Computational Methods for Finance
Week 9: Greeks

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November 2023

Learning Outcomes

At the end of this lecture you will be able to

- Use derivatives to hedge long and short positions.
- Differentiate between the different risks faced when taking option positions.

Review

- The BSM Pricing Formula

  The BSM formula (Black-Scholes model) for the price of a European *call* on a non-dividend paying stock is

$$c = SN(d_1) - Ke^{-rT}N(d_2), \tag{1}$$

where

$$d_1 = \frac{ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

where $N(.)$ is the *cumulative probability distribution* function for a *standardised normal distribution*. $N(d_2)$ is the probability that a call option will be exercised in a risk-neutral world. $Se^{rT}N(d_1)$ is the expected stock price at time T in a risk-neutral world when stock prices less than the strike price are counted as zero.

Review

- The BSM Pricing Formula
  The BSM formula for the price of a European *put* on a non-dividend paying stock is

$$p = Ke^{-rT}N(-d_2) - SN(-d_1), \tag{2}$$

where all previous notation is maintained.

Hedging Using Options

- Delta
  Delta (Δ) is defined as the rate of change of the option price with respect to the underlying asset; specifically,

$$\Delta = \frac{\partial c}{\partial S},$$

  where all previous notation is maintained.

- Delta Hedging
  This involves maintaining a delta neutral portfolio, in turn, achieved by purchasing Δ shares for every 1 share contained in the option position.
  - European Call Delta: for a European call on a non-dividend paying stock $\Delta = N(d_1)$, where $N(.)$ is the cumulative density function for a standard normal distribution.
  - European Put Delta: for a European put on a non-dividend paying stock $\Delta = N(d_1)$ -1.

Hedging Using Options

- Delta Hedging

Example

A bank has sold for \$300,000 a European call option on 100,000 shares of a non-dividend paying stock. Furthermore, assume that the current stock price is \$49, the strike price is \$50, the risk-free rate is 5% per annum, the stock return volatility is 20% per annum, the time-to-maturity is 20 weeks (0.3846 years), and the expected return on the stock is 13% per annum. Using this information we have

$$d1 = \frac{ln(49/50) + ((0.05 + 0.2^2/2) \times 0.3846)}{0.2 \times \sqrt{0.3846}} = 0.0542.$$

Thus, the delta is $N(d_1)$ or 0.522. When the stock price changes by ΔS, the option price changes by $0.522\Delta S$.

Hedging Using Options

- Delta Hedging
  Note that:
  - As delta changes over time, the investor's position remains delta neutral for only a relatively short period of time
  - Consequently, the hedge has to be frequently adjusted or rebalanced.
  - When hedges are rebalanced the process is referred to as dynamic hedging (as opposed to static hedging).

Hedging Using Options

- Gamma

  The gamma (Γ) of a portfolio of options on an underlying asset is the rate of change of the portfolio's delta with respect to the price of the underlying asset. It is given by

$$\Gamma = \left(\frac{\partial^2 \Pi}{\partial S^2} \right),$$

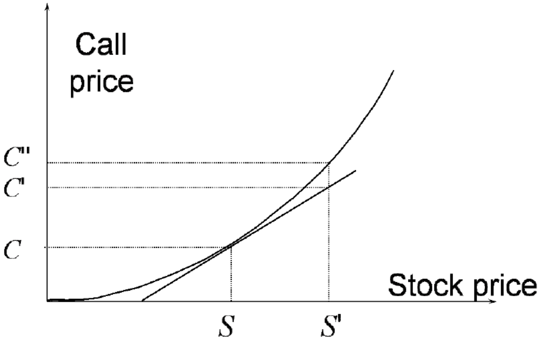
  where Π is the value of the portfolio of options.

Hedging Using Options

- Gamma
  Note that:
  - If gamma is small (large), then delta changes slowly (rapidly), and hence infrequent (frequent) adjustments are required to keep a portfolio delta neutral.
  - Gamma is greatest for options that are close to the money.
  - Gamma addresses delta hedging errors caused by curvature in the relationship between call premia and the underlying asset price.

Hedging Using Options

- Gamma
  The difference between C' and C'' in the following diagram leads to hedging error. The size of this error is measured by gamma.



Hedging Using Options

- Gamma

  For a European call **or** put option on a non-dividend paying stock, gamma is given by

$$\Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T}},$$

  where all previous notation is maintained.

Example

Using the parameters in the above example, the option's gamma is given by

$$\Gamma = \frac{N'(0.0542)}{490.2 \times \sqrt{0.3846}} = 0.066.$$

Thus, when the stock price changes by ΔS, the delta of the option changes by $0.066 \times \Delta S$.

- Theta

  The theta (Θ) of a portfolio of options is the rate of change of the value of the portfolio with respect to time (other things remaining equal).

  For a European option on a non-dividend-paying stock,

$$\Theta(call) = -\frac{SN'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2),$$

where $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ (the probability density function for a standard normal distribution).

$$\Theta(put) = -\frac{SN'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2),$$

Hedging Using Options

- Vega
  Vega (ν) is the rate of change of the value of a derivatives portfolio with respect to volatility.
  For a European call **or** put option on a non-dividend-paying stock, vega is given by

$$\nu = S\sqrt{T}N'(d_1)$$

- Rho
  The rho (ρ) of a porfolio of options is the rate of change of the value of the portfolio with respect to the interest rate.

$$rho(call) = KTe^{-rT}N(d_2)$$

$$rho(put) = -KTe^{-rT}N(-d_2)$$

Reading

- Chapter 19, Hull (2015)