March 6, 2024

Module: High Frequency Trading

Week 7: Optimal Control: deterministic case

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Introduction

Dynamic programming (DP)

- deterministic case
- general setups under uncertainty (i.e. Brownian motion)
- maximize utility/ return over a certain period
- subject to constraints

Deterministic Case Set-Up

Let y denote the wealth process which satisfies the state equation:

$$\frac{dy(t)}{dt} = ry(t) - c(t) \quad (1)$$

with the initial condition $y(t) = x$.

Here, r represents the risk-free rate, and $c(t)$ is the amount that the agent consumes at time t, while x is his initial wealth. This agent does not earn any other money in terms of wages. We also

assume that the agent's objective is to maximize:

$$\max_{c(t)} \int_t^T h(y(s), c(s))\, ds \quad (2)$$

Value Function

The agent is incentivized to postpone consumption because he earns the risk-free rate on the cash he holds. However, the agent is incentivized to consume to survive. Therefore, we expect h to be increasing in consumption, i.e., $\frac{dh}{dc} > 0$.

Additionally, we assume that at time T, the agent derives utility $B(y(T))$, which can be seen as a bequest. The value function is expressed as:

$$J(y,t) = \max_c \left[\int_t^T h(y(s), c(s))\, ds + B(y(T)) \right] \quad (3)$$

Here, T is the final time, t is the current time, and $h(y(s), c(s))$ represents the agent's utility function.

Principle of Optimality

The *Principle of Optimality* can be summarized as follows: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. (See Bellman, 1957, Chap. III.3.)

The value function can be expressed as:

$$J(y,t) = \max_c \left[\int_t^{t+\Delta t} h(y(s),c(s))\,ds + \max_c \left(\int_{t+\Delta t}^{T} h(y(s),c(s))\,ds + B(y(T)) \right) \right]$$

The control in this equation is $c(t)$ because it is what we can change optimally to obtain maximal utility.

Intermediate Value Theorem

To find the optimal policy (by this, we mean finding $c^*(t)$), we go through the motions in different steps by focusing on the two terms.

We use the Intermediate Value Theorem (IVT) to write the first term :

$$\int_t^{t+\Delta t} h(y(s), c(s))\, ds = h(y + \Delta y, c(t + \Delta t'))\Delta t$$

where $\Delta t' \in [0, \Delta t]$.

The second term:

$$\max_c \left[\int_{t+\Delta t}^{T} h(y(s), c(s))\, ds + B(y(T)) \right] = J(y + \Delta y, t + \Delta t)$$

By Taylor expansion

$$J(y + \Delta y, t + \Delta t) = J(y, t) + J_t(y, t)\Delta t + J_y(y, t)\Delta y + \text{error terms} \quad (\ldots$$

HJB

The value function $J(y, t)$ can be written as:

$$J(y,t) = \max_c \left[h(y + \Delta y, c(t + \Delta t'))\Delta t + J(y,t) + J_t(y,t)\Delta t + J_y(y,t)(ry(t) - c(t))\Delta t + \text{error terms} \right] \tag{6}$$

Dividing through by Δt and letting $\Delta t \to 0$:

$$0 = \max_c \left[h(y, c(t)) + J_t(y,t) + J_y(y,t)(ry(t) - c(t)) \right] \tag{7}$$

Here, J_t and J_y denote partial derivatives with respect to t and y respectively.
This is known as the Bellman equation or the Hamilton-Jacobi-Bellman Equation (HJB).

Utility Function

Specifying $h(y, c)$:

$$h(y, c(t)) = e^{-\rho t} U(c(t)) \quad (8)$$

where ρ is the discount rate, and the utility function U is given by:

$$U(c(t)) = \frac{c(t)^\gamma}{\gamma} \quad (9)$$

with $0 < \gamma < 1$ to ensure that $U'(c(t))$ is greater than 0 and $U''(c(t))$ is less than 0, i.e., the utility function is increasing and concave in consumption. For simplicity, we assume that there is no bequest, i.e., $B(y(T)) = 0$.

Then the Bellman equation becomes:

$$0 = \max_{c} \left[e^{-\rho t} c(t)^{\gamma} + J_t(y,t) + J_y(y,t)(ry(t) - c(t)) \right] \quad (10)$$

It is easy to calculate the optimal consumption:

$$c^* = \left[J_y(y,t) e^{\rho t} / \gamma \right]^{1/(\gamma - 1)} \quad (11)$$

Substituting back into the equation 10, we obtain:

$$0 = e^{-\rho t} \left[J_y(y,t) e^{\rho t} / \gamma \right]^{\gamma/(\gamma-1)} + J_t(y,t) + J_y(y,t)(ry(t) - \left[J_y(y,t) e^{\rho t} / \gamma \right]^{1/(\gamma-1)} \quad (12)$$

Solving for J looks like 'hocus pocus', but the usual way to proceed is to 'guess' the functional form of the value function $J(y, t)$. This is not an easy task. In our case we observe that if $y \to \lambda y$ and we also multiply the control $c \to \lambda c$ we can show that the value function is homogeneous of degree γ

$$J(\lambda y, t) = \lambda^{\gamma} J(y, t) . \tag{13}$$

But how does this 'symmetry' help us? Well, note that if we choose $\lambda = 1/y$

$$J(1, t) = y^{-\gamma} J(y, t) \tag{14}$$

which, after a simple rearrangement, leads us to the **trial solution** (also known as **Ansatz**)

$$J(y, t) = g(t) y^{\gamma} . \tag{15}$$

Later we also need to check that it is true that if we scale the initial condition of our wealth y then the optimal trajectory is the same as before but also scaled, i.e. that the optimal policy is to have λc. So let us recall that we need to solve (if possible) HJB (12). Our first step is to substitute (15) in (12) and calculate the derivatives

$$J_y(y,t) = \gamma g(t) y^{\gamma-1} \tag{16}$$

and

$$J_t(y,t) = g_t(t) y^{\gamma} \,. \tag{17}$$

And after substituting the trial solution $J(y,t)$ and its derivatives in (12) we obtain

$$g_t(t)+r\gamma g(t)+e^{-\rho t}\left(g(t)e^{\rho t}\right)^{\frac{\gamma}{\gamma-1}}-\gamma g(t)\left(g(t)e^{\rho t}\right)^{\frac{1}{\gamma-1}}=0 \quad (18)$$

with boundary condition $B(T)=0$ because there is no bequest. To solve (18) we first multiply it through by $e^{\rho t}$, and noting that

$$\frac{d}{dt}e^{\rho t}g(t)=\rho e^{\rho t}g(t)+e^{\rho t}g_t(t)\,, \quad (19)$$

We proceed and write

$$
\begin{aligned}
e^{\rho t} g_t(t) + e^{\rho t} r\gamma g(t) + \left(g(t)e^{\rho t}\right)^{\frac{\gamma}{\gamma-1}} - \gamma e^{\rho t} g(t)\left(g(t)e^{\rho t}\right)^{\frac{1}{\gamma-1}} &= 0 \\
\frac{d}{dt} e^{\rho t} g(t) + (r\gamma - \rho) e^{\rho t} g(t) + \left(g(t)e^{\rho t}\right)^{\frac{\gamma}{\gamma-1}} - \gamma e^{\rho t} g(t)\left(g(t)e^{\rho t}\right)^{\frac{1}{\gamma-1}} &= 0 \\
\frac{d}{dt} e^{\rho t} g(t) + (r\gamma - \rho) e^{\rho t} g(t) + \left(g(t)e^{\rho t}\right)^{\frac{\gamma}{\gamma-1}} - \gamma \left(g(t)e^{\rho t}\right)^{\frac{\gamma}{\gamma-1}} &= 0\,.
\end{aligned}
$$

Now let $G(t) = e^{\rho t} g(t)$

$$
G_t(t) + (r\gamma - \rho) G(t) + (1-\gamma) G(t)^{\frac{\gamma}{\gamma-1}} = 0
$$

and dividing through by $(1-\gamma)G(t)^{\frac{\gamma}{\gamma-1}}$

$$
\frac{G_t(t)}{(1-\gamma)G(t)^{\frac{\gamma}{\gamma-1}}} + \frac{r\gamma - \rho}{1-\gamma} G(t)^{1-\frac{\gamma}{\gamma-1}} + 1 = 0\,.
$$

Now we let $H(t) = G(t)^{\frac{1}{1-\gamma}}$ so that $G_t(t) = (1-\gamma)H(t)^{-\gamma}H_t(t)$ and write

$$\begin{aligned} H_t(t)(1-\gamma)H(t)^{-\gamma} + (r\gamma - \rho)H(t)^{1-\gamma} + (1-\gamma)H(t)^{-\gamma} &= 0 \\ H_t(t) + \frac{r\gamma - \rho}{(1-\gamma)}H(t) + 1 &= 0\,, \end{aligned}$$

with boundary condition $H(T) = 0$. This is an easy linear equation to solve. Let $v(t) = -(\mu H(t) + 1)$, with $v(T) = -1$ and $\mu = \frac{r\gamma-\rho}{1-\gamma}$, and write

$$\begin{aligned} v_t(t) &= -\mu v(t) \\ \int_t^T \frac{dv(s)}{v(s)} &= -\mu(T-t) \\ -1 &= v(t)e^{-\mu(T-t)}\,. \end{aligned}$$

Hence

$$H(t) = -\frac{1}{\mu}\left(1 - e^{\mu(T-t)}\right)\,.$$

Now we can write (note that instead of having $r\gamma - \rho$ we have written $-(\rho - r\gamma)$ since it is more natural for the discount rare ρ to appear in this way)

$$g(t) = e^{-\rho t}\left[\frac{1-\gamma}{\rho - r\gamma}\left(1 - e^{-\frac{(\rho - r\gamma)(T-t)}{1-\gamma}}\right)\right]^{1-\gamma},$$

and the value function

$$J(y,t) = e^{-\rho t}\left[\frac{1-\gamma}{\rho - r\gamma}\left(1 - e^{-\frac{(\rho - r\gamma)(T-t)}{1-\gamma}}\right)\right]^{1-\gamma} y^{\gamma}.$$

This allows us to calculate the optimal consumption

$$\begin{aligned} c^* &= \left(\frac{J_y(y,t)e^{\rho t}}{\gamma}\right)^{\frac{1}{\gamma-1}} \\ &= \left(\frac{\gamma e^{-\rho t}\left[\frac{1-\gamma}{\rho - r\gamma}\left(1 - e^{-\frac{(\rho - r\gamma)(T-t)}{1-\gamma}}\right)\right]^{1-\gamma} y^{\gamma-1}e^{\rho t}}{\gamma}\right)^{\frac{1}{\gamma-1}} \end{aligned}$$

$$
\begin{aligned}
&= \left(\left[\frac{1-\gamma}{\rho - r\gamma}\left(1 - e^{-\frac{(\rho)-r\gamma(T-t)}{1-\gamma}}\right)\right]^{1-\gamma} y^{\gamma-1}\right)^{\frac{1}{\gamma-1}} \\
&= \left[\frac{1-\gamma}{\rho - r\gamma}\left(1 - e^{-\frac{(\rho-r\gamma)(T-t)}{1-\gamma}}\right)\right]^{-1} y \\
&= \left(1 - e^{-\frac{(\rho-r\gamma)(T-t)}{1-\gamma}}\right)^{-1} \frac{\rho - r\gamma}{1-\gamma}\, y\,. \qquad (20)
\end{aligned}
$$

It is interesting (and useful) to stare at the optimal consumption to see if it makes sense. It is always good to check that the maths is OK but sometimes financial intuition is also a good common sense check... For example, we know that at expiry T it is optimal to have consumed everything because the agent derives no utility from leaving any bequests. Thus, one can check that $c^*(t) \to y(t)$ as $t \to T$. This makes sense because whatever wealth there is at time T it must be consumed.

Exercise

Let y denote the wealth process which satisfies the state equation:

$$\frac{dy(t)}{dt} = -c(t)$$

with the initial condition $y(t) = x$.

We also assume that the agent's objective is to maximize:

$$\max_{c(t)} \int_t^T c(s)^\gamma \, ds$$

HJB Equation

According to Equation 7, the HJB should be:

$$0 = \max_{c} \left[c(t)^{\gamma} + J_t(y,t) + J_y(y,t)(-c(t)) \right]$$

Then,

$$c^* = [J_y(y,t)/\gamma]^{1/(\gamma-1)}$$

and thus,

$$g_t(t) + (1-\gamma)g(t)^{\frac{\gamma}{\gamma-1}} = 0$$

$$\frac{dg_t(t)}{g(t)^{\frac{\gamma}{\gamma-1}}} = (\gamma - 1)dt$$

$$\int_t^T \frac{dg_s(s)}{g(s)^{\frac{\gamma}{\gamma-1}}} = \int_t^T (\gamma-1)ds$$

$$(1-\gamma)\left(g(T)^{\frac{1}{1-\gamma}} - g(t)^{\frac{1}{1-\gamma}} \right) = (\gamma-1)(T-t)$$

Solution

If

$$g(T) = 0$$

Then

$$g(t) = (T - t)^{1-\gamma}$$

$$J(y,t) = g(t)y^{\gamma} = (T - t)^{1-\gamma}y^{\gamma}$$

$$J_y(y,t) = \gamma(T - t)^{1-\gamma}y^{\gamma-1}$$

$$c^* = \frac{y}{T - t}$$