Computational Methods for Finance
Week2 : Mathematical Model I

Yang Yue

University of Westminster

September 2023

Learning Outcomes

At the end of this lecture you will be able to

- Distinguish between a variety of stochastic processes.
- Provide a mathematical description of the process followed by asset prices (in continuous time).

Review

- A random variable (See Math Boot Camp p.235)
- A discrete random variable
- A continuous random variable

Stochastic Process Types

- A stochastic Process
  Any variable whose value changes over time in an uncertain way.
  Stochastic processes can be categorised as follows:
  - Discrete-time stochastic process: the value of the variable can *change* only at certain fixed points in time.
  - Continuous-time stochastic process: the change of the value of the variable can take place at any time.
  - Discrete-variable stochastic process: the underlying variable can only take certain discrete *values*.
  - Continuous-variable stochastic process: the underlying variable can only take any value within a certain range.

Stochastic Process Types

- A Markov Process
  - A stochastic process where only the *current value* of a variable (x_t) is relevant for predicting the future. – Can you think of anything that you have learned that possesses this property?
  - Stock prices are usually assumed to follow the Markov Process. The Markov property of stock prices is consistent with the *weak form* of market efficiency.

Continuous-time Stochastic Processes

- A Wiener Process (aka Brownian Motion)
  Let z follow a type of Markov process called a Wiener process. Consider changes in z, denoted Δz, over short periods of time, denoted Δt.
  - Property 1: Δz and Δt are related by $\Delta z = \epsilon\sqrt{\Delta t}$, where ϵ has a standard normal distribution $N(0, 1)$.
  - Property 2: For any two different Δt's, the values of Δz are independent. This implies that z will follow a Markov process.
  - Strictly speaking, z is a Wiener process only when $\Delta t \to 0$, hence $dz = \epsilon\sqrt{dt}$.

Continuous-time Stochastic Processes

- A Wiener Process

$$E(\Delta z) = E(\epsilon\sqrt{\Delta t}) = E(\epsilon) \times \sqrt{\Delta t} = 0, \tag{1}$$

and

$$var(\Delta z) = var(\epsilon\sqrt{\Delta t}) = var(\epsilon) \times \Delta t = \Delta t. \tag{2}$$

Both of which imply that

$$\Delta z \sim N(0, \Delta t)$$

.

Continuous-time Stochastic Processes

- A Generalised Wiener Process
  A Wiener Process has a *drift rate* of zero and a *variance rate* of one (that is, $var(\Delta z) = 1\times$ time interval used). By contrast, a generalised Wiener process, x, has a *drift rate* of a and a *variance rate* of b^2. Such a process is given by the following equation:

$$\begin{aligned} dx &= a\,dt + b\,dz \\ &= a\,dt + b\,\epsilon\sqrt{dt}. \end{aligned} \tag{3}$$

  A discrete version of this process is given by

$$\begin{aligned} \Delta x &= a\,\Delta t + b\,\Delta z \\ &= a\,\Delta t + b\,\epsilon\sqrt{\Delta t}. \end{aligned} \tag{4}$$

- A Generalised Wiener Process

$$E(\Delta x) = E(a\,\Delta t + b\,\epsilon\sqrt{\Delta t}) = E(a\,\Delta t) = a\,\Delta t, \tag{5}$$

and

$$var(\Delta x) = var(a\,\Delta t + b\,\epsilon\sqrt{\Delta t}) = var(b\,\epsilon\sqrt{\Delta t}) = b^2\Delta t. \tag{6}$$

Both of which imply that

$$\Delta x \sim N(a\,\Delta t, b^2\Delta t)$$

.

- A Generalised Wiener Process
  The following graphs contain plots of generalised Wiener processes (See Python codes).

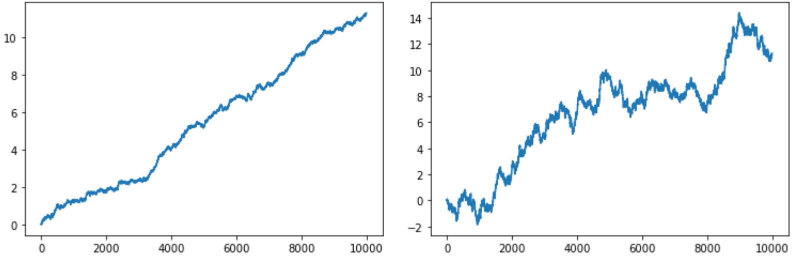
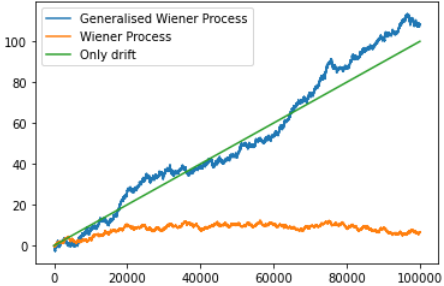


Figure 1: Generalised Wiener Process

The left plot assumes that $a = 1$ and $b = 0.5$, while the right plot assumes that $a = 1$, $b = 2$. Both plots are based on 10000 observations with $\Delta t = 1/1000$.

- Comparison

  The following graphs contain plots of generalised Wiener processes, Wiener process, and the process with only the drift part of the generalised Wiener process (See Python codes).



The generalised Wiener process assumes that $a = 1$ and $b = 2$, while for the Wiener process, $a = 0$ and $b = 1$. $a = 1$ and $b = 0$ for the process with only the drift part. Plots are based on 100,000 observations with $\Delta t = 1/1000$.

Reading

- Chapter 14, Hull (2015)