3D Transformation

TRANSLATION

- 3D translation $T_v = t_x i + t_y j + t_z k$

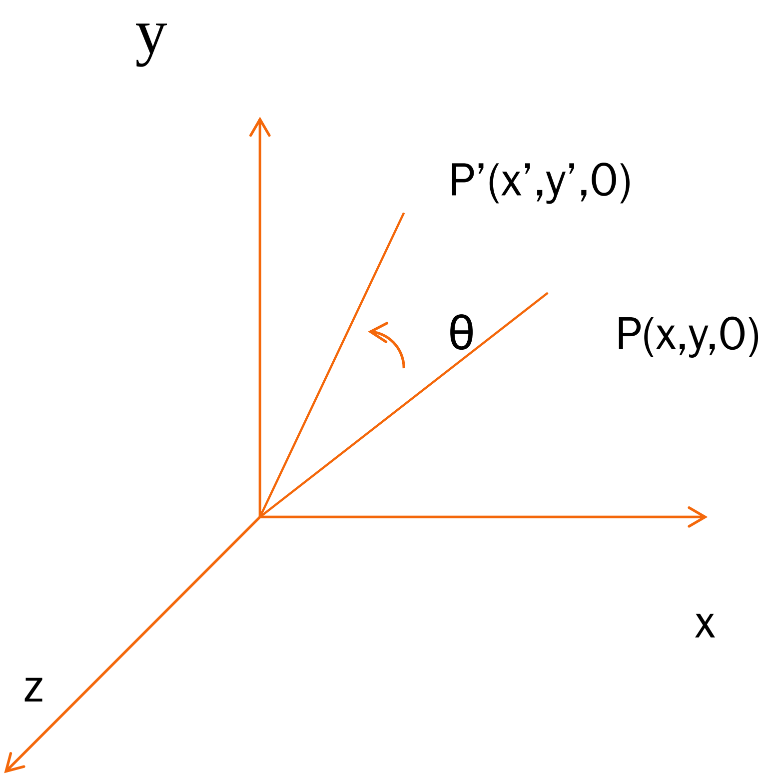
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

SCALING

$$S_{s_x, s_y, s_z} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ROTATION

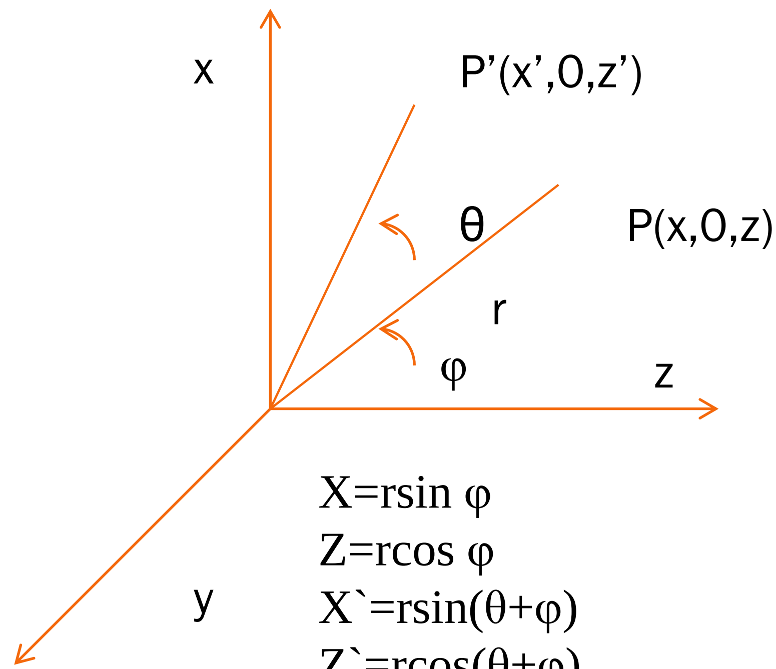
- Rotation in 3D is about an *axis* in 3D space passing through the origin



$$R_{\Theta,z} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 & 0 \\ \sin\Theta & \cos\Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{\Theta,x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta & 0 \\ 0 & \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{\Theta,y} \begin{bmatrix} \cos\Theta & 0 & \sin\Theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\Theta & 0 & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



X=rsin φ

Z=rcos φ

X\`=rsin(θ+φ)

Z\`=rcos(θ+φ)

X\`=rsin φcosθ+rsin θ cosφ

Z\`=rcos φ cos θ -rsin φ sin θ

X\`=xcos θ +zsin θ

Y\`=y

Z\`= -xsin θ +zcosθ

3D REFLECTION

$$M_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{yz} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{zx} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

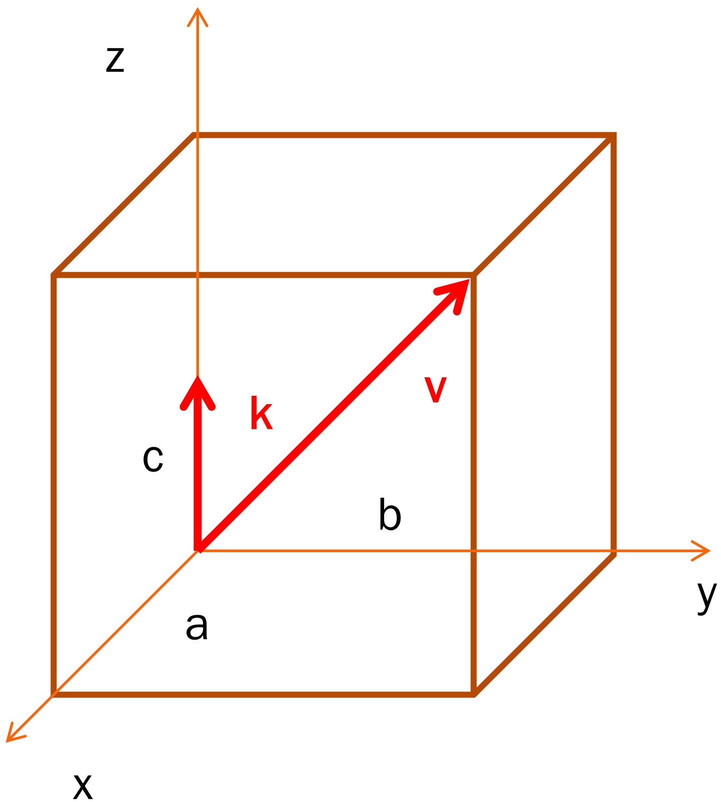
3D SHEARING

$$SH = \begin{bmatrix} 1 & a & b & 0 \\ c & 1 & d & 0 \\ e & f & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

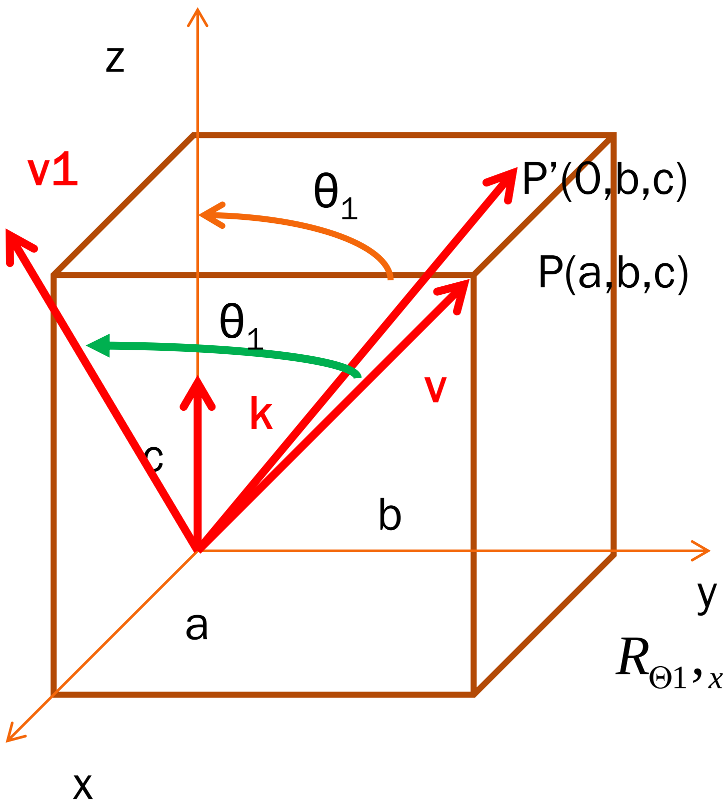
PROBLEM

Find the transformation A_V that aligns a given vector V with the vector k along the positive z-axis.

$V = ai + bj + ck$



- Steps
- Rotate about x-axis by an angle θ_1 so that v rotates in to the upper half of the xz plane as vector V_1



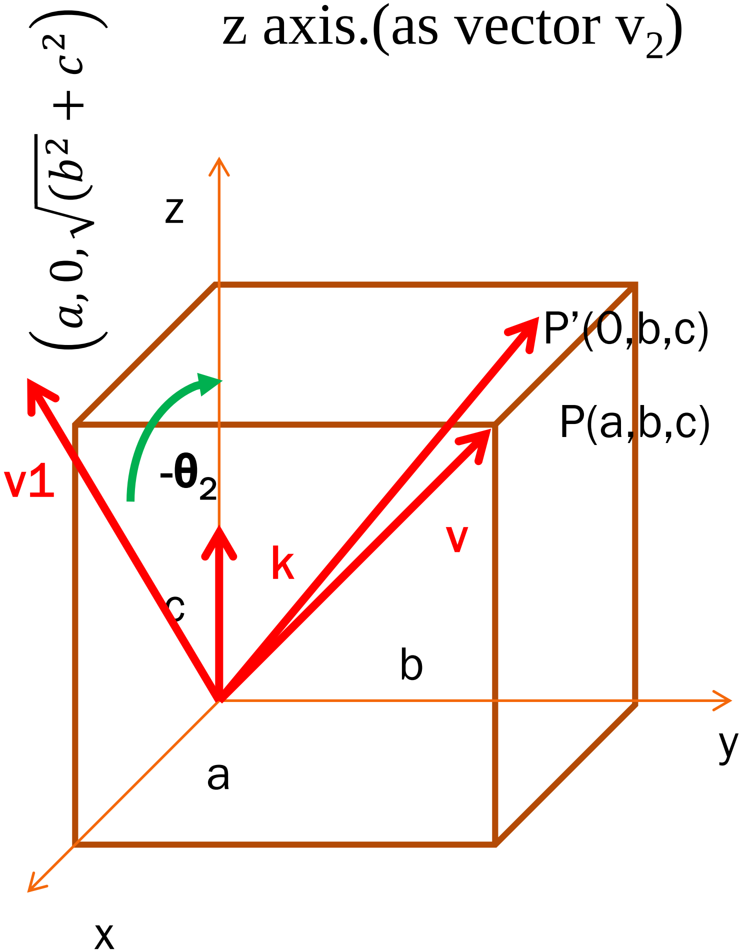
- Step-1
- Rotate about x-axis by an angle θ_1 so that v rotates in to the upper half of the yz plane as vector V_1

$$R_{\Theta 1,x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Theta_1 & -\sin\Theta_1 & 0 \\ 0 & \sin\Theta_1 & \cos\Theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos\Theta_1 = \frac{c}{\sqrt{b2+c2}}$$

$$\sin\Theta_1 = \frac{b}{\sqrt{b2+c2}}$$

- Applying $R_{\theta 1,x}$ on v produces V_1 with coordinates $\left(a, 0, \sqrt{(b^2} + c^2\right)$
- step 2
- Rotate vector v_1 by an angle $-\theta_2$ about y axis aligns it with positive z axis.(as vector v_2)



$$\sin(-\theta_2) = -\sin(\theta_2) = -\frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos(-\theta_2) = \cos(\theta_2) = \frac{\sqrt{b^2 + c^2}}{\sqrt{a^2 + b^2 + c^2}}$$

$$R_{-\Theta 2, y} = \begin{bmatrix} \cos\Theta_2 & 0 & -\sin\Theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ \sin\Theta_2 & 0 & \cos\Theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Since |v|=$\sqrt{a^2+b^2+c^2}$ and introducing λ=$\sqrt{b^2+c^2}$ we have
- $A_v = R_{-\theta 2,\, y} . R_{\theta 1,\, x}$

$$A_v=\begin{bmatrix} \frac{\lambda}{|v|} & \frac{-ab}{\lambda|v|} & \frac{-ac}{\lambda|v|} & 0 \\ 0 & \frac{c}{\lambda} & \frac{-b}{\lambda} & 0 \\ \frac{a}{|v|} & \frac{b}{|v|} & \frac{c}{|v|} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

INVERSE ALIGNMENT

$(A_v)^{-1} = (R\text{-}_{\theta 2,j} . R_{\theta 1,i})^{-1}$

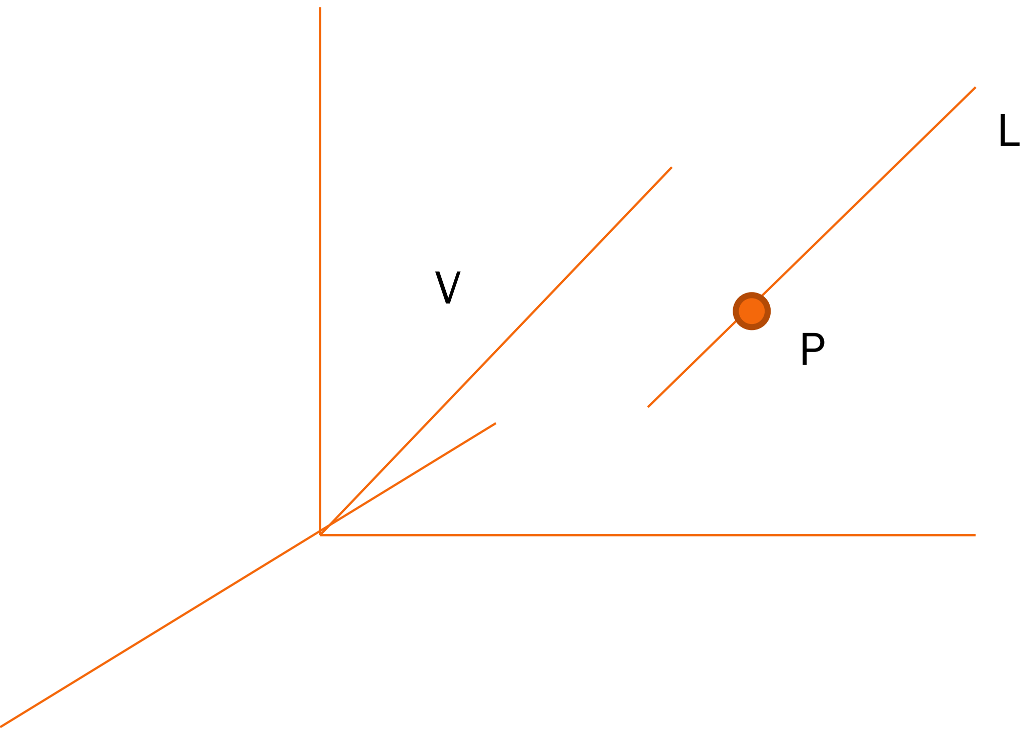
$= R_{\theta 1,i}{}^{-1} . R_{\theta 2,j}{}^{-1}$ Since $(A.B)^{-1} = B^{-1}.A^{-1}$

$= R\text{-}_{\theta 1,i} . R_{\theta 2,j}$

$$A_v^{-1} = \begin{bmatrix} \frac{\lambda}{|v|} & 0 & \frac{a}{|v|} & 0 \\ \frac{-ab}{\lambda|v|} & \frac{c}{\lambda} & \frac{b}{|v|} & 0 \\ \frac{-ac}{\lambda|v|} & -\frac{b}{\lambda} & \frac{c}{|v|} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ROTATION ABOUT AN ARBITRARY LINE

- Let axis of rotation be specified by given line L whose direction vector is V and passing through point P.



- $\mathbf{R_{\theta,L}=T_{vp}.A_{V}^{-1}R_{\theta,K}.A_{V}.T\text{-}_{VP}}$
- The required transformations include

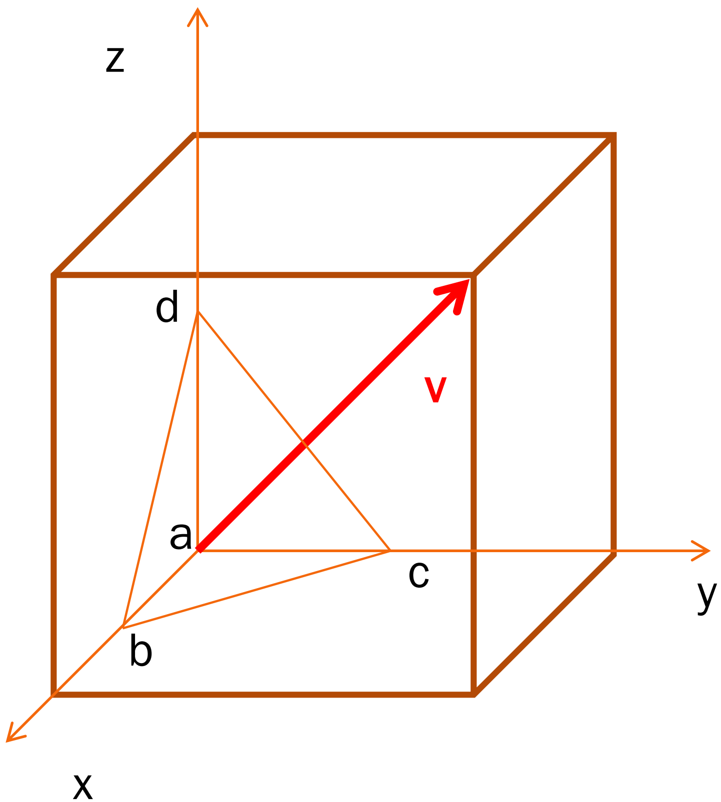
1. Translate P to origin
2. Align V with the normal vector k
3. Rotate by θ about k
4. Reverse step 2 and 1

MIRROR REFLECTION ABOUT AN ARBITRARY PLANE

- Let any plane is given whose normal vector N=ai+bj+ck
- The plane is passing through the point P(l,k,m)
- The steps involved are

1. Translate P to origin
2. Align the normal vector N to k
3. Perform mirror reflection with respect to xy
4. Reverse step 1 and 2
5. $M_{N,P}=T_{V.}A_N^{-1}.M_{XY}.A_N.T_{-V}$

PROBLEM



- Rotate the pyramid defined by the coordinates a(000) b(100) c(010) and d(001) by an angle of **90^o** about the line L that has the direction vector **v=i+j+k** and passing through the origin. Find the coordinates of the rotated figure.

$$\mathbf{R_{\theta,L}=T_{vp}.A_V^{-1}R_{\theta,K}.A_V.T\text{-}_{VP}}$$

$$\mathbf{R_{\theta,L}=A_V^{-1}R_{\theta,K}.A_V}$$

ANSWER

- A\`(000)
- B\`($\frac{1}{3}, \frac{1+\sqrt{3}}{3}, \frac{1-\sqrt{3}}{3}$)
- C\`($\frac{1-\sqrt{3}}{3}, \frac{1}{3}, \frac{1+\sqrt{3}}{3}$)
- D\`($\frac{1+\sqrt{3}}{3}, \frac{1-\sqrt{3}}{3}, \frac{1}{3}$)