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Module: High Frequency Trading

Week 11: Pairs Trading

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Introduction to Pairs Trading

- Pair trading is a market-neutral trading strategy that involves the simultaneous purchase of one stock while short selling another stock taking advantage of their co-integration.
- For a given period, we need to maximise the agent's terminal utility of wealth subject to budget constraints.

Stochastic Differential Equations for Co-integrated Stock Prices

Let S_1 and S_2 denote the co-integrated stock prices satisfying the stochastic differential equations:

$$dS_1 = (\mu_1 + \delta z(t))S_1 dt + \sigma_1 S_1 dB_1 \tag{1}$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 \left(\rho dB_1 + \sqrt{1-\rho^2} dB_2\right) \tag{2}$$

where B_1 and B_2 are independent Brownian Motions.

Instantaneous Co-integrating Vector and its Dynamics

The instantaneous co-integrating vector $z(t)$ is defined by:

$$z(t) = a + \ln S_1(t) + \beta \ln S_2(t) \tag{3}$$

The dynamics of $z(t)$ is a stationary Ornstein-Uhlenbeck process described by:

$$dz = \alpha(\eta - z)dt + \sigma_\beta dB_t \tag{4}$$

where $\alpha = -\delta$ is the speed of mean reversion, $\sigma_\beta = \sqrt{\sigma_1^2 + \beta^2\sigma_2^2 + 2\beta\sigma_1\sigma_2\rho}$, B_t is a Brownian motion adapted to F_t, and

$$B_t = \frac{\sigma_1 + \beta\sigma_2\rho}{\sigma_\beta} B_1 + \frac{\beta\sigma_2}{\sqrt{1-\rho^2}\sigma_\beta} B_2$$

$$\eta = -\frac{1}{\delta}\left(\mu_1 - \frac{\sigma_1^2}{2} + \beta\left(\mu_2 - \frac{\sigma_2^2}{2}\right)\right)$$

is the equilibrium level.

Dynamic of Wealth Value and Resulting SDE

The dynamics of the wealth value is given by:

$$dW = \pi_1(t)dS_1 + \pi_2(t)dS_2$$

Substituting equations (1) and (2) into the wealth value equation (5), we obtain the following stochastic differential equation (SDE):

$$\begin{aligned} dW &= \pi_1(t)(\mu_1 + \delta z(t))S_1 dt + \pi_2(t)\mu_2 S_2 dt \\ &\quad + \pi_1(t)\sigma_1 S_1 dB_1 + \pi_2(t)\sigma_2 S_2 \left(\rho dB_1 + \sqrt{1-\rho^2} dB_2\right) \end{aligned}$$

Agent's Objective and Utility Function

The agent's objective is given by:

$$J(t, W, S_1, S_2) = \max_{(\pi_1, \pi_2) \in \mathcal{A}_t} \mathbb{E}\left[U(W_T^{t,W,S_1,S_2}) \right]$$

where $J(t, W, S_1, S_2)$ denotes the value function. The agent seeks an admissible control pair (π_1, π_2) that maximizes the utility of wealth at time T.
Specifying the utility function as:

$$U(W) = -\exp(-\gamma W)$$

which is the CARA (Constant Absolute Risk Aversion) utility, where $\gamma > 0$ is constant and equal to the absolute risk aversion.

HJB Partial Differential Equation and Final Condition

We expect the value function $J(t, W, S_1, S_2)$ to satisfy the following HJB partial differential equation:

$$
\begin{aligned}
J_t + \max_{\pi_1,\pi_2} \Big[& (\pi_1(\mu_1 + \delta z)S_1 + \pi_2\mu_2 S_2)J_W \\
& + (\mu_1 + \delta z)S_1 J_{S_1} + \mu_2 S_2 J_{S_2} + \pi_1\sigma_1^2 S_1^2 J_{WS_1} \\
& + \pi_2\rho\sigma_1\sigma_2 S_1 S_2 J_{WS_1} + \pi_2\sigma_2^2 S_2^2 J_{WS_2} \\
& + \pi_1\rho\sigma_1\sigma_2 S_1 S_2 J_{WS_2} \\
& + \frac{1}{2}\left(\pi_1^2\sigma_1^2 S_1^2 + \rho\pi_1\pi_2\sigma_1\sigma_2 S_1 S_2 + \pi_2^2\sigma_2^2 S_2^2\right) J_{WW} \\
& + \frac{1}{2}\sigma_1^2 S_1^2 J_{S_1 S_1} + \rho\sigma_1\sigma_2 J_{S_1 S_2} + \frac{1}{2}\sigma_2^2 S_2^2 J_{S_2 S_2} \Big] = 0
\end{aligned}
$$

with the final condition:

$$J(T, W, S_1, S_2) = U(W_T) = -\exp(-\gamma W_T)$$

Solution Equations

The solution is given by:

$$\pi_1^* S_1 = \frac{(\mu_1 + \delta z)}{\gamma(1-\rho^2)\sigma_1^2} + \frac{\delta(-2a(t)(\mu_1 + \delta z) - b(t))}{\gamma} - \frac{\rho\mu_2}{\gamma(1-\rho^2)\sigma_1\sigma_2}$$
$$\pi_2^* S_2 = \frac{\mu_2}{\gamma(1-\rho^2)\sigma_2^2} + \frac{\delta\beta(-2a(t)(\mu_1 + \delta z) - b(t))}{\gamma} - \frac{\rho(\mu_1 + \delta z)}{\gamma(1-\rho^2)\sigma_1\sigma_2}$$

Coefficients

The coefficients $a(t)$, $b(t)$, and $c(t)$ are given by:

$$
\begin{aligned}
a(t) &= \frac{T-t}{2(1-\rho^2)\sigma_1^2}, \\
b(t) &= -\frac{1}{4}(\sigma_1^2 + \beta\sigma_2^2)\delta \frac{(T-t)^2}{(1-\rho^2)\sigma_1^2} - \frac{\rho\mu_2}{(1-\rho^2)\sigma_1\sigma_2}(T-t), \\
c(t) &= \frac{1}{2}\frac{\mu_2^2}{(1-\rho^2)\sigma_2^2}(T-t) + \frac{1}{4}(\sigma_1^2 + \beta\sigma_2^2 + 2\sigma_1\sigma_2\beta\rho)\delta^2 \frac{(T-t)^2}{(1-\rho^2)\sigma_1^2} \\
&\quad + \frac{1}{4}\frac{\mu_2(\sigma_1^2 + \beta\sigma_2^2)\delta\rho}{(1-\rho^2)\sigma_1\sigma_2}(T-t)^2 + \frac{1}{24}(\sigma_1^2 + \beta\sigma_2^2)^2\delta^2 \frac{(T-t)^3}{(1-\rho^2)\sigma_1^2}.
\end{aligned}
$$