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Module: High Frequency Trading

Week 6: Ito's lemma

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Brownian Motion

Definition

Brownian Motion, denoted as $W(t)$, is a Wiener process with the following properties:

- $W(0) = 0$.
- For $r < s < t < u$, $W(u) - W(t)$ and $W(s) - W(r)$ are independent stochastic variables.
- For $s < t$, $W(t) - W(s)$ follows a Gaussian distribution $\mathcal{N}(0, t - s)$.
- Brownian Motion has continuous sample paths.

Stochastic Differential Equations (SDE)

A Stochastic Differential Equation (SDE) is a differential equation with both deterministic and stochastic components, typically expressed as:

$$dX(t) = \mu(t)dt + \sigma(t)dW(t)$$

where:

- $X(t)$ is the stochastic process of interest.
- $\mu(t)$ is the deterministic drift term.
- $\sigma(t)$ is the stochastic diffusion term.
- $dW(t)$ represents the differential of the Wiener process.

Integral Formula for $X(t)$

Formula with Initial Condition

Given the stochastic differential equation (SDE):

$$dX(t) = \mu(t)dt + \sigma(t)dW(t)$$

with the initial condition $X(0) = a$, the integral formula for $X(t)$ is:

$$X(t) = a + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s)$$

Stock Price

Formulate return

$$\frac{dS_t}{S} = \mu dt + \sigma dW(t)$$

the integral formula for S_t is:

$$\int_0^t \frac{dS_t}{S_t} = \int_0^t \mu ds + \sigma \int_0^t dW(s)$$

Hence

$$ln \frac{S_t}{S_0} = \mu t + \sigma W_t$$

$$S_t = S_0 e^{\mu t + \sigma W_t}$$

Is it correct?

Taylor Expansion

No! It only holds if

$$\frac{d\ln S_t}{dt} = \frac{1}{S_t} \cdot \frac{dS_t}{dt}$$

However, According to Taylor expansion

$$d\ln(S_t) = \frac{dS_t}{S_t} - \frac{1}{2}\left(\frac{dS_t}{S_t}\right)^2 + \frac{1}{3}\left(\frac{dS_t}{S_t}\right)^3 - \cdots$$

Because

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Properties of the Wiener Process

Expectations and Variances of Increments

For the Wiener process $W(t)$ with Δt being a small time increment:

- $\mathbb{E}[\Delta W] = 0$: The expectation of the increment is zero.
- $\mathrm{Var}(\Delta W) = \Delta t$: The variance of the increment is also equal to the time increment.
- $\mathbb{E}[(\Delta W)^2] = \Delta t$: The expectation of the square of the increment is equal to the time increment.
- $\mathrm{Var}((\Delta W)^2) = 2(\Delta t)^2$: The variance of the square of the increment is twice the square of the time increment.

Behaviour of $(dW(t))^2$ as dt Approaches Zero

From the properties of Wiener process, we can find that as the time increment dt approaches zero, the variance of the process $(dW(t))^2$ goes to zero faster than the expected value. Therefore, the process $(dW(t))^2$ becomes deterministic.

$$(dW(t))^2 = dt$$

Itô's Isometry

Thereom

Itô's Isometry states that for a stochastic integral:

$$\mathbb{E}\left[\left(\int_0^t X(s)dW(s)\right)^2\right] = \int_0^t \mathbb{E}\left[X(s)^2\right] ds$$

Itô's Lemma

Theorem

Itô's Lemma relates the differential of a function $f(t, X(t))$ of a stochastic process $X(t)$ to its derivatives and the stochastic differential of $X(t)$:

$$df(t, X(t)) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X} dX(t) + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} (dX(t))^2$$

Stock Price Formula Using Itô's Lemma

Formula

By applying Itô's Lemma to the Geometric Brownian Motion model. Let $f = ln(S_t)$:

$$d(\ln S) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW$$

thus

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$$

SDE for Merton's Jump Diffusion (MJD)

Model Description

Merton's Jump Diffusion is an extension of the Geometric Brownian Motion model that incorporates jumps in the asset price. The SDE for MJD is given by:

$$\frac{dS_t}{S_t} = (\mu - \lambda k)dt + \sigma dW_t + (y_t - 1)dN_t$$

where:

- N_t is a Poisons process with intensity λ.
- y_t is absolute jump size.

Poisson Distribution and Process

Poisson Process

A Poisson process is a counting process that models the number of events occurring in a fixed interval of time. Key characteristics include:

- λ denotes the mean arrival rate of events during a time interval dt.
- The probability that an event will occur is given by λdt, and that it will not occur is $1 - \lambda dt$.

Interpretation

The event represents a price jump in MJD.

Probability Mass Function (PMF) of the Poisson Distribution

PMF of the Poisson Distribution

The Poisson distribution is characterized by its probability mass function (PMF), which describes the probability of observing a specific number of events (n) in a given time interval. The PMF is given by:

$$P(X = n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

where:

- $P(X = n)$ is the probability of observing n events. $n = 0, 1, 2, 3...$
- λ is the mean number of events in the interval.

Then, within time t,

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Memoryless Property of the Poisson Process

Interarrival Times

In a Poisson counting process, the interarrival times represent the time elapsed between consecutive Poisson events. If T_j is the time of the j-th arrival, then:

$$P[T_{n+1} - T_n < s | T_1, \ldots, T_n] = 1 - e^{-\lambda s}$$

This means that the interarrival times $T_1, T_2 - T_1, \ldots$ of a Poisson process are independent and identically distributed (i.i.d.) with a cumulative distribution function (CDF) of $1 - e^{-\lambda s}$.

Memoryless Property

The memoryless property can be expressed as:

$$P[\tau > t + s | \tau > t] = P[\tau > s]$$

This property states that the probability of observing an event does not depend on past events, making the Poisson process memoryless in terms of interarrival times.

Applying Itô's Lemma to MJD

Result

By applying Itô's Lemma to the Merton's Jump Diffusion model, we obtain the following stochastic differential equation for $d(\ln S_t)$:

$$d(\ln S_t) = (\mu - \lambda k)dt + \sigma dW_t + \ln y_t$$

Thus,

$$\ln S_t = \ln S_0 + (\mu - \sigma^2/2 - \lambda k)t + \sigma W_t + \sum_{i=1}^{N_T} y_i$$

or,

$$S_t = S_0 exp[(\mu - \sigma^2/2 - \lambda k)t + \sigma W_t + \sum_{i=1}^{N_T} lny_i]$$