Computational Methods for Finance
Week 11: Volatility

Yang Yue

University of Westminster

December 2023

Learning Outcomes

At the end of this lecture you will be able to

- Understand three types of volatilities.
- Estimate historical and implied volatilities.
- Understand the features of implied volatility.

Review

- The log return of a stock $ln(\frac{S_T}{S})$ follow the distribution below:
- $$N((\mu - \frac{\sigma^2}{2})T, \sigma^2 T),$$

  where μ is the expected return on stock per year, and σ is the volatility of the stock return per year.

Review

- The Black-Scholes model for the price of a European *call* on a non-dividend paying stock is

$$c = SN(d_1) - Ke^{-rT}N(d_2), \tag{1}$$

where

$$d_1 = \frac{ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

where $N(.)$ is the cumulative probability distribution function for a standardised normal distribution.

- The BSM formula for the price of a European *put* on a non-dividend paying stock is

$$p = Ke^{-rT}N(-d_2) - SN(-d_1), \tag{2}$$

where all previous notation is maintained.

Three Types of Volatilities

- The volatility of the stock return, σ, is a measure of our uncertainty about the returns provided by the stock.
- However, contrary to the raw return, actual realisations of return volatility are not directly observable.
- But we can use measures to proxy for them.
- Based on the time of estimation, there are three types of volatilities:
    - Notional volatility (Historical volatility): A volatility estimated from historical data. It corresponds to the *ex-post* sample-path return variability over a fixed time interval.
    - Instantaneous volatility: corresponds to the strength of the volatility process *at a point in time*.
    - Expected volatility: corresponds to the *ex-ante* sample-path return variability over a fixed time interval.

Historical Volatility

- Historical Volatility
  - The most basic estimate of historical volatility is the standard deviation of returns.
  - It can be estimated as follows:
    Let $u_i = ln(\frac{S_i}{S_{i-1}})$, for i = 1, 2, ..., n, where n+1 is the number of observations, and S_i is the stock price at the end of i^{th} interval (i=0, 1, 2, ...,n).
    Hence, the sample standard deviation

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})},$$

where $\bar{u}$ is the simple average of u_i.

Historical Volatility

- Historical Volatility
  - Since returns follow $N((\mu - \frac{\sigma^2}{2})T, \sigma^2 T)$, the standard deviation of returns

    $$s = \sqrt{\sigma^2 T},$$

    hence,

    $$\sigma = \frac{s}{\sqrt{T}},$$

    hence

    $$\hat{\sigma} = \frac{s}{\sqrt{T}},$$

    where T is the time interval in years between two stock prices (e.g. if the return is monthly return, then $\sigma = \sqrt{12}s$). We usually assume there are 252 trading days per year. Therefore, if the time interval between two prices is 15 trading days, then $T = \frac{15}{252}$.

Implied Volatility

- Implied Volatility
  - Implied volatility is one of ways to measure the *expected volatility*. It reflects market's opinion about future volatility.
  - It often is the volatility *implied* from an option price using the Black–Scholes or a similar model. Volatility is the only parameter in the BSM pricing formula that cannot be directly observed.
  - The SPX VIX index is an index of the implied volatility of 30-day options on the S&P 500 calculated using a range of calls and puts.

Implied Volatility

- Implied Volatility
  - However, it is not possible to invert the Black-Scholes model so that σ is expressed as a function of S, K, r, T, and c.
  - We can use the iterative search procedure to find the implied σ.

Example

The European call option price c is 1.875. We also know that $S = 21$, $K = 20$, $r = 0.1$, and $T = 0.25$. We can first try $\sigma = 0.2$, which gives $c = 1.76$, which is too low. Because c is an increasing function of σ, a higher value of σ is required. We can next try a value of 0.30 for σ. This gives a value of c equal to 2.10, which is too high and means that σ must lie between 0.20 and 0.30. Next, a value of 0.25 can be tried for σ. This also proves to be too high, showing that σ lies between 0.20 and 0.25. Proceeding in this way, we can halve the range for σ at each iteration and the correct value of σ can be calculated to any required accuracy.

Implied Volatility

- Implied Volatility
  - Another iteration procedure – Newton-Raphson method – can estimate the implied volatility σ.
  - It is an iterative procedure for solving nonlinear equations.
  - We start with an estimate σ_0 of the solution and produce successively better estimates σ_1, σ_2, σ_3, ... using the formula $\sigma_{i+1} = \sigma_i - \frac{f(\sigma_i) - c}{f'(\sigma_i)}$, where f is the European option formula, and $f'(\sigma_i)$ denotes the derivative of f with respect to σ, which is *Vega*. The iteration stops when $|f(\sigma_i) - c|$ is less than a tolerance.

Implied Volatility

- Volatility Smiles
  A plot of the implied volatility of an option as a function of its strike price should be flat if the option pricing model is accurate. However, in reality, such plots reveal *volatility smiles* or *volatility skews*.
  - Foreign Currency Options
    - Observation: implied volatility is lower for at-the-money options than for in-the-money and out-of-the-money options, i.e., volatility smiles.
    - Reason: exchange rates are not unconditionally lognormally distributed (due to time-varying volatility and jumps).
  - Equity Options
    - Observation: since 1987, implied volatility falls as the strike price increases, i.e., volatility skews.
    - Reason: equity returns are not unconditionally lognormally distributed (due to company's leverage).

Implied Volatility

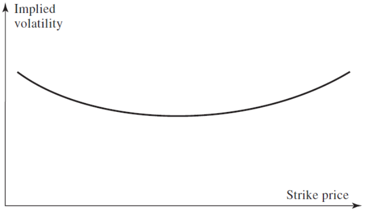

Figure 1: Volatility Smile

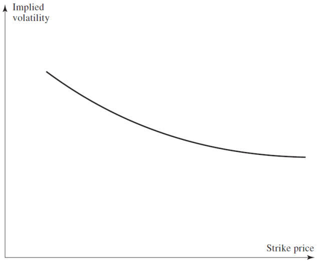

Figure 2: Volatility Skew

Implied Volatility

- Volatility Term Structure
  A plot of the implied volatility of an option as a function of its maturity. This does not have to be at; indeed, empirically it is upward sloping when short-term volatility is low (i.e., markets expect volatility to rise in the future), and downward sloping when short-term volatility is high (i.e., markets expect volatility to fall in the future).
- Volatility Surface
  Volatility surfaces combine volatility smiles with the volatility term structure to tabulate the volatilities appropriate for pricing an option with any strike price and any maturity. (The implied volatility is a function of both strike price and time to maturity.)

Implied Volatility

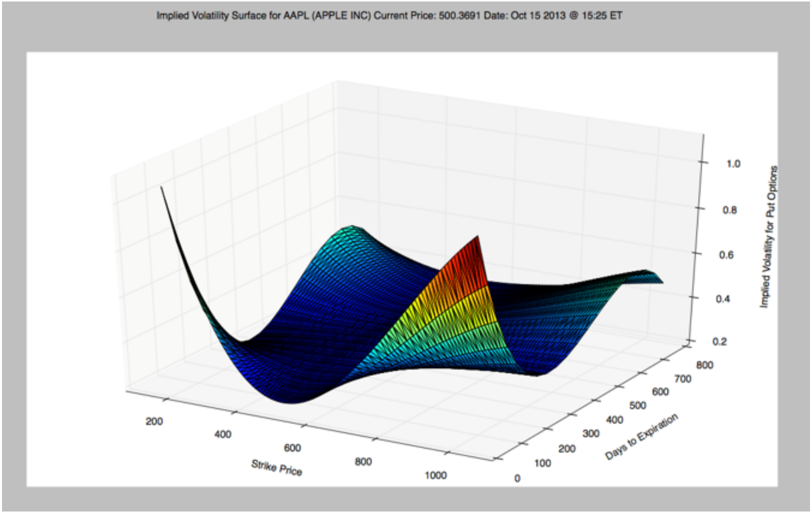

Figure 3: Volatility Surface

Reading

- Chapter 15 and 20, Hull (2015); Chapter 5 and 7, Hilpisch (2014)