Computational Methods for Finance
Week 6: Mathematical Model IV (Binomial Trees)

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Learning Outcomes

At the end of this lecture you will be able to

- Price options using the Binomial Trees method.

Review

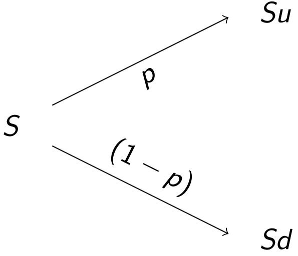
- Risk-Neutral Valuation
  - The BSM differential equation does not contain variables that are affected by investors' risk preferences (i.e. the drift parameter μ)
    $\Rightarrow$ Any set of risk preferences can be used when evaluating f.
  - We can make a simple assumption that all investors are risk-neutral.
    $\Rightarrow$ The expected return on *all* investment assets is the *risk-free rate* (including the underlying asset, and thus the drift parameter can be replaced by the risk-free rate wherever it appears in the derivation).
  - Under risk-neutral valuation, solutions obtained are valid in all worlds (not only the risk-neural world).
  - The economic argument for the risk-neutral valuation is that since we can perfectly hedge the option with the underlying (e.g. non-dividend-paying stock), we should not be rewarded for taking unnecessary risk; only the risk-free rate of return is in the equation. This means that if you and I agree on the *volatility* (i.e. σ) of an asset we will agree on the value of its derivatives even if we have differing estimates of the drift.

Binomial Tree

- One of numerical methods to price option.
- A method to price the American option.
- The Representation
  The movement of stock prices (as given by geometric Brownian motion) can be approximated by a binomial tree. Note that:
  - Each point in the tree is referred to as a node.
  - The first node corresponds to the current price.
  - The last set of nodes correspond to the possible prices at the maturity of the option.

Binomial Tree

- The Representation
  The one-step binomial model is given by



  - Starting at S, the stock price is allowed to move up (with probability p) to Su or down (with probability $(1-p)$) to Sd.
  - Increasing the number of steps in the tree improves accuracy.

- The Representation
  The three-step binomial model is given by

$$
\begin{tikzcd}
 & & & Su^3 \\
 & & Su^2 \arrow[ur] \arrow[dr] & \\
 & Su \arrow[ur] \arrow[dr] & & Su \\
S \arrow[ur] \arrow[dr] & & \mathrm{S} \arrow[ur] \arrow[dr] & \\
 & Sd \arrow[ur] \arrow[dr] & & Sd \\
 & & Sd^2 \arrow[ur] \arrow[dr] & \\
 & & & Sd^3
\end{tikzcd}
$$

The BSM Differential Equation

- The Representation

  Assuming that u $=1/$d (u$>$1) and using risk-neutral valuation, geometric Brownian motion can be approximated using the following parameter values:

$$p = \frac{a-d}{u-d},$$

$$u = e^{\sigma\sqrt{\Delta t}},$$

$$d = e^{-\sigma\sqrt{\Delta t}},$$

$$a = e^{r\Delta t}.$$

  These parameters are chosen so that the tree gives correct values for the mean and variance of the stock price changes in a risk-neutral world.

The BSM Differential Equation

- Backwards Induction The logic is as follows:
  - The value of the option at the final nodes is known.
  - Therefore, we can work back through the tree using risk-neutral valuation to calculate the value of the option at each node.
  - This process continues until the first (single) node is reached. At this point the price of the derivative is determined.

The BSM Differential Equation

- Backwards Induction (procedure) For an American put option on a non-dividend paying asset:
  - Step 1: Calculate the stock prices for *each* node.
  - Step 2: At T, calculate the option price ($\max(K - S_T\ , 0)$).
  - Step 3: The premium at the previous node will equal:
    (a) The expected value of the subsequent premia discounted by the risk-free rate.
    OR
    (b) The value of $\max(K - S_{i\Delta t}\ , 0)$, if this is greater than the value calculated in Step 3 (a) (i.e., early exercise).
  - Step 4: Continue backwards until the first node is reached.

The BSM Differential Equation

- Backwards Induction

Example

Consider an American put option with T $= 5/12$, S $= 50$, K $= 50$, r $=$ 0.1, $\sigma = 0.4$. Divide the life of the option into one month intervals $(\Delta t = 1/12)$. Under these conditions, what are a, u, d, p in the binomial tree model that I showed you just now?

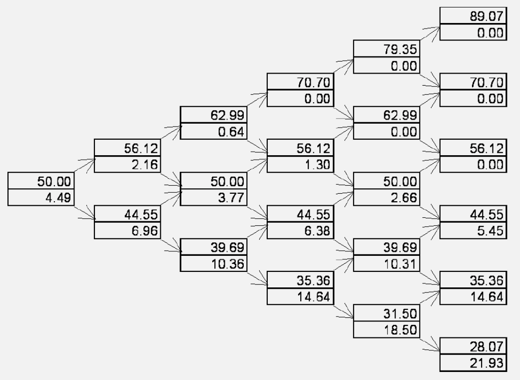
$$u = 1.1224, \ d = 0.8909, \ a = 1.0084, \ p = 0.5073.$$

Please plot the binomial tree which includes the possible stock prices and the corresponding option prices in each period.

The BSM Differential Equation

- Backwards Induction

Example



Reading

- Chapter 13, Hull (2015)