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Module: High Frequency Trading

Week 8: Optimal Liquidation

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Stochastic Case: Almgren-Chriss

Imagine that between time $t = 0$ and T an agent wants to liquidate Q shares. To find an optimal execution strategy, we must first assume:

- a model for the stock price dynamics
- how the agent moves prices against himself

We can then formulate the optimization problem faced by the agent. The solution will provide us with the optimal policy for the agent.

Stock Dynamics: Let the stock price $S(t)$ satisfy the Stochastic Differential Equation (SDE):

$$dS(t) = \sigma dW(t) \quad (1)$$

where $\sigma > 0$ and $W(t)$ is a standard Brownian motion. Note that we assume the dynamics of the price change follow an arithmetic Brownian motion, allowing prices to become negative. Although this is not a realistic feature, it makes the problem mathematically tractable.

Adverse Price Impact

Adverse price impact occurs when liquidating (acquiring) shares. We assume the agent's trade creates a temporary adverse move in prices, so the price at which the agent will transact ($\hat{S}(t)$) is given by:

$$\hat{S}(t) = S(t) + kv(t) \quad (2)$$

k denotes the adverse impact that the agent's trading action incurs (negative if liquidating or positive if acquiring), and $v(t) = -\frac{dq(t)}{dt}$ (3), where $v(t)$ is the positive liquidation rate, and $q(t)$ is the amount of outstanding inventory at time t.

Objective Function

The objective function, representing the cash value of liquidating Q shares, can be expressed as:

$$R = E\left[\int_0^T \hat{S}(t)v(t)\,dt\right] \quad (4)$$

where E denotes the expectation. This formula captures the expected amount of money that the agent receives over the time period $[0, T]$ when selling shares at the obtained prices.

Dynamic Optimal Control

The agent's objective is to choose the rate at which he liquidates Q shares to obtain the maximum revenue from the sale. The corresponding value function is given by:

$$J(0, S, q) = \max_{v} E\left[\int_0^T (S(t) + kv(t))v(t)\,dt\right] \quad (5)$$

subject to the stochastic differential equation $dS(t) = \sigma dW(t)$ and $v(t) = -\frac{dq}{dt}$, where $q(0) = Q$ is the initial amount of shares to be liquidated, and recall that $k < 0$.

HJB Equation

To solve this dynamic optimal control problem, we use the dynamic programming principle to write:

$$J(0,S,q) = \max_v E\left[\int_0^{\Delta t} (S(t) + kv(t))v(t)\,dt + J(t+\Delta t, S+\Delta S, q+\Delta q)\right]$$

$$J(0,S,q) = \max_v E\Big[S(\Delta t') + kv(\Delta t')v(\Delta t')\Delta t + J(0,S,q) + J_t(0,S,q)\Delta t + JS(0,S,q)\Delta S + Jq(0,S,q)\Delta q + \frac{1}{2}\sigma^2 J_S$$

$$0 = \max_v E\Big[S(\Delta t') + kv(\Delta t')v(\Delta t')\Delta t + J_t(0,S,q)\Delta t + \sigma JS(0,S,q)\Delta W - Jq(0,S,q)v(t)\Delta t + \frac{1}{2}\sigma^2 J_{SS}(0,S,q)\Delta t$$

$$0 = \max_v \left[(S(0) + kv(0))v(0) + J_t(0,S,q) - J_q(0,S,q)v(0) + \frac{1}{2}\sigma^2 J_{SS}(0,S,q)\right]$$

So, the Hamilton-Jacobi-Bellman (HJB) equation we need to solve is:

$$0 = \max_v (S + kv)v + J_t - vJ_q + \frac{1}{2}\sigma^2 J_{SS} \quad (6)$$

Optimal Solution

The optimal v is given by:

$$v^*(t) = \frac{J_q - S}{2k} \quad (7)$$

Plugging this into the HJB equation (8), we obtain:

$$0 = \left(S + \frac{J_q - S}{2}\right)\frac{J_q - S}{2k} + J_t - \frac{J_q - S}{2k}J_q + \frac{1}{2}\sigma^2 J_{SS}$$

$$0 = \frac{1}{4k}\left((J_q)^2 - S^2\right) + J_t + \frac{1}{2k}SJ_q - \frac{1}{2k}(J_q)^2 + \frac{1}{2}\sigma^2 J_{SS}$$

$$0 = \frac{1}{4k}\left((J_q)^2 - S^2\right) + \frac{2SJ_q - 2(J_q)^2}{4k} + 4kJ_t + 2k\sigma^2 J_{SS}$$

$$0 = (J_q)^2 - S^2 + 2SJ_q - 2(J_q)^2 + 4kJ_t + 2k\sigma^2 J_{SS}$$

$$0 = -(J_q - S)^2 + 4kJ_t + 2k\sigma^2 J_{SS} \quad (8)$$

Partial Differential Equation (PDE) and Trial Solution

In this case, we know that at time T, $J(T, S(T), q(T)) = q(T)S(T)$, hinting that the trial solution of the value function is linear in $S(t)$, which means $J_{SS} = 0$. Thus, our problem reduces to solving the PDE:

$$4kJ_t = (J_q - S)^2 \quad (9)$$

To solve this PDE, we look for a separable solution. First, we let $V(t, q) = J(t, q) - q(t)S(t)$, so that we can write:

$$4kV_t = (V_q)^2 \quad (10)$$

We try $V(t, q) = g(t)h(q)$. Substituting this trial solution into (10), we obtain:

$$4k\frac{g_t(t)}{g^2(t)} = \frac{h_q^2(q)}{h(q)}$$

$$4k\frac{g_t(t)}{g^2(t)} = c \quad (11)$$

$$\frac{h_q^2(q)}{h(q)} = c \quad (12)$$

$$4k\int_t^T \frac{dg}{g^2} = c\int_t^T dt \quad \text{and} \quad 4k\left(\frac{1}{g(t)} - \frac{1}{g(T)}\right) = c(T-t)$$

$$\int_t^T \frac{dh(q)}{\sqrt{h(q)}} = \int_t^T \sqrt{c}\,dq \quad \text{and} \quad 2\left(\sqrt{h(q(T))} - \sqrt{h(q(t))}\right) = -\sqrt{c}q(t$$

$$h(q(t)) = \frac{1}{4}cq^2(t)$$

$$V(q,t) = g(t)h(q) \quad \text{and} \quad V_q(q,t) = g(t)h_q(q)$$

$$\frac{J_q(q,t) - S(t)}{2k} = \frac{g(t)h_q(q)}{2k}$$

$$\frac{q(t)}{T-t} = \frac{g(t)h_q(q)}{2k}$$

$$\frac{q(t)}{T-t} = \frac{g(t)}{2k}\frac{cq(t)}{2} \quad \text{and} \quad g(t) = \frac{4k}{c(T-t)} \quad (13)$$

Hence, $\dfrac{1}{g(T)} = 0$ and we write (for all t) $g(t) = \dfrac{4k}{c(T-t)}$

$$V(t,q) = \frac{4k}{c(T-t)} \cdot \frac{1}{4} c q^2(t) = \frac{k q^2(t)}{T-t} \quad (14)$$

so that

$$J(t,S,q) = \frac{4k}{c(T-t)} \cdot \frac{1}{4} c q^2(t) + q(t)S(t) = \frac{k q^2(t)}{T-t} + q(t)S(t) \quad (15)$$

$$v^*(t) = \frac{J_q - S}{2k} = \frac{\frac{2kq(t)}{T-t} + S(t) - S(t)}{2k} = \frac{q(t)}{T-t} \quad (16)$$

The answer is quite simple and intuitive. The shares must be liquidated at a constant rate. This strategy is the same as that of the time-weighted average price (TWAP).