Objectives

- Matrix multiplication
- Matrix chain multiplication
- Deriving formula using DP and How to use the formula
- Example of matrix chain multiplication

Matrix Multiplication:

a. Column of first matrix is equal to the row of second matrix
b. Total number of multiplication required in matrix multiplication

Example

$$\begin{bmatrix} 2 & 4 & 3 \\ 1 & 5 & 6 \end{bmatrix}_{2\times 3} * \begin{bmatrix} 2 & 3 \\ 5 & 6 \\ 3 & 7 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 33 & 51 \\ 45 & 75 \end{bmatrix}_{2\times 2}$$

$$Total\ \ no.\ of\ \ multiplications = 12$$

$$2\times 2 + 4\times 5 + 3\times 3 = 3\ multiplications$$

$$Thus\,total\ \ 3\times 4 = 12\ multiplications$$

Problem statement:

Given a sequence of matrices, find the most efficient way to multiply these matrices together:

- No need to actually perform the multiplications
- Decide the order

$Eg.$

$A_1A_2A_3A_4$

$(A_1)(A_2A_3A_4)$

$(A_1A_2A_3)(A_4)$

$(A_1)(A_2A_3)(A_4)$

$(A_1(A_2(A_3A_4)))$

Objective is to find out the minimum cost of multiplication that is the ordering that gives the minimum cost.

Cost of multiplication:

Example

$$\begin{bmatrix} 2 & 4 & 3 \\ 1 & 5 & 6 \end{bmatrix}_{2\times 3} * \begin{bmatrix} 2 & 3 \\ 5 & 6 \\ 3 & 7 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 33 & 51 \\ 45 & 75 \end{bmatrix}_{2\times 2}$$

$$Total\ no.\ of\ multiplications = 12$$

$$2\times 2 + 4\times 5 + 3\times 3 = 3\ multiplications$$

$$Thus\ total\ 3\times 4 = 12\ multiplications$$

The input which is given in the following format:

$$M[5] = \{40,20,30,10,30\}$$

Which represents 4 matrices having dimension as?

$$A_1 = 40\times 20,\ A_2 = 20\times 30,\ A_3 = 30\times 10,\ A_4 = 10\times 30,$$

That is matrix A_i $has\ \dim ension = M[i-1] * M[i]$

Naïve solution is:

1. Place parentheses at all possible places
2. Calculate the cost of each placement
3. Return the minimum value

Eg.

$A_1 A_2 A_3 A_4$

$(A_1)(A_2 A_3 A_4)$

$(A_1 A_2)(A_3 (A_4)$

$(A_1 A_2 A_3)(A_4)$

That is the problem has optimal substructure property.

Also this problem has overlapping sub problem property. So matrix chain multiplication problem can be simplified using dynamic programming.

Consider the matrix chain sequence as:

A_1	A_2	A_3	A_4
3×2	2×4	4×2	2×5

$$M[i,j] = \min_{i \le k < j} \begin{cases} M[i,k] + M[k+1,j] + d_{i-1} d_k d_j \\ if\ i == j \quad M[i,j] = 0 \end{cases}$$

$$M = \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}_{4\times 4}, \quad K = \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}_{4\times 4}$$

$$M[1,1] = M[2,2] = M[3,3] = M[4,4] = 0$$

$$M[1,2] = \min_{1 \le k < 2} \begin{cases} M[1,1] + M[2,2] + d_0 d_1 d_2 \\ 0 + 0 + 3 * 2 * 4 = 24 \end{cases}$$

$$M[2,3] = \min_{2 \le k < 3} \begin{cases} M[2,2] + M[3,3] + d_1 d_2 d_3 \\ 0 + 0 + 2 * 4 * 2 = 16 \end{cases}$$

$$M[3,4] = \min_{3 \le k < 4} \begin{cases} M[3,3] + M[4,4] + d_2 d_3 d_4 \\ 0 + 0 + 4 * 2 * 5 = 40 \end{cases}$$

$$M = \begin{bmatrix} 0 & 24 & - & - \\ - & 0 & 16 & - \\ - & - & 0 & 40 \\ - & - & - & 0 \end{bmatrix}_{4 \times 4}, K = \begin{bmatrix} 0 & 1 & - & - \\ - & 0 & 2 & - \\ - & - & 0 & 3 \\ - & - & - & 0 \end{bmatrix}_{4 \times 4}$$

$$M[1,3] = \min_{1 \le k < 3} \begin{cases} M[1,1] + M[2,3] + d_0 d_1 d_3 \\ \quad 0 \quad + \quad 16 \quad + 3*2*2 = 28 \\ M[1,2] + M[3,3] + d_0 d_2 d_3 \\ \quad 24 \quad + \quad 0 \quad + 3*4*2 = 48 \end{cases}$$

$$M[2,4] = \min_{2 \le k < 4} \begin{cases} M[2,2] + M[3,4] + d_1 d_2 d_4 \\ \quad 0 \quad + \quad 40 \quad + 2*4*5 = 80 \\ M[2,3] + M[4,4] + d_1 d_3 d_4 \\ \quad 16 \quad + \quad 0 \quad + 2*2*5 = 36 \end{cases}$$

$$M = \begin{bmatrix} 0 & 24 & 28 & - \\ - & 0 & 16 & 36 \\ - & - & 0 & 40 \\ - & - & - & 0 \end{bmatrix}_{4\times 4}, K = \begin{bmatrix} 0 & 1 & 1 & - \\ - & 0 & 2 & 3 \\ - & - & 0 & 3 \\ - & - & - & 0 \end{bmatrix}_{4\times 4}$$

$$M[1,4] = \min_{1 \le k < 4} \begin{cases} M[1,1] + M[2,4] + d_0 d_1 d_4 \\ \quad 0 \quad + \quad 36 \quad + 3*2*5 = 66 \\ \\ M[1,2] + M[3,4] + d_0 d_2 d_4 \\ \quad 24 \quad + \quad 40 \quad + 3*4*5 = 124 \\ \\ M[1,3] + M[4,4] + d_0 d_3 d_4 \\ \quad 28 \quad + \quad 0 \quad + 3*2*5 = 58 \end{cases}$$

$$M = \begin{bmatrix} 0 & 24 & 28 & 58 \\ - & 0 & 16 & 36 \\ - & - & 0 & 40 \\ - & - & - & 0 \end{bmatrix}_{4\times4}, K = \begin{bmatrix} 0 & 1 & 1 & 3 \\ - & 0 & 2 & 3 \\ - & - & 0 & 3 \\ - & - & - & 0 \end{bmatrix}_{4\times4}$$

```
Matrix_Chain_Mul_Order(p, n)
{
int m[n][n];
for(i = 1; i < n; i++)
{m[i][i] = 0;}
for(l = 2; l < n; l++)
{
for(i = 1; i < n - l + 1; i++)
{
j = i + l - 1;
m[i][j] = ∞;
for(k = i; k ≤ j - 1; k++)
{
q = m[i][k] + m[k+1][j] + p[i-1]p[k]p[j];
if(q < m[i][j])
m[i][j] = q;
}
}
}
return m[1][n-1];
}
```

Time Complexity: $O(n^3)$

THANKS !