Computational Methods for Finance
Week 3: Mathematical Model III

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Learning Outcomes

At the end of this lecture you will be able to

- Derive the Black-Scholes-Merton differential equation.
- Understand the concept of risk-neutral valuation.
- Use the Black-Scholes-Merton option pricing model.

Review

- Wiener Process (Brownian Motion):
  $dz = \epsilon\sqrt{dt}$
- A Generalised Wiener Process:
  $dx = a\,dt + b\,dz = a\,dt + b\,\epsilon\sqrt{dt}.$
- Itô process (Geometric Brownian Motion):
  $\frac{dS}{S} = \mu\,dt + \sigma\,dz,\ or\ dS = \underbrace{\mu\,S}_{\text{a(S)}}\ dt + \underbrace{\sigma\,S}_{\text{b(S)}}\ dz.$
- Stock prices are assumed to follow the Itô process.

Review

- Law of one price: two identical assets cannot sell at different prices.
- Arbitrage process: traders buy and sell securities so that the law of one price is ensured. If arbitrage opportunities arise (i.e. the prices are different), a relatively few investors can act to restore equilibrium.

General Information

- We use the Black-Scholes-Merton (BSM henceforth) model to price and hedge European stock options.
- Defining features of the BSM differential equation
    - BSM differential equation is an equation that must be satisfied by the price of any derivative dependent on a non-dividend paying stock.
    - The derivation involves setting up a risk-free portfolio consisting of a position in the derivative and a position in the stock.
    - The return on this portfolio must be the risk-free rate – this insight ultimately leads to the BSM differential equation.

General Information

- Assumptions
  - Stock prices follow geometric Brownian Motion
  - Short-selling is allowed.
  - No transaction costs or taxes.
  - No dividends.
  - Arbitrage opportunities are taken immediately (i.e. no arbitrage opportunities exist).
  - Trading is continuous.
  - Risk-free rate is constant and equal across maturities.

The BSM Differential Equation

- Derivation of the BSM Differential Equation
  Assume that stock prices evolve as follows (geometric Brownian Motion):
  $$dS = \mu S dt + \sigma S dz. \tag{1}$$
  Moreover, the price of any security whose price (f) depends on S (i.e. the derivative) will follow the Itô process,
  $$df = (\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2)dt + \frac{\partial f}{\partial S}\sigma S dz, \tag{2}$$
  where all previous notation is maintained.

The BSM Differential Equation

- Derivation of the BSM Differential Equation
  Discrete versions of the previous two equations are given by:

$$\Delta S = \mu S \Delta t + \sigma S \Delta z, \tag{3}$$

  and

$$\Delta f = (\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2)\Delta t + \frac{\partial f}{\partial S}\sigma S \Delta z, \tag{4}$$

  respectively.

The BSM Differential Equation

- Derivation of the BSM Differential Equation
  - As the same Wiener process underlies the stock and the derivative, then it is possible to eliminate this process.
  - Why do we eliminate this process?
  - Wiener process is the source of the uncertainty of two price series (why?).
  - How?
  - Taking certain positions in the two securities.
  - What are the positions?
  - -1 unit of the derivative security;
    $\frac{\partial f}{\partial S}$ units of the underlying stock.
  - What is the portfolio value?
  - $$\Pi = -f + \frac{\partial f}{\partial S} S, \ \Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S,$$

    where the change in the portfolio value is measured over the period Δt.

The BSM Differential Equation

- Derivation of the BSM Differential Equation
  Substituting in the expressions for Δf and ΔS into the expression for $\Delta\Pi$,

$$
\begin{aligned}
\Delta\Pi &= -\Delta f + \frac{\partial f}{\partial S}\Delta S, \\
&= -\underbrace{\left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S\Delta z}_{\Delta f} \\
&+ \frac{\partial f}{\partial S}\underbrace{(\mu S\Delta t + \sigma S\Delta z)}_{\Delta S},
\end{aligned} \tag{5}
$$

Rearranging and cancelling out terms,

$$
= -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2\right)\Delta t.
$$

The BSM Differential Equation

- Derivation of the BSM Differential Equation

  As the constructed portfolio is risk-free (i.e., no z's), then the return on this portfolio must also be risk-free. Therefore,

$$\Delta\Pi = r\Pi\Delta t, \tag{6}$$

  where r is the risk-free rate. Substituting in the expressions for Π and $\Delta\Pi$, we obtain

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma S)^2\right)\Delta t = r\left(f - \frac{\partial f}{\partial S}S\right)\Delta t. \tag{7}$$

  Rearranging leads to the BSM differential equation,

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf. \tag{8}$$

  (any tradable derivatives should satisfy this equation)

The BSM Differential Equation

- Solve the BSM Differential Equation
  The BSM differential equation can be solved subject to certain boundary conditions (those boundary conditions can help us to identify the f we need, otherwise, the partial differential equation would have many solutions). For European options, we are interested in solving the equation in the presence of the following boundary conditions:
  - European Call: $f = \max(S - K,0)$, when t=T (expiration date!)
  - European Put: $f = \max(K - S,0)$, when t=T

The BSM Formula

- Risk-Neutral Valuation
  - The BSM differential equation does not contain variables that are affected by investors' risk preferences (i.e. the drift parameter μ)
    $\Rightarrow$ Any set of risk preferences can be used when evaluating f.
  - We can make a simple assumption that all investors are risk-neutral.
    $\Rightarrow$ The expected return on *all* investment assets is the *risk-free rate* (including the underlying asset, and thus the drift parameter can be replaced by the risk-free rate wherever it appears in the derivation).
  - Under risk-neutral valuation, solutions obtained are valid in all worlds (not only the risk-neural world).
  - The economic argument for the risk-neutral valuation is that since we can perfectly hedge the option with the underlying (e.g. non-dividend-paying stock), we should not be rewarded for taking unnecessary risk; only the risk-free rate of return is in the equation. This means that if you and I agree on the *volatility* (i.e. σ) of an asset we will agree on the value of its derivatives even if we have differing estimates of the drift.

The BSM Formula

- The BSM Pricing Formula
  The BSM formula (aka Black-Scholes model) for the price of a European *call* on a non-dividend paying stock is

$$c = SN(d_1) - Ke^{-rT}N(d_2), \tag{9}$$

where

$$d_1 = \frac{ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

where $N(.)$ is the *cumulative probability distribution* function for a *standardised normal distribution*. $N(d_2)$ is the probability that a call option will be exercised in a risk-neutral world. $Se^{rT}N(d_1)$ is the expected stock price at time T in a risk-neutral world when stock prices less than the strike price are counted as zero.

The BSM Formula

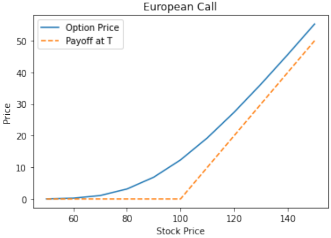
- The BSM Pricing Formula
  The BSM formula for the price of a European *put* on a non-dividend paying stock is

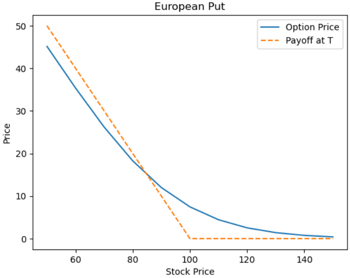
$$p = Ke^{-rT}N(-d_2) - SN(-d_1), \tag{10}$$

where all previous notation is maintained.

The BSM Formula

- The BSM Pricing Formula





The BSM formula

- The BSM Pricing Formula

Example

Consider a European call option on a stock with a strike price of \$ 40 and six months to maturity. The current stock price is \$ 42, the volatility of stock returns is 20% per annum, and the risk-free rate is 10%. The d_1 and d_2 parameter values are given by

$$d_1 = \frac{ln(42/40) + ((0.1 + 0.2^2/2) \times 0.5)}{0.2\sqrt{0.5}} = 0.7693,$$

$$d_2 = \frac{ln(42/40) + ((0.1 - 0.2^2/2) \times 0.5)}{0.2\sqrt{0.5}} = 0.6278,$$

what is the theoretical price of a European call option?

The BSM formula

- The BSM Pricing Formula

Example

The theoretical price of a European call is

$$c = 42N(0.7693) - 40e^{-0.10\times 0.5}N(0.6278) = \$4.76,$$

where $N(0.7693) = 0.7791$ and $N(0.6278) = 0.7349$.
What is the theoretical price of a European put option?

The BSM formula

- The BSM Pricing Formula

Example

The theoretical price of a European put is

$$p = 40e^{-0.10\times0.5}N(-0.6278) + 42N(-0.7693) = \$0.81,$$

where $N(-0.7693) = 1 - 0.7791 = 0.2209$ and
$N(-0.6278) = 1 - 0.7349 = 0.2651$.
What is the theoretical price of a European put option?

Reading

- Chapter 15, Hull (2015)