TIME RESPONSE OF SECOND ORDER SYSTEM

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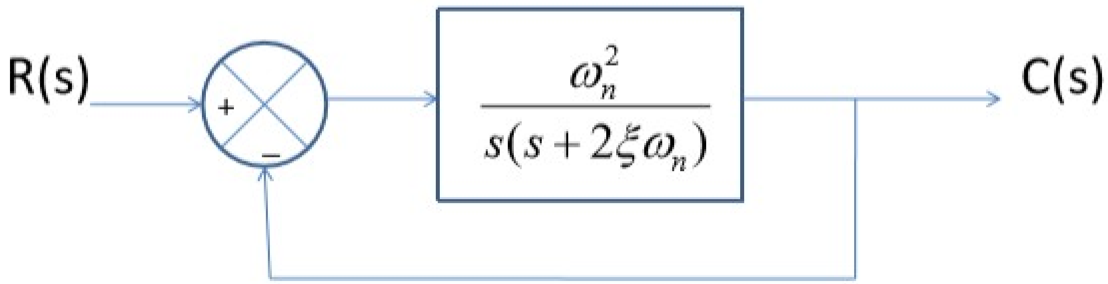
SECOND ORDER SYSTEM

- When the power of Laplace factor ‘s’ is 2 in the denominator of the general equation of the system, then it is called second order system



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

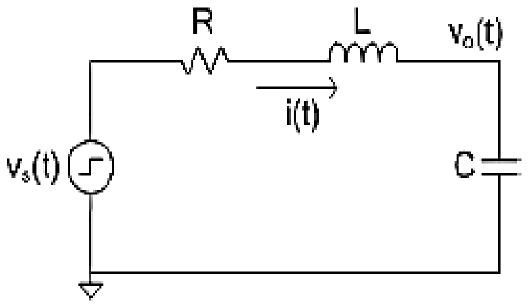
CONTINUED

ω_n the natural frequency and ξ the damping ratio.

ω_d called damping frequency. $\omega_d = \omega_n\sqrt{1-\zeta^2}$

- The value of ξ decides weather it is Underdamped, overdamped or critical damped system.
- For $\xi < 1$, underdamped
- For $\xi = 1$, critical damped
- For $\xi > 1$, overdamped system
- For $\xi = 0$, undamped system and response become oscillatory.
- **Thus ξ is a parameter which controls the oscillations in a system response.**
- *More the ξ is, lesser will be the oscillations .*

SECOND ORDER SYSTEM



Substituting in the expression for current, the KVL equation becomes

$$v_s(t) - C\frac{dv_o}{dt}R - LC\frac{d^2v_o}{dt^2} - v_o(t) = 0$$

The ***homogeneous equation*** is obtained by setting the forcing function (input) to zero

$$\frac{d^2v_o}{dt^2} + \frac{R}{L}\frac{dv_o}{dt} + \frac{1}{LC}v_o(t) = 0$$

Substituting back into the homogeneous ODE yields the ***characteristic equation***

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

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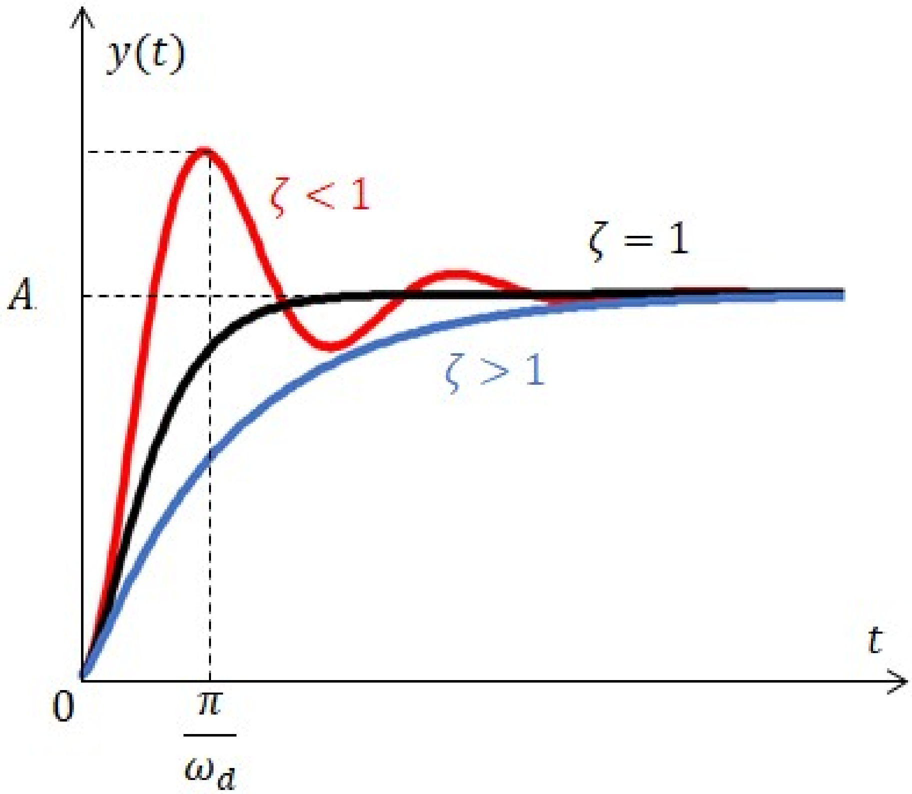
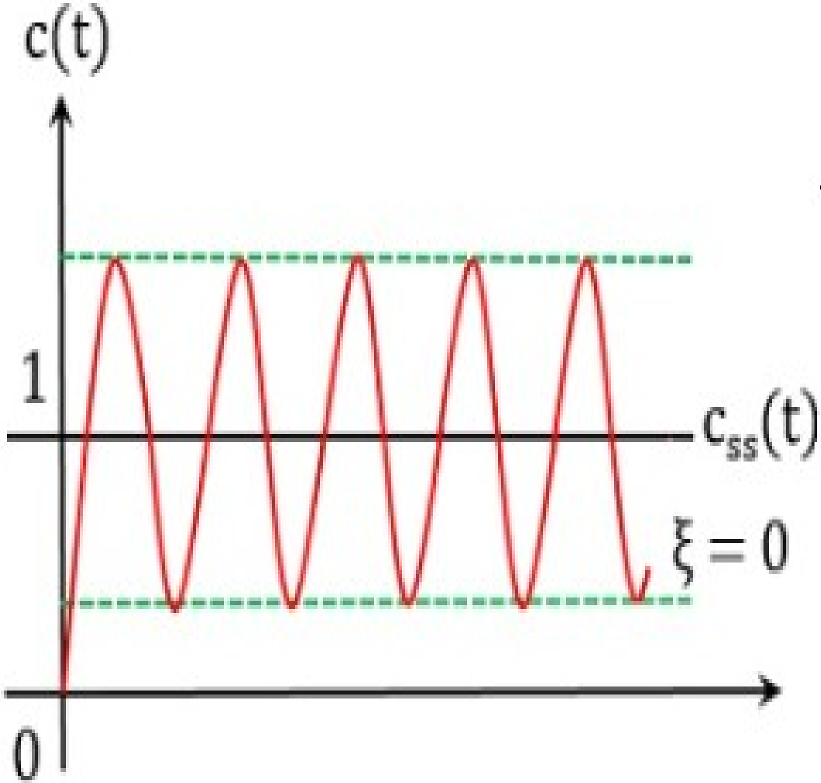
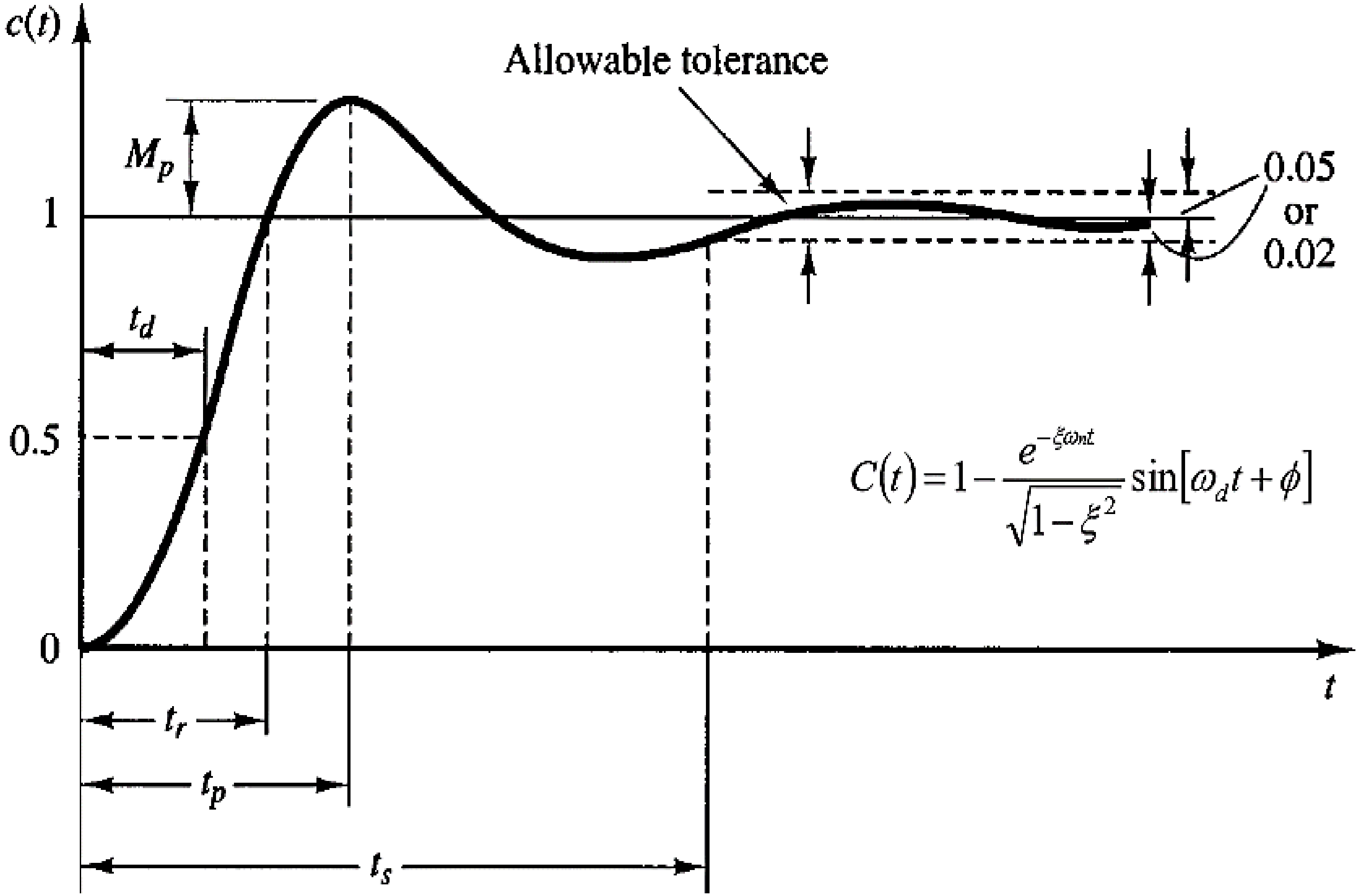




RESPONSE OF AN UNDERDAMPED SYSTEM SUBJECTED TO STEP INPUT



$$C(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$\mathcal{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t}\cos\omega_d t$$

$$\mathcal{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t}\sin\omega_d t$$

$$\mathcal{L}^{-1}[C(s)] = c(t)$$

$$= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}}\sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1 - \zeta^2}}{\zeta}\right), \qquad \text{for } t \geq 0$$

CONTINUED

RISE TIME (tr): It is time required for the response to rise from 10% to 90% of its final value for overdamped systems and 0 to 100% for under-damped systems.

$$t_r = \frac{\pi - \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}$$

PEAK TIME (t_p): The peak time is the time required for the response to reach the first peak of the time response or first peak overshoot.

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

CONTINUED

Maximum Overshoot: Maximum positive deviation of the response from the desired steady state value.

$$M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$$

Settling Time: The settling time is defined as the time required for the transient response to reach and stay within the prescribed percentage error.

$$t_s = \frac{4}{\xi\omega_n}$$

EXAMPLE

Let us now find the time domain specifications of a control system having the closed loop transfer function $\frac{4}{s^2+2s+4}$ when the unit step signal is applied as an input to this control system.

We know that the standard form of the transfer function of the second order closed loop control system as

$$\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

By equating these two transfer functions, we will get the un-damped natural frequency ω_n as 2 rad/sec and the damping ratio δ as 0.5.

CONTINUED

We know the formula for damped frequency ω_d as

$$\omega_d = \omega_n \sqrt{1 - \delta^2}$$

Substitute, ω_n and δ values in the above formula.

$$\Rightarrow \omega_d = 2\sqrt{1 - (0.5)^2}$$

$$\Rightarrow \omega_d = 1.732\, rad/sec$$

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Solve the Problem

When a second-order system is subjected to a unit step input, the values of $\xi = 0.5$ and $\omega n = 6$ rad/sec. Determine the rise time, peak time, settling time and peak overshoot.

- For critically damped case

$$C(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s}$$

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t), \qquad \text{for } t \geq 0$$

- For overdamped case

$$C(s) = \frac{\omega_n^2}{\left(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)\left(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)s}$$

$$c(t) = 1 + \frac{1}{2\sqrt{\zeta^2 - 1}\left(\zeta + \sqrt{\zeta^2 - 1}\right)} e^{-\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

$$- \frac{1}{2\sqrt{\zeta^2 - 1}\left(\zeta - \sqrt{\zeta^2 - 1}\right)} e^{-\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

$$= 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right), \qquad \text{for } t \geq 0$$

where $s_1 = \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$ and $s_2 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$.

THANK YOU