Bezier Curves and B-Spline Curves

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Bezier Curve and Surfaces

- This spline approximation method developed by the French engineer Pierre Bezier for use in the design of Renault automobile bodies.
- Easy to implement.
- Available in CAD system, graphic package, drawing and painting packages.

Bezier Curve

- A Bezier curve can be fitted to any number of control points.
- Given $n+1$ control points position

$$p_k=(x_k, y_k, z_k) \qquad 0 \leq k \leq n$$

- The coordinate positions are blended to produce the position vector *P(u)* which describes the path of the Bezier polynomial function between p_0 and p_n

$$P(u) = \sum_{k=0}^{n} p_k BEZ_{k,n}(u), \qquad 0 \le u \le 1$$

- The Bezier blending functions $BEZ_{k,n}(u)$ are the *Bernstein polynomials*

$$BEZ_{k,n}(u) = C(n,k)u^k(1-u)^{n-k}$$

where parameters *C(n,k)* are the binomial coefficients

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

- The individual curve coordinates can be given as follows

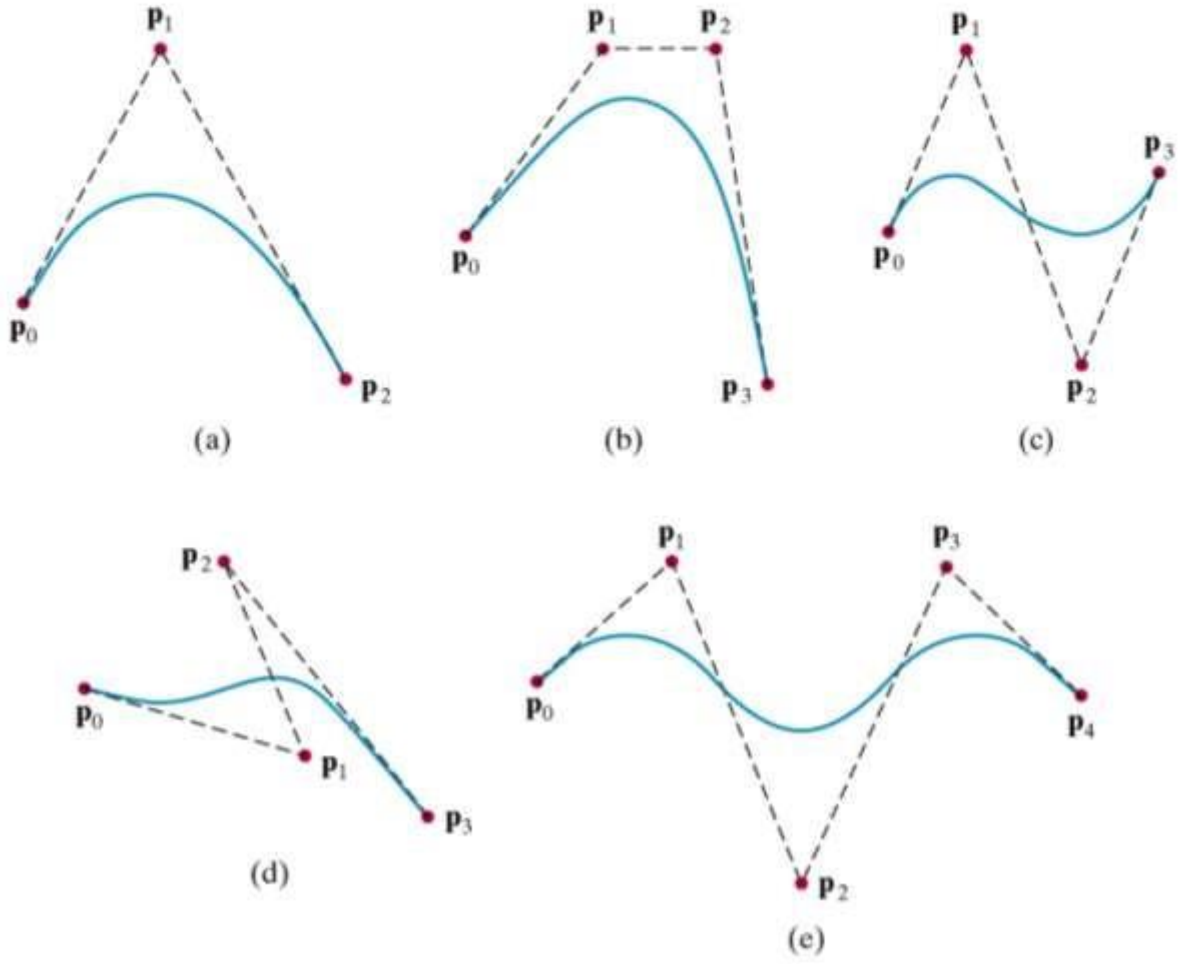
$$x(u) = \sum_{k=0}^{n} x_k \, BEZ_{k,n}(\mathrm{u})$$

$$y(u) = \sum_{k=0}^{n} y_k \, BEZ_{k,n}(\mathrm{u})$$

$$\mathrm{z}(u) = \sum_{k=0}^{n} z_k \, BEZ_{k,n}(\mathrm{u})$$

Properties Of Bezier Curves

- Bezier Curve is a polynomial of degree one less than the number of control points



- Bezier Curves always passes through the first and last control points.

$$P(0) = p_0$$

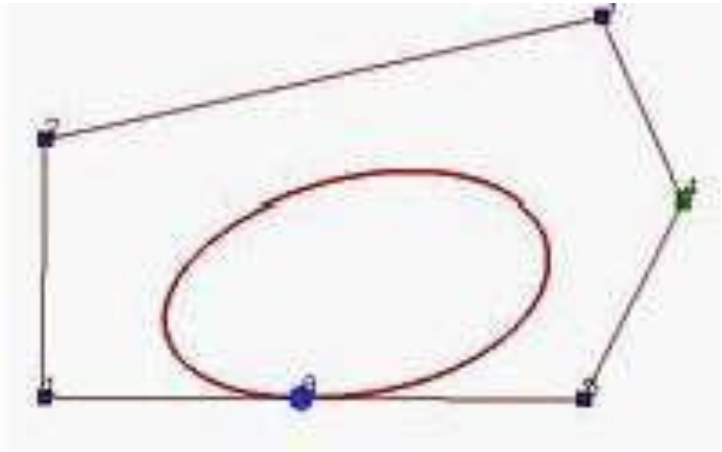
$$P(1) = p_n$$

- Bezier curves are tangent to their first and last edges of control garph.
- The curve lies within the convex hull as the Bezier blending functions are all positive and sum to 1

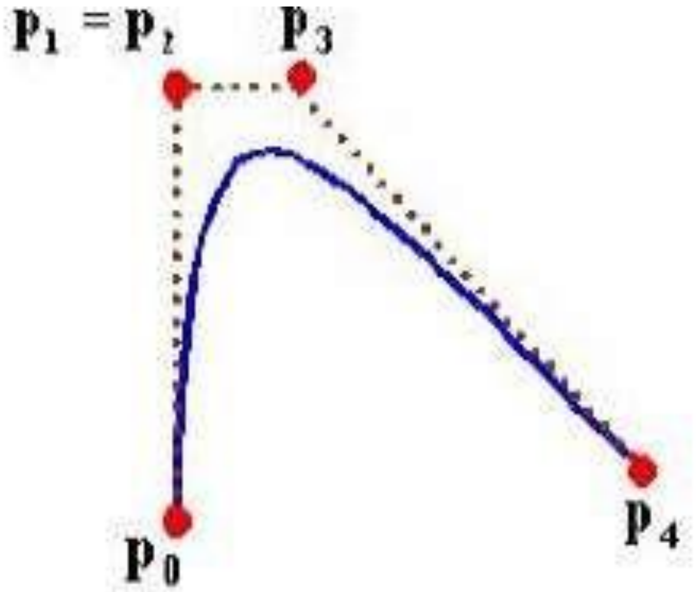
$$\sum_{k=0}^{n} BEZ_{k,n}(u) = 1$$

Design Techniques

- Closed Bezier curves are generated by specifying the first and last control points at same position.



- Specifying multiple control points at a single coordinate position gives more weight to that position.



Cubic Bezier Curve

- Cubic Bezier curves are generated with **4 control points.**
- Cubic Bezier curves gives reasonable design flexibility while avoiding the increased calculations needed with higher order polynomials.

The blending functions when n = 3

$$BEZ_{0,3} = (1-u)^3$$

$$BEZ_{1,3} = 3u(1-u)^2$$

$$BEZ_{2,3} = 3u^2(1-u)$$

$$BEZ_{3,3} = u^3$$

- At $u=0$, $BEZ_{0,3}=1$, and at $u=1$, $BEZ_{3,3}=1$. thus, the curve will always pass through control points P_0 and P_3.

- The functions $BEZ_{1,3}$ and $BEZ_{2,3}$, influence the shape of the curve at intermediate values of parameter u.

- The resulting curve tends toward points P1 and P3.

Bezier Surface

- Two sets of orthogonal Bezier curves are used to design surface.

$$P(u, v) = \sum_{j=0}^{m} \sum_{k=0}^{n} p_{j,k} BEZ_{j,m}(v) BEZ_{k,n}(u)$$

$P_{j,k}$ specify the location of the control points.

B-Spline Curves and Surfaces

1. The degree of a B-spline polynomial can be set independently of the number of control points.

2. B-splines allow local control over the shape of a spline curve (or surface)

- The point on the curve that corresponds to a knot is referred to as a *knot vector*.

- The knot vector divide a B-spline curve into curve subinterval, each of which is defined on a knot span.

- Given $n + 1$ control points $P_0, P_1, \ldots, P_n$
- Knot vector $U = \{ u_0, u_1, \ldots, u_{n+d} \}$
- The B-spline curve defined by these control points and knot vector

$$P(u) = \sum_{k=0}^{n} p_k B_{k,d}(u), \quad u_{min} \le u \le u_{max}, \quad 2 \le d \le n+1$$

P_k is kth control point

Blending function $B_{k,d}$ of degree d-1

- Blending functions defined with Cox-deBoox recursive form

$$B_{k,1}(u) = \begin{cases} 1, & if\, u_k \le u \le u_{k+1} \\ 0, & otherwise \end{cases}$$

$$B_{k,d}(u) = \frac{u - u_k}{u_{k+d-1}} B_{k,d-1}(u) + \frac{u_{k+d} - u}{u_{k+d} - u_{k+1}} B_{k+1,d-1}(u)$$

To change the shape of a B-spline curve, modify one or more of these control parameters:

1. The positions of control points
2. The positions of knots
3. The degree of the curve

Uniform B-Spline

- The spacing between knot values is constant.

Non-uniform B-spline

- Unequal spacing between the knot values.

Open uniform B-Spline

- This B-Spline is across between Uniform B-Spline and non-uniform B-Spline.
- The knot spacing is uniform except at the ends where knot values are repeated d times

B-Spline Surfaces

Similar to Bezier surface

$$P(u,v) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} p_{k_1,k_2} B_{k_1,d_1}(u) B_{k_2,d_2}(v)$$