7FNCE025 High Frequency Trading
Dr Hui Gong
Week 5 Seminar Solutions

The set-up of the agent's liquidation problem is similar to that above. The only difference is that the agent does not require $q(T) = 0$, i.e. he is not required to liquidate all shares by T. Instead, the agent picks up a penalty for all remaining shares $q(T)$. Thus, the expected revenues from sales is

$$R = \mathbb{E}\left[\int_0^T \hat{S}(t)v(t)dt + q(T)S(T) - \phi q^2(T)\right], \tag{1}$$

where $-\phi q^2(T)$ is the penalty received from not liquidating all the shares, $\phi > 0$ is a penalty parameter,

$$\hat{S}(t) = S(t) + kv(t) \tag{2}$$

with $k < 0$,

$$v(t) = -\frac{d}{dt}q(t), \tag{3}$$

and $q(t)$ is the amount of outstanding shares at time t.

Moreover, the stock price satisfies the SDE

$$dS(t) = \sigma dW(t)$$

where $\sigma > 0$ and $W(t)$ is a standard Brownian motion.

What is the optimal liquidation strategy?

Solution: The value function is given by

$$J(0, S, q) = \max_v \mathbb{E}\left[\int_0^T (S(t) + kv(t))\, v(t)dt + q(T)S(T) - \phi q^2(T)\right] \tag{4}$$

subject to

$$dS(t) = \sigma dW(t) \qquad \text{and} \qquad v(t) = -\frac{dq}{dt},$$

where $q(0) = Q$ is the initial amount of shares to be liquidated.

Using the dynamic programming principle allows us to write the HJB satisfied by the value function:

$$J(0, S, q) = \max_v \left\{(S(0) + kv(0))\, v(0) + J(0, S, q) + J_t(0, S, q) - J_q(0, S, q)v(0) + \frac{1}{2}\sigma^2 J_{SS}(0, S, q)\right\}$$

so the HJB we need to solve is

$$0 = \max_v \left\{(S + kv)\, v + J_t - vJ_q + \frac{1}{2}\sigma^2 J_{SS}\right\}. \tag{5}$$

Thus the optimal v is

$$v^*(t) = \frac{J_q - S}{2k}\,, \tag{6}$$

and plugging it in the HJB we obtain

$$0 \;=\; -\left(J_q - S\right)^2 + 4kJ_t + 2k\sigma^2 J_{SS}\,.$$

We know that at time T we have that $J(T, S(T), q(T)) = q(T)S(T) - \phi q^2(T)$. Note that in the problem above we required that $q(T) = 0$, but here, although we allow final inventory to deviate from zero, the term $-\phi q^2(T)$ makes these deviations costly. Recall that in this problem what the agent is worried about is that the market impact of his own orders will eat too much into the book and this is why he prefers to liquidate the total number of shares Q by smoothing it over time. However, if we allow him to look for an optimal strategy where he can leave some unfinished business (that is $q(T) \neq 0$) then depending on how high the penalty ϕ is compared to the cost k the agent might find it optimal to leave some shares until the end and then liquidate them at $S(T)$ and pick up a penalty. Intuitively we can interpret the penalty ϕ in the same way as k. From a mathematical viewpoint is nice to be able to see how the optimal strategy depends on ϕ because we know that a high ϕ will deter the agent from carrying over any inventory to time T.

We proceed as in the previous case and note that the trial solution is linear in S, therefore we know that the term $J_{SS} = 0$ so the PDE we solve is

$$4kJ_t = \left(J_q - S\right)^2\,. \tag{7}$$

To solve this PDE we look for a separable solution but first we let $V(t, q) = J(t, q) - qS$ so that we write

$$4kV_t = \left(V_q\right)^2\,, \tag{8}$$

and try $V(t, q) = g(t)h(q)$. Substituting the trial solution in (8) we obtain

$$4k\frac{g_t(t)}{g^2(t)} = \frac{h_q^2(q)}{h(q)}\,. \tag{9}$$

We observe that the left-hand side of equation (9) only depends on time and the right-hand side only depends on the amount of shares q, which can only be possible if both sides are equal to a constant, say c. Thus we need to solve two ODEs

$$4k\frac{g_t(t)}{g^2(t)} = c\,. \tag{10}$$

$$\frac{h_q^2(q)}{h(q)} = c\,. \tag{11}$$

Recall that at time T the value function $J(T, S(T), q(T)) = q(T)S(T) - \phi q^2(T)$. Therefore $V(T, q) = -\phi q^2(T)$. We also know that if $q(t) = 0$ then $J(t, S(t), 0) = 0$.

To solve (10) and (11) we integrate between t and T

$$\begin{aligned} 4k\int_t^T \frac{dg}{g^2} &= c\int_t^T dt \\ 4k\left(\frac{1}{g(t)}-\frac{1}{g(T)}\right) &= c(T-t)\,, \end{aligned}$$

and

$$\begin{aligned} \int_t^T \frac{dh(q)}{\sqrt{h(q)}} &= \int_t^T \sqrt{c}dq \\ 2\sqrt{h(q)}|_t^T &= \sqrt{c}(q(T)-q(t)) \\ 2\left(\sqrt{h(q(T))}-\sqrt{h(q(t))}\right) &= -\sqrt{c}(q(T)-q(t)) \\ h(q(t)) &= \left(\frac{1}{2}\sqrt{c}(q(T)-q(t))-\sqrt{h(q(T))}\right)^2 . \end{aligned}$$

Now we need to use the boundary conditions to determine $g(t)$ and $h(q(t))$. We know that $V(T,q(T)) = -\phi q^2(T)$ therefore $g(T)h(T) = -\phi q^2(T)$. Hence, $g(T) = -a\phi$ and $h(q(T)) = bq^2(T)$ where a, b are constants such that $ab = 1$. (We might as well just say that $g(T) = -\phi$ and $h(T) = q^2(T)$, but I have preferred to leave it as general as we possibly can). Hence,

$$\begin{aligned} \frac{1}{g(t)} &= \frac{c}{4k}(T-t)-\frac{1}{a\phi} \\ g(t) &= \left[\frac{c}{4k}(T-t)-\frac{1}{a\phi}\right]^{-1} \\ &= \left[\frac{a\phi c(T-t)-4k}{a\phi 4k}\right]^{-1} \\ &= \frac{a\phi 4k}{a\phi c(T-t)-4k}\,. \end{aligned} \tag{12}$$

And now we use the fact that $h(0) = 0$ and $h(q(T)) = bq^2(T)$ to show

$$\begin{aligned} 0 &= \left(\frac{1}{2}\sqrt{c}q(T)-\sqrt{b}q(T)\right)^2 \\ 0 &= q^2(T)\left(\frac{1}{2}\sqrt{c}-\sqrt{b}\right)^2 \\ c &= 4b\,, \end{aligned}$$

hence

$$\begin{aligned} h(q(t)) &= \left(\frac{1}{2}\sqrt{4b}(q(T)-q(t)) - \sqrt{h(q(T))}\right)^2 \\ &= \left(\sqrt{b}(q(T)-q(t)) - \sqrt{b}q(T)\right)^2 \\ &= bq^2(t)\,. \end{aligned}$$

Putting these results together we obtain

$$\begin{aligned} V(q,t) &= g(t)h(q) \\ &= \frac{a\phi 4k}{a\phi c(T-t)-4k} bq^2(t) \\ &= \frac{a\phi 4k}{a\phi 4b(T-t)-4k} bq^2(t) \\ &= \frac{ab\phi k}{a\phi b(T-t)-k} q^2(t) \\ &= \frac{\phi k}{\phi(T-t)-k} q^2(t)\,. \end{aligned}$$

And finally we write

$$J(q,t) = \frac{\phi k}{\phi(T-t)-k} q^2(t) + q(t)S(t)\,.$$

Now we use (6) to write

$$\begin{aligned} v^*(t) &= \frac{J_q - S}{2k} \\ &= \frac{2\frac{\phi k}{\phi(T-t)-k} q(t) + S(t) - S(t)}{2k} \\ &= \frac{q(t)}{T-t-k\phi^{-1}}\,. \end{aligned} \tag{13}$$

Note that this result is very intuitive. Compared to the case where the agent must reach time T with zero inventory left to liquidate, when there is a penalty ϕ the agent will always liquidate at a rate smaller than $q(t)/(T-t)$. Recall that $k<0$ so it is trivial to see that $T-t-k\phi^{-1} > T-t$ for $t \leq T$.

Moreover, it is easy to check that

$$\lim_{\phi\to\infty} v^*(t) = \frac{q(t)}{T-t}$$

which, as expected, is the solution we derived before when the agent was forced to end up with zero inventory $q(T)=0$. In other words, the penalisation arising from not liquidating the entire position is too costly so the agent guarantees (no matter what) that $q(T)=0$.