Singular Value Decomposition

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Singular Value Decomposition

The Singular-Value Decomposition (SVD) is a matrix decomposition method for reducing a matrix to its constituent parts in order to make certain subsequent matrix calculations simpler.

Any real ***m* x *n*** matrix *A* can be decomposed uniquely:

$$A = UDV^T$$

U is ***m* x *n*** and column orthogonal ($U^TU=I$)

D is ***n* x *n*** and diagonal

σ_i are called *singular* values of A

It is assumed that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$

V is ***n* x *n*** and orthogonal ($VV^T=V^TV=I$)

$$D = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$$

SVD (cont'd)

If ***m=n,*** then:

$$A = UDV^T$$

U is ***n* x *n*** and orthogonal ($U^TU=UU^T=I$)

D is ***n* x *n*** and diagonal

$$D = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$$

V is ***n* x *n*** and orthogonal ($VV^T=V^TV=I$)

SVD (cont'd)

- The columns of U are eigenvectors of AA^T

$$AA^T = UDV^T VDU^T = UD^2U^T$$

- The columns of V are eigenvectors of A^TA

$$A^T A = VDU^T UDV^T = VD^2V^T$$

If λ_i is an eigenvalue of A^TA (or AA^T), then $\lambda_i = \sigma_i^2$

Numerical Problem on SVD

A=
$$\begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

STEP 1: Find A.A’ in order to find U

$$A \cdot A' = \begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix}$$

STEP 2: Find Eigen Vector for A.A’

Find Eigen vector for $A \cdot A'$

$|A \cdot A' - \lambda I| = 0 \longrightarrow (1)$

$$\begin{vmatrix} (16-\lambda) & 12 \\ 12 & (34-\lambda) \end{vmatrix} = 0$$

$\therefore (16-\lambda) \times (34-\lambda) - 12 \times 12 = 0$

$\therefore \left(544 - 50\lambda + \lambda^2\right) - 144 = 0$

$\therefore \left(\lambda^2 - 50\lambda + 400\right) = 0$

$\therefore (\lambda - 10)(\lambda - 40) = 0$

$\therefore (\lambda - 10) = 0$ or $(\lambda - 40) = 0$

$\therefore$ The eigenvalues of the matrix A are given by $\lambda = 10, 40,$

STEP 3: Substitute the value of λ in (1)

1. Eigenvectors for $\lambda = 40$

$$A \cdot A' - \lambda I = \begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix} - 40 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix} - \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} -24 & 12 \\ 12 & -6 \end{bmatrix}$$

Now, reduce this matrix

$R_1 \leftarrow R_1 \div -24$

$$= \begin{bmatrix} 1 & -\frac{1}{2} \\ 12 & -6 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 12 \times R_1$

$$= \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

The system associated with the eigenvalue $\lambda = 40$

$$(A \cdot A' - 40I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 - \frac{1}{2}x_2 = 0$$

$$\Rightarrow x_1 = \frac{1}{2}x_2$$

$\therefore$ eigenvectors corresponding to the eigenvalue $\lambda = 40$ is

$$v = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix}$$

Let $x_2 = 1$

$$v_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

2. Eigenvectors for $\lambda = 10$

$$A \cdot A' - \lambda I = \begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix} - 10 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 12 \\ 12 & 34 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 12 \\ 12 & 24 \end{bmatrix}$$

Now, reduce this matrix
interchanging rows $R_1 \leftrightarrow R_2$

$$= \begin{bmatrix} 12 & 24 \\ 6 & 12 \end{bmatrix}$$

$R_1 \leftarrow R_1 \div 12$

$$= \begin{bmatrix} 1 & 2 \\ 6 & 12 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 6 \times R_1$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

The system associated with the eigenvalue $\lambda = 10$

$$(A \cdot A' - 10I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 = 0$

$\Rightarrow x_1 = -2x_2$

$\therefore$ eigenvectors corresponding to the eigenvalue $\lambda = 10$ is

$$v = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix}$$

Let $x_2 = 1$

$$v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

For Eigenvector-1 $\left(\frac{1}{2}, 1\right)$, Length L = $\sqrt{0.5^2 + 1^2} = 1.11803$

So, normalizing gives $u_1 = \left(\frac{0.5}{1.11803}, \frac{1}{1.11803}\right) = (0.4472, 0.8944)$

For Eigenvector-2 $(-2, 1)$, Length L = $\sqrt{(-2)^2 + 1^2} = 2.23607$

So, normalizing gives $u_2 = \left(\frac{-2}{2.23607}, \frac{1}{2.23607}\right) = (-0.8944, 0.4472)$

$$\therefore \Sigma = \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix} = \begin{bmatrix} 6.32456 & 0 \\ 0 & 3.16228 \end{bmatrix}$$

$$\therefore U = [u_1, u_2] = \begin{bmatrix} 0.44721 & -0.89443 \\ 0.89443 & 0.44721 \end{bmatrix}$$

(2)

V is found using formula $v_i = \frac{1}{\sigma_i} A^T \cdot u_i$

$$\therefore V = \begin{bmatrix} 0.70711 & -0.70711 \\ -0.70711 & -0.7071 \end{bmatrix}$$

Firstly take, u1 (column wise)and sigma 1, to find v1
Then take u2 and sigma2 , to find v2

Another Way to find V and U

$$\therefore A' \cdot A = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix}$$

To Find V using A.A’

Find Eigen vector for $A' \cdot A$

$A' \cdot A - \lambda I| = 0$

$$\begin{vmatrix} (25-\lambda) & -15 \\ -15 & (25-\lambda) \end{vmatrix} = 0$$

$$\therefore (25-\lambda) \times (25-\lambda) - (-15) \times (-15) = 0$$

$$\therefore \left(625 - 50\lambda + \lambda^2\right) - 225 = 0$$

$$\therefore \left(\lambda^2 - 50\lambda + 400\right) = 0$$

$$\therefore (\lambda - 10)(\lambda - 40) = 0$$

$$\therefore (\lambda - 10) = 0 \text{ or } (\lambda - 40) = 0$$

$\therefore$ The eigenvalues of the matrix A are given by $\lambda = 10, 40,$

1. Eigenvectors for $\lambda = 40$

$$A' \cdot A - \lambda I = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix} - 40 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix} - \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}$$

Now, reduce this matrix

$R_1 \leftarrow R_1 \div -15$

$$= \begin{bmatrix} 1 & 1 \\ -15 & -15 \end{bmatrix}$$

$R_2 \leftarrow R_2 + 15 \times R_1$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The system associated with the eigenvalue $\lambda = 40$

$$(A' \cdot A - 40I) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -x_2$$

$\therefore$ eigenvectors corresponding to the eigenvalue $\lambda = 40$ is

$$v = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}$$

Let $x_2 = 1$

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

2. Eigenvectors for $\lambda = 10$

$$A' \cdot A - \lambda I = \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix} - 10 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -15 \\ -15 & 25 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}$$

Now, reduce this matrix

$R_1 \leftarrow R_1 \div 15$

$$= \begin{bmatrix} 1 & -1 \\ -15 & 15 \end{bmatrix}$$

$R_2 \leftarrow R_2 + 15 \times R_1$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

The system associated with the eigenvalue $\lambda = 10$

$$(A' \cdot A - 10I) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

$\therefore$ eigenvectors corresponding to the eigenvalue $\lambda = 10$ is

$$v = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix}$$

Let $x_2 = 1$

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For Eigenvector-1 $(-1, 1)$, Length L = $\sqrt{(-1)^2 + 1^2} = 1.41421$

So, normalizing gives $v_1 = \left(\frac{-1}{1.41421}, \frac{1}{1.41421}\right) = (-0.7071, 0.7071)$

For Eigenvector-2 $(1, 1)$, Length L = $\sqrt{1^2 + 1^2} = 1.41421$

So, normalizing gives $v_2 = \left(\frac{1}{1.41421}, \frac{1}{1.41421}\right) = (0.7071, 0.7071)$

$$\therefore \Sigma = \begin{bmatrix} \sqrt{40} & 0 \\ 0 & \sqrt{10} \end{bmatrix} = \begin{bmatrix} 6.32456 & 0 \\ 0 & 3.16228 \end{bmatrix}$$

$$\therefore V = \left[v_1, v_2 \right] = \begin{bmatrix} -0.70711 & 0.70711 \\ 0.70711 & 0.70711 \end{bmatrix}$$

U is found using formula $u_i = \frac{1}{\sigma_i} A \cdot v_i$

$$\therefore U = \begin{bmatrix} -0.44722 & 0.89443 \\ -0.89443 & -0.44722 \end{bmatrix}$$

Verification by using first solution

$$U \times \Sigma = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix} \times \begin{bmatrix} 6.3246 & 0 \\ 0 & 3.1623 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4472 \times 6.3246 - 0.8944 \times 0 & 0.4472 \times 0 - 0.8944 \times 3.1623 \\ 0.8944 \times 6.3246 + 0.4472 \times 0 & 0.8944 \times 0 + 0.4472 \times 3.1623 \end{bmatrix}$$

$$= \begin{bmatrix} 2.8284 + 0 & 0 - 2.8284 \\ 5.6569 + 0 & 0 + 1.4142 \end{bmatrix}$$

$$= \begin{bmatrix} 2.8284 & -2.8284 \\ 5.6569 & 1.4142 \end{bmatrix}$$

$$(U \times \Sigma) \times \left(V^T\right) = \begin{bmatrix} 2.8284 & -2.8284 \\ 5.6569 & 1.4142 \end{bmatrix} \times \begin{bmatrix} 0.7071 & -0.7071 \\ -0.7071 & -0.7071 \end{bmatrix}$$

$$= \begin{bmatrix} 2.8284 \times 0.7071 - 2.8284 \times -0.7071 & 2.8284 \times -0.7071 - 2.8284 \times -0.7071 \\ 5.6569 \times 0.7071 + 1.4142 \times -0.7071 & 5.6569 \times -0.7071 + 1.4142 \times -0.7071 \end{bmatrix}$$

$$= \begin{bmatrix} 2 + 2 & -2 + 2 \\ 4 - 1 & -4 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

Verification by using second solution

$$U \times \Sigma = \begin{bmatrix} -0.4472 & 0.8944 \\ -0.8944 & -0.4472 \end{bmatrix} \times \begin{bmatrix} 6.3246 & 0 \\ 0 & 3.1623 \end{bmatrix}$$

$$= \begin{bmatrix} -0.4472 \times 6.3246 + 0.8944 \times 0 & -0.4472 \times 0 + 0.8944 \times 3.1623 \\ -0.8944 \times 6.3246 - 0.4472 \times 0 & -0.8944 \times 0 - 0.4472 \times 3.1623 \end{bmatrix}$$

$$= \begin{bmatrix} -2.8284 + 0 & 0 + 2.8284 \\ -5.6569 + 0 & 0 - 1.4142 \end{bmatrix}$$

$$= \begin{bmatrix} -2.8284 & 2.8284 \\ -5.6569 & -1.4142 \end{bmatrix}$$

$$(U \times \Sigma) \times \left(V^T\right) = \begin{bmatrix} -2.8284 & 2.8284 \\ -5.6569 & -1.4142 \end{bmatrix} \times \begin{bmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$$

$$= \begin{bmatrix} -2.8284 \times -0.7071 + 2.8284 \times 0.7071 & -2.8284 \times 0.7071 + 2.8284 \times 0.7071 \\ -5.6569 \times -0.7071 - 1.4142 \times 0.7071 & -5.6569 \times 0.7071 - 1.4142 \times 0.7071 \end{bmatrix}$$

$$= \begin{bmatrix} 2 + 2 & -2 + 2 \\ 4 - 1 & -4 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$

Dimensionality Reduction:

Consider first singular value only and U matrix (i.e. Equation (2)

$$\text{Projected Data} = \begin{pmatrix} 0.4472136 & -0.89442719 \\ 0.89442719 & 0.4472136 \end{pmatrix} \times \begin{pmatrix} 6.32456 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2.8284 \\ 5.6568 \end{pmatrix}$$

SVD properties

A square ($n \times n$) matrix A is singular iff at least one of its singular values $\sigma_1, \ldots, \sigma_n$ is zero.

The rank of matrix A is equal to the number of nonzero singular values σ_i


Thank
You