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**Module: High Frequency Trading**

# Week 3: Market Microstructure Theories

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# Introduction to Models

*"All models are wrong; some models are useful."*
*— George Box*

- Eliminate all but one or two major factors.
- Reproduce real-life features.
- Identify factors that matter.
- Understand how these factors matter.

# Market Microstructure Models

**Market microstructure studies liquidity and information.**

## Liquidity

- What influences liquidity provision?
- What influences price impact of orders?

If market makers are risk averse, they need compensation for bearing inventory risk.

# Grossman-Miller Market Making Model

**This model deals with inventory risk borne by market makers.**

1. There are $n$ identical risk-averse market makers in the market for a given asset.
2. There are three dates, $t \in \{1, 2, 3\}$.
3. At date $t = 1$, a liquidity trader, denoted by LT1, comes to the market to sell $i$ units of the asset.
4. At date $t = 2$, another liquidity trader LT2 arrives at the market to buy $i$ units of the asset.

# Grossman-Miller Market Making Model

**There are no direct costs for holding inventory.The focus is on price change:**

1. The asset has a cash value at $t = 3$ given by $S_3 = \mu + \epsilon_2 + \epsilon_3$, where $\mu$ is constant, $\epsilon_2$ and $\epsilon_3$ are independent, normally distributed random variables with mean zero and variance $\sigma^2$
2. For $j \in \{MM, LT1, LT2\}$, we denote the number of shares that trader $j$ is holding when exiting time $t$ as $q_t^j$.
3. The utility function for each risk-averse trader is defined as:

$$U(x) = -\exp(-\gamma x)$$

where $\gamma$ measures the risk averse of traders, and X is the wealth

# Optimal Holdings when exiting $t = 2$

For each trader, we want to maximize their utilities. For example, at t=2, each trader will hold $q_2^j$ shares to maximise the utility of their wealth $X_3^j$:

$$\max_{q_2^j} E[U(X_3^j)|\epsilon_2]$$

subject to

$$X_3^j = X_2^j + q_2^j S_3$$

and

$$X_2^j + q_2^j S_2 = X_1^j + q_1^j S_2$$

# Optimal Holdings when exiting $t = 2$

Given the normality assumption and the expected utility function, we get

$$E[U(X_3^j)|\epsilon_2] = -\exp\{-\gamma(X_2^j + q_2^j E[S_3|\epsilon_2]) + 1/2\gamma^2(q_2^j)^2\sigma^2\}$$

the problem is concave and the solution is characterised by

$$q_2^{j,*} = \frac{E[S_3|\epsilon_2] - S_2}{\gamma\sigma^2}$$

for all traders: $n$ MMs, LT1, and LT2.

# Optimal Holdings when exiting $t = 2$

At date t=2, demand and supply for the asset have to be equal to each other, therefore:

$$nq_1^{MM} + q_1^{LT1} + q_1^{LT2} = nq_2^{MM} + q_2^{LT1} + q_2^{LT2}$$

where

$$q_1^{LT2} = -i$$

and

$$nq_1^{MM} + q_1^{LT1} = i$$

Therefore,

$$nq_1^{MM} + q_1^{LT1} + q_1^{LT2} = nq_2^{MM} + q_2^{LT1} + q_2^{LT2} = 0$$

# Optimal Holdings when exiting $t = 2$

For each trader,

$$q_2^{j,*} = \frac{E[S_3|\epsilon_2] - S_2}{\gamma\sigma^2}$$

they should all hold $q_2^{j,*}$ each, and thus

$$(n+2)\frac{E[S_3|\epsilon_2] - S_2}{\gamma\sigma^2} = 0$$

Therefore,

$$q_2^j = 0$$

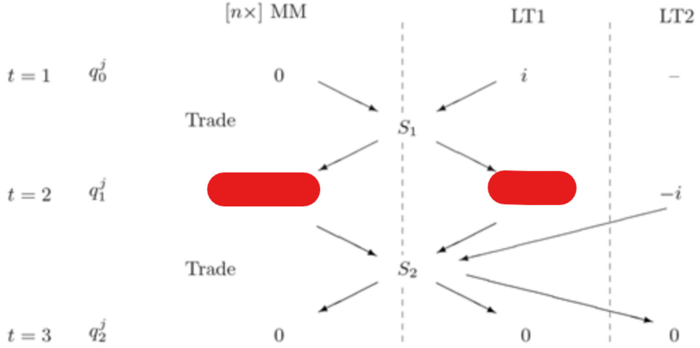
and

$$\frac{E[S_3|\epsilon_2] - S_2}{\gamma\sigma^2} = 0$$

$$S_2 = E[S_3] = \mu + \epsilon_2 + E[\epsilon_3] = \mu + \epsilon_2$$

# Optimal Holdings when exiting $t = 2$

We have worked half of the process, which can be shown as follows:



# Optimal Holdings at $t = 1$

Now let's see what the traders do at $t = 1$:

$$\max_{q_1^j} E[U(X_2^j)]$$

subject to

$$X_2^j = X_1^j + q_1^j S_2$$

and

$$X_1^j + q_1^j S_1 = X_0^j + q_0^j S_1$$

# Optimal Holdings at $t = 1$

The optimal solution is still:

$$q_1^{j,*} = \frac{E[S_2] - S_1}{\gamma\sigma^2}$$

and still supply and demand have to be equal, thus:

$$nq_0^{MM} + q_0^{LT1} = nq_1^{MM} + q_1^{LT1}$$

where

$$q_0^{LT1} = i$$

and

$$q_0^{MM} = 0$$

# Optimal Holdings at $t = 1$

Therefore, they should all hold $q_1^{j,*}$ each, and thus

$$(n+1)\frac{\mu - S_1}{\gamma\sigma^2} = i$$

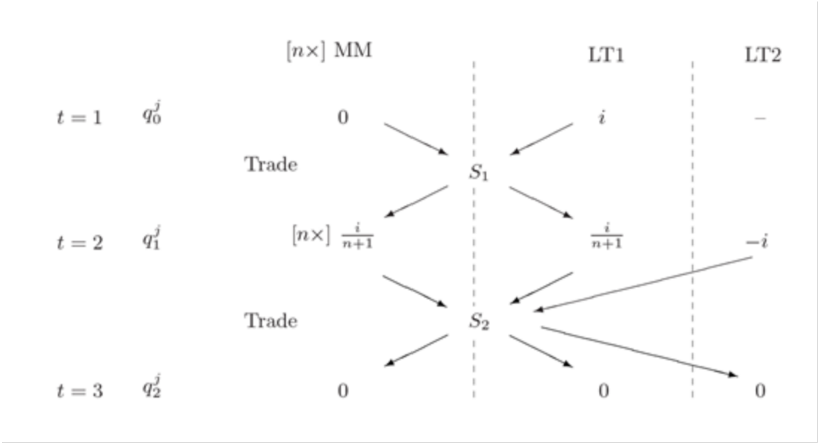
$$S_1 = \mu - \gamma\sigma^2 \frac{i}{n+1}$$

In equilibrium, $n$ MMs and LT1 end up holding

$$q_1^{j,*} = \frac{i}{n+1}$$

# Optimal Holdings

The shares and prices traded are shown below:



# Measuring Liquidity

We can write S as a linear function of the quantity traded $q^{LT1}$:

$$S_1 = \mu + \lambda q^{LT1}$$

According to Grossman-Miller:

$$\lambda = -\frac{1}{n}\gamma\sigma^2$$

and

$$q^{LT1} = i\frac{n}{n+1}$$

# Measuring Liquidity

Another popular way to measure liquidity is through autocovariance of price changes because according to Grossman-Miller:

$$S_1 = \mu_1 + \lambda q^{LT1}$$

$$S_2 = \mu_2$$

Aurocovariance of price changes $\Delta_1 = S_1 - S_0$ and $\Delta_2 = S_2 - S_1$ is:

$$\begin{aligned}
\text{Cov}[\Delta_1, \Delta_2] &= \text{Cov}[\mu_1 + \lambda q^{LT1} - \mu_0, \mu_2 - \mu_1 - \lambda q^{LT1}] \\
&= \text{Cov}[\epsilon_1 + \lambda q^{LT1}, \epsilon_2 - \lambda q^{LT1}] \\
&= -\lambda^2 \text{Var}[q^{LT1}]
\end{aligned}$$

The autocorrelation of price changes capture market liquidity just as price impact does.

# Glosten & Milgrom Model

**This model deals with information asymmetry**

- In the market, there are three types of traders: informed traders ('insiders'), liquidity traders and market makers.
- In contrast to Grossman & Miller, MMs are risk neutral, so they do not need a liquidity premium to compensate for the price risk from holding inventory.But they need liquidity premium to compensate for their informational disadvantage.
- The asset is traded at price $S$, and the future cash value of the asset is $v$.

# Glosten & Milgrom Model

- $v$ limits to two possible value $V_H > V_L$. There are many insiders who know the exact value of $v$. But for other $v = V_H$ with probability $p$, $v = V_L$ with probability $1 - p$
- MMs know that there are informed traders in the market but do not know who they are.
- Liquidity traders buy with probability $1/2$ and sell with probability $1/2$.
- MMs set the ask price as $a$ and bid price as $b$. When $v = V_H$, insiders buy one unit if $a < V_H$, while $v = V_L$, insiders sell one unit if $b > V_L$.
- The total population of liquidity and informed traders is normalised to one, and of these, a proportion $\alpha$ are informed and $1 - \alpha$ are uninformed liquidity traders.

# Market Maker's Problem

- We define $\Delta_a$ and $\Delta_b$ to be ask- and bid-halfspreads respectively. The sum of the two $\Delta_a + \Delta_b$ is the quoted spread. Therefore,

  $$a = \mu + \Delta_a$$

  and

  $$b = \mu - \Delta_b$$

- If a buy order comes in, the MM makes an expected profit of $a - \mu = \Delta_a$ if it comes from an uninformed liquidity trader, and makes an expected loss $a - V_H = \Delta_a - (V_H - \mu)$.

# Market Maker's Problem

The expected profit from posting a price $a = \mu + \Delta_a$ is:

$$\frac{(1-\alpha)/2}{\alpha p + (1-\alpha)/2}\Delta_a + \frac{p\alpha}{\alpha p + (1-\alpha)/2}(\Delta_a - (V_H - \mu))$$

The probability is calculated by Bayesian probability theory. The competition among market makers will make their profit converge to zero, but not negative. Set the price to be zero we obtain:

$$\Delta_a = \frac{1}{1 + \frac{1-\alpha}{\alpha}\frac{1/2}{p}}(V_H - \mu)$$

Following the same reasoning,

$$\Delta_b = \frac{1}{1 + \frac{1-\alpha}{\alpha}\frac{1/2}{1-p}}(\mu - V_L)$$

# Kyle(1985)

**This model deals with information asymmetry in a more generalised setting**

1. In the market, there are three types of traders: an informed trader, liquidity traders and market makers.
2. In contrast to Grossman & Miller, MMs are risk neutral, so they do not need a liquidity premium to compensate for the price risk from holding inventory.But they need liquidity premium to compensate for their informational disadvantage.
3. Liquidity traders demand a random quantity of the asset $u$ with mean zero and variance $\sigma_u^2$.
4. The asset is traded at price $S$, and the future cash value of the asset is $v$.

# Kyle(1985)

1. $v$ is assumed to be normally distributed with mean $\mu$ and variance $\sigma^2$. But the informed trader ('insider') knows the exact value of $v$.Let $x_(v)$ denotes the number of shares traded by the insider.
2. MMs know that there is an informed trader in the market but do not know who this trader is.
3. MMs observe the net order flow $x(v) + u$, based on the order flow, MMs compete to set the asset price as a function of net order flows $S(x + u)$.

# Kyle(1985)

**To solve the mode, we use the Bayesian Nash equilibrium, which means all agents optimise given the decisions of other players according to their beliefs.**

- The inside will buy if $v < \mu$, sell if $v > \mu$.
- The competition among market makers will make their profit converge to zero, but not negative. The zero profit forces equilibrium price to be:

$$S(x + u) = E[v|x + u]$$

- Now conditional on this price rule, we need to find an $x(v)$ that maximize insider's profit.

# Kyle(1985)

We Hypothesis that

$$S(x+u) = \mu + \lambda(x+u)$$

So the insider's problem now is:

$$\max_x E[x(v - \mu - \lambda(x+u))]$$

The objective function is concave, so the first order condition is :

$$x^*(v) = \frac{v-\mu}{2\lambda}$$

Now since

$$S = E[v|x+u]$$

then according to the projection theorem,

$$S = \mu + 2(x+u)\frac{\sigma_u}{\sigma}$$