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PARAMETER ESTIMATION Assingment

Ques1 let $x_1, x_2, \dots$ be random sample of size n taken from a Normal Population with Parameters mean $= \theta_1$ and variance $= \theta_2$ find the Maximum likelihood estimates of These Two Parameters

Sol:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (\because \text{PDF of Normal distribution})$$

$$\mu = \theta_1 ; \quad \sigma^2 = \theta_2$$

$$f(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$f(x_i) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

likelihood function

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{(x_i-\theta_1)^2}{2\theta_2}}$$

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} (\theta_2)^{-1/2} \prod_{i=1}^{n} (2\pi)^{-1/2} \prod_{i=1}^{n} e^{-\frac{(x_i - \theta_1)^2}{2\theta_2}}$$

$$L(\theta_1, \theta_2) = (\theta_2)^{n/2} (2\pi)^{-n/2} e^{-\frac{\prod_{i=1}^{n}(x_i - \theta_1)^2}{2\theta_2}}$$

$$L(\theta_1, \theta_2) = (\theta_2)^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{2\theta_2} \prod_{i=1}^{n} (x_i - \theta_1)^2} \quad -(1)$$

Taking log Both Sides

$$\ln(L(\theta_1, \theta_2)) = \ln\left[(\theta_2)^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{2\theta_2} \prod_{i=1}^{n} (x_i - \theta_1)^2}\right]$$

$$z = \ln(L(\theta_1, \theta_2)) = -\frac{n}{2} \ln\theta_2 - \frac{n}{2} \ln(2\pi) - \frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)^2 \quad (2)$$

Diff eqn (2) w.r.t θ_1

$$\frac{\partial z}{\partial \theta_1} = \frac{1}{\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)$$

now,

$$\frac{\partial z}{\partial \theta_1} = 0 \Rightarrow \frac{1}{\theta_2} \sum_{i=1}^{n} (x_i - \theta_1) = 0$$

$$\Rightarrow \frac{1}{\theta_2}\left(\sum_{i=1}^{n} x_i - n\theta_1\right) = 0 \Rightarrow \sum_{i=1}^{n} (x_i - n\theta_1) = 0$$

$$\Rightarrow n\theta_1 = \sum_{i=1}^{n} x_i \rightarrow \theta_1 = \frac{\sum_{i=1}^{n} x_i}{n} \Rightarrow \boxed{\theta_{1MLE} = \bar{x}_n} \quad -(3)$$

Diff (2) w.r.t θ_2

$$\frac{\partial z}{\partial \theta_2} = \frac{-n}{2\theta_2} + \frac{1}{2(\theta_2)^2} \sum_{i=1}^{n} (x_i - \theta_1)^2, \quad \frac{\partial z}{\partial \theta_2} = 0$$

$$\frac{-n}{2\theta_2} + \frac{1}{2\theta_2^2} \sum_{i=1}^{m} (x_i - \theta_1)^2 = 0$$

$$\theta_2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \theta_1)^2$$

From (3) $\theta_1 = \bar{x}_n$

$$\theta_2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x}_n)^2$$

Ques2 Let $x_1, x_2 \dots x_n$ be random Sample from $B(m, \theta)$ distribution where $\theta \in \Theta = (0,1)$ is unknown and 'm' is a known +ve integer. Compute value of θ using MLE

Sol: $f(x) = {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$ (PDF of Binomial Distribution)

here, $n = m$, $p = \theta$

$f(x) = {}^{m}C_{x}\, \theta^{x} (1-\theta)^{m-x}$ $\Rightarrow$ $f(x_i) = {}^{m}C_{x_i}\, \theta^{x_i} (1-\theta)^{m-x_i}$

Likelihood function

$$L(m, \theta) = \prod_{i=1}^{n} f(x_i)$$

$$L(m, \theta) = \prod_{i=1}^{n} {}^{m}C_{x_i}\, \theta^{x_i} (1-\theta)^{m-x_i}$$

$$L(m, \theta) = \prod_{i=1}^{n} {}^{m}C_{x_i}\, \theta^{x_i} (1-\theta)^{mn - \sum_{i=1}^{n} x_i}$$

Taking log Both sides.

$$\ln\left[L(m,\theta)\right] = \ln\left(\prod_{i=1}^{n} {}^{m}C_{x_i}\, \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{mn-\sum_{i=1}^{n} x_i}\right]$$

$$\ln\left(L(m,\theta)\right) = \ln\left(\prod_{i=1}^{n} {}^{m}C_{x_i}\right) + \ln\left(\theta^{\sum_{i=1}^{n} x_i}\right) + \ln\left[(1-\theta)^{mn-\sum_{i=1}^{n} x_i}\right]$$

$$z = \ln\left(L(m,\theta)\right) = \ln\left(\prod_{i=1}^{n} {}^{m}C_{x_i}\right) + \sum_{i=1}^{n} x_i \ln\theta + \left(mn - \sum_{i=1}^{n} x_i\right)\ln(1-\theta) \quad (1)$$

Diff (1) w.r.t θ

$$\frac{\partial z}{\partial \theta} = \frac{1}{\theta}\sum_{i=1}^{n} x_i + \left(\frac{\sum_{i=1}^{n}(x_i - mn)}{1-\theta}\right)$$

Now,

$$\frac{\partial z}{\partial \theta} = 0$$

$$\Rightarrow \frac{1}{\theta}\sum_{i=1}^{n} x_i + \left(\frac{\sum_{i=1}^{n} x_i - mn}{1-\theta}\right) = 0$$

$$\frac{\sum_{i=1}^{n} x_i - mn}{\theta - 1} = \frac{\sum_{i=1}^{n} x_i}{\theta}$$

$$1 - \frac{mn}{\sum_{i=1}^{n} x_i} = \frac{\theta - 1}{\theta}$$

$$+\frac{mn}{\sum x_i} = +\frac{1}{\theta}$$

$$\theta = \frac{\overline{x_n}}{m}$$

$$\theta_{MLE} \in (0,1) = \frac{\overline{x_n}}{m}$$