Computational Methods for Finance
Week 7: Mathematical Model IV (Binomial Trees)

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Learning Outcomes

At the end of this lecture you will be able to

- Price options using the Binomial Trees method.

Review

- Risk-Neutral Valuation
    - The BSM differential equation does not contain variables that are affected by investors' risk preferences (i.e. the drift parameter μ) $\Rightarrow$ Any set of risk preferences can be used when evaluating f.
    - We can make a simple assumption that all investors are risk-neutral. $\Rightarrow$ The expected return on *all* investment assets is the *risk-free rate* (including the underlying asset, and thus the drift parameter can be replaced by the risk-free rate wherever it appears in the derivation).
    - Under risk-neutral valuation, solutions obtained are valid in all worlds (not only the risk-neural world).
    - The economic argument for the risk-neutral valuation is that since we can perfectly hedge the option with the underlying (e.g. non-dividend-paying stock), we should not be rewarded for taking unnecessary risk; only the risk-free rate of return is in the equation. This means that if you and I agree on the *volatility* (i.e. σ) of an asset we will agree on the value of its derivatives even if we have differing estimates of the drift.

Binomial Tree

- One of numerical methods to price option.
- A method that can price the American option.
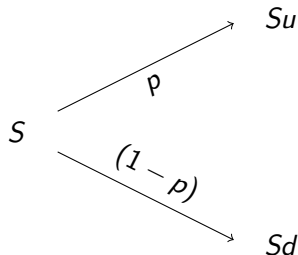- The Representation
  The movement of stock prices (as given by geometric Brownian motion) can be approximated by a binomial tree. Note that:
    - Each point in the tree is referred to as a node.
    - The first node corresponds to the current price.
    - The last set of nodes correspond to the possible prices at the maturity of the option.

Binomial Tree

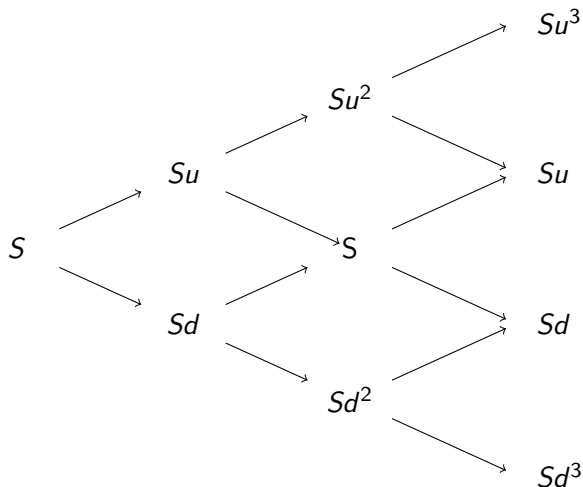- The Representation
  The one-step binomial model is given by

$$
\begin{array}{c}
\\
S \nearrow^{p} Su \\
\searrow_{(1-p)} Sd
\end{array}
$$

  - Starting at S, the stock price is allowed to move up (with probability p) to Su or down (with probability $(1-p)$) to Sd.
  - Increasing the number of steps in the tree improves accuracy.

Binomial Tree

- The Representation
  The three-step binomial model is given by

Binomial Tree

- The Representation
  Assuming that u = 1/d (u>1) and using risk-neutral valuation,
  geometric Brownian motion can be approximated using the following
  parameter values:

$$p = \frac{a - d}{u - d},$$
$$u = e^{\sigma\sqrt{\Delta t}},$$
$$d = e^{-\sigma\sqrt{\Delta t}},$$
$$a = e^{r\Delta t}.$$

These parameters are chosen so that the tree gives correct values for
the mean and variance of the stock price changes in a risk-neutral
world.

Binomial Tree

- Backwards Induction
  The logic is as follows:
  - The value of the option at the final nodes is known.
  - Therefore, we can work back through the tree using risk-neutral valuation to calculate the value of the option at each node.
  - This process continues until the first (single) node is reached. At this point the price of the derivative is determined.

Binomial Tree

- Backwards Induction (procedure)
  For an American put option on a non-dividend paying asset:
  - Step 1: Calculate the stock prices for *each* node.
  - Step 2: At T, calculate the option price ($\max(K - S_T, 0)$).
  - Step 3: The premium at the previous node will equal:
    (a) The expected value of the subsequent premia discounted by the risk-free rate.
    OR
    (b) The value of $\max(K - S_{i\Delta t}, 0)$, if this is greater than the value calculated in Step 3 (a) (i.e., early exercise).
  - Step 4: Continue backwards until the first node is reached.

Binomial Tree

- Backwards Induction

Example

Consider an American put option with $T = 5/12$, $S = 50$, $K = 50$, $r = 0.1$, $\sigma = 0.4$. Divide the life of the option into one month intervals ($\Delta t = 1/12$). Under these conditions, what are a, u, d, p in the binomial tree model that I showed you just now?
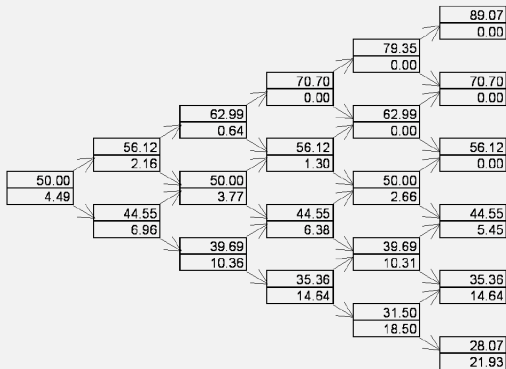
$$u = 1.1224, \ d = 0.8909, \ a = 1.0084, \ p = 0.5073.$$

Please plot the binomial tree which includes the possible stock prices and the corresponding option prices in each period.

Binomial Tree

- Backwards Induction

Example

- Chapter 13, Hull (2015)