7FNCE025 High Frequency Trading
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Week 3 Seminar Solutions

1. If $dS = \mu S dt + \sigma S dZ$, and $A$ and $n$ are constants, find the stochastic equations satisfied by
(a) $f(S) = AS$;
(b) $f(S) = S^n$.

2. By expanding $df$ in Taylor series to $\mathcal{O}dt$ and using that $(dZ)^2 = dt$, prove that

$$\int_{t_0}^{t} Z(\tau)dZ(\tau) = \frac{1}{2}\left(Z(t)^2 - Z(t_0)^2\right) - \frac{1}{2}(t - t_0).$$

3. Consider the general stochastic differential equation

$$dG = A(G,t)dt + B(G,t)dZ.$$

Use Itô's Lemma to show that it is theoretically possible to find a function $f(G)$ which itself follows a random walk but with zero drift.

4. There are $n$ assets satisfying the following stochastic differential equations

$$dS_i = \mu_i S_i dt + \sigma_i S_i dZ_i \qquad \text{for } i = 1, \ldots, n.$$

Recall that the Wiener process $dZ_i$ satisfies

$$\mathbb{E}[dZ_i] = 0, \qquad dZ_i^2 = dt$$

as usual, but the asset price changes are correlated with

$$dZ_i dZ_j = \rho_{ij} dt$$

where $-1 \leq \rho_{ij} = \rho_{ji} \leq 1$.[1]
Derive Itô's Lemma for a function $f(S_1, \ldots, S_n)$ of the $n$ assets $S_1, \ldots, S_n$.

## Exercise 1

We can write Itô's Lemma for the defined SDE as

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \mu S \frac{\partial f}{\partial S}\right) dt + \sigma S \frac{\partial f}{\partial S} dZ. \tag{1}$$

[1]Note that we do not need to take the expected value. In general if two Wiener processes $Z_1(t)$ and $Z_2(t)$ are correlated we have that $dZ_1(t)dZ_2(t) = \rho dt$ where $\rho$ is the correlation coefficient

(a) Replacing $f(S) = AS$ into (1) we obtain

$$\begin{aligned} df &= \mu(AS)dt + \sigma(AS)dZ \\ \frac{df}{f} &= \mu dt + \sigma dZ. \end{aligned} \tag{2}$$

(b) Replacing $f(S) = S^n$ into (1) we obtain

$$\begin{aligned} df &= \left(n\mu + \frac{\sigma^2}{2}n(n-1)\right) S^n dt + \sigma n S^n dZ \\ \frac{df}{f} &= \hat{\mu} dt + \hat{\sigma} dZ, \end{aligned} \tag{3}$$

where $\hat{\mu} = n\mu + \frac{\sigma^2}{2}n(n-1)$ and $\hat{\sigma} = \sigma n$.

## Exercise 2

Expanding $df$ in Taylor series to $\mathcal{O}dt$ and using that $(dZ)^2 = dt$ we may write

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial Z}dZ + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2}(dZ)^2 \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial Z}dZ + \frac{1}{2}\frac{\partial^2 f}{\partial Z^2}dt. \end{aligned} \tag{1}$$

Upon substituting $f = Z^2$ into (1) we obtain

$$d(Z)^2 = 2ZdZ + dt; \tag{2}$$

and integrating this last equation we obtain

$$\begin{aligned} \int_{t_0}^{t} d(Z)^2 &= 2\int_{t_0}^{t} Z(\tau)dZ(\tau) + \int_{t_0}^{t} dt \\ Z(t)^2 - Z(t_0)^2 &= 2\int_{t_0}^{t} Z(\tau)dZ(\tau) + (t - t_0). \end{aligned} \tag{3}$$

Hence, rearranging terms we finally obtain

$$\int_{t_0}^{t} Z(\tau)dZ(\tau) = \frac{1}{2}\left(Z(t)^2 - Z(t_0)^2\right) - \frac{1}{2}(t - t_0). \tag{4}$$

## Exercise 3

For the SDE

$$dG = A(G,t)dt + B(G,t)dZ \tag{1}$$

we can write Itô's Lemma as

$$df = \left( \frac{\partial f}{\partial t} + \frac{1}{2}B^2 \frac{\partial^2 f}{\partial G^2} + A\frac{\partial f}{\partial G} \right) dt + B\frac{\partial f}{\partial G} dZ. \tag{2}$$

Consequently, any function $f$ which satisfies, with appropriate boundary conditions, the differential equation

$$\frac{\partial f}{\partial t} + \frac{1}{2}B^2 \frac{\partial^2 f}{\partial G^2} + A\frac{\partial f}{\partial G} = 0 \tag{3}$$

will be such that $f$ follows itself a random walk with no drift, namely

$$df = B\frac{\partial f}{\partial G} dZ. \tag{4}$$

## Exercise 4

There are $n$ assets satisfying the following stochastic differential equations

$$dS_i = \mu_i S_i dt + \sigma_i S_i dZ_i \qquad \text{for } i = 1, \ldots, n. \tag{1}$$

where recall that

$$\mathbb{E}[dZ_i] = 0, \qquad dZ_i^2 = dt. \tag{2}$$

However, the asset price changes are correlated with

$$dZ_i dZ_j = \rho_{ij} dt \tag{3}$$

where $-1 \leq \rho_{ij} = \rho_{ji} \leq 1$.

Let us then derive Itô's multivariate Lemma. We can write the system of stochastic differential equation defined in (1) subject to the constraint in (2) and (3) as

$$\begin{pmatrix} dS_1 \\ dS_2 \\ \vdots \\ dS_n \end{pmatrix} = \begin{pmatrix} \mu_1 S_1 \\ \mu_2 S_2 \\ \vdots \\ \mu_n S_n \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} S_1 & \sigma_{12} S_2 & \cdots & \sigma_{1n} S_1 \\ \sigma_{21} S_2 & \sigma_{22} S_2 & \cdots & \sigma_{2n} S_2 \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{n1} S_n & \sigma_{n2} S_n & \cdots & \sigma_{nn} S_n \end{pmatrix} \begin{pmatrix} d\tilde{Z}1 \\ d\tilde{Z}_2 \\ \vdots \\ d\tilde{Z}_n \end{pmatrix} \tag{4}$$

where the Wiener increments are now independent, such that $d\tilde{Z}_i d\tilde{Z}_j = 0$. The correlation is now present in the $n \times n$ matrix defined by the $\sigma_{ij}$ components.

We may also write the system of correlated differential equations as

$$\begin{pmatrix} dS_1 \\ dS_2 \\ \vdots \\ dS_n \end{pmatrix} = \begin{pmatrix} \mu_1 S_1 \\ \mu_2 S_2 \\ \vdots \\ \mu_n S_n \end{pmatrix} dt + \begin{pmatrix} \sigma_1 S_1 \\ \sigma_2 S_2 \\ \vdots \\ \sigma_n S_n \end{pmatrix} \begin{pmatrix} dZ_1 & dZ_2 & \dots & dZ_n \end{pmatrix} \tag{5}$$

where $dZ_i dZ_j = \rho_{ij} dt$.

We can expand in Taylor series any function $f(S_1, \dots, S_n)$ as

$$\begin{aligned} df &= \frac{\partial f}{\partial t} dt + \left( \frac{\partial f}{\partial S_1} dS_1 + \frac{\partial f}{\partial S_2} dS_2 + \dots + \frac{\partial f}{\partial S_n} dS_n \right) \\ &\quad + \left( \frac{\partial f}{\partial S_1} dS_1 + \frac{\partial f}{\partial S_2} dS_2 + \dots + \frac{\partial f}{\partial S_n} dS_n \right)^2 + \cdots . \end{aligned} \tag{6}$$

Let us rewrite (6), up to second order, in a more compact form as

$$df = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial S_i} dS_i + \sum_{i=1}^{n} \frac{\partial^2 f}{\partial S_i^2} (dS_i)^2 + \sum_{i \neq j}^{n} \frac{\partial^2 f}{\partial S_i S_j} dS_i dS_j, \tag{7}$$

and replacing into (7) $dS_i$ as given by (1), with (2) and (3) we obtain (up to second order in $dt$)

$$(dS_i)^2 = \sigma_i^2 S_i^2 dt \tag{8}$$

and

$$\begin{aligned} dS_i dSj &= \sigma_i \sigma_j S_i S_j dZ_i dZ_j \\ &= \sigma_i \sigma_j S_i S_j \rho_{ij} dt. \end{aligned} \tag{9}$$

Finally, regrouping terms we may write Itô's Lemma for a function $f(S_1, \dots, S_n)$ of the $n$ assets $S_1, \dots, S_n$ as

$$df = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \left( \frac{\partial f}{\partial S_i} \mu_i S_i + \frac{\partial^2 f}{\partial S_i^2} \sigma_i^2 S_i^2 \right) dt + \sum_{i \neq j}^{n} \left( \frac{\partial^2 f}{\partial S_i S_j} \sigma_i \sigma_j S_i S_j \rho_{ij} \right) dt + \sum_{i=1}^{n} \frac{\partial f}{\partial S_i} \sigma_i S_i dZ_i. \tag{10}$$