Computational Methods for Finance
Week 2: Mathematical Model II

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Learning Outcomes

At the end of this lecture you will be able to

- Understand Itô's lemma and its connections with Wiener and generalised Wiener processes.
- Apply Itô's lemma.

Review

- A Generalised Wiener Process
  A Wiener Process has a draft rate of zero and a variance rate of one (that is, $var(\Delta z) = 1\times$ time interval used). By contrast, a generalised Wiener process, x, has a drift rate of a and a variance rate of b^2. Such a process is given by the following equation:

$$\begin{aligned} dx &= a\,dt + b\,dz \\ &= a\,dt + b\,\epsilon\sqrt{dt}. \end{aligned} \tag{1}$$

  A discrete version of this process is given by

$$\begin{aligned} \Delta x &= a\,\Delta t + b\,\Delta z \\ &= a\,\Delta t + b\,\epsilon\sqrt{\Delta t}. \end{aligned} \tag{2}$$

Review

- A Generalised Wiener Process

$$E(\Delta x) = E(a\,\Delta t + b\,\epsilon\sqrt{\Delta t}) = E(a\,\Delta t) = a\,\Delta t, \tag{3}$$

and

$$var(\Delta x) = var(a\,\Delta t + b\,\epsilon\sqrt{\Delta t}) = var(b\,\epsilon\sqrt{\Delta t}) = b^2\Delta t. \tag{4}$$

Both of which imply that $\Delta x \sim N(a\,\Delta t, b^2\Delta t)$.

Continuous-time Stochastic Processes

- A Process for Stock Price
  Consider the following *expected* price processes:

Time Period	P_A	P_B	R_A	R_B
1	100	100	NA	NA
2	110	110	10%	10%
3	120	121	9.1%	10%
4	130	133	8.3%	10%

Which process do you think is more appropriate?

Continuous-time Stochastic Processes

- A Process for Stock Price
  Regarding stock A and B:
  - The process underlying Stock A is a generalised Wiener process (constant drift rate and constant variance).
  - The process underlying Stock B is more appropriate. Investors normally expect a constant percentage return rather than constant profits (cash flows).
  - Return is expected drift divided by the stock price.

Continuous-time Stochastic Processes

- A Process for Stock Price
  - If the variance rate of stock B price is 0, then,

    $$dS = \mu\, S\, dt,\ or\ \frac{dS}{S} = \mu\, dt. \tag{5}$$

    This means that the percentage increase in stock price is constant.
  - Does it remind you something?
  - The rate of return is constant.
  - Integrating between time 0 and time T, we obtain

    $$S_T = S_0 e^{\mu t}, \tag{6}$$

    where S_0 and S_T are the stock prices at time 0 and time T, respectively.
  - Variance is also allowed to vary with the stock price.
  - Specifically, stock prices are assumed to follow the Itô process (aka Geometric Brownian Motion):

    $$\frac{dS}{S} = \mu\, dt + \sigma\, dz,\ or\ dS = \underbrace{\mu\, S}_{\text{a(S)}}\, dt + \underbrace{\sigma\, S}_{\text{b(S)}}\, dz. \tag{7}$$

- A Process for Stock Price

  A discrete-time version of the above process is given by

$$\frac{\Delta S}{S} = \mu\, \Delta t + \sigma\, \epsilon\, \sqrt{\Delta t}, \tag{8}$$

  where

$$E(\frac{\Delta S}{S}) = \mu\, \Delta t \text{ and } var(\frac{\Delta S}{S}) = \sigma\, \Delta t, \tag{9}$$

  imply that

$$\frac{\Delta S}{S} \sim N(\mu \Delta t, \sigma^2 \Delta t)$$

  .

Functions of Stochastic Variables

- Itô's Lemma
  Assuming that x follows the Itô process

$$dx = a(x,t)dt + b(x,t)dz, \tag{10}$$

  where a and b are functions of x and t, then Itô's lemma enables us to express a function of x, denoted by G, as an alternative Itô process. This alternative process is given by the following expression:

$$dG = (\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2)\,dt + \frac{\partial G}{\partial x}\,b\,dz, \tag{11}$$

  where dz is the same Wiener process as considered previously.

Functions of Stochastic Variables

- Itô's Lemma
  - It follows that G also follows an Itô process.
  - Why?
  - Drift rate:

$$\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2$$

  - Variance rate:

$$(\frac{\partial G}{\partial x})^2 \; b^2$$

.

Functions of Stochastic Variables

- Itô's Lemma

Example

Consider the following *logarithmic* transformation of stock prices:

$$G = ln\, S$$

. Associated partial differentials are given by

$$\frac{\partial G}{\partial S} = \frac{1}{S}, \ \frac{\partial^2 G}{\partial S} = -\frac{1}{S^2}, \ \frac{\partial G}{\partial t} = 0$$

. Therefore,

$$dG = (\mu - \frac{\sigma^2}{2})dt + \sigma dz$$

. Therefore, logarithmic stock prices follow a generalised Wiener process (as a and b do not depend on S or t).

- Mean? Variance?

Functions of Stochastic Variables

- Itô's Lemma

Example

It follows that ΔG has a normal distribution with a mean of

$$\mu - \frac{\sigma^2}{2},$$

and a variance of,

$$\sigma^2 \Delta t.$$

Moreover, if Δt equals T then:

$$ln\, S_T - ln\, S \sim N((\mu - \frac{\sigma^2}{2})T, \sigma^2 T),$$

$$\Rightarrow ln\, S_T \sim N(lnS + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T)$$

. Thus, stock prices have a *log-normal distribution*.

Summary of Week 2

- Continuous-time Stochastic Process:
  Wiener, generalised Wiener, and Itô processes.
- Transforming Stochastic Processes:
  Itô's lemma with an application.

Reading

- Chapter 14, Hull (2015)