7FNCE025 High Frequency Trading
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Week 4 Seminar Solutions

Let the state equation be

$$\frac{dy(t)}{dt} = ry(t) + w - c(t) \tag{1}$$

where $y(t)$ is wealth at time t, r is the risk-free rate, w denotes wages, and $c(t)$ consumption. The agents's problem is to

$$\max_c \int_t^T e^{-\rho s} h(c(s))ds\,, \tag{2}$$

and he faces the constraint $y(T) \geq 0$, i.e. the agent is restricted to end with non-negative cash at time T.

Before trying to solve the optimisation problem we ask ourselves:

1. What amount of initial cash does the restriction $y(T) \geq 0$ implies?
2. What is the maximum amount of cash that the agent can end up with at time T?

The answers to these two questions will help solve the optimisation problem.

Before getting down to the maths we can think about the second question in simple financial terms. First, if the agent does not consume any of the cash between t and T then he maximises the amount of cash that is accumulated up until and including time T. Thus, if we set $c(s) = 0$ for $s \in [t, T]$ then the state equation (1) becomes

$$\frac{dy(s)}{ds} = ry(s) + w\,. \tag{3}$$

To solve (3) we integrate it between t and T

$$\begin{aligned}
\int_t^T \frac{dy}{ry + w} &= \int_t^T ds \\
\int_t^T \frac{dv}{v} &= \int_t^T ds \\
J(T) &= J(t)e^{r(T-t)} \\
ry(T) + w &= (ry(t) + w)e^{r(T-t)} \\
y(T) &= y(t)e^{r(T-t)} + \frac{w}{r}\left(e^{r(T-t)} - 1\right)\,. \qquad (4)
\end{aligned}$$

The last expression, which tells us how much cash we end up with at time T if we start with $y(t)$ and do not consume anything, is very simple to understand. Note that the first term on the right-hand side of (4) is the cash that we started with, which is invested for a period of time $T - t$ in a bank account at the risk-free rate r. The second term is pretty much the same because it says that the wages w that we earn at every instant in time are also put in a bank account and left there earning interest – see that in this case the amount of cash we end up with, which comes from wages and interest, is given by $\int_t^T we^{r(T-s)}ds$.

Below we will see that knowing the maximum amount of cash the agent can end up with comes in handy when looking for a trial solution to the HJB we derive. So let us first turn the crank as in previous exercises and show that the value function $J(y,t)$ satisfies:

$$0 = \max_c \left[e^{-\rho t}c^\gamma - \rho J(y,t) + J_t(y,t) + J_y(y,t)(ry(t) + w - c(t))\right] . \tag{5}$$

To show this, we proceed as usual:

$$\begin{aligned}
J(y,t) &= \max_c \int_t^T e^{-\rho s}c^\gamma ds \\
&= \max_c \left[e^{-\rho(t+\Delta t')}c^\gamma \Delta t + \int_{t+\Delta t}^T e^{-\rho s}c^\gamma ds\right] \\
&= \max_c \left[e^{-\rho(t+\Delta t')}c^\gamma \Delta t + J(y+\Delta y, t+\Delta t)\right] \\
&= \max_c \left[e^{-\rho(t+\Delta t')}c^\gamma \Delta t + J(y,t) + J_t(y,t)\Delta t + J_y(y,t)\Delta y + \text{error terms}\right] \\
0 &= \max_c \left[e^{-\rho(t+\Delta t')}c^\gamma + J_t(y,t) + J_y(y,t)(ry(t) + w - c(t)) + \text{error terms}\right] \\
0 &= \max_c \left[e^{-\rho t}c^\gamma + J_t(y,t) + J_y(y,t)(ry(t) + w - c(t))\right] ,
\end{aligned}$$

where it is simple to calculate the optimal consumption

$$c^*(t) = \left(\frac{e^{\rho t}J_y(y,t)}{\gamma}\right)^{\frac{1}{\gamma-1}} .$$

As before, we are not there yet because we still do not know the value function $J(y,t)$. We need to guess the form of the value function. Above we solved a similar problem where the agent maximised utility but did not earn any wages. In that case the trial solution was of the form $g(t)y^\gamma$ which, unfortunately, does not work when the agent earns wages. The problem is that earning wages, which affects the state equation, does make a difference to the value function. Somehow, we need to see how wages appear in the trial solution.

To this end, we go back to our discussion of what was the maximum amount of cash we could have after $T-t$ which is given by (4). Now we can ask ourselves, what about if my initial endowment $y(t)$ of cash is such that

$$-y(t)e^{r(T-t)} + \frac{w}{r}\left(e^{r(T-t)} - 1\right) = 0 . \tag{6}$$

This means that I start owing money and should repay it. Also, recall that one of the conditions of the problem is that the agent cannot end up owing any money, i.e. cannot end up with $y(T) < 0$. This means that the agent must use all his wages to pay off the debt, consume nothing, and end up with $y(T) = 0$. Not consuming means that the agent will, at all times, derive zero utility, $J(y,t) = 0$ for all $t \leq T$.

So why do we want to know all of this? Well, this is important when writing down the trial solution. Note that the trial solution $J(y,t) = g(t)y^\gamma$ will not satisfy

$$g(t)y^\gamma(t) = 0\,, \qquad \text{for all } t \leq T\,,$$

if the initial endowment $y(t)$ is such that equation (6) holds. Thus all we need is to tweak the trial solution:

$$J(y,t) = g(t)\left(y + \frac{w}{r}\left(1 - e^{-r(T-t)}\right)\right)^\gamma\,, \tag{7}$$

where it is obvious that if $y(t)$ is such that equation (6) is satisfied, the value function is zero over the period $[t,T]$ as required.

Now we proceed to calculate the derivatives of the value function:

$$J_t(y,t) = g_t(t)\left(y(t) + \frac{w}{r}\left(1 - e^{-r(T-t)}\right)\right)^\gamma - \gamma w g(t) e^{-r(T-t)}\left(y(t) + \frac{w}{r}\left(1 - e^{-r(T-t)}\right)\right)^{\gamma-1}\,, \tag{8}$$

and

$$J_y(y,t) = \gamma g(t)\left(y(t) + \frac{w}{r}\left(1 - e^{-r(T-t)}\right)\right)^{\gamma-1}\,. \tag{9}$$

Thus, the optimal consumption is given by

$$c^*(t) = \left(e^{\rho t} g(t)\right)^{\frac{1}{\gamma-1}}\left(y(t) + \frac{w}{r}\left(1 - e^{-r(T-t)}\right)\right)\,. \tag{10}$$

Putting all of these into the HJB:

$$
\begin{aligned}
0 &= e^{-\rho t}\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}}\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma} \\
&\quad +g_t(t)\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma}-\gamma w g(t)e^{-r(T-t)}\left(y+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma-1} \\
&\quad +\gamma g(t)\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma-1} \\
&\quad \times\left(ry(t)+w-\left(e^{\rho t}g(t)\right)^{\frac{1}{\gamma-1}}\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)\right) \\
0 &= \left[g_t(t)+e^{-\rho t}\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}}\right]\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma} \\
&\quad +\gamma g(t)\left[\left(ry(t)+w-\left(e^{\rho t}g(t)\right)^{\frac{1}{\gamma-1}}\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)\right)-we^{-r(T-t)}\right] \\
&\quad \times\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma-1} \\
0 &= \left[g_t(t)+e^{-\rho t}\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}}\right]\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma} \\
&\quad +\gamma g(t)\left[ry(t)+w\left(1-e^{-r(T-t)}\right)-\left(e^{\rho t}g(t)\right)^{\frac{1}{\gamma-1}}\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)\right] \\
&\quad \times\left(y(t)+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma-1} \\
0 &= \left[g_t(t)+e^{-\rho t}\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}})\right]\left(y+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma} \\
&\quad +\gamma r g(t)\left[y+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right]\left(y+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma-1} \\
&\quad -\gamma g(t)\left(e^{\rho t}g(t)\right)^{\frac{1}{\gamma-1}}\left(y+\frac{w}{r}\left(1-e^{-r(T-t)}\right)\right)^{\gamma} \\
0 &= g_t(t)+e^{-\rho t}\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}}+\gamma r g(t)-\gamma g(t)\left(e^{\rho t}g(t)\right)^{\frac{1}{\gamma-1}} \\
0 &= e^{\rho t}g_t(t)+\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}}-\gamma\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}}+\gamma r e^{\rho t}g(t)\,.
\end{aligned}
$$

Using $\frac{d}{dt}e^{\rho t}g(t)=\rho e^{\rho t}g(t)+e^{\rho t}g_t(t)$

$$0=\frac{d}{dt}e^{\rho t}g(t)+(\gamma r-\rho)e^{\rho t}g(t)+(1-\gamma)\left(e^{\rho t}g(t)\right)^{\frac{\gamma}{\gamma-1}}\ ,$$

letting $G(t)=e^{\rho t}g(t)$

$$G_t(t)+(r\gamma-\rho)G(t)+(1-\gamma)G(t)^{\frac{\gamma}{\gamma-1}}=0\,,$$

and dividing through by $(1-\gamma)G(t)^{\frac{\gamma}{\gamma-1}}$

$$\frac{G_t(t)}{(1-\gamma)G(t)^{\frac{\gamma}{\gamma-1}}}+\frac{r\gamma-\rho}{(1-\gamma)}G(t)^{1-\frac{\gamma}{\gamma-1}}+1=0\,.$$

Now we let $H(t) = G(t)^{\frac{1}{1-\gamma}}$ so that $G_t(t) = (1-\gamma)H(t)^{-\gamma}H_t(t)$ and write

$$\begin{aligned}
H_t(t)(1-\gamma)H(t)^{-\gamma} + (r\gamma - \rho)H(t)^{1-\gamma} + (1-\gamma)H(t)^{-\gamma} &= 0 \\
H_t(t) + \frac{r\gamma - \rho}{(1-\gamma)}H(t) + 1 &= 0\,,
\end{aligned}$$

with boundary condition $H(T) = 0$. This is an easy linear equation to solve. Let $v(t) = -\left(\mu H(t) + 1\right)$, with $v(T) = 1$ and $\mu = \frac{r\gamma-\rho}{1-\gamma}$, and write

$$\begin{aligned}
v_t(t) &= -\mu v(t) \\
\int_t^T \frac{dv(s)}{v(s)} &= -\mu(T-t) \\
-1 &= v(t)e^{-\mu(T-t)}\,.
\end{aligned}$$

Hence

$$H(t) = -\frac{1}{\mu}\left(1 - e^{\mu(T-t)}\right)\,.$$

Now we can write (note that instead of having $r\gamma - \rho$ we have written $-(\rho - r\gamma)$ since it is more natural for the discount rare ρ to appear in this way)

$$g(t) = e^{-\rho t}\left[\frac{1-\gamma}{\rho - r\gamma}\left(1 - e^{-\frac{(\rho - r\gamma)(T-t)}{1-\gamma}}\right)\right]^{1-\gamma}\,,$$

and the value function

$$J(y,t) = e^{-\rho t}\left[\frac{1-\gamma}{\rho - r\gamma}\left(1 - e^{-\frac{(\rho - r\gamma)(T-t)}{1-\gamma}}\right)\right]^{1-\gamma}\left(y(t) + \frac{w}{r}\left(1 - e^{-r(T-t)}\right)\right)^{\gamma}\,,$$

and the optimal consumption

$$c^*(t) = \frac{1-\gamma}{\rho - r\gamma}\left(1 - e^{-\frac{(\rho - r\gamma-)(T-t)}{1-\gamma}}\right)^{-1}\left(y(t) + \frac{w}{r}\left(1 - e^{-r(T-t)}\right)\right)\,.$$

As a sanity check you can verify that if we let $w = 0$ in the equation above we obtain the same optimal consumption policy as in the previous lecture.