor*** < -1 and ***balance(Node→right)*** > 0
13. ***Node→right*** = ***rotateRight***(***Node→right***);
14. Return ***rotateLeft***(***Node***); // Left of Right.

Example – 4

BalFactor < -1 and ***balance(Node→right)*** > 0

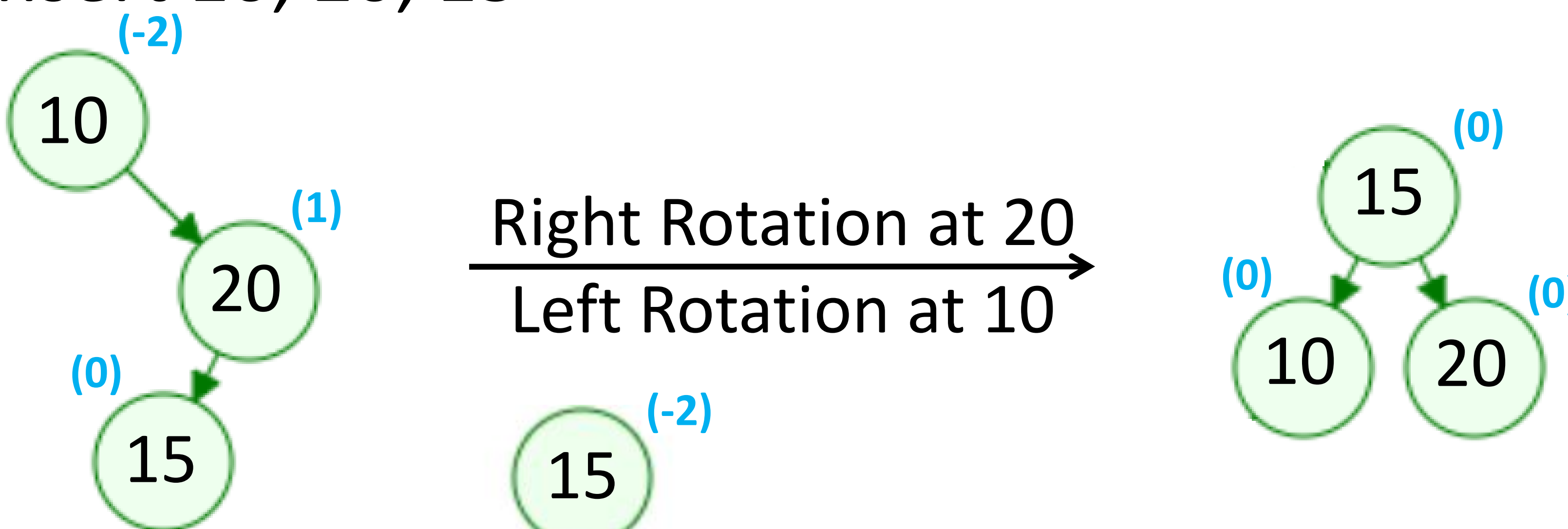


Node→right = rotateRight(Node→right);
rotateLeft(Node);

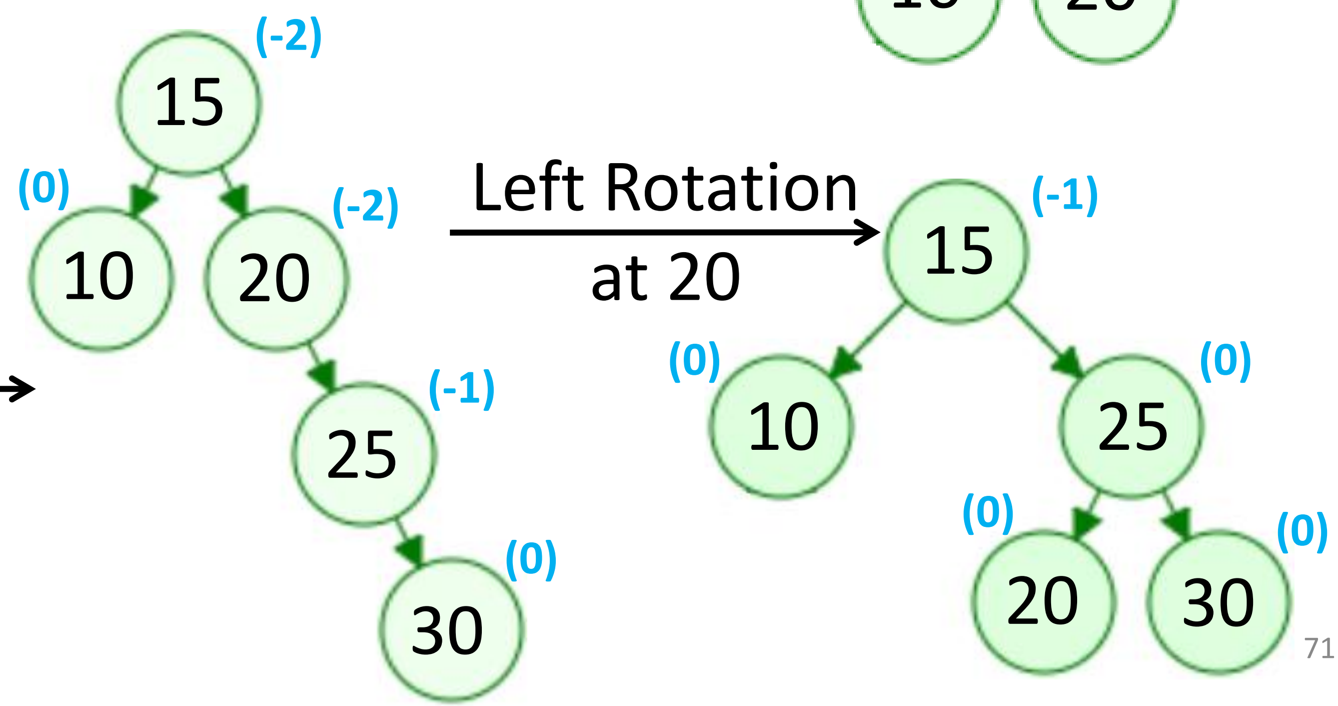
Create AVL: 10, 20, 15, 25, 30, 16, 18, 19.

Delete 30.

- Insert 10, 20, 15

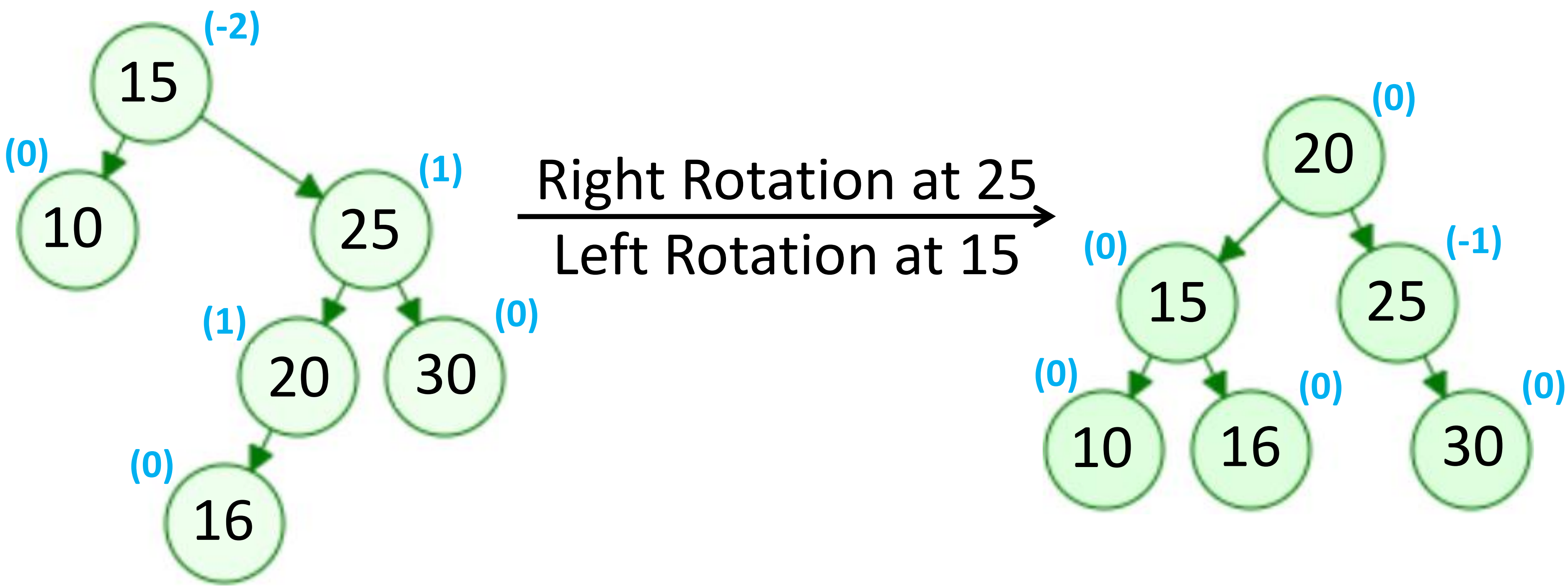


- Insert 25, 30



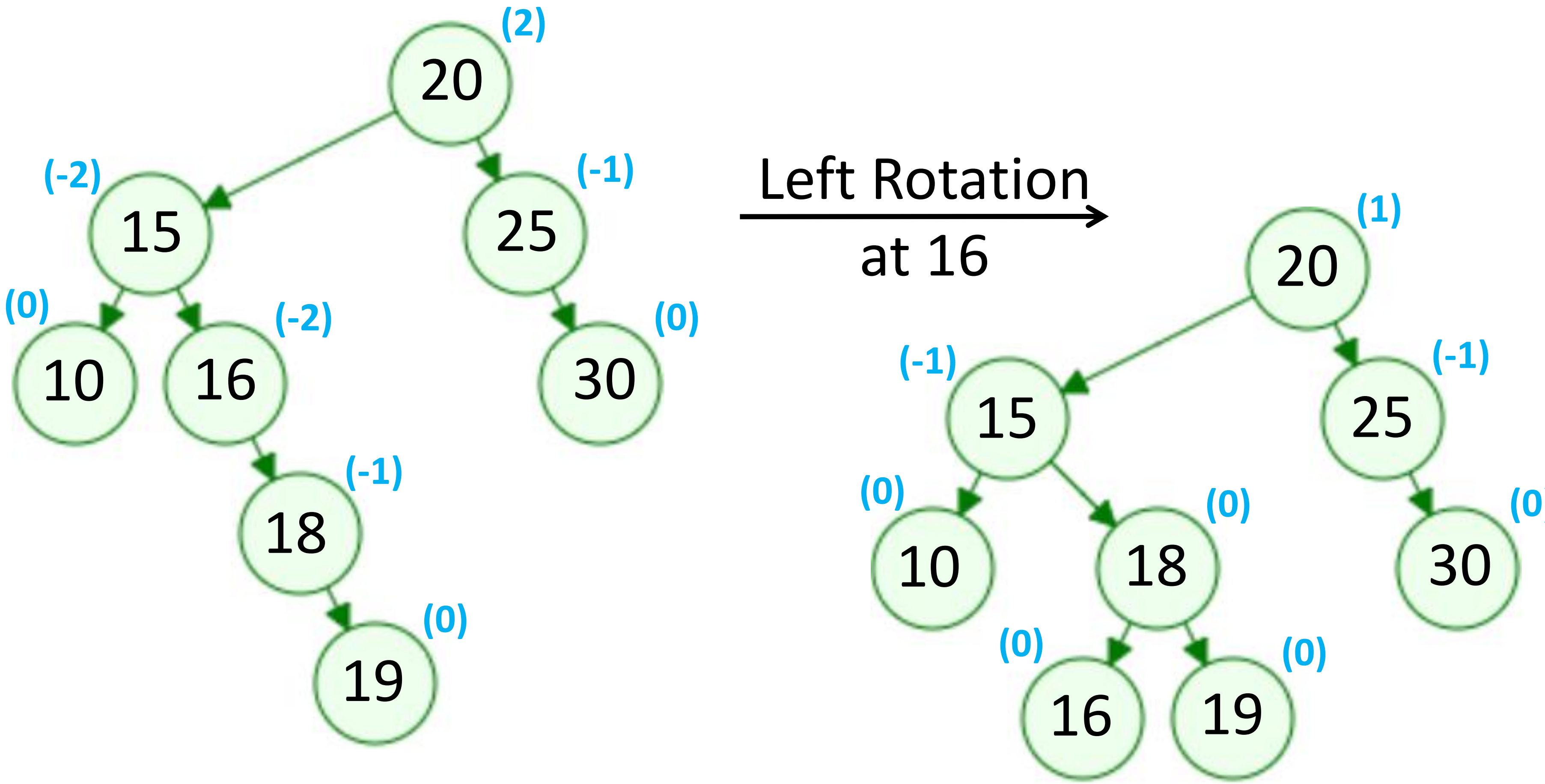
Contd…

- Insert 16



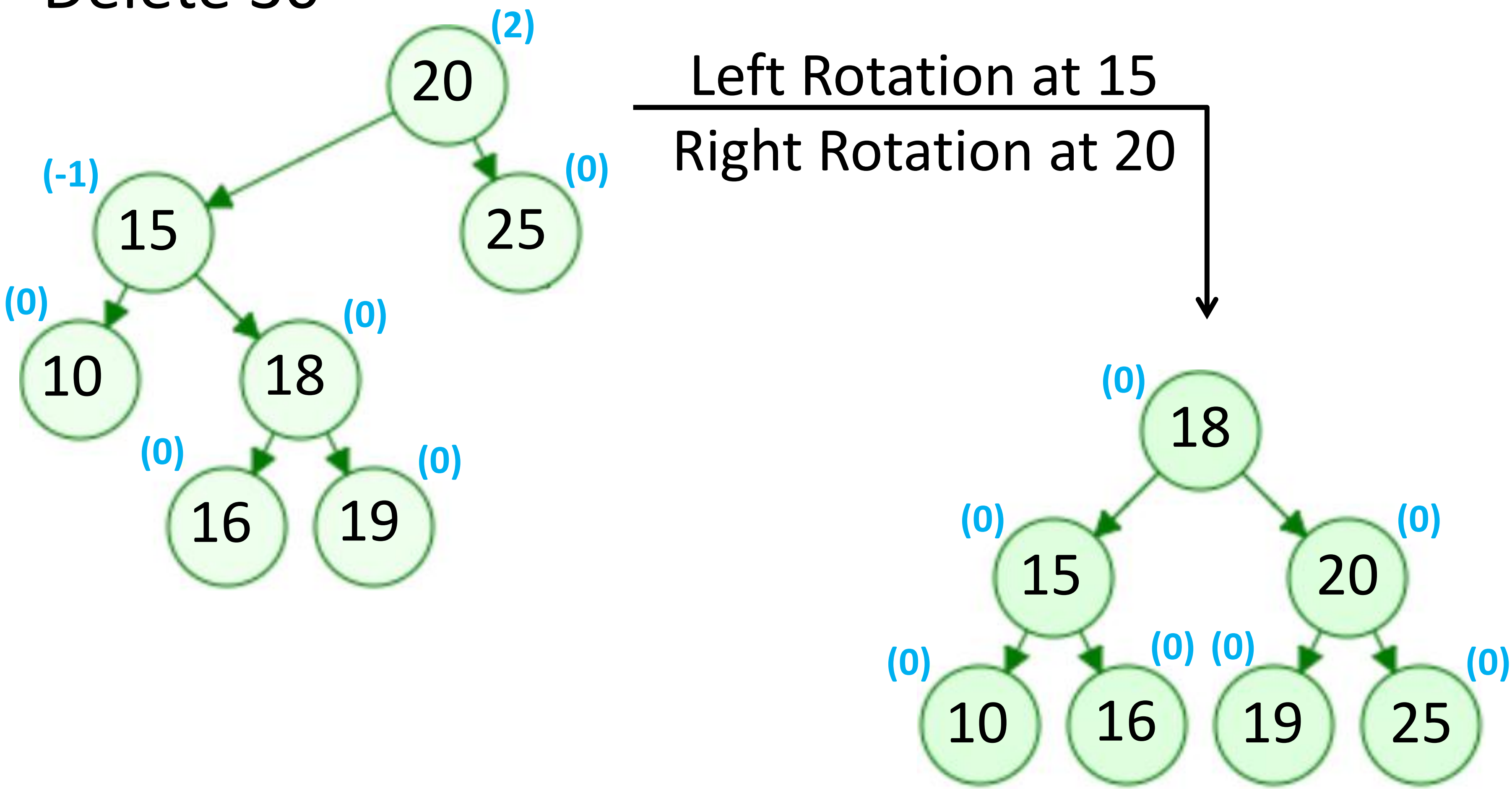
Contd…

- Insert 18, 19



Contd…

- Delete 30



Summary

- AVL trees are always balanced thus worst-case complexity of all operations (search, insert, and delete) is O(log n).
- Rotations performed for height balancing are constant time operations, but takes a little time.
- Difficult to program & debug.
- Needs space to store either a height or a balance factor.
- Suitable for applications where search or look-up is the most frequent operations as compared to insertion or deletion.