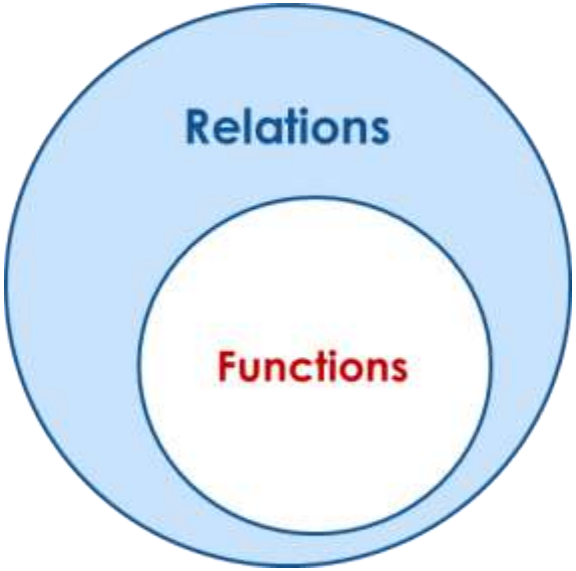


Relations

Contents

What is a Relation?

In discrete mathematics, relation is a way of showing a relationship between any two sets.

- Relationship between any program and its variable.
- Relationship between pair of cities linked by railway in a network.

Necessity for studying Relation

- Relational Database model is based on the concept of relation.

Cartesian Product

- Given two sets A and B, their ***cartesian product*** $A \times B$, is defined as

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Ordered Pairs

- The elements of $A \times B$ are called ***ordered pairs*** with the elements of A as the first entry and elements of B as the second entry.

- Order matters

Special Case:

$$A^2 = A \times A = \{(a_1, a_2) \mid a_1, a_2 \in A\}$$

Similarly,

$$A^n = A \times A \times \cdots \times A(n\, times) = \{(a_1, a_2, \dots, a_n) \mid a_1, a_2, \dots, a_n \in A\}$$

Relations

Relation is the subset of the cartesian product of the sets.

n-ary Relation

- Let $\{A_1, A_2, \dots, A_n\}$ be n sets.
- An n-ary relation R on $A_1 \times A_2 \times \dots \times A_n$ is a subset of $A_1 \times A_2 \times \dots \times A_n$.
- If $A_i = \mathrm{A}\,; \forall i$, then R is called the *n-ary relation on A*.

Empty and Universal Relation

- If $R = \emptyset$, then R is called the ***empty*** or ***void relation***.
- If $R = A_1 \times A_2 \times \ldots \times A_n$, then R is called the ***universal relation***.

Definition (Binary Relation)

- Given two sets A and B, a relation between A and B is a subset of $A \times B$.
- If R is a relation on $A \times B$ (i.e., $R \subseteq A \times B$) and $(a, b) \in R$, we say "a is related to b".
- It can also be written as aRb.

Example:

Let $\boldsymbol{A} = \{\boldsymbol{a}, \boldsymbol{b}\}$ and $\boldsymbol{B} = \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$

$\boldsymbol{R} = \{(\boldsymbol{a}, \mathbf{3}), (\boldsymbol{b}, \mathbf{2}), (\boldsymbol{b}, \mathbf{4})\}$ is a relation from A to B.

Binary Relation on a set

- A binary relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is a **subset of $\boldsymbol{A} \times \boldsymbol{A}$**.

Examples:

1. "Taller -than " is a relation on people.

   $(a, b) \in$ "Taller -than" if person a is taller than person b.

2. "$\geq$" is a relation on real set **R**.

   "$\geq$" $= \{(x, y) \in \mathbf{R} \mid x, y \in \mathbf{R}, x \geq y\}$

Examples (Cont..)

3. Let $\boldsymbol{A = \{1, 2, 3, 4, 5, 6\}}$.

If $R = \{(a, b) |\ a\ divides\ b)\}$ is a relation from A to B then ordered pairs in the relation R are

{(1,1,), (1,2), (1,3), (1,4), (1,5), (1,6), (2,4), (2,6), (3,6)}

Examples (Cont..)

Let A={1, 2, 3}

$$A \times A = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

- Here, $A \times A$ is an universal relation on A.
- $\emptyset$ is an empty relation on A.

Examples (Cont..)

“=” = $\Delta = \{(1,1), (2,2), (3,3)\}$

“<”= $\{(1,2), (1,3), (2,3)\}$

“>”= $\{(2,1), (3,1), (3,2)\}$

“≤”= $\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

“≥”= $\{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3)\}$

“|”= $\{(1,1), (1,2), (1,3), (2,2), (3,3)\}$

“multiple of”= $\{(1,1), (2,1), (2,2), (3,1), (3,3)\}$

Representing Relations

Relations can be represented in two ways:

- Matrix
- Graph

Representation of Relations as Matrix

- If R is a relation on set $A = \{a_1, a_2, \dots, a_n\}$ and $|A| = n$, then it can be represented as $n \times n$ Boolean Matrix M_R.

M_R can be defined as:

$$M_R = [m_{ij}]_{n\times n}$$

where, $m_{ij} = \begin{cases} 0 & ; if\ (a_i, a_j) \notin R \\ 1 & ; if\ (a_i, a_j) \in R \end{cases}$

Examples

- Let A={1, 2, 3}
- Let $R = \{(1,1), (1,2), (2,1), (2,3), (3,2), (3,3)\}$ be a relation on A.

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Examples (Cont..)

- Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ?$$

- Because R consists of those ordered pairs with $a_{ij} = 1$, it follows that:

$$\boldsymbol{R} = \{(\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{3}),(\mathbf{2},\mathbf{4}),(\mathbf{2},\mathbf{5}),(\mathbf{3},\mathbf{1}),(\mathbf{3},\mathbf{3}),(\mathbf{3},\mathbf{5})\}.$$

Representation of Relations as a Digraph (Directed Graph)

- The graph of a relation R over A is a directed graph with nodes corresponding to the elements of A. There is an edge from node x to y if and only if $(x, y) \in R$.
- An edge of the form (x, x) is called a self-loop.

Examples

- Let A={1, 2, 3}
- Let $R_1 = \{(1,2), (1,3), (2,3)\}$ be a $<$ relation on A.

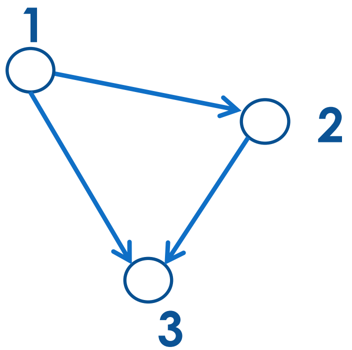


Figure 1

Examples (Cont..)

- Let $A = \{1,2,3\}$ and $B = \{a,b\}$
- Let $R_2 = \{(1, a), (1, b), (2,a), (3, b)\}$ be a relation from A to B.

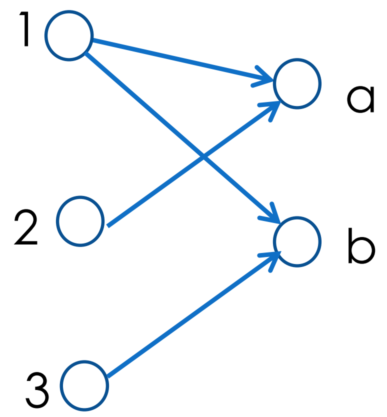


Figure 2

Domain and Range

Domain of Relation R= set of all first co-ordinates

Range of Relation R= set of all second co-ordinates

Example

"<"$= \{(1,2),(1,3),(2,3)\}$ on $A = \{1,2,3\}$

- Domain of "<"$= \{1,2\}$
- Range of "<"$= \{2,3\}$

Equality of Two relations

- Let R_1 be an n-ary relation on $A_1 \times A_2 \times \ldots \times A_n$.
- Let R_2 be an m-ary relation on $B_1 \times B_2 \times \cdots \times B_m$.

- Then, $\boldsymbol{R_1 = R_2}$
  If and only if

- **n=m**
- $\boldsymbol{A_i = B_i}$; $\forall i,\ 1 \leq i \leq n$
- and, $\boldsymbol{R_1}$ **&** $\boldsymbol{R_2}$ **are equal set of ordered pairs.**

Example

- Let $A = \{a, b\}, B = \{1, 2\}, C = \{1, 2, 3\}$

- Let $R_1 = \{(a, 1), (b, 2)\}$ is a relation on $A \times B$
- Let $R_2 = \{(a, 1), (b, 2)\}$ is a relation on $A \times C$

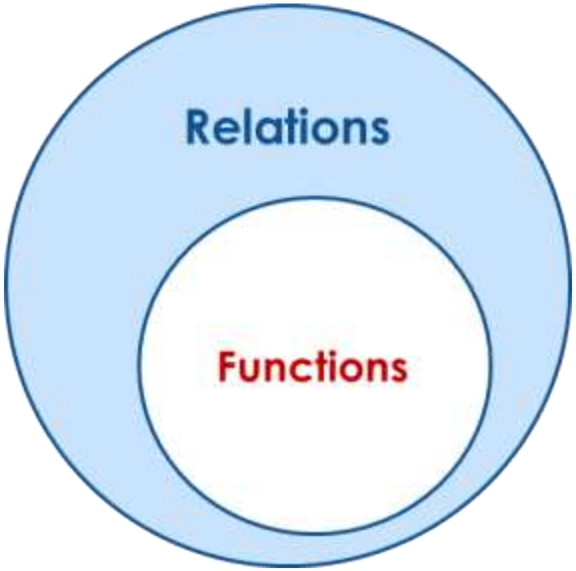
$$R_1 = R_2?$$

No

How many number of relations are there on a set A having n elements?

$$2^{n^2}$$


Thank you!!!



Relations

Contents

Properties of Relations

- Reflexive
- Symmetric
- Transitive
- Irreflexive
- Asymmetric
- Antisymmetric

Reflexive Relations

- R is ***reflexive*** iff $(x, x) \in R$ for every element $x \in A$.

Examples

1. Let A={1, 2, 3}

   Suppose $R_1 = \{(1,1), (2,2), (2,3)\}$ be a relation on A.

   Is R_1 reflexive?

   No

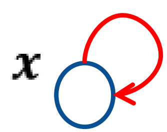
2. $=, A \times A, \leq, \geq, |, multiple\ of$ Reflexive? Yes
3. $\emptyset, <, >$ Reflexive? No

Reflexive Relation in Matrix and Graph

- If R is a **reflexive** relation, all the elements on the main diagonal of M_R are equal to 1.

$$M_R = \begin{bmatrix} 1 & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & 1 \end{bmatrix}$$

- A loop must be present at all vertices in the graph.



Symmetric Relations

- R is ***symmetric*** iff $(y, x) \in R$ whenever $(x, y) \in R$ for all $x, y \in A$.

Examples

1. Let A={1, 2, 3}

   Suppose R_2= {(1,2), (2,1), (2,3)} be a relation on A.

   Is R_1 Symmetric?

   No

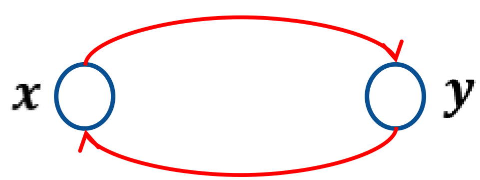
2. "sibling-of" is symmetric, but "sister-of" is not.
3. $A \times A, \emptyset, =$ Symmetric? Yes
4. $<, >, \leq, \geq, |, \textit{multiple of}$ Symmetric? No

Symmetric Relation in Matrix and Graph

- R is a symmetric relation if and only if $m_{ji} = 1$, whenever $m_{ij} = 1$.

$$M_R = \begin{bmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{bmatrix}$$

- If (x, y) is an edge in the graph, then there must be an edge (y, x) also.



Transitive Relations

- A relation R on a set A is called **transitive** if whenever $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$, for all $x, y, z \in A$.

Examples

1. Let A={1, 2, 3}

   Suppose R_3= {(1,3), (3,1)} be a relation on A.

   Is R_3 Transitive?

   No

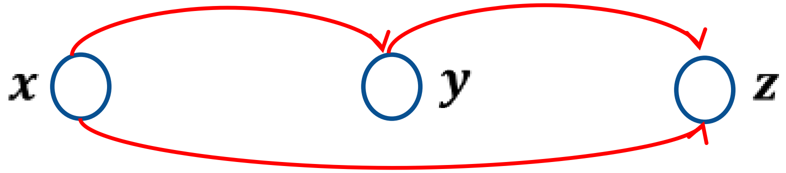
2. $A \times A,\ \emptyset, =, <, >, \leq,\ \geq,\ |,\ multiple\ of$

   Transitive?

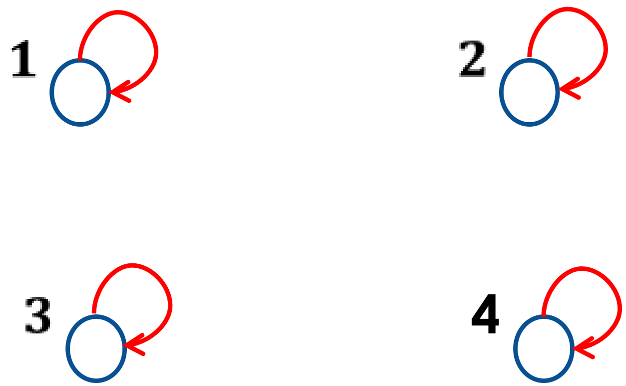
   Yes

Transitive Relations in Graph

- R is transitive iff in its graph, for any three nodes x, y and z such that there is an edge (x, y) and (y, z), there exists an edge (x, z).



Examples

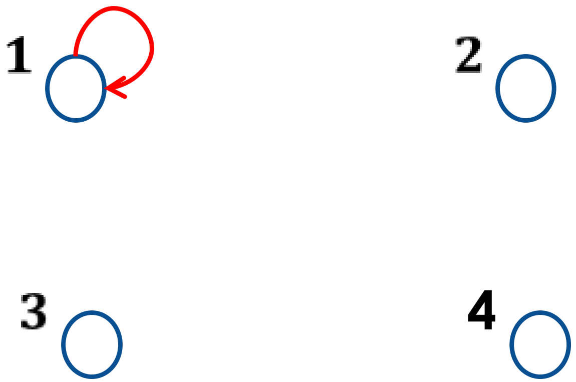


Equality Relation on A= {1, 2, 3, 4}

$$M_R = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

- Reflexive?
  Yes
- Symmetric?
  Yes
- Transitive?
  Yes

Examples (Cont..)



- Reflexive?
  No
- Symmetric?
  Yes
- Transitive?
  Yes

Examples (Cont..)

- Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- Reflexive?

  Yes
- Symmetric?

  Yes

Relations

How many number of Reflexive Relations are there on set A having n elements?

$$2^{n(n-1)}$$

Relations

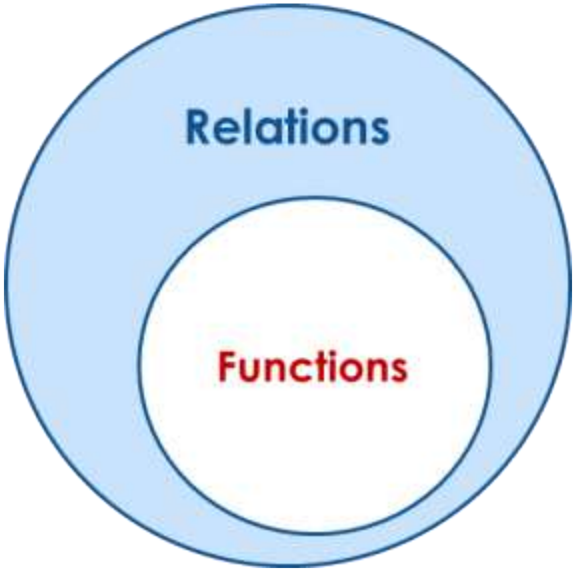
How many number of Symmetric Relations are there on set A having n elements?

$$2^{n(n+1)/2}$$

How many number of Transitive Relations are there on set A having n elements?

No closed form found


Thank you!!!
...



Relations

Properties of Relations

- **Reflexive**
- **Symmetric**
- **Transitive**
- Irreflexive
- Asymmetric
- Antisymmetric

Irreflexive Relations

- R is **irreflexive** iff $(x, x) \notin R$ for every element $x \in A$.
- No Reflexive ordered pair should belong to the relation.

Examples

1. Let A={1, 2, 3}

   Suppose $R_1 = \{(1,1), (2,2), (2,3)\}$ be a relation on A.

   Is R_1 Irreflexive?

   No

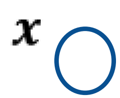
2. $\emptyset, <, >$ Irreflexive? Yes
3. $\Delta, A \times A, \leq, \geq, |, \; multiple\ of$ Irreflexive? No

Irreflexive Relation in Matrix and Graph

- If R is an irreflexive relation, all the elements on the main diagonal of M_R are equal to 0.

$$M_R = \begin{bmatrix} 0 & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & 0 \end{bmatrix}$$

- No vertex should contain self-loop in the graph.



Asymmetric Relations

- A relation R on a set A such that for all $x, y \in A$, if $(x, y) \in R$ then $(y, x) \notin R$, is called ***asymmetric.***

Examples

Let A={1, 2, 3}

1. Suppose $R_2 = \{(1,2)\}$ be a relation on A.

Is R_2 Asymmetric?

Yes

2. Suppose $R_3 = \{(1, 3), (3,1), (2,3)\}$ be another relation on A.

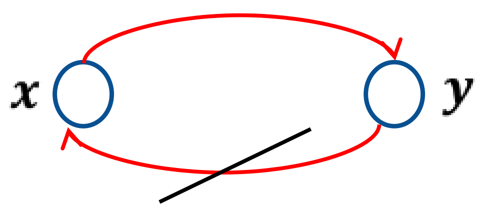
Is R_3 Asymmetric?

No

3. $\emptyset, <, >$ Asymmetric? Yes
4. $A \times A, \leq, \geq, |, \textit{multiple of}$ Asymmetric? No

Asymmetric Relations in Graph

- If (x, y) with $x \neq y$ is an edge, then (y, x) is not an edge.
- There must also be no self loop.



Antisymmetric Relations

- A relation R on a set A such that for all $x, y \in A$, if $(x, y) \in R$ and if $(y, x) \in R$, then $x = y$, is called ***antisymmetric.***

If $x \neq y$ and if (x, y) is present, then (y, x) should not be present there.

Examples

1. Let A={1, 2, 3}

   Suppose R_1= {(1,2), (2,1), (2,3)} be a relation on A.

   Is R_1 Antisymmetric?

   No

2. $\emptyset, \Delta, <, >, \leq, \geq, |, \textit{multiple of}$ Antisymmetric? Yes

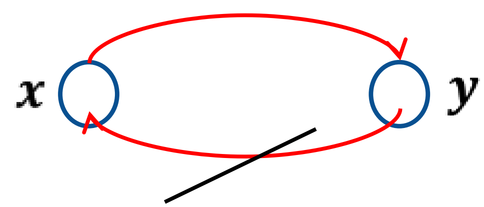
3. $A \times A$ Antisymmetric? No

Antisymmetric Relation in Matrix and Graph

- R is a antisymmetric relation if and only if $m_{ji} = 0$, or $m_{ij} = 0$, when $i \neq j$.

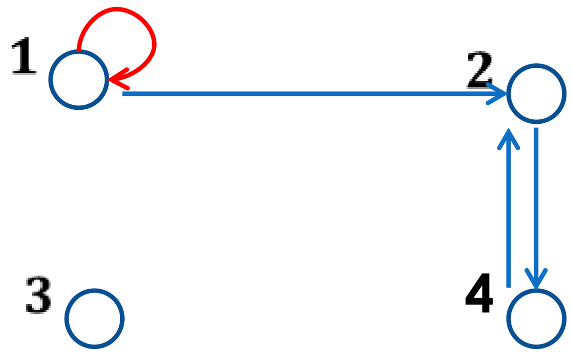
$$M_R = \begin{bmatrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{bmatrix}$$

- If (x, y) with $x \neq y$ is an edge, then (y, x) is not an edge.



Self-loops can be there.

Example



- **Reflexive?**
  No
- **Symmetric?**
  No
- **Transitive?**
  No
- **Irreflexive?**
  No
- **Antisymmetric?**
  No
- **Asymmetric ?**
  No

Some Points to remember

- There can be a relation which is neither reflexive nor irreflexive.

Example

1. Let A={1, 2, 3}

   Suppose R_3= {(1,1), (2,2), (2,3)} be a relation on A.

Neither Reflexive nor Irreflexive

Some Points to remember (Cont..)

- There can be a relation which is both symmetric and antisymmetric.

Example:

1. Let A = {1, 2, 3}

   Suppose R_4= {(1,1), (2,2), (3,3)} be a relation on A.

   both symmetric and antisymmetric

Some Points to remember (Cont..)

- There can be a relation which is neither symmetric nor antisymmetric.

Example

Let A={1, 2, 3}

Suppose R_5= {(1,2), (2,3), (3,2)} be a relation on A.

Neither Symmetric nor Antisymmetric

Some Points to remember (Cont..)

- Every asymmetric relation is antisymmetric but every antisymmetric relation need not be asymmetric.

Example:

Let A = {1, 2, 3}

1. Suppose R_6= {(1,2)} be a relation on A.

   both asymmetric and antisymmetric

2. Suppose R_7= {(1,1), (1,2)} be a relation on A.

   Antisymmetric but not asymmetric

Relations

How many number of Irreflexive Relations are there on set A having n elements?

$$2^{n(n-1)}$$

How many number of Asymmetric Relations are there on set A having n elements?

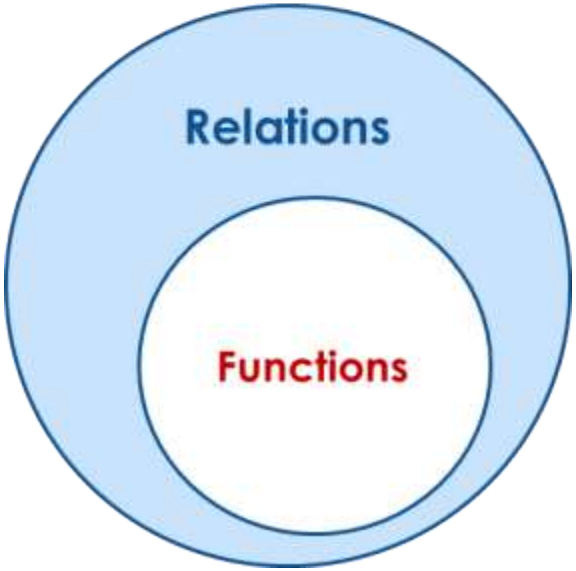
$$3^{n(n-1)/2}$$

Relations

How many number of Antisymmetric Relations are there on set A having n elements?

$$2^n 3^{n(n-1)/2}$$


Thank you!!!
...



Relations

Contents

Inverse Relation

- If $R \subseteq \mathrm{A} \times B$ then $R^{-1} \subseteq \mathrm{B} \times A$, and is defined as:

$$R^{-1} = \{(\mathrm{b}, \mathrm{a}) | (\mathrm{a}, \mathrm{b}) \in R\}$$

R	R^{-1}
$<$	$>$
$\leq$	$\geq$
divides	*multiple of*
subset	*superset*

Example

- Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$
  Suppose $R = \{(1,3), (1,5), (2,4), (3,5)\}$

$$R^{-1} = \{(3,1), (5,1), (4,2), (5,3)\}$$

Complementary Relations

- Let R be a relation from A to B, then complementary relation R^C is defined as:

$$R^C = \{(a,b) | (a,b) \notin R \; and \; (a,b) \in A \times B\}$$

Example

- Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$

  Suppose $R = \{(1,3), (1,5), (2,4), (3,5)\}$

$$A \times B = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5))\}$$

$$R^C = \{(1,4), (2,3), (3,3), (3,4), (2,5)\}$$

Combining Relation

- Given two relations R_1 and R_2, these can be combined by using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

- $R_1 \cup R_2 = \{(a,b) |\ (a,b) \in R_1 or\ (a,b) \in R_2\ or\ both\}$
- $R_1 \cap R_2 = \{(a,b) |\ (a,b) \in R_1\ and\ (a,b) \in R_2\}$
- $R_1 - R_2 = \{(a,b) |\ (a,b) \in R_1\ and\ (a,b) \notin R_2\}$
- $R_2 - R_1 = \{(a,b) |\ (a,b) \in R_2\ and\ (a,b) \notin R_1\}$

Example

- Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$
  Suppose $R = \{(1,3), (1,5), (2,4), (3,5)\}$,
  $R_1 = \{(1,4), (2,3), (2,5), (3,3), (3,5)\}$ and
  $R_2 = \{(1,3), (1,4), (2,3), (3,4), (3,5)\}$

- $R_1 \cup R_2 = \{(1,3), (1,4), (2,3), (2,5), (3,3), (3,4), (3,5)\}$
- $R_1 \cap R_2 = \{(1,4), (2,3), (3,5)\}$
- $R_1 - R_2 = \{(2,5), (3,3)\}$
- $R_2 - R_1 = \{(1,3), (3,4)\}$

Results

- Let R, R_1 and R_2 be relations on A.

R, R_1 and R_2 are	R^{-1}	$R_1 \cap R_2$	$R_1 \cup R_2$
Reflexive	Yes	Yes	Yes
Irreflexive	Yes	Yes	Yes
Symmetric	Yes	Yes	Yes
Asymmetric	Yes	Yes	Need not be, but cannot be assured
Antisymmetric	Yes	Yes	Need not be, but cannot be assured
Transitive	Yes	Yes	Need not be, but cannot be assured

Examples

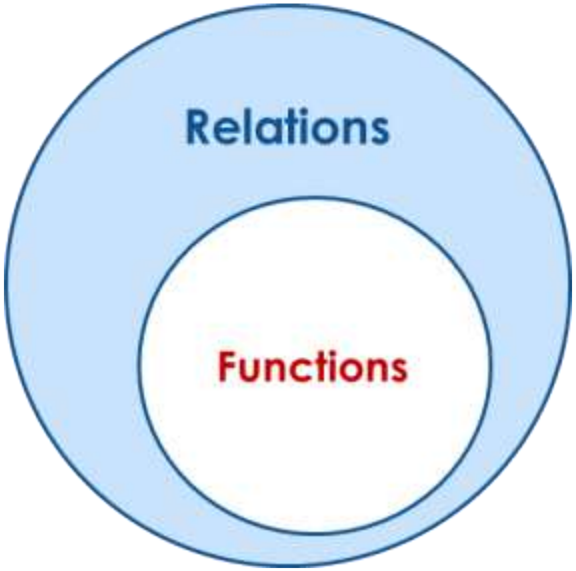
$A = \{1,2,3\}$

$R_1 = \{(1,2)\}$

$R_2 = \{(2,1)\}$

$R_1 \cup R_2 = \{(1,2), (2,1)\}$


Thank you!!!
...



Relations

Contents

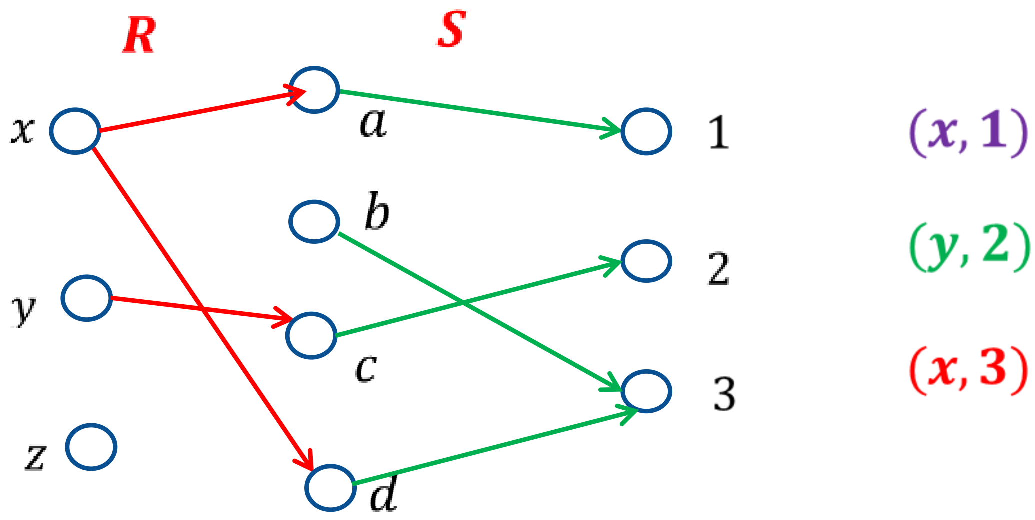
Composition of Relations

- If $R \subseteq \mathrm{A} \times B$ and $S \subseteq B \times C$ are two relations, then the composition (or composite) of S with R is a relation from A to C and is defined as:

$$SoR = \{(a,c) \mid \exists b \in B \text{ such that } (a,b) \in R \text{ and } (b,c) \in S\}$$

Representing the Composition of Relations

- Let $A = \{x, y, z\}$, $B = \{a, b, c, d\}$ and $C = \{1, 2, 3\}$.
- Suppose $R = \{(x, a), (x, d), (y, c)\}$ be a relation from A *to* B.
- Suppose $S = \{(a, 1), (b, 3), (c, 2), (d, 3)\}$ be a relation from B *to* C.



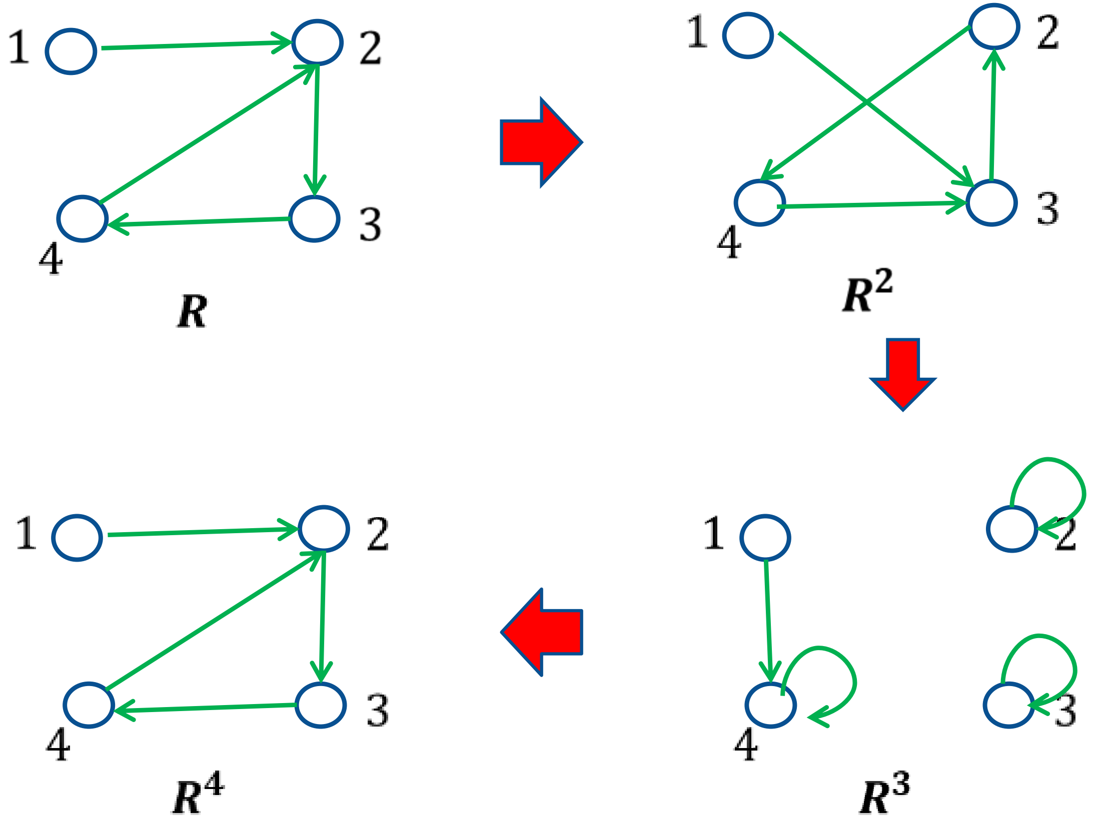
$\boldsymbol{SoR = \{(x, 1), (y, 2), (x, 3)\}}$

Power of Relations

- If $R \subseteq A \times A$, then

$$R^2 = RoR$$
$$R^3 = R^2 oR$$
$$\vdots$$
$$R^n = R^{n-1} oR$$

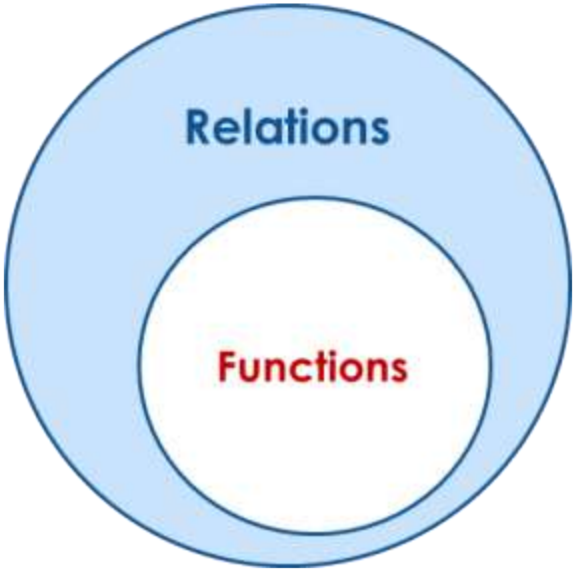
Example



R = {1, 2, 3, 4}
R ⊆ A X A

The pair (a, b) is in R^n if there is a path of length n from a to b in R.


Thank you!!!
...



Relations

Contents

Equivalence Relation

- Let R be a relation on set A, then R is called equivalence relation if it is:

- Reflexive
- Symmetric
- Transitive

Examples

- Let $A = \{1, 2, 3\}$

1. $\emptyset$ $i.e.$ **Empty Relation on A**

Reflexive?

Symmetric?

Transitive?

Not an Equivalence Relation

2. $\Delta = \{(1,1), (2,2), (3,3)\}$

Reflexive?

Symmetric?

Transitive?

Equivalence Relation on A

Smallest Equivalence Relation on A

Examples (Cont..)

- Let $A = \{1, 2, 3\}$

3. Universal Relation on A i.e. $A \times A$

Reflexive?

Symmetric?

Transitive?

Equivalence Relation

Largest Equivalence Relation on A

4. Let $A = \{1, 2, 3, 4\}$

$$\boldsymbol{R} = \{(\mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{2}), (\mathbf{2}, \mathbf{1}), (\mathbf{2}, \mathbf{2}), (\mathbf{3}, \mathbf{3}), (\mathbf{3}, \mathbf{4}), (\mathbf{4}, \mathbf{3}), (\mathbf{4}, \mathbf{4})\}$$

Reflexive?

Symmetric?

Transitive?

Equivalence Relation on A

Relations

If R_1 *and* R_2 are two equivalence relations on A, then which of the following is always true?

I. $R_1 \cap R_2$ is an Equivalence Relation.

II. $R_1 \cup R_2$ is an Equivalence Relation.

(a) Only I

(b) Only II

(c) Both are true

(d) Both are false

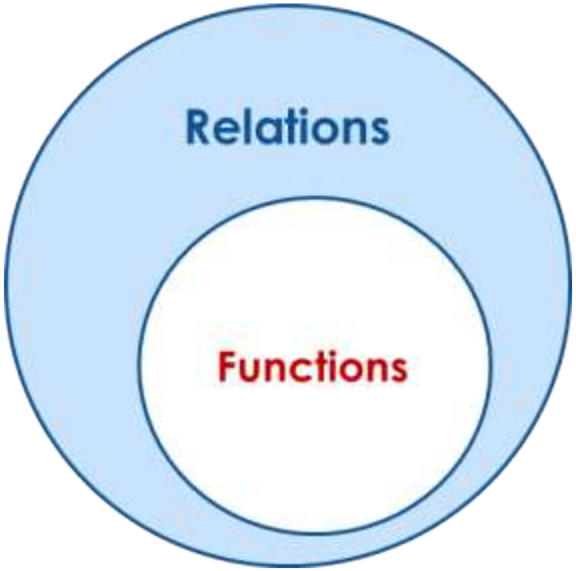
Exercise

- Let R be a relation defined on *set of integers* as:

$$xRy \text{ iff } x + y \ is \ even$$

Is R an equivalence relation?


Thank you!!!



Relations

Contents

Equivalence Class

- Let R be an equivalence relation on A, and $a \in A$.
- The equivalence class of a, denoted as $[a]$ or $\overline{a}$, is defined as:

$$\overline{a} = [a] = \{b \in A | (a, b) \in R\}$$

Examples

- Let $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$ on $A = \{1, 2, 3, 4\}$

First Check whether R is an equivalence relation on A or not.

Reflexive?

Symmetric?

Transitive?

Equivalence Classes:-

$[1] = \{1, 2\}$
$[2] = \{1, 2\}$
$[3] = \{3, 4\}$
$[4] = \{3, 4\}$

Examples

- Let $R = \{(1,1), (2,2), (3,3), (4,4)\}$ on $A = \{1, 2, 3, 4\}$

Equivalence Classes:-

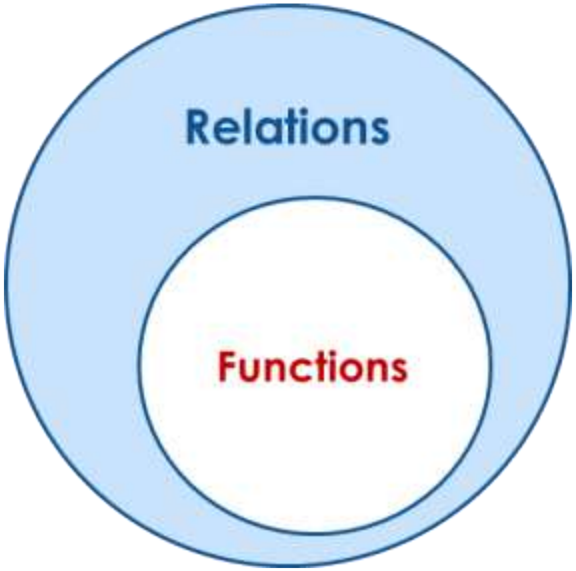
$[1] = \{1\}$
$[2] = \{2\}$
$[3] = \{3\}$
$[4] = \{4\}$

Properties

- Let R be an equivalence relation on A.

1. $a \in [a]$
2. If $b \in [a]$ then $a \in [b]$
3. If $b \in [a]$ then $[a] = [b]$
4. $[a] = [b]$ or $[a] \cap [b] = \emptyset$


Thank you!!!



Relations

Contents

Equivalence Relation to Partition

- Let $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$ on $A = \{1, 2, 3, 4\}$.
- R is an equivalence relation on A.

Equivalence Classes:-

$[1] = \{1, 2\} = [2]$

$[3] = \{3, 4\} = [4]$

- Partition $P = \{\{1, 2\}, \{3, 4\}\}$

Partition to Equivalence Relation

- Let $A = \{1, 2, 3, 4\}$ be a set and $P = \{\{1, 3\}, \{2, 4\}\}$ be a partition on A.
- Find Equivalence relation on A.

The parts of partition are distinct equivalence classes.

$\{1, 3\} \rightarrow \{(1, 1), (1,3), (3,1), (3,3)\}$
$\{2, 4\} \rightarrow \{(2, 2), (2,4), (4,2), (4,4)\}$

Therefore, the equivalence relation on A is:

- $\boldsymbol{R = \{(1, 1), (1, 3), (3, 1), (3, 3), (2, 2), (2, 4), (4, 2), (4, 4)\}}$

Result

- There is a one-to-one correspondence between partitions of A and Equivalence Relation on A.

- Therefore, if $|A| = n$, then

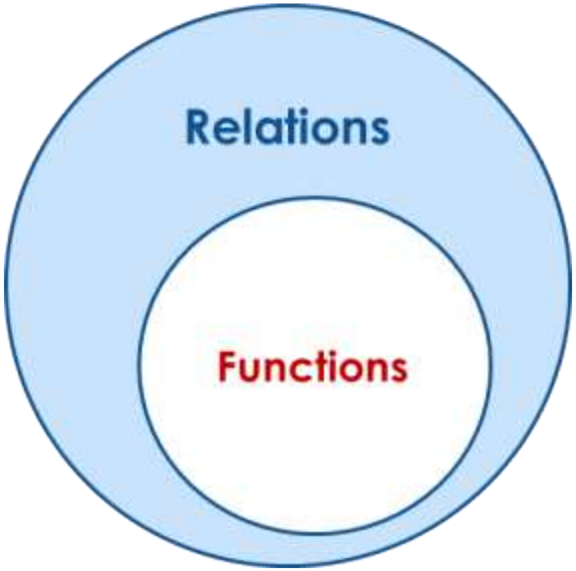
Number of Partitions of A= Number of Equivalence Relations on A
$= B_n$ (Bell Number)

Bell Number:

$$B_n = \sum_{k=0}^{n-1} n-1_{C_k} B_k$$

where, $B_0 = 1$


Thank you!!!
...



Relations

Contents

Closure of Relations

- Reflexive Closure
- Symmetric Closure
- Transitive Closure

Reflexive Closure

- A relation is called **reflexive closure R_r** of relation R if:

1) It is reflexive.
2) It contains R.
3) It is the minimal relation satisfying conditions (1) and (2).

Examples

1. Let A={1, 2, 3}

Suppose $R_1 = \{(1,1), (2,2), (2,3)\}$ be a relation on A.

$$\boldsymbol{R_r = \{(1,1), (2,2), (2,3), (3,3)\}}$$

Examples (Cont..)

2. R is a relation defined on set of positive integers such that aRb if $a < b$.

Reflexive Closure?

Result:

- $\boldsymbol{R_r = R \cup \Delta}$
- $\boldsymbol{R_r = R}$ **iff $\boldsymbol{R}$ is Reflexive.**

Symmetric Closure

- A relation is called **symmetric closure R_s** of relation R if:

1) It is symmetric.
2) It contains R.
3) It is the minimal relation satisfying conditions (1) and (2).

Examples

- Let A={1, 2, 3}

  Suppose $R_1 = \{(1,1), (2,2), (2,3)\}$ be a relation on A.

$$\boldsymbol{R_s = \{((1,1),(2,2),(2,3),(3,2))\}}$$

Result

- $R_s = R \cup R^{-1}$
- $R_r = R$ iff R is Symmetric.

Transitive Closure

- A relation is called Transitive closure R^* of relation R if:

1) It is transitive.
2) It contains R.
3) It is the minimal relation satisfying conditions (1) and (2).

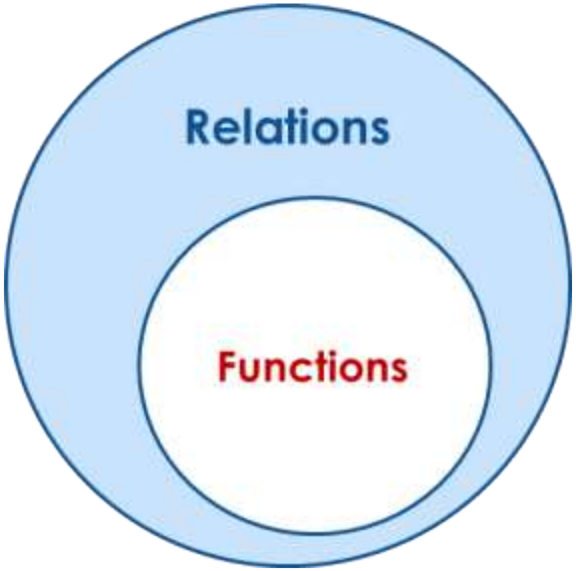
Result

1. Let $|A| = n$,

   $then, R^* = R^1 \cup R^2 \cup \cdots \cup R^n$

2. $R\ is\ transitive\ iff\ R^* = R$


Thank you!!!
...



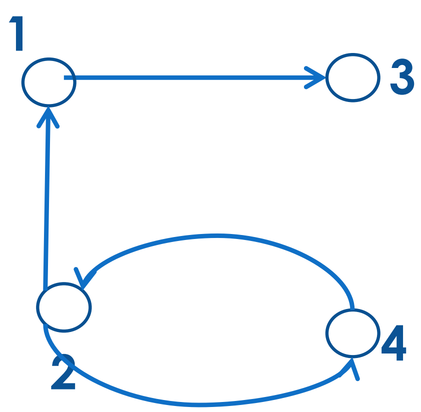
Relations

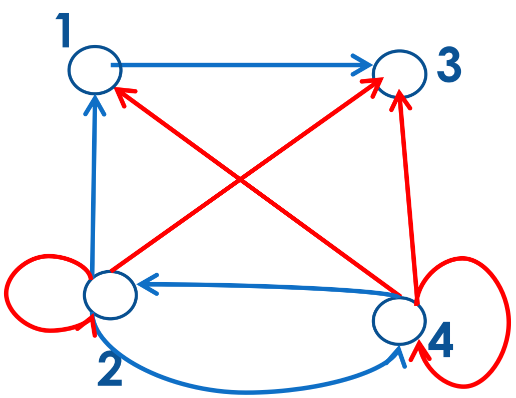
Contents

Warshall's Algorithm

- Computes the transitive closure of a relation

Example of transitive closure:





Warshall's Algorithm (Cont..)

- Main concept: a path exists between two vertices i, j, iff
  - there is an edge from i to j; or
  - there is a path from i to j going through vertex 1; or
  - there is a path from i to j going through vertex 1 and/or 2; or
  - there is a path from i to j going through vertex 1, 2, and/or 3; or
  - ...
  - there is a path from i to j going through any of the other vertices

Warshall's Algorithm (Cont..)

- On the k^{th} iteration, the algorithm determine if a path exists between two vertices i, j using vertices among $1, \ldots, k$ allowed as intermediate

$$\boldsymbol{W^{(k)}[i,j]} = \begin{cases} \boldsymbol{W^{(k-1)}[i,j]} \\ \boldsymbol{or} \\ \left(\boldsymbol{W^{(k-1)}[i,k]}\right) \boldsymbol{and} \left(\boldsymbol{W^{(k-1)}[k,j]}\right) \end{cases}$$

Warshall's Algorithm (Cont..)

- Recurrence relating elements $W^{(k)}$ to elements of $W^{(k-1)}$ is:

  $$\boldsymbol{W^{(k)}[i,j] = W^{(k-1)}[i,j]} \text{ or } \left(\boldsymbol{W^{(k-1)}[i,k] \; and \; W^{(k-1)}[k,j]}\right)$$

- It implies the following rules for generating $W^{(k)}$ from $W^{(k-1)}$ is:

1. If an element in row i and column j is 1 in $W^{(k-1)}$, it remains 1 in $W^{(k)}$.

2. If an element in row i and column j is 0 in $W^{(k-1)}$, it has to be changed to 1 in $W^{(k)}$ if and only if the element in its row i and column k and the element in its row k and column j are both $1's$ in $W^{(k-1)}$.

Warshall's Algorithm (Cont..)

- The procedure for computing $W^{(k)}$ from $W^{(k-1)}$ is as follows:

1. First transfer all $1's$ in $W^{(k-1)}$ to $W^{(k)}$.

2. List the locations $p_1, p_2, \dots,$ in column k of $W^{(k-1)}$, where the entry is 1, and the locations $q_1, q_2, \dots,$ in row k of $W^{(k-1)}$, where the entry is 1.

3. Put $1's$ in all the positions (p_i, q_i) of $W^{(k)}$ (if they are not already there).

Warshall's Algorithm (Cont..)

$$W^{(k-1)} = \begin{array}{cc} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} i \\ k \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & \end{bmatrix} \end{array} \Rightarrow W^{(k)} = \begin{array}{cc} & \begin{array}{cc} j & k \end{array} \\ \begin{array}{c} i \\ k \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & \end{bmatrix} \end{array}$$

Figure 1: Step for Changing zeros in Warshall's Algorithm

Example

1. Find transitive closure of relation R represented by following matrix (using Warshall's algorithm):

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Solution 1:

$$W^{(0)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

$$W^{(1)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

$$W^{(2)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

$$W^{(3)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

Relations

$$W^{(4)} = W^{(3)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array}$$

$$W^{(4)} = W^{(3)} = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Answer is :

$$M_{R^*} = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$R^* = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,4)\}$$


Thank you!!!
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